PARAMETERS AND DUALITY FOR THE METAPLECTIC GEOMETRIC LANGLANDS THEORY

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Abstract. This is a corrected version of the paper, and it differs substantially from the original one.

We introduce the space of parameters for the metaplectic Langlands theory as *factorization gerbes* on the affine Grassmannian, and develop metaplectic Langlands duality in the incarnation of the metaplectic geometric Satake functor.

We formulate a conjecture in the context of the global metaplectic Langlands theory, which is a metaplectic version of the "vanishing theorem" of [Ga5, Theorem 4.5.2].

Introduction

0.1. **What is this paper about?** The goal of this paper is to provide a summary of the metaplectic Langlands theory. Our main objectives are:

–Description of the set (rather, space) of parameters for the metaplectic Langlands theory;

–Construction of the *metaplectic Langlands dual* (see Sect. 0.1.6 for what we mean by this);

–The statement of the *metaplectic geometric Satake*.

0.1.1. *The metaplectic setting.* Let $\mathbf{F}$ be a local field and G an algebraic group over $\mathbf{F}$. The classical representation theory of locally compact groups studies (smooth) representations of the group $G(\mathbf{F})$ on vector spaces over another field E. Suppose now that we are given a central extension

$$1 \to E^\times \to \widetilde{G(\mathbf{F})} \to G(\mathbf{F}) \to 1. \tag{0.1}$$

We can then study representations of $\widetilde{G(\mathbf{F})}$ on which the central $E^\times$ acts by the tautological character. We will refer to (0.1) as a *local metaplectic extension* of $G(\mathbf{F})$, and to the above category of representations as *metaplectic representations* of $G(\mathbf{F})$ corresponding to the extension (0.1).

Let now $\mathbf{F}$ be a global field, and let $\mathbb{A}_\mathbf{F}$ be the corresponding ring of adèles. Let us be given a central extension

$$1 \to E^\times \to \widetilde{G(\mathbb{A}_\mathbf{F})} \to G(\mathbb{A}_\mathbf{F}) \to 1, \tag{0.2}$$

equipped with a splitting over $G(\mathbf{F}) \hookrightarrow G(\mathbb{A}_\mathbf{F})$.

We can then study the space of E-valued functions on the quotient $\widetilde{G(\mathbb{A}_\mathbf{F})}/G(\mathbf{F})$, on which the central $E^\times$ acts by the tautological character. We will refer to (0.2) as a *global metaplectic extension* of $G(\mathbf{F})$, and to the above space of functions as *metaplectic automorphic functions* on $G(\mathbf{F})$ corresponding to the extension (0.2).

There has been a renewed interest in the study of metaplectic representations and metaplectic automorphic functions, e.g., by B.Brubaker–D.Bump–S.Friedberg, P.McNamara, W.T.Gan–F.Gao.

M. Weissman has initiated a program of constructing the L-groups corresponding to metaplectic extensions, to be used in the formulation of the Langlands program in the metaplectic setting, see [We].

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0.1.2. *Parameters for metaplectic extensions.* In order to construct metaplectic extensions, in both the local and global settings, one starts with a datum of algebro-geometric nature. Namely, one usually takes as an input what we call a *Brylinski-Deligne datum*, by which we mean a central extension

$$1 \to (K_2)_{\mathrm{Zar}} \to \widetilde{G} \to G \to 1, \tag{0.3}$$

of sheaves of groups on the big Zariski site of $\mathbf{F}$, where $(K_2)_{\mathrm{Zar}}$ is the sheafification of the presheaf of abelian groups that assigns to an affine scheme $S = \mathrm{Spec}(A)$ the group $K_2(A)$.

For a local field $\mathbf{F}$, let $\mathbf{f}$ denote its residue field and let us choose a homomorphism

$$\mathbf{f}^\times \to E^\times. \tag{0.4}$$

Then taking the group of $\mathbf{F}$-points of $\widetilde{G}$ and pushing out with respect to

$$K_2(\mathbf{F}) \overset{\text{symbol}}{\longrightarrow} \mathbf{f}^\times \to E^\times,$$

we obtain a central extension (0.1). A similar procedure applies also in the global setting.

0.1.3. *The geometric theory.* Let k be a ground field and let G be a reductive group over k.

In the local geometric Langlands theory one considers the loop group $G(\!(t)\!)$ along with its action on various spaces, such as the affine Grassmannian $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$. Specfically one studies the behavior of categories of sheaves[1] on such spaces with respect to this action.

In the global geometric Langlands theory one considers a smooth proper curve X, and one studies the stack Bun_G that classifies principal G-bundles on X. The main object of investigation is the category of sheaves on Bun_G.

There are multiple ways in which the local and global theories interact. For example, given a (k-rational) point $x \in X$, and identifying the local ring $\mathcal{O}_x$ of X at x with $k[\![t]\!]$, we have the map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G, \tag{0.5}$$

where we interpret Gr_G as the moduli space of principal G-bundles on X, trivialized over $X - x$.

0.1.4. *The setting of metaplectic geometric Langlands theory.* Let E denote the field of coefficients of the sheaf theory that we consider. Recall (see Sect. 1.7.4) that if $\mathcal{Y}$ is a space[2] and $\mathcal{G}$ is a $E^\times$-gerbe on $\mathcal{Y}$, we can twist the category of sheaves on $\mathcal{Y}$, and obtain a new category, denoted

$$\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y}).$$

In the local metaplectic Langlands theory, the input datum (which is an analog of a central extension (0.1)) is an $E^\times$-gerbe over the loop group $G(\!(t)\!)$ that behaves *multiplicatively*, i.e., one that is compatible with the group-law on $G(\!(t)\!)$.

Similarly, whenever we consider an action of $G(\!(t)\!)$ on $\mathcal{Y}$, we equip $\mathcal{Y}$ with $E^\times$-gerbe that is compatible with the given multiplicative gerbe on $G(\!(t)\!)$. In this case we say that the category $\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y})$ carries a *twisted* action of $G(\!(t)\!)$, where the parameter of the twist is our gerbe on $G(\!(t)\!)$.

In the global setting we consider a gerbe $\mathcal{G}$ over Bun_G, and the corresponding category $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Bun}_G)$ of twisted sheaves.

Now, if we want to consider the local vs. global interaction, we need a compatibility structure on our gerbes. For example, we need that for every point $x \in X$, the pullback along (0.5) of the given gerbe on Bun_G be a gerbe compatible with some given multiplicative gerbe on $G(\!(t)\!)$.

So, it is natural to seek an algebro-geometric datum, akin to (0.3), that would provide such a compatible family of gerbes.

[1] See Sect. 1.5 for what we mean by the category of sheaves.

[2] By a "space" we mean a scheme, stack, ind-scheme, or more generally a *prestack*, see Sect. 1.2 for what the latter word means.

0.1.5. *Geometric metaplectic datum.* It turns out that such a datum (let us call it "the geometric metaplectic datum") is not difficult to describe, see Sect. 2.4.1 below. It amounts to the datum of a *factorization gerbe* with respect to $E^\times$ on the *affine Grassmannian*[3] Gr_G of the group G.

In a way, this answer is more elementary than (0.3) in that we are dealing with étale cohomology rather than K-theory.

Moreover, in the original metaplectic setting, if the global field $\mathbf{F}$ is the function field corresponding to the curve X over a finite ground field k, a geometric metaplectic datum gives rise directly to an extension (0.2).

Finally, a Brylinski-Deligne datum (i.e., an extension (0.3)) and a choice of a character $k^\times \to E^\times$ gives rise to a geometric metaplectic datum, see Sect. 3.4.

Thus, we could venture into saying that a geometric metaplectic datum is a more economical way, sufficient for most purposes, to encode also the datum needed to set up the classical metaplectic representation/automorphic theory.

0.1.6. *The metaplectic Langlands dual.* Given a geometric metaplectic datum, i.e., a factorization gerbe $\mathcal{G}$ on Gr_G, we attach to it a certain reductive group H, a gerbe $\mathcal{G}_{Z_H}$ on X with respect to the center Z_H of H, and a character $\epsilon : \pm 1 \to Z_H$. We refer to the triple

$$(H, \mathcal{G}_{Z_H}, \epsilon)$$

as the *metaplectic Langlands dual* datum corresponding to $\mathcal{G}$.

The datum of $\mathcal{G}_{Z_H}$ determines the notion of twisted H-local system of X. Such twisted local systems are supposed to play a role vis-à-vis metaplectic representations/automorphic functions of G parallel to that of usual $\check{G}$-local systems vis-à-vis usual representations/automorphic functions of G.

For example, in the context of the global geometric theory (in the setting of D-modules), we will propose a conjecture (namely, Conjecture 9.6.2) that says that the monoidal category $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_{Z_H}}\right)$ of quasi-coherent sheaves on the stack $\mathrm{LocSys}_H^{\mathcal{G}_{Z_H}}$ classifying such twisted local systems, *acts* on the category $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Bun}_G)$.

The geometric input for such an action is provided by the metaplectic geometric Satake functor, see Sect. 9.

Presumably, in the arithmetic context, the above notion of twisted H-local system coincides with that of homomorphism of the (arithmetic) fundamental group of X to Weissman's L-group.

0.2. "Metaplectic" vs "Quantum".

In the paper [Ga4], a program was proposed towards the *quantum Langlands theory.* Let us comment on the terminological difference between "metaplectic" and "quantum", and how the two theories are supposed to be related.

0.2.1. If $\mathcal{Y}$ is a scheme (resp., or more generally, a prestack) we can talk about $E^\times$-gerbes on it. As was mentioned above, such gerbes on various spaces associated with the group G and the geometry of the curve X are parameters for the metaplectic Langlands theory.

Let us now assume that k has characteristic 0, and let us work in the context of D-modules. Then, in addition to the notion of $E^\times$-gerbe on $\mathcal{Y}$, there is another one: that of *twisting* (see [GR1, Sect. 6]).

There is a forgetful map from twistings to gerbes. Roughly speaking, a gerbe $\mathcal{G}$ on $\mathcal{Y}$ defines the corresponding twisted category of sheaves (=D-modules) $\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y}) = \text{D-mod}_{\mathcal{G}}(\mathcal{Y})$, while if we lift our gerbe to a twsiting, we also have a forgetful functor

$$\text{D-mod}_{\mathcal{G}}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}).$$

[3] Here the affine Grassmannian appears in its factorization (a.k.a, Beilinson-Drinfeld) incarnation. I.e., it is a prestack mapping to the Ran space of X, rather than $G((t))/G[[t]]$, which corresponds to a particular point of X.

0.2.2. For the *quantum* Langlands theory, our parameter will be a factorizable *twisting* on the affine Grassmannian, which one can also interpret as a *Kac-Moody level*; we will denote it by κ.

Thus, for example, in the global quantum geometric Langlands theory, we consider the category

$$\text{D-mod}_\kappa(\text{Bun}_G),$$

which is the same as $\text{Shv}_{\mathcal{G}}(\text{Bun}_G)$, where $\mathcal{G}$ is the gerbe corresponding to κ.

As was mentioned above, the additional piece of datum that the twisting "buys" us is the forgetful functor

$$\text{D-mod}_\kappa(\text{Bun}_G) \to \text{QCoh}(\text{Bun}_G).$$

In the TQFT interpretation of geometric Langlands, this forgetful functor is called "the big brane". It allows us to relate the category $\text{D-mod}_\kappa(\text{Bun}_G)$ to representations of the Kac-Moody algebra attached to G and the level κ.

0.2.3. Consider the usual Langlands dual group $\check{G}$ of G, and if κ is non-degenerate, it gives rise to a twisting, denoted $-\kappa^{-1}$, on the affine Grassmannian $\text{Gr}_{\check{G}}$ of $\check{G}$.

In the global quantum geometric theory one expects to have an equivalence of categories

$$\text{D-mod}_\kappa(\text{Bun}_G) \simeq \text{D-mod}_{-\kappa^{-1}}(\text{Bun}_{\check{G}}). \tag{0.6}$$

We refer to (0.6) as the *global quantum Langlands equivalence*.

0.2.4. *How are the two theories related?* The relationship between the equivalence (0.6) and the metaplectic Langlands dual is the following:

Let $\mathcal{G}$ (resp., $\check{\mathcal{G}}$) be the gerbe on Gr_G (resp., $\text{Gr}_{\check{G}}$) corresponding to κ (resp., $-\kappa^{-1}$). We conjecture that the metaplectic Langlands dual data $(H, \mathcal{G}_{Z_H}, \epsilon)$ corresponding to $\mathcal{G}$ and $\check{\mathcal{G}}$ *are isomorphic*.

Furthermore, we conjecture that the resulting actions of

$$\text{QCoh}\left(\text{LocSys}_H^{\mathcal{G}_{Z_H}}\right)$$

on $\text{D-mod}_\kappa(\text{Bun}_G)$ and $\text{D-mod}_{-\kappa^{-1}}(\text{Bun}_{\check{G}})$, respectively (see Sect. 0.1.6 above) are intertwined by the equivalence (0.6).

0.3. **What is actually done in this paper?** Technically, our focus is on the geometric metaplectic theory, with the goal of constructing the *metaplectic geometric Satake* functor.

0.3.1. The mathematical content of this paper is the following:

–We define a geometric metaplectic datum to be a factorization gerbe on the (factorization version) of affine Grassmannian Gr_G. This is done in Sect. 2.

–We formulate the classification result that describes factorization gerbes on Gr_G in terms of étale cohomology on the classifying stack BG of G. This is done in Sect. 3.

This classification result is inspired by an analogous one in the topological setting, explained to us by J. Lurie.

–We make an explicit analysis of the space of factorization gerbes in the case when $G = T$ is a torus. This is done in Sect. 4.

–We study the relationship between factorization gerbes on Gr_G and those on Gr_M, where M is the Levi quotient of a parabolic $P \subset G$. This is done in Sect. 5.

The main point is that the naive map from factorization gerbes on Gr_G to those on Gr_M needs to be corrected by a gerbe that has to do with signs. It is this correction that is responsible for the fact that the usual geometric Satake does not quite produce the category $\text{Rep}(\check{G})$, but rather its modification where we alter the commutativity constraint by the element $2\rho(-1) \in Z(\check{G})$.

–We define the notion of *metaplectic Langlands dual* datum, denoted $(H, \mathcal{G}_{Z_H}, \epsilon)$, attached to a given geometric metaplectic datum $\mathcal{G}$. We introduce the notion of $\mathcal{G}_{Z_H}$-twisted H-local system on X; when we work with D-modules, these local systems are k-points of a (derived) algebraic stack, denoted $\mathrm{LocSys}_H^{\mathcal{G}_{Z_H}}$. This is done in Sect. 6.

–We show that a factorization gerbe on Gr_G gives rise to a *multiplicative* gerbe over the loop group $G(\!(t)\!)$ for every point $x \in X$. Moreover, these multiplicative gerbes also admit a natural factorization structure when instead of a single point x we consider the entire Ran space. This is done in Sect. 7.

–We introduce the various twisted versions of the category of representations of a reductive group, and the associated notion of twisted local system. This is done in Sect. 8.

–We define metaplectic geometric Satake as a functor between *factorization categories* over the Ran space. This is done in Sect. 9.

–We formulate a conjecture about the action of the monoidal category $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_{Z_H}}\right)$ on $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Bun}_G)$. This is also done in Sect. 9.

0.3.2. *A disclaimer.* Although most of the items listed in Sect. 0.3.1 have not appeared in the previously existing literature, this is mainly due to the fact that these earlier sources, specifically the paper [FL] of M. Finkelberg and the second-named author and the paper [Re] of R. Reich, did not use the language of ∞-categories, while containing most of the relevant mathematics.

So, one can regard the present paper as a summary of results that are "almost known", but formulated in the language that is better adapted to the modern take on the geometric Langlands theory[4].

We felt that there was a need for such a summary in order to facilitate further research in this area.

Correspondingly, our focus is on statements, rather than proofs. Most of the omitted proofs can be found in either [FL] or [Re], or can be obtained from other sources cited in the paper.

Below we give some details on the relation of contents of this paper and some of previously existing literature.

0.3.3. *Relation to other work: geometric theory.* As was just mentioned, a significant part of this paper is devoted to reformulating the results of [FL] and [Re] in a way tailored for the needs of the geometric metaplectic theory.

The paper [Re] develops the theory of factorization gerbes on Gr_G (in *loc. cit.* they are called "symmetric factorizable gerbes"). One caveat is that in the setting of [Re] one works with schemes over $\mathbb{C}$ and sheaves in the analytic topology, while in the present paper we work over a general ground field and étale sheaves.

The main points of the theory developed in [Re] are the description of the *homotopy groups* of the space of factorization gerbes (but not of the space itself; the latter is done in Sect. 3 of the present paper), and the fact that a factorization gerbe on Gr_G gives rise to a multiplicative gerbe on (the factorization version of) the loop group (we summarize this construction in Sect. 7 of the present paper).

The proofs of the corresponding results in [Re] are obtained by reducing assertions for a reductive group G to that for its Cartan subgroup, and an explicit analysis for tori. We do not reproduce these proofs in the present paper.

In both [FL] and [Re], metaplectic geometric Satake is stated as an equivalence of certain abelian categories. In [FL], this is an equivalence of symmetric monoidal categories (corresponding to a chosen point $x \in X$), for a particular class of gerbes (namely, ones obtained from the determinant line bundle).

In [Re] more general gerbes are considered and the factorization structure on both sides of the equivalence is taken into account. Our version of metaplectic geometric Satake is a statement at the level of DG categories; it is no longer an equivalence, but rather a functor in one direction, between *monoidal factorization categories*. In this form, our formulation is a simple consequence of that of [Re].

[4]This excludes, however, the material in Sect. 9.5 and the statement of Conjecture 9.6.2 (the latter is new, to the best of our knowledge)

0.3.4. *Relation to other work: arithmetic theory.* As was already mentioned above, our notion of the metaplectic Langlands dual datum is probably equivalent to the datum constructed by M. Weissman in [We] for his definition of the L-group.

0.4. **Conventions.**

0.4.1. *Algebraic geometry.* In the main body of the paper we will be working over a fixed ground field k, assumed algebraically closed.

For arithmetic applications one would also be interested in the case of k being a finite field $\mathbb{F}_q$. However, since all the constructions in this paper are canonical, the results over $\mathbb{F}_q$ can be deduced from those over $\overline{\mathbb{F}}_q$ by Galois descent.

We will denote by X a smooth connected algebraic curve over k (we *do not* need X to be complete).

For the purposes of this paper, we *do not need* derived algebraic geometry, with the exception of Sects. 8.4 and 9.6 (where we discuss the stack of local systems, which is a derived object).

In the last two sections of the paper we will make an extensive use of algebro-geometric objects more general than schemes, namely, prestacks. We recall the definition of prestacks in Sect. 1.2, and refer the reader to [GR2, Vol. 1, Chapter 2] for a more detailed discussion.

0.4.2. *Coefficients.* Gerbes, which constitute the object of study of this paper, can be used to *twist* categories of sheaves, see Sect. 1.7.

We will mostly work with the *sheaf theory* of D-modules. We will denote by E the field of coefficients of our sheaves (assumed algebraically closed and of characteristic 0).

0.4.3. *Groups.* We will work with a fixed connected algebraic group G over k; our main interest is the case when G is reductive.

We will denote by Λ the coweight lattice of G and by $\check{\Lambda}$ its dual, i.e., the weight lattice.

We will denote by $\alpha_i \in \Lambda$ (resp., $\check{\alpha}_i \in \check{\Lambda}$) the simple coroots (resp., roots), where i runs over the set of vertices of the Dynkin diagram of G.

If G is reductive, we denote by $\check{G}$ its Langlands dual, viewed as a reductive group over E.

0.4.4. *The usage of higher category theory.* Although, as we have said above, we do not need derived algebraic geometry, we do need higher category theory. However, we only really need ∞-categories for one type of manipulation: in order to define the notion of the *category of sheaves* on a given prestack (and a related notion of a *sheaf of categories* over a prestack); we will recall the corresponding definitions in Sects. 1.2 and 1.6), respectively. These definitions involve the procedure of taking the limit, and the language of higher categories is the adequate framework for doing so.

In their turn, sheaves of categories on prestacks appear for us as follows: the metaplectic spherical Hecke category, which is the recipient of the metaplectic geometric Satake functor (and hence is of primary interest for us), is a sheaf of categories over the Ran space.

Thus, the reader who is only interested in the notion of geometric metaplectic datum (and does not wish to proceed to metaplectic geometric Satake) *does not* need higher category theory either.

0.4.5. *Glossary of ∞-categories.* We will now recall several most common pieces of notation, pertaining to ∞-categories, used in this paper. We refer the reader to [Lu1, Lu2] for the foundations of the theory, or [GR2, Vol. 1, Chapter 1] for a concise summary.

We denote by Spc the ∞-category of spaces. We denote by $*$ the point-space. For a space $\mathcal{S}$, we denote by $\pi_0(\mathcal{S})$ its *set of connected components.* If $\mathcal{S}$ is a space we can view it as an ∞-category; its objects are also called the *points* of $\mathcal{S}$.

For an ∞-category $\mathbf{C}$ and two objects $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$, we let $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Spc}$ denote the mapping space between them.

For an object $\mathbf{c} \in \mathbf{C}$ we let $\mathbf{C}_{\mathbf{c}/}$ (resp., $\mathbf{C}_{/\mathbf{c}}$) denote the corresponding under-category (resp., over-category).

In several places in the paper we will need the notion of left (resp., right) Kan extension. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor, and let $\mathbf{E}$ is an ∞-category with colimits. Then the functor

$$\mathrm{Funct}(\mathbf{D}, \mathbf{E}) \xrightarrow{\circ F} \mathrm{Funct}(\mathbf{C}, \mathbf{E}) \tag{0.7}$$

admits a left adjoint, called the functor of *left Kan extension* along F.

For $\Phi \in \mathrm{Funct}(\mathbf{C}, \mathbf{E})$, the value of its left Kan extension on $\mathbf{d} \in \mathbf{D}$ is calculated by the formula

$$\underset{(\mathbf{c}, F(\mathbf{c}) \to \mathbf{d}) \in \mathbf{C} \underset{\mathbf{D}}{\times} \mathbf{D}_{/\mathbf{d}}}{\mathrm{colim}} \Phi(\mathbf{c}).$$

The notion of *right Kan extension* is obtained similarly: it is the right adjoint of (0.7); the formula for it is given by

$$\underset{(\mathbf{c}, \mathbf{d} \to F(\mathbf{c})) \in \mathbf{C} \underset{\mathbf{D}}{\times} \mathbf{D}_{\mathbf{d}/}}{\lim} \Phi(\mathbf{c}).$$

0.4.6. *DG categories.* We let DGCat denote the ∞-category of DG categories over E, see [GR2, Vol. 1, Chapter 1, Sect. 10.3.3] (in *loc.cit.* it is denoted $\mathrm{DGCat}_{\mathrm{cont}}$). I.e., we will assume all our DG categories to be *cocomplete* and we allow only colimit-preserving functors as 1-morphisms.

For example, let R be a DG associative algebra over k. Then we let R-mod denote the corresponding DG category of R-modules (i.e., its homotopy category is the usual derived category of the abelian category of R-modules, without any boundedness conditions).

For an algebraic group H over E, we let $\mathrm{Rep}(H)$ denote the DG category of representations of H, see, e.g., [DrGa, Sects. 6.4.3-6.4.4].

The piece of structure on DGCat that we will exploit extensively is the operation of tensor product, which makes DGCat into a symmetric monoidal category.

For a pair of DG associative algebras R_1 and R_2, we have:

$$(R_1\text{-mod}) \otimes (R_2\text{-mod}) \simeq (R_1 \otimes R_2)\text{-mod}.$$

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1. Preliminaries

This section is included for the reader's convenience: we review some constructions in algebraic geometry that involve higher category theory. The reader having a basic familiarity with this material should feel free to skip it.

1.1. **Some higher algebra.** To facilitate the reader's task, in this subsection we will review some notions from higher algebra that will be used in this paper. The main reference for this material is [Lu2].

We should emphasize that for the purposes of studying geometric metaplectic data, we only need higher algebra in ∞-categories that are $(n, 1)$-categories for small values of n. The corresponding objects can be studied in a hands-on way (i.e., we do not need the full extent of higher category theory).

The only place where we really need higher categories is for working with categories of sheaves on prestacks.

1.1.1. *Monoids and groups.* In any ∞-category $\mathbf{C}$ that contains finite products (including the empty finite product, i.e., a final object), it makes sense to consider the category $\mathrm{Monoid}(\mathbf{C})$ of monoid-objects in $\mathbf{C}$. This is a full subcategory in the category of *simplicial objects* of $\mathbf{C}$ (i.e., $\mathrm{Funct}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathbf{C})$) that consists of objects, satisfying the Segal condition.

One defines the category commutative monoids $\mathrm{ComMonoid}(\mathbf{C})$ in $\mathbf{C}$ similarly, but using the category Fin_s of pointed finite sets instead of $\boldsymbol{\Delta}^{\mathrm{op}}$.

For example, take $\mathbf{C} = \infty\text{-Cat}$. In this way we obtain the notion of monoidal (resp., symmetric monoidal) category.

1.1.2. The ∞-category $\mathrm{Monoid}(\mathbf{C})$ (resp., $\mathrm{ComMonoid}(\mathbf{C})$) contains the full subcategory of group-like objects, denoted $\mathrm{Grp}(\mathbf{C})$ (resp., $\mathrm{ComGrp}(\mathbf{C})$).

Let $\mathrm{Ptd}(\mathbf{C})$ be the category of pointed objects in $\mathbf{C}$, i.e., $\mathbf{C}_{*/}$, where $*$ denotes the final object in $\mathbf{C}$. We have the loop functor

$$\Omega : \mathrm{Ptd}(\mathbf{C}) \to \mathrm{Grp}(\mathbf{C}), \quad (* \to \mathbf{c}) \mapsto * \underset{\mathbf{c}}{\times} *.$$

The left adjoint of this functor (if it exists) is called the functor of the *classifying space* and is denoted

$$H \mapsto B(H).$$

1.1.3. For $\mathbf{C} = \mathrm{Spc}$ (or $\mathbf{C} = \mathrm{Funct}(\mathbf{D}, \mathrm{Spc})$ for some other category $\mathbf{D}$), the functor B does exist and is fully faithful. The essential image of $B : \mathrm{Grp}(\mathrm{Spc}) \to \mathrm{Ptd}(\mathrm{Spc})$ consists of *connected* spaces.

For an object $\mathcal{S} \in \mathrm{Ptd}(\mathrm{Spc})$, its i-th homotopy group $\pi_i(\mathcal{S})$ is defined to be

$$\pi_0(\Omega^i(\mathcal{S})),$$

where $\Omega^i(\mathcal{S})$ is viewed as a plain object of Spc.

1.1.4. For $k \geq 0$, we introduce the category $\mathbb{E}_k(\mathbf{C})$ of $\mathbb{E}_k$-objects in $\mathbf{C}$ inductively, by setting

$$\mathbb{E}_0(\mathbf{C}) = \mathrm{Ptd}(\mathbf{C})$$

and

$$\mathbb{E}_k(\mathbf{C}) = \mathrm{Monoid}(\mathbb{E}_{k-1}(\mathbf{C})).$$

Let $\mathbb{E}_k^{\text{grp-like}}(\mathbf{C}) \subset \mathbb{E}_k(\mathbf{C})$ the full subcategory of group-like objects, defined to be the preimage of

$$\mathrm{Grp}(\mathbf{C}) \subset \mathrm{Monoid}(\mathbf{C}) = \mathbb{E}_1(\mathbf{C})$$

under any of the k possible forgetful functors $\mathbb{E}_k(\mathbf{C}) \to \mathbb{E}_1(\mathbf{C})$.

The functor $B : \mathrm{Grp}(\mathbf{C}) \to \mathrm{Ptd}(\mathbf{C})$ (if it exists) induces a functor

$$B : \mathbb{E}_k^{\text{grp-like}}(\mathbf{C}) \rightleftarrows \mathbb{E}_{k-1}^{\text{grp-like}}(\mathbf{C}) : \Omega$$

for $k \geq 2$, which is the left adjoint of

$$\Omega : \mathbb{E}_{k-1}^{\text{grp-like}}(\mathbf{C}) \to \mathbb{E}_k^{\text{grp-like}}(\mathbf{C}).$$

For $i \leq k$ we let B^i denote the resulting functor

$$\mathbb{E}_k^{\text{grp-like}}(\mathbf{C}) \to \mathbb{E}_{k-i}^{\text{grp-like}}(\mathbf{C}).$$

1.1.5. One shows that the forgetful functor

$$\mathrm{Monoid}(\mathrm{ComMonoid}(\mathbf{C})) \to \mathrm{ComMonoid}(\mathbf{C})$$

is an equivalence.

This implies that for every k we have a canonically defined functor

$$\mathrm{ComMonoid}(\mathbf{C}) \to \mathbb{E}_k(\mathbf{C}),$$

and these functors are compatible with the forgetful functors $\mathbb{E}_k(\mathbf{C}) \to \mathbb{E}_{k-1}(\mathbf{C})$. Thus, we obtain a canonically defined functor

$$\mathrm{ComMonoid}(\mathbf{C}) \to \mathbb{E}_\infty(\mathbf{C}) := \underset{\longleftarrow}{\lim}\, \mathbb{E}_k(\mathbf{C}). \tag{1.1}$$

It is known (see [Lu2, Remark 5.2.6.26]) that the functor (1.1) is an equivalence.

1.1.6. The category

$$\mathrm{ComGrp}(\mathrm{Spc}) \simeq \mathbb{E}_\infty^{\text{grp-like}}(\mathrm{Spc})$$

identifies with that of connective spectra.

For any $i \geq 0$, we have the mutually adjoint endo-functors

$$B^i : \mathrm{ComGrp}(\mathrm{Spc}) \rightleftarrows \mathrm{ComGrp}(\mathrm{Spc}) : \Omega^i$$

with B^i being fully faithful.

1.1.7. Let A be an object of $\mathbb{E}_2^{\text{grp-like}}(\mathrm{Spc})$, so that $B(A)$ is an object of $\mathrm{Grp}(\mathrm{Spc})$.

By an action of A on an ∞-category $\mathbf{C}$ we shall mean an action of $B(A)$ on $\mathbf{C}$ as an object of ∞-Cat.

For example, taking $A = E^\times \in \mathrm{ComGrp}(\mathrm{Spc})$, we obtain an action of $E^\times$ on any DG category. Explicitly, we identify $B(E^\times)$ with the space of $E^\times$-torsors, i.e., lines, and the action in question sends a line ℓ to the endofunctor

$$\mathbf{c} \mapsto \ell \otimes \mathbf{c}.$$

1.2. **Prestacks.**

1.2.1. Let $\mathrm{Sch}^{\mathrm{aff}}$ be the category of *classical* affine schemes over k.

We let PreStk denote the category of all (accessible) functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

We shall say that an object of PreStk is n-truncated if it takes values in the full subcategory of Spc that consists of n-truncated spaces[5].

The ∞-category of n-truncated prestacks is in fact an $(n+1,1)$-category. For small values of n, one can work with it avoiding the full machinery of higher category theory.

Remark 1.2.2. There will be two types of prestacks in this paper: the "source" type and the "target" type. The source type will be various geometric objects associated to the group G and the curve X, such as the Ran space, affine Grassmannian Gr_G, the loop group $\mathfrak{L}(G)$, etc. These prestacks are 0-truncated, i.e., they take values in the full subcategory

$$\mathrm{Sets} \subset \mathrm{Spc}\,.$$

There will be a few other source prestacks (such as Bun_G or quotients of Gr_G by groups acting on it) and they will be 1-truncated (i.e., they take values in the full subcategory of Spc spanned by ordinary groupoids).

When we talk about the category of sheaves on a prestack, the prestack in question will be typically of the source type.

[5]An object of Spc is said to be truncated, if for any choice of a base point, its homotopy groups $\pi_{n'}$ vanish for $n' > n$.

The target prestacks will be of the form $B^n(\mathcal{A})$ (see Sect. 1.1.5), where $\mathcal{A}$ is a prestack that takes a constant value A, where A is a *discrete* abelian group (or its sheafification in, say, the étale topology, denoted $B^n_{\mathrm{et}}(\mathcal{A})$, see below). Such a prestack is n-truncated. When n is small, they can be described in a hands-on way by specifying objects, 1-morphisms, 2-morphisms, etc; in this paper n will be ≤ 4, and in most cases ≤ 2.

For example, we will often use the notion of a *multiplicative* A-gerbe on a group-prestack $\mathcal{H}$. Such an object is the same as a map of group-prestacks

$$\mathcal{H} \to B^2_{\mathrm{et}}(A).$$

1.2.3. Let $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \subset \mathrm{Sch}^{\mathrm{aff}}$ denote the full subcategory of affine schemes of finite type. Functorially, thus subcategory can be characterized as consisting of *co-compact* objects, i.e., $S \in \mathrm{Sch}^{\mathrm{aff}}$ if and only if the functor

$$S' \mapsto \mathrm{Hom}(S', S)$$

commutes with filtered limits.

Moreover, every object of $\mathrm{Sch}^{\mathrm{aff}}$ can be written as a filtered limit of objects of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

The two facts mentioned above combine to the statement that we can identify $\mathrm{Sch}^{\mathrm{aff}}$ with the pro-completion of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

1.2.4. We let

$$\mathrm{PreStk}_{\mathrm{lft}} \subset \mathrm{PreStk}$$

denote the full subcategory consisting of functors that preserve filtered colimits. I.e., $\mathcal{Y} \in \mathrm{PreStk}$ is locally of finite type if for

$$S = \lim_{\alpha} S_\alpha,$$

the map

$$\underset{\alpha}{\mathrm{colim}}\, \mathrm{Maps}(S_\alpha, \mathcal{Y}) \to \mathrm{Maps}(S, \mathcal{Y})$$

is an isomorphism in Spc.

The functors of restriction and left Kan extension along

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \tag{1.2}$$

define an equivalence between $\mathrm{PreStk}_{\mathrm{lft}}$ and the category of all functors

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

If $\mathcal{F} \in \mathrm{PreStk}_{\mathrm{lft}}$ is such that its restriction to $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ takes values in n-truncated spaces, then $\mathcal{Y}$ itself is n-truncated.

1.2.5. In this paper we will work with the étale topology on $\mathrm{Sch}^{\mathrm{aff}}$. Let

$$\mathrm{Stk} \subset \mathrm{PreStk}$$

be the full subcategory consisting of objects that satisfy descent for Čech nerves of étale morphisms, see [GR2, Vol. 1, Chapter 2, Sect. 2.3.1].

The inclusion $\mathrm{Stk} \hookrightarrow \mathrm{PreStk}$ admits a left adjoint, called the functor of étale sheafification, denoted L_{et}.

This functor sends n-truncated objects to n-truncated objects.

1.2.6. Denote

$$\mathrm{Stk}_{\mathrm{lft}} := \mathrm{Stk} \cap \mathrm{PreStk}_{\mathrm{lft}} \subset \mathrm{PreStk}\,.$$

However, we can consider a different subcategory of $\mathrm{PreStk}_{\mathrm{lft}}$, denoted $\mathrm{NearStk}_{\mathrm{lft}}$. Namely, identifying $\mathrm{PreStk}_{\mathrm{lft}}$ with $\mathrm{Funct}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}, \mathrm{Spc})$, we can consider the full subcategory consisting of functors that satisfy descent for Čech covers of étale morphisms (within $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$).

Restriction along (1.2) sends $\mathrm{Stk}_{\mathrm{lft}}$ to $\mathrm{NearStk}_{\mathrm{lft}}$. However, it is not true that the functor of left Kan extension along (1.2) sends $\mathrm{NearStk}_{\mathrm{lft}}$ to $\mathrm{Stk}_{\mathrm{lft}}$. However, the following weaker statement holds (see [GR2, Vol. 1, Chapter 2, Proposition 2.7.7]):

Lemma 1.2.7. *Assume that* $\mathcal{Y} \in \mathrm{Funct}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}, \mathrm{Spc})$ *is n-truncated for some n and belongs to* $\mathrm{NearStk}_{\mathrm{lft}}$. *Then the left Kan extension of $\mathcal{Y}$ along* (1.2) *belongs to* $\mathrm{Stk}_{\mathrm{lft}}$.

This formally implies:

Corollary 1.2.8. *If* $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ *is n-truncated for some n, then $L_{\mathrm{et}}(\mathcal{Y})$ belongs to* $\mathrm{Stk}_{\mathrm{lft}}$.

1.3. **Gerbes.**

1.3.1. Let $\mathcal{Y}$ be a prestack, and let $\mathcal{A}$ be a group-like $\mathbb{E}_n$-object in the category $\mathrm{PreStk}_{/\mathcal{Y}}$, for $n \geq 1$. In other words, for a given $(S \overset{y}{\to} \mathcal{Y}) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$, the space

$$\mathrm{Maps}(S, \mathcal{A}) \underset{\mathrm{Maps}(S,\mathcal{Y})}{\times} \{y\} \tag{1.3}$$

is a group-like $\mathbb{E}_n$-object of Spc, in a way functorial in (S, y).

We include the case of $n = \infty$, when we stipulate that $\mathcal{A}$ is a commutative group-object of $\mathrm{PreStk}_{/\mathcal{Y}}$. I.e., (1.3) should be a commutative group-object of Spc, i.e., a connective spectrum.

For any $0 \leq i \leq n$, we let $B^i(\mathcal{A})$ denote the i-fold classifying space of $\mathcal{A}$. This is a group-like $\mathbb{E}_{n-i}$-object in $\mathrm{PreStk}_{/\mathcal{Y}}$. For $i = 1$ we simply write $B(\mathcal{A})$ instead of $B^1(\mathcal{A})$.

1.3.2. We let $B^i_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})$ (resp., $B^i_{\mathrm{Zar},/\mathcal{Y}}(\mathcal{A})$) denote the étale (resp., Zariski) sheafification of $B^i(\mathcal{A})$ in the category $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$ (see [GR2, Vol. 1, Chapter 2, Sect. 2.3]). We will be interested in spaces of the form

$$\mathrm{Maps}_{/\mathcal{Y}}(\mathcal{Y}, B^i_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})), \tag{1.4}$$

where $\mathrm{Maps}_{/\mathcal{Y}}(-,-)$ is short-hand for $\mathrm{Maps}_{\mathrm{PreStk}_{/\mathcal{Y}}}(-,-)$.

Note that (1.4) is naturally a group-like $\mathbb{E}_{n-i}$-space (resp., a commutative group object in Spc if $n = \infty$).

1.3.3. In most examples, we will take $\mathcal{A}$ to be of the form $A \times \mathcal{Y}$, where A is a torsion abelian group, considered as a constant prestack. In this case

$$\mathrm{Maps}_{/\mathcal{Y}}(\mathcal{Y}, B^i_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})) \simeq \mathrm{Maps}(\mathcal{Y}, B^i_{\mathrm{et}}(A)).$$

Note that

$$\pi_j\left(\mathrm{Maps}(\mathcal{Y}, B^i_{\mathrm{et}}(A))\right) = \begin{cases} H^{i-j}_{\mathrm{et}}(\mathcal{Y}, A), & j \leq i; \\ 0, & j > i. \end{cases}$$

Here $H^\bullet_{\mathrm{et}}(\mathcal{Y}, A)$ refers to the étale cohomology of $\mathcal{Y}$ with coefficients in A. In other words, it is the cohomology of the object

$$\mathrm{C}^\bullet_{\mathrm{et}}(\mathcal{Y}, A) := \underset{(S,y) \in \mathrm{Sch}^{\mathrm{aff}}\,/\mathcal{Y}}{\underset{\longleftarrow}{\lim}} \mathrm{C}^\bullet_{\mathrm{et}}(S, A),$$

see [GL2, Construction 3.2.1.1].

1.3.4. Note also that in this case the functor

$$S \mapsto \mathrm{Maps}(S, B^i_{\mathrm{et}}(A)), \quad (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$$

identifies with the *left Kan extension* of its restriction to $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}$. I.e., if an affine scheme S is written as a filtered limit

$$S = \underset{\underset{\alpha}{\longleftarrow}}{\lim} S_\alpha, \quad S_\alpha \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},$$

then the map

$$\underset{\underset{\alpha}{\longrightarrow}}{\mathrm{colim}}\ \mathrm{Maps}(S_\alpha, B^i_{\mathrm{et}}(A)) \to \mathrm{Maps}(S, B^i_{\mathrm{et}}(A))$$

is an isomorphism (this latter assertion means that $B^i_{\mathrm{et}}(A)$ is locally of finite type as a prestack), see Corollary 1.2.8.

1.3.5. For $k = 1$, the points of the space

$$\mathrm{Tors}_{\mathcal{A}}(\mathcal{Y}) := \mathrm{Maps}_{/\mathcal{Y}}(\mathcal{Y}, B_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})) \tag{1.5}$$

are by definition $\mathcal{A}$-torsors on $\mathcal{Y}$.

1.3.6. Our primary interest is the cases of $k = 2$. We will call objects of the space

$$\mathrm{Ge}_{\mathcal{A}}(\mathcal{Y}) := \mathrm{Maps}_{/\mathcal{Y}}(\mathcal{Y}, B^2_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})). \tag{1.6}$$

$\mathcal{A}$-gerbes on $\mathcal{Y}$.

When $\mathcal{A}$ is of the form $A \times \mathcal{Y}$ (see Sect. 1.3.3 above), we will simply write $\mathrm{Ge}_A(\mathcal{Y})$.

1.4. **Gerbes coming from line bundles.** In this subsection we will be studying gerbes for a constant commutative group-prestack, corresponding to a torsion abelian group A. In what follows, we will be assuming that the orders of elements of A are co-prime to $\mathrm{char}(k)$.

1.4.1. Let $A(-1)$ denote the group

$$\underset{n \in \mathbb{N}}{\mathrm{colim}}\ \mathrm{Hom}(\mu_n, A).$$

In the above formula we regard $\mathbb{N}$ as a poset via

$$n' \geq n \Leftrightarrow n \mid n',$$

and in forming the above colimit the transition maps are given by

$$\mu_{n'} \overset{x \mapsto x^{\frac{n'}{n}}}{\twoheadrightarrow} \mu_n, \quad \text{for } n \mid n'. \tag{1.7}$$

For future reference, denote also

$$A(1) = \underset{n \in \mathbb{N}}{\mathrm{colim}} \left(\mu_{n'} \underset{\mathbb{Z}/n'\mathbb{Z}}{\otimes} A_{n\text{-tors}} \right),$$

where $A_{n\text{-tors}} \subset A$ is the subgroup of n-torsion elements, and in the above formula n' is any integer divisible by n.

1.4.2. We claim that to any line bundle $\mathcal{L}$ on a prestack $\mathcal{Y}$ and an element $a \in A(-1)$ one can canonically associate an A-gerbe, denoted $\mathcal{L}^a$, over $\mathcal{Y}$.

