Orthogonality relations between automorphic sheaves attached to 2-dimensional irreducible local systems on a curve

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Introduction générale

Motivation

0.1. L'objet principal de cette thèse est de donner une interprétation géométrique des résultats de Rankin et Selberg sur le produit scalaire de deux formes automorphes (cuspidales et partout non ramifiées) pour $GL(2)$ sur un corps de fonctions. Cette géométrisation fait partie du Programme de Langlands Géométrique initié par V.Drinfeld, A.Beilinson et G.Laumon. Comme première motivation, rappelons les résultats de G.Laumon ([14]) et M.Rothstein ([13]) dans le cas de $GL(1)$ qui nous sert de modèle.

Soit X une courbe projective, lisse et connexe de genre $g \geq 1$ sur $\mathbb{C}$. Notons M' le schéma de Picard de X qui paramètre les classes d'isomorphie de $\mathcal{O}_X$-modules inversibles L de degré 0. Soit M l'espace grossier de modules des faisceaux inversibles L sur X munis d'une connexion (integrable) $\nabla : L \to L \otimes_{\mathcal{O}_X} \Omega_X$. C'est un $\mathbb{C}$-schéma en groupes abéliens (pour le produit tensoriel), muni d'une structure naturelle de $\mathrm{H}^0(X, \Omega_X)$-torseur sur M'. Il existe un $\mathcal{O}_{M\times M'}$-module inversible Aut à connexion (relative à M) qui est le noyau de deux transformations de Fourier

$$\mathcal{F} : \mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{D}_{M'}) \to \mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{O}_M)$$

et

$$\mathcal{F}' : \mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{O}_M) \to \mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{D}_{M'})$$

Le théorème de Laumon-Rothstein dit que *ces foncteurs sont quasi-inverse l'un de l'autre (à un décallage et un automorphisme* $(-1)^*$ *près).* Ce resultat peut être obtenu comme une conséquence formelle de deux relations d'orthogonalité. L'une d'elles s'écrit sous la forme:

le complexe

$$\mathrm{R}(\mathrm{pr}_{12})_*(\mathrm{pr}_{13}^* \mathrm{Aut} \otimes \mathrm{pr}_{23}^* \mathrm{Aut})$$

est canoniquement isomorphe à $\triangle_* \mathcal{O}_M$ *dans* $\mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{O}_{M\times M})$ *(à un décallage et un automorphisme* $(-1)^*$ *près),*

où $\mathrm{pr}_{13}, \mathrm{pr}_{23} : M \times M \times M' \to M \times M'$ et $\mathrm{pr}_{12} : M \times M \times M' \to M \times M$ sont les projections, $\triangle$: $M \to M \times M$ est le morphisme diagonal, et l'image directe par rapport à pr_{12} est calculée au sens de $\mathcal{D}$-modules.

La variante ℓ-adique de cette relation d'orthogonalité est la suivante. Soit X une courbe projective, lisse et connexe de genre $g \geq 1$ sur un corps algébriquement clos de caractéristique positive p. On fixe un nombre premier ℓ different de p et une clôture algébrique $\bar{\mathbb{Q}}_\ell$ de $\mathbb{Q}_\ell$. Soit E_0 un $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1 sur X.

On ne dispose pas d'espace de modules des systèmes locaux ℓ-adiques sur X. Cependant, on peut considérer les déformations de E_0 (au sens suivante: si A est une $\bar{\mathbb{Q}}_\ell$-algèbre Artinienne locale à corps résiduel $\bar{\mathbb{Q}}_\ell$ alors une *A-déformation de E_0* est un A-faisceau lisse F de rang 1 sur X muni d'un isomorphisme $F \otimes_A \bar{\mathbb{Q}}_\ell \widetilde{\to} E_0$). Le système local E_0 admet la déformation universelle E (cf. Prop. 1). Notons R la base de cette déformtion universelle. En fait, R est un anneau des séries formelles sur $\bar{\mathbb{Q}}_\ell$ de dimension $2g$.

Pour tout $n \geq 0$ on a un R-faisceau lisse $E^{(n)}$ de rang 1 sur $X^{(n)}$ (cf. sect. 1.3.1). Notons $\underline{\mathrm{Pic}}^n X$ le schéma de Picard de X qui paramètre les classes d'isomorphie de $\mathcal{O}_X$-modules inversibles de degré n. Par la theorie de corps de classes abelien géométrique, $E^{(n)}$ est l'image inverse par rapport à un morphisme naturel $X^{(n)} \to \underline{\mathrm{Pic}}^n X$ d'un R-faisceau lisse E^n de rang 1 sur $\underline{\mathrm{Pic}}^n X$. Notons E_1^n, E_2^n les deux relèvements de E à $\mathrm{Spf}(R \hat{\otimes} R)$.

Pour tout n il existe un isomorphisme canonique de $R \hat{\otimes} R$-modules

$$\mathrm{H}^{2g}(\underline{\mathrm{Pic}}^n X, \mathcal{H}om(E_1^n, E_2^n)) \widetilde{\to} R(-g),$$

où la structure de $R \hat{\otimes} R$-module sur R est définie par le morphisme diagonal $R \hat{\otimes} R \to R$. De plus, on a $\mathrm{H}^i(\underline{\mathrm{Pic}}^n X, \mathcal{H}om(E_1^n, E_2^n)) = 0$ pour tout $i \neq 2g$.

C'est ce type de relations d'orthogonalité qu'on cherche à établir dans le cadre ℓ-adic pour GL(2) au lieu de GL(1).

0.2. Rappelons le résultat de Rankin et Selberg mentionné au début, notre deuxième source d'inspiration.

Soit X une courbe projective, lisse et géométriquement connexe de genre g sur le corps fini $\mathbb{F}_q$ à q elements. On fixe un nombre premier ℓ inversible dans $\mathbb{F}_q$, une clôture algébrique $\bar{\mathbb{Q}}_\ell$ de $\mathbb{Q}_\ell$, une racine carré de q dans $\bar{\mathbb{Q}}_\ell$, et une clôture algébrique $\bar{\mathbb{F}}_q$ de $\mathbb{F}_q$.

Soit E un $\bar{\mathbb{Q}}_\ell$-faisceau lisse et géométriquement irréductible de rang 2 sur X. V.Drinfeld (et D.Gaitsgory dans [2]) ont associé à E un $\bar{\mathbb{Q}}_\ell$-faisceau pervers irréductible Aut_E sur le champ de modules Bun_2 des fibrés vectoriels de rang 2 sur X, qui est en un sens convenable vecteur propre des opérateurs de Hecke par rapport à E.

Soit $\mathrm{Bun}_2(\mathbb{F}_q)$ l'ensemble de classes d'isomorphie de fibrés vectoriels de rang 2 sur X. On note $\varphi_E : \mathrm{Bun}_2(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$ la fonction "trace de Frobenius" associé au faisceau pervers Aut_E. C'est la forme automorphe (cuspidale et partout non ramifiée) associé à E au sens habituel de correspondance de Langlands. On note Bun_2^n la composante connexe de Bun_2 qui correspond aux fibrés de degré n.

Pour tout $n \in \mathbb{Z}$ *on a*

$$\sum_{L \in \mathrm{Bun}_2^n(\mathbb{F}_q)} \frac{1}{\# \operatorname{Aut} L} \varphi_{E^*}(L)\varphi_E(L) = \frac{q}{q-1} \det(1 - \mathrm{Fr}\, q^{-1} \mid \mathrm{H}^1(\bar{X}, \mathcal{E}nd E)),$$

où $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ *et* Fr *est le Frobenius géométrique. En plus, si* E_1 *et* E_2 *sont deux* $\bar{\mathbb{Q}}_\ell$*-faisceaux lisses et géométriquement irréductibles de rang 2 sur* X *qui sont non isomorphes, alors pour tout* $n \in \mathbb{Z}$

$$\sum_{L \in \mathrm{Bun}_2^n(\mathbb{F}_q)} \frac{1}{\# \operatorname{Aut} L} \varphi_{E_1^*}(L)\varphi_{E_2}(L) = 0$$

Remarque . Pour $a \in \mathbb{Z}/2\mathbb{Z}$ notons $\mathrm{Bun}^a_{\mathrm{PGL}_2}(\mathbb{F}_q)$ l'ensemble de classes d'isomorphie des PGL_2-fibrés sur X de degré a. Si $\det E_1 \tilde{\rightarrow} \det E_2$ alors la fonction $L \mapsto \varphi_{E_1^*}(L)\varphi_{E_2}(L)$ ne depend que de l'image de $L \in \mathrm{Bun}_2^n(\mathbb{F}_q)$ dans $\mathrm{Bun}_{\mathrm{PGL}_2}^{n \bmod 2}(\mathbb{F}_q)$ et on a

$$\sum_{L \in \mathrm{Bun}_2^n(\mathbb{F}_q)} \frac{1}{\# \operatorname{Aut} L} \varphi_{E_1^*}(L)\varphi_{E_2}(L) = \frac{\# \underline{\mathrm{Pic}}^0 X(\mathbb{F}_q)}{q-1} \sum_{L \in \mathrm{Bun}_{\mathrm{PGL}_2}^{n \bmod 2}(\mathbb{F}_q)} \frac{1}{\# \operatorname{Aut} L} \varphi_{E_1^*}(L)\varphi_{E_2}(L),$$

où $\underline{\mathrm{Pic}}^0 X(\mathbb{F}_q)$ est le group de Picard de faisceaux inversibles sur X de degré 0. Comme

$$\# \underline{\mathrm{Pic}}^0 X(\mathbb{F}_q) = q^g \det(1 - \mathrm{Fr}\, q^{-1}, \mathrm{H}^1(X, \bar{\mathbb{Q}}_\ell)),$$

on a finalement pour tout $a \in \mathbb{Z}/2\mathbb{Z}$

$$\sum_{L \in \mathrm{Bun}^a_{\mathrm{PGL}_2}(\mathbb{F}_q)} \frac{1}{\# \operatorname{Aut} L} \varphi_{E^*}(L)\varphi_E(L) = q^{1-g} \det(1 - \mathrm{Fr}\, q^{-1}, \mathcal{E}nd_0 E)$$

Résultats principaux

0.3. On fixe un corps k algébriquement clos de caractéristique positive, un nombre premier ℓ different de caractéristique de k. Soit X une courbe projective, lisse et connexe de genre $g \geq 1$ sur k. Choisissons un système local ℓ-adique irréductible E_0 de rang 2 sur X. Soit E la déformation universelle de E_0, R la base de cette déformation universelle (cf. sect. 1.1.1). En fait, R est un anneau des séries formelles sur $\bar{\mathbb{Q}}_\ell$ de dimension $8g-6$ (cf. Prop. 1). Notons E_i $(i=1,2)$ les deux relèvements de E à $\mathrm{Spf}(R\hat{\otimes} R)$.

D'après V.Drinfeld (et D.Gaitsgory, [2]), on a un faisceau pervers irréductible Aut_{E_0} sur le k-champ Bun_2 de modules des fibrés vectoriels de rank 2 sur X, qui est un vecteur propre des operateurs de Hecke par rapport à E_0. On note $\mathrm{Aut}^n_{E_0}$ la restriction de Aut_{E_0} à la composante connexe Bun^n_2 de Bun_2 qui correspond aux fibrés de degré n.

On montre que la construction de Gaitsgory reste valable pour les déformations de E_0, ce qui nous permet à définir un R-faisceau pervers Aut^n_E sur Bun^n_2 (cf. sect. 1.4.1). (Pour la definition d'un R-faisceau pervers cf. sect. 0.2.1).

Les automorphismes scalaires des fibrés vectoriels definissent une action de $\mathbb{G}_m$ sur Bun^n_2 par 2-automorphismes de champs. Notons $\overline{\mathrm{Bun}}^n_2$ le quotient de Bun^n_2 par cette action, de sorte que le mophisme naturel $\mathrm{Bun}^n_2 \to \overline{\mathrm{Bun}}^n_2$ est une $\mathbb{G}_m$-gerbe (cf. sect. 1.2.1). Du fait que Aut^n_E est pervers, il est un image réciproque d'un faisceau $\overline{\mathrm{Aut}}^n_E[-1]$ sur $\overline{\mathrm{Bun}}^n_2$, où $\overline{\mathrm{Aut}}^n_E$ est un R-faisceau pervers.

Le résultat pricipal de cet exposé est le

Théorème Principal Global . *Pour tout $n\in\mathbb{Z}$ il existe un isomorphisme canonique de $R\hat{\otimes} R$-modules*

$$\mathrm{H}^0_c(\overline{\mathrm{Bun}}^n_2, \overline{\mathrm{Aut}}^n_{E_1^*} \otimes_{R\hat{\otimes} R} \overline{\mathrm{Aut}}^n_{E_2}) \widetilde{\to} R(-1),$$

où la structure de $R\hat{\otimes} R$-module sur R est definie par le morphisme diagonal $R\hat{\otimes} R \to R$. De plus, on a $\mathrm{H}^i_c(\overline{\mathrm{Bun}}^n_2, \overline{\mathrm{Aut}}^n_{E_1^*} \otimes_{R\hat{\otimes} R} \overline{\mathrm{Aut}}^n_{E_2}) = 0$ pour tout $i \neq 0$.

Remarque . i) Le champ $\overline{\mathrm{Bun}}^n_2$ n'est pas de type fini. Cependant, Aut^n_E est le prolongement par zero de sa resriction à un sous-champ de $\overline{\mathrm{Bun}}^n_2$, qui lui est de type fini (cf. Cor. 1).
ii) Les cohomologies à support compact d'un champ ne sont à priori pas définies. Nous allons utiliser la définition du livre à venir de G.Laumon et L.Moret-Bailly ([7],(18.8)) communiquée à l'auteur par G.Laumon (cf. Appendice A). Leur définition est inspirée par une idée de J.Bernstein et V.Lunts ([8]). Une définition analogue a été aussi communiqué à l'auteur par V.Drinfeld.

0.4. On déduit le Théorème Principal Global d'un résultat local. Pour l'énoncer on a besoin de quelques notations.

Notons Sh^n_i le champ qui classifie les faisceaux cohérents sur X de rang générique i et de degré n (cf. sect. 1.2.2). Soit ${}^0\,\mathrm{Sh}'^n_1$ le champ qui classifie les couples $(\mathcal{O} \overset{s}{\hookrightarrow} F_1)$ sur X, où $F_1 \in \mathrm{Sh}^n_1$ (cf. sect. 1.2.2). On a un morphisme représentable ${}^0\pi_1 : {}^0\,\mathrm{Sh}'^n_1 \to \mathrm{Sh}^n_1$

qui envoie $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$ sur F_1. On a aussi un morphisme $\pi'_0 : {}^0\operatorname{Sh}'^n_1 \to \operatorname{Sh}^n_0$ qui envoie $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$ sur $F_1/\operatorname{Im} s$. On vérifie facilement que π'_0 est un fibré vectoriel de rang n. On note $\overline{\operatorname{Flag}}_n$ le champ qui paramètre les suites exactes

$$0 \to \Omega \to M \to F_1 \to 0, \tag{1}$$

où M est un fibré vectoriel de rang 2 et de degré n sur X, et Ω est le faisceau inversible canonique sur X (de sorte que F_1 est un faisceau cohérent sur X de rang générique 1).

Soit $\mathcal{E}xt$ le champ qui classifie les suites exactes $0 \to \Omega \to ? \to \mathcal{O} \to 0$ sur X. Le morphisme $\mathcal{E}xt \to \operatorname{Spec} k$ est donc un fibré vectoriel généralisé (non-représentable) de rang $1-g$ (la définition d'un fibré vectoriel généralisé est donnée dans sect. 0.2.2). On a un morphisme naturel $\mathcal{E}xt \to \mathbb{A}^1_k$.

Considerons le champ ${}^0\operatorname{Sh}'^n_1 \times_{\operatorname{Sh}^n_1} \overline{\operatorname{Flag}}_{n+2g-2}$, où le morphisme $\overline{\operatorname{Flag}}_{n+2g-2} \to \operatorname{Sh}^n_1$ envoie la suite (1) sur F_1. On a un morphisme représentable ${}^0\operatorname{Sh}'^n_1 \times_{\operatorname{Sh}^n_1} \overline{\operatorname{Flag}}_{n+2g-2} \to \mathcal{E}xt$ qui envoie la collection $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1, 0 \to \Omega \to M \to F_1 \to 0)$ sur le 'pull-back' de (1) par rapport à $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$. On définie le couplement $\mu : {}^0\operatorname{Sh}'^n_1 \times_{\operatorname{Sh}^n_1} \overline{\operatorname{Flag}}_{n+2g-2} \to \mathbb{A}^1_k$ comme le morphisme composé ${}^0\operatorname{Sh}'^n_1 \times_{\operatorname{Sh}^n_1} \overline{\operatorname{Flag}}_{n+2g-2} \to \mathcal{E}xt \to \mathbb{A}^1_k$. Notons $\mathcal{L}_\psi$ le faisceau d'Artin-Schreier sur $\mathbb{A}^1_k$ (cf. sect. 0.2.3).

Pour un intier positif n on note $X^{(n)}$ la puissance symétrique n-ième de X. C'est le schéma de modules des diviseurs effectifs D de degré n sur X. Soit $\operatorname{Pic}^n X$ le champ de Picard qui classifie les faisceaux inversibles sur X de degré n. On a un morphisme $\pi : X^{(n)} \to \operatorname{Pic}^n X$ qui envoie un diviseur $D \in X^{(n)}$ sur le faisceau inversible $\mathcal{O}(D)$.

Soit $\det : \operatorname{Sh}^n_1 \to \operatorname{Pic}^n X$ le morphisme naturel qui envoie F_1 sur $\det F_1$ (cf. [12]). On dispose d'une flèche ${}^0\operatorname{Sh}'^n_1 \to X^{(n)}$ qui envoie $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$ sur $(\mathcal{O} \stackrel{\det s}{\hookrightarrow} \det F_1) \in X^{(n)}$. (Ici $X^{(n)}$ est consideré comme le schéma de modules des couples $(\mathcal{O} \hookrightarrow \mathcal{A}, \mathcal{A} \in \operatorname{Pic}^n X)$).

On note

$$\varphi_n : {}^0\operatorname{Sh}'^n_1 \times_{\operatorname{Sh}^n_1} {}^0\operatorname{Sh}'^n_1 \times_{\operatorname{Sh}^n_1} \overline{\operatorname{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$$

le morphisme qui envoie la collection $(\mathcal{O} \stackrel{s_1}{\hookrightarrow} F_1, \mathcal{O} \stackrel{s_2}{\hookrightarrow} F_1, 0 \to \Omega \to M \to F_1 \to 0)$ sur $(\mathcal{O} \stackrel{\det s_1}{\hookrightarrow} \det F_1, \mathcal{O} \stackrel{\det s_2}{\hookrightarrow} \det F_1)$. (Ici $X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$ est identifié au schéma de modules des triplés $(s', s'' : \mathcal{O} \hookrightarrow \mathcal{A}, \mathcal{A} \in \operatorname{Pic}^n X)$).

Notons $i : X^{(n)} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$ l'immersion fermée qui envoie $(\mathcal{O} \stackrel{s}{\hookrightarrow} \mathcal{A})$ sur $(\mathcal{O} \stackrel{s}{\hookrightarrow} \mathcal{A}, \mathcal{O} \stackrel{-s}{\hookrightarrow} \mathcal{A})$.

Soit A une $\bar{\mathbb{Q}}_\ell$-algèbre locale d'Artin à corps résiduel $\bar{\mathbb{Q}}_\ell$. Pour un A-faisceau lisse E sur X (cf. sect. 0.2.1) on note $\mathcal{L}^n_E$ le A-faisceau pervers de Laumon sur Sh^n_0 (cf. sect. 1.3.1).

Théorème Principal Local . *Soient E_1, E_2 deux A-faisceaux lisses de rang 2 sur X. Le complexe*

$$(\varphi_n)_!(\mathrm{pr}_1^*\,\pi_0'^*\mathcal{L}^n_{E_1} \otimes_A \mathrm{pr}_2^*\,\pi_0'^*\mathcal{L}^n_{E_2} \otimes_{\bar{\mathbb{Q}}_\ell} \mathrm{pr}_{13}^*\,\mu^*\mathcal{L}_\psi \otimes_{\bar{\mathbb{Q}}_\ell} \mathrm{pr}_{23}^*\,\mu^*\mathcal{L}_\psi)[3n+2-2g]$$

est un A-faisceau pervers sur $X^{(n)} \times_{\mathrm{Pic}^n X} X^{(n)}$. Il est supporté par le sous-schéma fermé $i : X^{(n)} \to X^{(n)} \times_{\mathrm{Pic}^n X} X^{(n)}$ et est canoniquement isomorphe à

$$i_*(E_1 \otimes_A E_2)^{(n)}(-n-1+g)[n]$$

Remarque . i) Le morphisme φ_n peut être facilement decomposé en des morphismes représentables et des fibrés vectoriels généralisés, par example de façon suivante

$$\begin{aligned} {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to X^{(n)} \times_{\mathrm{Pic}^n X} ({}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2}) \to \\ X^{(n)} \times_{\mathrm{Pic}^n X} (X^{(n)} \times \mathcal{E}xt) \to X^{(n)} \times_{\mathrm{Pic}^n X} X^{(n)} \end{aligned}$$

Par suite, $(\varphi_n)_!$ peut être aussi defini comme le composé de foncteurs correspondants.
ii) On verra que l'énoncé du Théorème Local Principal est local par rapport à E_1, E_2. Pour cette raison on peut espérer établir un résultat analogue pour les systèmes locaux à coefficients de torsion.

La preuve au niveau de fonctions

0.5. Dans ce numero nous rappelons les traits marquants de la demonstration due à Rankin et Selberg de l'énoncé 0.2. On conserve les notations de 0.2.

La forme automorphe $\varphi_E : \mathrm{Bun}_2(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$ est uniquement déterminée par ses propriétés:

pour tout point fermé $x \in X$, tout $L \in \mathrm{Bun}_2(\mathbb{F}_q)$

1) $\varphi_E(L(x)) = \mathrm{tr}(\mathrm{Fr}_x, \det E)\varphi_E(L)$

2) $\sum\limits_{L' \in S} \varphi_E(L') = -q_x^{\frac{1}{2}}\,\mathrm{tr}(\mathrm{Fr}_x, E)\varphi_E(L)$,

où S est l'ensemble des fibrés vectoriels L' tels que $L \subset L' \subset L(x)$ et $\deg L' = \deg L + 1$, $q_x = q^{\deg x}$, Fr_x est le Frobenius géométrique agissant sur la fibre géométrique en x;

3) Étant donné une suite exacte $0 \to \Omega \to L \to \mathcal{A} \to 0$, où $\mathcal{A}$ est un $\mathcal{O}_X$-module inversible de degré n, on a

$$\varphi_E(L) = (-1)^{n+3g-3} q^{-\frac{n+3g-3}{2}} \sum_{s:\mathcal{O} \hookrightarrow \mathcal{A}} \mathrm{tr}(\mathrm{Fr}_{\mathrm{div}\, s}, E^{(n)})\psi(< u, s >),$$

où $u \in \mathrm{Ext}^1(\mathcal{A}, \Omega)$ correspond à $0 \to \Omega \to L \to \mathcal{A} \to 0$, $\psi : \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell^*$ est un caractère additif nontrivial, $\operatorname{div} s$ est consideré comme un $\mathbb{F}_q$-point du schéma $X^{(n)}$. (Pour la définition du faisceau $E^{(n)}$ sur $X^{(n)}$ cf. sect. 1.3.1).

Tout d'abord on démontre la formule suivante.

0.5.1. *Soient E_1, E_2 deux $\bar{\mathbb{Q}}_\ell$-faisceaux lisses et géométriquement irréductibles de rang 2 sur X. Étant donné un $\mathcal{O}_X$-module inversible $\mathcal{A}$ de degré n, on a*

$$\sum_{\Omega \hookrightarrow L, \alpha : (\det L) \otimes \Omega^{-1} \widetilde{\to} \mathcal{A}} \frac{1}{\# \mathrm{Aut}(\Omega \hookrightarrow L, \alpha)} \varphi_{E_1}(L) \varphi_{E_2}(L) = q^{4-4g} \sum_{s : \mathcal{O} \hookrightarrow \mathcal{A}} \mathrm{tr}(\mathrm{Fr}_{\operatorname{div} s}, (E_1 \otimes E_2)^{(n)}),$$

où la première somme est prise sur l'ensemble de classes d'isomorphie des triples $(\Omega \hookrightarrow L, \alpha)$, où $\Omega \hookrightarrow L$ est une inclusion de Ω dans un fibré vectoriel L de rang 2 sur X (on ne suppose pas que Ω est localement facteur direct de L), et $\alpha : (\det L) \otimes \Omega^{-1} \widetilde{\to} \mathcal{A}$ est un isomorphisme.

Le point essentiel est l'assertion suivante dont la verification est laissée au lecteur. Notons $\sigma_m : X^{(n-2m)} \times X^{(m)} \to X^{(n)}$ le morphisme qui envoie (D', D) sur $D' + 2D$.

0.5.2. *Soient E_1, E_2 deux $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de rang 2 sur X. Sur $(E_1 \otimes E_2)^{(n)}$ on a une filtration canonique*

$$0 \subset F_0 \subset F_1 \subset \ldots \subset F_{[\frac{n}{2}]} = (E_1 \otimes E_2)^{(n)}$$

par $\bar{\mathbb{Q}}_\ell$-faisceaux constructibles tels que

$$F_m / F_{m-1} \widetilde{\to} (\sigma_m)_* (E_1^{(n-2m)} \otimes E_2^{(n-2m)}) \boxtimes ((\det E_1)^{(m)} \otimes (\det E_2)^{(m)})$$

pour $0 \le m \le \frac{n}{2}$ $(F_{-1} = 0)$.

Remarque . La variante non-géométrisée de cette assertion s'ecrit: si E_1, E_2 sont deux espaces vectoriels de dimension 2, alors on a

$$\mathrm{Sym}^n(E_1 \otimes E_2) = \bigoplus_{m \ge 0} (\det E_1 \otimes \det E_2)^{\otimes m} \otimes \mathrm{Sym}^{n-2m} E_1 \otimes \mathrm{Sym}^{n-2m} E_2.$$

Voici la démonstration de 0.5.1. Étant donné un triplet $(\Omega \hookrightarrow L, \alpha)$ comme ci-dessus, on note D le diviseur effectif sur X tel que $\Omega(D) \hookrightarrow L$ est un sous-fibré vectoriel, $m = \deg D$. On a une suite exacte $0 \to \Omega(D) \to L \to \mathcal{A}(-D) \to 0$. Notons $u \in \mathrm{Ext}^1(\mathcal{A}(-D), \Omega(D))$ l'element correspondant. En plus, on a $\mathrm{Aut}(\Omega \hookrightarrow L, \alpha) = \mathrm{Hom}(\mathcal{A}(-D), \Omega(D))$.

Par suite, la première somme s'ecrit

$$\sum_{m\geq 0}\sum_{D\in X^{(m)}(\mathbb{F}_q)}\sum_{u\in \mathrm{Ext}^1(\mathcal{A}(-D),\Omega(D))} q^{-\dim \mathrm{Hom}(\mathcal{A}(-D),\Omega(D))}\varphi_{E_1}(L)\varphi_{E_2}(L) =$$

$$\sum_{m\geq 0}\sum_{D\in X^{(m)}(\mathbb{F}_q)} q^{-\dim \mathrm{Hom}(\mathcal{A}(-D),\Omega(D))}\,\mathrm{tr}(\mathrm{Fr}_D,(\det E_1)^{(m)}\otimes(\det E_2)^{(m)})\cdot$$

$$\sum_{v\in \mathrm{Ext}^1(\mathcal{A}(-2D),\Omega)} \varphi_{E_1}(L_v)\varphi_{E_2}(L_v),$$

où la suite exacte $0\to\Omega\to L_v\to\mathcal{A}(-2D)\to 0$ correspond à v.

