GEOMETRIC THETA-LIFTING FOR UNITARY GROUPS

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Abstract. In this note we define the geometric theta-lifting functors in the global noramified setting. They are expected to provide new cases of the geometric Langlands functoriality.

1. Introduction

1.1. In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality. At level of functions the theta-lifting for unitary groups was studied by many authors ([1, 3, 4, 13, 15]).

The definition uses a splitting of the metaplectic extension over the unitary groups. The splitting existing in the litterature at the level of functions ([6, 4, 14, 15]) are not entirely satisfactory. Namely, given a local field F and its degree two extension E, for the corresponding unitary group the splitting in *loc.cit.* is defined up to a multiplication by a character of E^*/F^* with values in $\mathbb{C}^1$ (the group of complex numbers of absolute value one). We give a more canonical geometric construction of this splitting.

1.2. Notations. We follow the conventions of [10]. So, we work over an algebraically closed field k of characteristic $p \neq 2$. We fix a prime $\ell \neq p$, write $\bar{\mathbb{Q}}_\ell$ for the algebraic closure of $\mathbb{Q}_\ell$. All our stacks are defined over k. For an algebraic stack locally of finite type S we have the derived categories $\mathrm{D}(S), \mathrm{D}^-(S)_!, \mathrm{D}^{\prec}(S)$ of complexes of $\bar{\mathbb{Q}}_\ell$-sheaves on S as in *loc.cit.*

Let X be a smooth projective connected curve. Write Ω for the canonical line bundle on X. Let $\pi : Y \to X$ be an étale degree 2 cover with Y connected. Write σ for the nontrivial automorphism of Y over X, $\pi_*\mathcal{O} = \mathcal{O} \oplus \mathcal{E}$, where $\mathcal{E}$ is the sheaf of σ-antiinvariants, so $\mathcal{E}^2 \widetilde{\to} \mathcal{O}$. Once and for all we pick a line bundles $\mathcal{E}'$, $\Omega^{\frac{1}{2}}$ on X with isomorphisms $\mathcal{E}'^2 \widetilde{\to} \mathcal{E}$, $(\Omega^{\frac{1}{2}})^2 \widetilde{\to} \Omega$.

For an algebraic group G write Bun_G for the stack of G-torsors on X. For $n \geq 1$ write Bun_n (resp., $\mathrm{Bun}_{n,Y}$) for the stack of G-torsors on X (resp., on Y).

2. Moduli of unitary bundles and geometric theta-lifting

2.1. Pick $\epsilon = \pm 1$. For $n \geq 1$ an ϵ-hermitian vector bundle on Y (with respect to π) is a datum of $V \in \mathrm{Bun}_{n,Y}$ with a nondegenerate form $\phi : V \otimes \sigma^* V \to \mathcal{O}_Y$ such that the

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diagram commutes

$$\begin{array}{ccc} \phi^*(V\otimes\sigma^*V) & \stackrel{\sigma^*\phi}{\to} & \phi^*\mathcal{O}=\mathcal{O} \\ \downarrow & & \downarrow \epsilon \\ V\otimes\sigma^*V & \stackrel{\phi}{\to} & \mathcal{O}, \end{array}$$

the left vertical arrow being a canonical isomorphism. Write Bun_{U_n} (resp., $\mathrm{Bun}_{U_n^-}$) for the stack of hermitian (resp., skew-hermitian) vector bundles on Y.

Here U_n is the group scheme on X, the quotient of $(\mathrm{GL}_n\times Y)/\{\mathrm{id},\sigma\}$ by the diagonal action. Here $\sigma(g)=({}^tg)^{-1}$ for $g\in\mathrm{GL}_n$. The stacks like Bun_{U_n} have been studied for example in [5, 12].