It suffices to perform this construction for $A = \mu_n$ and a coming from the identity map $\mu_n \to \mu_n$. In this case, the corresponding μ_n-gerbe will be denoted $\mathcal{L}^{\frac{1}{n}}$.

By definition, for an affine test scheme S over $\mathcal{Y}$, the value of $\mathcal{L}^{\frac{1}{n}}$ on S is the groupoid of pairs

$$(\mathcal{L}', (\mathcal{L}')^{\otimes n} \simeq \mathcal{L}|_S),$$

where $\mathcal{L}'$ is a line bundle on S.

Note that if $\mathcal{L}$ admits an n-th root $\mathcal{L}'$, then this $\mathcal{L}'$ determines a trivialization of $\mathcal{L}^{\frac{1}{n}}$.

Remark 1.4.3. We emphasize the notational difference between the μ_n-gerbe $\mathcal{L}^{\frac{1}{n}}$, and the line bundle $\mathcal{L}^{\otimes \frac{1}{n}}$, when the latter happens to exist. Namely, a choice of $\mathcal{L}^{\otimes \frac{1}{n}}$ defines a trivialization of the gerbe $\mathcal{L}^{\frac{1}{n}}$.

1.4.4. Let Y be a smooth scheme, and let $Z \subset Y$ be a subvariety of codimension one. Let $Z_i, i \in I$ denote the irreducible components of Z. For every i, let $\mathcal{O}(Z_i)$ denote the corresponding line bundle on Y, trivialized away from Z.

We obtain a homomorphism

$$\text{Maps}(I, A(-1)) \to \text{Ge}_A(Y) \underset{\text{Ge}_A(Y-Z)}{\times} *, \quad (I \mapsto a_i) \rightsquigarrow \bigotimes_i \mathcal{O}(Z_i)^{a_i} \tag{1.8}$$

Lemma 1.4.5. *Assume that the orders of elements in A are* prime *to* $\text{char}(k)$*, i.e., that A has no p-torsion, where $p = \text{char}(k)$. Then the map* (1.8) *is an isomorphism in* Spc.

Proof. The assertion follows from the fact that the étale cohomology group $H^i_{\text{et},Z}(Y, A)$ identifies with $\text{Maps}(I, A(-1))$ for $i = 2$ and vanishes for $i = 1, 0$. □

1.5. **The sheaf-theoretic context.** Most of this paper is devoted to the discussion of gerbes. However, in the last two sections, we will apply this discussion in order to formulate metaplectic geometric Satake. The latter involves *sheaves* and more generally *sheaves of categories* on various geometric objects.

When discussing sheaves (and sheaves of categories) we will only need to consider algebro-geometric objects that are *locally of finite type*, i.e., prestacks that belong to $\text{PreStk}_{\text{lft}}$, see Sect. 1.2.4. In what follows, in order to simplify the notation, we will omit the subscripts ft and lft.

1.5.1. There are several possible sheaf-theoretic contexts (for schemes of finite type):

(a) For any ground field k we can consider the ind-completion of the constructible derived category of ℓ-adic sheaves (viewed as a DG category);

(b) When the ground field is $\mathbb{C}$, then for an arbitrary algebraically closed field E of characteristic 0, we can consider the ind-completion of the constructible derived category of E-vector spaces (viewed as a DG category);

(c) When the ground field k has characteristic 0, we can consider the ind-completion of the derived category of holonomic D-modules (viewed as a DG category);

(c') In the setting of (c), we consider can consider the derived category of *all* D-modules (viewed as a DG category).

We will refer to contexts (a), (b) and (c) as *constructible*.

We will denote by E the field of coefficients of our sheaves. So, in the above cases, this is $\overline{\mathbb{Q}}_\ell$, E and k, respectively. For $Y \in \text{Sch}$ we will denote by

$$\text{Shv}(Y)$$

the resulting category of sheaves. This is an E-linear DG category.

Note that in cases (c) and (c'), $E = k$. Yet, we will keep the notational distinction between k (the ground field) and E (the field of coefficients) even in this case.

When we need to emphasize the distinction between case (c') and the other cases, we will use the notation

$$\text{D-mod}(Y).$$

1.5.2. We will denote by

$$\text{Shv} : (\text{Sch}^{\text{aff}})^{\text{op}} \to \text{DGCat} \tag{1.9}$$

the functor constructed that associates to an affine scheme S of finite type the DG category $\text{Shv}(S)$ and to a morphism $f : S_1 \to S_2$ the functor

$$f^! : \text{Shv}(S_2) \to \text{Shv}(S_1).$$

A basic feature of this functor is that the functor (1.9) carries a natural *lax symmetric monoidal structure*. In particular, for $S_1, S_2 \in \text{Sch}^{\text{aff}}$ we have a (fully faithful) functor

$$\text{Shv}(S_1) \otimes \text{Shv}(S_2) \to \text{Shv}(S_1 \times S_2). \tag{1.10}$$

When $\text{Shv}(-) = \text{D-mod}(-)$, i.e., in context (c'), this lax structure is *strict*, i.e., the functor (1.10) is an equivalence. This is *emphatically not the case* in the constructible contexts.

1.5.3. Yoneda embedding is a fully faithful functor

$$\text{Sch}^{\text{aff}} \hookrightarrow \text{PreStk}\,.$$

The right Kan extension of Shv along the (opposite of the) Yoneda embedding $(\text{Sch}^{\text{aff}})^{\text{op}} \to (\text{PreStk})^{\text{op}}$ defines a functor

$$\text{Shv} : (\text{PreStk})^{\text{op}} \to \text{DGCat}\,.$$

Thus, if $\mathcal{Y} \in \text{PreStk}$ is written as

$$\mathcal{Y} = \underset{i}{\underrightarrow{\text{colim}}}\, S_i, \quad S_i \in \text{Sch}^{\text{aff}},$$

we have by definition

$$\text{Shv}(\mathcal{Y}) = \underset{i}{\underleftarrow{\lim}}\, \text{Shv}(S_i).$$

1.6. **Sheaves of categories.** Sheaves of categories appear in this paper as a language in which we formulate the metaplectic geometric Satake functor. The reader can skip this subsection on the first pass, and return to it when necessary.

In this subsection we take our sheaf theory to be that of all D-modules, i.e., $\text{Shv}(-) = \text{D-mod}(-)$.

The discussion in this section is essentially borrowed from [Ga1].

1.6.1. Note that the diagonal morphism for affine schemes defines on every object of Sch^{aff} a canonical structure of co-commutative co-algebra.

Hence, the symmetric monoidal structure on Shv (see [GR2, Vol. 2, Chapter 3, Corollary 6.1.2]) naturally gives rise to a functor

$$(\text{Sch}^{\text{aff}})^{\text{op}} \to \text{ComAlg}(\text{DGCat}) =: \text{DGCat}^{\text{SymMon}}\,.$$

In particular, for every $S \in \text{Sch}^{\text{aff}}$, the category $\text{Shv}(S)$ has a natural symmetric monoidal structure, and for every $f : S_1 \to S_2$, the functor $f^! : \text{Shv}(S_2) \to \text{Shv}(S_1)$ is symmetric monoidal.

1.6.2. By a sheaf of DG categories $\mathcal{C}$ over $\mathcal{Y} \in \text{PreStk}$ we will mean a functorial assignment

$$(S \overset{y}{\to} \mathcal{Y}) \in ((\text{Sch}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}} \rightsquigarrow \mathcal{C}(S, y) \in \text{Shv}(S)\text{-}\mathbf{mod}, \tag{1.11}$$

where $\text{Shv}(S)$-**mod** denotes the category of modules in the (symmetric) monoidal category DGCat for the (commutative) algebra object $\text{Shv}(S)$. We impose the following quasi-coherence condition:

For a morphism of affine schemes $f : S_1 \to S_2$, $y_2 : S_2 \to \mathcal{Y}$ and $y_1 = y_2 \circ f$, consider the corresponding functor

$$\mathcal{C}(S_2, y_2) \to \mathcal{C}(S_1, y_1). \tag{1.12}$$

Part of the data of (1.11) is that the functor (1.12) should be $\text{Shv}(S_2)$-linear. Hence, it gives rise to a functor of $\text{Shv}(S_1)$-module categories

$$\text{Shv}(S_1) \underset{\text{Shv}(S_2)}{\otimes} \mathcal{C}(S_2, y_2) \to \mathcal{C}(S_1, y_1), \tag{1.13}$$

where $\otimes$ is the operation of tensor product of DG categories (see, e.g., [GR2, Vol. 1, Chapter 1, Sect. 10.4]).

We require that (1.13) should be an isomorphism.

Remark 1.6.3. What we defined as a sheaf of categories over $\mathcal{Y}$ would in the language of [Ga1] be rather called a *crystals of categories*. More precisely, [Ga1, Theorem 2.6.3] guarantees that our notion of a sheaf of categories over $\mathcal{Y}$ coincides with the notion of a sheaf of categories over $\mathcal{Y}_{\text{dR}}$ in the terminology of [Ga1].

1.6.4. A basic example of a sheaf of categories is denoted $\text{Shv}_{/\mathcal{Y}}$; it is defined by setting

$$\text{Shv}_{/\mathcal{Y}}(S, y) := \text{Shv}(S).$$

Let Z be a prestack locally of finite type over $\mathcal{Y}$. We define a sheaf of categories $\text{Shv}(Z)_{/\mathcal{Y}}$ over $\mathcal{Y}$ by setting for $S \overset{y}{\to} \mathcal{Y}$,

$$\text{Shv}(Z)_{/\mathcal{Y}}(S, y) = \text{Shv}(S \underset{\mathcal{Y}}{\times} Z).$$

The fact that for $f : S_1 \to S_2$, the functor

$$\text{Shv}(S_1) \underset{\text{Shv}(S_2)}{\otimes} \text{Shv}(S_2 \underset{\mathcal{Y}}{\times} Z) \to \text{Shv}(S_1 \underset{\mathcal{Y}}{\times} Z)$$

is an equivalence follows from [Ga1, Theorem 2.6.3].

1.6.5. *Descent.* Forgetting the module structure, a sheaf of DG categories $\mathcal{C}$ over $\mathcal{Y}$ defines a functor

$$((\text{Sch}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}} \to \text{DGCat}\,. \tag{1.14}$$

It follows from [Ga1, Corollary 1.5.2] that the assignment (1.14) satisfies étale descent (in fact, it satisfies h-descent).

1.6.6. Applying to the functor (1.14) the procedure of right Kan extension along

$$((\text{Sch}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}} \to ((\text{PreStk})_{/\mathcal{Y}})^{\text{op}},$$

we obtain that for every prestack Z over $\mathcal{Y}$ there is a well-defined DG category $\mathcal{C}(Z)$.

Namely, if

$$Z \simeq \underset{i}{\underset{\longrightarrow}{\text{colim}}}\, S_i, \quad (S_i, y_i) \in (\text{Sch}^{\text{aff}})_{/\mathcal{Y}},$$

then

$$\mathcal{C}(Z) = \underset{i}{\underset{\longleftarrow}{\lim}}\, \mathcal{C}(S_i, y_i).$$

We will refer to $\mathcal{C}(Z)$ as the "category of sections of $\mathcal{C}$ over Z". By construction the DG category $\mathcal{C}(Z)$ is naturally an object of $\text{Shv}(Z)$-**mod**.

When Z is $\mathcal{Y}$ itself, we will refer to $\mathcal{C}(\mathcal{Y})$ as the "category of global sections of $\mathcal{C}$".

1.6.7. *Example.* For $\mathcal{C} = \text{Shv}(Z)_{/Y}$ as in Sect. 1.6.4, we have

$$\mathcal{C}(\mathcal{Y}) \simeq \text{Shv}(Z).$$

1.6.8. The construction in Sect. 1.6.6 defines a functor

$$\{\text{Sheaves of categories over } \mathcal{Y}\} \to \text{Shv}(\mathcal{Y})\textbf{-mod}. \tag{1.15}$$

The functor (1.15) admits a left adjoint given by sending

$$\mathcal{C} \rightsquigarrow \left((S \to \mathcal{Y}) \mapsto \text{Shv}(S) \underset{\text{Shv}(\mathcal{Y})}{\otimes} \mathcal{C} \right). \tag{1.16}$$

We have the following assertion from [Ga1, Theorem 2.6.3] states:

Theorem 1.6.9. *For $\mathcal{Y}$ that is an ind-scheme of ind-finite type, the mutually adjoint functors* (1.15) *and* (1.16) *are equivalences.*

Remark 1.6.10. For the purposes of the present paper one can make do avoiding (the somewhat non-trivial) Theorem 1.6.9. However, allowing ourselves to use it simplifies a lot of discussions related to sheaves of categories.

1.7. **Some twisting constructions.** The material in this subsection may not have proper references in the literature, so we provide some details. The reader is advised to skip it and return to it when necessary.

When discussing sheaves of categories, we will be assuming that $\text{Shv}(-) = \text{D-mod}(-)$.

1.7.1. *Twisting by a torsor.* Let $\mathcal{Y}$ be a prestack, and let $\mathcal{H}$ (resp., $\mathcal{F}$) a group-like object in $\text{PreStk}_{/\mathcal{Y}}$ (resp., an object in $\text{PreStk}_{/\mathcal{Y}}$, equipped with an action of $\mathcal{H}$). In other words, these are functorial assignments

$$(S,y) \in (\text{Sch}^{\text{aff}})_{/\mathcal{Y}} \rightsquigarrow \mathcal{H}(S,y) \in \text{Grp}(\text{Spc}), \quad (S,y) \in (\text{Sch}^{\text{aff}})_{/\mathcal{Y}} \rightsquigarrow \mathcal{F}(S,y) \in \text{Spc},$$

and an action of $\mathcal{H}(S,y)$ on $\mathcal{F}(S,y)$.

Let $\mathcal{T}$ be an $\mathcal{H}$-torsor on $\mathcal{Y}$. In this case, we can form a $\mathcal{T}$-twist of $\mathcal{F}$, denoted $\mathcal{F}_{\mathcal{T}}$, and which is an étale *sheaf*. Here is the construction[6]:

Consider the subcategory $\text{Split}(\mathcal{T}) \subset (\text{Sch}^{\text{aff}})_{/\mathcal{Y}}$ formed by $(S,y) \in (\text{Sch}^{\text{aff}})_{/\mathcal{Y}}$ for which the torsor $\mathcal{T}|_S$ *admits a splitting*. This subcategory forms a basis of the étale topology, so it is sufficient to specify the restriction of $\mathcal{F}_{\mathcal{T}}$ to $\text{Split}(\mathcal{T})$.

The sought-for functor $\mathcal{F}_{\mathcal{T}}|_{\text{Split}(\mathcal{T})}$ is given by sending (S,y) to

$$\left(* \underset{\text{Maps}_{/\mathcal{Y}}(S, B_{\text{et}}(\mathcal{H}))}{\times} * \right) \overset{\text{Maps}_{/\mathcal{Y}}(S,\mathcal{H})}{\times} \mathcal{F}, \tag{1.17}$$

where the two maps

$$* \to \text{Maps}_{/\mathcal{Y}}(S, B_{\text{et}}(\mathcal{H})) \leftarrow *$$

are (i) the trivial map, and (ii) the one given by the composition

$$S \to \mathcal{Y} \overset{\mathcal{T}}{\to} B_{\text{et}}(\mathcal{H}).$$

Note that

$$* \underset{\text{Maps}_{/\mathcal{Y}}(S, B_{\text{et}}(\mathcal{H}))}{\times} *$$

is a groupoid equipped with a simply-transitive action of the group $\text{Maps}_{/\mathcal{Y}}(S,\mathcal{H})$. In formula (1.17), the notation $\overset{\text{Maps}_{/\mathcal{Y}}(S,\mathcal{H})}{\times}$ means "divide by the diagonal action of $\text{Maps}_{/\mathcal{Y}}(S,\mathcal{H})$".

[6]Note that when $\mathcal{T}$ is the trivial torsor, the output of this construction is the étale sheafification of $\mathcal{F}$.

1.7.2. *A twist of a sheaf of categories by a gerbe.* Let now $\mathcal{C}$ be a sheaf of DG categories over $\mathcal{Y}$, and let $\mathcal{A}$ be a group-like $\mathbb{E}_2$-object in $(\text{PreStk}_{\text{lft}})_{/\mathcal{Y}}$.

Let us be given an action of $\mathcal{A}$ on $\mathcal{C}$. In other words, we are given a functorial assignment for every $(S,y) \in (\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{Y}}$ of an action of $\mathcal{A}(S,y)$ on $\mathcal{C}(S,y)$, see Sect. 1.1.7.

Let $\mathcal{G}$ be an étale $\mathcal{A}$-gerbe on $\mathcal{Y}$. Repeating the construction of Sect. 1.7.1, we obtain that we can form the twist $\mathcal{C}_\mathcal{G}$ of $\mathcal{C}$ by $\mathcal{G}$, which is a new sheaf of DG categories over $\mathcal{Y}$.

In more detail, for $(S,y) \in (\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{Y}}$ such that $\mathcal{G}|_S$ *admits a splitting*, we define the value of $\mathcal{C}_\mathcal{G}$ on (S,y) to be

$$\left(* \underset{\text{Maps}_{/\mathcal{Y}}(S, B^2_{\text{et}}(\mathcal{A}))}{\times} * \right) \overset{\text{Maps}_{/\mathcal{Y}}(S, B_{\text{et}}(\mathcal{A}))}{\times} \mathcal{C}(S,y).$$

Concretely, for every $(S \overset{y}{\to} \mathcal{Y}) \in (\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{Y}}$ and a trivialization of $\mathcal{G}|_S$ we have an identification

$$\mathcal{C}_\mathcal{G}(S,y) \simeq \mathcal{C}(S,y).$$

The effect of change of trivialization by a point $a \in B_{\text{et}}(\mathcal{A})(S,y)$ has the effect of action of

$$a \in \text{Funct}(\mathcal{C}(S,y), \mathcal{C}(S,y)).$$

1.7.3. Let A be a torsion subgroup of $E^\times$.

Let us take $\mathcal{A}$ to be the constant group-prestack $\mathcal{Y} \times A$. In this case, the embedding $A \to E^\times$ gives rise to an action of $\mathcal{A}$ on *any* sheaf of DG categories.

Thus, for every $\mathcal{G} \in \text{Ge}_A(\mathcal{Y})$ and any sheaf of categories $\mathcal{C}$ over $\mathcal{Y}$, we can form its twisted version $\mathcal{C}_\mathcal{G}$.

1.7.4. *The category of sheaves twisted by a gerbe.* Let A and $\mathcal{G}$ be as in Sect. 1.7.3.

We apply the above construction to $\mathcal{C} := \text{Shv}_{/\mathcal{Y}}$. Thus, for any $(S,y) \in (\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{Y}}$ we have the twisted version of the category $\text{Shv}(S)$, denoted $\text{Shv}_\mathcal{G}(S)$.

As in Sect. 1.6.6, the procedure of Kan extension defines the category

$$\text{Shv}_\mathcal{G}(Z)$$

for any $Z \in \text{PreStk}_{/\mathcal{Y}}$.

2. Factorization gerbes on the affine Grassmannian

In this section we introduce our main object of study: factorization gerbes on the affine Grassmannian, which we stipulate to be the parameters for the metaplectic Langlands theory.

2.1. **The Ran space.**

The Ran space of a curve X is an algebro-geometric device (first suggested in [BD1]) that allows us to talk about *factorization structures* relative to our curve.

2.1.1. Let X be a fixed smooth algebraic curve. We let $\text{Ran} \in \text{PreStk}$ be the Ran space of X. By definition, for an affine test scheme S, the space $\text{Maps}(S, \text{Ran})$ is discrete (i.e., is a set), and equals the set of finite non-empty subsets of the (set) $\text{Maps}(S,X)$.

For a finite set J we have a map

$$\text{Ran}^J \to \text{Ran} \tag{2.1}$$

given by the union of the corresponding finite subsets.

This operation makes Ran into a (non-unital) semi-group object in $\text{PreStk}_{\text{lft}}$ (see [Lu2, Definition 5.4.1.1] for what this means).

2.1.2. The Ran space admits the following explicit description as a colimit (as an object of PreStk):

$$\mathrm{Ran} = \underset{\overrightarrow{I}}{\mathrm{colim}}\, X^I,$$

where I runs through the category opposite to that of non-empty finite sets and surjective maps[7]. For a surjection $\phi : I_1 \to I_2$, the corresponding map $X^{I_2} \to X^{I_1}$ is the corresponding diagonal morphism, denoted Δ_ϕ.

This presentation makes it manifest that $\mathrm{Ran} \in \mathrm{PreStk}_{\mathrm{lft}}$.

2.1.3. We denote by

$$(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}} \subset \mathrm{Ran} \times \mathrm{Ran}$$

the open substack corresponding to the following condition:

For an affine test scheme S, and two points

$$I_1, I_2 \in \mathrm{Maps}(S, \mathrm{Ran}),$$

the point $I_1 \times I_2 \in \mathrm{Maps}(S, \mathrm{Ran} \times \mathrm{Ran})$ belongs to $(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}$ if the corresponding subsets

$$I_1, I_2 \subset \mathrm{Maps}(S, X)$$

satisfy the following condition: for every $i_1 \in I_1, i_2 \in I_2$, the corresponding two maps $S \rightrightarrows X$ have non-intersecting images.

2.1.4. We give a similar definition for any power: for a finite set J we let

$$\mathrm{Ran}^J_{\mathrm{disj}} \subset \mathrm{Ran}^J$$

be the open substack corresponding to the following condition:

An S-point of Ran^J, given by

$$I_j \subset \mathrm{Maps}(S, X), \quad j \in J$$

belongs to $\mathrm{Ran}^J_{\mathrm{disj}}$ if for every $j_1 \neq j_2$ and $i_1 \in I_{j_1}$, $i_2 \in I_{j_2}$, the corresponding two maps $S \rightrightarrows X$ have non-intersecting images.

2.2. **Factorization patterns over the Ran space.** Let Z be a prestack over Ran. At the level of k-points, a factorization structure on Z is the following system of isomorphisms:

For a k-point $\underline{x}$ of Ran corresponding to a finite set $x_1, ..., x_n$ of k-points of X, the fiber $Z_{\underline{x}}$ of Z over the above point is supposed to be identified with

$$\prod_i Z_{\{x_i\}},$$

where $\{x_i\}$ are the corresponding singleton points of Ran.

We will now spell this idea, and some related notions, more precisely.

[7]We note that this category is *not filtered*, and hence Ran is *not* an ind-scheme.

2.2.1. By a factorization structure on Z we shall mean an assignment for any finite set J of an isomorphism

$$Z^J \underset{\operatorname{Ran}^J}{\times} \operatorname{Ran}^J_{\operatorname{disj}} \overset{\gamma_J}{\simeq} Z \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^J_{\operatorname{disj}}, \tag{2.2}$$

where the morphism $\operatorname{Ran}^J \to \operatorname{Ran}$ is given by (2.1).

We require the isomorphisms (2.2) to be compatible with surjections of finite sets in the sense that for $I \overset{\phi}{\twoheadrightarrow} J$ the diagram

$$\begin{array}{ccc}
Z^I \underset{\operatorname{Ran}^I}{\times} \operatorname{Ran}^I_{\operatorname{disj}} & \xrightarrow{\gamma_I} & Z \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^I_{\operatorname{disj}} \\
\sim\downarrow & & \uparrow\sim \\
\left(\prod\limits_{j\in J} Z^{I_j} \underset{\operatorname{Ran}^{I_j}}{\times} \operatorname{Ran}^{I_j}_{\operatorname{disj}}\right) \underset{\prod\limits_{j\in J} \operatorname{Ran}^{I_j}_{\operatorname{disj}}}{\times} \operatorname{Ran}^I_{\operatorname{disj}} & & (Z \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^J_{\operatorname{disj}}) \underset{\operatorname{Ran}^J_{\operatorname{disj}}}{\times} \operatorname{Ran}^I_{\operatorname{disj}} \\
\prod\limits_{j\in J}\gamma_{I_j}\downarrow & & \uparrow\gamma_J \\
\left(\prod\limits_{j\in J} Z \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{I_j}_{\operatorname{disj}}\right) \underset{\prod\limits_{j\in J} \operatorname{Ran}^{I_j}_{\operatorname{disj}}}{\times} \operatorname{Ran}^I_{\operatorname{disj}} & & (Z^J \underset{\operatorname{Ran}^J}{\times} \operatorname{Ran}^J_{\operatorname{disj}}) \underset{\operatorname{Ran}^J_{\operatorname{disj}}}{\times} \operatorname{Ran}^I_{\operatorname{disj}} \\
\sim\downarrow & & \uparrow\sim \\
\left(Z^J \underset{\operatorname{Ran}^J}{\times} \prod\limits_{j\in J} \operatorname{Ran}^{I_j}_{\operatorname{disj}}\right) \underset{\prod\limits_{j\in J} \operatorname{Ran}^{I_j}_{\operatorname{disj}}}{\times} \operatorname{Ran}^I_{\operatorname{disj}} & \xrightarrow{\sim} & Z^J \underset{\operatorname{Ran}^J}{\times} \operatorname{Ran}^I_{\operatorname{disj}},
\end{array} \tag{2.3}$$

where $I_j := \phi^{-1}(j)$, is required to commute. Furthermore, if Z takes values in ∞-groupoids (rather than sets), we require a homotopy-coherent system of compatibilities for higher order compositions, see [Ras1, Sect. 6].

2.2.2. Let $\mathcal{C}$ be a sheaf of DG categories over Ran (recall that this means that we are working over a ground field of characteristic 0 and in the context of D-modules).

By a *factorization structure* on $\mathcal{C}$ we shall mean a functorial assignment for any finite set J and an S-point of $\operatorname{Ran}^J_{\operatorname{disj}}$, given by

$$I_j \subset \operatorname{Maps}(S, X), \quad j \in J$$

of an identification

$$\bigotimes_{j, \operatorname{Shv}(S)} \mathcal{C}(S, I_j) \simeq \mathcal{C}(S, I), \tag{2.4}$$

where $I = \underset{j\in J}{\sqcup} I_j$.

We require the functors (2.4) to be compatible with surjections $J_1 \twoheadrightarrow J_2$ via the commutative diagrams analogous to (2.3). A precise formulation of these compatibilities is given in [Ras1, Sect. 6].

2.2.3. Let Z be a factorization prestack over Ran. Assume that for every finite set I, the category $\operatorname{Shv}(X^I \underset{\operatorname{Ran}}{\times} Z)$ is dualizable. We claim that in this case the sheaf of categories $\operatorname{Shv}(Z)_{/\operatorname{Ran}}$, i.e.,

$$(S, I \subset \operatorname{Maps}(S, X)) \rightsquigarrow \operatorname{Shv}(S \underset{\operatorname{Ran}}{\times} Z),$$

has a natural factorization structure.

Indeed, for any Z we have a canonically defined system of functors

$$\bigotimes_{j, \operatorname{Shv}(S)} \operatorname{Shv}(S \underset{I_j, \operatorname{Ran}}{\times} Z) \to \operatorname{Shv}\left(\prod_{j,S} (S \underset{I_j, \operatorname{Ran}}{\times} Z)\right) = \operatorname{Shv}(S \underset{\operatorname{Ran}^J}{\times} Z^J) \overset{(2.2)}{\simeq} \operatorname{Shv}(S \underset{I, \operatorname{Ran}}{\times} Z)$$

for a map $S \to \operatorname{Ran}^J_{\mathrm{disj}}$. We claim that the first arrow is an equivalence if each $\operatorname{Shv}(X^I \underset{\operatorname{Ran}}{\times} Z)$ is dualizable.

To prove this, it suffices to can consider the universal case when

$$S := X^I_{\mathrm{disj}} := X^I \underset{\operatorname{Ran}^J}{\times} \operatorname{Ran}^J_{\mathrm{disj}}$$

for a finite set I and a surjection $I \twoheadrightarrow J$.

We have

$$\bigotimes_{j,\operatorname{Shv}(S)} \operatorname{Shv}(S \underset{I_j,\operatorname{Ran}}{\times} Z) \simeq \left(\bigotimes_{j\in J} \operatorname{Shv}(X^{I_j} \underset{\operatorname{Ran}}{\times} Z)\right) \underset{\operatorname{Shv}(X^I)}{\otimes} \operatorname{Shv}(X^I_{\mathrm{disj}}) \to$$
$$\to \operatorname{Shv}\left(\prod_{j\in J} (X^{I_j} \underset{\operatorname{Ran}}{\times} Z)\right) \underset{\operatorname{Shv}(X^I)}{\otimes} \operatorname{Shv}(X^I_{\mathrm{disj}}) \simeq \operatorname{Shv}\left(\left(\prod_{j\in J} (X^{I_j} \underset{\operatorname{Ran}}{\times} Z)\right) \underset{X^I}{\times} X^I_{\mathrm{disj}}\right) = \operatorname{Shv}(S \underset{\operatorname{Ran}^J}{\times} Z^J),$$

where the second arrow is an isomorphism due to the assumption that the categories $\operatorname{Shv}(X^{I_j} \underset{\operatorname{Ran}}{\times} Z)$ are dualizable.

2.2.4. Let Z be a factorization prestack over Ran, and let A be a torsion abelian group. Let $\mathcal{G}$ be an A-gerbe on Z. By a factorization structure on $\mathcal{G}$ we shall mean a system of identifications

$$\mathcal{G}^{\boxtimes J}|_{Z^J \underset{\operatorname{Ran}^J}{\times} \operatorname{Ran}^J_{\mathrm{disj}}} \simeq \mathcal{G}|_{Z \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^J_{\mathrm{disj}}}, \tag{2.5}$$

where the underlying spaces are identified via (2.2).

The identifications (2.5) are required to be compatible with surjections $J_1 \twoheadrightarrow J_2$ via the commutative diagrams (2.3). Note that since gerbes form a 2-groupoid, we only need to specify the datum of (2.5) up to $|J| = 3$, and check the relations up to $|J| = 4$.

Factorization gerbes over Z naturally form a space (in fact, a 2-groupoid), equipped with a structure of commutative group in Spc (i.e., connective spectrum), to be denoted $\operatorname{FactGe}_A(Z)$.

Remark 2.2.5. Note that the diagrams (2.3) include those corresponding to automorphisms of finite sets. I.e., the datum of factorization gerbe includes equivariance with respect to the action of the symmetric group. For this reason what we call "factorization gerbe" in [Re] was called "symmetric factorizable gerbe".

2.2.6. *Variant.* Let Z be a factorization prestack over Ran, and let $\mathcal{G}$ be a factorization A-gerbe over it for $A \subset E^\times$. Assume that for every finite set I, the category $\operatorname{Shv}_{\mathcal{G}}(X^I \underset{\operatorname{Ran}}{\times} Z)$ is dualizable. Then the sheaf of categories $\operatorname{Shv}_{\mathcal{G}}(Z)_{/\operatorname{Ran}}$ defined by

$$(S, I \subset \operatorname{Maps}(S,X)) \rightsquigarrow \operatorname{Shv}_{\mathcal{G}}(S \underset{\operatorname{Ran}}{\times} Z)$$

has a natural factorization structure.

2.2.7. By a similar token, we can consider factorization line bundles over factorization prestacks, and also $\mathbb{Z}$- or $\mathbb{Z}/2\mathbb{Z}$-graded line bundles[8].

If $\mathcal{L}$ is a (usual, i.e., not graded) factorization line bundle and $a \in A(-1)$, we obtain a factorization gerbe $\mathcal{L}^a$.

2.3. **The Ran version of the affine Grassmannian.** In this subsection we introduce the Ran version of the affine Grassmannian, which plays a crucial role in the geometric Langlands theory.

[8]Note that in the latter case, the compatibility involved in the factorization structure (arising from the diagrams (2.3) for automorphisms of finite sets J) involves *sign rules.* I.e., a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle *does not* give rise to a factorization line bundle by forgetting the grading.

2.3.1. For an algebraic group G, we define the Ran version of the affine Grassmannian of G, denoted Gr_G, to be the following prestack.

For an affine test scheme S, the groupoid (in fact, set) $\mathrm{Maps}(S, \mathrm{Gr}_G)$ consists of triples

$$(I, \mathcal{P}_G, \alpha),$$

where I is an S-point of Ran, $\mathcal{P}_G$ is a G-bundle on $S \times X$, and α is a trivialization of $\mathcal{P}_G$ over the open subset $U_I \subset S \times X$ equal to the complement of the union Γ_I of the graphs of the maps $S \to X$ corresponding to the elements of $I \subset \mathrm{Maps}(S, X)$.

2.3.2. It is known that for every finite set I, the prestack $X^I \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G$ is an ind-scheme of ind-finite type. This implies, in partircular, that the dualizability assumptions in Sects. 2.2.3 and 2.2.6 are satisfied.

2.3.3. The basic feature of the prestack Gr_G is that it admits a natural factorization structure over Ran, obtained by gluing bundles.

Hence, for a torsion abelian group A, it makes sense to talk about factorization A-gerbes over Gr_G. We denote the the resulting space (i.e., in fact, a connective 2-truncated spectrum) by

$$\mathrm{FactGe}_A(\mathrm{Gr}_G).$$

2.3.4. *An example.* Let $\mathcal{L}$ be a factorization line bundle on Gr_G, and let a be an element of $A(-1)$. Then the A-gerbe

$$\mathcal{L}^a$$

of Sect. 1.4.1 is naturally a factorization gerbe on Gr_G.

This example is important because there is a canonical factorization line bundle on Gr_G, denoted $\det_{\mathfrak{g}}$; we will encounter it in Sect. 5.2.1.

2.3.5. Recall that we are not assuming that X be complete. Let $\overline{X}$ be its compactification. Let $D \subset \overline{X}$ be the complementary divisor. Let $\mathrm{Bun}_G(\overline{X}; D)$ be the moduli stack of G-bundles on $\overline{X}$ equipped with a trivialization along D. (When X is complete, we have $\mathrm{Bun}_G(\overline{X}; D)$ is the usual stack $\mathrm{Bun}_G(X)$ classifying G-bundles on X.

We have a naturally defined map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G(\overline{X}; D). \tag{2.6}$$

Recall now that [GL2, Theorem 3.2.13] says[9] that the map (2.6) is a *universal homological equivalence*. This implies that any gerbe on Gr_G uniquely descends to a gerbe on $\mathrm{Bun}_G(\overline{X}; D)$.

In particular, this is the case for factorization gerbes.

2.4. **The space of geometric metaplectic data.**

2.4.1. Let $E^{\times,\mathrm{tors}}$ denote the group of roots of unity in E of orders prime to $\mathrm{char}(k)$.

We stipulate that the space

$$\mathrm{FactGe}_{E^{\times,\mathrm{tors}}}(\mathrm{Gr}_G)$$

is the space of parameters for the metaplectic Langlands theory. We also refer to it as the space of *geometric metaplectic data*.

This includes both the global case (when X is complete), and the local case when we take X to be a Zariski neighborhood of some point x.

[9]This assertion was proved in *loc.cit.* under the additional assumption that G be semi-simple and simply connected. However, in the case of constant groups-schemes, the statement is known to hold in general: see [Ga3, Theorem 4.1.6].

2.4.2. Given an $E^{\times,\mathrm{tors}}$-factorization gerbe $\mathcal{G}$ on Gr_G, we can thus talk about the factorization sheaf of categories, denoted

$$\mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_G)_{/\,\mathrm{Ran}},$$

whose value on $S, I \subset \mathrm{Maps}(S, X)$ is

$$\mathrm{Shv}_{\mathcal{G}}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G).$$

3. Parameterization of factorization gerbes

From now on we let A be a torsion abelian group whose elements have orders prime to $\mathrm{char}(k)$. The main example is $A = E^{\times,\mathrm{tors}}$.

The goal of this section is to describe the set of isomorphism classes (and, more ambitiously, the *space*) of A-factorization gerbes on Gr_G in terms of more concise algebro-geometric objects.

3.1. **Parameterization via étale cohomology.** In this subsection we will create a space, provided by the theory of étale cohomology, that maps to the space $\mathrm{FactGe}_A(\mathrm{Gr}_G)$, thereby giving a parameterization of geometric metaplectic data.

3.1.1. Let $B_{\mathrm{et}}(G) =: \mathrm{pt}\,/G$ be the stack of G-torsors. I.e., this is the sheafification in the étale topology of the prestack $B(G)$ that attaches to an affine test scheme S the groupoid

$$*/\,\mathrm{Maps}(S, G).$$

3.1.2. Consider the space of maps

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(A(1))),$$

where the subscript Ptd stands for the the space of maps

$$B_{\mathrm{et}}(G) \times X \to B^4_{\mathrm{et}}(A(1)), \tag{3.1}$$

equipped with an identification of the composite map

$$X = \mathrm{pt} \times X \to B_{\mathrm{et}}(G) \times X \to B^4_{\mathrm{et}}(A(1)) \tag{3.2}$$

with

$$X \to \mathrm{pt} \to B^4_{\mathrm{et}}(A(1)).$$

We claim that there is a naturally defined map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(A(1))) \to \mathrm{FactGe}_A(\mathrm{Gr}_G). \tag{3.3}$$

3.1.3. The construction of the map (3.3) proceeds as follows. Let us be given a map (3.1) equipped with a trivialization of the composition (3.2).

For an affine test scheme S and an S-point $(I, \mathcal{P}_G, \alpha)$ of Gr_G, we need to construct a A-gerbe $\mathcal{G}_I$ on S.

Moreover, for $\phi : I \twoheadrightarrow J$, such that the point

$$\{\phi^{-1}(j) \subset \mathrm{Maps}(S, \mathrm{Ran}^J), \quad j \in J\}$$

hits $\mathrm{Ran}^J_{\mathrm{disj}}$, we need to be given an identification

$$\mathcal{G}_I \simeq \bigotimes_{j \in J} \mathcal{G}_{I_j}. \tag{3.4}$$

3.1.4. Let us interpret the datum of $\mathcal{P}_G$ as a map

$$S \times X \to B_{\mathrm{et}}(G) \times X.$$

Composing with (3.1), we obtain a map

$$S \times X \to B^4_{\mathrm{et}}(A(1)), \tag{3.5}$$

and a trivialization of the resulting map

$$U_I \to B^4_{\mathrm{et}}(A(1)), \tag{3.6}$$

where U_I is as in Sect. 2.3.1.

We claim that such a datum indeed gives rise to a A-gerbe $\mathcal{G}_I$ on S, equipped with identifications (3.4).

3.1.5. Recall that $\Gamma_I \subset S \times X$ denotes the union of the graphs of the maps that comprise $I \subset \mathrm{Hom}(S, X)$. Consider the maps

$$\begin{array}{ccc} \Gamma_I & \xrightarrow{\iota} & S \times X \\ {\scriptstyle \pi}\big\downarrow & & \\ S. & & \end{array}$$

We regard the datum of (3.5) together with a trivialization (3.6) as a 4-cocycle in

$$\mathrm{C}^\bullet_{\mathrm{et}}(\Gamma_I, \iota^!(A_{S\times X}(1))),$$

where for a scheme Y we denote by A_Y the constant étale sheaf with value A.

Thus, in order to construct the gerbe $\mathcal{G}_I$ on S from Sect. 3.1.4, it suffices construct a map

$$\mathrm{C}^\bullet_{\mathrm{et}}(\Gamma_I, \iota^!(A_{S\times X}(1)[2])) \to \mathrm{C}^\bullet_{\mathrm{et}}(S, A). \tag{3.7}$$

3.1.6. Let p_X denote the projection $X \to \mathrm{pt}$. We have a canonical identification

$$(p_X)^!(A) \simeq A_X(1)[2],$$

and hence

$$(\mathrm{id}_S \times p_X)^!(A_S) \simeq A_{S\times X}(1)[2]. \tag{3.8}$$

From here we obtain an isomorphism

$$\iota^!(A_{S\times X})(1)[2] \simeq \pi^!(A_S),$$

and by the $(\pi_*, \pi^!)$-adjunction, a morphism

$$\pi_* \circ \iota^!(A_{S\times X})(1)[2] \to A_S, \tag{3.9}$$

The sought-for morphism (3.7) is obtained from (3.9) by applying $\mathrm{C}^\bullet_{\mathrm{et}}(S, -)$.

3.1.7. The factorization structure on the assignment

$$I \rightsquigarrow \mathcal{G}_I$$

follows from the construction:

For $\phi : I \twoheadrightarrow J$ as in Sect. 3.1.3, we have

$$\Gamma_I = \bigsqcup_j \Gamma_{I_j},$$

and under this identification, the map (3.7) is the sum of the corresponding maps (3.7) with Γ_I replaced by Γ_{I_j}.

3.1.8. We have the following assertion:

Proposition 3.1.9. *The map* (3.3) *is an isomorphism.*

Remark 3.1.10. As was explained to us by J. Lurie, the assertion of Proposition 3.1.9 is nearly tautological if one works over the field of complex numbers and in the context of sheaves in the analytic topology.

3.1.11. From Proposition 3.1.9 we will obtain that

$$\pi_i(\mathrm{FactGe}_A(\mathrm{Gr}_G)) = H^{4-i}_{\mathrm{et}}(B_{\mathrm{et}}(G) \times X; \mathrm{pt} \times X, A(1)).$$

In Sect. 3.3 below we will analyze what these cohomology groups look like.

3.1.12. *Status of Proposition 3.1.9.* The assertion of Proposition 3.1.9 was initially claimed in [Re, Theorem II.7.3], but it was stated incorrectly, and the proof was erroneous.

Currently, a proof is available thanks to the paper [Zhao]. Namely, in Theorem 5.4 of *loc.cit.* an explicit description of $\mathrm{FactGe}_A(\mathrm{Gr}_G)$ is given in terms of what the author calls "enhanced Θ-data". In particular, this theorem gives an explicit description of the homotopy groups $\pi_i(\mathrm{FactGe}_A(\mathrm{Gr}_G))$, and those turn out to be isomorphic to the homotopy groups $\pi_i(\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(A(1)))$ (the Sect. 3.3.2 for the explicit description of the latter).

Further, the compatibility of the map (3.3) for G with that for its Cartan subgroup T, and an explicit description of the map (3.3) for T (see Sects. 4.2 and 4.3), show that the map (3.3) induces an isomorphism on homotopy groups. This implies that (3.3) is an equivalence, see also Remark 4.5.8.

We will supply a yet different argument in Sect. D.

3.2. Digression: étale cohomology of $B(G)$.

3.2.1. Let $\pi_{1,\mathrm{alg}}(G)$ denote the algebraic fundamental group of G. Explicitly, $\pi_{1,\mathrm{alg}}(G)$ can be described as follows:

Choose a short exact sequence

$$1 \to T_2 \to \widetilde{G}_1 \to G \to 1, \tag{3.10}$$

where T_2 is a torus and $[\widetilde{G}_1, \widetilde{G}_1]$ is simply connected. Set $T_1 = \widetilde{G}_1/[\widetilde{G}_1, \widetilde{G}_1]$. Let Λ_1 and Λ_2 be the coweight lattices of T_1 and T_2, respectively. Then $\pi_{1,\mathrm{alg}}(G) \simeq \Lambda_1/\Lambda_2$.