En utilisant la propriété 3) de φ_E, on obtient finalement

$$(q-1)q^{4-4g}\sum_{m\geq 0}\sum_{D\in X^{(m)}(\mathbb{F}_q)}\mathrm{tr}(\mathrm{Fr}_D,(\det E_1)^{(m)}\otimes(\det E_2)^{(m)})$$

$$\sum_{D'\in|\mathcal{A}(-2D)|}\mathrm{tr}(\mathrm{Fr}_{D'},E_1^{(n-2m)}\otimes E_2^{(n-2m)}),$$

où on a noté $|\mathcal{A}(-2D)|$ l'ensemble des diviseurs effectifs de la classe $\mathcal{A}(-2D)$. L'application de l'énoncé 0.5.2. achève la démonstration de 0.5.1.

Comme corollaire immédiat de 0.5.1 on obtient

0.5.3. *Soient E_1, E_2 deux $\bar{\mathbb{Q}}_\ell$-faisceaux lisses et géométriquement irréductible de rang 2 sur X. Pour tout $n\geq 0$ on a*

$$\sum_{\Omega\hookrightarrow L\,\deg L=n+2g-2}\frac{1}{\#\,\mathrm{Aut}(\Omega\hookrightarrow L)}\varphi_{E_1}(L)\varphi_{E_2}(L) = q^{4-4g}\sum_{D\in X^{(n)}(\mathbb{F}_q)}\mathrm{tr}(\mathrm{Fr}_D,(E_1\otimes E_2)^{(n)}),$$

où la première somme est prise sur l'ensemble de classes d'isomorphie des couples $(\Omega\hookrightarrow L)$, où $\Omega\hookrightarrow L$ est une inclusion de Ω dans un fibré vectoriel de rang 2 et de degré $n+2g-2$ sur X (on ne suppose pas que Ω est localement facteur direct de L).

Rappelons que la fonction L attachée au système local $E_1\otimes E_2$ sur X est définie par la série formelle

$$L(E_1\otimes E_2,t)=\sum_{n\geq 0}\sum_{D\in X^{(n)}(\mathbb{F}_q)}\mathrm{tr}(\mathrm{Fr}_D,(E_1\otimes E_2)^{(n)})t^n$$

dans $\bar{\mathbb{Q}}_\ell[[t]]$ et, d'après la formule des traces de Grothendieck (cf. [SGA5] (exp.15, par.3, N.2) et [SGA $4\frac{1}{2}$][Rapport](4.10)), on a

$$L(E_1\otimes E_2,t)=\prod_{r=0}^{2}\det(1-\mathrm{Fr}\ t,\mathrm{H}^r(\bar{X},E_1\otimes E_2))^{(-1)^{r+1}},$$

où Fr est le Frobenius géométrique, $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$.

Il resulte de 0.5.3 qu'on a l'égalité entre séries formelles

$$\sum_{n\geq 0} \sum_{L\in \mathrm{Bun}_2^{n+2g-2}(\mathbb{F}_q)} \frac{1}{\# \mathrm{Aut}\, L}(q^{\dim \mathrm{Hom}(\Omega, L)} - 1)\varphi_{E_1}(L)\varphi_{E_2}(L)t^n = q^{4-4g} L(E_1\otimes E_2, t)$$

En utilisant la cuspidalité de φ_E, on verifie que si $\varphi_E(L) \neq 0$ et $\deg L$ est assez grand, alors $\mathrm{Ext}^1(\Omega, L) = 0$ et $\dim \mathrm{Hom}(\Omega, L) = \deg L + 6 - 6g$. D'autre part, si $\det E_1^* \widetilde{\to} \det E_2$ alors la somme

$$\sum_{L\in \mathrm{Bun}_2^{n+2g-2}(\mathbb{F}_q)} \frac{1}{\# \mathrm{Aut}\, L}\varphi_{E_1}(L)\varphi_{E_2}(L)$$

ne depend que de $n \bmod 2$. Pour conclure, il reste à comparer le comportement asymptotique des séries ci-dessus lorsque t tend vers q^{-1}, en utilisant l'interprétation cohomologique de la fonction $L(E_1\otimes E_2, t)$.

Les aspects nouveaux au niveau géométrique

0.6. On rappelle en 1.1 la structure de la déformation universelle E d'un système local ℓ-adique E_0 sur la courbe X (sous la condition $\mathrm{End}(E_0) = \bar{\mathbb{Q}}_\ell$). La base R de cette déformation est un anneau des séries formelles sur $\bar{\mathbb{Q}}_\ell$ de dimension $2 + (2g-2)m^2$, où m est le rang de E_0. En plus, on calcule la cohomologie de $\mathcal{H}om(E_1, E_2)$ sur X, où E_1 et E_2 sont deux relevement de E au spectre formel de $R \hat{\otimes} R$ (cf. Prop.2).

On introduit en 1.2.1 le champ $\overline{\mathrm{Bun}}_2^n$ qui est le quotient de Bun_2^n par l'action de $\mathbb{G}_m$ (precisons que $\mathbb{G}_m$ adit sur Bun_2^n par 2-automorphismes de champs. Cette action est donnée par les automorphismes scalaires des fibrés vectoriels). Le morphisme naturel $\bar{\pi} : \mathrm{Bun}_2^n \to \overline{\mathrm{Bun}}_2^n$ est une $\mathbb{G}_m$-gerbe.

En 1.4 nous rappelons la construction due à D.Gaitsgory du faisceau pervers Aut_{E_0} pour le système local irréductible E_0 de rang 2 sur X. On montre que cette construction reste valable pour les déformations de E_0, ce qui permet à définir le R-faisceau pervers Aut_E sur Bun_2. En fait, en adaptant les arguments originaux de [2], on construit d'abord un R-faisceau pervers $\overline{\mathrm{Aut}}_E$ sur $\overline{\mathrm{Bun}}_2$ et on pose $\mathrm{Aut}_E = \bar{\pi}^* \overline{\mathrm{Aut}}_E[-1]$. En plus, on donne une nouvelle démonstration pour la deuxième propriètè de Hecke de $\overline{\mathrm{Aut}}_E$.

La partie gauche de la formule 0.2 s'interpréte géométriquement comme

$$\mathrm{R}\Gamma_c(\mathrm{Bun}_2^n, \mathrm{Aut}_{E_1^*}^n \otimes \mathrm{Aut}_{E_2}^n) \tag{2}$$

Notons $B(\mathbb{G}_m)$ le champ classifiant du group multiplicatif. D'une part, on observe par la formule de projection que les cohomologies $\mathrm{R}\Gamma_c(B(\mathbb{G}_m), \bar{\mathbb{Q}}_\ell)$ interviennent dans (2). D'autre part, ils interviennent dans la partie droite de 0.2 comme une géométrisation du dénominateur $\frac{1}{q-1}$ (cf. Appendice A, éxample 1). Puisqu'on veut une réponse comme en 0.1, on se propose de calculer

$$\mathrm{R}\Gamma_c(\overline{\mathrm{Bun}}_2^n, \overline{\mathrm{Aut}}_{E_1^*}^n \otimes \overline{\mathrm{Aut}}_{E_2}^n)$$

au lieu de (2).

Pour ça, on donne tout d'abord une interprétation géométrique de l'énoncé 0.5.1, à savoir Théorème 2, sect. 2.2.1. C'est un résultat local en quelque sorte qui n'est pas relié à l'irréducibilité des systèmes locaux évoqués. La difficulté principale est que la démonstration de 0.5.1 donné ci-dessus ne se géométrise pas directement. On obtient Théorème 2 autrement, comme une conséquence formelle du Théorème Principal Local qui, à connaissance de l'auteur, n'apparait pas explicitement dans la méthode classique de Rankin-Selberg pour GL(2).

Le but de la sect. 2 est de déduir le Théorème Principal Global du Théorème 2 en adaptant les arguments de 0.5 à la situation géométrique. On exploite le fait que au-dessus du support de Aut_E^n le morphisme naturel $\overline{\mathrm{Flag}}_n \to \mathrm{Bun}_2^n$ est universellement $a(n)$-acyclic, où $a(n)$ est une fonction de n telle que $a(n) \to \infty$ lorsque $n \to \infty$. On rancontre aussi une difficulté technique reliée au fait que l'image inverse de Aut_E^n à $\overline{\mathrm{Flag}}_n$ est donnée par la formule explicite (qui géométrise la propriété 3) de φ_E) seulement sur un ouvert "assez grand" de $\overline{\mathrm{Flag}}_n$ (cf. Remarque 8, sect. 1.4.1).

La sect. 3 est consacrée à la démonstration du Théorème Principal Local. L'idée de cette démonstration, qui est absente au niveau de fonctions, est *d'étudier les propriétés locales (pour la topologie étale de la base) de l'image directe en question.* On observe, en particulier, que la restriction de cette image directe à l'henselisé (strict) d'un point arbitraire de la base ne depend pas des systèmes locaux E_1, E_2.

En sect. 4, qui est essentiellement independante du reste de la thèse, on introduit les faisceaux $\mathcal{P}_E^n$ sur le champ $X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2}$. Le but est de démontrer qu'ils sont pervers. En plus, on calcule tous les faisceaux de cohomologies (pour la t-structure habituelle) de $\mathcal{P}_E^n$. Comme une application, on démontre la deuxième propriété de Hecke de $\overline{\mathrm{Aut}}_E$.

Dans l'appendice D on propose une conjecture qui renforce un résultat de Laumon (Théorème (3.3.8),[3]). On construit aussi un analog du $\bar{\mathbb{Q}}_\ell$-faisceau constructible correspondant pour un groupe reductif arbitraire.

0.1 Main results

0.1.1

Let X be a smooth complete connected curve of genus $g \geq 1$ (defined over k, cf. sect. 0.2.1). Let E_0 be an irreducible 2-dimensional ℓ-adic local system on X. Let E be the universal deformation of E_0 and R be the base of this universal deformation (cf. sect. 1.1.1). In fact, R is a ring of formal power series over $\bar{\mathbb{Q}}_\ell$ of dimension $8g-6$ (cf. Prop. 1). Denote by E_i ($i = 1, 2$) the two liftings of E to the formal spectrum of $R \hat{\otimes} R$.

D.Gaitsgory in [2] has associated to E_0 an irreducible perverse sheaf Aut_{E_0} on the moduli stack Bun_2 of 2-bundles on X, which is a Hecke eigen-sheaf with respect to E_0. Denote by $\mathrm{Aut}^n_{E_0}$ its restriction to the connected component Bun^n_2 of Bun_2 that corresponds to bundles of degree n. Gaitsgory's construction is still valid for deformations of E_0, so that we have an R-perverse sheaf Aut^n_E on Bun^n_2 (cf. sect. 1.4.1). (The definition of an R-perverse sheaf is given in sect. 0.2.1). Scalar automorphisms of 2-bundles provide an action of $\mathbb{G}_m$ on Bun^n_2 by 2-automorphisms of stacks. Denote by $\overline{\mathrm{Bun}}^n_2$ the quotient of Bun^n_2 under this action, so that the natural morphism $\mathrm{Bun}^n_2 \to \overline{\mathrm{Bun}}^n_2$ is a $\mathbb{G}_m$-gerb (cf. sect. 1.2.1). Since Aut^n_E is perverse, it is a pull-back of some sheaf $\overline{\mathrm{Aut}}^n_E[-1]$ on $\overline{\mathrm{Bun}}^n_2$, where $\overline{\mathrm{Aut}}^n_E$ is a perverse R-sheaf.

The main result of this work is the following

Main Global Theorem . *For any integer n there is a canonical isomorphism of $R \hat{\otimes} R$-modules*

$$\mathrm{H}^0_c(\overline{\mathrm{Bun}}^n_2, \overline{\mathrm{Aut}}^n_{E_1^*} \otimes_{R \hat{\otimes} R} \overline{\mathrm{Aut}}^n_{E_2}) \widetilde{\to} R(-1),$$

where the $R \hat{\otimes} R$-module structure on R is given via the diagonal mapping $R \hat{\otimes} R \to R$. Besides, we have $\mathrm{H}^i_c(\overline{\mathrm{Bun}}^n_2, \overline{\mathrm{Aut}}^n_{E_1^*} \otimes_{R \hat{\otimes} R} \overline{\mathrm{Aut}}^n_{E_2}) = 0$ for $i \neq 0$.

Remark 1. i) The stack $\overline{\mathrm{Bun}}^n_2$ is not of finite type. However, $\overline{\mathrm{Aut}}^n_E$ is the extension by zero of its restriction to an open substack of $\overline{\mathrm{Bun}}^n_2$ which is of finite type (cf. Corolary 1).
ii) The cohomology with compact support of a stack a pripori was not defined. We will use a definition from the forthcoming book of G.Laumon and L.Moret-Bailly ([7],(18.8)) communicated to the author by G.Laumon (cf. Appendix A). Their definition is inspired by an idea of J.Bernstein and V.Lunts ([8]). An analogous definition was also communicated to the author by V.Drinfeld.

0.1.2

We derive Main Global Theorem from some local result, which can be considered as a geometrization of the classical Rankin-Selberg method for GL(2) (cf.[9], [10]). To formulate it we need some notation.

Denote by Sh_i^n the stack that classifies coherent sheaves on X of generic rank i and of degree n (cf. our conventions in sect. 0.2.3, cf. also sect. 1.2.2). Let ${}^0\mathrm{Sh}_1'^n$ be the stack that classifies pairs $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$ on X, where $F_1 \in \mathrm{Sh}_1^n$ (cf. sect. 1.2.2). We have a representable map ${}^0\pi_1 : {}^0\mathrm{Sh}_1'^n \to \mathrm{Sh}_1^n$ that sends $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$ to F_1. We also have a map $\pi_0' : {}^0\mathrm{Sh}_1'^n \to \mathrm{Sh}_0^n$ that sends $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$ to $F_1/\operatorname{Im} s$. Notice that π_0' is a vector bundle of rank n. Denote by $\overline{\mathrm{Flag}}_n$ the stack that classifies exact sequences

$$0 \to \Omega \to M \to F_1 \to 0,$$

where M is a bundle of rank 2 and degree n on X, and Ω is the canonical invertible sheaf on X (so that F_1 is a coherent sheaf on X of generic rank 1).

Let $\mathcal{E}xt$ be the stack classifying the exact sequences $0 \to \Omega \to ? \to \mathcal{O} \to 0$ on X. Then $\mathcal{E}xt \to \operatorname{Spec} k$ is a generalized (non-representable) vector fibration of rank $1-g$ (the definition of a generalized vector fibration is given in sect. 0.2.2). We have a natural morphism $\mathcal{E}xt \to \mathbb{A}_k^1$.

Consider the stack ${}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2}$, where the morphism $\overline{\mathrm{Flag}}_{n+2g-2} \to \mathrm{Sh}_1^n$ sends a sequence (1) to F_1. We have a representable map ${}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to \mathcal{E}xt$ that sends a collection $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1, 0 \to \Omega \to M \to F_1 \to 0)$ to the pull-back of (1) w.r.t. $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$. Define the pairing $\mu : {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to \mathbb{A}_k^1$ as the composition ${}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to \mathcal{E}xt \to \mathbb{A}_k^1$. Denote by $\mathcal{L}_\psi$ the Artin-Schreier sheaf on $\mathbb{A}_k^1$ (cf. sect. 0.2.3).

For a positive integer n we denote by $X^{(n)}$ the n-th symmetric power of X. Let $\operatorname{Pic}^n X$ denote the Picard stack that classifies invertible sheaves on X of degree n. We have a morphism $\pi : X^{(n)} \to \operatorname{Pic}^n X$ that sends a divisor $D \in X^{(n)}$ to the invertible sheaf $\mathcal{O}(D)$.

Let $\det : \mathrm{Sh}_1^n \to \operatorname{Pic}^n X$ be the natural map that sends F_1 to $\det F_1$ (cf. [12]). We have a map ${}^0\mathrm{Sh}_1'^n \to X^{(n)}$ that sends $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$ to $(\mathcal{O} \stackrel{\det s}{\hookrightarrow} \det F_1) \in X^{(n)}$. (Here $X^{(n)}$ is considered as the moduli scheme of pairs $(\mathcal{O} \hookrightarrow \mathcal{A}, \mathcal{A} \in \operatorname{Pic}^n X)$). Denote by

$$\varphi_n : {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$$

the morphism that sends a collection $(\mathcal{O} \stackrel{s_1}{\hookrightarrow} F_1, \mathcal{O} \stackrel{s_2}{\hookrightarrow} F_1, 0 \to \Omega \to M \to F_1 \to 0)$ to $(\mathcal{O} \stackrel{\det s_1}{\hookrightarrow} \det F_1, \mathcal{O} \stackrel{\det s_2}{\hookrightarrow} \det F_1)$. (Here we consider $X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$ as the moduli scheme of triples $(s', s'' : \mathcal{O} \hookrightarrow \mathcal{A}, \mathcal{A} \in \operatorname{Pic}^n X))$. Let $i : X^{(n)} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$ be the closed immersion that sends $(\mathcal{O} \stackrel{s}{\hookrightarrow} \mathcal{A})$ to $(\mathcal{O} \stackrel{s}{\hookrightarrow} \mathcal{A}, \mathcal{O} \stackrel{-s}{\hookrightarrow} \mathcal{A})$.

Let A be a local Artin $\bar{\mathbb{Q}}_\ell$-algebra with residue field $\bar{\mathbb{Q}}_\ell$. For a smooth A-sheaf E on X (cf. sect. 0.2.1) we denote by $\mathcal{L}_E^n$ the Laumon's perverse A-sheaf on Sh_0^n (cf. sect. 1.3.1).

Main Local Theorem . *Let E_1, E_2 be any smooth A-sheaves on X of rank 2. The complex*

$$(\varphi_n)_!(\operatorname{pr}_1^* \pi_0'^* \mathcal{L}_{E_1}^n \otimes_A \operatorname{pr}_2^* \pi_0'^* \mathcal{L}_{E_2}^n \otimes_{\bar{\mathbb{Q}}_\ell} \operatorname{pr}_{13}^* \mu^* \mathcal{L}_\psi \otimes_{\bar{\mathbb{Q}}_\ell} \operatorname{pr}_{23}^* \mu^* \mathcal{L}_\psi)[3n + 2 - 2g]$$

is a perverse A-sheaf on $X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$. It is supported at the closed subscheme $i : X^{(n)} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$ and is canonically isomorphic to

$$i_*(E_1 \otimes_A E_2)^{(n)}(-n - 1 + g)[n]$$

Remark 2. i) The morphism φ_n is easily written as the composition of representable morphisms and generalized vector fibrations, for example as follows

$$\begin{aligned} {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} \overline{\operatorname{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} ({}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} \overline{\operatorname{Flag}}_{n+2g-2}) \to \\ X^{(n)} \times_{\operatorname{Pic}^n X} (X^{(n)} \times \mathcal{E}xt) \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)} \end{aligned}$$

So, $(\varphi_n)_!$ can also be defined as the composition of the corresponding functors.
ii) We will see that the assertion of Main Local Theorem is local w.r.t. E_1, E_2. For this reason one may hope that the same result is also true for local systems with torsion coefficients.

0.2 Conventions

0.2.1

We fix an algebraically closed ground field k of positive characteristic. In this paper all the schemes and stacks will be defineed over k. We also fix a prime number ℓ different from $\operatorname{char} k$ and an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Let $E_\lambda \subset \bar{\mathbb{Q}}_\ell$ be either a finite extension field of $\mathbb{Q}_\ell$ or $\bar{\mathbb{Q}}_\ell$. By a E_λ-sheaf we will always mean a constructible E_λ-sheaf.

We will be working with algebraic stacks in the smooth topology and with (perverse) E_λ-sheaves on them (cf. [6]).

Let $\mathcal{X}$ be an algebraic stack locally of finite type. The notion of a (perverse) E_λ-sheaf on $\mathcal{X}$ localizes in the smooth topology: for a scheme S locally of finite type and a smooth surjective morphism $S \to \mathcal{X}$ the category of (perverse) E_λ-sheaves on $\mathcal{X}$ is equivalent to the corresponding category of sheaves on S equipped with the descent datas.

If A is a local Artin E_λ-algebra with residue field E_λ then the category of (perverse) A-sheaves on $\mathcal{X}$ is the category of pairs (ρ, F), where F is a (perverse) E_λ-sheaf on $\mathcal{X}$ and $\rho : A \to \mathrm{End}(F)$ is an action of A on F. We say that a (perverse) A-sheaf F on $\mathcal{X}$ is A-flat if the functor $N \mapsto F \otimes_A N$ from the category of finite type A-modules to the category of (perverse) A-sheaves is exact. By a smooth A-sheaf we mean an A-sheaf (ρ, F) such that F is a smooth E_λ-sheaf and all the geometric fibres of (ρ, F) are free A-modules of finite type.

If R is a complete local noetherian E_λ-algebra with residue field E_λ and maximal ideal $\mathfrak{m}$ then we denote by $\mathcal{A}(\mathcal{X}, R)$ (resp., by $\mathcal{PA}(\mathcal{X}, R)$) the projective 2-limit of the categories of $R/\mathfrak{m}^n$-sheaves (resp., of perverse $R/\mathfrak{m}^n$-sheaves) on $\mathcal{X}$ (for the general notion of the categorical projective 2-limit cf. SGA4, éxposé 6 (6.10). In our case this notion coincides with that of sect. 1.1.3). In other words, an object of $\mathcal{A}(\mathcal{X}, R)$ (resp., of $\mathcal{PA}(\mathcal{X}, R)$) is a projective system $(F_n, \psi_n)_{n \in \mathbb{N}}$, where F_n is a $R/\mathfrak{m}^n$-sheaf (resp., a perverse $R/\mathfrak{m}^n$-sheaf) and $\psi_n : F_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \widetilde{\to} F_n$ is an isomorphism. Morphisms in $\mathcal{A}(\mathcal{X}, R)$ (resp., in $\mathcal{PA}(\mathcal{X}, R)$) are morphisms of the corresponding projective systems.

The formalism we need is not put into shape yet, and we will satisfy ourselves with the following definitions. If E_λ is a finite extension of $\mathbb{Q}_\ell$ then the category of R-sheaves (resp., of perverse R-sheaves) on $\mathcal{X}$ is, by definition, the category $\mathcal{A}(\mathcal{X}, R)$ (resp., $\mathcal{PA}(\mathcal{X}, R)$). If $E_\lambda = \bar{\mathbb{Q}}_\ell$ then we define the category of R-sheaves (resp., of perverse R-sheaves) on $\mathcal{X}$ as the following full subcategory of $\mathcal{A}(\mathcal{X}, R)$ (resp., of $\mathcal{PA}(\mathcal{X}, R)$). An object F of $\mathcal{A}(\mathcal{X}, R)$ (resp., of $\mathcal{PA}(\mathcal{X}, R)$) is an R-sheaf (resp., a perverse R-sheaf) on $\mathcal{X}$ if there exists a finite extension field E_μ of $\mathbb{Q}_\ell$, a complete local noetherian E_μ-algebra R' with residue field E_μ and an isomorphism $R' \hat{\otimes}_{E_\mu} \bar{\mathbb{Q}}_\ell \widetilde{\to} R$ such that F lies in the essensial image of the functor $\mathcal{A}(\mathcal{X}, R') \to \mathcal{A}(\mathcal{X}, R)$ (resp., of the functor $\mathcal{PA}(\mathcal{X}, R') \to \mathcal{PA}(\mathcal{X}, R)$) that sends M to $M \otimes_{E_\mu} \bar{\mathbb{Q}}_\ell$.

Remark 3. To use the geometric intuition, an R-sheaf on $\mathcal{X}$ can be thought of as a sheaf on $\mathcal{X} \times \mathrm{Spf}(R)$, i.e., a family of ℓ-adic sheaves on $\mathcal{X}$ parametrized by R.

0.2.2

Throughout the paper the phrase "complex on a stack" will mean an object of a suitable derived category on this stack (in general, it is neither a sheaf nor a perverse sheaf). However, the existence of such a derived category will never be used, any complex will appear only through its cohomologies (in the perverse or usual sense).

Following [2], we say that a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ of algebraic stacks is a generalized vector (resp., affine) fibration if there exists a homomorphism of smooth unipotent group schemes $\mathcal{A} \to \mathcal{A}'$ over $\mathcal{X}_2$ (resp., a homomorphism of smooth unipotent group schemes $\mathcal{A} \to \mathcal{A}'$ over $\mathcal{X}_2$ and an $\mathcal{A}'$-torsor $\mathcal{X}'_1$ over $\mathcal{X}_2$) such that $\mathcal{X}_1 \widetilde{\to} \mathcal{A}'/\mathcal{A}$ (resp., $\mathcal{X}_1 \widetilde{\to} \mathcal{X}'_1/\mathcal{A}$). If, in addition, $\mathcal{A}$ and $\mathcal{A}'$ can be chosen smooth over $\mathcal{X}_2$ of constant relative dimensions n and n' respectively then we say that f is a generalized (vector or affine) fibration of rank $n' - n$. Notice that the stack $\mathcal{A}'/\mathcal{A}$ depends in some sense only on the quasi-isomorphism class of the complex $\mathcal{A} \to \mathcal{A}'$ (cf. SGA4, t.3, éxp.18, 1.4.10).

For a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ of algebraic stacks we will use the functor f^* between the derived categories. When f is representable (resp., a generalized (vector or affine) fibration) we will use the functors $f_!$ and f_* (resp., the functor $f_!$). However, we will also need the functor $f_!$ for more general non-representable morphisms. For them we use a definition of Appendix A.

Remark 4. A definition of the derived category of smooth-étale sheaves on a stack is given in the forthcoming book [7] of G.Laumon and L.Moret-Bailly. They introduce a notion of a Bernstein-Lunts stack (cf.also Appendix A) and partially establish a formalism of 'six operations' à la Grothendieck for Bernstein-Lunts stacks of finite type and morphisms between them.

0.2.3

For example, when we write: "consider the stack that classifies pairs $M_1 \hookrightarrow M_2$ with M_1 (resp. M_2) being a coherent sheaf on X of generic rank i_1 and of degree d_1 (resp., of gen. rk i_2 and of degree d_2)", the reader should keep in mind that what we mean is the following k-stack. Its category fibre over a scheme S is the groupoïd whose objects are inclusions $M_1 \hookrightarrow M_2$ of coherent sheaves on $S \times X$ that are S-flat and such that the quotient M_2/M_1 is also S-flat, and for any point $s \in S$ the conditions on the generic rank and on the degree of $M_i \mid_{s \times X}$ $(i = 1, 2)$ hold. Morphisms from an object $M_1 \hookrightarrow M_2$ to an object $M'_1 \hookrightarrow M'_2$ are by definition the isomorphisms $M_1 \widetilde{\to} M'_1$ and $M_2 \widetilde{\to} M'_2$ making the natural diagram commutative.

By a local system we mean a smooth $\bar{\mathbb{Q}}_\ell$-sheaf. We fix a finite subfield $\mathbb{F}_q \subset k$ of q elements. We also fix a square root of q in $\bar{\mathbb{Q}}_\ell$ and define using it the sheaf $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$ over $\operatorname{Spec} \mathbb{F}_q$ and, hence, over $\operatorname{Spec} k$. We denote by Ω the canonical invertible sheaf on X.

We fix a nontrivial additive character $\psi : \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell^*$ and denote by $\mathcal{L}_\psi$ the Artin-Schreier sheaf on $\mathbb{A}^1_k$ associated to ψ (cf. SGA4$\frac{1}{2}$, [Sommes trig.], 1.7). If x is the coordinate on $\mathbb{A}^1_k$ then the equation $t^q - t = x$ defines an $\mathbb{F}_q$-torsor over $\mathbb{A}^1_k$, and $\mathcal{L}_\psi$ is a smooth $\bar{\mathbb{Q}}_\ell$-sheaf of rank 1 on $\mathbb{A}^1_k$ obtained from this torsor by extension of the structure group via ψ^{-1}. The Fourier transform functor is normalized to preserve the perversity and weights. (cf. the explicit formulae in sect. 1.4.1).

1 Reminders and complements

1.1 The universal deformation of a local system and cohomologies of $\mathcal{H}om(E_1, E_2)$.

1.1.1

Let E_λ be either a finite extension field of $\mathbb{Q}_\ell$ or $\bar{\mathbb{Q}}_\ell$. Fix a smooth E_λ-sheaf E_0 on X of rank m. In this subsection we recall the structure of the universal deformation of E_0. This construction is standard (cf.[1] for the definition of pro-representability, etc.).