2.1.1. For an ϵ-hermitian vector bundle V on Y viewing ϕ as an isomorphism $\phi: \sigma^*V\widetilde{\to}V^*$, one gets $(\sigma^*\phi)^*=\epsilon\phi$. Consider the isomorphism

$$(\sigma^*\phi,\phi): V\oplus\sigma^*V\to\sigma^*V^*\oplus V^* \tag{1}$$

on Y. Let $M=\pi_*C$. Let σ act on $V\oplus\sigma^*V$ such that the natural isomorphism $V\oplus\sigma^*V\widetilde{\to}\pi^*M$ is σ-invariant. Then (1) is σ-equivariant, so descends to an isomorphism $\bar\phi: M\widetilde{\to}M^*$ such that $\bar\phi^*=\epsilon\bar\phi$. We also view the latter as a map $\bar\phi: M\otimes M\to\mathcal{O}_X$. So, $\bar\phi$ is symmetric for $\epsilon=1$ (resp., anti-symmetric for $\epsilon=-1$).

Consider the isomorphism

$$(-\sigma^*\phi,\phi): V\oplus\sigma^*V\to\sigma^*V^*\oplus V^* \tag{2}$$

As above, there is a unique isomorphism $\phi': M\widetilde{\to}M^*\otimes\mathcal{E}$ such that $\pi^*(\phi')=(-\sigma^*\phi,\phi)$. Besides, ϕ' is symmetric for $\epsilon=-1$ (resp., antisymmetric for $\epsilon=1$).

Let $\mathrm{Bun}_{\mathbb{O}_{2n}}$ be the stack classifying $M\in\mathrm{Bun}_{2n}$ with a nondegerate symmetric form $\mathrm{Sym}^2 M\to\mathcal{O}_X$. Let $\mathfrak{q}_n:\mathrm{Bun}_{U_n}\to\mathrm{Bun}_{\mathbb{O}_{2n}}$ be the map sending (V,ϕ) to $(M=\pi_*V,\bar\phi)$. Let $\mathfrak{q}_n^-:\mathrm{Bun}_{U_n^-}\to\mathrm{Bun}_{\mathbb{O}_{2n}}$ be the map sending (V,ϕ) to $(M\otimes\mathcal{E}',\phi')$. Here we have viewed ϕ' as a map $\mathrm{Sym}^2(M\otimes\mathcal{E}')\to\mathcal{O}_X$.

The stack $\mathrm{Bun}_{\mathbb{S}\mathrm{p}_{2n}}$ classifies $M\in\mathrm{Bun}_{2n}$ with a symplectic form $\wedge^2M\to\mathcal{O}_X$. Let $\mathfrak{p}_n:\mathrm{Bun}_{U_n}\to\mathrm{Bun}_{\mathbb{S}\mathrm{p}_{2n}}$ be the map sending (V,ϕ) to $(M\otimes\mathcal{E}',\phi')$. Here $M=\pi_*V$. Let also $\mathfrak{p}_n^-:\mathrm{Bun}_{U_n^-}\to\mathrm{Bun}_{\mathbb{S}\mathrm{p}_{2n}}$ be the map sending (V,ϕ) to $(M,\bar\phi)$.

2.1.2. We have a canonical identification $\pi^*\mathcal{E}\widetilde{\to}\mathcal{O}$ such that the descent data for $\pi^*\mathcal{E}$ are given by the action of σ on $\mathcal{O}$ as -1. This gives an isomorphism $\delta:\mathrm{Bun}_{U_n}\widetilde{\to}\mathrm{Bun}_{U_n^-}$ sending (V,ϕ) to $(V\otimes\pi^*\mathcal{E}',\phi)$. The diagram commutes

$$\begin{array}{ccccc} & & \mathrm{Bun}_{U_n} & & \\ & \swarrow \mathfrak{q}_n & \downarrow\delta & \searrow \mathfrak{p}_n & \\ \mathrm{Bun}_{\mathbb{O}_{2n}} & \stackrel{\mathfrak{q}_n^-}{\leftarrow} & \mathrm{Bun}_{U_n^-} & \stackrel{\mathfrak{p}_n^-}{\to} & \mathrm{Bun}_{\mathbb{S}\mathrm{p}_{2n}} \end{array}$$