Equivalently, $\pi_{1,\mathrm{alg}}(G)$ is the quotient of Λ by the coroot lattice.

3.2.2. For an abelian group A, let $\mathrm{Quad}(\Lambda, A)^W$ denote the set of W-invariant quadratic forms on Λ with values in A. For any such form, denoted q, let b denote the associated symmetric bilinear form:

$$b(\lambda_1, \lambda_2) = q(\lambda_1 + \lambda_2) - q(\lambda_1) - q(\lambda_2). \tag{3.11}$$

Let $\mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}} \subset \mathrm{Quad}(\Lambda, A)^W$ be the subset consisting of forms q that satisfy the following additional condition: for every coroot $\alpha \in \Lambda$ and any $\lambda \in \Lambda$

$$b(\alpha, \lambda) = \langle \check{\alpha}, \lambda \rangle \cdot q(\alpha), \tag{3.12}$$

where $\check{\alpha}$ is the root corresponding to α.

Remark 3.2.3. Note that the identity

$$2b(\alpha, \lambda) = 2\langle \check{\alpha}, \lambda \rangle \cdot q(\alpha)$$

holds automatically.

Moreover, (3.12) itself holds automatically if $\frac{\alpha}{2} \in \Lambda$.

3.2.4. Note that we have injective maps

$$\mathrm{Quad}(\Lambda,\mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A \hookrightarrow \mathrm{Quad}(\Lambda, A)^W \hookleftarrow \mathrm{Quad}(\pi_{1,\mathrm{alg}}(G), A).$$

whose images belongs to $\mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}}$.

Assume for a moment that A is divisible. Then we have a surjection

$$\left(\mathrm{Quad}(\Lambda,\mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A\right) \oplus \mathrm{Quad}(\pi_{1,\mathrm{alg}}(G), A) \twoheadrightarrow \mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}}.$$

In particular, the inclusion

$$\mathrm{Quad}(\Lambda,\mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A \hookrightarrow \mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}}$$

is an equality when the derived group $[G,G]$ of G is simply connected.

3.2.5. We claim:

Theorem 3.2.6. *Let A be a torsion abelian group A whose elements have orders prime to* $\mathrm{char}(k)$. *Assume also that A is divisible. Then:*

$$H^i_{\mathrm{et}}(B_{\mathrm{et}}(G), A(1)) = 0 \textit{ for } i = 1, 3;$$

$$H^2_{\mathrm{et}}(B_{\mathrm{et}}(G), A(1)) \simeq \mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A);$$

$$H^4_{\mathrm{et}}(B_{\mathrm{et}}(G), A(1)) \simeq \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}.$$

Remark 3.2.7. When A is not divisible, the only difference will be that $H^3_{\mathrm{et}}(B_{\mathrm{et}}(G), A(1)) \simeq \mathrm{Ext}^1(\pi_{1,\mathrm{alg}}(G), A)$; in particular it will vanish if the derived group of G is simply-connected.

As we could not find a reference for this statement in the literature, we will supply the proof in Sect. B.

Remark 3.2.8. In fact, this is the same computation as in the context of algebraic topology, where we calculate singular cohomology with coefficients in $\mathbb{Q}/\mathbb{Z}$ of the classifying space of a compact connected Lie group, for which we could not find a reference either (the cohomology with $\mathbb{Z}$ coefficients is well-known of course).

3.3. **Analysis of homotopy groups of the space of factorization gerbes.** In this subsection we will assume that A is divisible (this assumption is only necessary when the derived group of G is not simply-connected, see Remark 3.2.7 above).

3.3.1. Consider the object

$$\mathcal{Q} := \tau^{\geq 1, \leq 4}\left(\mathrm{C}_{\mathrm{et}}(B_{\mathrm{et}}(G), A(1))\right). \tag{3.13}$$

In Sect. C.6.2 we will give a "hands-on" description of $\mathcal{Q}$.

By the Leray spectral sequence associated with the projection $B_{\mathrm{et}}(G) \times X \to X \to \mathrm{pt}$ and smooth base change along

$$\begin{array}{ccc} B_{\mathrm{et}}(G) \times X & \longrightarrow & B_{\mathrm{et}}(G) \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{p_X} & \mathrm{pt}, \end{array}$$

we have

$$H^{4-i}_{\mathrm{et}}(B_{\mathrm{et}}(G) \times X; \mathrm{pt} \times X, A(1)) \simeq H^{4-i}_{\mathrm{et}}(X, p_X^*(\mathcal{Q})).$$

3.3.2. By Theorem 3.2.6, we have a distinguished triangle

$$\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A)[-2]\to \mathcal{Q}\to \mathrm{Quad}(\Lambda,A(-1))^W_{\mathrm{restr}}[-4]. \tag{3.14}$$

In particular, we obtain that $H^4_{\mathrm{et}}(B_{\mathrm{et}}(G)\times X;\mathrm{pt}\times X,A(1))$ fits into *canonically split* short exact sequence

$$0\to H^2_{\mathrm{et}}(X,\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A))\to H^4_{\mathrm{et}}(B_{\mathrm{et}}(G)\times X;\mathrm{pt}\times X,A(1))\to \mathrm{Quad}(\Lambda,A(-1))^W_{\mathrm{restr}}\to 0,$$

while

$$H^3_{\mathrm{et}}(B_{\mathrm{et}}(G)\times X;\mathrm{pt}\times X,A(1))\simeq H^1_{\mathrm{et}}(X,\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A))$$

and

$$H^2_{\mathrm{et}}(B_{\mathrm{et}}(G)\times X;\mathrm{pt}\times X,A(1))\simeq H^0_{\mathrm{et}}(X,\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A)),$$

and

$$H^1_{\mathrm{et}}(B_{\mathrm{et}}(G)\times X;\mathrm{pt}\times X,A(1))=H^0_{\mathrm{et}}(B_{\mathrm{et}}(G)\times X;\mathrm{pt}\times X,A(1))=0.$$

3.3.3. Combining with Proposition 3.1.9, we obtain that $\pi_0(\mathrm{FactGe}_A(\mathrm{Gr}_G))$ fits into *canonically split* short exact sequence

$$0\to H^2_{\mathrm{et}}(X,\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A))\to \pi_0(\mathrm{FactGe}_A(\mathrm{Gr}_G))\to \mathrm{Quad}(\Lambda,A(-1))^W_{\mathrm{restr}}\to 0,$$

while

$$\pi_1(\mathrm{FactGe}_A(\mathrm{Gr}_G))\simeq H^1_{\mathrm{et}}(X,\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A))$$

and

$$\pi_2(\mathrm{FactGe}_A(\mathrm{Gr}_G))\simeq \mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A).$$

3.3.4. In particular, we obtain a map (of spectra)

$$\mathrm{FactGe}_A(\mathrm{Gr}_G)\to \mathrm{Quad}(\Lambda,A(-1))^W_{\mathrm{restr}}.$$

Let $\mathrm{FactGe}^0_A(\mathrm{Gr}_G)$ denote its fiber.

3.3.5. From Proposition 3.1.9 and the distinguished triangle (3.14), we obtain:

Corollary 3.3.6. *There is a canonical isomorphism*

$$\mathrm{Maps}(X,B^2_{\mathrm{et}}(\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A)))\simeq \mathrm{FactGe}^0_A(\mathrm{Gr}_G). \tag{3.15}$$

The subspace

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_G)\subset \mathrm{FactGe}_A(\mathrm{Gr}_G)$$

consists of objects that are trivial étale-locally on X.

3.4. **Parametrization of factorization line bundles.** This subsection is included for the sake of completeness, in order to make contact with the theory of metaplectic extensions developed in [We].

Recall from Sect. 2.3.4 that given a factorization line bundle $\mathcal{L}$ on Gr_G and an element $a\in A(-1)$ we can produce a factorization gerbe $\mathcal{L}^a$. In this subsection we will describe a geometric data that gives rise to factorization line bundles[10] on Gr_G.

[10] We emphasize that this construction produces just factorization line bundles, and *not* $\mathbb{Z}/2\mathbb{Z}$-graded ones.

3.4.1. Let K_2 denote the prestack over X that associates to an affine scheme $S = \operatorname{Spec}(A)$ mapping to X the abelian group $K_2(A)$. Let $(K_2)_{\mathrm{Zar}}$ be the sheafification of K_2 in the Zariski topology.

On the one hand, we consider the space $\mathrm{CExt}(G, (K_2)_{\mathrm{Zar}})$ (in fact, an ordinary groupoid) of *Brylinski-Deligne data*, which are by definition *central* extensions

$$1 \to (K_2)_{\mathrm{Zar}} \to \widetilde{G} \to G \times X \to 1$$

of the constant group-scheme $G \times X$ by $(K_2)_{\mathrm{Zar}}$.

The operation of Baer sum makes $\mathrm{CExt}(G, (K_2)_{\mathrm{Zar}})$ into a commutative group in spaces, i.e., into a Picard category.

On the other hand, consider the Picard category

$$\mathrm{FactPic}(\mathrm{Gr}_G)$$

of factorizable line bundles on Gr_G.

In the paper [Ga6] a map of Picard groupoids is constructed:

$$\mathrm{CExt}(G, (K_2)_{\mathrm{Zar}}) \to \mathrm{FactPic}(\mathrm{Gr}_G), \tag{3.16}$$

and the following conjecture is stated (this is Conjecture 6.1.2 in *loc.cit.*)[11]:

Conjecture 3.4.2. *The map* (3.16) *is an isomorphism.*

Remark 3.4.3. One can show that it follows from [BrDe, Theorem 3.16] combined with Sect. 4.1.5 that Conjecture 3.4.2 holds when $G = T$ is a torus.

3.4.4. Let us fix an integer ℓ of order prime to $\mathrm{char}(k)$. In [Ga6, Sect. 6.3.6] the following map was constructed

$$\mathrm{CExt}(G, (K_2)_{\mathrm{Zar}}) \to \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}\left(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(\mu_\ell^{\otimes 2}) \times X\right). \tag{3.17}$$

Let us take $A = \mu_\ell$, and note that $A(1) \simeq \mu_\ell^{\otimes 2}$. Note that the construction in Sect. 1.4.2 gives rise to a canonical map

$$\mathrm{FactPic}(\mathrm{Gr}_G) \to \mathrm{FactGe}_{\mu_\ell}(\mathrm{Gr}_G). \tag{3.18}$$

The following is equivalent to Conjecture 6.3.8 of *loc.cit.*:

Conjecture 3.4.5. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{CExt}(G, (K_2)_{\mathrm{Zar}}) & \xrightarrow{(3.17)} & \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}\left(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(\mu_\ell^{\otimes 2}) \times X\right) \\ {\scriptstyle (3.16)}\downarrow & & \downarrow{\scriptstyle (3.3)} \\ \mathrm{FactPic}(\mathrm{Gr}_G) & \xrightarrow{(3.18)} & \mathrm{FactGe}_{\mu_\ell}(\mathrm{Gr}_G). \end{array}$$

4. The case of tori

In this section we let $G = T$ be a torus. We will perform an explicit analysis of factorization gerbes on the affine Grassmannian Gr_T, and introduce related objects (multiplicative factorization gerbes) that will play an important role in the sequel.

4.1. **Factorization Grassmannian for a torus.** In this section we will show that the affine Grassmannian of a torus can be approximated by a prestack assembled from (=written as a colimit of) powers of X.

[11] Since the previous version of this paper, this conjecture has been proved in [TZ].

4.1.1. Recall that Λ denotes the coweight lattice of $G = T$. Consider the index category whose objects are pairs (I, λ^I), where I is a finite non-empty set and λ^I is a map $I \to \Lambda$; in what follows we will denote by $\lambda_i \in \Lambda$ is the value of λ^I on $i \in I$.

A morphism $(J, \lambda^J) \to (I, \lambda^I)$ is a surjection $\phi : I \twoheadrightarrow J$ such that

$$\lambda_j = \underset{i \in \phi^{-1}(j)}{\Sigma} \lambda_i. \tag{4.1}$$

Consider the prestack

$$\mathrm{Gr}_{T,\mathrm{comb}} := \underset{(I,\lambda^I)}{\mathrm{colim}}\, X^I.$$

The prestack $\mathrm{Gr}_{T,\mathrm{comb}}$ endowed with its natural forgetful map to Ran, also has a natural factorization structure.

There is a canonical map

$$\mathrm{Gr}_{T,\mathrm{comb}} \to \mathrm{Gr}_T, \tag{4.2}$$

compatible with the factorization structures.

Namely, for each (I, λ^I) the corresponding T-bundle on $X^I \times X$ is

$$\bigotimes_{i \in I} \lambda_i \cdot \mathcal{O}(\Delta_i),$$

where Δ_i is the divisor on $X^I \times X$ corresponding to the i-th coordinate being equal to the last one.

4.1.2. As in [Ga2, Sect. 8.1] one shows that the map (4.2) induces an isomorphism of the sheafifications with respect to the topology on the category of affine schemes of finite type, in which coverings are finite surjective maps.

In particular, for any $S \to \mathrm{Ran}$, the map

$$\mathrm{Ge}_A(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_T) \to \mathrm{Ge}_A(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{comb}})$$

is an isomorphism, and hence, so is the map

$$\mathrm{FactGe}_A(\mathrm{Gr}_T) \to \mathrm{FactGe}_A(\mathrm{Gr}_{T,\mathrm{comb}}).$$

Furthermore, for a given $\mathcal{G} \in \mathrm{FactGe}_{E^\times,\mathrm{tors}}(\mathrm{Gr}_T)$, the corresponding map of sheaves of categories

$$\mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_{T,\mathrm{comb}})_{/\,\mathrm{Ran}}$$

is also an isomorphism.

4.1.3. The datum of a factorization gerbe on $\mathrm{Gr}_{T,\mathrm{comb}}$ can be explicitly described as follows:

For a finite set I and a map

$$\lambda^I : I \to \Lambda$$

we specify a gerbe $\mathcal{G}^{\lambda^I}$ on X^I.

For a surjection of finite sets $I \overset{\phi}{\twoheadrightarrow} J$ such that (4.1) holds, we specify an identification

$$(\Delta_\phi)^*(\mathcal{G}^{\lambda^I}) \simeq \mathcal{G}^{\lambda^J}. \tag{4.3}$$

The identifications (4.3) must be compatible with compositions of maps of finite sets in the natural sense.

Let now $I \overset{\phi}{\twoheadrightarrow} J$ be a surjection of finite sets, and let

$$X^I_{\phi,\mathrm{disj}} \subset X^I, \quad x_{i_1} \neq x_{i_2} \text{ whenever } \phi(i_1) \neq \phi(i_2)$$

be the corresponding open subset. For $j \in J$, let λ^{I_j} be the restriction of λ^I to I_j.

We impose the structure of factorization that consists of isomorphisms

$$(\mathcal{G}^{\lambda^I})|_{X^I_{\phi,\mathrm{disj}}} \simeq \left(\bigotimes_{j \in J} \mathcal{G}^{\lambda^{I_j}}\right)|_{X^I_{\phi,\mathrm{disj}}}. \tag{4.4}$$

The isomorphisms (4.4) must be compatible with compositions of maps of finite sets in the natural sense.

In addition, the isomorphisms (4.4) and (4.3) must be compatible in the natural sense.

4.1.4. For a factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_{T,\mathrm{comb}}$, the value of the category $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_{T,\mathrm{comb}})_{/\,\mathrm{Ran}}$ on $X^I \to \mathrm{Ran}$ can be explicitly described as follows:

It is the limit over the index category

$$(J, \lambda^J, I \twoheadrightarrow J)$$

of the categories $\mathrm{Shv}_{\mathcal{G}^{\lambda^J}}(X^J)$.

4.1.5. *The case of factorization line bundles.* The datum of a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on $\mathrm{Gr}_{T,\mathrm{comb}}$ can be described in a way similar to that of factorization gerbes. This description recovers the notion of what in [BD1, Sect. 3.10.3] is called a θ-datum.

We note that a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle is evenly (i.e., trivially) graded if and only if the corresponding θ-datum is even, i.e., if the corresponding symmetric bilinear $\mathbb{Z}$-valued form on Λ comes from a $\mathbb{Z}$-valued quadratic form.

We also note that [BD1, Proposition 3.10.7] says that restriction along

$$\mathrm{Gr}_{T,\mathrm{comb}} \to \mathrm{Gr}_T$$

defines an equivalence between the Picard categories of factorization ($\mathbb{Z}/2\mathbb{Z}$-graded) line bundles.

4.2. **Making the parameterization explicit for tori.** In this subsection we will show explicitly how a factorization A-gerbe on Gr_T gives rise to an A-valued quadratic form

$$q : \Lambda \to A(-1).$$

For the duration of this section we assume that elements of A have orders prime to $\mathrm{char}(k)$.

4.2.1. We first describe the bilinear form

$$b : \Lambda \times \Lambda \to A(-1).$$

For an element $\lambda \in \Lambda$, take I to be the one-element set $\{*\}$, and consider the corresponding map

$$\lambda^I : I \to \Lambda, \quad * \mapsto \lambda.$$

Let $\mathcal{G}^\lambda$ denote the resulting A-gerbe on X.

Given two elements $\lambda_1, \lambda_2 \in \Lambda$, consider $I = \{1, 2\}$ and the map

$$\lambda^I : I \to \Lambda; \quad 1 \mapsto \lambda_1, 2 \mapsto \lambda_2.$$

Consider the corresponding gerbe

$$\mathcal{G}^{\lambda_1,\lambda_2} := \mathcal{G}^{\lambda^I}$$

over X^2.

By (4.4), $\mathcal{G}^{\lambda_1,\lambda_2}$ is identified with $\mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2}$ over $X^2 - \Delta$. By Lemma 1.4.5, there exists a well-defined element $a \in A(-1)$ such that

$$\mathcal{G}^{\lambda_1,\lambda_2} \simeq (\mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2}) \otimes \mathcal{O}(\Delta)^a. \tag{4.5}$$

We let

$$a =: b(\lambda_1, \lambda_2).$$

4.2.2. The fact that $b(-,-)$ is bilinear can be seen as follows. For a triple of elements $\lambda_1, \lambda_2, \lambda_3$ consider the corresponding gerbes

$$\mathcal{G}^{\lambda_1,\lambda_2,\lambda_3} \text{ and } (\mathcal{G}^{\lambda_1,\lambda_2} \boxtimes \mathcal{G}^{\lambda_3}) \otimes \mathcal{O}(\Delta_{1,3})^{\otimes b(\lambda_1,\lambda_3)} \otimes \mathcal{O}(\Delta_{2,3})^{\otimes b(\lambda_2,\lambda_3)}$$

over X^3.

They are identified away from the main diagonal $\Delta_{1,2,3}$, and hence this identification extends to all of X^3, since $\Delta_{1,2,3}$ has codimension 2. Restricting to $\Delta_{1,2}$, we obtain an identification

$$\mathcal{G}^{\lambda_1+\lambda_2,\lambda_3} \simeq (\mathcal{G}^{\lambda_1+\lambda_2} \boxtimes \mathcal{G}^{\lambda_3}) \otimes \mathcal{O}(\Delta)^{\otimes b(\lambda_1,\lambda_3)} \otimes \mathcal{O}(\Delta)^{\otimes b(\lambda_2,\lambda_3)}$$

as gerbes over X^2. Comparing with the identification

$$\mathcal{G}^{\lambda_1+\lambda_2,\lambda_3} \simeq (\mathcal{G}^{\lambda_1+\lambda_2} \boxtimes \mathcal{G}^{\lambda_3}) \otimes \mathcal{O}(\Delta)^{\otimes b(\lambda_1+\lambda_2,\lambda_3)},$$

we obtain the desired

$$b(\lambda_1,\lambda_3) + b(\lambda_2,\lambda_3) = b(\lambda_1+\lambda_2,\lambda_3).$$

4.2.3. It is easy to see that the resulting map

$$b : \Lambda \times \Lambda \to A(-1)$$

is symmetric. In fact, we have a canonical datum of commutativity for the diagram

$$\begin{array}{ccc} \sigma^*(\mathcal{G}^{\lambda_1,\lambda_2}) & \longrightarrow & \sigma^*((\mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2}) \otimes \mathcal{O}(\Delta)^{b(\lambda_1,\lambda_2)}) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{G}^{\lambda_2,\lambda_1} & \longrightarrow & (\mathcal{G}^{\lambda_2} \boxtimes \mathcal{G}^{\lambda_1}) \otimes \mathcal{O}(\Delta)^{b(\lambda_2,\lambda_1)} \end{array} \tag{4.6}$$

that extends the given one over $X \times X - \Delta$ (in the above formula, σ denotes the transposition acting on $X \times X$):

Indeed, the measure of *non-commutativity* of the above diagram is an étale A-torsor over $X \times X$, which is trivialized over $X \times X - \Delta$, and hence this trivialization uniquely extends to all of $X \times X$.

For the sequel we will need to understand in more detail the behavior of the restriction of diagram (4.6) to the diagonal.

4.2.4. We start with the following observation. We claim that to an element $a \in A(-1)$ one can canonically attach an A-torsor $(-1)^a$:

The Kummer cover

$$\mathbb{G}_m \overset{x \mapsto x^n}{\longrightarrow} \mathbb{G}_m$$

defines a group homomorphism

$$\mathbb{G}_m \to B_{\text{et}}(\mu_n). \tag{4.7}$$

From here we obtain a bilinear map

$$A(-1) \times \mathbb{G}_m \to B_{\text{et}}(A), \tag{4.8}$$

i.e., an element $a \in A(-1)$ defines an étale A-torsor χ_a over $\mathbb{G}_m$, which behaves multiplicatively.

4.2.5. We let $(-1)^a$ denote the fiber of χ_a at $-1 \in \mathbb{G}_m$.

The multiplicativity of (4.8) along $\mathbb{G}_m$ implies that we have a canonical trivialization

$$((-1)^a)^{\otimes 2} \simeq \text{triv}. \tag{4.9}$$

The multiplicativity of (4.8) along $A(-1)$ implies that a choice of $a' \in A(-1)$ such that $2a' = a$ defines a trivialization of $(-1)^a$. Moreover, this trivialization is compatible with (4.9).

This construction is a morphism (and hence an *isomorphism*) of $A_{2\text{-tors}}$-torsors:

$$\{a' \in A(-1)\,,\, 2a' = a\} \to \{\text{trivializations of } (-1)^a \text{ compatible with (4.9)}\}.$$

(By enlarging A if needed, one shows that the LHS is empty if and only if the RHS is.)

4.2.6. Consider now the A-gerbe $\mathcal{O}(\Delta)^a$ on $X \times X$, equipped with the natural identification

$$\sigma^*(\mathcal{O}(\Delta)^a) \simeq \mathcal{O}(\Delta)^a, \tag{4.10}$$

which uniquely extends the tautological one over $X \times X - \Delta$.

Restricting (4.10) to the diagonal, and using the fact that $\sigma|_\Delta$ is trivial, we obtain an identification of A-gerbes

$$\mathcal{O}(\Delta)^a|_\Delta \simeq \mathcal{O}(\Delta)^a|_\Delta, \tag{4.11}$$

whose square is the identity map.

The map (4.11) is given by tensoring by an A-torsor that squares to the trivial one. It is easy to see that this torsor is constant along X and identifies canonically with $(-1)^a$ in a way compatible with (4.9). This follows from the fact that the composite

$$\mathcal{O}(\Delta)|_\Delta \simeq \sigma^*(\mathcal{O}(\Delta))|_\Delta \simeq \sigma^*(\mathcal{O}(\Delta)|_\Delta) \simeq \mathcal{O}(\Delta)|_\Delta$$

acts as -1.

4.2.7. The identification (4.3) for the map $\{1, 2\} \to \{*\}$ yields an identification

$$\mathcal{G}^{\lambda_1,\lambda_2}|_\Delta \simeq \mathcal{G}^{\lambda_1+\lambda_2}, \tag{4.12}$$

compatible with the transposition of factors, i.e., the diagram

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1,\lambda_2}|_\Delta \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \sim} \\ \mathcal{G}^{\lambda_2+\lambda_1} & \longrightarrow & \mathcal{G}^{\lambda_2,\lambda_1}|_\Delta \end{array} \tag{4.13}$$

is endowed with a datum of commutativity that squares to one. In the above diagram the right vertical arrow is the map

$$\mathcal{G}^{\lambda_1,\lambda_2}|_\Delta \simeq \sigma^*(\mathcal{G}^{\lambda_1,\lambda_2})|_\Delta \simeq \mathcal{G}^{\lambda_2,\lambda_1}|_\Delta.$$

Restricting diagram (4.6) to the diagonal and concatenating with diagrams (4.13), we obtain that we have a datum of commutativity for the diagram

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & (\mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2}) \otimes \mathcal{O}(\Delta)^{b(\lambda_1,\lambda_2)}|_\Delta \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{tautological}\otimes(-1)^{b(\lambda_1,\lambda_2)}} \\ \mathcal{G}^{\lambda_2+\lambda_1} & \longrightarrow & (\mathcal{G}^{\lambda_2} \otimes \mathcal{G}^{\lambda_1}) \otimes \mathcal{O}(\Delta)^{b(\lambda_2,\lambda_1)}|_\Delta \end{array} \tag{4.14}$$

that squares to the tautological one.

4.2.8. We are finally ready to recover the quadratic form

$$q : \Lambda \to A(-1).$$

Namely, in (4.14), let us set $\lambda_1 = \lambda = \lambda_2$. We obtain a datum of commutativity of the diagram

$$\begin{array}{ccc} \mathcal{G}^{2\lambda} & \longrightarrow & (\mathcal{G}^{\lambda} \otimes \mathcal{G}^{\lambda}) \otimes \mathcal{O}(\Delta)^{b(\lambda,\lambda)}|_\Delta \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes(-1)^{b(\lambda,\lambda)}} \\ \mathcal{G}^{2\lambda} & \longrightarrow & (\mathcal{G}^{\lambda} \otimes \mathcal{G}^{\lambda}) \otimes \mathcal{O}(\Delta)^{b(\lambda,\lambda)}|_\Delta, \end{array} \tag{4.15}$$

where the upper and lower horizontal arrows are canonically identified, and which squares to the tautological one.

This datum is equivalent to that of trivialization of the A-torsor $(-1)^{b(\lambda,\lambda)}$, that squares to the identity. By Sect. 4.2.5, this datum is equivalent to that of an element $q(\lambda) \in A(-1)$ such that $2q(\lambda) = b(\lambda, \lambda)$. This is the value of our quadratic form on λ.

4.2.9. The relation
$$q(\lambda_1+\lambda_2)=q(\lambda_1)+q(\lambda_2)+b(\lambda_1,\lambda_2)$$
is verified in a way similar to Sect. 4.2.2.

4.3. **Matching the parameters.**

4.3.1. Let us start with a datum of a based map
$$B_{\mathrm{et}}(T)\times X\to B^4_{\mathrm{et}}(A(1)), \tag{4.16}$$
and produce an object $\mathcal{G}$ of $\mathrm{FactGe}_A(\mathrm{Gr}_T)$ by the map (3.3).

In Sect. 4.2, to $\mathcal{G}$ we have attached a quadratic form
$$q:\Lambda\to A(-1).$$

In this section we will show that q equals to the form attached to (4.16) via the map
$$\pi_0\left(\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T)\times X,B^4_{\mathrm{et}}(A(1)))\right)=H^4_{\mathrm{et}}(B_{\mathrm{et}}(T)\times X;\mathrm{pt}\times X,A(1))\to \\ \to H^4_{\mathrm{et}}(B_{\mathrm{et}}(T);\mathrm{pt},A(1))\simeq \mathrm{Quad}(\Lambda,A(-1)). \tag{4.17}$$

I.e., we want to establish the commutativity of the diagram
$$\begin{array}{ccc}\pi_0\left(\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T)\times X,B^4_{\mathrm{et}}(A(1)))\right) & \xrightarrow{(3.3)} & \pi_0(\mathrm{FactGe}_A(\mathrm{Gr}_T)) \\ \sim\downarrow & & \downarrow{\scriptstyle \text{Sect. 4.2}} \\ H^4_{\mathrm{et}}(B_{\mathrm{et}}(T)\times X;\mathrm{pt}\times X,A(1)) & \longrightarrow & \mathrm{Quad}(\Lambda,A(-1)).\end{array} \tag{4.18}$$

4.3.2. From the fiber sequence (3.14), we obtain a fiber sequence
$$\mathrm{Maps}(X,B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda,A)))\to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T)\times X,B^4_{\mathrm{et}}(A(1)))\to \mathrm{Quad}(\Lambda,A(-1)). \tag{4.19}$$

First, we claim that if $\mathcal{G}$ comes from an object in $\mathrm{Maps}(X,B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda,A)))$, then the form q, attached to $\mathcal{G}$ by Sect. 4.2, equals 0.

Indeed, in this case $\mathcal{G}$ is trivial étale-locally on X. And the form q is zero by construction.

Thus, it remains to exhibit a collection of objects $\mathcal{G}$ of $\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T)\times X,B^4_{\mathrm{et}}(A(1)))$, whose images under (4.17) span $\mathrm{Quad}(\Lambda,A(-1))$, on which the two circuits of the diagram (4.18) produce the same result.

4.3.3. With a future application in mind, we will choose our collection of objects $\mathcal{G}$ to be obtained as compositions
$$B_{\mathrm{et}}(T)\times X\to B_{\mathrm{et}}(T)\to B^4_{\mathrm{et}}(A(1))$$
for some particular collection of based maps $B_{\mathrm{et}}(T)\to B^4_{\mathrm{et}}(A(1))$.

Let A_1 and A_2 be a pair of finite abelian groups (of orders prime to $\mathrm{char}(k)$), equipped with homomorphisms
$$\chi_i:\Lambda\to A_i(-1)$$
and a bilinear map
$$b':A_1\times A_2\to A(1),$$
which we can also think of as a bilinear map
$$A_1(-1)\times A_2(-1)\to A(-1).$$

Let q be the quadratic form on Λ with values in $A(-1)$ given by
$$q(\lambda)=b'(\chi_1(\lambda),\chi_2(\lambda)). \tag{4.20}$$

Clearly, forms q obtained in this way span $\mathrm{Quad}(\Lambda,A(-1))$.

Let b denote the associated symmetric bilinear form. Explicitly,

$$b(\lambda_1,\lambda_2)=b'(\chi_1(\lambda_1),\chi_2(\lambda_2))+b'(\chi_1(\lambda_2),\chi_2(\lambda_1)). \tag{4.21}$$

4.3.4. We can regard χ_i as a map of pointed prestacks

$$B_{\text{et}}(T)\to B^2_{\text{et}}(A_i). \tag{4.22}$$

The map b' and cup-product give rise to a (based) map

$$B^2(A_1)\times B^2(A_2)\to B^4(A(1)),$$

which in turn gives rise to a (based) map

$$B^2_{\text{et}}(A_1)\times B^2_{\text{et}}(A_2)\to B^4_{\text{et}}(A(1)).$$

Precomposing with the χ_i's we obtain a (based) map

$$B_{\text{et}}(T)\overset{\chi_1,\chi_2}{\longrightarrow} B^2_{\text{et}}(A_1)\times B^2_{\text{et}}(A_2)\to B^4_{\text{et}}(A(1)). \tag{4.23}$$

The class of the map (4.23) in $H^4_{\text{et}}(B_{\text{et}}(T);\text{pt},A(1))$ is the cup product of the classes of $\chi_i\in H^2_{\text{et}}(B_{\text{et}}(T);\text{pt},A_i)$, $i=1,2$.

The corresponding element in

$$H^4_{\text{et}}(B_{\text{et}}(T);\text{pt},A(1))\simeq \text{Quad}(\Lambda,A(-1))$$

is the form q from (4.20).

4.3.5. Let $\mathcal{G}$ be the object of $\text{FactGe}_A(\text{Gr}_T)$ corresponding via (3.3) to the composition

$$B_{\text{et}}(T)\times X\to B_{\text{et}}(T)\overset{(4.23)}{\longrightarrow} B^4_{\text{et}}(A(1)).$$

We are going to show that the quadratic form attached to $\mathcal{G}$ by the procedure of Sect. 4.2 equals q. To show this, we will have to unwind the construction in Sects. 3.1.3-3.1.7.

4.3.6. First, we identify explicitly the corresponding gerbes $\mathcal{G}^\lambda$. We claim that we have

$$\mathcal{G}^\lambda\simeq(\omega_X^{\otimes -1})^{q(\lambda)}, \tag{4.24}$$

where ω_X is the sheaf of 1-forms on X.

We take $S=X$ with I being the one-element set corresponding to the identity map $X\to X$. Consider the map

$$X\times X\to B_{\text{et}}(T), \tag{4.25}$$

corresponding to the T-bundle $\lambda\cdot\mathcal{O}(\Delta)$, equipped with its natural trivialization over

$$U_I=X\times X-\Delta.$$

We identify $\Gamma_I=X$ with the map ι being the diagonal map. The composition of (4.25) with (4.22) (say, for $i=1$) defines a 2-cocycle in

$$\text{C}^\bullet_{\text{et}}(X,\Delta^!((A_1)_{X\times X})). \tag{4.26}$$

We identify

$$\Delta^!((A_1)_{X\times X})\simeq A_1(-1)[-2].$$

Thus, the above 2-cocycle corresponds to an element of $A_1(-1)$. We claim that the resulting element of $A_1(-1)$ equals $\chi_1(\lambda)$.

Indeed, the object

$$\chi_1\in \text{Maps}_{\text{Ptd(PreStk)}}(B_{\text{et}}(T),B^2_{\text{et}}(A_1))$$

is the A_1-gerbe over $B_{\text{et}}(T)$ attached by the procedure in Sect. 1.4.2 to the tautological T-bundle and the map $\chi_1:\Lambda\to A_1(-1)$. Hence, the 2-cocyle in (4.26) is obtained by the procedure Sect. 1.4.2 applied to the T-bundle $\lambda\cdot\mathcal{O}(\Delta)$ equipped with its trivialization over $X\times X-\Delta$. The required assertion follows now from Sect. 1.4.4.

4.3.7. Now, the composition of (4.25) with (4.23), viewed as a 4-cocycle in

$$\mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^!(A_{X\times X})(1))$$

is obtained from the above 2-cocycle in (4.26) by multiplication with the 2-cocycle in

$$\mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^*((A_2)_{X\times X})) \tag{4.27}$$

under the cup-product map

$$\mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^!((A_1)_{X\times X})) \otimes \mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^*((A_2)_{X\times X})) \xrightarrow{b'} \mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^!(A_{X\times X}(1))) \tag{4.28}$$

where the 2-cocycle in (4.27) is

$$X \overset{\Delta}{\to} X \times X \overset{(4.25)}{\longrightarrow} B_{\mathrm{et}}(T) \overset{(4.22) \text{ for } i=2}{\longrightarrow} B^2_{\mathrm{et}}(A_2).$$

4.3.8. The map (4.28) fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^!((A_1)_{X\times X})) \otimes \mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^*((A_2)_{X\times X})) & \xrightarrow{b'} & \mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^!(A_{X\times X}(1))) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{C}^\bullet_{\mathrm{et}}(X, (A_1)_X(-1)[-2]) \otimes \mathrm{C}^\bullet_{\mathrm{et}}(X, (A_2)_X) & \xrightarrow{b'} & \mathrm{C}^\bullet_{\mathrm{et}}(X, A_X[-2]). \end{array}$$

Hence, in order to prove (4.24), it suffices to show that the 2-cocycle in (4.27), thought of as a 2-cocycle in

$$\mathrm{C}^\bullet_{\mathrm{et}}(X, (A_2)_X), \tag{4.29}$$

interpreted as a A_2-gerbe on X, identifies with

$$(\omega_X^{\otimes -1})^{\chi_2(\lambda)}.$$

4.3.9. Note that the map (4.22) (say, for $i = 2$) is obtained by the procedure of Sect. 1.4.2 applied to the tautological T-bundle on $B_{\mathrm{et}}(T)$ and $\chi_2 : \Lambda \to A_2(-1)$.

Hence, the above class in (4.29) is attached by the procedure of Sect. 1.4.2 applied to the T-bundle $\lambda \cdot \mathcal{O}(\Delta)|_\Delta$ and $\chi_2 : \Lambda \to A_2(-1)$.

The required assertion follows now from the fact that

$$\mathcal{O}(\Delta)|_\Delta \simeq \omega_X^{\otimes -1}.$$

Remark 4.3.10. The above description of the resulting A-gerbe $\mathcal{G}^\lambda$ relied on breaking the symmetry in the roles of A_1 and A_2. We claim that if we swap the roles, the resulting automorphism of the gerbe $(\omega_X^{\otimes -1})^{q(\lambda)}$ will be given by the $A_{2\text{-tors}}$-torsor $(-1)^{q(\lambda)}$.

Indeed, the 4-cocycle we obtained

$$\mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^!(A_{X\times X}(1)))$$

equals the image of the 4-cocycle in

$$\chi_1(\lambda) \otimes \chi_2(\lambda) \in \mathrm{C}^\bullet_{\mathrm{et}}(X, (A_1)_X(-1)[-2] \otimes (A_2)_X(-1)[-2]) \simeq \mathrm{C}^\bullet_{\mathrm{et}}(X, \Delta^!((A_1)_{X\times X}) \otimes \Delta^!((A_2)_{X\times X})),$$

along the map

$$\Delta^!((A_1)_{X\times X}) \otimes \Delta^!((A_2)_{X\times X}) \to \Delta^!((A_1)_{X\times X}) \otimes (A_2)_{X\times X} \to$$
$$\to \Delta^!((A_1)_{X\times X}) \otimes \Delta^*((A_2)_{X\times X}) \to \Delta^!((A_1 \otimes A_2)_{X\times X}) \xrightarrow{b'} \Delta^!(A_{X\times X}(1)).$$

We note that the latter map is canonically identified with

$$\Delta^!((A_1)_{X\times X}) \otimes \Delta^!((A_2)_{X\times X}) \to (A_1)_{X\times X} \otimes \Delta^!((A_2)_{X\times X}) \to$$
$$\to \Delta^*((A_1)_{X\times X}) \otimes \Delta^!((A_2)_{X\times X}) \to \Delta^!((A_1 \otimes A_2)_{X\times X}) \xrightarrow{b'} \Delta^!(A_{X\times X}(1)).$$

The exchange of roles of A_1 and A_2 thus acts as the swap of the two factors in $X \times X$, resulting in the automorphism (-1) on $\omega_X^{-1} \simeq \mathcal{O}(\Delta)|_\Delta$.

4.3.11. Fix now two elements $\lambda_1, \lambda_2 \in \Lambda$, and let us describe explicitly the resulting gerbe $\mathcal{G}^{\lambda_1,\lambda_2}$. We claim that we will obtain a canonical identification

$$\mathcal{G}^{\lambda_1,\lambda_2} \simeq \left((\omega_X^{\otimes -1})^{q(\lambda_1)} \boxtimes (\omega_X^{\otimes -1})^{q(\lambda_2)}\right) \otimes \mathcal{O}(\Delta)^{b'(\chi_1(\lambda_1),\chi_2(\lambda_2))} \otimes \mathcal{O}(\Delta)^{b'(\chi_1(\lambda_2),\chi_2(\lambda_1))}. \tag{4.30}$$

This would imply that the symmetric bilinear form attached to $\mathcal{G}$ by the procedure of Sect. 4.2 equals b of (4.21).

4.3.12. We take $S = X \times X$ and $I = \{1, 2\}$, where the two maps $S \to X$ are the two projections. The subset $\Gamma_I \subset S \times X$ identifies with $X \times X \underset{\Delta}{\sqcup} X \times X$. It admits a normalization

$$\widetilde{\Gamma}_I \simeq (X \times X) \sqcup (X \times X) \xrightarrow{s} \Gamma_I.$$

Denote the composite map

$$\widetilde{\Gamma}_I \to \Gamma_I \to S \times X$$

by $\widetilde{\iota}$.

Denote the two maps

$$X \times X \to \Gamma_I$$

by s_i. We have

$$\iota \circ s_1 = \Delta_{1,3} \text{ and } \iota \circ s_2 = \Delta_{2,3}.$$

Let $s_{1,2}$ denote the diagonal map

$$X \to \Gamma_I,$$

so that $\iota \circ s_{1,2}$ is the main diagonal $\Delta_{1,2,3}$.

4.3.13. We consider the map

$$(X \times X) \times X \to B_{\text{et}}(T), \tag{4.31}$$

corresponding to the T-bundle $(\lambda_1 \cdot \mathcal{O}(\Delta_{1,3})) \otimes (\lambda_2 \cdot \mathcal{O}(\Delta_{2,3}))$, equipped with its natural trivialization on $U_I = (X \times X) \times X - (\Delta_{1,3} \cup \Delta_{2,3})$.

The composition of (4.31) with (4.22) (for $i = 1$) defines a 2-cocycle

$$\mathrm{C}^\bullet_{\text{et}}(\Gamma_I, \iota^!((A_1)_{(X\times X)\times X})). \tag{4.32}$$

We have a distinguished triangle

$$(s_{1,2})_!((A_1)_X(-2)[-4]) \to (s_1)_!((A_1)_{X\times X}(-1)[-2]) \oplus (s_2)_!((A_1)_{X\times X}(-1)[-2]) \to \iota^!((A_1)_{(X\times X)\times X}).$$

Hence, the above 2-cocycle canonically comes from a 2-cocycle in

$$\mathrm{C}^\bullet_{\text{et}}(\widetilde{\Gamma}_I, \widetilde{\iota}^!((A_1)_{(X\times X)\times X})) \simeq \mathrm{C}^\bullet_{\text{et}}(X \times X, (A_1)_{X\times X}(-1)[-2]) \oplus \mathrm{C}^\bullet_{\text{et}}(X \times X, (A_1)_{X\times X}(-1)[-2]).$$

I.e., this cocycle corresponds to a pair of elements in $A_1(-1)$. The computation in Sect. 4.3.6 implies that this pair of elements is given by

$$(\chi_1(\lambda_1), \chi_1(\lambda_2)) \in A_1(-1) \oplus A_1(-1). \tag{4.33}$$

4.3.14. From here, using the cup-product manipulation as in Sect. 4.3.7, we obtain that the 4-cocycle in

$$\mathrm{C}^\bullet_{\mathrm{et}}(\Gamma_I, \iota^!(A_{(X\times X)\times X}(1))), \tag{4.34}$$

corresponding to the composition of (4.31) with (4.23), also comes from a 4-cocycle in

$$\mathrm{C}^\bullet_{\mathrm{et}}(\widetilde{\Gamma}_I, \widetilde{\iota}^!(A_{(X\times X)\times X})(1)) \simeq \mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2]) \oplus \mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2]), \tag{4.35}$$

with components obtained via

$$\mathrm{C}^\bullet_{\mathrm{et}}(X\times X, (A_2)_{X\times X}[-2]) \overset{b'(\chi_1(\lambda_1),-)}{\longrightarrow} \mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2])$$

and

$$\mathrm{C}^\bullet_{\mathrm{et}}(X\times X, (A_2)_{X\times X}[-2]) \overset{b'(\chi_1(\lambda_2),-)}{\longrightarrow} \mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2]),$$

respectively, from the 2-cocycles in

$$\mathrm{C}^\bullet_{\mathrm{et}}(X\times X, (A_2)_{X\times X}[-2])$$

attached by the procedure of Sect. 1.4.2 to the T-bundles

$$X\times X \overset{\Delta_{i,3}}{\longrightarrow} (X\times X)\times X \overset{(4.31)}{\longrightarrow} B_{\mathrm{et}}(T), \quad i=1,2 \tag{4.36}$$

and $\chi_2 : \Lambda \to A_2(-1)$.