Let $\eta \in X$ be the generic point of X and $\bar{\eta} \to \eta \to X$ be a geometric point over η. Set $G = \pi_1(X, \bar{\eta})$. Denote by $\mathcal{C}_{E_\lambda}$ the category of local Artin E_λ-algebras with residue field E_λ. ($\mathcal{C}_{\bar{\mathbb{Q}}_\ell}$ will also be denoted symply by $\mathcal{C}$). Recall that for $A \in \mathrm{Ob}(\mathcal{C}_{E_\lambda})$ the functor that sends E to $E_{\bar{\eta}}$ is an equivalence of the category of smooth A-sheaves on X of rank m with the category of pairs (E, ρ), where E is a free A-module of rank m and $\rho : G \to \mathrm{Aut}_A E$ is a representation (it is required that, as a representation over E_λ, ρ is already defined over a finite extension field of $\mathbb{Q}_\ell$ and is continuous in the ℓ-adic topology).

Definition 1. Let $A \in \mathrm{Ob}(\mathcal{C}_{E_\lambda})$. An *$A$-deformation of* E_0 is a pair (E, ψ), where E is a smooth A-sheaf on X of rank m and $\psi : E \otimes_A E_\lambda \tilde{\to} E_0$ is an isomorphism of E_λ-sheaves on X.

Define the functor $F_{E_0} : \mathcal{C}_{E_\lambda} \to \mathrm{Sets}$ by $F_{E_0}(A) =$ the set of isomorphism classes of A-deformations of E_0.

Proposition 1. *Suppose that* $\mathrm{End}(E_0) = E_\lambda$ *then* F_{E_0} *is pro-representable by a pro-pair* (R, E), *where* R *is a ring of formal power series over* E_λ *of dimension* $2 + (2g - 2)m^2$. *If* $\mathfrak{m} \subset R$ *is the maximal ideal of* R *then* $\mathrm{Hom}_{E_\lambda}(\mathfrak{m}/\mathfrak{m}^2, E_\lambda) \tilde{\to} \mathrm{H}^1(X, \mathcal{E}nd E_0)$ *canonically. If, in addition,* E_λ *is finite over* $\mathbb{Q}_\ell$ *then the pro-pair* $(R \hat{\otimes}_{E_\lambda} \bar{\mathbb{Q}}_\ell, E \otimes_{E_\lambda} \bar{\mathbb{Q}}_\ell)$ *pro-represents the functor* $F_{E_0 \otimes_{E_\lambda} \bar{\mathbb{Q}}_\ell}$.

Lemma 1. *Suppose that* $\mathrm{End}(E_0) = E_\lambda$
1) If E *is an* A*-deformation of* E_0 *then* $\mathrm{End}(E) = A$.
2) Let $A' \to A, A'' \to A$ *be two morphisms in* $\mathcal{C}_{E_\lambda}$. *Suppose that* $A'' \to A$ *is surjective then the natural morphism* $F_{E_0}(A' \times_A A'') \to F_{E_0}(A') \times_{F_{E_0}(A)} F_{E_0}(A'')$ *is a bijection.*

Proof 2) The surjectivity is easy. To prove the injectivity use point 1) and Corollary 3.6,p.217 of [1]. □

Lemma 2. *Let $A' \to A$ be a surjective morphism in $\mathcal{C}_{E_\lambda}$, whose kernel is a 1-dimensional E_λ-vector space. Let V' be a free A'-module of rank m. Put $V = V' \otimes_{A'} A$. Then the natural morphism*

$$\mathrm{GL}(V') \to \mathrm{PGL}(V') \times_{\mathrm{PGL}(V)} \mathrm{GL}(V) \times_{\mathrm{GL}(\det V)} \mathrm{GL}(\det V')$$

is an isomorphism of groups. □

Proof of Proposition 1

Consider the ring $E_\lambda[\varepsilon]/(\varepsilon^2)$ of dual numbers. The groupoid of $E_\lambda[\varepsilon]/(\varepsilon^2)$-deformations of E_0 is naturally equivalent to the category of extensions $0 \to E_0 \to ? \to E_0 \to 0$ on X. It follows that the tangent space to F_{E_0} is identified with $\mathrm{Ext}^1_X(E_0, E_0) = \mathrm{H}^1(X, \mathcal{E}nd E_0)$. Now combining Lemma 1 with Theorem 2.11,p.212 of [1] we get the pro-representability of F_{E_0} by a pro-pair (R, E).

Let us show that the morphism of functors associating to an A-deformation E of E_0 the A-deformation $\det E$ of $\det E_0$ is a formally smooth morphism from the universal deformation of E_0 to the universal deformation of $\det E_0$. Let $A' \to A$, V' and V be as in Lemma 2. Suppose that V is equipped with a structure of A-deformation of E_0. Let $\rho : G \to \mathrm{Aut}_A V$ be the corresponding representation of G. Since $\mathrm{H}^0(X, \mathcal{E}nd_0 E_0) = 0$, we get $\mathrm{H}^2(X, \mathcal{E}nd_0 E_0) = 0$ by Poincaré duality. It follows that the corresponding representation of G in $\mathrm{PGL}(V)$ can be lifted to a representation $\rho' : G \to \mathrm{PGL}(V')$. Now our assertion follows from Lemma 2.

Notice that the universal deformation of $\det E_0$ is formally smooth, because it is isomorphic to the universal deformation of the trivial 1-dimensional local system, which is an infinitesimal formal E_λ-group (cf. SGA3,t.1, éxposé 7_B, 3.3).

It follows that R is formally smooth (i.e., by Prop. 2.5(i) of [1], is a ring of formal power series over E_λ).

Since $\chi(\mathcal{E}nd_0 E_0) = (2-2g)(m^2-1)$, we have $\dim \mathrm{H}^1(X, \mathcal{E}nd E_0) = (2g-2)m^2+2$.

If E_λ is finite over $\mathbb{Q}_\ell$ then $(R \hat{\otimes} \bar{\mathbb{Q}}_\ell, E \otimes \bar{\mathbb{Q}}_\ell)$ is a pro-pair for the functor $F_{E_0 \otimes \bar{\mathbb{Q}}_\ell} : \mathcal{C} \to \mathrm{Sets}$, so that it defines a morphism of functors $h_{R \hat{\otimes} \bar{\mathbb{Q}}_\ell} \to F_{E_0 \otimes \bar{\mathbb{Q}}_\ell}$ and we have to show that this is an isomorphism. (Here $h_{R \hat{\otimes} \bar{\mathbb{Q}}_\ell} : \mathcal{C} \to \mathrm{Sets}$ is the functor represented by $R \hat{\otimes} \bar{\mathbb{Q}}_\ell$, in other words, $h_{R \hat{\otimes} \bar{\mathbb{Q}}_\ell}(A) = \mathrm{Hom}_{local\ \bar{\mathbb{Q}}_\ell - alg}(R \hat{\otimes} \bar{\mathbb{Q}}_\ell, A)$). Since we already know that $F_{E_0 \otimes \bar{\mathbb{Q}}_\ell}$ can be represented by a ring of formal power series over $\bar{\mathbb{Q}}_\ell$, our assertion follows from the fact that the induced map on the tangent spaces is an isomorphism. □

By definition, $E = (E_n)_{n \in \mathbb{N}}$, $E_n \in F_{E_0}(R/\mathfrak{m}^n)$ are such that the image of E_{n+1} under $F_{E_0}(R/\mathfrak{m}^{n+1}) \to F_{E_0}(R/\mathfrak{m}^n)$ is E_n. Fix a $R/\mathfrak{m}^n$-deformation of E_0 in the isomorphism class E_n and denote it by the same symbol E_n. For any n fix an isomorphism of $R/\mathfrak{m}^n$-deformations of E_0: $E_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \widetilde{\to} E_n$. Then the projective system

$(E_n)_{n\in\mathbb{N}}$ is an object E of $\mathcal{A}(X,R)$ equipped with an isomorphism $E\otimes_R E_\lambda \widetilde{\to} E_0$ of E_λ-sheaves on X. From Proposition 1 it also follows that if $E_\lambda=\bar{\mathbb{Q}}_\ell$ then $E\in\mathcal{A}(X,R)$ is, in fact, an R-sheaf.

Notice that R is defined up to a canonical isomorphism, and the R-sheaf E is defined up to a non-canonical isomorphism.

Definition 2. For a local system E_0 (i.e., for $E_\lambda=\bar{\mathbb{Q}}_\ell$) the R-sheaf E will be reffered to as *the universal deformation of* E_0, and the ring R will be reffered to as *the base of the universal deformation of* E_0.

1.1.2

In the rest of sect. 1.1 we set $E_\lambda=\bar{\mathbb{Q}}_\ell$. Put $R\hat{\otimes}_{\bar{\mathbb{Q}}_\ell}R=\varprojlim (R/\mathfrak{m}^n)\otimes_{\bar{\mathbb{Q}}_\ell}(R/\mathfrak{m}^n)$. This is a ring of formal power series over $\bar{\mathbb{Q}}_\ell$ of dimension $2\dim R$. Put $E_i=E\otimes_R(R\hat{\otimes}R)$, where the R-module structure on $R\hat{\otimes}R$ is given by $p_i:R\to R\hat{\otimes}R$ (i=1,2). Now $\mathcal{H}om(E_1,E_2)$ is a smooth $R\hat{\otimes}R$-sheaf on X.

In this subsection we establish some properties of $\mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2))$ we need in section 2.

Consider the morphism $\mathcal{H}om(E_1,E_2)\to\mathcal{H}om(E_1,E_2)\otimes_{R\hat{\otimes}R}R\widetilde{\to}\mathcal{E}nd(E)\overset{\mathrm{tr}}{\to}R$. Applying the functor $\mathrm{H}^2(X,\,.\,)$, we get a canonical morphism $\mathrm{H}^2(X,\mathcal{H}om(E_1,E_2))\to \mathrm{H}^2(X,R)\widetilde{\to}R(-1)$.

Proposition 2. *1)* $\mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2))$ *is well-defined as an object of* $\mathrm{D}_{\mathrm{parf}}(R\hat{\otimes}R)$.
2) It can be represented in $\mathrm{D}_{\mathrm{parf}}(R\hat{\otimes}R)$ *by a complex* $(V^0\to V^1\to V^2)$ *of free* $R\hat{\otimes}R$*-modules with* $\mathrm{rk}\,V^0=\mathrm{rk}\,V^2=1$, $\mathrm{rk}\,V^1=(2g-2)m^2+2$ *such that the differential in* V *is zero modulo the maximal ideal of* $R\hat{\otimes}R$.
3) The canonial morphism $\mathrm{H}^2(X,\mathcal{H}om(E_1,E_2))\to R(-1)$ *is an isomorphism of* $R\hat{\otimes}R$*-modules.*

Remark 5. ¿From the geometric point of view, we calculate the direct image under the projection $X\times\mathrm{Spf}(R)\times_{\mathrm{Spec}\,\bar{\mathbb{Q}}_\ell}\mathrm{Spf}(R)\to\mathrm{Spf}(R)\times_{\mathrm{Spec}\,\bar{\mathbb{Q}}_\ell}\mathrm{Spf}(R)$, and the point 3) of the above proposition says that the second highest direct image is canonically isomorphic to the (Tate twisted) constant sheaf on the diagonal $\mathrm{Spf}(R)\to\mathrm{Spf}(R)\times_{\mathrm{Spec}\,\bar{\mathbb{Q}}_\ell}\mathrm{Spf}(R)$.

1.1.3

To prove Proposition 2 we need some general lemmas about perfect derived categories. They are collected in this subsection.

Lemma 3. *Let Λ be a local noetherian ring with maximal ideal $\mathfrak{m}$.*
1) If K is a perfect complex of Λ-modules then it can be represented as a direct sum $K = K_0 \oplus K_1$ of two perfect complexes, where K_0 is acyclic and the differential in $K_1 \otimes_\Lambda \Lambda/\mathfrak{m}$ is zero.
2) If K_1 and K_2 are perfect complexes of Λ-modules such that the differential in $K_i \otimes_\Lambda \Lambda/\mathfrak{m}$ is zero $(i = 1, 2)$ and $f : K_1 \to K_2$ is a homotopical equivalence then f is an isomorphism of complexes.

□

Suppose we are given a projective system of categories $(\mathcal{C}_n)_{n\in\mathbb{N}}, f_{n+1} : \mathcal{C}_{n+1} \to \mathcal{C}_n$. Then $2\text{–}\varprojlim \mathcal{C}_n$ is the category defined as follows. An object of $2\text{–}\varprojlim \mathcal{C}_n$ is a collection $(c_n, \alpha_n)_{n\in\mathbb{N}}$, where $c_n \in \mathrm{Ob}\,\mathcal{C}_n$, $\alpha_n : f_{n+1}c_{n+1} \widetilde{\to} c_n$. A morphism from $(c_n, \alpha_n)_{n\in\mathbb{N}}$ to $(c'_n, \alpha'_n)_{n\in\mathbb{N}}$ is a collection $(\beta_n)_{n\in\mathbb{N}}$, where $\beta_n : c_n \to c'_n$ is a morphism in $\mathcal{C}_n$ such that the diagram

$$\begin{array}{ccc} f_{n+1}c_{n+1} & \xrightarrow{f_{n+1}(\beta_{n+1})} & f_{n+1}c'_{n+1} \\ \downarrow \alpha_n & & \downarrow \alpha'_n \\ c_n & \xrightarrow{\beta_n} & c'_n \end{array}$$

commutes for any $n \in \mathbb{N}$.

Lemma 4. *If Λ is a complete local noetherian ring with maximal ideal $\mathfrak{m}$ then the natural functor $\mathrm{D}_{\mathrm{parf}}(\Lambda) \to 2\text{–}\varprojlim \mathrm{D}_{\mathrm{parf}}(\Lambda/\mathfrak{m}^n)$ is an equivalence of categories.*

□

1.1.4

Denote by $\mathcal{A}$ the category whose objects are surjective local homomorphisms $R \to A$ of $\bar{\mathbb{Q}}_\ell$-algebras with $A \in \mathrm{Ob}\,\mathcal{C}$. A morphism from $R \to A$ to $R \to A'$ is a morphism $A \to A'$ in $\mathcal{C}$ making the natural diagram commute.
Proof of Proposition 2 If $A \in \mathrm{Ob}\,\mathcal{C}$ and $\mathcal{F}$ is a flat A-sheaf on X then we suppose known that $\mathrm{R}\Gamma(X, \mathcal{F}) \in \mathrm{D}_{\mathrm{parf}}(A)$. Since its tor-dimension is easily estimated using the projection formulae, it follows that $\mathrm{R}\Gamma(X, \mathcal{F})$ can be represented by a perfect complex $(U^0 \to U^1 \to U^2)$ of A-modules.

For any $R \to A$ in $\mathcal{A}$ the complex

$$\mathrm{R}\Gamma(X, \mathcal{H}om(E_1, E_2) \otimes_{R \hat{\otimes} R} (A \otimes A))$$

lies in $\mathrm{D}_{\mathrm{parf}}(A\otimes A)$. For any morphism in $\mathcal{A}$ as above we have a canonical isomorphism

$$\mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2)\otimes_{R\hat{\otimes} R}(A\otimes A))\overset{L}{\otimes}_{A\otimes A}(A'\otimes A')\widetilde{\to} \mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2)\otimes_{R\hat{\otimes} R}(A'\otimes A'))$$

By definition, $\mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2)) = \varprojlim \mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2)\otimes_{R\hat{\otimes} R}(R\hat{\otimes} R/\mathfrak{m}^n))$. Now 1) of Proposition 2 follows from Lemma 4. Besides, for any $R\to A$ in $\mathcal{A}$ we get a canonical isomorphism

$$\mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2))\overset{L}{\otimes}_{R\hat{\otimes} R}(A\otimes A)\widetilde{\to}\mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2)\otimes_{R\hat{\otimes} R}(A\otimes A))$$

Chose a perfect complex V of $R\hat{\otimes} R$-modules that represents $\mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2))$ and such that the differential in $V\otimes_{R\hat{\otimes} R}\bar{\mathbb{Q}}_\ell$ is zero. By 2) of Lemma 3 this complex is defined up to a non-canonical isomorphism of complexes of $R\hat{\otimes} R$-modules. However, the complex $V\otimes_{R\hat{\otimes} R}\bar{\mathbb{Q}}_\ell$ is defined up to a canonical isomorphism. More precisely, we have canonically $V^i\otimes_{R\hat{\otimes} R}\bar{\mathbb{Q}}_\ell\widetilde{\to}\mathrm{H}^i(X,\mathcal{E}nd E_0)$ for every i. The point 2) of Proposition 2 follows.

Notice that $\mathrm{R}\Gamma(X,\mathcal{H}om(E_1,E_2))\overset{L}{\otimes}_{R\hat{\otimes} R} R\widetilde{\to}\mathrm{R}\Gamma(X,\mathcal{E}nd E)$ which is the direct sum $\mathrm{R}\Gamma(X,R)\oplus\mathrm{R}\Gamma(X,\mathcal{E}nd_0 E)$. Since $\mathrm{H}^i(X,\mathcal{E}nd_0 E)=0$ for $i\neq 1$, we can conclude that the differential in $V\otimes_{R\hat{\otimes} R} R$ is zero.

In the rest of the proof $R\to A$ will denote an object of $\mathcal{A}$, $I\subset A\otimes A$ will be the ideal of the diagonal and $J\subset I$ will be another ideal of $A\otimes A$.

The next assertion is an immediate consequence of the universal property of (R,E).

Lemma 5. *Suppose that the images of $E_1\otimes_{R\hat{\otimes} R}(A\otimes A)$ and $E_2\otimes_{R\hat{\otimes} R}(A\otimes A)$ in $F_{E_0}(A\otimes A/J)$ coincide then $J=I$.* $\square$

Lemma 6. *Let $B\in\mathrm{Ob}\,\mathcal{C}$ and M_1,M_2 be two non-isomorphic B-deformations of E_0. Then $\mathrm{H}^0(X,\mathcal{H}om(M_1,M_2))=\mathrm{Hom}_B(M_1,M_2)^G$ and for any $f\in\mathrm{Hom}_B(M_1,M_2)^G$ the morphism $f\otimes\mathrm{id}: M_1\otimes_B\bar{\mathbb{Q}}_\ell\to M_2\otimes_B\bar{\mathbb{Q}}_\ell$ vanishes.*

Proof We have $f\otimes\mathrm{id}\in\mathrm{End}_G(E_0)=\bar{\mathbb{Q}}_\ell$. Suppose that $f\otimes\mathrm{id}\neq 0$ then $f\otimes\mathrm{id}$ is an isomorphism of $\bar{\mathbb{Q}}_\ell$-vector spaces. It follows that f is an isomorphism of $B[G]$-modules. Multiplying f by an element of $\bar{\mathbb{Q}}_\ell^*$ we get an isomorphism of B-deformations of E_0, a contradiction. $\square$

Lemma 7. *Put $B=A\otimes A/J$. Suppose that the differential $d^0: V^0\otimes_{R\hat{\otimes} R} B\to V^1\otimes_{R\hat{\otimes} R} B$ vanishes. Then $J=I$.*

Proof Consider the B-deformations $M_i = E_i \otimes_{R\hat{\otimes}R} B$ of E_0 ($i = 1,2$). By our assumption, $\mathrm{H}^0(X, \mathcal{H}om(M_1, M_2)) \widetilde{\to} V^0 \otimes_{R\hat{\otimes}R} B$ is a free B-module of rank 1. Suppose that $J \neq I$ then M_1 and M_2 are non-isomorphic by Lemma 5. Denote by $\mathfrak{n}$ the maximal ideal of B and set $\operatorname{Ann}\mathfrak{n} = \{b \in B \mid b\mathfrak{n} = 0\}$. By Lemma 6, $\operatorname{Ann}\mathfrak{n}$ annihilates $\mathrm{H}^0(X, \mathcal{H}om(M_1, M_2))$. Since $\operatorname{Ann}\mathfrak{n} \neq 0$, we get a contradiction. $\square$

Consider the complex $V \otimes_{R\hat{\otimes}R} (A \otimes A)$. Combining Lemma 7 with the Poincaré duality, one proves that the image of the differential $d^1 : V^1 \otimes_{R\hat{\otimes}R} (A \otimes A) \to V^2 \otimes_{R\hat{\otimes}R} (A \otimes A)$ is $I(V^2 \otimes_{R\hat{\otimes}R} (A \otimes A))$. In other words, the natural morphism $\mathrm{H}^2(X, \mathcal{H}om(E_1, E_2) \otimes_{R\hat{\otimes}R} (A \otimes A)) \to A(-1)$ is an isomorphism.

Passing to the limit we get the desired assertion.

$\square$ (of Proposition 2)

1.2 Definitions of stacks

1.2.1

Denote by Bun_i the moduli stack of vector bundles on X of rank i. So, its category fibre at a scheme S is the groupoïd whose objects are rank i vector bundles on $S \times X$, and morphisms are isomorphisms of such bundles. This is an algebraic stack whose connected components are indexed by $n \in \mathbb{Z}$: the component Bun_i^n classifies vector bundles on X of rank i and degree n. The stack Bun_i^n is locally of finite type, smooth and of dimension $i^2(g-1)$ over $\operatorname{Spec} k$. The stack Bun_1 (resp., Bun_1^n) will also be denoted by $\operatorname{Pic} X$ (resp., $\operatorname{Pic}^n X$). This is the Picard stack of X.

Consider the k-prestack whose category fibre at a scheme S is the following groupoïd. Its objects are vector bundles L on $S \times X$ of rank i and degree n. A morphism from L_1 to L_2 is an equivalence class $\in \{(\mathcal{A}, f)\}/\sim$, where $\mathcal{A}$ is an invertible sheaf on S, $f : L_1 \widetilde{\to} L_2 \otimes \mathcal{A}$ is an isomorphism of $\mathcal{O}_{S\times X}$-modules, and the pairs $(\mathcal{A}, f)$ and $(\mathcal{A}', f')$ are equivalent if there exists an isomorphism $\mathcal{A} \widetilde{\to} \mathcal{A}'$ making commute the diagram

$$\begin{array}{ccc} L_1 & \xrightarrow{f} & L_2 \otimes \mathcal{A} \\ & \searrow f' & \downarrow \wr \\ & & L_2 \otimes \mathcal{A}' \end{array}$$

We define $\overline{\mathrm{Bun}}_i^n$ as the stack associated to this prestack. Then the natural morphism $\bar{\pi} : \mathrm{Bun}_i^n \to \overline{\mathrm{Bun}}_i^n$ is a $\mathbb{G}_m$-gerb.

Lemma 8. $\overline{\mathrm{Bun}}_i^n$ *is an algebraic stack locally of finite type. It is smooth of dimension* $i^2(g-1)+1$.

Proof Let S be a scheme, L_1 and L_2 be vector bundles on $S \times X$ of rank i and degree n. Consider the morphism $\underline{\mathrm{Isom}}(L_1, L_2) \to S$ obtained by the base change $S \stackrel{(L_1,L_2)}{\to} \mathrm{Bun}_i^n \times \mathrm{Bun}_i^n$ from the diagonal mapping $\mathrm{Bun}_i^n \to \mathrm{Bun}_i^n \times \mathrm{Bun}_i^n$. Recall that $\underline{\mathrm{Isom}}(L_1, L_2)$ is an open subscheme of some affine S-scheme $\mathbb{V}(\mathcal{F})$, where $\mathcal{F}$ is a coherent $\mathcal{O}_S$-module, and $\underline{\mathrm{Isom}}(L_1, L_2)$ is of finite type over S (cf.[6],Th.4.14.2.1). We have a free action of $\mathbb{G}_m$ on $\underline{\mathrm{Isom}}(L_1, L_2)$, and the square is cartesian

$$\begin{array}{ccc} \mathrm{Bun}_i^n \times_{\overline{\mathrm{Bun}}_i^n} \mathrm{Bun}_i^n & \to & \mathrm{Bun}_i^n \times \mathrm{Bun}_i^n \\ \uparrow & & \uparrow (L_1, L_2) \\ \underline{\mathrm{Isom}}(L_1, L_2)/\mathbb{G}_m & \to & S \end{array}$$

It follows that $\underline{\mathrm{Isom}}(L_1, L_2)/\mathbb{G}_m$ is an open subscheme of $\mathbb{P}(\mathcal{F})$, in particular it is separated over S. So, the diagonal mapping $\overline{\mathrm{Bun}}_i^n \to \overline{\mathrm{Bun}}_i^n \times \overline{\mathrm{Bun}}_i^n$ is representable, separated and quasi-compact.

If $Y \to \mathrm{Bun}_i^n$ is a presentation of Bun_i^n then the composition $Y \to \mathrm{Bun}_i^n \to \overline{\mathrm{Bun}}_i^n$ is a presentation of $\overline{\mathrm{Bun}}_i^n$. □

We denote by $\overline{\mathrm{Bun}}_i$ the disjoint union of the stacks $\overline{\mathrm{Bun}}_i^n$ for $n \in \mathbb{Z}$.

The morphism $\bar{\pi} : \mathrm{Bun}_1^n \to \overline{\mathrm{Bun}}_1^n$ is, in fact, the canonical morphism $\mathrm{Pic}^n X \to \underline{\mathrm{Pic}}^n X$, where $\underline{\mathrm{Pic}}^n X$ is the Picard scheme of X. Any closed point x of X defines a section $\alpha_x : \underline{\mathrm{Pic}}^n X \to \mathrm{Pic}^n X$ of this $\mathbb{G}_m$-gerb. Namely, if one consider $\underline{\mathrm{Pic}}^n X$ as the moduli scheme of pairs $(\mathcal{A}, t)$ with $\mathcal{A} \in \mathrm{Pic}^n X$, $t : \mathcal{A}_x \widetilde{\to} k$ being a trivialization of the geometric fibre of $\mathcal{A}$ at x then α_x sends $(\mathcal{A}, t)$ to $\mathcal{A}$.

Remark 6. Denote by $\mathrm{Bun}_{\mathrm{PGL}_i}$ the moduli stack of PGL_i-bundles on X. There exists a morphism $\tilde{\pi} : \overline{\mathrm{Bun}}_i^n \to \mathrm{Bun}_{\mathrm{PGL}_i}$ such that the composition $\mathrm{Bun}_i^n \stackrel{\bar{\pi}}{\to} \overline{\mathrm{Bun}}_i^n \stackrel{\tilde{\pi}}{\to} \mathrm{Bun}_{\mathrm{PGL}_i}$ is the canonical map $\mathrm{Bun}_i^n \to \mathrm{Bun}_{\mathrm{PGL}_i}$. The morphism $\tilde{\pi}$ is representable, smooth and separated. Let $t_X : \mathrm{Bun}_i^n \times \mathrm{Pic}^0 X \to \mathrm{Bun}_i^n$ be the map that sends $(L, \mathcal{A})$ to $L \otimes \mathcal{A}$. We have a map $\bar{t}_X : \overline{\mathrm{Bun}}_i^n \times \underline{\mathrm{Pic}}^0 X \to \overline{\mathrm{Bun}}_i^n$ such that the diagram

$$\begin{array}{ccc} \mathrm{Bun}_i^n \times \mathrm{Pic}^0 X & \stackrel{t_X}{\to} & \mathrm{Bun}_i^n \\ \downarrow & & \downarrow \\ \overline{\mathrm{Bun}}_i^n \times \underline{\mathrm{Pic}}^0 X & \stackrel{\bar{t}_X}{\to} & \overline{\mathrm{Bun}}_i^n \end{array}$$

commutes, and the following two squares are cartesian

$$\begin{array}{ccc} \mathrm{Bun}_i^n & \to & \mathrm{Bun}_{\mathrm{PGL}_i} \\ \uparrow t_X & & \uparrow \\ \mathrm{Bun}_i^n \times \mathrm{Pic}^0 X & \stackrel{\mathrm{pr}_1}{\to} & \mathrm{Bun}_i^n \end{array} \qquad \begin{array}{ccc} \overline{\mathrm{Bun}}_i^n & \stackrel{\tilde{\pi}}{\to} & \mathrm{Bun}_{\mathrm{PGL}_i} \\ \uparrow \bar{t}_X & & \uparrow \tilde{\pi} \\ \overline{\mathrm{Bun}}_i^n \times \underline{\mathrm{Pic}}^0 X & \stackrel{\mathrm{pr}_1}{\to} & \overline{\mathrm{Bun}}_i^n \end{array}$$

1.2.2

By Sh_i we denote the moduli stack of coherent sheaves of generic rank i on X. This stack is algebraic, its connected components are numbered by $n \in \mathbb{Z}$: the component Sh_i^n classifies coherent sheaves of rank i and degree n on X. The stack Sh_i^n is locally of finite type and smooth of dimension $i^2(g-1)$.