2.1.3. The stack Bun_{U_1} is a group stack described in ([8], Appendix A). We have the norm map $N:\mathrm{Bun}_{1,Y}\to\mathrm{Bun}_1$ sending $\mathcal{L}$ to $\mathcal{E}\otimes\det(\pi_*\mathcal{L})$, this is a homomorphism of group stacks. The stack Bun_{U_1} classifies $V\in\mathrm{Bun}_{1,Y}$ together with a trivialization $N(V)\widetilde{\to}\mathcal{O}_X$. The stack Bun_{U_1} has connected components $\mathrm{Bun}_{U_1}^a$ indexed by $a\in\mathbb{Z}/2\mathbb{Z}$, $\mathrm{Bun}_{U_1}^0$ being the connected component of unity. The stack $\mathrm{Bun}_{U_1^-}$ classifies $V\in\mathrm{Bun}_{1,Y}$ with an isomorphism $N(V)\widetilde{\to}\mathcal{E}$.

The group stack Bun_{U_1} acts on $\mathrm{Bun}_{U_n^-}$ sending $(\mathcal{L},\phi_{\mathcal{L}})\in \mathrm{Bun}_{U_1}, (V,\phi)\in \mathrm{Bun}_{U_n^-}$ to $(V\otimes\mathcal{L}, \phi\otimes\phi_{\mathcal{L}})\in \mathrm{Bun}_{U_n^-}$. Let $\rho_n:\mathrm{Bun}_{U_n}\to\mathrm{Bun}_{U_1}$ be the map sending (V,ϕ) to $(\det V, \det\phi)$.

2.1.4. For $n>1$ let $(V,\phi)\in\mathrm{Bun}_{U_n}$. Then $\det\phi:\sigma^*\det V\,\widetilde{\to}\,\det V^*$ can be seen as a trivialization $\xi: N(\det V)\,\widetilde{\to}\,\mathcal{O}$. Let Bun_{SU_n} be the stack classifying $V\in\mathrm{Bun}_{U_n}$ with an isomorphism $\det V\,\widetilde{\to}\,\mathcal{O}_Y$ such that the induced isomorphism $N(\det V)\,\widetilde{\to}\,N(\mathcal{O}_Y)\,\widetilde{\to}\,\mathcal{O}_X$ is ξ. So, Bun_{SU_n} is the fibre of the map $\rho_n:\mathrm{Bun}_{U_n}\to\mathrm{Bun}_{U_1}$ over the unit $\mathcal{O}_Y\in\mathrm{Bun}_{U_1}$ of this group stack.

By ([5], Theorem 2) the stack Bun_{SU_n} is connected. By ([5], Theorem 3), if $n>1$ then $\mathrm{Pic}(\mathrm{Bun}_{SU_n})\,\widetilde{\to}\,\mathbb{Z}$. The line bundle on Bun_{SU_n} sending $V\in\mathrm{Bun}_{U_n}$ to $\det\mathrm{R}\Gamma(Y,V)$ is twice a generator of $\mathrm{Pic}(\mathrm{Bun}_{SU_n})$ by ([12], Remark 3.6(2)).

2.2. Twist by Ω. As in [8] write Bun_{G_n} for the stack classifying $M\in\mathrm{Bun}_{2n}$ with a symplectic form $\wedge^2 M\to\Omega_X$. Write $\mathrm{Bun}_{\mathbb{O}_{2n},\Omega}$ for the stack classifying $M\in\mathrm{Bun}_{2n}$ with a nondegenerate symmetric bilinear form $\mathrm{Sym}^2 M\to\Omega$.

Write $\mathrm{Bun}_{U_n,s}$ for the stack classifying $V\in\mathrm{Bun}_{n,Y}$ together with an isomorphism

$$\phi:\sigma^*V\,\widetilde{\to}\,V^*\otimes\pi^*\Omega$$

such that ϕ is skew-hermitian. In other words, $(\sigma^*\phi)^*=-\phi$. Here s stands for 'skew-hermitian'. Write Ω_Y for the canonical line bundle on Y, one has $\pi^*\Omega\,\widetilde{\to}\,\Omega_Y$.