4.3.15. The T-bundles in (4.36) are

$$\lambda_1\cdot(\omega_X^{\otimes -1}\boxtimes \mathcal{O}_X)\otimes \lambda_2\cdot \mathcal{O}(\Delta) \text{ and } \lambda_2\cdot(\mathcal{O}_X\boxtimes \omega_X^{\otimes -1})\otimes \lambda_1\cdot \mathcal{O}(\Delta),$$

respectively.

Hence, we obtain that the 4-cocycle in (4.35) has components in

$$\mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2]) \oplus \mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2]),$$

thought of as a pair of A-gerbes on $X\times X$, equal to

$$(\omega_X^{\otimes -1}\boxtimes \mathcal{O}_X)^{b'(\chi_1(\lambda_1),\chi_2(\lambda_1))}\otimes \mathcal{O}(\Delta)^{b'(\chi_1(\lambda_1),\chi_2(\lambda_2))}$$

and

$$(\mathcal{O}_X\boxtimes \omega_X^{\otimes -1})^{b'(\chi_1(\lambda_2),\chi_2(\lambda_2))}\otimes \mathcal{O}(\Delta)^{b'(\chi_1(\lambda_2),\chi_2(\lambda_1))},$$

respectively.

Finally, the trace map

$$\mathrm{C}^\bullet_{\mathrm{et}}(\widetilde{\Gamma}_I, \widetilde{\iota}^!(A_{S\times X}(1))) \simeq \mathrm{C}^\bullet_{\mathrm{et}}(\widetilde{\Gamma}_I, s^!\circ \pi^!(A_S)[-2]) \to \mathrm{C}^\bullet_{\mathrm{et}}(S, A_S[-2]),$$

thought of as a map

$$\mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2]) \oplus \mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2]) \to \mathrm{C}^\bullet_{\mathrm{et}}(X\times X, A_{X\times X}[-2]),$$

is the sum of two identity maps.

This implies the specified description of (4.30).

4.3.16. We have showed that the symmetric bilinear form corresponding to $\mathcal{G}$ via the procedure of Sect. 4.2 equals the form b from (4.21). We will now proceed to showing that the quadratic form equals q. For that we will have to describe explicitly the identifications

$$\mathcal{G}^{\lambda_1,\lambda_2}|_\Delta \simeq \mathcal{G}^{\lambda_1+\lambda_2}$$

of (4.12) and the datum of commutativity of the corresponding diagram (4.15)

Unwinding the definitions, we obtain that the resulting map

$$\begin{aligned}(\omega_X^{\otimes -1})^{q(\lambda_1+\lambda_2)} &\simeq (\omega_X^{\otimes -1})^{q(\lambda_1)+q(\lambda_2)+b(\lambda_1,\lambda_2)} \simeq \\ &\simeq (\omega_X^{\otimes -1})^{b'(\chi_1(\lambda_1),\chi_2(\lambda_1))+b'(\chi_1(\lambda_1),\chi_2(\lambda_2))+b'(\chi_1(\lambda_2),\chi_2(\lambda_2))+b'(\chi_1(\lambda_2),\chi_2(\lambda_1))} \simeq \\ &\simeq \Big((\omega_X^{\otimes -1} \boxtimes \mathcal{O}_X)^{b'(\chi_1(\lambda_1),\chi_2(\lambda_1))} \otimes \mathcal{O}(\Delta)^{b'(\chi_1(\lambda_1),\chi_2(\lambda_2))} \otimes \\ &\otimes (\mathcal{O}_X \boxtimes \omega_X^{\otimes -1})^{b'(\chi_1(\lambda_2),\chi_2(\lambda_2))} \otimes \mathcal{O}(\Delta)^{b'(\chi_1(\lambda_2),\chi_2(\lambda_1))}\Big)|_\Delta \simeq \\ &\simeq \mathcal{G}^{\lambda_1,\lambda_2}|_\Delta \simeq \mathcal{G}^{\lambda_1+\lambda_2} \simeq (\omega_X^{\otimes -1})^{q(\lambda_1+\lambda_2)}\end{aligned}$$

equals the tautological map tensored with the $A_{2\text{-tors}}$-torsor

$$(-1)^{b'(\chi_1(\lambda_1),\chi_2(\lambda_2))}.$$

This torsor comes from the fact that the map

$$\iota \circ s_1 : X \times X \to X \times X \times X$$

acts as

$$(x_1, x_2) \mapsto (x_1, x_2, x_1),$$

which involves the transposition of the last two factors.

Furthermore, the datum of commutativity of the diagram (4.14) is given by the identification of the $A_{2\text{-tors}}$-torsors

$$\begin{aligned}(-1)^{b'(\chi_1(\lambda_1),\chi_2(\lambda_2))} \otimes (-1)^{b(\lambda_1,\lambda_2)} &\simeq (-1)^{-b'(\chi_1(\lambda_1),\chi_2(\lambda_2))} \otimes (-1)^{b(\lambda_1,\lambda_2)} \simeq \\ &\simeq (-1)^{b'(\chi_1(\lambda_2),\chi_2(\lambda_1))},\end{aligned}$$

where the last isomorphism comes from

$$b(\lambda_1, \lambda_2) = b'(\chi_1(\lambda_1), \chi_2(\lambda_2)) + b'(\chi_1(\lambda_2), \chi_2(\lambda_1)).$$

From here we obtain that the datum of commutativity of the diagram (4.15) is given by the identification

$$(-1)^{q(\lambda)} \otimes (-1)^{b(\lambda,\lambda)} \simeq (-1)^{-q(\lambda)} \otimes (-1)^{b(\lambda)} \simeq (-1)^{q(\lambda)},$$

where the last isomorphism comes from

$$b(\lambda, \lambda) = 2q(\lambda),$$

as required.

4.4. **Proof of Proposition 3.1.9 for tori.** We will now give an alternative proof of Proposition 3.1.9 in the special case of tori.

4.4.1. From the commutative diagram (4.18), we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T)\times X, B^4_{\mathrm{et}}(A(1))) & \xrightarrow{(3.3)} & \mathrm{FactGe}_A(\mathrm{Gr}_T) \\ \downarrow & & \downarrow {\scriptstyle \text{Sect. 4.2}} \\ \mathrm{Quad}(\Lambda, A(-1)) & \xrightarrow{\mathrm{Id}} & \mathrm{Quad}(\Lambda, A(-1)), \end{array}$$

where the left vertical arrow corresponds to the projection

$$H^4_{\mathrm{et}}(B_{\mathrm{et}}(T)\times X; \mathrm{pt}\times X, A(1)) \to H^4_{\mathrm{et}}(B_{\mathrm{et}}(T); \mathrm{pt}, A(1)) \simeq \mathrm{Quad}(\Lambda, A(-1)).$$

Let $\mathrm{FactGe}^0_A(\mathrm{Gr}_T)$ denote the fiber of the right vertical arrow. From the distinguished triangle (4.19) we obtain that the fiber of the left vertical arrow identifies canonically with $\mathrm{Maps}\left(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))\right)$.

Hence, it remains to show that the induced map

$$\mathrm{Maps}\left(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))\right) \to \mathrm{FactGe}^0_A(\mathrm{Gr}_T) \tag{4.37}$$

is an isomorphism.

We will deduce this from the description of $\mathrm{FactGe}_A(\mathrm{Gr}_T)$ in Sect. 4.1.3.

4.4.2. Namely, the groupoid $\mathrm{FactGe}^0_A(\mathrm{Gr}_T)$ is isomorphic to that of assignments

$$\lambda \mapsto \mathcal{G}^\lambda \in \mathrm{Ge}_A(X),$$

equipped with the following pieces of data:

- One is *multiplicativity*, i.e., we must be given isomorphisms of gerbes
$$\mathcal{G}^{\lambda_1+\lambda_2} \simeq \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2}$$
that are associative in the natural sense.
- The other one is that of *commutativity*, i.e., we must be given the data of commutativity for the squares
$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_2+\lambda_1} & \longrightarrow & \mathcal{G}^{\lambda_2} \otimes \mathcal{G}^{\lambda_1} \end{array} \tag{4.38}$$
that satisfy the hexagon axiom.

In addition, the following conditions must be satisfied:

(1) The datum of commutativity for the outer square in
$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_2+\lambda_1} & \longrightarrow & \mathcal{G}^{\lambda_2} \otimes \mathcal{G}^{\lambda_1} \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \end{array} \tag{4.39}$$
is the identity one.

(2) The datum of commutativity in (4.38) for $\lambda_1 = \lambda = \lambda_2$
$$\begin{array}{ccc} \mathcal{G}^{2\lambda} & \longrightarrow & \mathcal{G}^{\lambda} \otimes \mathcal{G}^{\lambda} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{G}^{2\lambda} & \longrightarrow & \mathcal{G}^{\lambda} \otimes \mathcal{G}^{\lambda} \end{array} \tag{4.40}$$
is the identity one.

4.4.3. The above description implies that for $T \simeq T_1 \times T_2$, the natural map

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{T_1}) \times \mathrm{FactGe}^0_A(\mathrm{Gr}_{T_2}) \to \mathrm{FactGe}^0_A(\mathrm{Gr}_T)$$

is an isomorphism.

Hence, it is sufficient to show that the map (4.37) is an equivalence for $T = \mathbb{G}_m$.

4.4.4. Now, the description in Sect. 4.4.2 implies that for $T = \mathbb{G}_m$ (so $\Lambda = \mathbb{Z}$) we have the obvious equivalence

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\mathbb{G}_m}) \simeq \mathrm{Ge}_A(X),$$

given by

$$\mathcal{G} \mapsto \mathcal{G}^1, \quad 1 \in \mathbb{Z},$$

and the composition

$$\mathrm{Ge}_A(X) \simeq \mathrm{Maps}\big(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\mathbb{Z}, A))\big) \to \mathrm{FactGe}^0_A(\mathrm{Gr}_{\mathbb{G}_m}) \to \mathrm{Ge}_A(X)$$

is the identity map.

Hence, (4.37) is an isomorphism for $\mathbb{G}_m$.

4.5. **Relation to Θ-data.** In this section we will make contact with the paper [Zhao], and describe the category $\mathrm{FactGe}_A(\mathrm{Gr}_T)$ in terms of what the author of *loc. cit.* calls Θ data for the lattice Λ.

4.5.1. Following [Zhao, Sect. 5.3.5], we let $\Theta(\Lambda)$ be the space of the following data:

- A quadratic form $q : \Lambda \to A(-1)$; we denote the associated symmetric bilinear form by b;
- An assignment $\lambda \in \Lambda \rightsquigarrow \mathcal{G}^\lambda \in \mathrm{Ge}_A(X)$;
- A system of isomorphisms

$$\mathcal{G}^{\lambda_1+\lambda_2} \overset{c_{\lambda_1,\lambda_2}}{\simeq} \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \otimes (\omega_X^{-1})^{b(\lambda_1,\lambda_2)},$$

  endowed with an associativity constraint;
- A datum h_{λ_1,λ_2} of commutativity for the squares

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \xrightarrow{c_{\lambda_1,\lambda_2}} & \big(\mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2}\big) \otimes (\omega_X^{-1})^{b(\lambda_1,\lambda_2)} \\ \mathrm{id}\big\downarrow & & \big\downarrow \mathrm{id}\otimes(-1)^{b(\lambda_1,\lambda_2)} \\ \mathcal{G}^{\lambda_2+\lambda_1} & \xrightarrow{c_{\lambda_2,\lambda_1}} & \big(\mathcal{G}^{\lambda_2} \otimes \mathcal{G}^{\lambda_1}\big) \otimes (\omega_X^{-1})^{b(\lambda_2,\lambda_1)}, \end{array}$$

  where $(-1)^{b(\lambda_1,\lambda_2)}$ is as in Sect. 4.2.5, equipped with a datum of compatibility with the associativity constraint, and that squares to the identity;
- For $\lambda_1 = \lambda = \lambda_2$, the datum of identification of $h_{\lambda,\lambda}$ in the diagram

$$\begin{array}{ccc} \mathcal{G}^{\lambda+\lambda} & \xrightarrow{c_{\lambda,\lambda}} & \big(\mathcal{G}^{\lambda} \otimes \mathcal{G}^{\lambda}\big) \otimes (\omega_X^{-1})^{b(\lambda,\lambda)} \\ \mathrm{id}\big\downarrow & & \big\downarrow \mathrm{id}\otimes(-1)^{b(\lambda,\lambda)} \\ \mathcal{G}^{\lambda+\lambda} & \xrightarrow{c_{\lambda,\lambda}} & \big(\mathcal{G}^{\lambda} \otimes \mathcal{G}^{\lambda}\big) \otimes (\omega_X^{-1})^{b(\lambda,\lambda)}, \end{array}$$

  with the trivialization of $(-1)^{b(\lambda,\lambda)}$ resulting from the identity $b(\lambda,\lambda) = 2q(\lambda)$.

Let $\Theta^0(\Lambda)$ denote the fiber of the natural projection

$$\Theta(\Lambda) \to \mathrm{Quad}(\Lambda, A(-1)).$$

We have

$$\Theta^0(\Lambda) \simeq \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))).$$

4.5.2. Note that if Λ is equipped with a basis $e_1, ..., e_n$, then evaluation on basis elements defines a map

$$\Theta(\Lambda) \to (\mathrm{Ge}_A(X))^{\times n},$$

and the resulting map

$$\Theta(\Lambda) \to \mathrm{Quad}(\Lambda, A(-1)) \times (\mathrm{Ge}_A(X))^{\times n}$$

is an equivalence.

Indeed, it fits into a map of fiber sequences

$$\begin{array}{ccccc} \Theta^0(\Lambda) & \longrightarrow & \Theta(\Lambda) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)) \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ (\mathrm{Ge}_A(X))^{\times n} & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)) \times (\mathrm{Ge}_A(X))^{\times n} & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)), \end{array}$$

where the left vertical arrow is the isomorphism

$$\Theta^0(\Lambda) \simeq \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) \simeq (\mathrm{Ge}_A(X))^{\times n}.$$

4.5.3. Following [Zhao, Lemma 5.6], we claim that there is a canonical equivalence

$$\mathrm{FactGe}_A(\mathrm{Gr}_T) \simeq \Theta(\Lambda). \tag{4.41}$$

Indeed, the description of $\mathrm{FactGe}_A(\mathrm{Gr}_T)$ in Sect. 4.2 provides a functor

$$\mathrm{FactGe}_A(\mathrm{Gr}_T) \to \Theta(\Lambda).$$

Now, we have a morphism of fiber sequences

$$\begin{array}{ccccc} \mathrm{FactGe}^0_A(\mathrm{Gr}_T) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_T) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)) \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \Theta^0(\Lambda) & \longrightarrow & \Theta(\Lambda) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)), \end{array}$$

where the left vertical arrow is an equivalence by Sect. 4.4.2.

4.5.4. Let now G be a reductive group with T as its Cartan subgroup. Assume that A is divisible. We define the category $\Theta(\Lambda)_G$ as follows[12]:

An object of $\Theta(\Lambda)_G$ is an object of $\Theta(\Lambda)$, whose bilinear form q belongs to

$$\mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}} \subset \mathrm{Quad}(\Lambda, A(-1)),$$

and which is endowed with isomorphisms

$$\mathcal{G}^{\alpha_i} \simeq (\omega_X^{\otimes -1})^{q(\alpha_i)} \tag{4.42}$$

for every simple coroot α_i.

4.5.5. Restriction along the embedding $T \hookrightarrow G$ defined a map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{FactGe}_A(\mathrm{Gr}_T).$$

We claim that we have a naturally defined map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \Theta(\Lambda)_G \tag{4.43}$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{FactGe}_A(\mathrm{Gr}_G) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_T) \\ \downarrow & & \downarrow \\ \Theta(\Lambda)_G & \longrightarrow & \Theta(\Lambda) \end{array}$$

commute.

Indeed, by Sect. 3.3.2 we know that the composition

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{FactGe}_A(\mathrm{Gr}_T) \to \mathrm{Quad}(\Lambda, A(-1))$$

[12] Our definition is different, yet equivalent to that in [Zhao, Sect.5.3.6].

takes values in $\mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}$.

Hence, it remains to construct the data of (4.42). The latter reduces to the case of $G = SL_2$.

4.5.6. Note that by Sect. 3.3.3, any factorizable A-gerbe on Gr_{SL_2} is *canonically* of the form $(\det_{SL_2,\mathrm{St}})^a$ for some element $a \in A(-1)$, where $\det_{SL_2,\mathrm{St}}$ is the determinant line bundle on Gr_{SL_2} corresponding to the action on the *standard* representation.

For an integer k let $\det_{\mathbb{G}_m,k}$ denote the determinant line bundle on $\mathrm{Gr}_{\mathbb{G}_m}$ associated with the action of $\mathbb{G}_m$ on the one-dimensional vector space given by the k-th power of the tautological character. This is a $\mathbb{Z}$-graded factorization line bundle, and we note that the grading is even if k is even.

The restriction of $\det_{SL_2,\mathrm{St}}$ to $\mathrm{Gr}_{\mathbb{G}_m}$ identifies with $\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1}$, and hence the restriction of $(\det_{SL_2,\mathrm{St}})^a$ to $\mathrm{Gr}_{\mathbb{G}_m}$ identifies with $(\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^a$. The associated quadratic form

$$q : \mathbb{Z} \to A(-1)$$

takes value a on the generator $1 \in \mathbb{Z}$.

In order to construct (4.42), we have to show that the value of $(\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^a$ on the generator $1 \in \mathbb{Z} = \Lambda$ identifies canonically with $(\omega_X^{\otimes -1})^a$. For that, it suffices to construct an isomorphism between the restriction of the line bundle $\det_{SL_2,\mathrm{St}}$ to the section

$$X \to \mathrm{Gr}_{\mathbb{G}_m,X}$$

corresponding to $1 \in \mathbb{Z}$ identifies canonically with $\omega_X^{\otimes -1}$.

However, we indeed have a canonical isomorphism

$$\det.\mathrm{rel.}(\mathcal{O}(x) \oplus \mathcal{O}(-x), \mathcal{O} \oplus \mathcal{O}) \simeq \omega_X^{\otimes -1}|_x, \quad x \in X.$$

4.5.7. Thus, the map (4.43) has been constructed. It fits into a map of fiber sequences

$$\begin{array}{ccccc} \mathrm{FactGe}^0_A(\mathrm{Gr}_G) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_G) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}} \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \Theta^0(\Lambda)_G & \longrightarrow & \Theta(\Lambda)_G & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}, \end{array} \tag{4.44}$$

where

$$\Theta^0(\Lambda)_G \simeq \mathrm{Fib}\left(\Theta^0(\Lambda) \to \prod_{i\in I} \mathrm{Ge}_A(X)\right) \simeq \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A))).$$

By unwinding the construction, one obtains that in terms of the above identification, the left vertical arrow in (4.44) is the identity map on $\mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A)))$.

From here we obtain that the map (4.43) is also an isomorphism.

Remark 4.5.8. In [Zhao, Theorem 5.4], it is shown directly that the map (4.43) are isomorphism. This, in turn, can be used to deduce Proposition 3.1.9 by reversing the steps.

4.6. **The notion of *multiplicative* factorization gerbe.** In order to be able to state the metaplectic version of geometric Satake, we will need to discuss the notion of *multiplicative* factorization gerbe, first on Gr_T, and then when the lattice $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$ is replaced by a general finitely generated abelian group.

4.6.1. Note that since T is commutative, Gr_T is naturally a factorization *group*-prestack over Ran. Hence, along with $\mathrm{FactGe}_A(\mathrm{Gr}_T)$, we can consider the corresponding space (in fact, commutative group in spaces)

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_T) \tag{4.45}$$

that corresponds to gerbes that respect the group structure on Gr_T over Ran.

We have the evident forgetful map

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_T) \to \mathrm{FactGe}_A(\mathrm{Gr}_T). \tag{4.46}$$

Explicitly, a multiplicative structure on a gerbe $\mathcal{G}$ is an identification

$$\text{mult}^*(\mathcal{G}) \simeq \mathcal{G} \boxtimes \mathcal{G}$$

as *factorization gerbes* on $\text{Gr}_T \underset{\text{Ran}}{\times} \text{Gr}_T$ (in the above formula mult denotes the multiplication map $\text{Gr}_T \underset{\text{Ran}}{\times} \text{Gr}_T \to \text{Gr}_T$), equipped with a compatibility datum over triple product $\text{Gr}_T \underset{\text{Ran}}{\times} \text{Gr}_T \underset{\text{Ran}}{\times} \text{Gr}_T$, and an identity satisfied over the quadruple product.

We will prove:

Proposition 4.6.2. *The forgetful map*

$$\text{FactGe}_A^{\text{mult}}(\text{Gr}_T) \to \text{FactGe}_A(\text{Gr}_T)$$

is fully faithful. Its essential image is the preimage under

$$\text{FactGe}_A(\text{Gr}_T) \to \text{Quad}(\Lambda, A(-1))$$

of the subset consisting of those quadratic forms, whose associated bilinear form is zero.

Proof. We will use the description of factorization on gerbes on Gr_T given in Sect. 4.1.3. In these terms, the multiplicative structure on $\mathcal{G}$ amounts to specifying isomorphisms

$$\mathcal{G}^{\lambda_1,\lambda_2} \simeq \mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2} \tag{4.47}$$

equipped with an associativity constraint, and equipped with the datum of the identification of (4.47) with the factorization isomorphism over $X \times X - \Delta$.

In other words, we need that the factorization isomorphisms

$$\mathcal{G}^{\lambda_1,\lambda_2}|_{X\times X-\Delta} \simeq \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2}|_{X\times X-\Delta}$$

extend to all of $X \times X$. If they extend, they do so uniquely, and the extended isomorphisms are automatically equipped with an associativity constraint.

Thus, by Sect. 4.2.1, we obtain that the category $\text{FactGe}_A^{\text{mult}}(\text{Gr}_T)$ identifies with the full subcategory of $\text{FactGe}_A(\text{Gr}_T)$, consisting of objects for which the bilinear form $b(-,-)$ vanishes. □

Remark 4.6.3. Note that the set of quadratic forms $q : \Lambda \to A$ whose associated bilinear form vanishes, is in bijection with the set of *linear* maps $\Lambda \to A_{2\text{-tors}}$.

Note also that we have a tautological identification

$$A(-1)_{2\text{-tors}} \simeq A_{2\text{-tors}}, \tag{4.48}$$

since $\mu_2 \simeq \pm 1 \simeq \mathbb{Z}/2\mathbb{Z}$ canonically.

4.6.4. Note that it follows from Sect. 4.5.3 that we can describe the space

$$\text{FactGe}_A^{\text{mult}}(\text{Gr}_T) \simeq \text{Fib}\Big(\text{FactGe}_A(\text{Gr}_T) \to \text{SymBilin}(\Lambda, A(-1))\Big)$$

as follows:

It consists of

- A linear map
$$q : \Lambda \to A_{2\text{-tors}}$$
- An assignment
$$\lambda \in \Lambda \rightsquigarrow \mathcal{G}^\lambda \in \text{Ge}_A(X),$$
- A system of isomorphisms
$$\mathcal{G}^{\lambda_1+\lambda_2} \overset{c_{\lambda_1,\lambda_2}}{\simeq} \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2},$$
equipped with an associativity constraint;

- A datum h_{λ_1,λ_2} of commutativity for the squares

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \xrightarrow{c_{\lambda_1,\lambda_2}} & \mathcal{G}^{\lambda_1}\otimes\mathcal{G}^{\lambda_2} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{G}^{\lambda_2+\lambda_1} & \xrightarrow{c_{\lambda_2,\lambda_1}} & \mathcal{G}^{\lambda_2}\otimes\mathcal{G}^{\lambda_1}, \end{array}$$

  compatible with the associativity constraint and that square to the identity;
- For $\lambda_1=\lambda=\lambda_2$, we require that the datum of $h_{\lambda,\lambda}$, which in this case is the trivialization of the trivial $A_{2\text{-tors}}$-torsor, i.e., an element of 2-tors, equals $q(\lambda)$.

4.6.5. Note that Gr_T is not just a group-prestack over Ran, but a *commutative* group-prestack. Hence, along with

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T)=:\mathrm{FactGe}_A^{\mathbb{E}_1}(\mathrm{Gr}_T),$$

we can consider the spaces $\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T)$ for any $k\geq 1$ and also

$$\mathrm{FactGe}_A^{\mathrm{com}}(\mathrm{Gr}_T):=\mathrm{FactGe}_A^{\mathbb{E}_\infty}(\mathrm{Gr}_T):=\lim_{k}\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T).$$

We claim, however, that the forgetful maps

$$\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T)\to\mathrm{FactGe}_A^{\mathbb{E}_1}(\mathrm{Gr}_T)$$

are all equivalences.

First off, the maps $\mathrm{FactGe}_A^{\mathbb{E}_{k+1}}(\mathrm{Gr}_T)\to\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T)$ are automatically equivalences for $k\geq 3$ because A-gerbes are 2-categorical objects. Similarly, the forgetful map $\mathrm{FactGe}_A^{\mathbb{E}_3}(\mathrm{Gr}_T)\to\mathrm{FactGe}_A^{\mathbb{E}_2}(\mathrm{Gr}_T)$ is automatically fully faithful.

An $\mathbb{E}_2$-structure on a multiplicative gerbe $\mathcal{G}$ translates as a datum of commutativity for the squares

$$\begin{array}{ccc} \sigma^*(\mathcal{G}^{\lambda_1,\lambda_2}) & \xrightarrow{\sigma^*(4.47)} & \sigma^*(\mathcal{G}^{\lambda_1}\boxtimes\mathcal{G}^{\lambda_2}) \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_2,\lambda_1} & \xrightarrow{(4.47)} & \mathcal{G}^{\lambda_2}\boxtimes\mathcal{G}^{\lambda_1} \end{array} \tag{4.49}$$

that coincides with the one coming from factorization over $X\times X-\Delta$.

Thus, we are already given the datum of commutation of (4.49) over $X\times X-\Delta$. Therefore, this datum automatically uniquely extends to all of $X\times X$. This implies that

$$\mathrm{FactGe}_A^{\mathbb{E}_2}(\mathrm{Gr}_T)\to\mathrm{FactGe}_A^{\mathbb{E}_1}(\mathrm{Gr}_T)$$

is an equivalence.

An object in $\mathrm{FactGe}_A^{\mathbb{E}_2}(\mathrm{Gr}_T)$ comes from $\mathrm{FactGe}_A^{\mathbb{E}_3}(\mathrm{Gr}_T)$ if and only if the diagrams (4.49) square to the identity, in the sense that the datum of commutativity for the outer square in

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1,\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1}\boxtimes\mathcal{G}^{\lambda_2} \\ \downarrow & & \downarrow \\ \sigma^*\circ\sigma^*(\mathcal{G}^{\lambda_1,\lambda_2}) & \longrightarrow & \sigma^*\circ\sigma^*(\mathcal{G}^{\lambda_1}\boxtimes\mathcal{G}^{\lambda_2}) \\ \downarrow & & \downarrow \\ \sigma^*(\mathcal{G}^{\lambda_2,\lambda_1}) & \longrightarrow & \sigma^*(\mathcal{G}^{\lambda_2}\boxtimes\mathcal{G}^{\lambda_1}) \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_1,\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1}\boxtimes\mathcal{G}^{\lambda_2} \end{array}$$

is the tautological one. But this is automatic because this condition holds over $X\times X-\Delta$.

Remark 4.6.6. Note that from Proposition 3.1.9 we obtain the following a priori description of the groupoid $\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T)$ (here $k \geq 1$) as

$$\mathrm{Maps}^{\mathbb{E}_k}_{\mathrm{PreStk}}(B_{\mathrm{et}}(T) \times X, B^4_{\mathrm{et}}(A(1)) \simeq \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B^{1+k}_{\mathrm{et}}(T) \times X, B^{4+k}_{\mathrm{et}}(A(1)).$$

From Sect. 4.6.5 we obtain that the looping map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B^{1+k}_{\mathrm{et}}(T) \times X, B^{4+k}_{\mathrm{et}}(A(1)) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T) \times X, B^4_{\mathrm{et}}(A(1))$$

has the following properties:

- Both sides have vanishing homotopy groups π_i for $i \geq 3$, i.e.,
$$H^{i+k}_{\mathrm{et}}(B^{1+k}_{\mathrm{et}}(T) \times X; \mathrm{pt} \times X, A(1)) = H^i_{\mathrm{et}}(B_{\mathrm{et}}(T) \times X; \mathrm{pt} \times X, A(1)) = 0$$
for $i \leq 1$.
- It induces an isomorphism on π_2 for any $k \geq 1$, i.e., the map
$$H^{2+k}_{\mathrm{et}}(B^{1+k}_{\mathrm{et}}(T) \times X; \mathrm{pt} \times X, A(1)) \to H^2_{\mathrm{et}}(B_{\mathrm{et}}(T) \times X; \mathrm{pt} \times X, A(1))$$
is an isomorphism (note that the RHS identifies with $\mathrm{Hom}(\Lambda, A)$).
- It induces an isomorphism on π_1 for any $k \geq 1$, i.e., the map
$$H^{3+k}_{\mathrm{et}}(B^{1+k}_{\mathrm{et}}(T) \times X; \mathrm{pt} \times X, A(1)) \to H^3_{\mathrm{et}}(B_{\mathrm{et}}(T) \times X; \mathrm{pt} \times X, A(1))$$
is an isomorphism; in fact both sides are isomorphic to $H^1_{\mathrm{et}}(X, \mathrm{Hom}(\Lambda, A))$.
- For any $k \geq 1$, the induced map on π_0, i.e., the map
$$H^{4+k}_{\mathrm{et}}(B^{1+k}_{\mathrm{et}}(T) \times X; \mathrm{pt} \times X, A(1)) \to H^4_{\mathrm{et}}(B_{\mathrm{et}}(T) \times X; \mathrm{pt} \times X, A(1))$$
is injective with the image corresponding to the subset of $\mathrm{Quad}(\Lambda, A(-1))$, consisting of those quadratic forms, whose associated bilinear form is zero.

Remark 4.6.7. The isomorphisms of Remark 4.6.6 are the étale counterparts of the corresponding isomorphisms in the context of *algebraic topology*, which we will now explain. (We will come back and do a similar analysis in the étale setting in Sect. C.3.)

Let T be a topological torus with coweight lattice Λ. We can think of $B(T)$ as $B^2(\Lambda)$.

We start with the groupoid

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B(\Lambda), B^3(A)) \simeq \mathrm{Maps}_{\mathrm{Grp(Spc)}}(\Lambda, B^2(A)).$$

We can think of its objects as monoidal categories $\mathcal{C}$ that are groupoids such that $\pi_0(\mathcal{C}) = \Lambda$ (as monoids) and $\pi_1(\mathbf{1}_{\mathcal{C}}) = A$ (as groups).

A datum of lifting of such a point to a point of

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)) \simeq \mathrm{Maps}_{\mathbb{E}_1(\mathrm{Spc})}(B(\Lambda), B^3(A))$$

amounts to endowing the monoidal category $\mathcal{C}$ with a braiding. A further lift to an object of

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^{2+k}(\Lambda), B^{4+k}(A)) \simeq \mathrm{Maps}_{\mathbb{E}_{k+1}(\mathrm{Spc})}(B(\Lambda), B^3(A))$$

for $k \geq 1$ amounts to the *condition* that the resulting braided monoidal category be *symmetric*. This already implies that the forgetful map

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^{2+k+1}(\Lambda), B^{4+k+1}(A)) \to \mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^{2+k}(\Lambda), B^{4+k}(A))$$

is an isomorphism for $k \geq 1$ and is fully faithful for $k = 0$.

Moreover, for $k \geq 0$, the group $\pi_2 \left(\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^{2+k}(\Lambda), B^{4+k}(A))\right)$ identifies with

$$\mathrm{Maps}_{\mathrm{Grp(Spc)}}(\Lambda, A) = \mathrm{Hom}_{\mathrm{Ab}}(\Lambda, A),$$

and $\pi_1 \left(\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^{2+k}(\Lambda), B^{4+k}(A))\right)$ identifies with

$$\mathrm{Maps}_{\mathbb{E}_2(\mathrm{Spc})}(\Lambda, B(A)) \simeq \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B(A)) = \mathrm{Maps}_{\mathrm{Ab}}(\Lambda, B(A)) = 0,$$

where Ab denotes the ∞-category of chain complexes of abelian groups.

Finally, the set of isomorphism classes of braided monoidal categories as above is in bijection with $\mathrm{Quad}(\Lambda, A)$. Indeed, for a given $\mathcal{C}$, the corresponding bilinear form $b(\lambda_1, \lambda_2)$ is recovered as the square of the braiding

$$c^{\lambda_1} \otimes c^{\lambda_2} \to c^{\lambda_2} \otimes c^{\lambda_1} \to c^{\lambda_1} \otimes c^{\lambda_2},$$

and the quadratic form $q(\lambda)$ is recovered as the value of the braiding

$$c^{\lambda} \otimes c^{\lambda} \to c^{\lambda} \otimes c^{\lambda}.$$

In particular, this braided monoidal category is symmetric if and only if $b(-,-) = 0$.

4.6.8. Consider the connective spectrum

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)).$$

It follows from Remark 4.6.7 that it fits into a fiber sequence

$$B^2(\mathrm{Hom}(\Lambda, A)) \to \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \to \mathrm{Hom}(\Lambda, A_{2\text{-tors}}). \tag{4.50}$$

Remark 4.6.9. Note that since Λ is projective in the category of abelian groups, we have

$$B^2(\mathrm{Hom}(\Lambda, A)) \simeq \mathrm{Maps}_{\mathrm{Ab}}(\Lambda, B^2(A)).$$

So the map $B^2(\mathrm{Hom}(\Lambda, A)) \to \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A))$ can be interpreted as a map

$$\mathrm{Maps}_{\mathrm{Ab}}(\Lambda, B^2(A)) \to \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A))$$

given by the Dold-Kan functor $\mathrm{Ab}^{\leq 0} \to \mathbb{E}_\infty(\mathrm{Spc})$.

4.6.10. Similarly, we have a fiber sequence

$$\mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) \to \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \to \mathrm{Hom}(\Lambda, A_{2\text{-tors}}), \tag{4.51}$$

so that the diagram

$$\begin{array}{ccc} B^2(\mathrm{Hom}(\Lambda, A)) & \longrightarrow & \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \\ \downarrow & & \downarrow \\ \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) & \longrightarrow & \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \end{array}$$

is a push-out square.

4.6.11. The explicit description of $\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_T)$ in Sect. 4.6.4 implies that we have a canonical identification

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \simeq \mathrm{FactGe}^{\mathbb{E}_\infty}_A(\mathrm{Gr}_T) \simeq \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_T) \tag{4.52}$$

that fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) & \longrightarrow & \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{FactGe}^0_A(\mathrm{Gr}_T) & \longrightarrow & \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_T). \end{array}$$

Finally, the fiber sequence (4.51) is compatible with the fiber sequence

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_T) \to \mathrm{FactGe}_A(\mathrm{Gr}_T) \to \mathrm{Quad}(\Lambda, A(-1))$$

via the commutative diagram

$$\begin{array}{ccccc} \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) & \longrightarrow & \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) & \longrightarrow & \mathrm{Hom}(\Lambda, A_{2\text{-tors}}) \\ \sim\downarrow & & \downarrow\sim & & \downarrow= \\ \mathrm{FactGe}^0_A(\mathrm{Gr}_T) & \longrightarrow & \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_T) & \longrightarrow & \mathrm{Hom}(\Lambda, A_{2\text{-tors}}) \\ =\downarrow & & \downarrow & & \downarrow \\ \mathrm{FactGe}^0_A(\mathrm{Gr}_T) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_T) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)). \end{array}$$

4.6.12. In Sect. 4.8 we will see that if $A_{2\text{-tors}} \simeq \mathbb{Z}/2\mathbb{Z}$, there a *canonical* identification

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \simeq B^2(\mathrm{Hom}(\Lambda, A)) \times \mathrm{Hom}(\Lambda, \mathbb{Z}/2\mathbb{Z}).$$

This implies that for $A_{2\text{-tors}} \simeq \mathbb{Z}/2\mathbb{Z}$, we have a canonical identification

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_T) \simeq \mathrm{FactGe}^0_A(\mathrm{Gr}_T) \times \mathrm{Hom}(\Lambda, \mathbb{Z}/2\mathbb{Z}).$$

4.7. **More general abelian groups.** In this section we generalize the discussion of Sect. 4.6 to the case when instead of a lattice Λ (thought of as a lattice of cocharacters of a torus) we take a general finitely generated abelian group. We need this in order to state the metaplectic version of geometric Satake.

4.7.1. Let Γ be a finitely generated abelian group whose torsion part is of order prime to char(k). Let

$$\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$$

be the group-prestack defined as in Sect. A.

It s basic feature is that if Γ is written as Λ_1/Λ_2, where $\Lambda_2 \subset \Lambda_1$ are lattices, then we have a map

$$\mathrm{Gr}_{T_1}/\mathrm{Gr}_{T_2} \to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}, \tag{4.53}$$

which becomes an isomorphism after sheafification in the topology generated by finite surjective maps (in the above formula T_i is the torus $\Lambda_i \otimes \mathbb{G}_m$.

In particular, pullback with respect to (4.53) defines an equivalence on the categories of A-gerbes, A-torsors, etc.

4.7.2. The group-prestack $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ has a natural factorization structure over Ran, so we can talk about the space $\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$.

Since $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ is a (commutative) group-prestack over Ran, along with $\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$, we can consider the space (in fact, commutative group in spaces)

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}), \tag{4.54}$$

that correspond to gerbes that respect that group structure on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ over Ran.

Remark 4.7.3. Along with $\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$, one can also consider its variants

$$\mathrm{FactGe}^{\mathbb{E}_k}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}),\ \mathrm{FactGe}^{\mathbb{E}_\infty}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \simeq \mathrm{FactGe}^{\mathrm{com}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}).$$

However, as in Sect. 4.6.5, one shows that the forgetful maps

$$\mathrm{FactGe}^{\mathbb{E}_k}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{FactGe}^{\mathbb{E}_1}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) = \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$$

are equivalences for all $k \geq 1$.

4.7.4. The following results from Proposition 4.6.2:

Corollary 4.7.5. *Let Γ be written as a quotient of two lattices as in Sect. 4.7.1. Let $\mathcal{G}_1$ be a factorization A-gerbe on Gr_{T_1}, and let b_1 and q_1 be the associated bilinear and quadratic forms on Λ_1, respectively.*

(a) *The gerbe $\mathcal{G}_1$ can be descended to a factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$* only if $b_1(\Lambda_2, -) = 0$.

(a') *In the situation of* (a)*, a descent exists étale-locally on X if and only if the restriction of q_1 to Λ_2 is trivial.*

(a") *In the situation of* (a')*, a descent datum is equivalent to the trivialization of $\mathcal{G}_2 := \mathcal{G}_1|_{\mathrm{Gr}_{T_2}}$ as a factorization gerbe on Gr_{T_2}.*

(b) *In the situation of* (a")*, the descended gerbe $\mathcal{G}$ admits a multiplicative structure if and only if b_1 is trivial. In the latter case, the multiplicative structure is unique up to a unique isomorphism.*

From here, we obtain:

Corollary 4.7.6. *We have a canonically defined map*

$$\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{Quad}(\Gamma, A(-1)) \tag{4.55}$$

such that:

(a) *The forgetful map*

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$$

fits in the fiber square

$$\begin{array}{ccc} \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\Gamma, A)_{2\text{-tors}} & \longrightarrow & \mathrm{Quad}(\Gamma, A(-1)). \end{array} \tag{4.56}$$

(b) *The fiber of the map* (4.55)*, denoted* $\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$*, consists of objects that are étale-locally trivial on* X*.*

(c) *We have a push-out square*

$$\begin{array}{ccc} \mathrm{Maps}_{\mathrm{Ab}}(\Gamma, B^2(A)) & \longrightarrow & \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A)) \\ \downarrow & & \downarrow \\ \mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) & \longrightarrow & \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \end{array}$$

Remark 4.7.7. Note that it follows from Corollary 4.7.6(b) that we have a canonical isomorphism

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \simeq \mathrm{Maps}_{\mathrm{Ab}}(\Gamma, \mathrm{C}^\bullet_{\mathrm{et}}(X, A)[2]) \simeq \mathrm{Maps}(X, \mathrm{Maps}_{\mathrm{Ab}}(\Gamma, B^2(A))_{\mathrm{et}}),$$

where we denote by $\mathrm{Maps}_{\mathrm{Ab}}(\Gamma, B^2(A))_{\mathrm{et}}$ the étale localization of the constant presheaf with value $\mathrm{Maps}_{\mathrm{Ab}}(\Gamma, B^2(A))$.

Note also that $\mathrm{Maps}_{\mathrm{Ab}}(\Gamma, B^2(A))$ has non-trivial homotopy groups in degrees 2 and 1, equal to $\mathrm{Hom}(\Gamma, A)$ and $\mathrm{Ext}^1(\Gamma, A)$, respectively.

Similarly, it follows from Corollary 4.7.6(c) that we have

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \simeq \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, \mathrm{Ge}_A(X)) \simeq \mathrm{Maps}(X, \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))_{\mathrm{et}}),$$

where we denote by $\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))_{\mathrm{et}})$ the étale localization of the constant presheaf with value $\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))$.

As we have seen above, the non-trivial homotopy groups of $\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))$ are in degrees 2, 1 and 0, where the homotopy groups in degrees -2 and -1 are the same as for $\mathrm{Maps}_{\mathrm{Ab}}(\Gamma, B^2(A))$, while in degree 0 we have $\mathrm{Hom}(\Gamma, A_{2\text{-tors}})$.

4.7.8. Let now G be a connective reductive group. By Sect. A.2, we have a canonically defined map

$$\mathrm{Gr}_G \to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}, \tag{4.57}$$

compatible with the factorization structure.