For two pairs of integers (n_1, n_2) and (i_1, i_2) we denote by $\mathcal{F}l_{i_1,i_2}^{n_1,n_2}$ the stack that classifies pairs $(M_1 \hookrightarrow M_2)$, where M_1 and M_2 are coherent sheaves on X with M_j/M_{j-1} being of generic rank i_j and of degree n_j. We have a map $p_{i_1,i_2}^{n_1,n_2} : \mathcal{F}l_{i_1,i_2}^{n_1,n_2} \to \mathrm{Sh}_{i_1+i_2}^{n_1+n_2}$ that sends the above pair to M_2. This map is representable and proper. In addition, we have a map denoted by $q_{i_1,i_2}^{n_1,n_2} : \mathcal{F}l_{i_1,i_2}^{n_1,n_2} \to \mathrm{Sh}_{i_1}^{n_1} \times \mathrm{Sh}_{i_2}^{n_2}$ that sends an object $(M_1 \hookrightarrow M_2)$ to $(M_1, M_2/M_1)$. It is easy to see that $q_{i_1,i_2}^{n_1,n_2}$ is a generalized vector fibration. This, in particular, implies that $\mathcal{F}l_{i_1,i_2}^{n_1,n_2}$ is smooth.

We denote by $\mathrm{Sh}_1^{n;\le m}$ (resp., by $\mathrm{Sh}_1^{n;m}$) the open (resp., locally closed) substack of Sh_1^n that corresponds to those coherent sheaves whose maximal torsion subsheaf has length at most m (exactly m). To be precise, the stack structure on $\mathrm{Sh}_1^{n;m}$ is defined as follows. If $\widetilde{\mathcal{F}l}_{0,1}^{m,n-m}$ is the preimage of $\mathrm{Sh}_0^m \times \mathrm{Pic}^{n-m} X$ under $q_{0,1}^{m,n-m}$ then the restriction of $p_{0,1}^{m,n-m}$ to $\widetilde{\mathcal{F}l}_{0,1}^{m,n-m}$ is a locally closed immersion $\widetilde{\mathcal{F}l}_{0,1}^{m,n-m} \to \mathrm{Sh}_1^n$, and this substack is denoted $\mathrm{Sh}_1^{n;m}$.The stack $\mathrm{Sh}_1^{n;m}$ is smooth (and can also be defined as the complement of $\mathrm{Sh}_1^{n;\le m-1}$ to $\mathrm{Sh}^{n;\le m}$ with the reduced stack structure).

For $i \ge 1$ we denote by $\mathrm{Sh}_i'^n$ (resp., by ${}^0\mathrm{Sh}_i'^n$) the stack that classifies pairs $(F_i, s : \Omega^{\otimes i-1} \to F_i)$ (resp., $(F_i, s : \Omega^{\otimes i-1} \hookrightarrow F_i)$) with $F_i \in \mathrm{Sh}_i^n$. We have an obvious open embedding $j_i : {}^0\mathrm{Sh}_i'^n \to \mathrm{Sh}_i'^n$. By $\pi_i : \mathrm{Sh}_i'^n \to \mathrm{Sh}_i^n$ is denoted the natural projection. The morphism $\pi_{i-1}' : {}^0\mathrm{Sh}_i' \to \mathrm{Sh}_{i-1}$ sends a pair $(F_i, s : \Omega^{\otimes i-1} \hookrightarrow F_i)$ to $F_i/\mathrm{Im}(s)$. This is a generalized vector fibration. Set also ${}^0\pi_i = \pi_i \circ j_i$.

By $\mathrm{Sh}_0'^n$ we denote the stack of pairs $(F_0, s : \mathcal{O} \to F_0)$ with $F_0 \in \mathrm{Sh}_0^n$. (This notation does not agree with [2]!)

We denote by $\mathcal{W}^n$ the stack that classifies exact sequences $0 \to \Omega \to ? \to \mathcal{A} \to 0$ on X with $\mathcal{A} \in \mathrm{Pic}^n X$. The morphism $\mathcal{W}^n \to \mathrm{Pic}^n X$ that sends the above exact sequence to $\mathcal{A}$ is a generalized vector fibration of rank $n+1-g$.

By $X^{(n)} \to \mathrm{Sh}_0^n$ we will always mean the morphism that sends a divisor D to $\Omega(D)/\Omega$. By $\mathrm{div} : \mathrm{Sh}_0^n \to X^{(n)}$ is denoted the morphism norm (cf. [11],6). If $D_1, \dots, D_s$ are effective divisors on X then it sends the $\mathcal{O}_X$-module $\mathcal{O}_{D_1+\ldots+D_s} \oplus \mathcal{O}_{D_2+\ldots+D_s} \oplus \ldots \oplus \mathcal{O}_{D_s}$ to $D_1 + 2D_2 + \ldots + sD_s$.

1.2.3

The following notations are used in sect. 3 and 4.

We denote by $\mathrm{Pic}'^n X$ the preimage under $\pi_1 : \mathrm{Sh}_1'^n \to \mathrm{Sh}_1^n$ of the open substack $\mathrm{Pic}^n X \subset \mathrm{Sh}_1^n$. Let $i_0 : \mathrm{Pic}^n X \to \mathrm{Pic}'^n X$ be the zero section of the natural projection $\mathrm{Pic}'^n X \to \mathrm{Pic}^n X$. We have an open immersion $X^{(n)} \hookrightarrow \mathrm{Pic}'^n X$, and $i_0 : \mathrm{Pic}^n X \to \mathrm{Pic}'^n X$ is the complement of $X^{(n)}$ to $\mathrm{Pic}'^n X$ with the reduced stack structure.

Lemma 9. *If S is a locally noetherian scheme and $F_1 \in (\mathrm{Sh}_1^n)_S$ then there exists a natural morphism $F_1 \to \det F_1$ of $\mathcal{O}_{S\times X}$-modules. It depends functorially on S.*

Proof Since F_1 is S-flat and $\dim X = 1$, locally on S in Zarisky topology there exists a resolution $0 \to F^{-1} \to F^0 \to F_1 \to 0$ of F_1, where F^{-1}, F^0 are locally free coherent sheaves of constant ranks, say $m-1$ and m respectively. Instead of constructing a morphism $F_1 \to (\det F^0) \otimes (\det F^{-1})^*$, we define a mapping $F_1 \otimes (\det F^{-1}) \to \det F^0$ as follows. If $s \in F_1$ and $s_i \in F^{-1}$ $(0 \le i \le m-1)$ then it sends $s \otimes (s_1 \wedge \cdots \wedge s_{m-1})$ to $\tilde{s} \wedge (s_1 \wedge \cdots \wedge s_{m-1}) \in \det F^0$, where $\tilde{s} \in F^0$ is a (local) lifting of s.

The mapping $F_1 \to \det F_1$ constructed locally on S is compatible with the descent datas. $\square$

The above lemma shows that there is a natural morphism $\mathrm{Sh}_1'^n \to \mathrm{Pic}'^n X$. Namely, if $S \in (\mathrm{Aff} / \mathrm{Spec}\, k)$ and $F_1 \in (\mathrm{Sh}_1^n)_S$ then every section $\mathcal{O} \to F_1$ defines a section of $\det F_1$ as the composition $\mathcal{O} \to F_1 \to \det F_1$.

The following square is cartesian

$$\begin{array}{ccc} \mathrm{Sh}_1'^n & \to & \mathrm{Pic}'^n X \\ \uparrow {\scriptstyle j_1} & & \uparrow \\ {}^0\,\mathrm{Sh}_1'^n & \to & X^{(n)} \end{array}$$

We define the stack $\mathrm{Sh}_1''^n$ from the cartesian square

$$\begin{array}{ccc} \mathrm{Sh}_1'^n & \to & \mathrm{Pic}'^n X \\ \uparrow & & \uparrow {\scriptstyle i_0} \\ \mathrm{Sh}_1''^n & \to & \mathrm{Pic}^n X \end{array} \qquad (3)$$

(The author does not know whether $\mathrm{Sh}_1''^n$ is reduced).

Notice that $\mathrm{Sh}_1''^n \times_{\mathrm{Sh}_1^n} \mathrm{Sh}_1^{n;m}$ is the stack that clasifies the collections: $F_0 \in \mathrm{Sh}_0^m$, $\mathcal{A} \in \mathrm{Pic}^{n-m} X$, an extension $0 \to F_0 \to F_1 \to \mathcal{A} \to 0$, and a section $\mathcal{O} \to F_0$. In particular, this stack is smooth of dimension $g-1$. This, in particular, implies that $\mathrm{Sh}_1''^n$ is reducible. Its irreducible components are numbered by $m \ge 0$. The component corresponding to m is the closure of $\mathrm{Sh}_1''^n \times_{\mathrm{Sh}_1^n} \mathrm{Sh}_1^{n;m}$ in $\mathrm{Sh}_1''^n$.

1.3 Definitions of sheaves

1.3.1

Let $A \in \mathrm{Ob}\,\mathcal{C}$ and E be a smooth A-sheaf on X. Denote by $\mathrm{sym} : X^n \to X^{(n)}$ the natural map, and consider the smooth A-sheaf $E^{\boxtimes n}$ on X^n (the tensoring is taken over A). The A-sheaf $\mathrm{sym}_!(E^{\boxtimes n})$ carries a natural S^n-action. Using the projection formulae for sym, we see that it is flat over A. Notice that $\mathrm{sym}_!(E^{\boxtimes n})[n]$ is a perverse A-sheaf (flat over A).

We set $E^{(n)} = (\mathrm{sym}_!(E^{\boxtimes n}))^{S^n}$. Since $E^{(n)}$ is a direct summand of $\mathrm{sym}_!(E^{\boxtimes n})$, $E^{(n)}[n]$ is also a perverse A-sheaf (flat over A). For a morphism $A \to B$ in $\mathcal{C}$ we have $(E \otimes_A B)^{(n)} \widetilde{\to} E^{(n)} \otimes_A B$ naturally.

Consider the open substacks ${}^{rss}\,\mathrm{Sh}_0^n \subset {}^{r}\,\mathrm{Sh}_0^n$ of Sh_0^n that correspond to regular semi-simple and regular sheaves respectively. In other words, ${}^{r}\,\mathrm{Sh}_0^n$ is the image of the (smooth) map $X^{(n)} \to \mathrm{Sh}_0^n$, and ${}^{rss}\,\mathrm{Sh}_0^n$ is identified with $(X^{(n)} - \triangle)/\mathbb{G}_m^n$, where $\triangle$ is the divisor of coinciding points. We have a cartesian square

$$\begin{array}{ccc} X^n & \to & \mathcal{F}l_{0,\dots,0}^{1,\dots,1} \\ \downarrow \mathrm{sym} & & \downarrow p_{0,\dots,0}^{1,\dots,1} \\ X^{(n)} & \to & \mathrm{Sh}_0^n, \end{array}$$

and $p_{0,\dots,0}^{1,\dots,1}$ is an S^n-Galois étale covering over ${}^{rss}\,\mathrm{Sh}_0^n$.

Recall that the Springer's sheaf $\mathcal{S}pr_E^n$ on Sh_0^n is defined by

$$\mathcal{S}pr_E^n = p_{0,\dots,0\,!}^{1,\dots,1} \circ q_{0,\dots,0}^{1,\dots,1\,*} \circ \mathrm{div}^{\times n\,*}(E \boxtimes \cdots \boxtimes E)$$

This is a perverse A-sheaf (flat over A), and it coincides with the Goresky-MacPherson extension of its restriction to ${}^{rss}\,\mathrm{Sh}_0^n$. In addition, it carries a natural S^n-action (cf. Lemma 3,p.15 of [2] or Theorem 3.3.1 of [3]).

Definition 3. For a smooth A-sheaf E on X we define the Laumon's perverse A-sheaf $\mathcal{L}_E^n$ on Sh_0^n as $\mathrm{Hom}_{S^n}(\mathrm{triv}, \mathcal{S}pr_E^n)$, where triv denotes the trivial representation of the symmetric group S^n.

$\mathcal{L}_E^n$ is a direct summand of $\mathcal{S}pr_E^n$, hence it is an A-flat perverse A-sheaf. For a morphism $A \to B$ in $\mathcal{C}$ we have $\mathcal{L}_E^n \otimes_A B \widetilde{\to} \mathcal{L}_{(E \otimes_A B)}^n$ naturally. The pull-back of $\mathcal{L}_E^n$ to $X^{(n)}$ is identified with $E^{(n)}$. Besides, $\mathcal{L}_E^n$ is the Goresky-MacPherson extension of its restriction to ${}^{rss}\,\mathrm{Sh}_0^n$.

If E_0 is a local system on X such that $\mathrm{End}(E_0) = \bar{\mathbb{Q}}_\ell$ and the pair (R, E) is constructed out of E_0 as in sect.1.1.1 then the above definition yields a perverse R-sheaf $\mathcal{L}_E^n$ on Sh_0^n equipped with an isomorphism $\mathcal{L}_E^n \otimes_R \bar{\mathbb{Q}}_\ell \widetilde{\to} \mathcal{L}_{E_0}^n$.

1.3.2

For two non-negative integers c_1, c_2 consider the diagram

$$\begin{array}{ccc} \mathcal{F}l_{0,0}^{c_1,c_2} & \stackrel{q_{0,0}^{c_1,c_2}}{\rightarrow} & \mathrm{Sh}_0^{c_1} \times \mathrm{Sh}_0^{c_2} \\ \downarrow p_{0,0}^{c_1,c_2} & & \\ \mathrm{Sh}_0^{c_1+c_2} & & \end{array}$$

Let $A \in \mathrm{Ob}\,\mathcal{C}$ and E_1, E_2 be a pair of smooth A-sheaves on X. Following [2], we define the perverse A-sheaf $\mathcal{L}_{E_1,E_2}^{c_1,c_2}$ on $\mathrm{Sh}_0^{c_1+c_2}$ by

$$\mathcal{L}_{E_1,E_2}^{c_1,c_2} = p_{0,0!}^{c_1,c_2} \circ q_{0,0}^{c_1,c_2\,*}(\mathcal{L}_{E_1}^n \boxtimes \mathcal{L}_{E_2}^n)$$

Again, this sheaf is a Goresky-MacPherson extension of its restriction to ${}^{rss}\,\mathrm{Sh}_0^{c_1+c_2}$ (cf. Prop.2 of [2]).

1.4 Gaitsgory's construction of Aut_E and the Hecke property

Let E_0 be an irreducible local system on X of rank 2. The purpose of this subsection is to recall the Gaitsgory's definition of the perverse sheaf Aut_{E_0} on Bun_2 (cf. [2]) and explain that this construction is still valid for deformations of E_0. This leads to the definition of the perverse R-sheaf Aut_E on Bun_2, where E is the universal deformation of E_0, and R is the base of its universal deformation. In fact, we adapt the original argument of Gaitsgory ([2]) to first construct the perverse R-sheaf $\overline{\mathrm{Aut}}_E$ on $\overline{\mathrm{Bun}}_2$ and define Aut_E by $\mathrm{Aut}_E = \bar{\pi}^* \overline{\mathrm{Aut}}_E[-1]$, where $\bar{\pi} : \mathrm{Bun}_2 \to \overline{\mathrm{Bun}}_2$ is the natural map.

A new proof of the second Hecke property of $\overline{\mathrm{Aut}}_E$ is also given.

1.4.1

Gaitsgory's construction of Aut_E is given in termes of the following 'fundamental diagram' (first introduced by G.Laumon in [4], exposé 1, sect.1, cf. also [2] sect. 2.1.1):

$$\begin{array}{ccccccccc} & & {}^0\mathrm{Sh}_1'^n & \stackrel{j_1}{\hookrightarrow} & \mathrm{Sh}_1'^n & & & {}^0\mathrm{Sh}_2'^{n+2g-2} & \\ & \swarrow \pi_0' & & & & \searrow \pi_1 & \swarrow \pi_1' & & \searrow {}^0\pi_2 \\ \mathrm{Sh}_0^n & & & & & \mathrm{Sh}_1^n & & & \mathrm{Sh}_2^{n+2g-2} \end{array}$$

(The corresponding stacks and morphisms are defined in sect. 1.2.2). Notice that π_0' is a vector bundle of rank n, and π_1' is a generalized vector fibration of rank $n+1-g$.

Following [2], for an integer c we denote by ${}_c\mathrm{Sh}_i \subset \mathrm{Sh}_i$ the open substack that corresponds to sheaves $M \in \mathrm{Sh}_i$ such that $\mathrm{Ext}^1(L, M) = 0$ for any $L \in \mathrm{Pic}^{c'} X$ with $c' \le c$.

Notice that over ${}_{2g-2}\mathrm{Sh}_2^n$ the morphism $\pi_2 : \mathrm{Sh}_2'^n \to \mathrm{Sh}_2^n$ is a vector bundle of rank $n - 6g + 6$.

For the convenience of the reader we recall the definition of the Fourier transform functor Four between the derived categories on $\mathrm{Sh}_1'^n$ and on ${}^0\mathrm{Sh}_2'^{n+2g-2}$. If $A \in \mathcal{C}$ and K is an A-complex on $\mathrm{Sh}_1'^n$ then

$$\mathrm{Four}(K) = \mathrm{pr}'_!(\mathrm{pr}^* K \otimes_{\bar{\mathbb{Q}}_\ell} \mu^* \mathcal{L}_\psi)[n+1-g](\frac{n+1-g}{2})$$

is an A-complex on ${}^0\mathrm{Sh}_2'^{n+2g-2}$. If $\tilde{K}$ is an A-complex on ${}^0\mathrm{Sh}_2'^{n+2g-2}$ then

$$\mathrm{Four}(\tilde{K}) = \mathrm{pr}_!(\mathrm{pr}'^* \tilde{K} \otimes_{\bar{\mathbb{Q}}_\ell} \mu^* \mathcal{L}_\psi)[n+1-g](\frac{n+1-g}{2})$$

is an A-complex on $\mathrm{Sh}_1'^n$. Here $\mu : \mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_2'^{n+2g-2} \to \mathbb{A}_k^1$ is the natural pairing (defined as in sect. 0.1.2), and the projections pr and pr′ are given by the diagram

$$\begin{array}{ccc} \mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_2'^{n+2g-2} & \stackrel{\mathrm{pr}'}{\to} & {}^0\mathrm{Sh}_2'^{n+2g-2} \\ \downarrow \mathrm{pr} & & \downarrow \pi_1' \\ \mathrm{Sh}_1'^n & \stackrel{\pi_1}{\to} & \mathrm{Sh}_1^n \end{array}$$

The stack $\overline{\mathrm{Flag}}_n$ introduced in sect. 0.1.2 is the preimage of Bun_2^n under ${}^0\pi_2 : {}^0\mathrm{Sh}_2'^n \to \mathrm{Sh}_2^n$. Let $j : \overline{\mathrm{Flag}}_n \to {}^0\mathrm{Sh}_2'^n$ be the natural open immersion. Put

$$\overline{\mathrm{Flag}}_{n+2g-2}^{\le m} = \overline{\mathrm{Flag}}_{n+2g-2} \cap \pi_1'^{-1}(\mathrm{Sh}_1^{n;\le m})$$

Set also

$$\begin{aligned} {}_c\overline{\mathrm{Flag}}_n &= \overline{\mathrm{Flag}}_n \cap {}^0\pi_2^*({}_c\mathrm{Sh}_2^n), \\ {}_c\overline{\mathrm{Flag}}_n^{\le m} &=_c \overline{\mathrm{Flag}}_n \cap \overline{\mathrm{Flag}}_n^{\le m} \end{aligned}$$

Denote by ${}_c\overline{\mathrm{Bun}}_2^n$ the image of ${}_c\mathrm{Bun}_2^n$ under $\bar{\pi}$.

Definition 4. For $A \in \mathcal{C}$, E a smooth A-sheaf of rank 2, $n \ge 0$ define the A-complex $\mathcal{K}_E^{n+2g-2}$ on $\overline{\mathrm{Flag}}_{n+2g-2}$ by

$$\mathcal{K}_E^{n+2g-2} = j^* \circ \mathrm{Four} \circ j_{1!} \circ \pi_0'^*(\mathcal{L}_E^n)[n]$$

Notice that for a morphism $A \to B$ in $\mathcal{C}$ we have $\mathcal{K}_E^n \overset{L}{\otimes}_A B \widetilde{\to} \mathcal{K}_{E \otimes_A B}^n$.

We have a smooth map $m_X : \mathrm{Bun}_2 \times X \to \mathrm{Bun}_2$ that sends $(M, x \in X)$ to $M(x)$. We have also a map $\bar{m}_X : \overline{\mathrm{Bun}}_2 \times X \to \overline{\mathrm{Bun}}_2$ included into the commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}_2 \times X & \overset{m_X}{\to} & \mathrm{Bun}_2 \\ \downarrow \bar{\pi}\times\mathrm{id} & & \downarrow \bar{\pi} \\ \overline{\mathrm{Bun}}_2 \times X & \overset{\bar{m}_X}{\to} & \overline{\mathrm{Bun}}_2 \end{array}$$

Let $A \in \mathcal{C}$, E be a smooth A-sheaf of rank 2. We put $E_0 := E \otimes_A \bar{\mathbb{Q}}_\ell$.

Theorem 1. *Let E_0 be irreducible. There exists an A-flat perverse A-sheaf $\overline{\mathrm{Aut}}_E$ on $\overline{\mathrm{Bun}}_2$ that satisfies:*
(1) if $n > 6g-4$ then the sheaves ${}^0\pi_2^*\bar{\pi}^*(\overline{\mathrm{Aut}}_E^n)[n-6g+5](-2g+2)$ and $\mathcal{K}_E^n$ are canonically isomorphic over ${}_{2g}\overline{\mathrm{Flag}}_n^{\le 1}$. (Here $\overline{\mathrm{Aut}}_E^n$ is the restriction of $\overline{\mathrm{Aut}}_E$ to $\overline{\mathrm{Bun}}_2^n$).
(2) $\bar{m}_X^*(\overline{\mathrm{Aut}}_E) \widetilde{\to} \overline{\mathrm{Aut}}_E \boxtimes \wedge^2 E$ canonically
(3) for a morphism $A \to B$ in $\mathcal{C}$ we have $\overline{\mathrm{Aut}}_E \otimes_A B \widetilde{\to} \overline{\mathrm{Aut}}_{E\otimes_A B}$ canonically.

Remark 7. (i) The map ${}_{2g}\overline{\mathrm{Flag}}_n^{\le 1} \to {}_{2g}\overline{\mathrm{Bun}}_2^n$ is smooth and surjective with connected fibres. It easily follows that by the properties (1)-(2) the perverse A-sheaf $\overline{\mathrm{Aut}}_E$ is defined up to a canonical isomorphism.
(ii) Put $\mathrm{Aut}_E = \bar{\pi}^* \overline{\mathrm{Aut}}_E[-1]$. Then Aut_E is a perverse A-sheaf on Bun_2. For $A = \bar{\mathbb{Q}}_\ell$ it was constructed by D.Gaitsgory (cf. Main Theorem C, [2]).

Definition 5. Let E_0 be an irreducible local system on X of rank 2. Let E be its universal deformation and R be the base of its universal deformation. By Theorem 1, to the pair (R, E) is associated the object $\overline{\mathrm{Aut}}_E$ of $\mathcal{PA}(\overline{\mathrm{Bun}}_2, R)$ such that for any morphism $R \to A$ in $\mathcal{A}$ (cf. sect. 1.1.4) we have $\overline{\mathrm{Aut}}_E \otimes_R A \widetilde{\to} \overline{\mathrm{Aut}}_{E\otimes_R A}$ canonically. It is easy to see that $\overline{\mathrm{Aut}}_E$ is, in fact, a perverse R-sheaf flat over R.

The proof of Theorem 1 (modulo the results of D.Gaitsgory ([2],ch.3)) is given in sect. 1.4.2-1.4.3. Now we formulate some additional properties of $\overline{\mathrm{Aut}}_E^n$ we need in sect. 2.

Proposition 3. *For any $m \ge 0$ there exists a constant $const_1(m)$ such that for $n > const_1(m)$ we have*

$${}^0\pi_2^*\bar{\pi}^*(\overline{\mathrm{Aut}}_E^n)[n-6g+5](-2g+2) \widetilde{\to} \mathcal{K}_E^n$$

canonically over $\overline{\mathrm{Flag}}_n^{\le m}$.

Remark 8. The conjecture of Laumon (cf.[4]) states that this isomorphism holds for any $n \geq 2g-2$ over the whole of $\overline{\mathrm{Flag}}_n$. At the level of functions 'trace of Frobenius' this conjecture is true.

Proposition 4. *There exists a constant $const_2$ such that for any invertible sheaves $\mathcal{A}, \mathcal{B}$ on X with $\deg \mathcal{A} - \deg \mathcal{B} > const_2$ the fibre of $\overline{\mathrm{Aut}}_E$ at $\mathcal{A} \oplus \mathcal{B}$ vanishes.*

The proof of both Proposition 3 and Proposition 4 will be given in sect. 1.4.4. Now we derive an important corolary.

Corolary 1. *There exists a constant $const_3$ such that for any integers n, m with $n > 2m + const_3$ the perverse A-sheaf $\overline{\mathrm{Aut}}^n_E$ is the extension by zero of its restriction to the open substack ${}_m\overline{\mathrm{Bun}}^n_2$ of $\overline{\mathrm{Bun}}^n_2$.*

Proof We can suppose that $const_2 \geq 2g-2$. It is well known that there exists a constant, say c, such that any rank 2 bundle M on X can be included into an exact sequence $0 \to \mathcal{A} \to M \to \mathcal{B} \to 0$, where $\mathcal{A}, \mathcal{B}$ are invertible sheaves on X with $\deg \mathcal{A} - \deg \mathcal{B} \geq c$.

Pick $const_3$ such that $const_3 \geq 4g-4+const_2$ and $const_3 \geq 4g-4-c$. Let M be a rank 2 bundle on X of degree n. It defines a k-point of $\overline{\mathrm{Bun}}^n_2$. Suppose that the fibre of $\overline{\mathrm{Aut}}^n_E$ does not vanish at this point and chose an exact sequence $0 \to \mathcal{A} \to M \to \mathcal{B} \to 0$ as above. By Proposition 4, we have $c \leq \deg \mathcal{A} - \deg \mathcal{B} \leq const_2$. Let m be an integer such that $n > 2m + const_3$. Now for any invertible sheaf L of degree $\leq m$ we have $\mathrm{Ext}^1(L, \mathcal{A}) = \mathrm{Ext}^1(L, \mathcal{B}) = 0$ and, hence, $\mathrm{Ext}^1(L, M) = 0$. $\square$

1.4.2

In this subsection we prove the following result which is a part of Theorem 1.

Proposition 5. *Let E_0 be irreducible and let n satisfy: $n > 6g-4$. There exists an A-flat perverse A-sheaf $\bar{\mathcal{S}}^n_E$ on ${}_{2g}\overline{\mathrm{Bun}}^n_2$ such that the sheaves ${}^0\pi_2^*\bar{\pi}^*(\bar{\mathcal{S}}^n_E)[n-6g+5](-2g+2)$ and $\mathcal{K}^n_E$ are canonically isomorphic over ${}_{2g}\overline{\mathrm{Flag}}^{\leq 1}_n$. For a morphism $A \to B$ in $\mathcal{C}$ we have naturally $\bar{\mathcal{S}}^n_E \otimes_A B \widetilde{\to} \bar{\mathcal{S}}^n_{E \otimes_A B}$.*

Remark 9. The perverse A-sheaf $\bar{\mathcal{S}}^n_E$ is defined up to a canonical isomorphism, because the map ${}_{2g}\overline{\mathrm{Flag}}^{\leq 1}_n \to {}_{2g}\overline{\mathrm{Bun}}^n_2$ is smooth and surjective with connected fibres.