As in Section 2.1.1, for $(V,\phi)\in\mathrm{Bun}_{U_n,s}$ let $M=\pi_*V$. There is a unique symplectic form $\bar\phi: M\,\widetilde{\to}\,M^*\otimes\Omega$ such that

$$\pi^*\bar\phi=(\sigma^*\phi,\phi): V\oplus\sigma^*V\to(\sigma^*V^*\oplus V^*)\otimes\pi^*\Omega$$

There is also a unique symmetric bilinear form $\phi': M\to M^*\otimes\Omega\otimes\mathcal{E}$ such that

$$\pi^*\phi'=(-\sigma^*\phi,\phi): V\oplus\sigma^*V\to(\sigma^*V^*\oplus V^*)\otimes\pi^*(\Omega\otimes\mathcal{E})$$

This defines a diagram

$$\mathrm{Bun}_{\mathbb{O}_{2n},\Omega}\stackrel{\mathfrak{q}_{n,s}}{\leftarrow}\mathrm{Bun}_{U_n,s}\stackrel{\mathfrak{p}_{n,s}}{\to}\mathrm{Bun}_{G_n},$$

where $\mathfrak{p}_{n,s}$ sends (V,ϕ) to $(M=\pi_*V,\bar\phi)$, and $\mathfrak{q}_{n,s}$ sends (V,ϕ) to $(M\otimes\mathcal{E}',\phi')$.

2.3. Square root. Denote by $\mathcal{A}_n$ the line bundle on Bun_{G_n} whose fibre at M is $\det\mathrm{R}\Gamma(X,M)$. Write $\widetilde{\mathrm{Bun}}_{G_n}$ for the gerbe of square roots of $\mathcal{A}_n$ over Bun_{G_n}.

Our immediate purpose is to construct a distinguished square root of the line bundle $\mathfrak{p}_{n,s}^*\mathcal{A}_n$. Our construction will depend only on the choices of $\Omega^{\frac{1}{2}}$, $\mathcal{E}'$ that we made in Section 1.2, and also on a choice of $i\in k$ with $i^2=-1$.

2.3.1. For $W\in\mathrm{Bun}_n$ write for brevity $d(W)=\det\mathrm{R}\Gamma(X,W)$, we view it as a $\mathbb{Z}/2\mathbb{Z}$-graded line.

For $\mathcal{A}_i\in\mathrm{Bun}_1$ set

$$K(\mathcal{A}_1,\mathcal{A}_2)=\frac{d(\mathcal{A}_1\otimes\mathcal{A}_2)\otimes d(\mathcal{O})}{d(\mathcal{A}_1)\otimes d(\mathcal{A}_2)}$$

We view this line as a $\mathbb{Z}/2\mathbb{Z}$-graded. Then K is bilinear up to a canonical isomorphism by ([9], Section 4.2.1-4.2.2).

One has a canonical Pfaffian line bundle Pf on $\mathrm{Bun}_{\mathbb{O}_{2n},\Omega}$ defined in ([2], Section 4.2.1). Pick $i \in k$ with $i^2 = -1$. This choice yields a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $Pf(W)^2 \widetilde{\to} d(W)$ for $W \in \mathrm{Bun}_{\mathbb{O}_{2n},\Omega}$ as in *loc.cit.* An alternative construction of Pf is given in [7].

Given $(V,\phi) \in \mathrm{Bun}_{U_n,s}$ let $M = \phi_* V$. The symplectic form $\bar\phi : \wedge^2 M \to \Omega$ induces an isomorphism $\det M \widetilde{\to} \Omega^n$. By ([8], Lemma 1) one has canonically for $(V,\phi) \in \mathrm{Bun}_{U_n,s}$ and $M = \pi_* V$

$$d(M \otimes \mathcal{E}') \widetilde{\to} d(M) \otimes K(\Omega^n, \mathcal{E}') \otimes \frac{d(\mathcal{E}')^{2n}}{d(\mathcal{O})^{2n}} \tag{3}$$