From here we obtain a map

$$\mathrm{FactGe}_A(\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G)\otimes\mathbb{G}_m}) \to \mathrm{FactGe}_A(\mathrm{Gr}_G). \tag{4.58}$$

By construction, the diagram

$$\begin{array}{ccc} \mathrm{FactGe}_A(\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G)\otimes\mathbb{G}_m}) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_G) \\ \downarrow & & \downarrow \\ \mathrm{Quad}(\pi_{1,\mathrm{alg}}(G), A(-1)) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}} \end{array}$$

commutes. Hence, we obtain a map

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G)\otimes\mathbb{G}_m}) \to \mathrm{FactGe}^0_A(\mathrm{Gr}_G). \tag{4.59}$$

From Corollaries 3.3.6 and 4.7.6(b) and Remark 4.7.7 we obtain:

Corollary 4.7.9. *The map* (4.59) *is an isomorphism.*

From here we obtain:

Corollary 4.7.10. *The map* (4.58) *is fully faithful.*

4.8. **Splitting multiplicative gerbes.** In this subsection we will assume that $A_{2\text{-tors}} = \mathbb{Z}/2\mathbb{Z}$. (Note that this happens, e.g., if $A \subset E^\times$.)

We will show that in this case the fiber sequence

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{Hom}(\Gamma, A_{2\text{-tors}})$$

admits a canonical splitting, functorial in Γ.

4.8.1. According to Corollary 4.7.6(c), it suffices to construct a splitting of the fiber sequence

$$B^2(\mathrm{Hom}(\Gamma, A)) \to \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A)) \to \mathrm{Hom}(\Gamma, A_{2\text{-tors}}).$$

By functoriality, it suffices to treat the universal case: i.e., when $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and we need to construct an object of

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\mathbb{Z}/2\mathbb{Z}, B^2(\mathbb{Z}/2\mathbb{Z}))$$

that projects to the identity map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

4.8.2. We will construct the sought-for object in $\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\mathbb{Z}/2\mathbb{Z}, B^2(\mathbb{Z}/2\mathbb{Z}))$ as a symmetric monoidal groupoid $\mathcal{C}$ with $\pi_0(\mathcal{C}) \simeq \pi_1(\mathcal{C}) \simeq \mathbb{Z}/2\mathbb{Z}$.

As a monoidal groupoid, we set $\mathcal{C}$ be the product $\mathbb{Z}/2\mathbb{Z} \times B(\mathbb{Z}/2\mathbb{Z})$. A braided monoidal structure on such a $\mathcal{C}$ is equivalent to a choice of a bilinear form b' on $\mathbb{Z}/2\mathbb{Z}$ with values in $\mathbb{Z}/2\mathbb{Z}$. We set it to be

$$b'(\bar{1}, \bar{1}) = \bar{1}.$$

The resulting braided monoidal structure is automatically symmetric, and the associated quadratic form $q : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the identity map, as required.

4.8.3. In what follows, for a given element $\epsilon \in \mathrm{Hom}(\Gamma, A_{2\text{-tors}})$, we will denote by $\mathcal{G}^\epsilon$ the resulting multiplicative factorization gerbe on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$.

For $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and the identity map, we will denote this gerbe by $\mathcal{G}^{\epsilon_{\mathrm{taut}}}$. We will refer to it as the *sign gerbe*.

Remark 4.8.4. Note that $\mathcal{G}^\epsilon$, viewed as a gerbe on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$, equipped with the multiplicative structure, admits a canonical trivialization. However, this trivialization is *not* compatible with the factorization structure.

4.8.5. For a given object $\mathcal{G} \in \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$ let us denote by ϵ the map

$$\Gamma \to A_{2\text{-tors}} \simeq \mathbb{Z}/2\mathbb{Z}$$

that measures the obstruction of $\mathcal{G}$ to belong to $\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$.

We obtain that, canonically attached to $\mathcal{G}$, there exists an object

$$\mathcal{G}^0 \in \mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \simeq \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Gamma, A))),$$

such that

$$\mathcal{G} \simeq \mathcal{G}^0 \otimes \mathcal{G}^\epsilon,$$

where $\mathcal{G}^\epsilon$ is as in Sect. 4.8.3.

4.9. **More on the sign gerbe.** In this subsection we will make a digression, needed for the sequel, in which we will discuss several manipulations with the gerbe $\mathcal{G}^{\epsilon_{\mathrm{taut}}}$ introduced in Sect. 4.8.

4.9.1. Let Z be a prestack over Ran, equipped with a factorization structure, and equipped with a map to $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$, compatible with the factorization structures.

Let $\mathcal{L}$ be a line bundle on Z. We equip it with a $\mathbb{Z}/2\mathbb{Z}$-graded structure as follows: for an affine test scheme S and a map $S \to Z$ such that the composite $S \to Z \to \mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$ maps to the even/odd connected component, the grading on $\mathcal{L}|_S$ is even/odd.

Let us be given a factorization structure on $\mathcal{L}$, viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle. Note that $\mathcal{L}^{\otimes 2}$ is then a plain factorization line bundle. Assume that $\mathrm{char}(k) \neq 2$, and consider the $\mathbb{Z}/2\mathbb{Z}$-gerbe on Z given by $(\mathcal{L}^{\otimes 2})^{\frac{1}{2}}$, equipped with its natural factorization structure.

It is easy to see, however, that $(\mathcal{L}^{\otimes 2})^{\frac{1}{2}}$ identifies canonically with $\mathcal{G}^{\epsilon_{\mathrm{taut}}}|_Z$. Indeed, both gerbes are canonically trivialized as plain gerbes, and the factorization structures on both are given by the sign rules.

4.9.2. *An example.* Let us take $Z = \mathrm{Gr}_{\mathbb{G}_m}$. We can take as $\mathcal{L}$ the *determinant line bundle* on $\mathrm{Gr}_{\mathbb{G}_m}$, denoted $\det_{\mathbb{G}_m,\mathrm{St}}$, corresponding to the standard one-dimensional representation of $\mathbb{G}_m$.

4.9.3. Let $\mathcal{C}$ be a sheaf of categories over $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$, and let $\mathcal{C}$ be equipped with a factorization structure, compatible with the factorization structure on $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$.

Viewing $\mathbb{Z}/2\mathbb{Z}$ as 2-torsion in $E^\times$, and using the twisting construction of Sect. 1.7.2, we can twist $\mathcal{C}$ by $\mathcal{G}^{\epsilon_{\mathrm{taut}}}$ and obtain a new factorization sheaf of categories, denoted $\mathcal{C}^{\epsilon_{\mathrm{taut}}}$.

Suppose that in the above situation $\mathcal{C}$ is endowed with a monoidal (resp., symmetric monoidal) structure, compatible with the group structure on $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$. Then $\mathcal{C}^{\epsilon_{\mathrm{taut}}}$ also acquires a monoidal (resp., symmetric monoidal) structure.

Remark 4.9.4. Note that for any $S \to \mathrm{Ran}$, the corresponding categories $\mathcal{C}(S)$ and $\mathcal{C}^{\epsilon_{\mathrm{taut}}}(S)$ are canonically identified (since $\mathcal{G}^{\epsilon_{\mathrm{taut}}}$, viewed as a plain gerbe, is trivial). However, the factorization structures on $\mathcal{C}(S)$ and $\mathcal{C}^{\epsilon_{\mathrm{taut}}}(S)$ are different. The same applies to the monoidal (resp., symmetric monoidal) situation.

4.9.5. Assume for a moment that the structure on $\mathcal{C}$ of sheaf of categories over $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$ has been refined to that of sheaf over $\mathrm{Gr}_{\mathbb{G}_m}$, also compatible with the factorization structures.

Note that $\mathrm{Gr}_{\mathbb{G}_m}$ carries a locally constant function, denoted d, given by the degree. Hence, we have a well-defined endo-functor on $\mathcal{C}$,

$$c \mapsto c[d]. \tag{4.60}$$

Note that this functor is *not* compatible with the factorization structure, due to sign rules. However, when we view (4.60) as a functor

$$\mathcal{C} \to \mathcal{C}^{\epsilon_{\mathrm{taut}}},$$

it is an equivalence of factorization categories.

5. Jacquet functors for factorization gerbes

In this section we take G to be reductive. We will study the interaction between factorization gerbes on Gr_G and those on Gr_M, where M is the Levi quotient of a parabolic of G.

5.1. **The naive Jacquet functor.** Let P be a parabolic subgroup of G, and we let $P \twoheadrightarrow M$ be its Levi quotient. Let N_P denote the unipotent radical of P.

5.1.1. Consider the diagram of the Grassmannians

$$\mathrm{Gr}_G \xleftarrow{\mathsf{p}} \mathrm{Gr}_P \xrightarrow{\mathsf{q}} \mathrm{Gr}_M.$$

We claim that pullback along q defines an equivalence,

$$\mathrm{Ge}_A(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_M) \to \mathrm{Ge}_A(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_P) \tag{5.1}$$

for any $S \to \mathrm{Ran}$, in particular, inducing an equivalence

$$\mathrm{FactGe}_A(\mathrm{Gr}_M) \to \mathrm{FactGe}_A(\mathrm{Gr}_P).$$

5.1.2. To show that (5.1) is an equivalence, let us choose a splitting $M \hookrightarrow P$ of the projection $P \twoheadrightarrow M$. In particular, we obtain an adjoint action of M on N_P. Hence, we obtain an action of the group-prestack $\mathfrak{L}^+(M)$ (see Sect. 7.1.3 for the definition of this group-prestack) over Ran on Gr_{N_P}.

We can view Gr_M as a quotient $\mathfrak{L}(M)/\mathfrak{L}^+(M)$ (see Sect. 7.2.2), and hence we can view the map

$$\mathfrak{L}(M) \to \mathrm{Gr}_M$$

as a $\mathfrak{L}^+(M)$-torsor. Then Gr_P, when viewed as a prestack over Gr_M is obtained by twisting Gr_{N_P} by the above $\mathfrak{L}^+(M)$-torsor.

Now, the equivalence in (5.1) follows from the fact that for any $S \to \mathrm{Ran}$, pullback defines an isomorphism

$$H^i_{\mathrm{et}}(S, A) \to H^i_{\mathrm{et}}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{N_P}, A)$$

for all i.

5.1.3. In terms of the parameterization given by Proposition 3.1.9, the map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{FactGe}_A(\mathrm{Gr}_M)$$

can be interpreted as follows:

It corresponds to the map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(A(1))) \to$$
$$\to \mathrm{Maps}_{\mathrm{Ptd(PreStk}_{/X})}(B(P) \times X, B^4_{\mathrm{et}}(A(1))) \xleftarrow{\sim} \mathrm{Maps}_{\mathrm{Ptd(PreStk}_{/X})}(B(M) \times X, B^4_{\mathrm{et}}(A(1))),$$

where the second arrow is an isomorphism since the map $B(P) \to B(M)$ induces an isomorphism in étale cohomology with constant coefficients.

Thus, if $\mathcal{G}^G$ is a factorization A-gerbe on Gr_G, and $\mathcal{G}^M$ is the corresponding the factorization A-gerbe on Gr_M, the corresponding quadratic forms

$$q : \Lambda \to A(-1)$$

coincide.

5.1.4. We now take $A := E^{\times,\mathrm{tors}}$. Given a factorization $E^{\times,\mathrm{tors}}$-gerbe $\mathcal{G}^G$ over Gr_G, consider its pullback to Gr_P, denoted $\mathcal{G}^P$. We let $\mathcal{G}^M$ denote the canonically defined factorization gerbe on Gr_M, whose pullback to Gr_P gives $\mathcal{G}^P$.

By construction, for any $S \to \mathrm{Ran}$, we have a well-defined pullback functor

$$\mathsf{p}^! : \mathrm{Shv}_{\mathcal{G}^G}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G) \to \mathrm{Shv}_{\mathcal{G}^P}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_P).$$

Furthermore, since the morphism q is ind-schematic, we have a well-defined push-forward functor

$$\mathsf{q}_* : \mathrm{Shv}_{\mathcal{G}^P}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_P) \to \mathrm{Shv}_{\mathcal{G}^M}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_M).$$

Thus, the composite $\mathsf{q}_* \circ \mathsf{p}^!$ defines a map between factorization sheaves of categories

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^M}(\mathrm{Gr}_M)_{/\,\mathrm{Ran}}. \tag{5.2}$$

We will refer to (5.2) as the *naive Jacquet functor*.

5.2. **The critical twist.** The functor (5.2) is not quite what we need for the purposes of geometric Satake. Namely, we will need to correct this functor by a cohomological shift that depends on the connected component of Gr_M (this is needed in order to arrange that the corresponding functor on the spherical categories maps perverse sheaves to perverse sheaves). However, this cohomological shift will destroy the compatibility of the Jacquet functor with factorization, due to sign rules. In order to compensate for this, we will apply an additional twist of our categories by the square root of the determinant line bundle.

The nature of this additional twist will be explained in the present subsection.

For the rest of this subsection we will assume that $\mathrm{char}(k) \neq 2$.

5.2.1. Let $\det_{\mathfrak{g}}$ denote the determinant line bundle on Gr_G, corresponding to the adjoint representation. It is constructed as follows. For an affine test scheme S and an S-point $I \subset \mathrm{Maps}(S, X)$ of Ran, consider the corresponding G-bundle $\mathcal{P}_G$ on $S \times X$, equipped with an isomorphism

$$\alpha : \mathcal{P}_G \simeq \mathcal{P}_G^0$$

over $U_I \subset S \times X$. Consider the corresponding vector bundles associated with the adjoint representation

$$\mathfrak{g}_{\mathcal{P}_G}|_{U_I} \simeq \mathfrak{g}_{\mathcal{P}_G^0}|_{U_I}.$$

Then

$$\det.\mathrm{rel.}(\mathfrak{g}_{\mathcal{P}_G}, \mathfrak{g}_{\mathcal{P}_G^0}) \tag{5.3}$$

is a well-defined line bundle[13] on S.

This construction is compatible with pullbacks under $S' \to S$, thereby giving rise to the sought-for line bundle $\det_{\mathfrak{g}}$ on Gr_G.

It is easy to see that $\det_{\mathfrak{g}}$ is equipped with a factorization structure over Ran.

5.2.2. Consider the factorization $\mathbb{Z}/2\mathbb{Z}$-gerbe $\det_{\mathfrak{g}}^{\frac{1}{2}}$ over Gr_G.

From now on we will choose a square root, denoted $\omega_X^{\otimes \frac{1}{2}}$ of the canonical line bundle ω_X on X (see again Remark 1.4.3 for our notational conventions).

Let P be again a parabolic of G. Consider the factorization gerbes $\det_{\mathfrak{g}}^{\frac{1}{2}}|_{\mathrm{Gr}_P}$ and $\det_{\mathfrak{m}}^{\frac{1}{2}}|_{\mathrm{Gr}_P}$ over Gr_P. We claim that the choice of $\omega_X^{\otimes \frac{1}{2}}$ gives rise to an identification of the gerbes

$$\det_{\mathfrak{g}}^{\frac{1}{2}}|_{\mathrm{Gr}_P} \simeq \det_{\mathfrak{m}}^{\frac{1}{2}}|_{\mathrm{Gr}_P} \otimes \mathcal{G}^{\epsilon_P}|_{\mathrm{Gr}_P}, \tag{5.4}$$

where $\mathcal{G}^{\epsilon_P}$ is the $\mathbb{Z}/2\mathbb{Z}$-gerbe on $\mathrm{Gr}_{M/[M,M]}$ corresponding to the map ϵ_P

$$\Lambda_{M/[M,M]} \overset{2\check{\rho}_{G,M}}{\longrightarrow} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z},$$

where $2\check{\rho}_{G,M} : M/[M,M] \to \mathbb{G}_m$ is the determinant of the action of M on $\mathfrak{n}(P)$.

In fact, we claim that the ratio of the line bundles $\det_{\mathfrak{g}}|_{\mathrm{Gr}_P}$ and $\det_{\mathfrak{m}}|_{\mathrm{Gr}_P}$, i.e.,

$$\det_{\mathfrak{g}}|_{\mathrm{Gr}_P} \otimes (\det_{\mathfrak{m}}|_{\mathrm{Gr}_P})^{\otimes -1},$$

admits a square root, to be denoted $\det_{\mathfrak{n}(P)}$, which is a $\mathbb{Z}$-graded (and, in particular, $\mathbb{Z}/2\mathbb{Z}$-graded) factorization line bundle on Gr_P, with the grading given by the map

$$\mathrm{Gr}_P \to \mathrm{Gr}_M \to \mathrm{Gr}_{M/[M,M]} \overset{2\check{\rho}_{G,M}}{\longrightarrow} \mathrm{Gr}_{\mathbb{G}_m}, \tag{5.5}$$

see Sects. 4.9.5 and 4.9.1.

[13]Note that the line bundle (5.3) is a priori $\mathbb{Z}$-graded, but since G is reductive, and in particular, unimodular, this grading is actually trivial (i.e., concentrated in degree 0).

Remark 5.2.3. In fact, more is true: the construction of [BD2, Sect. 4] defines a square root of $\det_{\mathfrak{g}}$ itself, again viewed as a graded factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle, where the grading is given by the map

$$\mathrm{Gr}_G \to \mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G)\otimes\mathbb{G}_m} \to \mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m},$$

where $\pi_{1,\mathrm{alg}}(G) \to \mathbb{Z}/2\mathbb{Z}$ is is the canonical map that fits into the diagram

$$\begin{array}{ccc} \Lambda & \longrightarrow & \pi_{1,\mathrm{alg}}(G) \\ {\scriptstyle 2\check{\rho}}\big\downarrow & & \big\downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z}, \end{array}$$

where $2\check{\rho}$ is the sum of positive roots.

5.2.4. The graded line bundle $\det_{\mathfrak{n}(P)}$ is constructed as follows. For an S-point $(I, \mathcal{P}_P, \mathcal{P}_G|_{U_I} \simeq \mathcal{P}^0_G|_{U_I})$ of Gr_P we set the value of $\det_{\mathfrak{n}(P)}$ on S to be

$$\mathrm{rel.det.}(\mathfrak{n}(P)_{\mathcal{P}_P} \otimes \omega_X^{\otimes\frac{1}{2}}, \mathfrak{n}(P)_{\mathcal{P}^0_P} \otimes \omega_X^{\otimes\frac{1}{2}}).$$

Let us construct the isomorphism

$$(\det_{\mathfrak{n}(P)})^{\otimes 2} \otimes \det_{\mathfrak{m}}|_{\mathrm{Gr}_P} \simeq \det_{\mathfrak{g}}|_{\mathrm{Gr}_P}.$$

Let us identify the vector space $\mathfrak{g}/\mathfrak{p}$ with the dual of $\mathfrak{n}(P)$ (say, using the Killing form). For an S-point $(I, \mathcal{P}_P, \mathcal{P}_G|_{U_I} \simeq \mathcal{P}^0_G|_{U_I})$ of Gr_P, denote

$$\mathcal{E} := \mathfrak{n}(P)_{\mathcal{P}_P} \text{ and } \mathcal{E}_0 := \mathfrak{n}(P)_{\mathcal{P}^0_P}.$$

Then the ratio of $\det_{\mathfrak{g}}|_S$ and $\det_{\mathfrak{m}}|_S$ identifies with the line bundle

$$\mathrm{rel.det.}(\mathcal{E}, \mathcal{E}_0) \otimes \mathrm{rel.det.}(\mathcal{E}^\vee, \mathcal{E}_0^\vee) \simeq \mathrm{rel.det.}(\mathcal{E}, \mathcal{E}_0) \otimes \mathrm{rel.det.}(\mathcal{E}_0^\vee, \mathcal{E}^\vee)^{\otimes -1}.$$

.

Note, however, that for any line bundle $\mathcal{L}$ on $S \times X$, we have

$$\mathrm{rel.det.}(\mathcal{E}, \mathcal{E}_0) \otimes \mathrm{rel.det.}(\mathcal{E}_0^\vee, \mathcal{E}^\vee)^{\otimes -1} \simeq \mathrm{rel.det.}(\mathcal{E} \otimes \mathcal{L}, \mathcal{E}_0 \otimes \mathcal{L}) \otimes \mathrm{rel.det.}(\mathcal{E}_0^\vee \otimes \mathcal{L}, \mathcal{E}^\vee \otimes \mathcal{L})^{\otimes -1}.$$

Thus, letting $\mathcal{L}$ be the pullback of $\omega_X^{\otimes\frac{1}{2}}$, we thus need to construct an isomorphism

$$\mathrm{rel.det.}(\mathcal{E} \otimes \omega_X^{\otimes\frac{1}{2}}, \mathcal{E}_0 \otimes \omega_X^{\otimes\frac{1}{2}}) \simeq \mathrm{rel.det.}(\mathcal{E}_0^\vee \otimes \omega_X^{\otimes\frac{1}{2}}, \mathcal{E}^\vee \otimes \omega_X^{\otimes\frac{1}{2}})^{-1}.$$

However, this follows from the (relative to S) local Serre duality on $S \times X$:

$$\mathbb{D}^{\mathrm{Serre}}_{/S}(\mathcal{E} \otimes \omega_X^{\otimes\frac{1}{2}}) \simeq \mathcal{E}^\vee \otimes \omega_X^{\otimes\frac{1}{2}}[1] \text{ and } \mathbb{D}^{\mathrm{Serre}}_{/S}(\mathcal{E}_0 \otimes \omega_X^{\otimes\frac{1}{2}}) \simeq \mathcal{E}_0^\vee \otimes \omega_X^{\otimes\frac{1}{2}}[1].$$

Remark 5.2.5. The construction of the isomorphism (5.4) depended on the choice of $\omega_X^{\otimes\frac{1}{2}}$.

By analyzing the above construction one can show that the discrepancy between the two sides in (5.4) is canonically isomorphic to the $\mathbb{Z}/2\mathbb{Z}$-gerbe pulled back via

$$\mathrm{Gr}_P \to \mathrm{Gr}_M \to \mathrm{Gr}_{M/[M,M]} \overset{2\check{\rho}_{G,M}}{\to} \mathrm{Gr}_{\mathbb{G}_m}$$

from the object in

$$\mathrm{FactGe}^0_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Gr}_{\mathbb{G}_m}) \subset \mathrm{FactGe}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Gr}_{\mathbb{G}_m})$$

attached to the *gerbe*

$$\omega_X^{\frac{1}{2}} \in \mathrm{Ge}_{\mathbb{Z}/2\mathbb{Z}}(X)$$

via

$$\mathrm{Ge}_{\mathbb{Z}/2\mathbb{Z}}(X) \simeq \mathrm{FactGe}^0_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Gr}_{\mathbb{G}_m}).$$

5.3. **The corrected Jacquet functor.** We will now use the gerbe $\mathcal{G}^{\epsilon_P}$ from the previous subsection in order to introduce a correction to the naive Jacquet functor from Sect. 5.1.4.

5.3.1. Let $d_{G,M} : \mathrm{Gr}_P \to \mathbb{Z}$ be locally constant function on Gr_P corresponding to the map (5.5), see Sect. 4.9.5.

Given a factorization $E^{\times,\mathrm{tors}}$-gerbe $\mathcal{G}^G$ on Gr_G and the corresponding factorization gerbe $\mathcal{G}^M$ on Gr_M (see Sect. 5.1.4), we will now define the corrected Jacquet functor as a map between factorization sheaves of categories:

$$J^G_M : \mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Gr}_G)_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^M\otimes \det_{\mathfrak{m}}^{\frac{1}{2}}}(\mathrm{Gr}_M)_{/\,\mathrm{Ran}}. \tag{5.6}$$

5.3.2. Namely, J^G_M is the composition of the following four factorizable operations:

(i) The pullback functor

$$\mathsf{p}^! : \mathrm{Shv}_{\mathcal{P}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Gr}_G)_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{(\mathcal{P}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}};$$

(ii) The identification

$$\mathrm{Shv}_{(\mathcal{P}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}} \simeq \mathrm{Shv}_{(\mathcal{G}^M\otimes \det_{\mathfrak{m}}^{\frac{1}{2}}\otimes \mathcal{G}^{\epsilon_P})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}},$$

given by the isomorphism of gerbes (5.4);

(iii) The cohomological shift functor $\mathcal{F} \mapsto \mathcal{F}[-d_{G,M}]$

$$\mathrm{Shv}_{(\mathcal{G}^M\otimes \det_{\mathfrak{m}}^{\frac{1}{2}}\otimes \mathcal{G}^{\epsilon_P})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{(\mathcal{G}^M\otimes \det_{\mathfrak{m}}^{\frac{1}{2}})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}},$$

see Sect. 4.9.5.

(iv) The pushforward functor

$$\mathsf{q}_* : \mathrm{Shv}_{(\mathcal{G}^M\otimes \det_{\mathfrak{m}}^{\frac{1}{2}})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^M\otimes \det_{\mathfrak{m}}^{\frac{1}{2}}}(\mathrm{Gr}_M)_{/\,\mathrm{Ran}}.$$

6. The metaplectic Langlands dual datum

In section we take G to be reductive[14].

Given a factorization gerbe $\mathcal{G}$ on Gr_G, we will define the *metaplectic Langlands dual datum* attached to $\mathcal{G}$, and the corresponding notion of twisted local system on X.

6.1. **The metaplectic Langlands dual *root datum*.** The first component of the metaplectic Langlands dual datum is purely combinatorial and consists of a certain root datum that only depends on the root datum of G and q. This is essentially the same as the root datum defined by G. Lusztig as a recipient of the quantum Frobenius.

6.1.1. Given a factorization A-gerbe $\mathcal{G}^G$ on Gr_G, let

$$q : \Lambda \to A(-1)$$

$$b : \Lambda \times \Lambda \to A(-1)$$

be the associated quadratic and bilinear forms, respectively. Let $\Lambda^\sharp \subset \Lambda$ be the kernel of b. Let $\check{\Lambda}^\sharp$ be the dual of $\Lambda^\sharp$. Note that the inclusions

$$\Lambda^\sharp \subset \Lambda \text{ and } \check{\Lambda} \subset \check{\Lambda}^\sharp$$

induce isomorphisms after tensoring with $\mathbb{Q}$.

Let

$$(\Delta \subset \Lambda,\, \check{\Delta} \subset \check{\Lambda})$$

be the root datum for G. Following [Lus], we will now define a new root datum

$$(\Delta^\sharp \subset \Lambda^\sharp, \check{\Delta}^\sharp \subset \check{\Lambda}^\sharp). \tag{6.1}$$

[14]We will assume that the algebraic fundamental group of the derived group of G, i.e., the torsion part of $\pi_{1,\mathrm{alg}}(G)$, is of order prime to $\mathrm{char}(k)$.

6.1.2. We let $\Delta^\sharp$ be equal to Δ as an *abstract set*. For each element $\alpha \in \Delta$, we let the corresponding element $\alpha^\sharp \in \Delta^\sharp$ be equal to

$$\mathrm{ord}(q(\alpha)) \cdot \alpha \in \Lambda,$$

and the corresponding element $\check{\alpha}^\sharp \in \check{\Delta}^\sharp$ be

$$\frac{1}{\mathrm{ord}(q(\alpha))} \cdot \check{\alpha} \in \check{\Lambda} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}.$$

The fact that q lies in $\mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}$ implies that $\alpha^\sharp$ and $\check{\alpha}^\sharp$ defined in this way indeed belong to $\Lambda^\sharp \subset \Lambda$ and $\check{\Lambda}^\sharp \subset \check{\Lambda} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, respectively.

6.1.3. Since q is W-invariant, the action of W on Λ preserves $\Lambda^\sharp$. Moreover, for each $\alpha \in \Delta$, the action of the corresponding reflection $s_\alpha \in W$ on $\Lambda^\sharp$ equals that of $s_{\alpha^\sharp}$.

This implies that restriction defines an isomorphism from W to the group $W^\sharp$ of automorphisms of $\Lambda^\sharp$ generated by the elements $s_{\alpha^\sharp}$.

Hence, (6.1) is a finite root system with Weyl group $W^\sharp$, isomorphic to the original Weyl group W.

It follows from the construction that if α_i are the simple coroots of Δ, then the corresponding elements $\alpha_i^\sharp \in \Lambda^\sharp$ form a set of simple coroots of $\Delta^\sharp$.

6.1.4. We let $G^\sharp$ denote the reductive group (over k) corresponding to (6.1).

6.2. **The "π_1-gerbe".** Let $\mathcal{G}^G$ be as above. In this subsection we will show that in addition to the reductive group $G^\sharp$, the datum of $\mathcal{G}^G$ defines a certain *multiplicative* factorization gerbe on the affine Grassmannian attached to the abelian group $\pi_{1,\mathrm{alg}}(G^\sharp)$.

6.2.1. Let $\mathcal{G}^T$ be the factorization gerbe on Gr_T, corresponding to $\mathcal{G}^G$ via Sect. 5.1.4. Consider the corresponding torus $T^\sharp$.

Let $\mathcal{G}^{T^\sharp}$ be the factorization gerbe on $\mathrm{Gr}_{T^\sharp}$ equal to the pullback of $\mathcal{G}^T$ under $T^\sharp \to T$. By Proposition 4.6.2, the gerbe $\mathcal{G}^{T^\sharp}$ carries a canonical multiplicative structure.

Consider the algebraic fundamental group $\pi_{1,\mathrm{alg}}(G^\sharp)$ of $G^\sharp$, and the projection $\Lambda^\sharp \to \pi_{1,\mathrm{alg}}(G^\sharp)$. Consider the corresponding map

$$\mathrm{Gr}_{T^\sharp} \to \mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}. \tag{6.2}$$

We claim that there exists a canonically defined multiplicative factorization A-gerbe $\mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$ on $\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$, whose pullback under (6.2) identifies with $\mathcal{G}^{T^\sharp}$.

6.2.2. By Corollary 4.7.5, we need to show that for every simple coroot α_i, the pullback of $\mathcal{G}^T$ to $\mathrm{Gr}_{\mathbb{G}_m}$ under

$$\mathbb{G}_m \xrightarrow{\alpha_i^\sharp} T \tag{6.3}$$

is trivialized.

By the transitivity of the construction in Sect. 5.1.4, we can replace G by its Levi subgroup M_i of semi-simple rank 1, corresponding to α_i. Furthermore, using the map $SL_2 \to M_i$, we can assume that $G = SL_2$.

6.2.3. Let $\det_{SL_2,\mathrm{St}}$ and $\det_{\mathbb{G}_m,k}$ be as in Sect. 4.5.6. We can assume that object in $\mathrm{FactGe}_A(\mathrm{Gr}_{SL_2})$ is of the form $(\det_{SL_2,\mathrm{St}})^a$ for some element $a \in A(-1)$. Its restriction to $\mathrm{FactGe}_A(\mathrm{Gr}_{\mathbb{G}_m})$ identifies with $(\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^a$.

The associated quadratic form

$$q : \mathbb{Z} \to A(-1)$$

takes value a on the generator $1 \in \mathbb{Z}$. Let $n := \mathrm{ord}(a)$.

We need to show that the pullback of $(\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^a$ under the isogeny

$$\mathbb{G}_m \overset{x\mapsto x^n}{\longrightarrow} \mathbb{G}_m \tag{6.4}$$

is canonically trivial as a factorization gerbe on $\mathrm{Gr}_{\mathbb{G}_m}$. For this, it suffices to show that the pullback of the factorization line bundle $\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1}$ under the above isogeny admits a canonical n-th root.

We note that the pullback of $\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1}$ under (6.4) identifies with $\det_{\mathbb{G}_m,n} \otimes \det_{\mathbb{G}_m,-n}$. However, a local calculation shows that we have a canonical isomorphism

$$\det_{\mathbb{G}_m,n} \otimes \det_{\mathbb{G}_m,-n} \simeq (\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^{\otimes n^2},$$

which implies the desired assertion.

Remark 6.2.4. One can see explicitly what $\mathcal{G}^{T^\sharp}$ is in terms of $\mathcal{G}^G$ by interpreting both as Θ-data, see Sect. 4.5.5. In terms of *loc.cit.*, the Θ-data corresponding to $\mathcal{G}^{T^\sharp}$ is obtained from one for $\mathcal{G}^G$ by restriction along

$$\Lambda^\sharp \to \Lambda.$$

6.3. **The metaplectic Langlands dual datum as a triple.** In this subsection we take $A := E^{\times,\mathrm{tors}}$.

6.3.1. By Sect. 4.8.5, to $\mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$ we can canonically attach an object

$$(\mathcal{G}^{\pi_{1,\mathrm{alg}}\otimes\mathbb{G}_m})^0 \in \mathrm{FactGe}^0_A(\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}) \simeq \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G^\sharp), E^{\times,\mathrm{tors}})))$$

and a map

$$\epsilon : \pi_{1,\mathrm{alg}}(G^\sharp) \to \mathbb{Z}/2\mathbb{Z}.$$

6.3.2. Let H denote the Langlands dual of $G^\sharp$, viewed as an algebraic group over E. Note that

$$\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G^\sharp), E^\times)$$

identifies with $Z_H(E)$, where Z_H denotes the center of H.

Hence, we can think of $(\mathcal{G}^{\pi_{1,\mathrm{alg}}\otimes\mathbb{G}_m})^0$ as a Z_H-gerbe on X, to be denoted $\mathcal{G}_Z$. Furthermore, we interpret the above map ϵ as a homomorphism

$$\epsilon : \mathbb{Z}/2\mathbb{Z} \to Z_H(E). \tag{6.5}$$

6.3.3. We will refer to the triple

$$(H, \mathcal{G}_Z, \epsilon) \tag{6.6}$$

as the *metaplectic Langlands dual datum* corresponding to $\mathcal{G}^G$.

6.3.4. *Example.* Suppose that $\mathcal{G}^G$ is trivial. Then $T^\sharp = T$ and $G^\sharp = G$, so $H = \check{G}$. In this case $\mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$ and ϵ are trivial.

6.3.5. *Example.* Take now $\mathcal{G}^G = (\det_{\mathfrak{g}})^{\frac{1}{2}}$. In this case we will have $T^\sharp = T$ and $G^\sharp = G$, so $H = \check{G}$.

However, the element ϵ equals now the image of $-1 \in \mathbb{Z}/2\mathbb{Z}$ under the homomorphism

$$\mathbb{Z}/2\mathbb{Z} \to Z_{\check{G}} \tag{6.7}$$

given by

$$\mathbb{Z}/2\mathbb{Z} \to \mathbb{G}_m \overset{2\check{\rho}}{\to} \check{T} \to \check{G}.$$

Further, by Remark 5.2.5, the Z_H-gerbe $\mathcal{G}_Z$ is unduced by means of (6.7) from the $\mathbb{Z}/2\mathbb{Z}$-gerbe $\omega_X^{\frac{1}{2}}$.

7. Factorization gerbes on loop groups

In this section we will perform a crucial geometric construction that will explain why our definition of geometric metaplectic datum was "the right thing to do":

We will show that a factorization gerbe on Gr_G give rise to a (factorization) gerbe on (the factorization version of) the loop group of G.

7.1. **Digression: factorization loop and arc spaces.** Up until this point, the geometric objects that have appeared in this paper were all locally of finite type, considered as prestacks. However, the objects that we will introduce below *do not* have this property.

7.1.1. For an affine test scheme S and an S-point of Ran, given by a finite set $I \subset \mathrm{Maps}(S, X)$, let $\hat{\mathcal{D}}_I$ be the corresponding relative formal disc:

By definition, $\hat{\mathcal{D}}_I$ is the formal scheme equal to the completion of $S \times X$ along the union Γ_I of the graphs of the maps $S \to X$ corresponding to the elements of I.

Note that for a finite set J and a point

$$\{I_j, j \in J\} \in \mathrm{Ran}^J_{\mathrm{disj}},$$

we have

$$\hat{\mathcal{D}}_I \simeq \underset{j}{\sqcup} \hat{\mathcal{D}}_{I_j}, \tag{7.1}$$

where $I = \underset{j}{\sqcup} I_j$.

7.1.2. Since S was assumed affine, $\hat{\mathcal{D}}_I$ is an ind-object in the category $\mathrm{Sch}^{\mathrm{aff}}$. Let $\mathcal{D}_I$ be the affine scheme corresponding to the formal scheme $\hat{\mathcal{D}}_I$, i.e., the image of $\hat{\mathcal{D}}_I$ under the functor

$$\mathrm{colim} : \mathrm{Ind}(\mathrm{Sch}^{\mathrm{aff}}) \to \mathrm{Sch}^{\mathrm{aff}}.$$

In other words, if

$$\hat{\mathcal{D}}_I \simeq \underset{\alpha}{\mathrm{colim}}\, Z_\alpha,$$

where $Z_\alpha = \mathrm{Spec}(A_\alpha)$ and the colimit is taken in PreStk, then $\mathcal{D}_I = \mathrm{Spec}(A)$, where

$$A = \lim_{\alpha} A_\alpha.$$

Let $\overset{\circ}{\mathcal{D}}_I$ be the open subscheme of $\mathcal{D}_I$, obtained by removing the closed subscheme Γ_I equal to the union of the graphs of the maps $S \to X$ corresponding to the elements of I.

7.1.3. Let Z be a prestack. We define the prestacks $\mathfrak{L}^+(Z)$ (resp., $\mathfrak{L}(Z)$) over Ran as follows.

For an affine test scheme S and an S-point of Ran, given by a finite set $I \subset \mathrm{Maps}(S, X)$, its lift to an S-point of $\mathfrak{L}^+(Z)$ (resp., $\mathfrak{L}(Z)$) is the datum of a map $\mathcal{D}_I \to Z$ (resp., $\overset{\circ}{\mathcal{D}}_I \to Z$).

The isomorphisms (7.1) imply that $\mathfrak{L}^+(Z)$ and $\mathfrak{L}(Z)$ are naturally factorization prestacks over Ran.

7.1.4. Assume for a moment that Z is an affine scheme. Note that in this case the definition of $\mathfrak{L}^+(Z)$, the datum of a map $\mathcal{D}_I \to Z$ is equivalent to that of a map of prestacks $\hat{\mathcal{D}}_I \to Z$.

Assume now that Z is a smooth scheme of finite type (but not necessarily affine). Then one shows that for every $S \to \mathrm{Ran}$, the fiber product

$$S \underset{\mathrm{Ran}}{\times} \mathfrak{L}^+(Z)$$

is a projective limit (under smooth maps) of smooth affine schemes over S.

7.2. **Factorization loop and arc groups.**

7.2.1. Let us recall that the Beauville-Laszlo Theorem says that the definition of Gr_G can be rewritten in terms of the pair $\overset{\circ}{\mathcal{D}}_I \subset \mathcal{D}_I$.

Namely, given I as above, the datum of its lift to a point of Gr_G is a pair $(\mathcal{P}_G, \alpha)$, where $\mathcal{P}_G$ is a G-bundle on $\mathcal{D}_I$, and α is the trivialization of $\mathcal{P}_G|_{\overset{\circ}{\mathcal{D}}_I}$. (Note that restriction along $\hat{\mathcal{D}}_I \to \mathcal{D}_I$ induces an equivalence between the category of G-bundles on $\mathcal{D}_I$ and that on $\hat{\mathcal{D}}_I$.)

In other words, the Beauville-Laszlo says that restriction along

$$(\overset{\circ}{\mathcal{D}}_I \subset \mathcal{D}_I) \to (U_I \subset S \times X)$$

induces a bijection on the corresponding pairs $(\mathcal{P}_G, \alpha)$. In the above formula, the notation U_I is as in Sect. 2.3.1.

7.2.2. This interpretation of Gr_G shows that the group-prestack $\mathfrak{L}(G)$ acts naturally on Gr_G, with the stabilizer of the unit section being $\mathfrak{L}^+(G)$. Furthermore, the natural map

$$\mathfrak{L}(G)/\mathfrak{L}^+(G) \to \mathrm{Gr}_G, \tag{7.2}$$

is an isomorphism, where the quotient is understood in the sense of stacks in the étale topology.

The isomorphism (7.2) implies that for every $S \to \mathrm{Ran}$, the fiber product

$$S \underset{\mathrm{Ran}}{\times} \mathfrak{L}(G),$$

is an ind-scheme over S.

7.2.3. Recall that given a group-prestack $\mathcal{H}$ over a base Z, we can talk about a gerbe over $\mathcal{H}$ being *multiplicative*, i.e., compatible with the group-structure.

In particular, we can consider the spaces

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}(G)) \text{ and } \mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}^+(G))$$

of multiplicative factorization gerbes on $\mathfrak{L}(G)$ and $\mathfrak{L}^+(G)$, respectively.

7.2.4. The isomorphism (7.2) defines a map

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}(G)) \underset{\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}^+(G))}{\times} * \to \mathrm{FactGe}_A(\mathrm{Gr}_G). \tag{7.3}$$

We have the following result[15]:

Proposition 7.2.5. *The map* (7.3) *is an isomorphism.*

We will sketch the proof of this proposition in Sect. 7.3. It consists of explicitly constructing the inverse map.

7.2.6. Let us restate Proposition 7.2.5 in words. It says that, given a factorization gerbe on Gr_G, its pullback under the projection

$$\mathfrak{L}(G) \to \mathrm{Gr}_G,$$

carries a uniquely defined multiplicative structure that is compatible with that of factorization and the trivialization of the further restriction of our gerbe to $\mathfrak{L}^+(G)$.

7.3. **Sketch of proof of Proposition 7.2.5.** The proof we will sketch was suggested to us by Y. Zhao. We will produce an inverse map to (7.3) by appealing to Proposition 3.1.9.

Namely, using a variant of the construction described in Sects. 3.1.3-3.1.7, we will show that a based map

$$B_{\mathrm{et}}(G) \times X \to B^4_{\mathrm{et}}(A(1)), \tag{7.4}$$

gives rise to an object of $\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}(G)) \underset{\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}^+(G))}{\times} *$.

[15]This result was claimed in [Re, Theorem III.2.10], but the proof was incomplete; specifically the argument in Proposition III.2.8 of *loc.cit.* is incomplete.

7.3.1. Let
$$\mathrm{Hecke}^{\mathrm{loc}}_G$$
denote the version of the local Hecke stack over Ran.

Namely, an S-points of $\mathrm{Hecke}^{\mathrm{loc}}_G$ is a triple
$$(I, \mathcal{P}'_G, \mathcal{P}''_G, \alpha),$$
where:

–$I \subset \mathrm{Hom}(S, X)$ is a S-point of Ran;

–$\mathcal{P}'_G$ and $\mathcal{P}''_G$ are G-bundles on $\mathcal{D}_I$;

–α is an isomorphism between $\mathcal{P}'_G$ and $\mathcal{P}''_G$ over $\overset{\circ}{\mathcal{D}}_I$.

We can identify
$$\mathrm{Hecke}^{\mathrm{loc}}_G \simeq \mathfrak{L}^+(G)\backslash \mathrm{Gr}_G \simeq \mathfrak{L}^+(G)\backslash \mathfrak{L}(G)/\mathfrak{L}^+(G).$$

Consider also the prestack
$$\mathfrak{L}^+(G)\backslash \mathrm{Ran},$$
whose S-points are pairs $(I, \mathcal{P}_G)$, where I is as above and $\mathcal{P}_G$ is a G-bundle on $\mathcal{D}_I$.