Proposition 5 will be a consequence of (Theorem 5,[2]) and of the following assertion:

Proposition 6. *Let n satisfy $n > 4g-3$. The irreducibility of E_0 implies that the canonical maps*

$$j_{1!} \circ \pi_0'^*(\mathcal{L}_E^n) \to j_{1!*} \circ \pi_0'^*(\mathcal{L}_E^n) \to j_{1*} \circ \pi_0'^*(\mathcal{L}_E^n)$$

are isomorphisms over $\mathrm{Sh}_1'^{n;\le 1}$. In particular, over $\mathrm{Sh}_1'^{n;\le 1}$ the perverse A-sheaf $j_{1!} \circ \pi_0'^*(\mathcal{L}_E^n)[n]$ is A-flat.

Lemma 10. *Let $\mathcal{Y}$ be an algebraic stack, F be an A-flat perverse A-sheaf on $\mathcal{Y}$. Put $F_0 = F \otimes_A \bar{\mathbb{Q}}_\ell$. There exists a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_k = F$ of F by perverse A-subsheaves such that $F_{i+1}/F_i \widetilde{\to} F_0$ as perverse A-sheaves ($0 \le i < k$). If, in addition, $U \hookrightarrow \mathcal{Y}$ is an open substack, and F_0 is the Goresky-MacPherson extension of its restriction to U then the same holds for F.*

Proof Let $\mathfrak{m} \subset A$ be the maximal ideal. Choose $t \in A, t \ne 0$ such that $t\mathfrak{m} = 0$. Then the sequence $0 \to F \otimes_A (t) \to F \to F \otimes_A A/(t) \to 0$ is exact, and $F \otimes_A (t) \widetilde{\to} F_0$. The first assertion follows by induction on $\dim_{\bar{\mathbb{Q}}_\ell} A$.

If $0 \to G_1 \to G \to G_2 \to 0$ is an exact sequence of perverse sheaves on $\mathcal{Y}$, and G_i ($i = 1, 2$) is the Goresky-MacPherson extension of its restriction to U then the same is true for G. The second assertion follows. $\square$

Proof of Proposition 6
It suffices to show that $j_{1!} \circ \pi_0'^*(\mathcal{L}_E^n) \to j_{1*} \circ \pi_0'^*(\mathcal{L}_E^n)$ is an isomorphism over $\mathrm{Sh}_1'^{n;\le 1}$. For this combine (Theorem 6,[2]) with the first assertion of the previous lemma. $\square$

Lemma 11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks, with $\mathcal{Y}$ locally of finite type. Suppose that there exists a presentation $\pi : S \to \mathcal{Y}$ and an isomorphism $\mathcal{X} \times_{\mathcal{Y}} S \widetilde{\to} \mathbb{P}^N \times S$ over S. Then the functor f^* is an equivalence between the categories of smooth A-sheaves on $\mathcal{Y}$ and on $\mathcal{X}$ respectively.*

Proof The notion of a smooth A-sheaf on $\mathcal{Y}$ localizes in the smooth topology, so that we are reduced to the case, where $\mathcal{Y}$ is a scheme. In this case our assertion follows from SGA1,éxp.9,Cor.6.11 and éxp.11, Prop.1.1. $\square$

Lemma 12. *For $m \ge 1$ any fibre of the morphism ${}_{m+2g-2}\overline{\mathrm{Flag}}_n^{\le m-1} \to {}_{m+2g-2}\mathrm{Bun}_2^n$ is isomorphic to $\mathbb{A}_k^{n-6g+6} \setminus S$, where $S \subset \mathbb{A}_k^{n-6g+6}$ is a closed subscheme of codimension $\ge m$. Besides, S is $\mathbb{G}_m$-equivariant.*

Proof For $M \in {}_{m+2g-2}\mathrm{Bun}_2^n$ and $D \in X^{(m)}$ we have $\dim \mathrm{Hom}(\Omega, M(-D)) = \chi(M \otimes \Omega^{-1}(-D)) = (n - 6g + 6) - 2m$ and $\dim \mathrm{Hom}(\Omega, M) = \chi(M \otimes \Omega^{-1}) = n - 6g + 6$. Since $\dim X^{(m)} = m$, our assertion follows. $\square$

The next result is straightforward.

Lemma 13. *Let $\mathcal{Y}$ be a smooth algebraic stack, F be a perverse A-sheaf on $\mathcal{Y}$ flat over A. Suppose that $F_0 = F \otimes_A \bar{\mathbb{Q}}_\ell$ is a local system on $\mathcal{Y}$ (appropriately shifted). Then F is a smooth A-sheaf on $\mathcal{Y}$ (appropriately shifted).*
$\square$

Proof of Proposition 5
By Proposition 6, over $\overline{\mathrm{Flag}}_n^{\le 1}$ the A-complex $\mathcal{K}_E^n$ is an A-flat perverse A-sheaf. Since $\mathcal{K}_{E_0}^n$ is an irreducible perverse sheaf over $\overline{\mathrm{Flag}}_n^{\le 1}$, by Lemma 10 over this stack $\mathcal{K}_E^n$ is the Goresky-MacPherson extension of its restriction to any nonempty open substack $U \subset \overline{\mathrm{Flag}}_n^{\le 1}$.

Consider the restriction of $\pi_2 : \mathrm{Sh}_2'^n \to \mathrm{Sh}_2^n$ to ${}_{2g-2}\mathrm{Bun}_2^n \subset \mathrm{Sh}_2^n$. This is a vector bundle, and its projectivization coincides with

$$\mathrm{pr}_2 : {}_{2g-2}\overline{\mathrm{Flag}}_n \times_{{}_{2g-2}\overline{\mathrm{Bun}}_2^n} {}_{2g-2}\mathrm{Bun}_2^n \to {}_{2g-2}\mathrm{Bun}_2^n$$

It follows that the natural morphism ${}_{2g-2}\overline{\mathrm{Flag}}_n \to {}_{2g-2}\overline{\mathrm{Bun}}_2^n$ satisfies the conditions of Lemma 11.

By Theorem 5 of [2], there exists a perverse sheaf $\mathcal{S}_{E_0}^n$ on ${}_{2g}\mathrm{Bun}_2^n$ such that $\mathcal{K}_{E_0}^n \widetilde{\to}{}^0\pi_2^*\mathcal{S}_{E_0}^n[n-6g+6](-2g+2)$ over ${}_{2g}\overline{\mathrm{Flag}}_n^{\le 1}$. Let U be a nonempty open substack of ${}_{2g}\mathrm{Bun}_2^n$, where $\mathcal{S}_{E_0}^n$ is smooth. Denote by U' (resp., by U'') the preimage of U under ${}_{2g}\overline{\mathrm{Flag}}_n^{\le 1} \to {}_{2g}\mathrm{Bun}_2^n$ (resp., under ${}_{2g}\overline{\mathrm{Flag}}_n \to {}_{2g}\mathrm{Bun}_2^n$). Now, by Lemma 13, $\mathcal{K}_E^n$ is a smooth A-sheaf over U' (appropriately shifted). Since $U'' \setminus U'$ is of codimension 2 in U'', $\mathcal{K}_E^n$ can be extended as a smooth A-sheaf from U' to U''. Let $\bar{U}$ be the image of U under $\bar{\pi}$.

Applying Lemma 11 to the morphism $U'' \to \bar{U}$ we learn that there exists a perverse A-sheaf F on $\bar{U}$ such that the inverse image of $F[n-6g+5](-2g+2)$ to U' is isomorphic to $\mathcal{K}_E^n$ over U'. Define the perverse A-sheaf $\bar{\mathcal{S}}_E^n$ as the Goresky-MacPherson extension of F under $\bar{U} \hookrightarrow {}_{2g}\overline{\mathrm{Bun}}_2^n$.

Since the Goresky-MacPherson extension commutes with a smooth base change, we get the isomorphism $\mathcal{K}_E^n \widetilde{\to}{}^0\pi_2^*\bar{\pi}^*(\bar{\mathcal{S}}_E^n)[n-6g+5](-2g+2)$ over ${}_{2g}\overline{\mathrm{Flag}}_n^{\le 1}$.

Since the morphism ${}_{2g}\overline{\mathrm{Flag}}_n^{\le 1} \to {}_{2g}\overline{\mathrm{Bun}}_2^n$ is smooth and surjective, the A-flatness of $\bar{\mathcal{S}}_E^n$ follows from the A-flatness of $\mathcal{K}_E^n$ over ${}_{2g}\overline{\mathrm{Flag}}_n^{\le 1}$. Similarly, given a morphism $A \to B$ in $\mathcal{C}$, the isomorphism $\bar{\mathcal{S}}_E^n \otimes_A B \widetilde{\to} \bar{\mathcal{S}}_{E\otimes_A B}^n$ is obtaned from the analogous isomorphism for $\mathcal{K}_E^n$. $\square$

1.4.3

In this subsection we finish the proof of Theorem 1.

For $m \geq 0$ put $\overline{\mathrm{Flag}}^m_{n+2g-2} = \overline{\mathrm{Flag}}_{n+2g-2} \times_{\mathrm{Sh}^n_1} \mathrm{Sh}^{n;m}_1$. The stack $\overline{\mathrm{Flag}}^0_{n+2g-2}$ is also denoted by $\mathcal{W}^n$ (cf. sect. 1.2.2) . For $m \geq 0$ there exists a natural isomorphism

$$\mathcal{W}^{n-2m} \times X^{(m)} \widetilde{\to} \overline{\mathrm{Flag}}^m_{n+2g-2}$$

that sends a collection $(0 \to \Omega \to M \to \mathcal{A} \to 0, D \in X^{(m)})$ to $((0 \to \Omega \to M(D) \to M(D)/\Omega \to 0) \in \overline{\mathrm{Flag}}_{n+2g-2}, (0 \to \Omega(D)/\Omega \to M(D)/\Omega \to \mathcal{A}(D) \to 0) \in \mathrm{Sh}^{n;m}_1)$.

Denote by ${}^m\mathcal{K}^{n+2g-2}_E$ the restriction of $\mathcal{K}^{n+2g-2}_E$ to $\overline{\mathrm{Flag}}^m_{n+2g-2}$.

Our next result will be a corolary of our considerations in sect.4.1.2. (We would like to underline that in the following assertion E_0 is not supposed irreducible).

Proposition 7. *Let $A \in \mathcal{C}$, E be a smooth A-sheaf of rank 2 on X. Then for $m \geq 0$, $n \geq 2m$ there is a canonical isomorphism*

$${}^m\mathcal{K}^{n+2g-2}_E \widetilde{\to} ({}^0\mathcal{K}^{n-2m+2g-2}_E) \boxtimes (\wedge^2 E)^{(m)}[2m]$$

over $\mathcal{W}^{n-2m} \times X^{(m)}$. Besides, ${}^m\mathcal{K}^{n+2g-2}_E = 0$ for $n < 2m$.

The proof will be given in sect. 4.2.

For $m \geq 0$ we again denote by $m_X : \mathrm{Bun}^n_2 \times X^{(m)} \to \mathrm{Bun}^{n+2m}_2$ the smooth map that sends $(L, D \in X^{(m)})$ to $L(D)$. Similarly, we have the map $\bar{m}_X : \overline{\mathrm{Bun}}^n_2 \times X^{(m)} \to \overline{\mathrm{Bun}}^{n+2m}_2$. Define $p_n : \mathcal{W}^{n-2g+2} \to \overline{\mathrm{Bun}}^n_2$ as the composition $\mathcal{W}^{n-2g+2} \hookrightarrow \overline{\mathrm{Flag}}_n \to \mathrm{Bun}^n_2 \to \overline{\mathrm{Bun}}^n_2$.

Corolary 2. *Suppose that E_0 is irreducible. Let $n > 6g-4, m \geq 0$. Then over ${}_{2g}\overline{\mathrm{Bun}}^n_2 \times X^{(m)}$ we have canonically*

$$\bar{m}^*_X(\bar{\mathcal{S}}^{n+2m}_E) \widetilde{\to} \bar{\mathcal{S}}^n_E \boxtimes (\wedge^2 E)^{(m)}$$

Proof 1) Since $\wedge^2 E$ is of rank 1, one easily reduces this assertion to the case $m = 1$.
2) Let ${}_{2g}\mathcal{W}^{n-2g+2}$ be the preimage of ${}_{2g}\overline{\mathrm{Bun}}^n_2$ under p_n. Denote the restriction $p_n : {}_{2g}\mathcal{W}^{n-2g+2} \to {}_{2g}\overline{\mathrm{Bun}}^n_2$ by the same symbol. We have a commutative diagram

$$\begin{array}{ccc} {}_{2g}\mathcal{W}^{n-2g+2} \times X & \widetilde{\to} & {}_{2g+1}\overline{\mathrm{Flag}}^1_{n+2} \\ \downarrow {\scriptstyle p_n \times \mathrm{id}} & & \downarrow \\ {}_{2g}\overline{\mathrm{Bun}}^n_2 \times X & \stackrel{\bar{m}_X}{\to} & {}_{2g+1}\overline{\mathrm{Bun}}^{n+2}_2 \end{array}$$

Combining Proposition 7 with Proposition 5, we get a canonical isomorphism

$$(p_n \times \mathrm{id})^* \bar{m}^*_X \bar{\mathcal{S}}^{n+2}_E \stackrel{\gamma}{\to} (p_n \times \mathrm{id})^*(\bar{\mathcal{S}}^n_E \boxtimes \wedge^2 E)$$

Since $p_n : {}_{2g}\mathcal{W}^{n-2g+2} \to {}_{2g}\overline{\mathrm{Bun}}_2^n$ is smooth and surjective with connected fibres, there is a unique isomorphism $\bar{m}_X^*(\bar{\mathcal{S}}_E^{n+2}) \widetilde{\to} \bar{\mathcal{S}}_E^n \boxtimes \wedge^2 E$ that induces γ. □

Proof of Theorem 1
For $n > 6g-4$ define the perverse A-sheaf $\overline{\mathrm{Aut}}_E^n$ as the Goresky-MacPherson extension of $\bar{\mathcal{S}}_E^n$ to $\overline{\mathrm{Bun}}_2^n$. Since $\bar{m}_X$ is smooth, from Corolary 2 we conclude that for $n > 6g-4$ and $m \geq 0$ we have $\bar{m}_X^*(\overline{\mathrm{Aut}}_E^{n+2m}) \widetilde{\to} \overline{\mathrm{Aut}}_E^n \boxtimes (\wedge^2 E)^{(m)}$ over $\overline{\mathrm{Bun}}_2^n \times X^{(m)}$.

For any n pick a divisor $D : \mathrm{Spec}\, k \to X^{(m)}$ of degree m such that $n+2m > 6g-4$. We define $\overline{\mathrm{Aut}}_E^n$ as

$$\overline{\mathrm{Aut}}_E^n = \bar{m}_D^*(\overline{\mathrm{Aut}}_E^{n+2m}) \otimes (\wedge^2 E^*)_D^{(m)},$$

where $\bar{m}_D$ is the composition $\overline{\mathrm{Bun}}_2^n \stackrel{\mathrm{id}\times D}{\to} \overline{\mathrm{Bun}}_2^n \times X^{(m)} \stackrel{\bar{m}_X}{\to} \overline{\mathrm{Bun}}_2^{n+2m}$. By Corolary 2, this definition is independent of D. Moreover, the property (2) is automatically satisfied.

To show that $\overline{\mathrm{Aut}}_E^n$ is A-flat it is enough to prove the flatness over ${}_c\overline{\mathrm{Bun}}_2^n$ for each $c \in \mathbb{Z}$. Given $n, c \in \mathbb{Z}$ pick again $D : \mathrm{Spec}\, k \to X^{(m)}$ such that $n + 2m > 6g-4$ and $c + m > 2g$. We have $\bar{m}_D({}_c\overline{\mathrm{Bun}}_2^n) \subset {}_{2g}\overline{\mathrm{Bun}}_2^{n+2m}$, so that the A-flatness of $\overline{\mathrm{Aut}}_E^n$ over ${}_c\overline{\mathrm{Bun}}_2^n$ follows from the A-flatness of $\bar{\mathcal{S}}_E^n$ over ${}_{2g}\overline{\mathrm{Bun}}_2^{n+2m}$.

Given a morphism $A \to B$ in $\mathcal{C}$, the isomorphism $\overline{\mathrm{Aut}}_E^n \otimes_A B \widetilde{\to} \overline{\mathrm{Aut}}_{E\otimes_A B}^n$ is obtained by the same argument from the analogous isomorphism for $\bar{\mathcal{S}}_E^n$. □

1.4.4

In this subsection we prove Proposition 3 and Proposition 4.

Let $A \in \mathcal{C}$, E be a smooth A-sheaf of rank 2 on X. In this subsection the local system $E_0 = E \otimes_A \bar{\mathbb{Q}}_\ell$ is assumed irreducible.

The proof is based on the following result of D.Gaitsgory, which is a strengthened version of Proposition 6.

Proposition 8. *For any $m \geq 0$ there exists a constant $const_4(m)$ such that for $n > const_4(m)$ the canonical maps*

$$j_{1!} \circ \pi_0'^*(\mathcal{L}_E^n) \to j_{1!*} \circ \pi_0'^*(\mathcal{L}_E^n) \to j_{1*} \circ \pi_0'^*(\mathcal{L}_E^n)$$

are ismorphisms over $\mathrm{Sh}_1'^{n;\leq m}$.

Proof For $A = \bar{\mathbb{Q}}_\ell$ this is (footnote 4,[2]). The general case is reduced to $A = \bar{\mathbb{Q}}_\ell$ as in Proposition 6. □

Corolary 3. *Let n satisfy: $n > 6g-4$ and $n-2g+2 > const_4(m)$. Then we have canonically*

$$^0\pi_2^*\bar{\pi}^*(\overline{\mathrm{Aut}}_E^n)[n-6g+5](-2g+2)\widetilde{\to}\mathcal{K}_E^n$$

over ${}_{2g-2}\overline{\mathrm{Flag}}_n^{\le m}$.

Proof By Proposition 8, under our assumptions $\mathcal{K}_E^n$ is a perverse A-sheaf over $\overline{\mathrm{Flag}}_n^{\le m}$ and is the Goresky-MacPherson extension of its restriction to any nonempty open substack of $\overline{\mathrm{Flag}}_n^{\le m}$. Since ${}_{2g-2}\overline{\mathrm{Flag}}_n^{\le m} \to {}_{2g-2}\,\overline{\mathrm{Bun}}_2^n$ is smooth, our assertion follows from (1) of Theorem 1. □

For any n the natural map $\mathcal{W}^n \to \mathrm{Pic}^n X$ (obtained by restricting π_1') is a generalized vector fibration. Denote by $i_0 : \mathrm{Pic}^n X \to \mathcal{W}^n$ its zero section.

The proof of the following result was communicated to the author by V.Drinfeld (and is due to D.Gaitsgory).

Lemma 14. *The irreducibility of E_0 implies that $i_0^*({}^0\mathcal{K}_E^{n+2g-2}) = 0$ for $n > 4g-4$.*

Proof Denote by $\mathrm{Pic}'^n X$ the preimage of $\mathrm{Pic}^n X$ under $\pi_1 : \mathrm{Sh}_1'^n \to \mathrm{Sh}_1^n$. Consider the morphisms $X^{(n)} \overset{\tilde{j}_1}{\to} \mathrm{Pic}'^n X \overset{\tilde{\pi}_1}{\to} \mathrm{Pic}^n X$ obtained from ${}^0\,\mathrm{Sh}_1'^n \overset{j_1}{\to} \mathrm{Sh}_1'^n \overset{\pi_1}{\to} \mathrm{Sh}_1^n$ by the base change $\mathrm{Pic}^n X \hookrightarrow \mathrm{Sh}_1^n$. By Deligne's theorem, for $n > 4g-4$ the direct image of $E^{(n)}$ under the Abel-Jacoby map $X^{(n)}/\mathbb{G}_m \to \mathrm{Pic}^n X$ vanishes. Now from (Lemma 16,[2]) it follows that for $n > 4g-4$ we have $(\tilde{\pi}_1 \circ \tilde{j}_1)_! E^{(n)} = 0$. Applying now (Theorem 1.2.2.4,[5]) we get the desired assertion. □

Proof of Proposition 4
Pick $const_2$ such that $const_2 \ge 2g-2$ and $const_2 + 6g - 4 \ge const_4(2g-1)$. Let $\mathcal{A}, \mathcal{B}$ be invertible sheaves on X with $\deg \mathcal{A} - \deg \mathcal{B} > const_2$ and $\deg \mathcal{A} + \deg \mathcal{B} = n$. By (2) of Theorem 1, we may assume $\mathcal{B} = \Omega(D)$, where D is an effective divisor on X of degree $2g-1$. Consider the k-point $(\Omega \overset{(i,0)}{\hookrightarrow} \Omega(D) \oplus \mathcal{A})$ of ${}_{2g-2}\overline{\mathrm{Flag}}_n^{\le 2g-1}$, where $i : \Omega \hookrightarrow \Omega(D)$ is the canonical inclusion. By Corolary 3, it is enough to show that the fibre of $\mathcal{K}_E^n$ at this point vanishes. By Proposition 7, we can replace $(\Omega \overset{(i,0)}{\hookrightarrow} \Omega(D) \oplus \mathcal{A})$ by $(\Omega \overset{(\mathrm{id},0)}{\hookrightarrow} \Omega \oplus \mathcal{A}(-D))$. Now our assertion follows from Lemma 14. □

Proof of Propsition 3
Step 1. Pick $const_1(0)$ such that $const_1(0) \ge 4g-5, const_1(0) \ge 2g-2, const_1(0) \ge -2g + const_4(2g-1)$. If $n > const_1(0)$ then $\overline{\mathrm{Flag}}_{n+2(2g-1)}^{2g-1} \subset {}_{2g-2}\overline{\mathrm{Flag}}_{n+2(2g-1)}^{\le 2g-1}$ and, by Corolary 3, we have an isomorphism

$$^0\pi_2^*\bar{\pi}^*(\overline{\mathrm{Aut}}_E^{n+2(2g-1)})[n-2g+3](-2g+2)\widetilde{\to}\mathcal{K}_E^{n+2(2g-1)}$$

over $\overline{\mathrm{Flag}}^{2g-1}_{n+2(2g-1)}$. Now using the commutative diagram

$$\begin{array}{ccc} \mathcal{W}^{n-2g+2}\times X^{(2g-1)} & \widetilde{\to} & \overline{\mathrm{Flag}}^{2g-1}_{n+2(2g-1)} \\ \downarrow p_n\times\mathrm{id} & & \downarrow \\ \overline{\mathrm{Bun}}^n_2\times X^{(2g-1)} & \stackrel{\bar{m}_X}{\to} & \overline{\mathrm{Bun}}^{n+2(2g-1)}_2 \end{array}$$

and Proposition 7, we get the isomorphism $p_n^*(\overline{\mathrm{Aut}}^n_2)[n-6g+5](-2g+2)\widetilde{\to}{}^0\mathcal{K}^n_E$. So, for $m=0$ our assertion is proved.

Step 2. We claim that if $n-2m>4g-4+const_3$ and $n-2m>const_1(0)$ then for any $0\le t\le m$ the complex ${}^t\mathcal{K}^n_E$ is the extension by zero of its restriction to ${}_{2g-2}\overline{\mathrm{Flag}}^t_n$. Indeed, by Proposition 7, it is enough to show that the complex of ${}^0\mathcal{K}^{n-2t}_E$ is the extension by zero of its resriction to ${}_{2g-2}\overline{\mathrm{Flag}}^0_{n-2t}$. By the assertion of Step 1 and Corolary 1, this is the case.

Step 3. Pick $const_1(m)$ such that $const_1(m)\ge 4g-4+2m+const_3$, $const_1(m)\ge 6g-4$, $const_1(m)\ge 2g-2+const_4(m)$, $const_1(m)\ge 2m+const_1(0)$. Let $n>const_1(m)$. By Corolary 1, $\overline{\mathrm{Aut}}^n_E$ is the extension by zero of its restriction to ${}_{2g-2}\overline{\mathrm{Bun}}^n_2$. By Corolary 3, we have the desired isomorphism over ${}_{2g-2}\overline{\mathrm{Flag}}^{\le m}_n$. Now our assertion follows from Step 2.

□(Proposition 3)

Remark 10. Using Proposition 8 and Lemma 14, it is not difficult to prove the cuspidality of Aut_E, where $\mathrm{Aut}_E=\bar{\pi}^*\overline{\mathrm{Aut}}_E[-1]$. We will not need this fact.

2 Orthogonality relations between automorphic sheaves attached to 2-dimensional irreducible local systems

2.1 Cohomology of ${}^x(\mathcal{H}om(E_1, E_2))^{(n)}$

In this subsection we make the assumptions and use the notations of sect.1.1 (with $E_\lambda = \bar{\mathbb{Q}}_\ell$). So, E_0 is a local system on X of rank m such that $\operatorname{End}(E_0) = \bar{\mathbb{Q}}_\ell$, E is its universal deformation and R is the base of this universal deformation. Recall that $E_i = E \otimes_R (R \hat{\otimes} R)$ $(i = 1, 2)$, where the R-module structure on $R \hat{\otimes} R$ is given by $p_i : R \to R \hat{\otimes} R$. So, $(\mathcal{H}om(E_1, E_2))^{(n)}$ is an $R \hat{\otimes} R$-sheaf on $X^{(n)}$.

Chose a closed point $x \in X$. Recall that it defines the $\mathbb{G}_m$-torsor $\alpha_x : \underline{\operatorname{Pic}}^n X \to \operatorname{Pic}^n X$ (cf. sect. 1.2.1). Define the scheme ${}^x X^{(n)}$ from the cartesian square

$$\begin{array}{ccc} {}^x X^{(n)} & \stackrel{\alpha'_x}{\to} & X^{(n)} \\ \downarrow & & \downarrow \pi \\ \underline{\operatorname{Pic}}^n X & \stackrel{\alpha_x}{\to} & \operatorname{Pic}^n X \end{array}$$

Denote by ${}^x(\mathcal{H}om(E_1, E_2))^{(n)}$ the inverse image of $(\mathcal{H}om(E_1, E_2))^{(n)}$ to ${}^x X^{(n)}$.

The purpose of this subsection is to present a proof of the following result.

Proposition 9. *If $n > 0$ then we have*

$$\mathrm{H}_c^{2n+2}({}^x X^{(n)}, {}^x(\mathcal{H}om(E_1, E_2))^{(n)}) \widetilde{\to} R(-n-1)$$

canonically, where the $R \hat{\otimes} R$-module structure on R is given via the diagonal mapping $R \hat{\otimes} R \to R$. Besides,

$$\mathrm{H}_c^{2n+2-i}({}^x X^{(n)}, {}^x(\mathcal{H}om(E_1, E_2))^{(n)}) = 0$$

for $0 < i < n$.

This will be done using the results of sect. 1.1 and of Appendices B and C.

2.1.1

To prove Proposition 9 we need the following linear algebra lemma.

Let A be a (commutative) ring of characteristic 0. Consider a complex of A-modules $M = (A \to M^{-1} \stackrel{d}{\to} A)$, where M^{-1} is a free A-module of rank r. (So,

$M^{-2} = M^0 = A$ and $M^i = 0$ for $i > 0$ and $i < -2$). Suppose that there exists a basis $e_1, \dots, e_r \in M^{-1}$ such that $d(e_1), \dots, d(e_r)$ is a regular sequence for A. Put $I = \operatorname{Im} d$. Let $\varphi_1 : M \to M[2]$ be a morphism of complexes such that the induced map $M^{-2} \to M^0$ is an isomorphism. Define the morphism $\varphi : \bigotimes_{i=1}^{n} M \to (\bigotimes_{i=1}^{n} M)[2]$ as $\varphi_1 \otimes \operatorname{id} \otimes \dots \otimes \operatorname{id} + \dots + \operatorname{id} \otimes \dots \otimes \operatorname{id} \otimes \varphi_1$. Then there is a (unique) morphism $\varphi_n : \operatorname{Sym}^n(M) \to \operatorname{Sym}^n(M)[2]$ such that the diagram commutes

$$\begin{array}{ccc} \bigotimes_{i=1}^{n} M & \stackrel{\varphi}{\to} & \bigotimes_{i=1}^{n} M[2] \\ \cup & & \cup \\ \operatorname{Sym}^n(M) & \stackrel{\varphi_n}{\to} & \operatorname{Sym}^n(M)[2] \end{array}$$

Lemma B.1. *Define the object $K \in \mathrm{D}_{\mathbf{parf}}(A)$ from the distinguished triangle $K \to \operatorname{Sym}^n(M) \stackrel{\varphi_n}{\to} \operatorname{Sym}^n(M)[2]$. Then $\mathrm{H}^0(K) \widetilde{\to} A/I$ and $\mathrm{H}^i(K) = 0$ for $-n < i < 0$ and for $i > 0$.*

The proof is given in Appendix B.