Our choice of $\Omega^{\frac{1}{2}}$, bilinearity of K, and (3) yield an isomorphism

$$Pf(M \otimes \mathcal{E}')^2 \widetilde{\to} d(M) \otimes \left(K(\Omega^{\frac{1}{2}}, \mathcal{E}') \otimes \frac{d_X(\mathcal{E}')}{d_X(\mathcal{O})} \right)^{2n} \tag{4}$$

Denote by $\mathcal{L}_n$ the line bundle on $\mathrm{Bun}_{U_n,s}$ whose fibre at (V,ϕ) is

$$Pf(M \otimes \mathcal{E}) \otimes \left(\frac{d_X(\mathcal{O})}{K_X(\Omega^{\frac{1}{2}}, \mathcal{E}') \otimes d_X(\mathcal{E}')} \right)^n$$

Then (4) yields the desired isomorphism over $\mathrm{Bun}_{U_n,s}$

$$\mathcal{L}_n^2 \widetilde{\to} \mathfrak{p}_{n,s}^* \mathcal{A}_n$$

Let $\tilde{\mathfrak{p}}_{n,s} : \mathrm{Bun}_{U_n,s} \to \widetilde{\mathrm{Bun}}_{G_n}$ be the map sending (V,ψ) to $\mathfrak{p}_{n,s}(V,\phi) = (M,\bar\phi)$ and a line $\mathcal{L}_n(V,\phi)$ equipped with the above isomorphism $\mathcal{L}_n(V,\phi)^2 \widetilde{\to} d(M)$.

2.4. Dual pair U_n, U_m. Let $n, m \ge 1$. Write

$$\tau : \mathrm{Bun}_{U_n} \times \mathrm{Bun}_{U_m,s} \to \mathrm{Bun}_{U_{nm},s}$$

for the map sending $(V_1,\phi_1) \in \mathrm{Bun}_{U_n}$, $(V_2,\phi_2) \in \mathrm{Bun}_{U_m,s}$ to $V_1 \otimes V_2$ with the induced isomorphism

$$\phi_1 \otimes \phi_2 : \sigma^*(V_1 \otimes V_2) \widetilde{\to} (V_1 \otimes V_2)^* \otimes \pi^*\Omega$$

The groups U_n, U_m form a dual pair in G_{nm} essentially via the map $\mathfrak{p}_{nm,s}\tau$. Let Aut be the theta-sheaf on $\widetilde{\mathrm{Bun}}_{G_n}$ given in ([11], Definition 1). Define the theta-lifting functors

$$F_s : \mathrm{D}^-(\mathrm{Bun}_{U_n})_! \to \mathrm{D}^{\prec}(\mathrm{Bun}_{U_m,s}), \qquad F : \mathrm{D}^-(\mathrm{Bun}_{U_m,s})_! \to \mathrm{D}^{\prec}(\mathrm{Bun}_{U_n})$$

following the framework of the geometric Langlands functoriality proposed in ([10], Section 2) for the kernel

$$\mathcal{M} = \tau^* \tilde{\mathfrak{p}}_{nm,s}^* \,\mathrm{Aut}[\dim \mathrm{Bun}_{U_n} \times \mathrm{Bun}_{U_m,s} - \dim \mathrm{Bun}_{G_{nm}}]$$

That is, for $K \in \mathrm{D}^-(\mathrm{Bun}_{U_n})_!$ and $K' \in \mathrm{D}^-(\mathrm{Bun}_{U_m,s})_!$ we let

$$F_s(K) = (q_s)_!(q^*K \otimes \mathcal{M})[-\dim \mathrm{Bun}_{U_n}] \quad \text{and} \quad F(K') = q_!(q_s^*K' \otimes \mathcal{M})[-\dim \mathrm{Bun}_{U_m,s}]$$

for the diagram of projections

$$\mathrm{Bun}_{U_n} \stackrel{q}{\leftarrow} \mathrm{Bun}_{U_n} \times \mathrm{Bun}_{U_m,s} \stackrel{q_s}{\to} \mathrm{Bun}_{U_m,s}$$

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