We have the two projections
$$\mathrm{Hecke}^{\mathrm{loc}}_G \rightrightarrows \mathfrak{L}^+(G)\backslash \mathrm{Ran}$$
that remember the data of $\mathcal{P}'_G$ and $\mathcal{P}''_G$, respectively.

Furthermore, composition of isomorphisms defines on $\mathrm{Hecke}^{\mathrm{loc}}_G$ a natural structure of groupoid acting on $\mathfrak{L}^+(G)\backslash \mathrm{Ran}$. This structure is compatible with factorization in a natural way.

Hence, it makes sense to talk about factorization gerbes on $\mathrm{Hecke}^{\mathrm{loc}}_G$, equipped with a *multiplicative structure* with respect to the groupoid operation.

It is clear that the space $\mathrm{FactGe}^{\mathrm{mult}}_A(\mathfrak{L}(G)) \underset{\mathrm{FactGe}^{\mathrm{mult}}_A(\mathfrak{L}^+(G))}{\times} *$ is canonically equivalent to the space of such gerbes.

We will now show how to associate to a based map (7.4) such a gerbe on $\mathrm{Hecke}^{\mathrm{loc}}_G$.

7.3.2. Consider the diagram
$$\begin{array}{ccccc} \Gamma_I & \overset{\hat{\iota}}{\longrightarrow} & \mathcal{D}_I & \overset{\hat{\jmath}}{\longleftarrow} & \overset{\circ}{\mathcal{D}}_I \\ {\scriptstyle \pi}\big\downarrow & & & & \\ S. & & & & \end{array}$$

Given a based map (7.4) and the data of $(I, \mathcal{P}'_G, \mathcal{P}''_G)$, we obtain two maps
$$\mathcal{D}_I \rightrightarrows B^4_{\mathrm{et}}(A(1)). \tag{7.5}$$

The datum of α defines a trivialization of the difference between these two maps to $\overset{\circ}{\mathcal{D}}_I$. Hence, this difference can be viewed as a 4-cochain in
$$\mathrm{C}^\bullet_{\mathrm{et}}(\Gamma_I, \hat{\iota}^!(A_{\mathcal{D}_I}(1))).$$

To this data we need to associate a 2-cochain in
$$\mathrm{C}^\bullet_{\mathrm{et}}(S, A_S).$$

7.3.3. We have a commutative diagram

$$\begin{array}{ccccc} \Gamma_I & \xrightarrow{\hat{\iota}} & \mathcal{D}_I & \xleftarrow{\hat{\jmath}} & \overset{\circ}{\mathcal{D}}_I \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ \Gamma_I & \xrightarrow{\iota} & S\times X & \xleftarrow{\jmath} & U_I \end{array}$$

Not it follows from the Fujiwara-Gabber comparison theorem[16], (see [Fu, Corollary 6.6.4]), that the natural map

$$\hat{\iota}^!(A_{\mathcal{D}_I}(1)) \to \iota^!(A_{S\times X}(1))$$

is an isomorphism.

Hence, from (3.8), we obtain an isomorphism

$$\hat{\iota}^!(A_{\mathcal{D}_I}(1)) \simeq \pi^!(A_S)[-2].$$

The rest of the construction proceeds as in Sects. 3.1.5-3.1.7. The multiplicative structure on the resulting gerbe on $\mathrm{Hecke}^{\mathrm{loc}}_G$ follows from the construction.

7.3.4. *An alternative construction.* We will now sketch a construction of the map from

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G)\times X, B^4_{\mathrm{et}}(A(1)))$$

to the space of multiplicative factorization gerbes on $\mathrm{Hecke}^{\mathrm{loc}}_G$ that avoids the use of the Gabber-Fujiwara theorem (see Sect. 7.3.3). Its main geometric ingredient is of independent interest.

Let $\mathrm{Bun}^{\mathrm{loc,Zar}}_G$ be the prestack that attaches to a test affine scheme S the groupoid of pairs $(I,\mathcal{P}_G)$, where $I\subset \mathrm{Hom}(S,X)$ is an S-point of Ran, and $\mathcal{P}_G$ is a G-bundle defined on *some* Zariski-open subset

$$\mathcal{D}^{\mathrm{Zar}}_I \subset S\times X$$

that contains Γ_I, and such that $\mathcal{P}_G$ can be trivialiazed étale-locally on S. Morphisms in this groupoid are isomorphisms of G-bundles defined over the intersection of their domains of definition.

Let $\mathrm{Hecke}^{\mathrm{loc,Zar}}_G$ be the following version of $\mathrm{Hecke}^{\mathrm{loc}}_G$. Its S-points are quadruples $(I,\mathcal{P}'_G,\mathcal{P}''_G,\alpha)$, where I is as above, $\mathcal{P}'_G$ and $\mathcal{P}''_G$ are G-bundles defined on some $\mathcal{D}^{\mathrm{Zar}}_I$ as above, and α is an isomorphism between $\mathcal{P}'_G$ and $\mathcal{P}''_G$ defined over

$$\overset{\circ}{\mathcal{D}}{}^{\mathrm{Zar}}_I := \mathcal{D}^{\mathrm{Zar}}_I - \Gamma_I.$$

The prestack $\mathrm{Hecke}^{\mathrm{loc,Zar}}_G$ has a natural structure of groupoid acting on $\mathrm{Bun}^{\mathrm{loc,Zar}}_G$. So we can talk about mutiplicative factorization gerbes on $\mathrm{Hecke}^{\mathrm{loc,Zar}}_G$. The construction in Sects. 3.1.5-3.1.7 defines a map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G)\times X, B^4_{\mathrm{et}}(A(1))) \to \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Hecke}^{\mathrm{loc,Zar}}_G).$$

Now, we claim:

Proposition 7.3.5. *The forgetful map* $\mathrm{Hecke}^{\mathrm{loc,Zar}}_G \to \mathrm{Hecke}^{\mathrm{loc}}_G$ *defines equivalences*

$$\mathrm{FactGe}_A(\mathrm{Hecke}^{\mathrm{loc}}_G) \to \mathrm{FactGe}_A(\mathrm{Hecke}^{\mathrm{loc,Zar}}_G)$$

and

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Hecke}^{\mathrm{loc}}_G) \to \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Hecke}^{\mathrm{loc,Zar}}_G)$$

Proof. Note that the map $\mathrm{Hecke}^{\mathrm{loc,Zar}}_G \to \mathrm{Hecke}^{\mathrm{loc}}_G$ fits into a Cartesian square

$$\begin{array}{ccc} \mathrm{Hecke}^{\mathrm{loc,Zar}}_G & \longrightarrow & \mathrm{Hecke}^{\mathrm{loc}}_G \\ \downarrow & & \downarrow \\ \mathrm{Bun}^{\mathrm{loc,Zar}}_G & \longrightarrow & \mathfrak{L}^+(G)\backslash \mathrm{Ran}, \end{array}$$

where the vertical maps can be taken to be *either* of the projections. The same is true for the other terms of the Čech nerve of $\mathrm{Hecke}^{\mathrm{loc}}_G$ (resp., $\mathrm{Hecke}^{\mathrm{loc,Zar}}_G$) over $\mathfrak{L}^+(G)\backslash \mathrm{Ran}$ (resp., $\mathrm{Bun}^{\mathrm{loc,Zar}}_G$).

[16]We are grateful to Ofer Gabber for explaining this to us.

Hence, to prove the proposition, it suffices to show that the map

$$\mathrm{Bun}_G^{\mathrm{loc,Zar}} \to \mathfrak{L}^+(G)\backslash \mathrm{Ran}$$

is *universally homologically contractible* in the sense of [Ga7, Sect. 2.5].

Fix a map $S \to \mathrm{Ran}$ corresponding to a finite non-empty set $I \subset \mathrm{Hom}(S, X)$. It suffices to show that the map

$$S \underset{\mathrm{Ran}}{\times} \mathrm{Bun}_G^{\mathrm{loc,Zar}} \to S \underset{\mathrm{Ran}}{\times} (\mathfrak{L}^+(G)\backslash \mathrm{Ran})$$

is universally homologically contractible. Let us view Γ_I as subscheme in $S \times X$, which is finite and flat over S, and let G_I be the group-scheme over S that classifies maps from Γ_I to G. Evaluation defines a map of group-schemes over S

$$S \underset{\mathrm{Ran}}{\times} \mathfrak{L}^+(G) \to G_I$$

Denote by $\mathcal{K}_I$ its kernel.

It suffices to show that the map

$$(S \underset{\mathrm{Ran}}{\times} \mathrm{Bun}_G^{\mathrm{loc,Zar}}) \underset{G_I\backslash S}{\times} S \to (S \underset{\mathrm{Ran}}{\times} (\mathfrak{L}^+(G)\backslash \mathrm{Ran})) \underset{G_I\backslash S}{\times} S \simeq \mathcal{K}_I\backslash S$$

is universally homologically contractible.

Since $\mathcal{K}_I$ is pro-unipotent, it suffices to show that the projection

$$(S \underset{\mathrm{Ran}}{\times} \mathrm{Bun}_G^{\mathrm{loc,Zar}}) \underset{G_I\backslash S}{\times} S \to S$$

is universally homologically contractible.

Note that $(S \underset{\mathrm{Ran}}{\times} \mathrm{Bun}_G^{\mathrm{loc,Zar}}) \underset{G_I\backslash S}{\times} S$ is the étale sheafification of the prestack $B(\mathcal{K}_I^{\mathrm{Zar}})$, where $\mathcal{K}_I^{\mathrm{Zar}}$ is the group-prestack over S that attaches to $S' \to S$ the group of maps

$$g : \mathcal{D}_I^{\mathrm{Zar}} \to G, \quad S' \underset{S}{\times} \Gamma_I \subset \mathcal{D}_I^{\mathrm{Zar}} \subset S' \times X,$$

such that

$$g|_{S' \underset{S}{\times} \Gamma_I} = 1.$$

Hence, it suffices to show that

$$\mathcal{K}_I^{\mathrm{Zar}} \to S$$

is universally homologically contractible. However, this is the assertion of (the easy case of) [GL2, Theorem 3.3.2], see Proposition 3.5.3 in *loc. cit.*

□

7.4. **What does Proposition 7.2.5 say in concrete terms?** In this subsection we will give a "hands-on" explanation of the concrete meaning of Proposition 7.2.5.

7.4.1. Denote

$$\mathrm{Gr}_{G,X} := X \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G,\ \mathrm{Gr}_{G,X^2} := X^2 \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G,$$
$$\mathrm{Hecke}_{G,X}^{\mathrm{loc}} := X \underset{\mathrm{Ran}}{\times} \mathrm{Hecke}_G^{\mathrm{loc}},\ \mathrm{Hecke}_{G,X^2}^{\mathrm{loc}} := X^2 \underset{\mathrm{Ran}}{\times} \mathrm{Hecke}_G^{\mathrm{loc}},$$
$$\mathfrak{L}^+(G)_X := X \underset{\mathrm{Ran}}{\times} \mathfrak{L}^+(G),\ \mathfrak{L}^+(G)_{X^2} := X^2 \underset{\mathrm{Ran}}{\times} \mathfrak{L}^+(G),$$

etc.

Consider the fiber product

$$\mathrm{Gr}_{G,X^2} \underset{\mathfrak{L}^+(G)_{X^2}\backslash X^2}{\times} \mathrm{Hecke}_{G,X^2}^{\mathrm{loc}}. \tag{7.6}$$

It classifies the data of $(x_1, x_2, \mathcal{P}_G^1, \mathcal{P}_G^2, \alpha_1, \alpha_2)$, where
–x_1, x_2 are S-points of X;
–$\mathcal{P}_G^1$ and $\mathcal{P}_G^2$ are G-bundles on $\mathcal{D}_{x_1 \cup x_2}$;

–α_1 is an isomorphism between $\mathcal{P}^0_G$ (the trivial bundle) and $\mathcal{P}^1_G$ over $\overset{\circ}{\mathcal{D}}_{x_1\cup x_2}$;

–α_2 is an isomorphism between $\mathcal{P}^1_G$ (the trivial bundle) and $\mathcal{P}^2_G$ over $\overset{\circ}{\mathcal{D}}_{x_1\cup x_2}$.

Let

$$\widetilde{\mathrm{Gr}}_{G,X^2} \subset \mathrm{Gr}_{G,X^2} \underset{\mathfrak{L}^+(G)_{X^2}\backslash X^2}{\times} \mathrm{Hecke}^{\mathrm{loc}}_{G,X^2}$$

be the "convolution diagram". I.e., this is a closed subfunctor of (7.6) defined by the conditions that:

–α_1 extends to an isomorphism over

$$(\mathcal{D}_{x_1\cup x_2} - \Gamma_{x_1}) \supset \overset{\circ}{\mathcal{D}}_{x_1\cup x_2} = (\mathcal{D}_{x_1\cup x_2} - \Gamma_{x_1\cup x_2}).$$

–α_2 extends to an isomorphism over

$$(\mathcal{D}_{x_1\cup x_2} - \Gamma_{x_2}) \supset \overset{\circ}{\mathcal{D}}_{x_1\cup x_2} = (\mathcal{D}_{x_1\cup x_2} - \Gamma_{x_1\cup x_2}).$$

The groupoid structure on $\mathrm{Hecke}^{\mathrm{loc}}_{G,X^2}$ gives rise a map

$$\mathrm{conv} : \widetilde{\mathrm{Gr}}_{G,X^2} \to \mathrm{Gr}_{G,X^2}, \tag{7.7}$$

defined by sending

$$(x_1, x_2, \mathcal{P}^1_G, \mathcal{P}^2_G, \alpha_1, \alpha_2) \mapsto (x_1, x_2, \mathcal{P}^2_G, \alpha_2 \circ \alpha_1).$$

7.4.2. We also have a projection

$$\mathrm{pr} : \widetilde{\mathrm{Gr}}_{G,X^2} \to \mathrm{Gr}_{G,X} \times X,$$

defined by sending

$$(x_1, x_2, \mathcal{P}^1_G, \mathcal{P}^2_G, \alpha_1, \alpha_2) \mapsto (x_1, \mathcal{P}^1_G, \alpha_1, x_2).$$

This projection allows to view $\widetilde{\mathrm{Gr}}_{G,X^2}$ as a *twisted product*

$$(\mathrm{Gr}_{G,X} \times X) \underset{X^2}{\widetilde{\times}} (X \times \mathrm{Gr}_{G,X}), \tag{7.8}$$

by which we mean that it is associated to a $\mathfrak{L}^+(G)_{X^2}$-torsor over $\mathrm{Gr}_{G,X} \times X$ (thought of as the first factor in (7.8)) and the ind-scheme $X \times \mathrm{Gr}_{G,X}$ (thought of as the second factor in (7.8)), equipped with an action of $\mathfrak{L}^+(G)_{X^2}$.

The above torsor and action are obtained as follows. The $\mathfrak{L}^+(G)_{X^2}$-torsor over $\mathrm{Gr}_{G,X} \times X$ corresponds to the map

$$\mathrm{Gr}_{G,X} \times X \hookrightarrow \mathrm{Gr}_{G,X^2} \to (\mathfrak{L}^+(G)_{X^2}\backslash X^2),$$

and the $\mathfrak{L}^+(G)_{X^2}$-action on $X \times \mathrm{Gr}_{G,X}$ is induced by the $\mathfrak{L}^+(G)_X$-action on $\mathrm{Gr}_{G,X}$ via the projection

$$\mathfrak{L}^+(G)_{X_2} \overset{p_2}{\to} X \times \mathfrak{L}^+(G)_X,$$

obtained by restriction along $\mathcal{D}_{x_1\cup x_2} \hookleftarrow \mathcal{D}_{x_2}$.

Note, in particular, that the induced $(X \times \mathfrak{L}^+(G)_X)$-torsor on $\mathrm{Gr}_{G,X} \times X$ is canonocally trivialized over

$$X^2 - \Delta \subset X^2.$$

In particular, we have a canonical identification

$$\widetilde{\mathrm{Gr}}_{G,X^2}|_{X^2-\Delta} \simeq (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_{X^2-\Delta}. \tag{7.9}$$

Under this identification, the projection conv of (7.7) identifies with the factorization isomorphism

$$(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})|_{X^2-\Delta} \simeq \mathrm{Gr}_{G,X^2}|_{X^2-\Delta}$$

for Gr.

7.4.3. Now, (part of) the assertion of Proposition 7.2.5 can be formulated as follows. Let $\mathcal{G}$ be an object of $\mathrm{FactGe}_A(\mathrm{Gr}_G)$. Let $\mathcal{G}_X$ (resp., $\mathcal{G}_{X^2}$) be its restriction to $\mathrm{Gr}_{G,X}$ (resp., Gr_{G,X^2}).

Then the claim is that $\mathcal{G}_X$ *admits a unique structure of equivariance* with resect to $\mathfrak{L}^+(G)_X$ such that the following holds:

On the one hand, having a structure of $\mathfrak{L}^+(G)_X$-equivariance on $\mathcal{G}_X$, we can form the twisted product

$$\mathcal{G}_X \widetilde{\boxtimes} \mathcal{G}_X \in \mathrm{Ge}_A(\widetilde{\mathrm{Gr}}_{G,X^2}),$$

see formula (7.8).

On the other hand, we can consider the object

$$\mathrm{conv}^*(\mathcal{G}_{X^2}) \in \mathrm{Ge}_A(\widetilde{\mathrm{Gr}}_{G,X^2}),$$

see formula (7.7).

Now, the factorization structure on $\mathcal{G}$ and the identification (7.9) imply that we have a canonical isomorphism

$$\mathcal{G}_X \widetilde{\boxtimes} \mathcal{G}_X|_{X^2-\Delta} \simeq \mathrm{conv}^*(\mathcal{G}_{X^2})|_{X^2-\Delta}. \tag{7.10}$$

The requirement on the $\mathfrak{L}^+(G)_X$-equivariance on $\mathcal{G}_X$ is that the identification (7.10) extends to an identification

$$\mathcal{G}_X \widetilde{\boxtimes} \mathcal{G}_X \simeq \mathrm{conv}^*(\mathcal{G}_{X^2}) \tag{7.11}$$

over all of X^2.

7.5. **The case of tori.** We will now explain how the contents of Sect. 7.4 play out in the case when $G = T$ is a torus.

7.5.1. Up to nilpotents, we can identify

$$\mathrm{Gr}_{T,X} \simeq \Lambda \times X,$$

and the action of $\mathfrak{L}^+(T)_X$ on it is trivial.

Hence, the a datum of $\mathfrak{L}^+(T)_X$-equivariance on $\mathcal{G}_X$ amounts to a map from Λ to the set of multiplicative A-torsors on $\mathfrak{L}^+(T)_X$.

Since elements of A have orders prime to $\mathrm{char}(k)$, such torsors are pulled back via the evaluation map

$$\mathfrak{L}^+(T)_X \to X \times T \tag{7.12}$$

from multiplicative A-torsors on T. The latter are, by Kummer theory, in bijection with maps $\Lambda \to A(-1)$.

We will show that the resulting map

$$\Lambda \to \mathrm{Hom}_{\mathrm{Ab}}(\Lambda, A(-1)),$$

which can be viewed as a map

$$\widetilde{b} : \Lambda \times \Lambda \to A(-1),$$

is given by $b(-,-)$, the bilinear form corresponding to $\mathcal{G}$.

7.5.2. Fix a connected component of $\mathrm{Gr}_{G,X}$ corresponding to $\lambda \in \Lambda$, and the corresponding connected component in $\mathrm{Gr}_{G,X} \times X$, thought if as the base of the fibration in (7.8). Consider the induced $X \times \mathfrak{L}^+(T)_X$-torsor, and further the $X \times T$-torsor, induced by (7.12).

This torsor is trivialized over $X^2 - \Delta$, and over all of X^2 it identifies with $\lambda \cdot \mathcal{O}(\Delta)$.

Consider a connected component

$$\widetilde{\mathrm{Gr}}^{\lambda,\mu}_{T,X^2} := (\mathrm{Gr}^{\lambda}_{T,X} \times X) \underset{X^2}{\widetilde{\times}} (X \times \mathrm{Gr}^{\mu}_{T,X})$$

of $\widetilde{\mathrm{Gr}}_{T,X^2}$, corresponding $\lambda, \mu \in \Lambda$.

Since the action of $X \times \mathfrak{L}^+(T)_X$ on $X \times \mathrm{Gr}^{\mu}_{T,X}$ is trivial, we obtain an identification

$$\widetilde{\mathrm{Gr}}^{\lambda,\mu}_{T,X^2} \simeq \mathrm{Gr}^{\lambda}_{T,X} \times \mathrm{Gr}^{\mu}_{T,X} \simeq X \times X. \tag{7.13}$$

We obtain that with respect to this identification, the gerbe

$$\mathcal{G}^{\lambda} \widetilde{\boxtimes} \mathcal{G}^{\mu} := \mathcal{G}_X \widetilde{\boxtimes} \mathcal{G}_X|_{\widetilde{\mathrm{Gr}}^{\lambda,\mu}_{T,X^2}}$$

identifies with

$$\mathcal{O}(\Delta)^{\widetilde{b}(\lambda,\mu)}.$$

7.5.3. Consider the restriction of the map conv to the connected component $\widetilde{\mathrm{Gr}}^{\lambda,\mu}_{T,X^2}$. This map is an isomorphism onto an irreducible component $\mathrm{Gr}^{\lambda,\mu}_{T,X^2}$ of Gr_{T,X^2}.

Furthermore, the factorization isomorphism

$$\mathrm{Gr}^{\lambda,\mu}_{T,X^2}|_{X^2-\Delta} \simeq (\mathrm{Gr}^{\lambda}_{T,X} \times \mathrm{Gr}^{\mu}_{T,X})|_{X^2-\Delta}$$

also extends to an isomorphism over all of X^2.

The isomorphism

$$\mathcal{G}^{\lambda} \widetilde{\boxtimes} \mathcal{G}^{\mu} \simeq \mathcal{G}^{\lambda,\mu}$$

of (7.11) extends the tautological identification of the two sides with

$$\mathcal{G}^{\lambda} \boxtimes \mathcal{G}^{\mu}$$

over X. Hence, by (4.5), we obtain

$$\widetilde{b}(\lambda, \mu) = b(\lambda, \mu),$$

as claimed.

8. Factorization category of representations

From now on, until the end of the paper we will assume that $k = E$ and we will work in the context of D-modules.

8.1. **Digression: factorization categories arising from symmetric monoidal categories.** In this subsection we will explain a procedure that produces a factorization sheaf of categories from a sheaf symmetric monoidal categories on X. The source of the metaplectic geometric Satake functor will be a factorization sheaf of categories obtained in this way.

For a more detailed discussion see [Ras2, Sect. 6].

8.1.1. Let $\mathcal{C}$ be a sheaf of symmetric monoidal categories over X. To it we will associate a sheaf of symmetric monoidal categories over Ran, equipped with a factorization structure, denoted $\mathrm{Fact}(\mathcal{C})$.

We will construct $\mathrm{Fact}(\mathcal{C})$ as a family of sheaves of symmetric monoidal categories over X^I for all finite non-empty sets I, compatible under surjections $I_1 \twoheadrightarrow I_2$. We will use Theorem 1.6.9 that says that the datum of sheaf of categories over X^I is equivalent to that of a category acted on by $\mathrm{Shv}(X^I)$. So, we will produce system of symmetric monoidal categories $\mathrm{Fact}(\mathcal{C})(X^I)$, compatible under

$$\mathrm{Fact}(\mathcal{C})(X^{I_2}) \simeq \mathrm{Shv}(X^{I_2}) \underset{\mathrm{Shv}(X^{I_1})}{\otimes} \mathrm{Fact}(\mathcal{C})(X^{I_1}). \tag{8.1}$$

8.1.2. Let $\mathcal{C}(X)$ denote the category of sections of $\mathcal{C}$ over X; this is a symmetric monoidal category over $\operatorname{Shv}(X)$. For a finite set J we let $\mathcal{C}^{\otimes J}(X)$ the J-fold tensor product of $\mathcal{C}(X)$ *over* $\operatorname{Shv}(X)$.

Note that for a surjection of finite sets $I \twoheadrightarrow J$ we have a canonical isomorphism

$$\mathcal{C}^{\otimes I}(X) \simeq \left(\bigotimes_{j\in J} \mathcal{C}^{\otimes I_j}(X)\right) \underset{\operatorname{Shv}(X^J)}{\otimes} \operatorname{Shv}(X), \tag{8.2}$$

where I_j denotes the preimage of $j \in J$ under $I \to J$.

In addition, for $I \twoheadrightarrow J$, the symmetric monoidal structure on $\mathcal{C}(X)$ gives rise to the functors

$$\mathcal{C}^{\otimes I}(X) \to \mathcal{C}^{\otimes J}(X). \tag{8.3}$$

Remark 8.1.3. We will be particularly interested in the case when $\mathcal{C}$ is constant, i.e.,

$$\mathcal{C}(X) \simeq \mathcal{C}_{\mathrm{pt}} \otimes \operatorname{Shv}(X)$$

for a symmetric monoidal category $\mathcal{C}_{\mathrm{pt}}$. Note that in this case $\mathcal{C}^{\otimes J}(X)$ is just $\mathcal{C}_{\mathrm{pt}}^{\otimes J} \otimes \operatorname{Shv}(X)$.

8.1.4. For a given I, let $\operatorname{Tw}(I)$ be the category whose objects are pairs

$$I \twoheadrightarrow J \twoheadrightarrow K \tag{8.4}$$

(here J and K are sets (automatically, finite and non-empty)), and where morphisms from (J, K) to (J', K') are commutative diagrams

$$\begin{array}{ccccc} I & \longrightarrow & J & \longrightarrow & K \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \uparrow \\ I & \longrightarrow & J' & \longrightarrow & K'. \end{array} \tag{8.5}$$

(Note that the arrows between the K's go in the opposite direction.)

8.1.5. Consider the functor

$$\operatorname{Tw}(I) \to \operatorname{DGCat} \tag{8.6}$$

that sends an object (8.4) to

$$\bigotimes_{k\in K} \mathcal{C}^{\otimes J_k}(X),$$

where J_k is the preimage under $J \to K$ of the element $k \in K$. The above tensor product is naturally a symmetric monoidal category over $\operatorname{Shv}(X^K)$.

For a morphism (8.5) in $\operatorname{Tw}(I)$, we let the corresponding functor

$$\bigotimes_{k\in K} \mathcal{C}^{\otimes J_k}(X) \to \bigotimes_{k'\in K'} \mathcal{C}^{\otimes J'_{k'}}(X)$$

be given by the composition

$$\bigotimes_{k\in K} \mathcal{C}^{\otimes J_k}(X) \overset{(8.3)}{\longrightarrow} \bigotimes_{k\in K} \mathcal{C}^{\otimes J'_k}(X) \overset{(8.2)}{\simeq} \bigotimes_{k\in K} \left(\Big(\bigotimes_{k'\in K'_k} \mathcal{C}^{\otimes J'_{k'}}(X)\Big) \underset{\operatorname{Shv}(X^{K'_k})}{\otimes} \operatorname{Shv}(X)\right) =$$

$$= \left(\bigotimes_{k'\in K'} \mathcal{C}^{\otimes J'_{k'}}(X)\right) \underset{\operatorname{Shv}(X^{K'})}{\otimes} \operatorname{Shv}(X^K) \to \bigotimes_{k'\in K'} \mathcal{C}^{\otimes J'_{k'}}(X),$$

where the the last arrow is given by the direct image functor along $X^K \to X^{K'}$.

8.1.6. We let $\operatorname{Fact}(\mathcal{C})(X^I)$ on be the object of DGCat equal to the colimit of the functor (8.6) over $\operatorname{Tw}(I)$.

The compatibilities (8.1), as well as the factorization structure on $\operatorname{Fact}(\mathcal{C})$ follow from the construction.

8.1.7. Let $\text{Fact}(\mathcal{C})(\text{Ran})$ denote the category of global sections of $\text{Fact}(\mathcal{C})$ over Ran.

As in [Ga5, Sect. 4.2], the (symmetric) monoidal structure on $\text{Fact}(\mathcal{C})$ as a sheaf of categories over Ran and the operation of union of finite sets makes $\text{Fact}(\mathcal{C})(\text{Ran})$ into a *non-unital* (symmetric) monoidal category.

8.2. **Twisting procedures on the Ran space.** In this subsection we will start with a symmetric monoidal category $\mathcal{C}$ and some twisting data, and associate to it a sheaf of categories on the Ran space.

8.2.1. First, to $\mathcal{C}$ we associate the constant sheaf of symmetric monoidal categories over X, which, by a slight abuse of notation we denote by the same symbol $\mathcal{C}$; we have $\mathcal{C}(X) = \mathcal{C} \otimes \text{Shv}(X)$, see Remark 8.1.3.

Consider the corresponding factorization sheaf $\text{Fact}(\mathcal{C})$ of symmetric monoidal categories over Ran.

8.2.2. Let now A be a torsion abelian group that acts by automorphisms of the identity functor on $\mathcal{C}$ (viewed as a symmetric monoidal category), and let $\mathcal{G}_A$ be an A-gerbe on X.

Using Sect. 1.7.2, we can twist $\mathcal{C}$ by $\mathcal{G}_A$ and obtain a new sheaf of symmetric monoidal categories over X, denoted $\mathcal{C}_{\mathcal{G}_A}$.

In particular, we have the symmetric monoidal category $\mathcal{C}_{\mathcal{G}_A}(X)$ over $\text{Shv}(X)$.

8.2.3. Applying to $\mathcal{C}_{\mathcal{G}_A}$ the construction from Sect. 8.1, we obtain a new sheaf of symmetric monoidal categories over Ran, denoted $\text{Fact}(\mathcal{C})_{\mathcal{G}_A}$.

In particular, we obtain the non-unital symmetric monoidal category $\text{Fact}(\mathcal{C})_{\mathcal{G}_A}(\text{Ran})$.

Note that the value of $\text{Fact}(\mathcal{C})_{\mathcal{G}_A}$ on X under the canonical map $X \to \text{Ran}$ is the symmetric monoidal category $\mathcal{C}_{\mathcal{G}_A}(X)$

8.2.4. Let now ϵ be a 2-torsion element of A. Then we can further twist $\text{Fact}(\mathcal{C})_{\mathcal{G}_A}$ to obtain a factorization sheaf of symmetric monoidal DG categories, denoted $\text{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$.

Namely, the element ϵ can be used to modify the braiding on $\mathcal{C}$ and thereby obtain a *new* symmetric monoidal category, denoted $\mathcal{C}^{\epsilon}$. We let

$$\text{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A} := \text{Fact}(\mathcal{C}^{\epsilon})_{\mathcal{G}_A}.$$

A key feature of the latter twist is that we have a canonical isomorphism

$$\text{Fact}(\mathcal{C})_{\mathcal{G}_A} \simeq \text{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}, \tag{8.7}$$

as sheaves of *monoidal* categories over Ran. But this identification is *not* compatible with either the symmetric monoidal nor factorization structure.

Remark 8.2.5. At the level of underlying triangulated categories, the modification

$$\text{Fact}(\mathcal{C})_{\mathcal{G}_A} \rightsquigarrow \text{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$$

can be described as follows[17]:

We let $\text{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$ be the same as $\text{Fact}(\mathcal{C})_{\mathcal{G}_A}$ as a plain sheaf of monoidal categories. We define the factorization structure on $\text{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$ as follows:

The action of ϵ on $\mathcal{C}$ defines a direct sum decomposition

$$\text{Fact}(\mathcal{C})_{\mathcal{G}_A}(S) \simeq \text{Fact}(\mathcal{C})_{\mathcal{G}_A}(S)^1 \oplus \text{Fact}(\mathcal{C})_{\mathcal{G}_A}(S)^{-1}$$

for any $S \to \text{Ran}$.

Hence, for $S \to \text{Ran}^J$ we have a direct sum decomposition

$$(\text{Fact}(\mathcal{C})_{\mathcal{G}_A})^{\otimes J}(S) \simeq \underset{\gamma^J : J \to \pm 1}{\oplus} (\text{Fact}(\mathcal{C})_{\mathcal{G}_A})^{\otimes J}(S)^{\gamma^J}. \tag{8.8}$$

[17]However, it may not be so straightforward to perform this construction at the level of ∞-categories as it involves "explicit formulas".

For a given γ^J, let $J_{-1} \subset J$ be the preimage of the element $-1 \in \pm 1$.

We define the factorization functor for $\mathrm{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}(S)$ and $S \to \mathrm{Ran}^J_{\mathrm{disj}}$ to be equal to the one for $\mathrm{Fact}(\mathcal{C})_{\mathcal{G}_A}(S)$ on each factor of (8.8), for every choice of an ordering on J_{-1}. A change of ordering will result in multiplication by the sign character of the group of permutations of J_{-1}.

Remark 8.2.6. A general framework that performs both twistings

$$\mathrm{Fact}(\mathcal{C}) \rightsquigarrow \mathrm{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$$

in one shot is explained in Sect. E.

The construction in *loc.cit.* also makes the identification (8.7) as plain sheaves of monoidal categories over Ran, manifest. In particular, we have an identification

$$\mathrm{Fact}(\mathcal{C})_{\mathcal{G}_A}(\mathrm{Ran}) \simeq \mathrm{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}(\mathrm{Ran}), \tag{8.9}$$

as monoidal (but *not* symmetric monoidal) categories.

8.3. **Twisting the category of representations.** In this subsection we will introduce a factorization sheaf of symmetric monoidal categories on the Ran space, which will appear as the source of the metaplectic geometric Satake functor.

8.3.1. Let H be an algebraic group. We apply the discussion in Sect. 8.2 to the pair

$$\mathcal{C} = \mathrm{Rep}(H), \quad A = Z_H(E)^{\mathrm{tors}}.$$

Thus, let $\mathcal{G}_Z$ be a gerbe (of finite order) on X with respect to Z_H, and let ϵ be an element of order 2 in Z_H.

8.3.2. Thus, we obtain the symmetric monoidal category $\mathrm{Rep}(H)_{\mathcal{G}_Z}(X)$, and sheaves of symmetric monoidal categories over Ran:

$$\mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z} \text{ and } \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z},$$

and a *monoidal* equivalence

$$\mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z}(\mathrm{Ran}) \simeq \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}). \tag{8.10}$$

The case of interest for us is when the triple $(H, \mathcal{G}_Z, \epsilon)$ is the metaplectic datum attached to a geometric metaplectic datum of a reductive group G.

8.3.3. *Example of tori.* Consider the particular case when $G = T$ is a torus, and we start with a factorization gerbe $\mathcal{G}^T$ on Gr_T that is multiplicative[18]. In this case,

$$\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}}$$

is naturally a sheaf of symmetric monoidal DG categories on Ran, equipped with a factorization structure.

Note also that by Proposition 4.6.2(a), we have $T^\sharp = T$, and so $H \simeq \check{T}$. Let $(\mathcal{G}_Z, \epsilon)$ be as in Sect. 6.3.3.

One shows explicitly (see, e.g., [Re, Proposition IV.5.2]) that there is a canonical isomorphism

$$\mathrm{Fact}(\mathrm{Rep}(\check{T}))^{\epsilon}_{\mathcal{G}_Z} \simeq \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}} \tag{8.11}$$

as sheaves of factorization symmetric monoidal categories.

8.4. **Twisted local systems.** Let $(H, \mathcal{G}_Z)$ be as in Sect. 8.3. In this subsection we will introduce the notion of *twisted local system* for $(H, \mathcal{G}_Z)$.

[18]Recall that "multiplicative"="commutative", see Remark 4.7.3.

8.4.1. Note that the category $\operatorname{Rep}(H)_{\mathcal{G}_Z}(X)$ is naturally equipped with a t-structure. Namely, it is one for which the functor

$$\operatorname{Rep}(H)_{\mathcal{G}_Z}(X') \simeq \operatorname{Rep}(H)_{\mathcal{G}_Z}(X) \simeq \operatorname{Rep}(H)(X') \overset{\text{forget}}{\longrightarrow} \operatorname{Shv}(X')$$

is t-exact for every étale $X' \to X$ over which $\mathcal{G}_Z$ admits a trivialization.

By definition, a $\mathcal{G}_Z$-twisted local system on X with respect to H is a t-exact $\operatorname{Shv}(X)$-linear symmetric monoidal functor

$$\operatorname{Rep}(H)_{\mathcal{G}_Z}(X) \to \operatorname{Shv}(X).$$

In Sect. 9.5 we will formulate a precise relationship between twisted local systems in the above sense and objects appearing in the global metaplectic geometric theory.

Remark 8.4.2. Presumably, twisted local systems as defined above are the same as Galois representations into the metaplectic L-group, as defined in [We].

8.4.3. Let σ be a twisted local system on X as defined as above. The functoriality of the construction in Sect. 8.1 defines a symmetric monoidal functor

$$\operatorname{Fact}(\operatorname{Rep}(H))_{\mathcal{G}_Z}(\operatorname{Ran}) \to \operatorname{Shv}(\operatorname{Ran}).$$

In particular, we obtain a *monoidal* functor

$$\operatorname{Fact}(\operatorname{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\operatorname{Ran}) \to \operatorname{Shv}(\operatorname{Ran}).$$

8.4.4. Assume now that X is complete. Composing with the functor of direct image

$$\operatorname{Shv}(\operatorname{Ran}) \to \operatorname{Vect},$$

we thus obtain a functor

$$\operatorname{Ev}_\sigma : \operatorname{Fact}(\operatorname{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\operatorname{Ran}) \to \operatorname{Vect}. \tag{8.12}$$

We will use the functor (8.12) for the definition of the notion of *twisted Hecke eigensheaf* with respect to σ.

8.4.5. Assume that X is complete. We will now construct the *derived* stack $\operatorname{LocSys}_H^{\mathcal{G}_Z}$ of $\mathcal{G}_Z$-twisted local systems on X. Its k-points will be the twisted local systems as defined in Sect. 8.4.1.

We follow the strategy of [AG, Sect. 10.2]. For a derived affine scheme S, we set

$$\operatorname{Maps}(S, \operatorname{LocSys}_H^{\mathcal{G}_Z})$$

to be the space of *right t-exact* $\operatorname{Shv}(X)$-linear symmetric monoidal functors

$$\operatorname{Rep}(H)_{\mathcal{G}_Z}(X) \to \operatorname{QCoh}(S) \otimes \operatorname{Shv}(X).$$

One shows that $\operatorname{LocSys}_H^{\mathcal{G}_Z}$ defined in this way is representable by a quasi-smooth derived algebraic stack (see [AG, Sect. 8.1] for what this means).

8.4.6. As in [Ga5, Sect. 4.3], we have a canonically defined (symmetric) monoidal functor

$$\operatorname{Loc} : \operatorname{Fact}(\operatorname{Rep}(H))_{\mathcal{G}_Z}(\operatorname{Ran}) \to \operatorname{QCoh}\left(\operatorname{LocSys}_H^{\mathcal{G}_Z}\right). \tag{8.13}$$

The following is proved in the same way as [Ga5, Proposition 4.3.4][19]:

Proposition 8.4.7. *The functor* (8.13) *is a* localization*, i.e., it admits a fully faithful right adjoint.*

9. Metaplectic geometric Satake

We take G to be a reductive group. We will continue to assume that the order of the algebraic fundamental group of the derived group of G is prime to $\operatorname{char}(k)$.

We will define the metaplectic geometric Satake functor and formulate the "metaplectic vanishing conjecture" about the global Hecke action.

[19]The proof is reproduced in [Ro, Sect. 1.3].

9.1. **The metaplectic spherical Hecke category.** In this subsection we introduce the metaplectic spherical Hecke category, which is the recipient of the metaplectic geometric Satake functor.

9.1.1. Let $\mathcal{G}^G$ be a factorization $E^{\times,\mathrm{tors}}$-gerbe on Gr_G. We define the sheaf of categories $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ as follows. For an affine test scheme S and an S-point of Ran, we define the corresponding category by

$$\mathrm{Sph}_{\mathcal{G}^G}(S) := \mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}|_S}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G)^{\mathfrak{L}^+(G)|_S}. \tag{9.1}$$

In the above formula, $\mathfrak{L}^+(G)|_S$ denotes the value on S of the factorization group-scheme $\mathfrak{L}^+(G)$. The superscript $\mathfrak{L}^+(G)|_S$ indicates the equivariant category with respect to that group-scheme. Note that the latter makes sense due to the structure of equivariance on the gerbe $\mathcal{G}^G \otimes \det_{\mathfrak{g}}^{\frac{1}{2}}|_S$ with respect to $\mathfrak{L}^+(G)|_S$, which was constructed in Sect. 7.3.

By Proposition 7.2.5, we obtain that the operation of convolution product defines on $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ a structure of factorization sheaf of *monoidal* categories over Ran[20].

By construction, $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ carries a natural factorization structure, see Sect. 2.2.3.

9.1.2. Let P be a parabolic subgroup of G with Levi quotient M. Let us denote by $\mathcal{G}^M$ the factorization gerbe on Gr_M corresponding to $\mathcal{G}^G$.

The functor (5.6) naturally upgrades to a functor between sheaves of categories

$$J^G_M : (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}} \to (\mathrm{Sph}_{\mathcal{G}^M})_{/\,\mathrm{Ran}}. \tag{9.2}$$

By construction, (9.2) respects the factorization structures, i.e., it is a functor between factorization sheaves of categories.

Remark 9.1.3. We note that the functor (9.2) is *not at all* compatible with the monoidal structures!

9.2. **The metaplectic geometric Satake functor.** Let $(H, \mathcal{G}_Z, \epsilon)$ be the triple of Sect. 6.3.3 corresponding to the factorization gerbe $\mathcal{G}^G \otimes \det_{\mathfrak{g}}^{\frac{1}{2}}$.

Metaplectic geometric Satake is a canonically defined functor between factorization sheaves of monoidal DG categories

$$\mathrm{Sat} : \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z} \to (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}. \tag{9.3}$$

We will now explain how to obtain this functor from [Re, Theorem IV.8.3][21].

9.2.1. By Sect. 2.1.2, the datum of a functor (9.3) amounts to a compatible collection of functors

$$\mathrm{Sat}(I) : \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I) \to (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}(X^I), \tag{9.4}$$

where I runs over the category of finite non-empty sets and surjective morphisms.

Both sides in (9.4) are equipped with t-structures; moreover one shows that $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)$ identifies with the *derived category* of the heart of its t-structure[22], i.e., the canonical map of [Lu2, Theorem 1.3.3.2]

$$D\left(\left(\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)\right)^{\heartsuit}\right) \to \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)$$

is an equivalence.