2.1.2

Denote by $Y_x : X^{(n-1)} \hookrightarrow X^{(n)}$ the closed immersion that sends D to $D + x$. We consider Y_x as a divisor on $X^{(n)}$ and write sometimes $Y_x \hookrightarrow X^{(n)}$ for the same closed subscheme. Denote by $'Y_x$ the inverse image of Y_x under $\operatorname{sym} : X^n \to X^{(n)}$. (In other words, the closed immersion $'Y_x \hookrightarrow X^n$ is obtained from $Y_x \hookrightarrow X^{(n)}$ by the base change $\operatorname{sym} : X^n \to X^{(n)}$). Denote by $'Y_x^i$ the inverse image of x under $\operatorname{pr}_i : X^n \to X$. So, $'Y_x^i$ and $'Y_x$ are divisors on X^n, and we have $'Y_x = {'Y_x^1} + \dots + {'Y_x^n}$.

Consider the invertible sheaf $\mathcal{O}(Y_x)$ on $X^{(n)}$.

Lemma 15. *${}^xX^{(n)}$ is naturally isomorphic to the total space of $\mathcal{O}(Y_x)$ with removed zero section.*

Proof Denote by $Y^{univ} \hookrightarrow X^{(n)} \times X$ the universal divisor. Clearly, the inverse image of Y^{univ} under the closed immersion $X^{(n)} \times x \hookrightarrow X^{(n)} \times X$ is the divisor $Y_x \times x$ on $X^{(n)} \times x$ with some multiplicity $r > 0$. It is enough to show that $r = 1$, i.e., to show that the following square is cartesian

$$\begin{array}{ccc} Y^{univ} & \hookrightarrow & X^{(n)} \times X \\ \uparrow & & \uparrow \\ X^{(n-1)} \times x & \stackrel{Y_x \times \mathrm{id}}{\hookrightarrow} & X^{(n)} \times x \end{array}$$

To do so, denote by $'Y^{univ}$ the inverse image of Y^{univ} under $X^n \times X \overset{\mathrm{sym}\times\mathrm{id}}{\to} X^{(n)} \times X$. It is enough to show that the inverse image of $'Y^{univ}$ under the closed immersion $X^n \times x \to X^n \times X$ is $'Y_x \times x$ with multiplicity one. But this is obvious. □

The following lemma follows immediately from the formalism of 6 functors.

Lemma 16. *Let $f : Y \to Z$ be a proper morphism of separeted schemes of finite type. Let $F \in \mathrm{D}^b_c(Y, \bar{\mathbb{Q}}_\ell), G \in \mathrm{D}^b_c(Z, \bar{\mathbb{Q}}_\ell)$. Denote by $a_Z : \mathrm{R}\Gamma_c(Z, f_! F) \overset{L}{\otimes} \mathrm{R}\Gamma_c(Z, G) \to \mathrm{R}\Gamma_c(Z, f_! F \overset{L}{\otimes} G)$ the $\smile$-product on Z and by $a_Y : \mathrm{R}\Gamma_c(Y, F) \overset{L}{\otimes} \mathrm{R}\Gamma_c(Y, f^* G) \to \mathrm{R}\Gamma_c(Y, F \overset{L}{\otimes} f^* G)$ the $\smile$-product on Y. Then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{R}\Gamma_c(Y, F) \overset{L}{\otimes} \mathrm{R}\Gamma_c(Z, G) & \overset{a_X}{\to} & \mathrm{R}\Gamma_c(Z, f_! F \overset{L}{\otimes} G) \\ \downarrow \mathrm{id}\otimes b & \nearrow a_Y & \\ \mathrm{R}\Gamma_c(Y, F) \overset{L}{\otimes} \mathrm{R}\Gamma_c(Y, f^* G), & & \end{array}$$

where $b : \mathrm{R}\Gamma_c(Z, G) \to \mathrm{R}\Gamma_c(Y, f^* G)$ is the natural morphism. □

Proof of Proposition 9

By Proposition C.1, the complex $(\alpha'_x)_! \bar{\mathbb{Q}}_\ell$ is included into a distinguished triangle $(\alpha'_x)_! \bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_\ell(-1)[-2] \overset{c}{\to} \bar{\mathbb{Q}}_\ell$ on $X^{(n)}$, where $c \in \mathrm{H}^2(X^{(n)}, \bar{\mathbb{Q}}_\ell(1))$ is the Chern class of $\mathcal{O}(Y_x)$. By the Kunneth formulae, we have

$$\mathrm{H}^2(X^n, \bar{\mathbb{Q}}_\ell) = \underset{i_1+\cdots+i_n=2}{\oplus} \mathrm{H}^{i_1}(X, \bar{\mathbb{Q}}_\ell) \otimes \cdots \otimes \mathrm{H}^{i_n}(X, \bar{\mathbb{Q}}_\ell)$$

and $\mathrm{H}^2(X^{(n)}, \bar{\mathbb{Q}}_\ell) = \mathrm{H}^2(X^n, \bar{\mathbb{Q}}_\ell)^{S_n}$. Denote by c' the image of c in $\mathrm{H}^2(X^n, \bar{\mathbb{Q}}_\ell(1))$. The construction of the Chern class is functorial, so that c' is the Chern class of $\mathcal{O}('Y_x)$. Since $'Y_x = 'Y^1_x + \ldots + 'Y^n_x$, we get

$$\begin{aligned} c' = c_1 \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes c_1 \in & \\ & \mathrm{H}^2(X, \bar{\mathbb{Q}}_\ell(1)) \otimes \mathrm{H}^0(X, \bar{\mathbb{Q}}_\ell) \otimes \cdots \otimes \mathrm{H}^0(X, \bar{\mathbb{Q}}_\ell) \oplus \ldots \\ & \oplus \mathrm{H}^0(X, \bar{\mathbb{Q}}_\ell) \otimes \cdots \otimes \mathrm{H}^0(X, \bar{\mathbb{Q}}_\ell) \otimes \mathrm{H}^2(X, \bar{\mathbb{Q}}_\ell(1)) \subset \mathrm{H}^2(X^n, \bar{\mathbb{Q}}_\ell(1)), \end{aligned}$$

where $c_1 \in \mathrm{H}^2(X, \bar{\mathbb{Q}}_\ell(1))$ is the Chern class of the invertible sheaf $\mathcal{O}(x)$ on X. Since $\deg \mathcal{O}(x) = 1$, we have $c_1 \neq 0$. On $X^{(n)}$ we get the distinguished triangle

$$(\alpha'_x)_!{}^x(\mathcal{H}om(E_1, E_2))^{(n)}(1)[2] \;\to\; (\mathcal{H}om(E_1, E_2))^{(n)} \;\overset{c}{\to}\; (\mathcal{H}om(E_1, E_2))^{(n)}(1)[2]$$

Denote by

$$\varphi_n : \mathrm{R}\Gamma(X^{(n)}, (\mathcal{H}om(E_1, E_2))^{(n)}) \to \mathrm{R}\Gamma(X^{(n)}, (\mathcal{H}om(E_1, E_2))^{(n)})(1)[2]$$

the morphism obtained from $(\mathcal{H}om(E_1, E_2))^{(n)} \xrightarrow{c} (\mathcal{H}om(E_1, E_2))^{(n)}(1)[2]$ by applying the functor $\mathrm{R}\Gamma(X^{(n)}, \ . \ )$. We get the distinguished triangle in $\mathrm{D}_{\mathrm{parf}}(R \hat{\otimes} R)$

$$\mathrm{R}\Gamma_c({}^x X^{(n)}, {}^x(\mathcal{H}om(E_1, E_2))^{(n)})(1)[2] \to \mathrm{R}\Gamma(X^{(n)}, (\mathcal{H}om(E_1, E_2))^{(n)}) \xrightarrow{\varphi_n} \mathrm{R}\Gamma(X^{(n)}, (\mathcal{H}om(E_1, E_2))^{(n)})(1)[2]$$

Since $\mathrm{sym}_!(\mathcal{H}om(E_1, E_2))^{\boxtimes n}$ is a direct sum over the irreducible representations of S_n, the same holds for $\mathrm{R}\Gamma(X^n, (\mathcal{H}om(E_1, E_2))^{\boxtimes n}) = \bigotimes_{i=1}^{n} \mathrm{R}\Gamma(X, \mathcal{H}om(E_1, E_2))$, and we have naturally

$$\mathrm{R}\Gamma(X^{(n)}, (\mathcal{H}om(E_1, E_2))^{(n)}) \widetilde{\to} (\bigotimes_{i=1}^{n} \mathrm{R}\Gamma(X, \mathcal{H}om(E_1, E_2))\ )^{S_n}$$

Denote also by $\varphi : \bigotimes_{i=1}^{n} \mathrm{R}\Gamma(X, \mathcal{H}om(E_1, E_2)) \to \bigotimes_{i=1}^{n} \mathrm{R}\Gamma(X, \mathcal{H}om(E_1, E_2))(1)[2]$ the morphism obtained from $\mathrm{sym}_!(\mathcal{H}om(E_1, E_2))^{\boxtimes n} \xrightarrow{c} \mathrm{sym}_!(\mathcal{H}om(E_1, E_2))^{\boxtimes n}(1)[2]$ by applying the functor $\mathrm{R}\Gamma(X^{(n)}, \ . \ )$.

The morphism φ is a $\smile$-product by an element $c \in \mathrm{H}^2(X^{(n)}, \bar{\mathbb{Q}}_\ell(1))$. By Lemma 16, we can replace the $\smile$-product on $X^{(n)}$ by that on X^n. It follows that

$$\varphi = \varphi_1 \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \ldots + \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \varphi_1$$

Pick a perfect complex M of $R \hat{\otimes} R$-modules that represents $\mathrm{R}\Gamma(X, \mathcal{H}om(E_1, E_2))$. We suppose that M is chosen as in 2) of Proposition 2. Pick a morphism $\tilde{\varphi}_1 : M \to M(1)[2]$ that represents φ_1 in $\mathrm{D}_{parf}(R \hat{\otimes} R)$ (so, $\tilde{\varphi}_1$ is defined up to a homotopy). Since $c_1 \neq 0$, it follows that $\tilde{\varphi}_1$ is given by the diagram

$$\begin{array}{ccccc} & & M^0 & \to M^1 \to & M^2 \\ & & \downarrow & & \\ M^0(1) \to & M^1(1) \to & M^2(1), & & \end{array}$$

where the vertical arrow is an isomorphism of $R \hat{\otimes} R$-modules.

Notice that M^1 is a free $R \hat{\otimes} R$-module of rank $\dim R$, and the ideal of the diagonal in $R \hat{\otimes} R$ is generated by a regular sequence of $\dim R$ elements. Now combining 3) of Proposition 2 with Lemma B.1 we get the desired assertion.
□(Proposition 9)

2.2 Proof of Main Global Theorem

2.2.1

Denote by $f : \overline{\mathrm{Flag}}_{n+2g-2} \to \operatorname{Pic}^n X$ the morphism that sends $(\Omega \hookrightarrow L)$ to $(\det L) \otimes \Omega^{-1}$. Recall that $\pi : X^{(n)} \to \operatorname{Pic}^n X$ sends a divisor D to $\mathcal{O}(D)$.

The following assertion, which is a corolary of Main Local Theorem, will be a key point in our proof of Main Global Theorem.

Theorem 2. *Let $A \in \mathcal{C}$, E_1 and E_2 be smooth A-sheaves on X of rank 2. For any $n \geq 0$ there is a canonical isomorphism*

$$f_!(\mathcal{K}_{E_1}^{n+2g-2} \overset{L}{\otimes}_A \mathcal{K}_{E_2}^{n+2g-2}) \widetilde{\to} \pi_!((E_1 \otimes_A E_2)^{(n)})[2n]$$

Remark 11. The morphism f is not of finite type. However, by Proposition 7, the complex $\mathcal{K}_E^{n+2g-2}$ is the extension by zero of its restriction to the open substack $\overline{\mathrm{Flag}}_{n+2g-2}^{\leq[\frac{n}{2}]}$ of $\overline{\mathrm{Flag}}_{n+g-2}$ and the composition $\overline{\mathrm{Flag}}_{n+2g-2}^{\leq[\frac{n}{2}]} \hookrightarrow \overline{\mathrm{Flag}}_{n+g-2} \overset{f}{\to} \operatorname{Pic}^n X$ is of finite type.

Proof of Theorem 2

For the pojection $\mathrm{pr}_3 : {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to \overline{\mathrm{Flag}}_{n+2g-2}$ we have

$$\mathcal{K}_{E_1}^{n+2g-2} \overset{L}{\otimes}_A \mathcal{K}_{E_2}^{n+2g-2}[-4n-2+2g](-n-1+g) \widetilde{\to}$$
$$\mathrm{pr}_{3!}(\mathrm{pr}_1^* \pi_0'^* \mathcal{L}_{E_1}^n \otimes_A \mathrm{pr}_2^* \pi_0'^* \mathcal{L}_{E_2}^n \otimes_{\bar{\mathbb{Q}}_\ell} \mathrm{pr}_{13}^* \mu^* \mathcal{L}_\psi \otimes_{\bar{\mathbb{Q}}_\ell} \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi)$$

The following diagram commutes

$$\begin{array}{ccc} {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} & \overset{\mathrm{pr}_3}{\to} & \overline{\mathrm{Flag}}_{n+2g-2} \\ \downarrow \varphi_n & & \downarrow f \\ X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)} & \to & \operatorname{Pic}^n X \end{array} \qquad (4)$$

and the composition $X^{(n)} \overset{i}{\hookrightarrow} X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)} \to \operatorname{Pic}^n X$ coincides with π. Therefore, our assertion follows from Main Local Theorem.

□(Theorem 2)

Remark 12. Since $(\varphi_n)_!$ can be defined as in (Remark 2, sect. 0.1.2), the diagram (4) also shows that the complex $f_!(\mathcal{K}_{E_1}^{n+2g-2} \overset{L}{\otimes} \mathcal{K}_{E_2}^{n+2g-2})$ can be defined without using Appendix A. (In particular, this complex is bounded).

2.2.2

Fix a closed point $x \in X$ and define the stack ${}^x\operatorname{Bun}_2^n$ from the cartesian square

$$\begin{array}{ccc} {}^x\operatorname{Bun}_2^n & \to & \operatorname{Bun}_2^n \\ \downarrow & & \downarrow \\ \underline{\operatorname{Pic}}^{n-2g+2} X & \stackrel{\alpha_x}{\to} & \operatorname{Pic}^{n-2g+2} X, \end{array}$$

where $\operatorname{Bun}_2^n \to \operatorname{Pic}^{n-2g+2} X$ is the composition $\operatorname{Bun}_2^n \stackrel{\det}{\to} \operatorname{Pic}^n X \stackrel{\otimes\Omega^{-1}}{\to} \operatorname{Pic}^{n-2g+2} X$. Then the composition ${}^x\operatorname{Bun}_2^n \to \operatorname{Bun}_2^n \stackrel{\bar\pi}{\to} \overline{\operatorname{Bun}}_2^n$ is a μ_2-gerb. Define also the stack ${}^x\overline{\operatorname{Flag}}_n$ from the cartesian square

$$\begin{array}{ccc} {}^x\overline{\operatorname{Flag}}_n & \to & \overline{\operatorname{Flag}}_n \\ \downarrow & & \downarrow \\ {}^x\operatorname{Bun}_2^n & \to & \operatorname{Bun}_2^n, \end{array}$$

where the right vertical arrow is the restriction of ${}^0\pi_2$. For $m \geq 0$ we denote by ${}^x\overline{\operatorname{Flag}}_n^{\leq m}$ the preimage of $\overline{\operatorname{Flag}}_n^{\leq m}$ under ${}^x\overline{\operatorname{Flag}}_n \to \overline{\operatorname{Flag}}_n$. Let $i_m : \overline{\operatorname{Flag}}_n^{\geq m} \hookrightarrow \overline{\operatorname{Flag}}_n$ be the complement of $\overline{\operatorname{Flag}}_n^{\leq m-1}$ to $\overline{\operatorname{Flag}}_n$. Let ${}^x i_m : {}^x\overline{\operatorname{Flag}}_n^{\geq m} \hookrightarrow {}^x\overline{\operatorname{Flag}}_n$ be the complement of ${}^x\overline{\operatorname{Flag}}_n^{\leq m-1}$ to ${}^x\overline{\operatorname{Flag}}_n$.

Denote by ${}^x\mathcal{K}_E^n$ the inverse image of $\mathcal{K}_E^n[1]$ under ${}^x\overline{\operatorname{Flag}}_n \to \overline{\operatorname{Flag}}_n$.

Lemma 17. *Let $A \in \mathcal{C}$, E_1 and E_2 be smooth A-sheaves on X of rank 2. If $n \geq 0$ and $m \geq 0$ then the following three complexes are placed in (usual) degrees $\leq -2m$*

1) $\mathrm{R}\Gamma_c(\overline{\operatorname{Flag}}^m_{n+2g-2}, {}^m\mathcal{K}_{E_1}^{n+2g-2} \overset{L}{\otimes} {}^m\mathcal{K}_{E_2}^{n+2g-2})$

2) $\mathrm{R}\Gamma_c(\overline{\operatorname{Flag}}^{\geq m}_{n+2g-2}, i_m^*(\mathcal{K}_{E_1}^{n+2g-2} \overset{L}{\otimes} \mathcal{K}_{E_2}^{n+2g-2}))$

3) $\mathrm{R}\Gamma_c({}^x\overline{\operatorname{Flag}}^{\geq m}_{n+2g-2}, {}^x i_m^*({}^x\mathcal{K}_{E_1}^{n+2g-2} \overset{L}{\otimes} {}^x\mathcal{K}_{E_2}^{n+2g-2}))$

Proof 1) First, we show this for $m = 0$. For the projection $\operatorname{pr}_2 : X^{(n)} \times_{\operatorname{Pic}^n X} \mathcal{W}^n \to \mathcal{W}^n$ we have

$$\operatorname{pr}_{2!}(\operatorname{pr}_1^* E^{(n)} \otimes \mu^*\mathcal{L}_\psi)[2n+1-g](\frac{n+1-g}{2}) \widetilde{\to} {}^0\mathcal{K}_E^{n+2g-2},$$

where $\mu : X^{(n)} \times_{\operatorname{Pic}^n X} \mathcal{W}^n \to \mathbb{A}^1_k$ is the natural pairing. Since $\dim(X^{(n)} \times_{\operatorname{Pic}^n X} \mathcal{W}^n) = 2n+1-g$, our assertion follows.

The case of an arbitrary m is reduced to $m = 0$ as follows. By Proposition 7, if $n < 2m$ then our complex vanishes otherwise it is isomorphic to

$$\mathrm{R}\Gamma_c(\overline{\operatorname{Flag}}^0_{n-2m+2g-2}, {}^0\mathcal{K}_{E_1}^{n-2m+2g-2} \overset{L}{\otimes} {}^0\mathcal{K}_{E_2}^{n-2m+2g-2}) \overset{L}{\otimes} \mathrm{R}\Gamma(X^{(m)}, (\det E_1 \otimes \det E_2)^{(m)})[4m]$$

Since $\mathrm{R}\Gamma(X^{(m)}, (\det E_1 \otimes \det E_2)^{(m)})$ is placed in degrees $\le 2m$, we are reduced to the case $m = 0$.

2)Using the fact that ${}^m\mathcal{K}_E^{n+2g-2} = 0$ for $n < 2m$ and using the stratification of $\overline{\mathrm{Flag}}_{n+2g-2}$ by the locally closed substacks $\overline{\mathrm{Flag}}^m_{n+2g-2}$, $m \ge 0$, one reduces the desired assertion to the point 1).

3) follows immediately from 2). □

2.2.3

In this subsection we suppose that E_0 is an irreducible local system on X of rank 2, E is its universal deformation, and R is the base of this universal deformation. Let E_1, E_2 be the smooth $R\hat{\otimes} R$-sheaves on X defined as in section 1.1.2.

Proposition 10. *Let $m \ge 0, n > 2m$. Then we have*

$$\mathrm{H}^0_c({}^x\overline{\mathrm{Flag}}^{\le m}_{n+2g-2}, {}^x\mathcal{K}^{n+2g-2}_{E_1^*} \overset{L}{\otimes} {}^x\mathcal{K}^{n+2g-2}_{E_2}) \widetilde{\to} R(-n-1)$$

canonically. Besides, $\mathrm{H}^i_c({}^x\overline{\mathrm{Flag}}^{\le m}_{n+2g-2}, {}^x\mathcal{K}^{n+2g-2}_{E_1^*} \overset{L}{\otimes} {}^x\mathcal{K}^{n+2g-2}_{E_2}) = 0$ for $-2m \le i < 0$ and $i > 0$.

Proof Consider the diagram consisting of cartesian squares

$$\begin{array}{ccccc} X^{(n)} & \overset{\pi}{\to} & \mathrm{Pic}^n X & \overset{f}{\leftarrow} & \overline{\mathrm{Flag}}_{n+2g-2} \\ \uparrow & & \uparrow \alpha_x & & \uparrow \\ {}^xX^{(n)} & \to & \underline{\mathrm{Pic}}^n X & \leftarrow & {}^x\overline{\mathrm{Flag}}_{n+2g-2} \end{array}$$

By Theorem 2, we have canonically

$$\mathrm{R}\Gamma_c({}^xX^{(n)}, {}^x(\mathcal{H}om(E_1, E_2))^{(n)})[2n+2] \widetilde{\to} \mathrm{R}\Gamma_c({}^x\overline{\mathrm{Flag}}_{n+2g-2}, {}^x\mathcal{K}^{n+2g-2}_{E_1^*} \overset{L}{\otimes} {}^x\mathcal{K}^{n+2g-2}_{E_2})$$

So, our assertion follows from Proposition 9 and 3) of Lemma 17. □

Denote by $h : {}^x\overline{\mathrm{Flag}}_n \to \overline{\mathrm{Bun}}^n_2$ the composition ${}^x\overline{\mathrm{Flag}}_n \to \overline{\mathrm{Flag}}_n \to \mathrm{Bun}^n_2 \overset{\bar{\pi}}{\to} \overline{\mathrm{Bun}}^n_2$. Notice that h is representable. If L is a rank 2 vector bundle on X of degree n then the fibre of h at the k-point $\mathrm{Spec}\, k \overset{L}{\to} \mathrm{Bun}^n_2 \to \overline{\mathrm{Bun}}^n_2$ of $\overline{\mathrm{Bun}}^n_2$ is identified with $(\mathrm{Hom}(\Omega, L) \setminus \{0\})/\mu_2$.

Let $h^{\le m} : {}^x\overline{\mathrm{Flag}}^{\le m}_n \to \overline{\mathrm{Bun}}^n_2$ be the restriction of h to ${}^x\overline{\mathrm{Flag}}^{\le m}_n$. Our next result follows easily from Lemma 12.

Lemma 18. *For any $m \geq 0$ over ${}_{m+2g-1}\overline{\mathrm{Bun}}_2^n$ we have*
1) $(\mathrm{R}^{2n-12g+12}(h^{\leq m})_!)\bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell(-n+6g-6)$ *canonically*
2) $(\mathrm{R}^{2n-12g+12-i}(h^{\leq m})_!)\bar{\mathbb{Q}}_\ell = 0$ *for* $0 < i \leq 2m$. □

Proposition 11. *There exists a constant $const_5 \geq 0$ with the following property. If $n - 4g + 2 > 2m + const_3, 2m \geq const_5$, and $n > const_1(m)$ then*

$$\mathrm{H}^0_c(\overline{\mathrm{Bun}}_2^n, \overline{\mathrm{Aut}}^n_{E_1^*} \overset{L}{\otimes} \overline{\mathrm{Aut}}^n_{E_2}) \widetilde{\to} R(-1)$$

canonically and $\mathrm{H}^i_c(\overline{\mathrm{Bun}}_2^n, \overline{\mathrm{Aut}}^n_{E_1^*} \overset{L}{\otimes} \overline{\mathrm{Aut}}^n_{E_2}) = 0$ *for* $-2m + const_5 \leq i < 0$ *and* $i > 0$.

Proof By Proposition 3, we have $(h^{\leq m})^*(\overline{\mathrm{Aut}}^n_E)[n-6g+6](-2g+2) \widetilde{\to} {}^x\mathcal{K}^n_E$ canonically over ${}^x\overline{\mathrm{Flag}}_n^{\leq m}$. By Corolary 1, $\overline{\mathrm{Aut}}^n_E$ is the extension by zero of its restriction to ${}_{m+2g-1}\overline{\mathrm{Bun}}_2^n$. Since $h^{\leq m} : {}^x\overline{\mathrm{Flag}}_n^{\leq m} \to \overline{\mathrm{Bun}}_2^n$ is smooth over ${}_{m+2g-1}\overline{\mathrm{Bun}}_2^n$ of relative dimension $n - 6g + 6$, we have a canonical map

$$(h^{\leq m})_!({}^x\mathcal{K}^n_{E_1^*} \overset{L}{\otimes} {}^x\mathcal{K}^n_{E_2}) \to \overline{\mathrm{Aut}}^n_{E_1^*} \overset{L}{\otimes} \overline{\mathrm{Aut}}^n_{E_2}(-n+2g-2)$$

Define the complex K from the distinguished triangle

$$K \to (h^{\leq m})_!({}^x\mathcal{K}^n_{E_1^*} \overset{L}{\otimes} {}^x\mathcal{K}^n_{E_2}) \to \overline{\mathrm{Aut}}^n_{E_1^*} \overset{L}{\otimes} \overline{\mathrm{Aut}}^n_{E_2}(-n+2g-2)$$

Pick a constant $const_6$ such that for any n the perverse R-sheaf $\overline{\mathrm{Aut}}^n_E$ has cohomology sheaves (w.r.t. the usual t-structure) in degrees $\leq const_6$. (The existence of such constant follows from 2) of Theorem 1). ¿From Lemma 18 we see that K has cohomology sheaves only in (usual) degrees $\leq -2m - 1 + 2const_6$.

Since $\dim(\overline{\mathrm{Bun}}_2^n) = 4g-3$ does not depend on n, our assertion follows from Proposition 10. □

Proof of Main Global Theorem
Given a pair of integers $n, i \in \mathbb{Z}$, we calculate $\mathrm{H}^i_c(\overline{\mathrm{Bun}}_2^n, \overline{\mathrm{Aut}}^n_{E_1^*} \otimes \overline{\mathrm{Aut}}^n_{E_2})$ as follows. Pick m such that $2m \geq const_5$ and $-2m + const_5 \leq i$. Pick a divisor $D : \mathrm{Spec}\, k \to X^{(c)}$ of degree c such that $n + 2c - 4g + 2 > 2m + const_3$, $n + 2c > const_1(m)$. Denote by $\bar{m}_D$ the composition $\overline{\mathrm{Bun}}_2^n \overset{\mathrm{id}\times D}{\to} \overline{\mathrm{Bun}}_2^n \times X^{(c)} \overset{\bar{m}_X}{\to} \overline{\mathrm{Bun}}_2^{n+2c}$. By 2) of Theorem 1, we have $\bar{m}_D^*(\overline{\mathrm{Aut}}^{n+2c}_{E_1^*} \otimes \overline{\mathrm{Aut}}^{n+2c}_{E_2}) \widetilde{\to} (\overline{\mathrm{Aut}}^n_{E_1^*} \otimes \overline{\mathrm{Aut}}^n_{E_2}) \otimes (\wedge^2 E_1^*)^{(c)}_D \otimes (\wedge^2 E_2)^{(c)}_D$. Since $(\wedge^2 E_1^*)^{(c)}_D \otimes (\wedge^2 E_2)^{(c)}_D$ is a free $R \hat{\otimes} R$-module (of rank 1), we have

$$\mathrm{H}^i_c(\overline{\mathrm{Bun}}_2^{n+2c}, \overline{\mathrm{Aut}}^{n+2c}_{E_1^*} \otimes \overline{\mathrm{Aut}}^{n+2c}_{E_2}) \widetilde{\to} \mathrm{H}^i_c(\overline{\mathrm{Bun}}_2^n, \overline{\mathrm{Aut}}^n_{E_1^*} \otimes \overline{\mathrm{Aut}}^n_{E_2}) \otimes (\wedge^2 E_1^*)^{(c)}_D \otimes (\wedge^2 E_2)^{(c)}_D$$

Since $((\wedge^2 E_1^*)_D^{(c)} \otimes (\wedge^2 E_2)_D^{(c)}) \otimes_{R \hat{\otimes} R} R \widetilde{\to} R$ canonically, applying Proposition 11 one concludes the proof.
□ (Main Global Theorem)

3 Local geometrized Rankin-Selberg method for GL(2)

The purpose of this section is to prove Main Local Theorem.

All the results of section 3 (with obvious changes) are still true if one replaces $\bar{\mathbb{Q}}_\ell$ by any $A \in \mathcal{C}$. However, to underline the essential points of proofs, we work over $\bar{\mathbb{Q}}_\ell$. The reader will easily pass from $\bar{\mathbb{Q}}_\ell$ to A.

3.1 The complex $\mathcal{F}^n_{E_1,E_2}$.

3.1.1

For the projection $\mathrm{pr}_1 : \mathrm{Sh}'^n_1 \times_{\mathrm{Sh}^n_1} \overline{\mathrm{Flag}}_{n+2g-2} \to \mathrm{Sh}'^n_1$ put

$$K^n = (\mathrm{pr}_1)_! \mu^* \mathcal{L}_\psi [2n+2-2g]$$

In other words, $K^n = \mathrm{Four}(j_! \mathbb{Q}_\ell)(\frac{-n-1+g}{2})[n+1-g]$, where $j : \overline{\mathrm{Flag}}_{n+2g-2} \to {}^0\mathrm{Sh}'^{n+2g-2}_2$ is an open immersion.

Recall that the stack Sh''^n_1 was defined in sect. 1.2.3.

Lemma 19. *The complex K^n is supported at the closed substack Sh''^n_1 of Sh'^n_1.*

For $m \geq 0$ denote by $\gamma_{n,m} : (\mathrm{Pic}^{n-m} X) \times \mathrm{Sh}^m_0 \to \mathrm{Sh}^n_1$ the morphism that sends $(\mathcal{A}, F_0)$ to $(\mathcal{A} \oplus F_0)$. The square is cartesian

$$\begin{array}{ccc} \overline{\mathrm{Flag}}_{n+2g-2} & \leftarrow & \mathcal{W}^{n-m} \times X^{(m)} \\ \downarrow & & \downarrow \\ \mathrm{Sh}^n_1 & \stackrel{\gamma_{n,m}}{\leftarrow} & (\mathrm{Pic}^{n-m} X) \times \mathrm{Sh}^m_0 \end{array}$$

Proof of Lemma 19

In fact, we will show that the restriction of K^n to ${}^0\mathrm{Sh}'^n_1$ vanishes.

The following diagram consists of cartesian squares

$$\begin{array}{ccc} {}^0\mathrm{Sh}'^n_1 & \leftarrow & X^{(n-m)} \times \mathrm{Sh}'^m_0 \\ \downarrow & & \downarrow \\ \mathrm{Sh}'^n_1 & \leftarrow & (\mathrm{Pic}'^{n-m} X) \times \mathrm{Sh}'^m_0 \\ \downarrow & & \downarrow \\ \mathrm{Sh}^n_1 & \stackrel{\gamma_{n,m}}{\leftarrow} & (\mathrm{Pic}^{n-m} X) \times \mathrm{Sh}^m_0 \end{array}$$

It suffices to show that for any m the restriction of K^n to $X^{(n-m)} \times \mathrm{Sh}_0'^m$ vanishes. Consider the Fourier transform for the diagram

$$\begin{array}{ccc} \mathrm{Pic}'^{n-m} X & & \mathcal{W}^{n-m} \\ & \searrow \quad \swarrow & \\ & \mathrm{Pic}^{n-m} X & \end{array}$$

As is easy to see, the Fourier transform of the constant sheaf on $\mathcal{W}^{n-m}$ vanishes when restricted to $X^{(n-m)} \hookrightarrow \mathrm{Pic}'^{n-m} X$. Since Four commutes with the base change

$$(\mathrm{Pic}^{n-m} X) \times \mathrm{Sh}_0^m \stackrel{\gamma_{n,m}}{\to} \mathrm{Sh}_1^n,$$

and one can do the Fourier transform variable by variable, our assertion follows. □

Let ${}^0\nu : {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_1'^n \to \mathrm{Sh}_1'^n$ be the morphism that sends a pair of sections to their sum. Define the stack ${}^{00}\mathrm{Sh}_1''^n$ from the cartesian square

$$\begin{array}{ccc} {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_1'^n & \stackrel{{}^0\nu}{\to} & \mathrm{Sh}_1'^n \\ \uparrow & & \uparrow \\ {}^{00}\mathrm{Sh}_1''^n & \to & \mathrm{Sh}_1''^n \end{array}$$

Lemma 20. *There exists a (unique) morphism $\eta : {}^{00}\mathrm{Sh}_1''^n \to X^{(n)}$ making commute the diagram*

$$\begin{array}{ccc} {}^{00}\mathrm{Sh}_1''^n & \to & {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_1'^n \\ \downarrow \eta & & \downarrow \\ X^{(n)} & \stackrel{i}{\to} & X^{(n)} \times_{\mathrm{Pic}^n X} X^{(n)} \end{array}$$

Proof We have a cartesian square

$$\begin{array}{ccc} X^{(n)} & \stackrel{i}{\to} & X^{(n)} \times_{\mathrm{Pic}^n X} X^{(n)} \\ \downarrow & & \downarrow \\ \mathrm{Pic}^n X & \stackrel{i_0}{\to} & \mathrm{Pic}'^n X, \end{array}$$

where the right vertical arrow is the summation of two sections. So, our assertion follows from the fact that the square (3) is cartesian (cf. sect. 1.2.3). □

To prove Main Local Theorem, we have to calculate the direct image (with compact support) of the sheaf

$$({}^0\nu)^* K^n \otimes (\pi_0'^* \mathcal{L}^n_{E_1} \boxtimes \pi_0'^* \mathcal{L}^n_{E_2})[n]$$

under the morphism

$${}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_1'^n \to X^{(n)} \times_{\mathrm{Pic}^n X} X^{(n)}$$

By Lemma 19, the latter sheaf is supported at the closed substack ${}^{00}\mathrm{Sh}_1''^n \to {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_1'^n$, and we can give the following definition.

Definition 6. For any local systems E_1, E_2 on X of rank 2 put

$$\mathcal{F}^n_{E_1,E_2} = \eta_!(({}^0\nu)^* K^n \otimes (\pi_0'^* \mathcal{L}^n_{E_1} \boxtimes \pi_0'^* \mathcal{L}^n_{E_2}))[n]$$

This is a complex on $X^{(n)}$.

So, Main Local Theorem is reduced to the following result.

Theorem 3. *For any local systems E_1, E_2 on X of rank 2 there exists a canonical isomorphism*

$$\mathcal{F}^n_{E_1,E_2} \widetilde{\to} (E_1 \otimes E_2)^{(n)}(-n-1+g)[n]$$

in the derived category on $X^{(n)}$.

3.1.2 Plan of the proof of Theorem 3

Our proof of Theorem 3 consists of the following parts:

0) We notice that the sheaf $(E_1 \otimes E_2)^{(n)}$ on $X^{(n)}$ has the property: for any open subscheme $j: U \to X^{(n)}$ the natural map $(E_1 \otimes E_2)^{(n)} \to (\mathrm{R}^0 j_*) j^*(E_1 \otimes E_2)^{(n)}$ is an isomorphism.
1) We establish the isomorphism of Theorem 3 over the open subscheme $X^{(n)} - \triangle$, where $\triangle$ is the divisor of coinciding points (this is the purpose of sect. 3.1.3).
2) We show that $\mathcal{F}^n_{E_1,E_2}[-n]$ is a $\bar{\mathbb{Q}}_\ell$-sheaf, and for any open subscheme $j: U \to X^{(n)}$ the map $\mathcal{F}^n_{E_1,E_2}[-n] \to (\mathrm{R}^0 j_*) j^*(\mathcal{F}^n_{E_1,E_2}[-n])$ is an isomorphism.

The point 2) is local w.r.t. the étale topology. Its proof, which is the purpose of sect. 3.2, is divided into several steps:

2.1) We prove Theorem 3 under the additional assumption: $E_i = E_{i1} \oplus E_{i2}$ $(i = 1, 2)$, where E_{ij} is a rank 1 local system on X. (this is done in sect. 3.2.1-3.2.6).
2.2) If E_1, E_2, E_1', E_2' are local systems on X of rank 2, $\operatorname{Spec} k \stackrel{D}{\to} X^{(n)}$ is a k-point and $X^{(n),D} \to X^{(n)}$ is the (strict) henselization of $X^{(n)}$ at D then we show that the restrictions of $\mathcal{F}^n_{E_1,E_2}$ and of $\mathcal{F}^n_{E_1',E_2'}$ to $X^{(n),D}$ are isomorphic (cf. Lemma 30, sect. 3.2.7). (In particular, combining 2.1) and 2.2) we see that $\mathcal{F}^n_{E_1,E_2}[-n]$ is a $\bar{\mathbb{Q}}_\ell$-sheaf).
2.3) We conclude with the following simple observation (cf. Lemma 31, sect. 3.2.7): Let Y be a k-scheme of finite type, $\mathcal{F}_1, \mathcal{F}_2$ be constructable $\bar{\mathbb{Q}}_\ell$-sheaves on Y. Suppose that for any k-point $\operatorname{Spec} k \stackrel{y}{\to} Y$ the restrictions of $\mathcal{F}_1$ and of $\mathcal{F}_2$ to Y^y are isomorphic, where Y^y is the (strict) henselization of Y at y. If for an open subscheme $j: U \hookrightarrow Y$ the natural map $\mathcal{F}_1 \to (\mathrm{R}^0 j_*) j^* \mathcal{F}_1$ is an isomorphism then the same holds for $\mathcal{F}_2$.

3.1.3

Consider the natural morphism ${}^0\operatorname{Sh}_1'^n \to X^{(n)} \times_{\operatorname{Pic}^n X} \operatorname{Sh}_1^n$ that sends $(\mathcal{O} \stackrel{s}{\hookrightarrow} F_1)$ to $(\mathcal{O} \stackrel{\det s}{\hookrightarrow} \det F_1,\ F_1)$. It is representable (and later we will see that it is affine (cf. Lemma 34)). Denote by

$$\phi_n : {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} \overline{\operatorname{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2}$$

the morphism obtained from the previous one by the base change $\overline{\operatorname{Flag}}_{n+2g-2} \to \operatorname{Sh}_1^n$.

Recall that $\mathcal{W}^n$ is the stack that can be considered as an open substack of $\overline{\operatorname{Flag}}_{n+2g-2}$, namely the preimage of $\operatorname{Pic}^n X \subset \operatorname{Sh}_1^n$ under $\overline{\operatorname{Flag}}_{n+2g-2} \to \operatorname{Sh}_1^n$.The following lemma is straightforward.

Lemma 21. *For the morphism*

$$\phi_n : {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} \overline{\operatorname{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2}$$

the complex $(\phi_n)_!(\mu^*\mathcal{L}_\psi)$ is the extension by zero of its restriction to the open substack $X^{(n)} \times_{\operatorname{Pic}^n X} \mathcal{W}^n$.

□

Proposition 12. *Let E_1, E_2 be local sytems on X of rank 2. Then there is a canonical isomorphism*

$$\mathcal{F}^n_{E_1,E_2} \widetilde{\to} (E_1 \otimes E_2)^{(n)}(-n-1+g)[n]$$

over $X^{(n)} - \triangle$, where $\triangle$ is the divisor of coinciding points.

Proof Consider the natural morphism

$$\varphi_n \times \operatorname{id} : {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} \overline{\operatorname{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2}$$

The preimage of $X^{(n)} - \triangle$ under $\operatorname{div} : \operatorname{Sh}_0^n \to X^{(n)}$ is ${}^{rss}\operatorname{Sh}_0^n$, and over ${}^{rss}\operatorname{Sh}_0^n$ we have $\mathcal{L}_E^n \widetilde{\to} \operatorname{div}^* E^{(n)}$ canonically. Therefore, by the projection formulae, the complex

$$(\varphi_n \times \operatorname{id})_!(\operatorname{pr}_1^* \pi_0'^* \mathcal{L}^n_{E_1} \otimes \operatorname{pr}_2^* \pi_0'^* \mathcal{L}^n_{E_2} \otimes \operatorname{pr}_{13}^* \mu^* \mathcal{L}_\psi \otimes \operatorname{pr}_{23}^* \mu^* \mathcal{L}_\psi)$$

is isomorphic to

$$\operatorname{pr}_1^* E_1^{(n)} \otimes \operatorname{pr}_2^* E_2^{(n)} \otimes ((\varphi_n \times \operatorname{id})_! \operatorname{pr}_{13}^* \mu^* \mathcal{L}_\psi \otimes \operatorname{pr}_{23}^* \mu^* \mathcal{L}_\psi)$$

over $(X^{(n)} - \triangle) \times_{\operatorname{Pic}^n X} (X^{(n)} - \triangle) \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2}$. Furthermore, by Lemma 21, $(\varphi_n \times \operatorname{id})_! \operatorname{pr}_{13}^* \mu^* \mathcal{L}_\psi \otimes \operatorname{pr}_{23}^* \mu^* \mathcal{L}_\psi$ is the extension by zero of its restriction to

$$X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)} \times_{\operatorname{Pic}^n X} \mathcal{W}^n$$

Since the natural map $E_1^{(n)} \otimes E_2^{(n)} \to (E_1 \otimes E_2)^{(n)}$ is an isomorphism over $X^{(n)} - \triangle$, our assertion follows from the fact that $\mathcal{W}^n \to \operatorname{Pic}^n X$ is a generalized vector fibration of rank $n+1-g$. □

3.2 Local properties of $\mathcal{F}^n_{E_1,E_2}$.

3.2.1

Denote by J_n the set of 2×2-matrices $c=(c_{ij})$, where $c_{ij}\in\mathbb{Z}_+$ and $\sum_j c_{ij}=n$ for any i. For $c\in J_n$ put

$$Y^c=(X^{(c_{11})}\times X^{(c_{12})})\times_{X^{(n)}}(X^{(c_{21})}\times X^{(c_{22})})$$

So, Y^c classifies the 2×2-matrices (D_{ij}) of divisors $D_{ij}\in X^{(c_{ij})}$ such that $\sum_j D_{ij}$ does not depend on i. Denote by J'_n the set of 2×2-matrices $c'=(c'_{ij})$, where $c'_{ij}\in\mathbb{Z}_+$ and $\sum_{i,j}c'_{ij}=n$. We have a map $h: J'_n\to J_n$ that sends c' to $c=(c_{ij})$ with

$$c_{1i}=\sum_j c'_{ij},\ c_{2i}=\sum_j c'_{ji}$$

Put $Y_n=\bigsqcup\limits_{c\in J_n}Y^c$. For $c'\in J'_n$ put $'Y^{c'}=\prod\limits_{i,j}X^{(c'_{ij})}$, so $'Y^{c'}$ parametrizes the matrices (D'_{ij}) of divisors $D'_{ij}\in X^{(c'_{ij})}$. Put $'Y_n=\bigsqcup\limits_{c'\in J'_n}{}'Y^{c'}$. Denote by $norm: {}'Y_n\to Y_n$ the morphism that sends (D'_{ij}) to (D_{ij}), where

$$D_{1i}=\sum_j D'_{ij},\ D_{2i}=\sum_j D'_{ji}$$

Clearly, $norm$ maps $'Y^{c'}$ to Y^c, where $c=h(c')$. Denote by $norm^c: \bigsqcup\limits_{c'\in h^{-1}(c)}{}'Y^{c'}\to Y^c$ the restriction of $norm$.

The scheme Y^c admits a stratification by locally closed subschemes ${}_{c'}Y^c\subset Y^c$. The strata are numbered by $c'\in h^{-1}(c)$. First, define ${}_{c'}Y^c$ as the open subscheme of $'Y^{c'}$ given by the condition $D'_{12}\cap D'_{21}=\emptyset$. Then the composition ${}_{c'}Y^c\to{}'Y^{c'}\stackrel{norm}{\to}Y^c$ is a locally closed immersion. As a subscheme of Y^c, ${}_{c'}Y^c$ can be defined by imposing the condition: $\deg(D_{1i}\cap D_{2i})=c'_{ii}$ for $i=1,2$.

Our next result is straightforward.

Lemma 22. *1) Y^c is of pure dimension n. The irreducible components of Y^c are numbered by the set $h^{-1}(c)$. Namely, to $c'\in h^{-1}(c)$ there corresponds the component $norm('Y^{c'})=$ (the closure of ${}_{c'}Y^c$ in Y^c).*
2) $norm: {}'Y_n\to Y_n$ is the normalization of Y_n (more precisely, it is a finite morphism, an isomorphism over an open dense subscheme of Y_n, and $'Y_n$ is smooth). In particular,

$$(norm^c)_*\mathbb{Q}_\ell[n]\widetilde{\to}\mathrm{IC}_{Y^c},$$

where IC_{Y^c} is the intersection cohomology sheaf on Y^c.
□

3.2.2

Let $c \in J_n$. We have a cartesian square

$$\begin{array}{ccc} (X^{(c_{11})} \times X^{(c_{12})}) \times_{\mathrm{Pic}^n X} (X^{(c_{21})} \times X^{(c_{22})}) & \to & X^{(n)} \times_{\mathrm{Pic}^n X} X^{(n)} \\ \uparrow i^c & & \uparrow i \\ Y^c & \to & X^{(n)}, \end{array}$$

and i^c is a closed immersion. Put

$$\mathcal{M}^c = ({}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_0^n} \mathcal{F}l_{0,0}^{c_{11},c_{12}}) \times_{\mathrm{Sh}_1^n} ({}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_0^n} \mathcal{F}l_{0,0}^{c_{21},c_{22}})$$

So, $\mathcal{M}^c$ is the stack that classifies the collections $(\mathcal{O} \overset{s_1}{\hookrightarrow} G_1 \hookrightarrow F_1, \mathcal{O} \overset{s_2}{\hookrightarrow} G_2 \hookrightarrow F_1)$, where $F_1 \in \mathrm{Sh}_1^n, G_i \in \mathrm{Sh}_1^{c_{i1}}$ for $i = 1, 2$. We have a natural morphism $\mathcal{M}^c \to (X^{(c_{11})} \times X^{(c_{12})}) \times_{\mathrm{Pic}^n X} (X^{(c_{21})} \times X^{(c_{22})})$ that sends the above collection to $(\mathcal{O} \overset{\det s_1}{\hookrightarrow} \det G_1 \hookrightarrow \det F_1,\ \mathcal{O} \overset{\det s_2}{\hookrightarrow} \det G_2 \hookrightarrow \det F_1)$. Denote by

$$\varphi^c : \mathcal{M}^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to (X^{(c_{11})} \times X^{(c_{12})}) \times_{\mathrm{Pic}^n X} (X^{(c_{21})} \times X^{(c_{22})})$$

the composition of the above morphism with the projection pr_1. Denote by

$$p^c : \mathcal{M}^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2}$$

the natural projection.

Proposition 13. *For any $c \in J_n$ there is a canonical isomorphism*

$$(\varphi^c)_!(p^c)^*(\mathrm{pr}_{13}^*\mu^*\mathcal{L}_\psi \otimes \mathrm{pr}_{23}^*\mu^*\mathcal{L}_\psi)[3n+2-2g] \widetilde{\to} (i^c)_*(\mathrm{IC}_{Y^c})(-n-1+g),$$

The proof occupies sect. 3.2.3-3.2.6. Now we derive an important corolary.

Corolary 4. *Theorem 3 holds under the additional assumption: $E_i = E_{i1} \oplus E_{i2}$ $(i = 1, 2)$, where E_{ij} is a rank 1 local sytem on X.*

Proof Let $c \in J_n$. From Proposition 13 it follows that

$$(\varphi_n)_!(\mathrm{pr}_1^*\pi_0'^*\mathcal{L}_{E_{11},E_{12}}^{c_{11},c_{12}} \otimes \mathrm{pr}_2^*\pi_0'^*\mathcal{L}_{E_{21},E_{22}}^{c_{21},c_{22}} \otimes \mathrm{pr}_{13}^*\mu^*\mathcal{L}_\psi \otimes \mathrm{pr}_{23}^*\mu^*\mathcal{L}_\psi)[3n+2-2g]$$

is canonically isomorphic to the direct image of

$$\mathrm{pr}_{12}^*(E_{11}^{(c_{11})} \boxtimes E_{12}^{(c_{12})}) \otimes \mathrm{pr}_{34}^*(E_{21}^{(c_{21})} \boxtimes E_{22}^{(c_{22})}) \otimes (i^c)_*(\mathrm{IC}_{Y^c})(-n-1+g)$$

under

$$(X^{(c_{11})}\times X^{(c_{12})})\times_{\mathrm{Pic}^n X}(X^{(c_{21})}\times X^{(c_{22})})\to X^{(n)}\times_{\mathrm{Pic}^n X}X^{(n)}$$

Clearly, it is supported at $i: X^{(n)}\to X^{(n)}\times_{\mathrm{Pic}^n X}X^{(n)}$, and the corresponding complex on $X^{(n)}$ is a perverse sheaf which is a Goresky-MacPherson extension of its restriction to a sufficiently small open subscheme.

So, under our assumptions $\mathcal{F}^n_{E_1,E_2}$ is a direct sum of perverse sheaves, every of them is a Goresky-MacPherson extension of its restriction to a sufficiently small open subscheme. Our assertion follows now from Proposition 12. □

3.2.3 Plan of the proof of Proposition 13

Our proof of Proposition 13 will consist of the following steps:
Step 1. Let ${}^0\mathcal{M}^c\subset\mathcal{M}^c$ be the open substack defined by the condition: G_1 and G_2 are invertible. Denote by ${}^0p^c$ (resp., by ${}^0\varphi^c$) the restriction of p^c (resp., of φ^c) to ${}^0\mathcal{M}^c\times_{\mathrm{Sh}^n_1}\overline{\mathrm{Flag}}_{n+2g-2}$. Using Lemma 21 we will show that

$$(\varphi^c)_!(p^c)^*(\mathrm{pr}^*_{13}\,\mu^*\mathcal{L}_\psi\otimes\mathrm{pr}^*_{23}\,\mu^*\mathcal{L}_\psi)[3n+2-2g]\widetilde{\to}$$
$$({}^0\varphi^c)_!({}^0p^c)^*(\mathrm{pr}^*_{13}\,\mu^*\mathcal{L}_\psi\otimes\mathrm{pr}^*_{23}\,\mu^*\mathcal{L}_\psi)[3n+2-2g]$$

naturally.
Step 2. Define the closed substack $\mathcal{X}^c$ of ${}^0\mathcal{M}^c$ from the cartesian square

$$\begin{array}{ccc} {}^0\mathcal{M}^c & \stackrel{\mathrm{pr}}{\to} & {}^0\,\mathrm{Sh}'^n_1\times_{\mathrm{Sh}^n_1}{}^0\,\mathrm{Sh}'^n_1\\ \uparrow & & \uparrow\\ \mathcal{X}^c & \to & {}^{00}\,\mathrm{Sh}''^n_1\end{array}$$

We will prove that for the projection $\mathrm{pr}_1: {}^0\mathcal{M}^c\times_{\mathrm{Sh}^n_1}\overline{\mathrm{Flag}}_{n+2g-2}\to{}^0\mathcal{M}^c$ the complex $(\mathrm{pr}_1)_!({}^0p^c)^*(\mathrm{pr}^*_{13}\,\mu^*\mathcal{L}_\psi\otimes\mathrm{pr}^*_{23}\,\mu^*\mathcal{L}_\psi)$ is supported at $\mathcal{X}^c$.

By Lemma 20, there is a (unique) morphism $\eta^c:\mathcal{X}^c\to Y^c$ making commute the diagram

$$\begin{array}{lcc} \mathcal{X}^c & \to & {}^0\mathcal{M}^c\\ \downarrow\eta^c & & \downarrow\\ Y^c & \stackrel{i^c}{\to} & (X^{(c_{11})}\times X^{(c_{12})})\times_{\mathrm{Pic}^n X}(X^{(c_{21})}\times X^{(c_{22})})\end{array}$$

Denote by ${}'p^c$ the restriction of p^c to $\mathcal{X}^c\times_{\mathrm{Sh}^n_1}\overline{\mathrm{Flag}}_{n+2g-2}$. Denote by ${}'\varphi^c$ the composition $\mathcal{X}^c\times_{\mathrm{Sh}^n_1}\overline{\mathrm{Flag}}_{n+2g-2}\stackrel{\mathrm{pr}_1}{\to}\mathcal{X}^c\stackrel{\eta^c}{\to}Y^c$. So, it will remain to show that

$$({}'\varphi^c)_!({}'p^c)^*(\mathrm{pr}^*_{13}\,\mu^*\mathcal{L}_\psi\otimes\mathrm{pr}^*_{23}\,\mu^*\mathcal{L}_\psi)[3n+2-2g]\widetilde{\to}\mathrm{IC}_{Y^c}(-n-1+g)$$

naturally.
Step 3. We will prove that the complex

$$('\varphi^c)_!('p^c)^*(\mathrm{pr}_{13}^*\mu^*\mathcal{L}_\psi\otimes\mathrm{pr}_{23}^*\mu^*\mathcal{L}_\psi)(n+1-g)[2n+2-2g] \tag{5}$$

is a $\bar{\mathbb{Q}}_\ell$-sheaf (i.e., it is placed in usual degree 0).
Step 4. Denote by $\mathcal{Z}^c\subset\mathcal{X}^c$ the closed substack given by the condition $s_1+s_2=0$. Let $\tau:\mathcal{Z}^c\to Y^c$ be the composition $\mathcal{Z}^c\to\mathcal{X}^c\stackrel{\eta^c}{\to}Y^c$.

We will show that (5) is, in fact, the highest (i.e. the $(2n+2-2g)$-th) direct image with compact support of

$$('p^c)^*(\mathrm{pr}_{13}^*\mu^*\mathcal{L}_\psi\otimes\mathrm{pr}_{23}^*\mu^*\mathcal{L}_\psi)(n+1-g)$$

under the morphism $'\varphi^c$. This will allow us to reduce the assertion of Proposition 13 to establishing a canonical isomorphism

$$\mathrm{R}^0\tau_!\bar{\mathbb{Q}}_\ell\widetilde{\to}(norm^c)_*\bar{\mathbb{Q}}_\ell \tag{6}$$

Step 5. Using 2) of Lemma 25 we construct the canonical isomorphism (6).

3.2.4

For a pair of integers $c_1,c_2\in\mathbb{Z}_+$ such that $c_1+c_2=n$ denote by $\mathcal{N}^{c_1,c_2}$ the stack that classifies the collections $(\mathcal{O}\stackrel{s_1}{\hookrightarrow}G_1\hookrightarrow F_1)$, where $F_1\in\mathrm{Sh}_1^n$, $G_1\in\mathrm{Pic}^{c_1}X$. Let $m\ge 0$. Recall that the stack $\overline{\mathrm{Flag}}^m_{n+2g-2}$ was defined in sect. 1.4.3. The stack $\mathcal{N}^{c_1,c_2}\times_{\mathrm{Sh}_1^n}\overline{\mathrm{Flag}}^m_{n+2g-2}$ classifies the collections: $D\in X^{(m)}, D_1\in X^{(c_1)}, D_2\in X^{(c_2-m)}$, an extension $0\to\Omega(D)\to L\to\mathcal{O}(\bar{D})\to 0$, where $\bar{D}=D_1+D_2$, and a mophism $\xi:\mathcal{O}(D_1)\to L/\Omega$ making commute the diagram

$$\begin{array}{ccc}\mathcal{O}(D_1) & \to & L/\Omega\\ & \searrow & \downarrow\\ & & \mathcal{O}(\bar{D})\end{array}$$

(Here $G_1\hookrightarrow F_1$ is the inclusion $\mathcal{O}(D_1)\stackrel{\xi}{\hookrightarrow}L/\Omega$, and $\mathcal{O}\stackrel{s}{\hookrightarrow}G_1$ is the canonical section $\mathcal{O}\hookrightarrow\mathcal{O}(D_1)$).