Now, [Re, Theorem IV.8.3] constructs an *equivalence* of abelian categories

$$\left(\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)\right)^{\heartsuit} \to \left((\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}(X^I)\right)^{\heartsuit}. \tag{9.5}$$

[20] We wish to emphasize that the above monoidal structure $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ *cannot* be promoted to a symmetric monoidal structure in a way compatible with factorization, even when G is a torus and $\mathcal{G}^G$ is trivial.

[21] For a more detailed discussion on how to carry out this extension see [Ras2, Sect. 6], where the classical (i.e., non-metaplectic situation) is considered, but for this step, there is no difference between the two cases.

[22] Here, the derived category is understood as a DG category, see [Lu2, Sect. 1.3.2].

Applying [Lu2, Theorem 1.3.3.2] again, we obtain a canonically defined functor

$$D\left(\left(\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)\right)^{\heartsuit}\right) \to (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}(X^I),$$

thus giving rise to the desired functor (9.4).

The functoriality with respect to the finite sets I, as well as compatibility with factorization is built into the construction.

9.2.2. *Example.* Take $\mathcal{G} = \det^{\frac{1}{2}}_{\mathfrak{g}}$, so that $\mathrm{Sph}_{\mathcal{G}^G}$ corresponds to the *untwisted* category of sheaves on Gr_G.

Note that in this case, the source of geometric Satake is a *twisted* category of representations of $\check{G}$, see Sect. 6.3.5.

9.3. **Example: metaplectic geometric Satake for tori.** In this subsection we let $G = T$ be a torus[23].

9.3.1. Let $\Lambda^\sharp \subset \Lambda$ denote the kernel of b.

Direct image along the inclusion

$$\mathrm{Gr}_{T^\sharp} \to \mathrm{Gr}_T \tag{9.6}$$

is a fully faithful functor

$$\mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp})_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}}, \tag{9.7}$$

where we denote by $\mathcal{G}^{T^\sharp}$ the restriction of $\mathcal{G}^T$ along (9.6).

In this case, it follows from Sect. 7.5 that the forgetful functor

$$(\mathrm{Sph}_{\mathcal{G}^T})_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}}$$

factors through the essential image of (9.7), thereby giving rise to a functor

$$(\mathrm{Sph}_{\mathcal{G}^T})_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp})_{/\,\mathrm{Ran}}, \tag{9.8}$$

compatible with the factorization structures.

9.3.2. Furthermore, since the action of $\mathfrak{L}^+(T)$ on Gr_T is trivial, the functor (9.8) admits a canonically defined right inverse

$$\mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp})_{/\,\mathrm{Ran}} \to (\mathrm{Sph}_{\mathcal{G}^T})_{/\,\mathrm{Ran}}, \tag{9.9}$$

which is *monoidal* and compatible with the factorization structures.

9.3.3. By Proposition 4.6.2(b), the factorization gerbe $\mathcal{G}^{T^\sharp}$ carries a canonical multiplicative structure. Recall the equivalence

$$\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z} \simeq \mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp})_{/\,\mathrm{Ran}} \tag{9.10}$$

of (8.11).

The geometric Satake functor for T is the composite of (9.10) and (9.9).

9.4. Compatibility with Jacquet functors.

[23] We will omit the gerbe $\det^{\frac{1}{2}}_{\mathfrak{t}}$ from the notation since it is trivial in the case of tori.

9.4.1. A key feature of the assignment

$$\mathcal{G}^G \rightsquigarrow \mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$$

of Sect. 6.2.1 is compatibility with parabolics in the following sense.

Note that for a parabolic P of G with Levi quotient M, the corresponding reductive group $M^\sharp$ identifies with the Levi subgroup of $G^\sharp$, attached to the same subset of the Dynkin diagram.

We have a canonical surjection

$$\pi_{1,\mathrm{alg}}(M^\sharp) \to \pi_{1,\mathrm{alg}}(G^\sharp), \tag{9.11}$$

and the corresponding map of factorization Grassmannians

$$\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(M^\sharp)\otimes\mathbb{G}_m} \to \mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}. \tag{9.12}$$

Let $\mathcal{G}^M$ be the factorization gerbe on Gr_M such that $\mathcal{G}^M \otimes \det_{\mathfrak{m}}^{\frac{1}{2}}$ corresponds to $\mathcal{G}^G \otimes \det_{\mathfrak{g}}^{\frac{1}{2}}$ under the map of Sect. 5.1.4.

Then the multiplicative gerbe $\mathcal{G}^{\pi_{1,\mathrm{alg}}(M^\sharp)\otimes\mathbb{G}_m}$ on $\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(M^\sharp)\otimes\mathbb{G}_m}$ attached to $\mathcal{G}^M \otimes \det_{\mathfrak{m}}^{\frac{1}{2}}$ by Sect. 6.2.1 identifies with the pullback with respect to (9.12) of the multiplicative gerbe $\mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$ on $\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$ attached to $\mathcal{G}^G \det_{\mathfrak{g}}^{\frac{1}{2}}$.

9.4.2. Let M_H be the standard Levi quotient in H corresponding to standard Levi $M^\sharp$ of $G^\sharp$. Corresponding to (9.11) we have the inclusion

$$Z_H \to Z_{M_H}.$$

By the above, this inclusion is compatible with the corresponding datum of

$$\epsilon : \pm 1 \to Z_H(E),\ \epsilon : \pm 1 \to Z_{M_H}(E)$$

and the corresponding Z_H- and Z_{M_H}-gerbes on X (we denote both by $\mathcal{G}_Z$).

Therefore, restriction along $M_H \to H$ defines a monoidal functor

$$\mathrm{Res}^G_M : \mathrm{Fact}(\mathrm{Rep}(H))^\epsilon_{\mathcal{G}_Z} \to \mathrm{Fact}(\mathrm{Rep}(M_H))^\epsilon_{\mathcal{G}_Z},$$

compatible with the factorization structures.

9.4.3. The key feature of the monoidal functor (9.3) is that it makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Fact}(\mathrm{Rep}(H))^\epsilon_{\mathcal{G}_Z} & \xrightarrow{\mathrm{Sat}} & (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}} \\ {\scriptstyle \mathrm{Res}^G_M}\downarrow & & \downarrow{\scriptstyle J^G_M} \\ \mathrm{Fact}(\mathrm{Rep}(M_H))^\epsilon_{\mathcal{G}_Z} & \xrightarrow{\mathrm{Sat}} & (\mathrm{Sph}_{\mathcal{G}^M})_{/\,\mathrm{Ran}}, \end{array}$$

where J^G_M is the Jacquet functor of (9.2).

9.5. **Global Hecke action.** In this subsection we will assume that X is complete. We will define the notion of Hecke eigensheaf on Bun_G with respect to a given twisted local system.

9.5.1. Consider category of *global sections* of $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ over Ran (see Sect. 1.6.6), denote it by

$$\mathrm{Sph}_{\mathcal{G}^G}(\mathrm{Ran}),$$

and note that it identifies with

$$\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(G)}.$$

As in [Ga5, Sect. 4.4], the monoidal structure on $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$, and the operation of union of finite sets, define a (non-unital) monoidal structure on $\mathrm{Sph}_{\mathcal{G}^G}(\mathrm{Ran})$.

Moreover, the Hecke action defines a monoidal action of $\mathrm{Sph}_{\mathcal{G}^G}(\mathrm{Ran})$ on $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$, where by a slight abuse of notation we denote by the same symbols $\mathcal{G}^G$ and $\det_{\mathfrak{g}}^{\frac{1}{2}}$ the corresponding $E^{\times,\mathrm{tors}}$-gerbes on Bun_G, see Sect. 2.3.5.

9.5.2. Passing to global sections over Ran in (9.3), we obtain a monoidal functor

$$\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Sph}_{\mathcal{G}^G}(\mathrm{Ran}),$$

where we remind that $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ denotes the monoidal category of global sections of $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}$.

Thus, we obtain a monoidal action of $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ on $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$.

9.5.3. *Hecke eigensheaves.* Let σ be a twisted local system on X, as defined in Sect. 8.4.1. Recall (see Sect. 8.4.3) that σ gives rise to a (symmetric) monoidal functor

$$\mathrm{Ev}_\sigma : \mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Vect},$$

and hence, via the monoidal equivalence (8.10) to a monoidal functor

$$\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Vect},$$

which we will denote by the same symbol Ev_σ.

We define the category of twisted Hecke eigensheaves with respect to σ to be the DG category of functors of $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$-module categories

$$\mathrm{Vect} \to \mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G),$$

where $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ acts on Vect via Ev_σ and on $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ as in Sect. 9.5.2.

9.6. **The metaplectic vanishing conjecture.** We continue to assume that X is complete. Recall the (derived) stack $\mathrm{LocSys}_H^{\mathcal{G}_Z}$, see Sect. 8.4

We will state a conjecture to the effect that the (non-unital) monoidal category

$$\mathrm{QCoh}(\mathrm{LocSys}_H^{\mathcal{G}_Z})$$

acts on the category

$$\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G).$$

9.6.1. Recall (see Proposition 8.4.7) that we have a (symmetric) monoidal functor

$$\mathrm{Loc} : \mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$$

of (8.13) with a fully faithful right adjoint. Hence, by (8.10), we obtain a monoidal functor, denoted by the same symbol

$$\mathrm{Loc} : \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right),$$

also with a fully faithful right adjoint.

The following is an analog of [Ga5, Theorem 4.5.2] in the metaplectic case:

Conjecture 9.6.2. *If an object of* $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ *lies in the kernel of the functor* Loc*, then this object acts by zero on* $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$.

This conjecture can be restated as follows:

Conjecture 9.6.3. *The action of* $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ *on* $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ *(uniquely) factors through an action of* $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$.

Remark 9.6.4. Using Fourier-Mukai transform, one can show that Conjecture 9.6.2 holds when $G = T$ is a torus, see [Lys].

9.6.5. Let us assume Conjecture 9.6.3, so that $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ becomes a module category over $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$.

As in the classical (i.e., non-metaplectic case), one expects that $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ is "almost" free of rank one, and the "almost" has to do with temperedness.

More precisely, one expects that the metaplectic geometric Satake functor (9.3) extends to a *derived metaplectic geometric Satake equivalence*, generalizing [Ga5, Sects. 4.6 and 4.7], which one can use in order to define the *tempered part* of $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$, as in [AG, Sect. 12.8].

Now, one expects that the tempered subcategory of $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ *is* free of rank one as a module over $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$.

However, it is not clear whether this module admits a distinguished generator.

9.6.6. Furthermore, one expects that the entire $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ is *non-canonically* equivalent to the category $\mathrm{IndCoh}_{\mathrm{nilp}}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$, where we refer the reader to [AG, Sect. 11.1] for the $\mathrm{IndCoh}_{\mathrm{nilp}}$ notation.

9.6.7. When $G = T$ is a torus, we have

$$\mathrm{IndCoh}_{\mathrm{nilp}}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right) = \mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right).$$

In particular, the equivalence of Sect. 9.6.6 says that for each $\sigma \in \mathrm{LocSys}_H^{\mathcal{G}_Z}$, the corresponding category of Hecke eigensheaves is non-canonically equivalent to Vect. This equivalence can be made explicit as follows (see [Lys] for more details):

A point $\sigma \in \mathrm{LocSys}_H^{\mathcal{G}_Z}$ gives rise to a trivialization of the pullback of the gerbe $\mathcal{G}^T$ from Bun_T to $\mathrm{Bun}_{T^\sharp}$. Hence, it gives rise to a central extension

$$1 \to E^\times \to \mathrm{Heis}_\sigma \to \mathrm{Bun}_{\ker(T^\sharp \to T)} \to 1,$$

which is easily seen to be of Heinsenberg type, i.e., corresponding to a *non-degenerate* symplectic form on $\ker(T^\sharp \to T)$ with values in $E^\times$.

The category of Hecke eigensheaves with respect to σ is *canonically* equivalent to

$$(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Bun}_T))^{\mathrm{Bun}_{T^\sharp}},$$

where the $\mathrm{Bun}_{T^\sharp}$-equivariance makes sense due to the above trivialization of $\mathcal{G}|_{\mathrm{Bun}_{T^\sharp}}$. This category is *canonically* equivalent to the category of representations of Heis_σ, on which $E^\times$ acts by the standard character.

Since Heis_σ is of Heinsenberg type, the above category is *non-canonically* equivalent to Vect.

9.6.8. At the moment, we *do not* have a conjecture as to how to explicitly describe the category of Hecke eigensheaves in the tempered subcategory of $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ with respect to a given σ for a general reductive G.

Appendix A. The affine Grassmannian attached to finitely generated abelian groups

A.1. **The group-stack attached to a finitely generated abelian group.** Let Γ be a finitely generated abelian group, whose torsion part has order prime to $\mathrm{char}(k)$. We attach to it the group-stack

$$\Gamma\otimes\mathbb{G}_m,$$

as follows:

Write Γ as a quotient of two lattices Λ_1/Λ_2 with associated tori T_i. We set

$$\Gamma\otimes\mathbb{G}_m := T_1/T_2.$$

It is easy to see that this definition is canonically independent of the presentation of Γ as a quotient.

Explicitly, writing Γ as $\Gamma^{\mathrm{free}}\oplus\Gamma^{\mathrm{tors}}$, we have

$$\Gamma\otimes\mathbb{G}_m \simeq T\times B_{\mathrm{et}}(\Gamma^{\mathrm{tors}}(1)),$$

where T is the torus whose lattice of cocharacters is Γ^{free}.

A.2. **Maps from an algebraic group.** Let G be a reductive group (such that the torsion part of the fundamental group has order prime to $\mathrm{char}(k)$).

We claim that we have a canonically defined map

$$G\to \pi_{1,\mathrm{alg}}\otimes\mathbb{G}_m.$$

Indeed, write G as

$$1\to T_2\to \widetilde{G}_1\to G\to 1 \tag{A.1}$$

as in (3.10). Recall that T_1 denotes the torus $\widetilde{G}_1/[\widetilde{G}_1,\widetilde{G}_1]$.

From here we obtain a canonical map

$$G\simeq \widetilde{G}_1/T_2\to T_1/T_2\simeq \pi_{1,\mathrm{alg}}\otimes\mathbb{G}_m.$$

A.3. **The affine Grassmannian.** Since $\Gamma\otimes\mathbb{G}_m$ is a (commutative) group-object of PreStk, we can consider its classifying space

$$B_{\mathrm{et}}(\Gamma\otimes\mathbb{G}_m)\in \mathrm{Ptd}(\mathrm{PreStk}),$$

and the corresponding affine Grassmannian,

$$\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}.$$

A.3.1. Note that if Γ is finite, we have

$$B_{\mathrm{et}}(\Gamma \otimes \mathbb{G}_m) = B^2_{\mathrm{et}}(\Gamma(1)).$$

So in this case $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ classifies $\Gamma(1)$-gerbes on the curve with a trivialization of the punctured curves.

In other words, for a test scheme S and an S-point of Ran given by $I \subset \mathrm{Hom}(S, X)$, its lift to a point of $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ is the same as a section of

$$\mathrm{C}^\bullet(\Gamma_I, \iota^!(\Gamma_{S\times X}(1)[2])) \simeq \mathrm{C}^\bullet(\Gamma_I, \pi^!(\Gamma_S)),$$

where we recall that π denotes the projection $\Gamma_I \to S$.

A.3.2. Assume again that Γ is written as a quotient of two lattices Γ_1/Γ_2. Consider the corresponding map

$$\mathrm{Gr}_{T_1} \to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}. \tag{A.2}$$

We claim:

Theorem A.3.3.
(a) *The map* (A.2) *is ind-finite (i.e., its base change by an affine scheme S yields an ind-scheme, which is ind-finite over S).*
(b) *The resulting map*

$$\mathrm{Gr}_{T_1}/\mathrm{Gr}_{T_2} \to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$$

is an isomorphism in the topology generated by finite surjective maps.

The rest of this subsection is devoted to the proof of this theorem. It is easy to see that it is sufficient to consider the case of the short exact sequence

$$0 \to \mathbb{Z} \overset{n\cdot}{\to} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

A.3.4. First, we claim that the map

$$\mathrm{coFib}(\mathrm{Gr}_{\mathbb{G}_m} \overset{n\cdot}{\to} \mathrm{Gr}_{\mathbb{G}_m}) \to \mathrm{Gr}_{\mathbb{Z}/n\mathbb{Z}\otimes\mathbb{G}_m}$$

is bijective at the level of field-valued points, where $\mathrm{coFib}(-)$ is taken in the category ComGrp(PreStk).

Indeed, for a curve X with a point x, the space of its lifts to a point of $\mathrm{Gr}_{\mathbb{Z}/n\mathbb{Z}\otimes\mathbb{G}_m}$ is the space of μ_n-gerbes on X, equipped with a trivialization over $X - x$. The Kummer sequence

$$0 \to \mu_n \to \mathcal{O}^\times \overset{n}{\to} \mathcal{O}^\times \to 0. \tag{A.3}$$

identifies this space with $\mathbb{Z}/n\mathbb{Z}$, as required.

A.3.5. It follows from the definitions that the fiber of the map

$$\mathrm{Gr}_{T_1} \to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$$

(as group prestacks over Ran) identifies canonically with Gr_{T_2}. Hence, in order to prove point (b), it suffices to show that the map (A.2) is surjective after sheafification in the topology generated by finite surjective maps.

Given Sect. A.3.4, we obtain that point (b) of the theorem follows from point (a).

Furthermore, by Sect. A.3.4, for point (a), it suffices to show that the morphism (A.2) is ind-proper (i.e., its base change by a scheme S yields an ind-scheme, which is ind-proper over S).

We proceed with the proof of the latter fact.

A.3.6. For an affine test-scheme S, let us be given an S-point of $\mathrm{Gr}_{\mathbb{Z}/n\mathbb{Z}\otimes\mathbb{G}_m}$. I.e., we have a finite non-empty subset

$$I \subset \mathrm{Hom}(S, X),$$

a μ_n-gerbe $\mathcal{G}$ on $S \times X$ equipped with a trivialization α over U_I. The fiber product

$$S \underset{\mathrm{Gr}_{\mathbb{Z}/n\mathbb{Z}\otimes\mathbb{G}_m}}{\times} \mathrm{Gr}_{\mathbb{G}_m}$$

is the functor on schemes over S that attaches to $S' \to S$ the space of $(\mathcal{L}', \alpha', \beta)$, where $\mathcal{L}'$ is a line bundle on $S' \times X$, α' is its trivialization over $U'_I := S' \underset{S}{\times} U_I$, and β is an identification of $(\mathcal{G}, \alpha)|_{S'}$ with the datum induced by $(\mathcal{L}', \alpha')$ by the Kummer sequence (A.3). We need to show that this functor s representable by an ind-scheme locally of finite type over S.

With no restriction of generality, we can assume that X is affine. Then, after étale localization with respect to S, we can assume that $\mathcal{G}$ can be trivialized. In this case, we can assume that the pair $(\mathcal{G}, \alpha)$ also comes from the Kummer sequence. I.e., it is given by a line bundle $\mathcal{L}_U$ on U_I whose n-th power is extended over $S \times X$.

In this case, for $S' \to S$, the datum of $(\mathcal{L}, \alpha', \beta)$ as above is equivalent to the datum of extension of $\mathcal{L}_U|_{S'}$ to a line bundle over $S' \times X$. We wish to show that this functor is representable by an ind-scheme which is ind-proper over S. This is a particular case of the following general assertion:

Theorem A.3.7. *Let G be an algebraic group. Then for S and $I \subset \mathrm{Hom}(S, X)$ as above, for a G-bundle $\mathcal{P}_U$ over U_I, the functor $\mathrm{Ext}(\mathcal{P}_U)$ that attaches to $S' \to S$ the set of extensions of $\mathcal{P}_U|_{U'_I}$ to $S' \times X$ is representable by an ind-scheme locally of finite type over S. If G is reductive, then this ind-scheme is ind-proper.*

□[Theorem A.3.3]

A.4. **Proof of Theorem A.3.7.**

A.4.1. First off, it is easy to see that we can assume that S is itself locally of finite type over the ground field. Further, it is easy to see that in this case, the functor $\mathrm{Ext}(\mathcal{P}_U)$ is locally of finite type (see [GR1, Chapter 3, Sect.1.5.2]).

Below we will show that $\mathrm{Ext}(\mathcal{P}_U)$ is an ind-scheme[24]. Once this is done, the proof that it is ind-proper (for G reductive) would follow from the valuative criterion:

Indeed, choose an embedding $G \hookrightarrow GL_n$. Since G is reductive, the quotient GL_n/G is affine. A standard argument thus reduces the proof to the case $G = GL_n$.

Let $S' = C$ be a curve with a marked point c, and let be given a map $C \to S$ and $(C - c) \to \mathrm{Ext}(\mathcal{P}_U)$. We wish to be able to extend the above map to a map $C \to \mathrm{Ext}(\mathcal{P}_U)$.

By definition, we are given a vector bundle on $(C - c) \times X$ and $C \underset{S}{\times} U_I$ with a datum of compatibility on the overlap. Now, the scheme $C \times X$ is a regular surface and the open subscheme

$$((C - c) \times X) \cup (C \underset{S}{\times} U_I) \subset C \times X$$

has a complement of codimension 2. This implies that our G-bundle indeed admits a unique extension.

[24]It is automatically of ind-finite type because $\mathrm{Ext}(\mathcal{P}_U)$ is locally of finite type as a functor.

A.4.2. To prove ind-representability, with no restriction of generality, we can assume that X is complete and that U_I contains a subset of the form $S \times \{x_0\}$ for a point $x_0 \in X$.

After localization with respect to S we may assume that $\mathcal{P}_U|_{S\times x_0}$ is trivial.

Let $\mathrm{Bun}_G^{x_0}$ be the modui space of G-bundles on X with a *full* level structure at x_0. It is known to be a *scheme* (but of infinite type). We have a forgetful map

$$\mathrm{Ext}(\mathcal{P}_U) \to S \times \mathrm{Bun}_G^{x_0},$$

and it is sufficient to show that that it is a relative ind-scheme.

Given $S' \to S$ and an S'-point of $\mathrm{Bun}_G^{x_0}$, corresponding to a G-bundle $\mathcal{P}'$ on $S' \times X$, the datum of its lift to an S'-point of $\mathrm{Ext}(\mathcal{P}_U)$ is a choice of an isomorphism

$$\mathcal{P}'|_{U'_I} \simeq \mathcal{P}_U|_{U'_I}$$

compatible with the level structures at x_0.

Let Isom be the (affine) scheme over U'_I that attaches to $T \to U'_I$ the set of isomorphisms

$$\mathcal{P}'|_T \simeq \mathcal{P}_U|_T$$

compatible with the level structures at x_0.

We claim that the functor on schemes over S' that sends $S'' \to S'$ to the set of maps of S'-schemes

$$S'' \to \mathrm{Isom}$$

is ind-representable.

A.4.3. We claim that the latter fact holds for Isom replaced by any affine scheme Z over U'_I.

Indeed, by a standard argument we can replace Z by the affine line $\mathbb{A}^1$ over S'. Then the functor in question attaches to $S'' \to S'$ the set of regular functions on U'_I.

Let $\mathcal{F}$ be the direct image of the structure sheaf along $U'_I \to S'$. We can write it as a union of vector bundles $\mathcal{E}_i$. The the above functor is the direct limit of the total spaces of these vector bundles.

□[Theorem A.3.7]

Appendix B. Calculation of the étale cohomology of $B_{\mathrm{et}}(G)$

B.1. **The Leray spectral sequence.** The calculation is based on considering the Leray spectral sequence associated with the projection

$$\pi : B(B) \to B_{\mathrm{et}}(G),$$

where $B \subset G$ is the Borel subgroup.

Namely, let $\underline{A}$ denote the constant étale sheaf on either $B_{\mathrm{et}}(G)$ or $B(B)$ with coefficients in A, and let us consider the exact triangle

$$\underline{A} \to \pi_*(\underline{A}) \to \tau^{\geq 1}(R\pi_*(\underline{A})). \tag{B.1}$$

We note that each individual cohomology sheaf $R^i\pi_*(\underline{A})$ is constant with fiber $H^i_{\mathrm{et}}(G/B, A)$.

Note also that the projection $B(B) \to B_{\mathrm{et}}(T)$ defines an isomorphism an étale cohomology, so we obtain:

$$H^i_{\mathrm{et}}(B(B), A) \simeq H^i_{\mathrm{et}}(B_{\mathrm{et}}(T), A) \simeq \begin{cases} 0 \text{ for } i \text{ odd;} \\ \mathrm{Hom}(\Lambda, A(-1)) \text{ for } i = 2; \\ \mathrm{Quad}(\Lambda, A(-2)) \text{ for } i = 4. \end{cases} \tag{B.2}$$

B.2. **Cohomology in degrees ≤ 3.** From the long exact cohomology sequence associated with (B.1) we immediately obtain that $H^1_{\mathrm{et}}(B_{\mathrm{et}}(G), A) = 0$.

Next, the fact that $H^1_{\mathrm{et}}(G/B, A) = 0$ implies that the map

$$H^2_{\mathrm{et}}(B_{\mathrm{et}}(G), A) \to H^2_{\mathrm{et}}(B(B), A)$$

is injective with image equal to the kernel of the map

$$H^2_{\mathrm{et}}(B(B), A) \to H^2_{\mathrm{et}}(G/B, A). \tag{B.3}$$

We identify $H^2_{\mathrm{et}}(G/B, A) = \mathrm{Hom}(\Lambda_{\mathrm{sc}}, A(-1))$, where Λ_{sc} is the coroot lattice in Λ, and the map (B.3) becomes the restriction map

$$\mathrm{Hom}(\Lambda, A(-1)) \to \mathrm{Hom}(\Lambda_{\mathrm{sc}}, A(-1)). \tag{B.4}$$

Since $\pi_{1,\mathrm{alg}}(G) = \Lambda/\Lambda_{\mathrm{sc}}$, we obtain the desired identification

$$H^2_{\mathrm{et}}(B_{\mathrm{et}}(G), A) \simeq \mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A).$$

Now, since A was assumed divisible, the map (B.4) is surjective. Hence, the map

$$H^3_{\mathrm{et}}(B_{\mathrm{et}}(G), A) \to H^3_{\mathrm{et}}(B(B), A)$$

is injective. Since $H^3_{\mathrm{et}}(B(B), A) = 0$, we obtain the desired $H^3_{\mathrm{et}}(B_{\mathrm{et}}(G), A) = 0$.

B.3. **Cohomology in degree 4: injectivity.** We will now show that the map

$$H^4_{\mathrm{et}}(B_{\mathrm{et}}(G), A) \to H^4_{\mathrm{et}}(B(B), A)$$

is injective.

For this, it suffices to show that

$$H^3_{\mathrm{et}}(B_{\mathrm{et}}(G), \tau^{\geq 1}(R\pi_*(\underline{A}))) = 0.$$

Since, $H^3_{\mathrm{et}}(G/B, A) = 0$, we have

$$H^3_{\mathrm{et}}(B_{\mathrm{et}}(G), \tau^{\geq 1}(R\pi_*(\underline{A}))) = H^1_{\mathrm{et}}(B_{\mathrm{et}}(G), H^2_{\mathrm{et}}(G/B, A)),$$

and the latter vanishes as $H^1_{\mathrm{et}}(B_{\mathrm{et}}(G), -) = 0$.

Thus, we obtain an injection

$$H^4_{\mathrm{et}}(B_{\mathrm{et}}(G), A) \hookrightarrow H^4_{\mathrm{et}}(B(B), A) \simeq H^4_{\mathrm{et}}(B_{\mathrm{et}}(T), A) \simeq \mathrm{Quad}(\Lambda, A(-2)),$$

and out task is to show that its image equals $\mathrm{Quad}(\Lambda, A(-2))^W_{\mathrm{restr}}$.

B.4. **Containment in one direction.** We will first show that the image of $H^4_{\mathrm{et}}(B_{\mathrm{et}}(G), A)$ in $\mathrm{Quad}(\Lambda, A(-2))$ is contained in $\mathrm{Quad}(\Lambda, A(-2))^W_{\mathrm{restr}}$.

Note that realizing T is a Cartan *subgroup* of G, we obtain a commutative diagram

$$\begin{array}{ccc} H^4_{\mathrm{et}}(B_{\mathrm{et}}(T), A) & \xrightarrow{\simeq} & \mathrm{Quad}(\Lambda_G, A(-2)) \\ \uparrow & & \uparrow \mathrm{id} \\ H^4_{\mathrm{et}}(B_{\mathrm{et}}(G), A) & \xrightarrow{\simeq} & \mathrm{Quad}(\Lambda_G, A(-2)), \end{array}$$

from which it follows that the image of $H^4_{\mathrm{et}}(B_{\mathrm{et}}(G), A)$ in $\mathrm{Quad}(\Lambda, A(-2))$ is a priori contained in $\mathrm{Quad}(\Lambda, A(-2))^W$.

Thus, it remains to show that for any $q \in \mathrm{Quad}(\Lambda, A(-2))$ that lies in the image of the above map, and any *simple* coroot α_i, we have

$$b(\alpha_i, \lambda) = \langle \check{\alpha}_i, \lambda \rangle \cdot q(\alpha_i) \text{ for any} \lambda \in \Lambda.$$

Let P_i be the subminimal parabolic associated with i, and let M_i be its Levi quotient. We have a commutative diagram

$$\begin{array}{ccc} H^4_{\rm et}(B(M_i),A) & \longrightarrow & H^4_{\rm et}(B(B),A) \\ \sim\downarrow & & \downarrow = \\ H^4_{\rm et}(B(P_i),A) & \longrightarrow & H^4_{\rm et}(B(B),A) \\ \uparrow & & \uparrow = \\ H^4_{\rm et}(B_{\rm et}(G),A) & \longrightarrow & H^4_{\rm et}(B(B),A), \end{array}$$

which implies that it is sufficient to prove our claim for G replaced by M_i, which is a reductive group of semi-simple rank 1.

B.5. **Calculation for groups of semi-simple rank 1.** Any group G of semi-simple rank 1 is of the form

$$G'\times T',$$

where G' is SL_2, PGL_2 or GL_2 and T' is a torus.

If $G'=SL_2$, then

$$H^4_{\rm et}(B_{\rm et}(G),A)\simeq H^4_{\rm et}(B(G'),A)\oplus H^4_{\rm et}(B(T'),A).$$

Similarly, in this case, it is easy to see that in this case[25]

$$\operatorname{Quad}(\Lambda_G,A(-2))^W_{\rm restr}=\operatorname{Quad}(\Lambda_{G'},A(-2))^W_{\rm restr}\oplus\operatorname{Quad}(\Lambda_{T'},A(-2)).$$

So the assertion reduces to the case $G=G'=SL_2$. Note, however, that in this case, the inclusions

$$\operatorname{Quad}(\Lambda_{SL_2},A(-2))^W_{\rm restr}\subseteq\operatorname{Quad}(\Lambda_{SL_2},A(-2))^W\subseteq\operatorname{Quad}(\Lambda_{SL_2},A(-2))$$

are equalities, and the assertion follows.

In the two cases of $G'=PGL_2$ or $G'=GL_2$, since the unique positive coroot is divisible by 2, the inclusion

$$\operatorname{Quad}(\Lambda_G,A(-2))^W_{\rm restr}\subseteq\operatorname{Quad}(\Lambda_G,A(-2))^W$$

is an equality, and the assertion follows.

B.6. **The opposite containment.** It remains to show that any element $q\in\operatorname{Quad}(\Lambda,A(-2))^W_{\rm restr}$ lies in the image of $H^4_{\rm et}(B_{\rm et}(G),A)$ in $\operatorname{Quad}(\Lambda,A(-2))$.

According to Sect. 3.2.4, we have to consider the following two cases:

(I) q lies in the image of the map

$$\operatorname{Quad}(\Lambda,\mathbb{Z})^W\underset{\mathbb{Z}}{\otimes}A(-2)\to\operatorname{Quad}(\Lambda,A(-2))^W_{\rm restr}.$$

(II) q comes from a quadratic form on $\pi_{1,\rm alg}(G)$.

We first deal with case II. Consider the (2)-stack $B_{\rm et}(\pi_{1,\rm alg}(G)\otimes\mathbb{G}_m)$, see Sect. A.1.

We have a canonical isomorphism

$$H^4_{\rm et}(B_{\rm et}(\pi_{1,\rm alg}(G)\otimes\mathbb{G}_m),A)\simeq\operatorname{Quad}(\pi_{1,\rm alg}(G),A(-2))$$

(see Corollary 4.7.6) which fits into the commutative diagram

$$\begin{array}{ccc} H^4_{\rm et}(B_{\rm et}(T),A) & \xrightarrow{\sim} & \operatorname{Quad}(\Lambda,A(-2)) \\ \uparrow & & \uparrow \\ H^4_{\rm et}(B_{\rm et}(\pi_{1,\rm alg}(G)\otimes\mathbb{G}_m),A) & \xrightarrow{\sim} & \operatorname{Quad}(\pi_{1,\rm alg}(G),A(-2)). \end{array}$$

[25]Note that the formula below would be *false* if we considered $\operatorname{Quad}(\Lambda_G,A(-2))^W$ instead of $\operatorname{Quad}(\Lambda_G,A(-2))^W_{\rm restr}$; for example it would be false for the group $SL_2\times\mathbb{G}_m$.

Now the desired containment follows from the commutative diagram

$$\begin{array}{ccccc} H^4_{\text{et}}(B_{\text{et}}(G), A) & \longrightarrow & H^4_{\text{et}}(B(B), A) & \xleftarrow{\sim} & H^4_{\text{et}}(B_{\text{et}}(T), A) \\ \uparrow & & & & \uparrow \text{id} \\ H^4_{\text{et}}(B_{\text{et}}(\pi_{1,\text{alg}}(G) \otimes \mathbb{G}_m), A) & & \longrightarrow & & H^4_{\text{et}}(B_{\text{et}}(T), A), \end{array}$$

where the left vertical arrow comes from the canonical projection

$$B_{\text{et}}(G) \to B_{\text{et}}(\pi_{1,\text{alg}}(G) \otimes \mathbb{G}_m),$$

see Sect. A.2.

In order to deal with case I, it suffices to show that for any ℓ comprime with $\text{char}(k)$, the map

$$H^4_{\text{et}}(B_{\text{et}}(G), \mathbb{Z}_\ell) \to \text{Quad}(\Lambda, \mathbb{Z}_\ell(-2))^W$$

is an isomorphism.

B.7. **Computation of the integral cohomology.** From the long exact cohomology sequence associated with (B.1), we obtain that the image of

$$H^4_{\text{et}}(B_{\text{et}}(G), \mathbb{Z}_\ell) \to H^4_{\text{et}}(B_{\text{et}}(T), \mathbb{Z}_\ell) \tag{B.5}$$

equals

$$\ker\left(\ker(H^4_{\text{et}}(B_{\text{et}}(T), \mathbb{Z}_\ell) \to H^4_{\text{et}}(G/B, \mathbb{Z}_\ell)) \to H^2_{\text{et}}(B_{\text{et}}(G), H^2_{\text{et}}(G/B, \mathbb{Z}_\ell))\right).$$

Since both groups $H^4_{\text{et}}(G/B, \mathbb{Z}_\ell)$ and

$$H^2_{\text{et}}(B_{\text{et}}(G), H^2_{\text{et}}(G/B, \mathbb{Z}_\ell)) \simeq \text{Hom}(\pi_{1,\text{alg}}(G), H^2_{\text{et}}(G/B, \mathbb{Z}_\ell))$$

are torsion-free, we obtain that the image of (B.5) equals

$$H^4_{\text{et}}(B_{\text{et}}(T), \mathbb{Z}_\ell) \cap \text{Im}\left(H^4_{\text{et}}(B_{\text{et}}(G), \mathbb{Q}_\ell) \to H^4_{\text{et}}(B_{\text{et}}(T), \mathbb{Q}_\ell)\right).$$

However,

$$H^4_{\text{et}}(B_{\text{et}}(T), \mathbb{Z}_\ell) \simeq \text{Quad}(\Lambda, \mathbb{Z}_\ell(-2)),$$

and rationally, we know that

$$H^4_{\text{et}}(B_{\text{et}}(G), \mathbb{Q}_\ell) \simeq \text{Quad}(\Lambda, \mathbb{Q}_\ell(-2))^W.$$

Hence, the image of (B.5) equals

$$\text{Quad}(\Lambda, \mathbb{Z}_\ell(-2)) \cap \text{Quad}(\Lambda, \mathbb{Q}_\ell(-2))^W = \text{Quad}(\Lambda, \mathbb{Z}_\ell(-2))^W,$$

as desired.

Appendix C. Factorization gerbes via bilinear forms

The material in this section is informed by the contents of Deligne's letter to Lusztig, [Del, P.S.]. It will allow us to obtain an even more explicit parameterization of factorization gerbes.

C.1. **The complex of bilinear forms.** Let Λ be a lattice and A a divisible torsion abelian group. We consider the following complex, to be denoted $\mathcal{D}(\Lambda)$, placed in degrees $[-2, 0]$:

$$\text{Quad}(\Lambda, A(-1)) \overset{d_2}{\to} \text{Bilin}(\Lambda, A(-1)) \overset{d_1}{\to} \text{Bilin}(\Lambda, A(-1)), \tag{C.1}$$

where

–the map $\text{Quad}(\Lambda, A(-1)) \overset{d_2}{\to} \text{Bilin}(\Lambda, A(-1))$ is the usual map from quadratic forms to (symmetric) bilinear forms;

–the map $\text{Bilin}(\Lambda, A(-1)) \overset{d_1}{\to} \text{Bilin}(\Lambda, A(-1))$ is

$$b'' \mapsto b', \quad b'(\lambda, \mu) = b''(\lambda, \mu) - b''(\mu, \lambda).$$

C.1.1. This complex is acyclic in degree -1. Its 0th cohomology identifies with

$$\mathrm{Quad}(\Lambda, A(-1))$$

via

$$b' \mapsto q, \quad q(\lambda) = b'(\lambda, \lambda).$$

Its cohomology in degree (-2) identifies with

$$\mathrm{Hom}(\Lambda, A(-1)_{2\text{-tors}}) \simeq \mathrm{Hom}(\Lambda, A_{2\text{-tors}}).$$

C.1.2. We note that the complex $\mathcal{D}(\Lambda)$ contains a quasi-isomorphic sub-complex, denoted $\widetilde{\mathcal{D}}(\Lambda)$:

$$\mathrm{Quad}(\Lambda, A_{2\text{-tors}}) \overset{\widetilde{d}_2}{\to} \mathrm{Alt}(\Lambda, A(-1)) \overset{\widetilde{d}_1}{\to} \mathrm{Bilin}(\Lambda, A(-1)), \tag{C.2}$$

where

$$\mathrm{Alt}(\Lambda, A(-1)) \subset \mathrm{Bilin}(\Lambda, A(-1))$$

is the subset of alternating forms b'', i.e., those forms for which $b''(\lambda, \lambda) = 0$, and we identify

$$\mathrm{Quad}(\Lambda, A_{2\text{-tors}}) = \mathrm{Quad}(\Lambda, A(-1)_{2\text{-tors}})$$

with the preimage of $\mathrm{Alt}(\Lambda, A(-1))$ under the differential d_2.

Note that the differential $\widetilde{d}_1$ is given by multiplication by 2.

C.1.3. Note also that have an inclusion

$$\mathrm{Hom}(\Lambda, A_{2\text{-tors}}) = \ker(\widetilde{d}_2) \subset \mathrm{Quad}(\Lambda, A_{2\text{-tors}}),$$

corresponding to those quadratic forms, whose associated symmetric bilinear form vanishes.

C.1.4. The object fundamental for this subsection will be the push-out

$$\mathcal{R}(\Lambda) := B^2(\mathrm{Hom}(\Lambda, A)) \underset{B^2(\mathrm{Hom}(\Lambda, A_{2\text{-tors}}))}{\sqcup} \widetilde{\mathcal{D}}(\Lambda). \tag{C.3}$$

We have:

$$\pi_i(\mathcal{R}(\Lambda)) = \begin{cases} \mathrm{Quad}(\Lambda, A(-1)), \text{ for } i = 0, \\ 0, \text{ for } i = 1, \\ \mathrm{Hom}(\Lambda, A), \text{ for } i = 2. \end{cases}$$

C.1.5. We claim that a choice of a basis $\lambda_1, ..., \lambda_n$ of Λ gives rise to a splitting

$$\mathcal{R}(\Lambda) \simeq \mathrm{Quad}(\Lambda, A(-1)) \times B^2(\mathrm{Hom}(\Lambda, A)).$$

Indeed, the choice of a basis gives rise to a left inverse

$$\mathrm{Hom}(\Lambda, A) \underset{\mathrm{Hom}(\Lambda, A_{2\text{-tors}})}{\sqcup} \mathrm{Quad}(\Lambda, A_{2\text{-tors}}) \to \mathrm{Hom}(\Lambda, A) \tag{C.4}$$

of the inclusion

$$\mathrm{Hom}(\Lambda, A) \hookrightarrow \mathrm{Hom}(\Lambda, A) \underset{\mathrm{Hom}(\Lambda, A_{2\text{-tors}})}{\sqcup} \mathrm{Quad}(\Lambda, A_{2\text{-tors}}).$$

Namely, the restriction of (C.4) to $\mathrm{Quad}(\Lambda, A_{2\text{-tors}})$ sends a given $q'' \in \mathrm{Quad}(\Lambda, A_{2\text{-tors}})$ to the map $\Lambda \to A$ defined by $\lambda_i \mapsto q''(\lambda_i)$.

C.2. **Relation to braided monoidal categories.** Let $\mathcal{D}_{\mathrm{top}}(\Lambda, A)$ (resp., $\mathcal{R}_{\mathrm{top}}(\Lambda, A)$) be a version of $\mathcal{R}_{\mathrm{top}}(\Lambda)$, in which we replace $A(-1)$ by A.

The key assertion in [Del, P.S.] is that the complex $\mathcal{R}_{\mathrm{top}}(\Lambda, A)$, viewed as a connective spectrum, is naturally isomorphic to

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)),$$

i.e., it classifies braided monoidal groupoids $\mathcal{C}$ with $\pi_0(\mathcal{C}) \simeq \Lambda$ and $\pi_1(\mathcal{C}) \simeq A$, see Remark 4.6.7.

C.2.1. Let us construct the corresponding map

$$\mathcal{D}_{\mathrm{top}}(\Lambda, A) \to \mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)). \tag{C.5}$$

For a bilinear form b' we let $\mathcal{C}_{b'}$ be $\Lambda \times B(A)$ as a *monoidal* groupoid, with the braiding defined by b'.

When we modify b' by the coboundary of $b'' \in \mathrm{Bilin}(\Lambda, A)$, we let

$$\phi_{b''} : \mathcal{C}_{b'} \simeq \mathcal{C}_{b'+d_1(b'')}$$

be the identity functor at the level of plain groupoids, but with the monoidal structure given by b'':

$$\begin{array}{ccc} \phi_{b''}(\lambda_1) \otimes \phi_{b''}(\lambda_2) & \longrightarrow & \phi_{b''}(\lambda_1+\lambda_2) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \lambda_1+\lambda_2 & \xrightarrow{b''(\lambda_1,\lambda_2)} & \lambda_1+\lambda_2. \end{array}$$

When we modify b'' by the coboundary of $q'' \in \mathrm{Quad}(\Lambda, A)$, we construct an isomorphism

$$\phi_{b''} \simeq \phi_{b''+d_2(q'')}$$

by letting its value on λ be equal to $q''(\lambda)$.

C.2.2. Note that $\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A))$ is the same as

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B(T_{\mathrm{top}}), B^4(A)),$$

where T_{top} is the topological torus corresponding to Λ. Hence, its homotopy groups are given by

$$\pi_2(\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A))) \simeq H^2_{\mathrm{top}}(B(T_{\mathrm{top}}), A) \simeq \mathrm{Hom}(\Lambda, A),$$

$$\pi_1(\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A))) \simeq H^3_{\mathrm{top}}(B(T_{\mathrm{top}}), A) = 0,$$

$$\pi_0(\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A))) \simeq H^4_{\mathrm{top}}(B(T_{\mathrm{top}}), A) = \mathrm{Quad}(\Lambda, A),$$

and the other homotopy groups vanish.

At the level of homotopy groups, the map (C.5) induces the maps

$$\pi_2(\mathcal{D}_{\mathrm{top}}(\Lambda, A)) \simeq \mathrm{Hom}(\Lambda, A_{2\text{-tors}}) \to \mathrm{Hom}(\Lambda, A) \simeq \pi_2(\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)))$$

$$\pi_1(\mathcal{D}_{\mathrm{top}}(\Lambda, A)) = 0 = \pi_1(\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)))$$

and

$$\pi_0(\mathcal{D}_{\mathrm{top}}(\Lambda, A)) \simeq \mathrm{Quad}(\Lambda, A) \simeq \pi_0(\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A))).$$

Hence, the map (C.5) induces an *isomorphism*

$$\mathcal{R}_{\mathrm{top}}(\Lambda, A) \simeq \mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)). \tag{C.6}$$

C.2.3. Let $\lambda_1, ..., \lambda_n$ be a basis of Λ, and recall (see Sect. C.1.5) that in this case we have a canonically defined splitting

$$\mathcal{R}_{\mathrm{top}}(\Lambda, A) \simeq \mathrm{Quad}(\Lambda, A) \times B^2(\mathrm{Hom}(\Lambda, A)).$$

The resulting splitting

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)) \simeq \mathrm{Quad}(\Lambda, A) \times B^2(\mathrm{Hom}(\Lambda, A))$$

is defined as follows: the corresponding map

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)) \to B^2(\mathrm{Hom}(\Lambda, A)) \simeq (B^2(A))^{\times n}$$

is given by

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)) \to \mathrm{Maps}_{\mathrm{Ptd(Spc)}}(\Lambda, B^2(A)) \xrightarrow{\lambda_i} B^2(A).$$

C.2.4. *Variant.* Let Γ be a finitely generated abelian group. Write it as $\Gamma \simeq \Lambda/\Lambda'$, where Λ is a lattice. Set

$$\mathcal{R}_{\mathrm{top}}(\Gamma, A) := \mathrm{Fib}\left(\mathcal{R}_{\mathrm{top}}(\Lambda, A) \to \mathcal{R}_{\mathrm{top}}(\Lambda', A)\right) \underset{\mathrm{Quad}(\Lambda,A)}{\times} \mathrm{Quad}(\Gamma, A).$$

It is easy to see that $\mathcal{R}_{\mathrm{top}}(\Gamma, A)$ is canonically independent of the presentation of Γ as a quotient.

It follows from (C.6) that we have a canonical idomorphism

$$\mathcal{R}_{\mathrm{top}}(\Gamma, A) \simeq \mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Gamma), B^4(A)). \tag{C.7}$$

C.3. **Relation to the étale theory.**

C.3.1. Let T be the algebro-geometric torus over k with cocharacter lattice Λ.

We are going to construct a canonical isomorphism

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1))) \simeq \mathcal{R}(\Lambda). \tag{C.8}$$

C.3.2. Let $\Lambda' \subset \Lambda$ be a sub-lattice such that

$$\Gamma := \Lambda/\Lambda'$$

is torsion (of order prime to $\mathrm{char}(k)$). Let T' be the (algebro-geometric) torus corresponding to the lattice Λ'.

Note that we have a short exact sequence

$$0 \to \Gamma(1) \to T' \to T \to 0,$$

and corresponding to it the fiber sequence

$$B_{\mathrm{et}}(\Gamma(1)) \to B_{\mathrm{et}}(T') \to B_{\mathrm{et}}(T)$$

in group-objects in Stk.

From here we obtain a map

$$B_{\mathrm{et}}(T) \to B^2_{\mathrm{et}}(\Gamma(1)),$$

and thus a map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B^2_{\mathrm{et}}(\Gamma(1)), B^4_{\mathrm{et}}(A(1))) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1))).$$

A computation of cohomology groups shows that the resulting map

$$\underset{\Lambda' \subset \Lambda}{\mathrm{colim}}\, \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B^2_{\mathrm{et}}(\Gamma(1)), B^4_{\mathrm{et}}(A(1))) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1)))$$

is an isomorphism.

C.3.3. Note that since $\Gamma(1)$ is discrete (as an object of algebraic geometry), the map

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Gamma(1)), B^4(A(1))) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B^2_{\mathrm{et}}(\Gamma(1)), B^4_{\mathrm{et}}(A(1)))$$

is an isomorphism.

Hence, in order to establish (C.8), it suffices to construct an isomorphism

$$\underset{\Lambda' \subset \Lambda}{\mathrm{colim}}\, \mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Gamma(1)), B^4(A(1))) \simeq \mathcal{R}(\Lambda). \tag{C.9}$$

C.3.4. By (C.7), we can identify

$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Gamma(1)), B^4(A(1))) \simeq \mathcal{R}_{top}(\Gamma(1), A(1)).$$

Note that we have a canonical identification

$$\mathcal{R}_{\mathrm{top}}(\Gamma(1), A(1)) \simeq \mathcal{R}(\Gamma, A(-1)).$$

Hence, the colimit in the left-hand side of (C.9) can be rewritten as

$$\underset{\Lambda' \subset \Lambda}{\mathrm{colim}}\, \mathcal{R}(\Gamma, A),$$

which does indeed map isomorphically to $\mathcal{R}(\Lambda, A)$.

C.4. **Relation to factorization gerbes.** We now claim that there exists a canonically defined map

$$\mathcal{R}(\Lambda) \to \Theta(\Lambda), \tag{C.10}$$

where $\Theta(\Lambda)$ is as in Sect. 4.5.1.

C.4.1. We start by constructing a map

$$\mathcal{D}(\Lambda) \to \Theta(\Lambda).$$

Namely, given $b' \in \mathrm{Bilin}(\Lambda, A(-1))$, we attach to it the quadratic form

$$q(\lambda) := b'(\lambda, \lambda),$$

and the system of A-gerbes

$$\mathcal{G}^\lambda := (\omega_X^{\otimes -1})^{q(\lambda)}.$$

The isomorphisms c_{λ_1,λ_2} are defined as follows: they are obtained by tensoring the tautological isomorphism

$$(\omega_X^{\otimes -1})^{q(\lambda_1+\lambda_2)} \simeq (\omega_X^{\otimes -1})^{(q(\lambda_1)} \otimes (\omega_X^{\otimes -1})^{q(\lambda_2)} \otimes (\omega_X^{\otimes -1})^{b(\lambda_1,\lambda_2)},$$

by the A-torsor $(-1)^{b'(\lambda_1,\lambda_2)}$. The datum of h_{λ_1,λ_2} comes from the identification

$$(-1)^{b'(\lambda_2,\lambda_1)} \simeq (-1)^{b'(\lambda_1,\lambda_2)} \otimes (-1)^{b(\lambda_1,\lambda_2)},$$

which in turn results from

$$(-1)^{b'(\lambda_1,\lambda_2)} \simeq (-1)^{-b'(\lambda_1,\lambda_2)} \text{ and } b(\lambda_1,\lambda_2) = b'(\lambda_1,\lambda_2) + b'(\lambda_2,\lambda_1).$$

C.4.2. When we modify b' by the coboundary of $b'' \in \mathrm{Bilin}(\Lambda, A(-1))$, we apply the system of automorphisms of $\mathcal{G}^\lambda$, given by $(-1)^{b''(\lambda,\lambda)}$.

Finally, the result of the modification of b'' by

$$q'' \in \mathrm{Quad}(\Lambda, A(-1))$$

acts as an automorphism of the identity map on $\mathcal{G}^\lambda$ given by $q''(\lambda)$.

C.4.3. By construction, we have a map of fiber sequences

$$\begin{array}{ccccc}
B^2(\mathrm{Hom}(\Lambda, A_{2\text{-tors}})) & \longrightarrow & \widetilde{\mathcal{D}}(\Lambda, A(-1)) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)) \\
\downarrow & & \downarrow & & \downarrow \mathrm{id} \\
\Theta^0(\Lambda) & \longrightarrow & \Theta(\Lambda) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)),
\end{array}$$

where the left vertical arrow is the map

$$B^2(\mathrm{Hom}(\Lambda, A_{2\text{-tors}})) \to B^2(\mathrm{Hom}(\Lambda, A)) \to \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) \simeq \Theta^0(\Lambda).$$

From here, we obtain the desired map (C.10), which fits in the diagram of fiber sequences

$$\begin{array}{ccccc}
B^2(\mathrm{Hom}(\Lambda, A)) & \longrightarrow & \mathcal{R}(\Lambda) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)) \\
\downarrow & & \downarrow & & \downarrow \mathrm{id} \\
\Theta^0(\Lambda) & \longrightarrow & \Theta(\Lambda) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)).
\end{array}$$

In particular, we obtain that the induced map

$$\mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) \underset{B^2(\mathrm{Hom}(\Lambda,A))}{\sqcup} \mathcal{R}(\Lambda) \to \Theta(\Lambda) \tag{C.11}$$

is an equivalence.

C.4.4. We claim

Theorem C.4.5. *The following diagram commutes*

$$
\begin{array}{ccc}
\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1))) & \xrightarrow{\text{pullback}} & \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T)\times X, B^4_{\mathrm{et}}(A(1))) \\
\sim\downarrow (C.8) & & \sim\downarrow (3.3) \\
\mathcal{R}(\Lambda) & & \mathrm{FactGe}_A(\mathrm{Gr}_T) \\
\downarrow (C.10) & & \sim\downarrow (4.41) \\
\Theta(\Lambda) & \xrightarrow{\text{id}} & \Theta(\Lambda).
\end{array}
\tag{C.12}
$$

The proof of this theorem will be given in the next subsection.

C.4.6. Before we launch the proof of Theorem C.4.5, let us note that the pre-composition of the diagram (C.12) with the map

$$B^2(\mathrm{Hom}(\Lambda, A)) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1)))$$

commutes, and so does the post-composition with the map

$$\Theta(\Lambda) \to \mathrm{Quad}(\Lambda, A(-1)).$$

In other words, the two circuits of the diagrams give rise to canonically isomorphic maps

$$B^2(\mathrm{Hom}(\Lambda, A)) \to \Theta^0(\Lambda).$$

C.4.7. Consider now the full subspace of $\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1)))$ equal to

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1))) \underset{\mathrm{Quad}(\Lambda, A(-1))}{\times} \mathrm{Hom}(\Lambda, A_{2\text{-tors}}). \tag{C.13}$$

Recall (see Remark 4.6.7) that the forgetful functor

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \simeq \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda(1), B^2(A(1))) \simeq \\ \simeq \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(B^2(\Lambda(1)), B^4(A(1))) \to \mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda(1)), B^4(A(1))) \tag{C.14}$$

is fully faithful with essential image equal to (C.13).

Let us note that we have three(!) *a priori* different ways of mapping the space $\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A))$ to $\Theta(\Lambda)$:

(i) The first way is the composition of (C.14) and clockwise circuit in (C.12).

(ii) The second way is the composition of (C.14) and counter-clockwise circuit in (C.12).

(iii) The third way comes from (4.52). In more detail, a point of $\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A))$ can be seen as a system of assignments

$$\lambda \in \Lambda \rightsquigarrow \mathcal{G}^\lambda \in B^2(A) \simeq \mathrm{Ge}_A(\mathrm{pt}),$$

endowed with a associativity and commutativity constraints, and we can pull it back along $X \to \mathrm{pt}$ to form an object of $\Theta(\Lambda)$ as in Sect. 4.6.4.

The assertion of Theorem C.4.5 implies, in particular, that (i) and (ii) are canonically isomorphic. We claim that (ii) and (iii) differ by a map

$$\mathrm{Hom}(\Lambda, A_{2\text{-tors}}) \to \Theta^0(\Lambda) \tag{C.15}$$

that can be described explicitly as follows.

The map (C.15) is obtained as a tensor product of the following two maps:

One is the map that sends $q \in \mathrm{Hom}(\Lambda, A_{2\text{-tors}})$ to the object of $\Theta^0(\Lambda)$ given by

$$\lambda \mapsto (\omega_X^{\otimes -1})^{q(\lambda)}. \tag{C.16}$$

The second map factors as

$$\mathrm{Hom}(\Lambda, A_{2\text{-tors}}) \to B^2(\mathrm{Hom}(\Lambda, A)) \to \Theta^0(\Lambda),$$

where the first map is described as follows:

For a given

$$q \in \mathrm{Hom}(\Lambda, A_{2\text{-tors}}),$$

a choice of

$$\widetilde{q} \in \mathrm{Hom}(\Lambda, A),$$

with $2\widetilde{q} = q$ trivializes the resulting point of $B^2(\mathrm{Hom}(\Lambda, A))$. Two choices for $\widetilde{q}$, which differ by

$$\widetilde{\widetilde{q}} \in \mathrm{Hom}(\Lambda, A_{2\text{-tors}}),$$

result in the change of trivialization by the map

$$\Lambda \to B(\mathrm{Hom}(\Lambda, A)), \quad \lambda \mapsto (-1)^{\widetilde{\widetilde{q}}(\lambda)}.$$

C.5. **Proof of Theorem C.4.5.**

C.5.1. By Sect. C.4.6, the obstruction to the commutativity of (C.12) is a map

$$\mathrm{Quad}(\Lambda, A(-1)) \to \Theta^0(\Lambda). \tag{C.17}$$

We wish to show that this map vanishes.

C.5.2. Consider the canonical maps

$$\mathrm{Cone}\left(\mathrm{Bilin}(\Lambda, A(-1)) \stackrel{d_1}{\to} \mathrm{Bilin}(\Lambda, A(-1))\right) \to \mathcal{D}(\Lambda) \to \mathrm{Quad}(\Lambda, A(-1)).$$

We claim that their composition with (C.17) does vanish.

Indeed, this calculation has been performed in Sect. 4.3: we factor the given bilinear form b' via a finite quotient

$$\Lambda \to \Gamma,$$

and we take $A_1 = A_2 = \Gamma(1)$.

Thus, mapping

$$\mathrm{Cone}\left(\mathrm{Alt}(\Lambda, A(-1)) \stackrel{d_1}{\to} \mathrm{Bilin}(\Lambda, A(-1))\right) \to \mathrm{Cone}\left(\mathrm{Bilin}(\Lambda, A(-1)) \stackrel{d_1}{\to} \mathrm{Bilin}(\Lambda, A(-1))\right),$$

we obtain that the obstruction to the commutativity of (C.12) is a map

$$\ker(d_1) \to \mathrm{Hom}(\Lambda, A),$$

where $\mathrm{Hom}(\Lambda, A)$ is the group of automorphisms of the identity map of the unit object of $\Theta^0(\Lambda)$.

We identify

$$\ker(d_1) \simeq \mathrm{Alt}(\Lambda, A_{2\text{-tors}}).$$

Thus, we have refined the obstruction to a map of abelian groups

$$\mathrm{Alt}(\Lambda, A_{2\text{-tors}}) \to \mathrm{Hom}(\Lambda, A). \tag{C.18}$$

The map (C.18) automatically factors as

$$\mathrm{Alt}(\Lambda/2\Lambda, A_{2\text{-tors}}) \to \mathrm{Hom}(\Lambda/2\Lambda, A_{2\text{-tors}}). \tag{C.19}$$

C.5.3. By functoriality with respect to A, we can assume that $A_{2\text{-tors}} \simeq \mathbb{Z}/2\mathbb{Z}$. Hence, we can interpret (C.19) as a map

$$\Lambda^2(V) \to V \tag{C.20}$$

for a finite-dimensional vector space V over $\mathbb{Z}/2\mathbb{Z}$ (our $V = (\Lambda/2\Lambda)^\vee$), functorial in V.

However, we claim any map as in (C.20), functorial in V, is necessarily zero. Indeed, $\Lambda^2(V)$ is an irreducible representation of $GL(V)$, non-isomorphic to V.

□[Theorem C.4.5]

C.6. **The reductive case.** Let G be a reductive group with Cartan group T.

Let $\mathcal{R}(\Lambda)_G$ denote the following subcomplex of $\mathcal{R}(\Lambda)$:

- In the 0th term we replace $\mathrm{Bilin}(\Lambda, A(-1))$ with
$$\mathrm{Bilin}(\Lambda, A(-1)) \underset{\mathrm{Quad}(\Lambda, A9-1))}{\times} \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}},$$
- The (-1)st term remains the same;
- In the (-2)nd term we replace
$$\mathrm{Hom}(\Lambda, A) \underset{\mathrm{Hom}(\Lambda, A_{2\text{-tors}})}{\sqcup} \mathrm{Quad}(\Lambda, A_{2\text{-tors}})$$
with the kernel of the map
$$\mathrm{Hom}(\Lambda, A) \underset{\mathrm{Hom}(\Lambda, A_{2\text{-tors}})}{\sqcup} \mathrm{Quad}(\Lambda, A_{2\text{-tors}}) \to \underset{i\in I}{\Pi} A, \tag{C.21}$$
where:
  - I is the set of simple roots;
  - The restriction of the map (C.21) to $\mathrm{Hom}(\Lambda, A)$ is given by evaluation on the simple roots;
  - The restriction of the map (C.21) to $\mathrm{Quad}(\Lambda, A_{2\text{-tors}})$ is given by evaluation on the simple roots followed by $A_{2\text{-tors}} \hookrightarrow A$.

C.6.1. The 0-th cohomology of $\mathcal{R}(\Lambda)_G$ identifies with $\mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}$, the cohomology in degree (-1) vanishes, and the cohomology in degree (-2) identifies with
$$\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A).$$

C.6.2. Our current goal is to prove the following assertion:

Theorem C.6.3. *There exists a canonical equivalence*
$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk})}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1))) \simeq \mathcal{R}(\Lambda)_G. \tag{C.22}$$

Note that the assertion of this theorem identifies explicitly the complex $\mathcal{Q}$ from (3.13). The rest of this subsection is devoted to the proof of the theorem.

C.6.4. Consider the diagram
$$\begin{array}{ccc} \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk})}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1))) & & \mathcal{R}(\Lambda)_G \\ \downarrow & & \downarrow \\ \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk})}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1))) & \xrightarrow{\sim} & \mathcal{R}(\Lambda). \end{array} \tag{C.23}$$

By Sect. B.4, we obtain that at the level of π_0, the image of the left vertical arrow indeed hits
$$\mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}} \subset \mathrm{Quad}(\Lambda, A(-1)).$$

Hence, in order to construct the upper horizontal arrow in (C.23), it suffices to show that for each simple root, the map
$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk})}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1))) \to \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk})}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1))) \to \\ \to \mathcal{R}(\Lambda) \to B^2(A) \tag{C.24}$$
admits a canonical null-homotopy, where the last arrow is
$$\mathcal{R}(\Lambda, A) = B^2(\mathrm{Hom}(\Lambda, A)) \underset{B^2(\mathrm{Hom}(\Lambda, A_{2\text{-tors}}))}{\sqcup} \widetilde{\mathcal{D}}(\Lambda) \to \\ \to B^2(\mathrm{Hom}(\Lambda, A)) \underset{B^2(\mathrm{Hom}(\Lambda, A_{2\text{-tors}}))}{\sqcup} B^2(\mathrm{Quad}(\Lambda, A_{2\text{-tors}})) \to B^2(A),$$
with the first arrow given by the projection on the term in degree -2
$$\widetilde{\mathcal{D}}(\Lambda) \to B^2(\mathrm{Quad}(\Lambda, A_{2\text{-tors}})),$$

and the last arrow is given by evaluation on the simple root α_i.

Restricting to the copy of SL_2 mapping to G corresponding to i, we reduce the last assertion to the case when $G = SL_2$.

C.6.5. For $G = SL_2$ and $\Lambda = \mathbb{Z}$ we note that the projection

$$\mathrm{Bilin}(\mathbb{Z}, A(-1)) \to \mathrm{Quad}(\mathbb{Z}, A(-1)) \simeq A(-1)$$

is an isomorphism, while $\mathrm{Alt}(\mathbb{Z}, A(-1)) = 0$.

Hence, the complex $\widetilde{\mathcal{D}}(\Lambda)$ equals, as a complex, to

$$B^2(A_{2\text{-tors}}) \oplus A(-1),$$

so

$$\mathcal{R}(\Lambda) \simeq B^2(A) \oplus A(-1). \tag{C.25}$$

In terms of this identification, the map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(-1))) \to \mathcal{R}(\Lambda) \overset{\text{evaluation on root}}{\longrightarrow} B^2(A)$$

that appears in (C.24) is the projection

$$B^2(A) \oplus A(-1) \to B^2(A).$$

Thus, we need to show that the map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1))) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A(1))) \simeq \mathcal{R}(\Lambda) \simeq B^2(A) \oplus A(-1).$$

factors via

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1))) \to A(-1) \to B^2(A) \oplus A(-1).$$

C.6.6. For an integer n, we have

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A_{n\text{-tors}}(1))) \simeq \mathrm{Fib}(\mathcal{R}(\Lambda) \overset{n\cdot}{\to} \mathcal{R}(\Lambda)) \simeq B^2(A_{n\text{-tors}}) \oplus A_{n\text{-tors}}(-1).$$

It suffices to show that for all n, the map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A_{n\text{-tors}}(1))) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(A_{n\text{-tors}}(1))) \to \\ \to B^2(A_{n\text{-tors}}) \oplus A_{n\text{-tors}}(-1)$$

factors via

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A_{n\text{-tors}}(1))) \to A_{n\text{-tors}}(-1) \to B^2(A_{n\text{-tors}}) \oplus A_{n\text{-tors}}(-1).$$

With no restriction of generality we can replace $A_{n\text{-tors}}$ by $\mathbb{Z}/n\mathbb{Z}$. Passing to the inverse limit over n, its suffices to show that the map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1))) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1))) \simeq B^2(\widehat{\mathbb{Z}}) \oplus \widehat{\mathbb{Z}}(-1) \tag{C.26}$$

factors via

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1))) \to \widehat{\mathbb{Z}}(-1) \to B^2(\widehat{\mathbb{Z}}) \oplus \widehat{\mathbb{Z}}(-1).$$

C.6.7. Note now that the map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1))) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1)))$$

factors via

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1))) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1)))^W \to$$
$$\to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1))),$$

where $W \simeq S_2$ is the Weyl group, which acts by inversion on T.

In terms of the identification

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(T), B^4_{\mathrm{et}}(\widehat{\mathbb{Z}}(1))) \simeq B^2(\widehat{\mathbb{Z}}) \oplus \widehat{\mathbb{Z}}(-1),$$

the Weyl group acts as identity on $\widehat{\mathbb{Z}}(-1)$ and as inversion on $B^2(\widehat{\mathbb{Z}})$.

Now, the required factorization of (C.26) follows from the fact that the forgetful map

$$(B^2(\widehat{\mathbb{Z}}))^W \to B^2(\widehat{\mathbb{Z}})$$

is zero, which in turn follows from the corresponding fact for the map

$$(\widehat{\mathbb{Z}})^W \to \widehat{\mathbb{Z}},$$

where $W = S_2$ acts on $\widehat{\mathbb{Z}}$ by inversion.

C.6.8. Thus, we have constructed a map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1))) \to \mathcal{R}(\Lambda)_G. \tag{C.27}$$

By construction, the maps on the homotopy groups are

$$\pi_2\left(\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1)))\right) \simeq \mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A) \simeq \pi_2(\mathcal{R}(\Lambda)_G),$$
$$\pi_1\left(\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1)))\right) = 0 = \pi_1(\mathcal{R}(\Lambda)_G)$$

and

$$\pi_0\left(\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1)))\right) \simeq \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}} \simeq \pi_0(\mathcal{R}(\Lambda)_G).$$

Hence, (C.27) is an isomorphism.

□

C.7. **Factorization gerbes in the reductive case.**

C.7.1. Recall the space $\Theta(\Lambda)_G$, see Sect. 4.5.4. It follows from the construction of the map (C.10) that we have a canonically defined map

$$\mathcal{R}(\Lambda)_G \to \Theta(\Lambda)_G \tag{C.28}$$

that fits into the commutative diagram

$$\begin{array}{ccc} \mathcal{R}(\Lambda)_G & \longrightarrow & \Theta(\Lambda)_G \\ \downarrow & & \downarrow \\ \mathcal{R}(\Lambda)_T & \longrightarrow & \Theta(\Lambda)_T \end{array} \tag{C.29}$$

and the commutative diagram of fiber sequences:

$$\begin{array}{ccccc} B^2(\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A)) & \longrightarrow & \mathcal{R}(\Lambda)_G & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}} \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \Theta^0(\Lambda)_G & \longrightarrow & \Theta(\Lambda)_G & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}. \end{array}$$

In particular, we obtain that the induced map

$$\mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A))) \underset{B^2(\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G),A))}{\sqcup} \mathcal{R}(\Lambda)_G \to \Theta(\Lambda)_G$$

is an equivalence.

C.7.2. We claim:

Theorem C.7.3. *The following diagram commutes*

$$\begin{array}{ccc}
\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1))) & \xrightarrow{\text{pullback}} & \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G)\times X, B^4_{\mathrm{et}}(A(1))) \\
\sim\downarrow \text{Theorem C.6.3} & & \sim\downarrow (3.3) \\
\mathcal{R}(\Lambda)_G & & \mathrm{FactGe}_A(\mathrm{Gr}_G) \\
\downarrow (C.28) & & \sim\downarrow (4.43) \\
\Theta(\Lambda)_G & \xrightarrow{\text{id}} & \Theta(\Lambda)_G.
\end{array}$$

Proof. The composites of both circuits with the forgetful map $\Theta(\Lambda)_G \to \Theta(\Lambda)$ are isomorphic by Theorem C.4.5. Hence, it suffices to show that for every simple coroot α_i, the composites of both circuits with the evaluation map

$$\Theta(\Lambda)_G \mapsto \mathrm{Ge}_A(X)$$

produce isomorphic results.

This reduces the verification to the case of $G = SL_2$. However, in the latter case, $\Theta(\Lambda)_G$ is *discrete*, i.e., the map

$$\Theta(\Lambda)_G \to \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}$$

is an isomorphism, and the assertion follows. □

Appendix D. Proof of Proposition 3.1.9 by global methods

In this section we will supply a proof of Proposition 3.1.9, which is different from both the original (but containing gaps) proof in [Re] and the recent proof in [Zhao].

D.1. **The simply connected case.** In this subsection we will assume that G is simply-connected, and we will prove that (3.3) is an isomorphism in this case.

D.1.1. Recall the notations from Sect. 2.3.5. A variant of the construction in Sects. 3.1.4-3.1.6 defines a map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G)\times X, B^4_{\mathrm{et}}(A(1)) \to \mathrm{Ge}_A(\mathrm{Bun}_G(\overline{X}; D); \mathrm{unit}), \tag{D.1}$$

(the notation "$-$; unit" means "gerbes trivialized at the unit"), so that we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G)\times X, B^4_{\mathrm{et}}(A(1)) & \xrightarrow{(3.3)} & \mathrm{FactGe}_A(\mathrm{Gr}_G) \\
\downarrow & & \downarrow \\
\mathrm{Ge}_A(\mathrm{Bun}_G(\overline{X}; D); \mathrm{unit}) & \xrightarrow{\sim} & \mathrm{Ge}_A(\mathrm{Gr}_G; \mathrm{unit}).
\end{array} \tag{D.2}$$

We will show that all arrows in the diagram (D.2) are isomorphisms.

D.1.2. First, recall that for G simply connected, the pullback map

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G), B^4_{\mathrm{et}}(A(1)) \to \mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G)\times X, B^4_{\mathrm{et}}(A(1))$$

is an isomorphism, and both spaces are discrete with π_0 given by $H^4_{\mathrm{et}}(B_{\mathrm{et}}(G), A(1))$.

D.1.3. Next, we claim that $\mathrm{Ge}_A(\mathrm{Bun}_G(\overline{X}; D); \mathrm{unit})$ is also discrete that the left vertical arrow in (D.2) induces an isomorphism on π_0.

Indeed, it follows from the Atiyah-Bott formula for the cohomology of $\mathrm{Bun}_G(\overline{X}; D), A)$ (see [GL1, Theorem 5.4.5] or [Ga7, Theorem 19.1.4]) that

$$H^0_{\mathrm{et}}(\mathrm{Bun}_G(\overline{X}; D), A) \simeq A, \;\; H^1_{\mathrm{et}}(\mathrm{Bun}_G(\overline{X}; D), A) = 0,$$

while the map

$$H^0_{\mathrm{et}}\left(X, H^4_{\mathrm{et}}(B_{\mathrm{et}}(G), A(1))\right) \to H^2_{\mathrm{et}}(\mathrm{Bun}_G(\overline{X}; D), A)$$

induced by (D.1), is an isomorphism.

D.1.4. Thus, we obtain that the forgetful map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{Ge}_A(\mathrm{Gr}_G; \mathrm{unit}) \tag{D.3}$$

realizes $\mathrm{Ge}_A(\mathrm{Gr}_G; \mathrm{unit})$ as a retract of$\mathrm{FactGe}_A(\mathrm{Gr}_G)$. Hence, it remains to prove that the map (D.3) is fully faithful.

I.e., we have to show that if a gerbe $\mathcal{G}$ on Gr_G admits a factorization structure, then this structure is unique. However, this follows from the fact that the projection

$$\mathrm{Gr}_G \to \mathrm{Ran}$$

is ind-proper with connected and simply-connected fibers.

D.2. **Retraction for an arbitrary group.** In this subsection we let G be an arbitrary reductive group.

D.2.1. Consider the forgetful map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{FactGe}_A(\mathrm{Gr}_T) \to \Theta(\Lambda).$$

We claim that this map canonically factors via a map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \Theta(\Lambda)_G \to \Theta(\Lambda).$$

First, we claim that the quadratic form q associated to an object of $\mathrm{FactGe}_A(\mathrm{Gr}_G)$ belongs to $\mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}$. This follows as in Sect. B.5.

To construct the identifications

$$\mathcal{G}^{\alpha_i} \simeq (\omega_X^{-1})^{q(\alpha_i)},$$

it is enough to consider the case of $G = SL_2$, and the assertion follows from the simply-connected case established above.

D.2.2. Consider the composition

$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(A(1))) \to \mathrm{FactGe}_A(\mathrm{Gr}_G) \to \Theta(\Lambda)_G.$$

It induces an isomorphism on homotopy groups, and hence is an equivalence. Hence, we obtain that the map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \Theta(\Lambda)_G \tag{D.4}$$

realizes $\Theta(\Lambda)_G$ as a retract of $\mathrm{FactGe}_A(\mathrm{Gr}_G)$.

D.2.3. Hence, in order to show that (3.3) is an isomorphism, it remains to show that the map (D.4) is fully faithful.

D.3. **The case of a simply-connected derived group.** In this subsection we will assume that the derived group G' of G is simply-connected.

D.3.1. Let T_0 be the quotient of G by its derived group G'. By the assumption on G, the map

$$\pi_{1,\mathrm{alg}}(G) \to \pi_{1,\mathrm{alg}}(T_0)$$

is an isomorphism.

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{FactGe}_A(\mathrm{Gr}_{G'}) & \xrightarrow{\sim} & \Theta(\Lambda)_{G'} \\ \uparrow & & \uparrow \\ \mathrm{FactGe}_A(\mathrm{Gr}_G) & \longrightarrow & \Theta(\Lambda)_G \\ \uparrow & & \uparrow \\ \mathrm{FactGe}_A(\mathrm{Gr}_{T_0}) & \xrightarrow{\sim} & \Theta(\Lambda)_{T_0} \end{array}$$

Hence, in order to prove that (3.3) is an isomorphism, it suffices to show that the pullback map

$$\mathrm{FactGe}_A(\mathrm{Gr}_{T_0}) \to \mathrm{Fib}(\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{FactGe}_A(\mathrm{Gr}_{G'})$$

is an isomorphism.

D.3.2. Note that the map

$$\mathrm{Gr}_G \to \mathrm{Gr}_{T_0}$$

is ind-proper with fibers that are connected and simply-connected. Hence, the map

$$\mathrm{FactGe}_A(\mathrm{Gr}_{T_0}) \to \mathrm{Ge}_A(\mathrm{Gr}_{T_0}) \underset{\mathrm{Ge}_A(\mathrm{Gr}_G)}{\times} \mathrm{FactGe}_A(\mathrm{Gr}_G)$$

is an isomorphism.

We claim that

$$\mathrm{Ge}_A(\mathrm{Gr}_{T_0}) \to \mathrm{Ge}_A(\mathrm{Gr}_G) \to \mathrm{Ge}_A(\mathrm{Gr}_{G'}) \tag{D.5}$$

is a fiber sequence.

Once we prove this, we can identify

$$\mathrm{Ge}_A(\mathrm{Gr}_{T_0}) \underset{\mathrm{Ge}_A(\mathrm{Gr}_G)}{\times} \mathrm{FactGe}_A(\mathrm{Gr}_G) \simeq \mathrm{Fib}(\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{Ge}_A(\mathrm{Gr}_{G'})),$$

and since

$$\mathrm{FactGe}_A(\mathrm{Gr}_{G'}) \to \mathrm{Ge}_A(\mathrm{Gr}_{G'})$$

is an equivalence (see Sect. D.1), further with

$$\mathrm{Fib}(\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{FactGe}_A(\mathrm{Gr}_{G'})),$$

as desired.

D.3.3. To prove that (D.5) is a fiber sequence, it is enough to prove that

$$\mathrm{Ge}_A(\mathrm{Bun}_{T_0}(\overline{X}; D)) \to \mathrm{Ge}_A(\mathrm{Bun}_G(\overline{X}; D)) \to \mathrm{Ge}_A(\mathrm{Bun}_{G'}(\overline{X}; D))$$

is a fiber sequence.

The latter follows from the fact that

$$\mathrm{Bun}_G(\overline{X}; D) \to \mathrm{Bun}_{T_0}(\overline{X}; D)$$

is a fibration, locally trivial in the smooth topology with typical fiber $\mathrm{Bun}_{G'}(\overline{X}; D)$, which is simply-connected.

D.4. **The general case.** We now let G be a general reductive group.

D.4.1. Let
$$1 \to T \to \widetilde{G} \to G \to 1$$
be as in (3.10). Let $\widetilde{G}^\bullet$ be the Čech nerve of the map $\widetilde{G} \to G$.

Note that the map
$$B_{\mathrm{et}}(\widetilde{G}) \to B_{\mathrm{et}}(G)$$
is an étale surjection, and
$$\mathrm{Gr}_{\widetilde{G}} \to \mathrm{Gr}_G$$
becomes a surjection after sheafification with respect to the topology generated by finite surjective maps.

Hence the maps
$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(A(1))) \to \mathrm{Tot}\left(\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(\widetilde{G}^\bullet) \times X, B^4_{\mathrm{et}}(A(1)))\right)$$
and
$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{Tot}\left(\mathrm{FactGe}_A(\mathrm{Gr}_{\widetilde{G}^\bullet})\right)$$
are both isomorphisms.

Now, the map
$$\mathrm{Maps}_{\mathrm{Ptd(PreStk)}}(B_{\mathrm{et}}(\widetilde{G}^\bullet) \times X, B^4_{\mathrm{et}}(A(1))) \to \mathrm{FactGe}_A(\mathrm{Gr}_{\widetilde{G}^\bullet})$$
is a term-wise isomorphism by Sect. D.3.

Hence, (3.3) is also an isomorphism, as desired.

Appendix E. Twisting of factorization categories by gerbes

E.1. **The context.** Let $\mathcal{C}$ be a symmetric monoidal category, and let A be a finite abelian group that acts by automorphisms of the identity functor on $\mathcal{C}$ (viewed as a symmetric monoidal functor). We will assume that the orders of elements in A are prime to $\mathrm{char}(k)$.

Up to passing to a colimit, we can write
$$A = \mathrm{Hom}(\Gamma, E^{\times,\mathrm{tors}}),$$
where Γ is the dual finite abelian group.

By Sect. 4.8.5, we can think of a pair $(\mathcal{G}_A \in \mathrm{Ge}_A(X), \epsilon \in A_{2\text{-tors}})$ as a multiplicative factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ with respect to $E^{\times,\mathrm{tors}}$. (Recall also that the multiplicative structure on $\mathcal{G}$ automatically lifts to a commutative one, see Remark 4.7.3.)

We will show how to perform a twist of $\mathrm{Fact}(\mathcal{C})$ by means of $\mathcal{G}$ and obtain a new sheaf of symmetric monoidal categories over Ran, denoted $\mathrm{Fact}(\mathcal{C})_{\mathcal{G}}$, equipped with a factorization structure.

This construction will contain both twisting constructions, mentioned in Sects. 8.2.3 and 8.2.4, respectively.

E.2. **Two symmetric monoidal structures on** Vect^Γ**.** Consider the category Vect^Γ of Γ-graded E-vector spaces. For $\gamma \in \Gamma$, let E^γ denote the corresponding 1-dimensional object in Vect^Γ.

The category Vect^Γ has two symmetric monoidal structures. One, denoted by $\otimes$, is given by convolution along γ:
$$E^{\gamma_1} \otimes E^{\gamma_2} = E^{\gamma_1+\gamma_2},$$

The other one, denoted $*$, is given by component-wise tensor product. Note that this symmetric is non-unital, unless Γ is finite.

Note that these two symmetric monoidal structures are *lax-compatible* in the sense that there exists a natural transformation
$$(V_1 \star W_1) \otimes (V_2 \star W_2) \to (V_1 \otimes V_2) \star (W_1 \otimes W_2)$$
satisfying a homotopy-coherent system of compatibilities.

E.3. **Interpreting the action.** Let us be given an action of A on $\mathcal{C}$. The datum of such an action is equivalent to that of an action on $\mathcal{C}$ of Vect^Γ, equipped with the $*$ monoidal structure.

Explicitly, this means that every object $c \in \mathcal{C}$ can be canonically written as

$$c \simeq \underset{\gamma\in\Gamma}{\oplus} c^\gamma,$$

in a way compatible with the symmetric monoidal structure on $\mathcal{C}$. In the above notations,

$$E^\gamma \star c = c^\gamma.$$

The $\star$-action of Vect^Γ on $\mathcal{C}$ is lax-compatible with the symmetric monoidal structure on $\mathcal{C}$ in the sense that we have a natural transformation

$$(V_1 \star c_1) \otimes (V_2 \star c_2) \to (V_1 \otimes V_2) \star (c_1 \otimes c_2),$$

satisfying a homotopy-coherent system of compatibilities.

E.4. **Creating factorization categories.** The assignment

$$\mathcal{C} \mapsto \mathrm{Fact}(\mathcal{C}) \tag{E.1}$$

is functorial with respect to lax symmetric monoidal functors.

Hence, we obtain that $\mathrm{Fact}(\mathrm{Vect}^\Gamma_\otimes)$ (i.e., the sheaf of symmetric monoidal categories obtained from Vect^Γ in the $\otimes$ symmetric monoidal structure) acquires another symmetric monoidal structure given by $*$, which is lax-compatible with one given by $\otimes$.

Similarly, the $*$-action of Vect^Γ on $\mathcal{C}$ implies that $\mathrm{Fact}(\mathrm{Vect}^\Gamma_\otimes)$, viewed as a sheaf of monoidal categories over Ran with respect to $*$, acts on $\mathrm{Fact}(\mathcal{C})$. This action is lax-compatible with $\otimes$-symmetric monoidal structure on $\mathrm{Fact}(\mathrm{Vect}^\Gamma_\otimes)$, and the given one on $\mathrm{Fact}(\mathcal{C})$.

E.5. **Relation to the affine Grassmannian.** We note now that there exists a canonical equivalence of sheaves of categories over Ran

$$\mathrm{Fact}(\mathrm{Vect}^\Gamma_\otimes) \simeq \mathrm{Shv}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\mathrm{Ran}}, \tag{E.2}$$

compatible with the factorization structures.

Under this equivalence, the $\otimes$-symmetric monoidal structure on $\mathrm{Fact}(\mathrm{Vect}^\Gamma_\otimes)$ corresponds to the symmetric monoidal structure on $\mathrm{Shv}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\mathrm{Ran}}$ given by convolution along the group structure on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$. The $*$-symmetric monoidal structure on $\mathrm{Fact}(\mathrm{Vect}^\Gamma_\otimes)$ corresponds to the symmetric monoidal structure on $\mathrm{Shv}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\mathrm{Ran}}$ given by pointwise !-tensor product.

E.6. **The twisting construction.** We obtain that $\mathrm{Fact}(\mathcal{C})$ acquires an action of $\mathrm{Shv}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\mathrm{Ran}}$, viewed as a symmetric monoidal category with respect to the pointwise !-tensor product.

Now, using Theorem 1.6.9, we obtain that we can upgrade $\mathrm{Fact}(\mathcal{C})$ to a sheaf of categories over $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$, compatible with the factorization structure.

Hence, the construction of Sect. 1.7.2 allows to twist $\mathrm{Fact}(\mathcal{C})$ by any factorization $E^{\times,\mathrm{tors}}$-gerbe $\mathcal{G}$ on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$, and obtain another factorization sheaf of categories, to be denoted $\mathrm{Fact}(\mathcal{C})_\mathcal{G}$.

Since the action of $\mathrm{Shv}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\mathrm{Ran}}$ on $\mathrm{Fact}(\mathcal{C})$ is lax-compatible with the symmetric monoidal structure on $\mathrm{Shv}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\mathrm{Ran}}$ given by convolution and the existing symmetric monoidal structure on $\mathrm{Fact}(\mathcal{C})$, if $\mathcal{G}$ carries a *commutative* structure with respect to the group structure on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$, the twisted sheaf of categories $\mathrm{Fact}(\mathcal{C})_\mathcal{G}$ carries a symmetric monoidal structure.

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