Denote by $\tilde{\mu}:\mathcal{N}^{c_1,c_2}\times_{\mathrm{Sh}_1^n}\overline{\mathrm{Flag}}^m_{n+2g-2}\to\mathbb{A}^1_k$ the composition

$$\mathcal{N}^{c_1,c_2}\times_{\mathrm{Sh}_1^n}\overline{\mathrm{Flag}}^m_{n+2g-2}\to{}^0\mathrm{Sh}_1'^n\times_{\mathrm{Sh}_1^n}\overline{\mathrm{Flag}}_{n+2g-2}\stackrel{\mu}{\to}\mathbb{A}^1_k,$$

where the first arrow is the natural map. Denote by $b_m^{c_1,c_2} : \mathcal{N}^{c_1,c_2} \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}^m_{n+2g-2} \to (X^{(c_1)} \times X^{(c_2-m)}) \times_{\mathrm{Pic}^{n-m} X} \overline{\mathrm{Flag}}^m_{n+2g-2}$ the morphism that sends the above collection to

$$((D_1, D_2), D \in X^{(m)}, 0 \to \Omega(D) \to L \to \mathcal{O}(\bar{D}) \to 0), \tag{7}$$

where $\bar{D} = D_1 + D_2$. In other words, $b_m^{c_1,c_2}$ is the morphism that forgets ξ. Notice that $b_m^{c_1,c_2}$ is an affine bundle of rank m.

Denote by $\mathcal{V}_m^{c_1,c_2} \hookrightarrow (X^{(c_1)} \times X^{(c_2-m)}) \times_{\mathrm{Pic}^{n-m} X} \overline{\mathrm{Flag}}^m_{n+2g-2}$ the closed substack defined by the condition $D_1 \geq D$. In other words, this stack is defined from the cartesian square

$$\begin{array}{ccc} \mathcal{V}_m^{c_1,c_2} & \hookrightarrow & (X^{(c_1)} \times X^{(c_2-m)}) \times_{\mathrm{Pic}^{n-m} X} \overline{\mathrm{Flag}}^m_{n+2g-2} \\ \downarrow & & \downarrow \mathrm{pr} \\ X^{(m)} \times X^{(c_1-m)} \times X^{(c_2-m)} & \hookrightarrow & X^{(m)} \times X^{(c_1)} \times X^{(c_2-m)}, \end{array}$$

where the lowest horizontal arrow is the closed immersion that sends $(D \in X^{(m)}, D_1' \in X^{(c_1-m)}, D_2 \in X^{(c_2-m)})$ to $(D, D + D_1', D_2)$, and pr is the natural projection.

Lemma 23. *The complex $(b_m^{c_1,c_2})_! \tilde{\mu}^* \mathcal{L}_\psi$ is supported at $\mathcal{V}_m^{c_1,c_2}$.*

Proof If $\mathrm{Spec}\, k \to (X^{(c_1)} \times X^{(c_2-m)}) \times_{\mathrm{Pic}^{n-m} X} \overline{\mathrm{Flag}}^m_{n+2g-2}$ is a k-point defined by a collection (7) then the restriction of $\tilde{\mu}^* \mathcal{L}_\psi$ to the fibre of $b_m^{c_1,c_2}$ over this point is a constant sheaf if and only if $D_1 \geq D$. □

3.2.5

For $n > 0$ denote by $G_n(k)$ the group of units of the ring $k[t]/(t^n)$ (t is a variable). Clearly, $G_n(k)$ is the set of k-points of the commutative affine algebraic group G_n (over k). For example, $G_1 = \mathbb{G}_m$. Denote by $B(G_n)$ the classifying stack of G_n.

Lemma 24. *There is a canonical isomorphism $\mathrm{H}_c^{-2n}(B(G_n), \bar{\mathbb{Q}}_\ell) \widetilde{\to} \bar{\mathbb{Q}}_\ell(n)$. Besides, $\mathrm{H}_c^{-2n-1}(B(G_n), \bar{\mathbb{Q}}_\ell) = 0$.*

Proof 1) For $n = 1$, i.e., for $\mathbb{G}_m$ this is known. (cf. Example 1, Appendix A)
2) For any n we have an exact sequence $0 \to U_n \to G_n \to \mathbb{G}_m \to 0$, where U_n is unipotent. The natural map $B(G_n) \xrightarrow{f} B(\mathbb{G}_m)$ is a generalized vector fibration of rank $1 - n$ (it is a U_n-gerb). So, $f_! \bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell(n-1)[2n-2]$. Our assertion follows now from the point 1).
□

Recall that ${}^r\mathrm{Sh}_0^m \subset \mathrm{Sh}_0^m$ is the open substack that classifies regular torsion sheaves (cf. sect. 1.3.1). Let ${}^{nr}\mathrm{Sh}_0^m$ be the complement of ${}^r\mathrm{Sh}_0^m$ to Sh_0^m (with the reduced stack structure).

Lemma 25. *1) All the fibres of* $\mathrm{div} : \mathrm{Sh}_0^m \to X^{(m)}$ *are stacks of dimension* $-m$*. The fibres of the restriction* $\mathrm{div} : {}^{nr}\mathrm{Sh}_0^m \to X^{(m)}$ *are of dimension* $< -m$*.*
2) There is a canonical isomorphism

$$\mathrm{R}^{-2m}\,\mathrm{div}_!\,\bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell(m)$$

Besides, $\mathrm{R}^{-2m-1}\,\mathrm{div}_!\,\bar{\mathbb{Q}}_\ell = 0$*.*

Proof 1) is straightforward. 2) Let ${}^r\mathrm{div}$ be the restriction of div to ${}^r\mathrm{Sh}_0^m$. By 1), the natural morphism $\mathrm{R}^{-2m}({}^r\mathrm{div})_!\bar{\mathbb{Q}}_\ell \to \mathrm{R}^{-2m}\,\mathrm{div}_!\,\bar{\mathbb{Q}}_\ell$ is an isomorphism. Since ${}^r\mathrm{div}$ is smooth of relative dimension $-m$, we have a natural morphism $({}^r\mathrm{div})_!\bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_\ell(m)[2m]$. Let $\mathrm{Spec}\,k \overset{D}{\to} X^{(m)}$ be a k-point with $D = \sum_{x\in X} d_x[x]$, $d_x \geq 0$. Then the fibre of ${}^r\mathrm{div}$ over D is isomorphic to $\prod_{x\in X} B(G_{d_x})$. So, our assertion follows from Lemma 24 by the Kunneth formulae. $\square$

3.2.6

Proof of Proposition 13.
Step 1. Denote by $\mathcal{M}_1^c$ the stack that classifies the collections $(\mathcal{O} \hookrightarrow \det G_1, G_1 \hookrightarrow F_1, \mathcal{O} \overset{s_2}{\hookrightarrow} G_2 \hookrightarrow F_1)$, where $F_1 \in \mathrm{Sh}_1^n$, $G_i \in \mathrm{Sh}_1^{c_{i1}}$ for $i = 1, 2$. Then φ^c is the composition

$$\mathcal{M}^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \overset{\varphi_1^c}{\to} \mathcal{M}_1^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to (X^{(c_{11})} \times X^{(c_{12})}) \times_{\mathrm{Pic}^n X} (X^{(c_{21})} \times X^{(c_{22})}),$$

where φ_1^c is obtained by the obvious base change from the map

$${}^0\,\mathrm{Sh}_1'^{c_{11}} \to X^{(c_{11})} \times_{\mathrm{Pic}^{c_{11}} X} \mathrm{Sh}_1^{c_{11}}$$

that sends $(\mathcal{O} \overset{s_1}{\hookrightarrow} G_1)$ to $(\mathcal{O} \overset{\det s_1}{\hookrightarrow} \det G_1, G_1)$. Let ${}^0\mathcal{M}_1^c \subset \mathcal{M}_1^c$ be the open substack defined by the condition: G_1 is invertible. By Lemma 21,

$$(\varphi_1^c)_!(p^c)^*(\mathrm{pr}_{13}^*\,\mu^*\mathcal{L}_\psi \otimes \mathrm{pr}_{23}^*\,\mu^*\mathcal{L}_\psi)$$

is the extension by zero of its restriction to the open substack ${}^0\mathcal{M}_1^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \subset \mathcal{M}_1^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2}$. Notice that over ${}^0\mathcal{M}_1^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2}$ the map φ_1^c is an isomorphism. Applying an analogous argument with G_2 instead of G_1, one "replaces" the stack ${}^0\mathcal{M}_1^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2}$ by its open substack ${}^0\mathcal{M}^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2}$.

Step 2. ¿From the cartesian square

$$\begin{array}{ccc} {}^0\mathcal{M}^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} & \stackrel{{}^0p^c}{\to} & {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \\ \downarrow \mathrm{pr}_1 & & \downarrow \\ {}^0\mathcal{M}^c & \stackrel{\mathrm{pr}}{\to} & {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\,\mathrm{Sh}_1'^n \end{array}$$

we get an isomorphism

$$(\mathrm{pr}_1)_!({}^0p^c)^*(\mathrm{pr}_{13}^*\,\mu^*\mathcal{L}_\psi \otimes \mathrm{pr}_{23}^*\,\mu^*\mathcal{L}_\psi)[3n+2-2g] \widetilde{\to} \mathrm{pr}^*({}^0\nu)^*K^n[n],$$

where K^n is the complex defined in sect. 3.1.1. ¿From Lemma 19 we conclude that $(\mathrm{pr}_1)_!({}^0p^c)^*(\mathrm{pr}_{13}^*\,\mu^*\mathcal{L}_\psi \otimes \mathrm{pr}_{23}^*\,\mu^*\mathcal{L}_\psi)$ is supported at $\mathcal{X}^c$.

Step 3. Let $m \geq 0$. Denote by $'\varphi^c_m$ (resp., by $'p^c_m$) the restriction of $'\varphi^c$ (resp., of $'p^c$) to the stack $\mathcal{X}^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}^m_{n+2g-2}$. Put

$$c(m) = \left(\begin{array}{cc} c_{11}-m & c_{12}-m \\ c_{21}-m & c_{22}-m \end{array} \right) \in J_{n-2m},$$

and denote by $a^c_m : X^{(m)} \times Y^{c(m)} \to Y^c$ the finite morphism that sends $(D,(D^{ij}))$ to

$$\left(\begin{array}{cc} D^{11}+D & D^{12}+D \\ D^{21}+D & D^{22}+D \end{array} \right)$$

Lemma 26. *There is a canonical isomorphism*

$$('\varphi^c_m)_!('p^c_m)^*(\mathrm{pr}_{13}^*\,\mu^*\mathcal{L}_\psi \otimes \mathrm{pr}_{23}^*\,\mu^*\mathcal{L}_\psi) \widetilde{\to} (a^c_m)_!\bar{\mathbb{Q}}_\ell(-n-1+g)[-2n-2+2g]$$

over Y^c.

Proof The stack $\mathcal{X}^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}^m_{n+2g-2}$ classifies the collections: $D \in X^{(m)}$, a matrix $(D^{ij}) \in Y^{\bar{c}}$, where

$$\bar{c} = \left(\begin{array}{cc} c_{11} & c_{12}-m \\ c_{21} & c_{22}-m \end{array} \right) \in J_{n-m},$$

an extension $0 \to \Omega(D) \to L \to \mathcal{O}(\bar{D}) \to 0$, where $\bar{D} = D_{i1} + D_{i2} \in X^{(n-m)}$, and two morphisms $\xi_i : \mathcal{O}(D_{i1}) \to L/\Omega$ making commute the diagram

$$\begin{array}{ccc} \mathcal{O}(D_{i1}) & \stackrel{\xi_i}{\to} & L/\Omega \\ & \searrow & \downarrow \\ & & \mathcal{O}(\bar{D}) \end{array}$$

Denote by $f^c_m : \mathcal{X}^c \times_{\mathrm{Sh}^n_1} \overline{\mathrm{Flag}}^m_{n+2g-2} \to X^{(m)} \times Y^{\bar{c}}$ the morphism that sends the above collection to $(D, (D^{ij}))$. This is a composition of an affine bundle of rank $2m$ and a generalized vector fibration of rank $n - 2m + 1 - g$. (The point is that the sum of ranks does not depend on m!)

We have a closed immersion $X^{(m)} \times Y^{c(m)} \hookrightarrow X^{(m)} \times Y^{\bar{c}}$ that sends $(D, (D^{ij}))$ to

$$\left(D, \left(\begin{array}{cc} D^{11} + D & D^{12} \\ D^{21} + D & D^{22} \end{array}\right)\right)$$

Using Lemma 23 one shows that $(f^c_m)_!({}'p^c_m)^*(\mathrm{pr}^*_{13}\,\mu^*\mathcal{L}_\psi \otimes \mathrm{pr}^*_{23}\,\mu^*\mathcal{L}_\psi)$ is supported at $X^{(m)} \times Y^{c(m)} \hookrightarrow X^{(m)} \times Y^{\bar{c}}$. (The argument is analogous to that of Step 1).

Denote temporary by $\mathcal{Y}$ the stack given by the cartesian square

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & \mathcal{X}^c \times_{\mathrm{Sh}^n_1} \overline{\mathrm{Flag}}^m_{n+2g-2} \\ \downarrow & & \downarrow f^c_m \\ X^{(m)} \times Y^{c(m)} & \hookrightarrow & X^{(m)} \times Y^{\bar{c}} \end{array}$$

Since the restriction of $({}'p^c_m)^*(\mathrm{pr}^*_{13}\,\mu^*\mathcal{L}_\psi \otimes \mathrm{pr}^*_{23}\,\mu^*\mathcal{L}_\psi)$ to $\mathcal{Y}$ is the constant sheaf $\bar{\mathbb{Q}}_\ell$, our assertion follows. $\square$

So, calculating the direct image (5) w.r.t the stratification of $\mathcal{X}^c \times_{\mathrm{Sh}^n_1} \overline{\mathrm{Flag}}_{n+2g-2}$ by locally closed substacks $\mathcal{X}^c \times_{\mathrm{Sh}^n_1} \overline{\mathrm{Flag}}^m_{n+2g-2}$, $m \geq 0$, we conclude that (5) is a $\bar{\mathbb{Q}}_\ell$-sheaf.

Remark 13. In addition, on the sheaf (5) we get a canonical filtration by $\bar{\mathbb{Q}}_\ell$-subsheaves

$$0 \subset F_0 \subset F_1 \subset F_2 \subset \ldots$$

such that $F_m / F_{m-1} \widetilde{\to} (a^c_m)_! \bar{\mathbb{Q}}_\ell$ for $m \geq 0$. (Here $F_{-1} = 0$). Since for $m \gg 0$ we have $Y^{c(m)} = \emptyset$, this filtration is finite.

Step 4. Denote by ${}^r\mathrm{Sh}^n_1 \subset \mathrm{Sh}^n_1$ the image of the (smooth) morphism $\overline{\mathrm{Flag}}_{n+2g-2} \to \mathrm{Sh}^n_1$. Notice that ${}^r\mathrm{Sh}^n_1 \times_{\mathrm{Sh}^n_1} \mathrm{Sh}^{n;m}_1$ is the preimage under the natural morphism $\mathrm{Sh}^{n;m}_1 \to \mathrm{Sh}^m_0$ of the open substack ${}^r\mathrm{Sh}^m_0 \subset \mathrm{Sh}^m_0$.

Put ${}^r\mathcal{X}^c = \mathcal{X}^c \times_{\mathrm{Sh}^n_1} {}^r\mathrm{Sh}^n_1$. Denote by ${}^{nr}\mathcal{X}^c$ the complement of ${}^r\mathcal{X}^c$ to $\mathcal{X}^c$.

Lemma 27. *The morphism $\eta^c : \mathcal{X}^c \to Y^c$ is of finite type and all its fibres are stacks of dimension ≤ 0. The fibres of the restriction $\eta^c : {}^{nr}\mathcal{X}^c \to Y^c$ are of dimension < 0.*

Proof For $m \geqslant 0$ the stack $\mathcal{X}^c \times_{\mathrm{Sh}^n_1} \mathrm{Sh}^{n;m}_1$ classifies the collections: $F_0 \in \mathrm{Sh}^m_0$, a matrix $(D_{ij}) \in Y^{\bar{c}}$, where

$$\bar{c} = \left(\begin{array}{cc} c_{11} & c_{12} - m \\ c_{21} & c_{22} - m \end{array}\right) \in J_{n-m},$$

an extension $0 \to F_0 \to F_1 \to \mathcal{O}(\bar{D}) \to 0$, where $\bar{D} = D_{i1} + D_{i2} \in X^{(n-m)}$, and two morphisms $\xi_i : \mathcal{O}(D_{i1}) \to F_1$ making commute the diagram

$$\begin{array}{ccc} \mathcal{O}(D_{i1}) & \stackrel{\xi_i}{\to} & F_1 \\ & \searrow & \downarrow \\ & & \mathcal{O}(\bar{D}) \end{array}$$

The morphism $\mathcal{X}^c \times_{\mathrm{Sh}_1^n} \mathrm{Sh}_1^{n;m} \to \mathrm{Sh}_0^m \times Y^{\bar{c}}$ that sends the above collection to $(F_0, (D_{ij}))$ is a composition of an affine bundle of rank $2m$ and a generalized vector fibration of rank $-m$. Since $X^{(m)} \times Y^{\bar{c}} \to Y^c$ is finite, using 1) of Lemma 25 we conclude that all the fibres of the resriction of η^c to $\mathcal{X}^c \times_{\mathrm{Sh}_1^n} \mathrm{Sh}_1^{n;m}$ (resp., to ${}^{nr}\mathcal{X}^c \times_{\mathrm{Sh}_1^n} \mathrm{Sh}_1^{n;m}$) are of dimension 0 (resp., of dimension < 0). Besides, $\mathcal{X}^c \times_{\mathrm{Sh}_1^n} \mathrm{Sh}_1^{n;m} = \emptyset$ for $m > n$, because $J_{n-m} = \emptyset$ in this case. Our assertion follows.
□ (Lemma 27)

Since $\pi_1' : {}^0\mathrm{Sh}_2'^{n+2g-2} \to \mathrm{Sh}_1^n$ is a generalized vector fibration of rank $n+1-g$, and $\overline{\mathrm{Flag}}_{n+2g-2} \hookrightarrow {}^0\mathrm{Sh}_2'^{n+2g-2}$ is an open immersion, from the above lemma it follows that the fibres of ${}'\varphi^c$ are stacks of dimension at most $n+1-g$. So, the sheaf (5) is, in fact, the highest direct image with compact support of

$$({}'p^c)^*(\mathrm{pr}_{13}^* \mu^* \mathcal{L}_\psi \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi)(n+1-g)$$

under the morphism ${}'\varphi^c$.

Denote by ${}^r\varphi^c$ the composition ${}^r\mathcal{X}^c \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_2'^{n+2g-2} \stackrel{\mathrm{pr}_1}{\to} {}^r\mathcal{X}^c \hookrightarrow \mathcal{X}^c \stackrel{\eta^c}{\to} Y^c$. Let

$${}^r p^c : {}^r\mathcal{X}^c \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_2'^{n+2g-2} \to {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_2'^{n+2g-2}$$

be the natural morphism. By abuse of notation, $\mu : {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_2'^{n+2g-2} \to \mathbb{A}_k^1$ will also denote the natural pairing. Clearly,

$$\mathcal{X}^c \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \subset {}^r\mathcal{X}^c \times_{\mathrm{Sh}_1^n} {}^0\mathrm{Sh}_2'^{n+2g-2}$$

is an open substack. Now we state that the natural morphism

$$\mathrm{R}^{2n+2-2g}({}'\varphi^c)_!({}'p^c)^*(\mathrm{pr}_{13}^* \mu^* \mathcal{L}_\psi \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi) \to \mathrm{R}^{2n+2-2g}({}^r\varphi^c)_!({}^r p^c)^*(\mathrm{pr}_{13}^* \mu^* \mathcal{L}_\psi \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi)$$

is an isomorphism. Indeed, if V is the complement of $\overline{\mathrm{Flag}}_{n+2g-2}$ to ${}^0\mathrm{Sh}_2'^{n+2g-2}$ then the fibres of the morphism $\mathrm{pr}_1 : {}^r\mathrm{Sh}_1^n \times_{\mathrm{Sh}_1^n} V \to {}^r\mathrm{Sh}_1^n$ are of dimension $< n+1-g$.

Recall that $\mathcal{Z}^c \subset \mathcal{X}^c$ is the closed substack given by the condition $s_1 + s_2 = 0$. Let ${}^r\mathcal{Z}^c$ be the preimage of ${}^r\mathcal{X}^c$ under $\mathcal{Z}^c \to \mathcal{X}^c$. Recall that for the projection

$$\mathrm{pr}_1 : \mathrm{Sh}'^n_1 \times_{\mathrm{Sh}^n_1} {}^0\mathrm{Sh}'^{n+2g-2}_2 \to \mathrm{Sh}'^n_1$$

the complex $(\mathrm{pr}_1)_! \mu^* \mathcal{L}_\psi$ is supported at the zero section $\mathrm{Sh}^n_1 \to \mathrm{Sh}'^n_1$ and is canonically isomorphic to $\mathbb{Q}_\ell(-n-1+g)[-2n-2+2g]$ over Sh^n_1. It follows that

$$(\mathrm{pr}_1)_!({}^r p^c)^*(\mathrm{pr}^*_{13}\, \mu^* \mathcal{L}_\psi \otimes \mathrm{pr}^*_{23}\, \mu^* \mathcal{L}_\psi)$$

is supported at ${}^r\mathcal{Z}^c$ and is naturally isomorphic to $\mathbb{Q}_\ell(-n-1+g)[-2n-2+2g]$ over ${}^r\mathcal{Z}^c$.

Recall that $\tau : \mathcal{Z}^c \to Y^c$ is the composition $\mathcal{Z}^c \to \mathcal{X}^c \stackrel{\eta^c}{\to} Y^c$. Let ${}^r\tau$ be the restriction of τ to ${}^r\mathcal{Z}^c$. ¿From Lemma 27 it follows that $\mathrm{R}^0({}^r\tau)_!\mathbb{Q}_\ell \widetilde{\to} \mathrm{R}^0\tau_!\mathbb{Q}_\ell$ naturally. So, to finish the proof it remains to establish the canonical isomorphism (6).

Step 5. The stack $\mathcal{Z}^c$ classifies the commutative diagrams

$$\begin{array}{ccc} G_1 & \to & F_1 \\ \uparrow s_1 & & \uparrow \\ \mathcal{O} & \stackrel{-s_2}{\to} & G_2 \end{array}$$

where $G_i \in \mathrm{Pic}^{c_{i1}} X, F_1 \in \mathrm{Sh}^n_1$ (and all the arrows are inclusions). We stratify $\mathcal{Z}^c$ by locally closed substacks ${}_{c'}\mathcal{Z}^c \subset \mathcal{Z}^c$ as follows. The strata are indexed by $c' \in h^{-1}(c)$. The stratum ${}_{c'}\mathcal{Z}^c$ is defined by imposing the condition: $\deg(G_1 \bigcap G_2) = c'_{11}$.

Lemma 28. *There is a commutative diagram*

$$\begin{array}{ccc} {}_{c'}\mathcal{Z}^c & \to & \mathcal{Z}^c \\ \downarrow {}_{c'}\tau & & \downarrow \tau \\ {}'Y^{c'} & \stackrel{norm}{\to} & Y^c, \end{array} \qquad (8)$$

and we have $\mathrm{R}^0({}_{c'}\tau)_!\bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell$ *canonically. Besides,* $\mathrm{R}^{-1}({}_{c'}\tau)_!\bar{\mathbb{Q}}_\ell = 0$.

Proof Let $c' \in h^{-1}(c)$. For a triple of divisors $(D'_{11}, D'_{12}, D'_{21})$ with $D'_{ij} \in X^{(c'_{ij})}$ define $F_{D'_{11}, D'_{12}, D'_{21}} \in \mathrm{Sh}^{n-c'_{22}}_1$ from the cocartesian square

$$\begin{array}{ccc} \mathcal{O}(D'_{11} + D'_{12}) & \to & F_{D'_{11}, D'_{12}, D'_{21}} \\ \uparrow & & \uparrow \\ \mathcal{O}(D'_{11}) & \to & \mathcal{O}(D'_{11} + D'_{21}) \end{array}$$

Then ${}_{c'}\mathcal{Z}^c$ is the stack classifying the collections: $(D'_{11}, D'_{12}, D'_{21})$ with $D'_{ij} \in X^{(c'_{ij})}$, $F_0 \in \mathrm{Sh}_0^{c'_{22}}$ and an exact sequence $0 \to F_{D'_{11},D'_{12},D'_{21}} \to F_1 \to F_0 \to 0$.

The morphism ${}_{c'}\mathcal{Z}^c \to X^{(c'_{11})} \times X^{(c'_{12})} \times X^{(c'_{21})} \times \mathrm{Sh}_0^{c'_{22}}$ that sends the above collection to $(D'_{11}, D'_{12}, D'_{21}, F_0)$ is a generalized vector fibration of rank c'_{22}. Let ${}_{c'}\tau$ be the composition of the latter morphism with $X^{(c'_{11})} \times X^{(c'_{12})} \times X^{(c'_{21})} \times \mathrm{Sh}_0^{c'_{22}} \stackrel{\mathrm{id}\times\mathrm{div}}{\to} {}'Y^{c'}$. Then the diagram (8) commutes. Our assertion follows now from 2) of Lemma 25.
□ (Lemma 28)

By Lemma 28, $\mathrm{R}^0(norm \circ {}_{c'}\tau)_!\bar{\mathbb{Q}}_\ell \widetilde{\to} (norm^{c'})_*\bar{\mathbb{Q}}_\ell$, where $norm^{c'} : {}'Y^{c'} \to Y^c$ is the restriction of $norm$, and $\mathrm{R}^{-1}(norm \circ {}_{c'}\tau)_!\bar{\mathbb{Q}}_\ell = 0$. Now the spectral sequence that computes $\tau_!\bar{\mathbb{Q}}_\ell$ w.r.t. the stratification of $\mathcal{Z}^c$ by ${}_{c'}\mathcal{Z}^c$'s shows the following. On $\mathrm{R}^0\tau_!\bar{\mathbb{Q}}_\ell$ there is a filtration parametrized by the partially ordered set $h^{-1}(c)$ (the order is that of adjunction of the ${}_{c'}\mathcal{Z}^c$'s), with successive quotients being $(norm^{c'})_*\bar{\mathbb{Q}}_\ell$. Since the different successive quotients are supported on different irreducible components of Y^c (cf. Lemma 22), this filtration degenerates over some open dense subscheme of Y^c into a direct sum. Now it suffices to show that $\mathrm{R}^0\tau_!\bar{\mathbb{Q}}_\ell[n]$ is a perverse sheaf which is a Goresky-MacPherson extension of its restriction to a sufficiently small open subscheme. Since the same is true for $(norm^{c'})_*\bar{\mathbb{Q}}_\ell[n]$, we are done.
□ (Proposition 13)

3.2.7

Our next result is closely related to (Lemma 18, [2]).

Lemma 29. *Let E_1, E_2 be local systems on X of the same rank. Let* $\mathrm{Spec}\, k \stackrel{D}{\to} X^{(n)}$ *be a k-point. Denote by $X^{(n),D} \to X^{(n)}$ the (strict) henselization of $X^{(n)}$ at D. Then the restrictions of $\mathcal{L}^n_{E_1}$ and of $\mathcal{L}^n_{E_2}$ to $X^{(n),D} \times_{X^{(n)}} \mathrm{Sh}^n_0$ are isomorphic.*

Proof Let $\delta : X' \to X$ be an étale Galois covering. Denote by $X'^{(n)}(X) \subset X'^{(n)}$ the open subscheme parametrizing the divisors $D' \in X'^{(n)}$ for which $\sigma(D') \bigcap D' = \emptyset$ for any $\sigma \in \mathrm{Gal}(X'/X), \sigma \neq 1$. The morphism $X'^{(n)} \to X^{(n)}$ is not étale for $n > 1$, however its restriction $X'^{(n)}(X) \to X^{(n)}$ is étale and surjective. We have a cartesian square

$$\begin{array}{ccc} X'^n(X) & \to & X^n \\ \downarrow & & \downarrow \\ X'^{(n)}(X) & \to & X^{(n)}, \end{array}$$

where $X'^n(X)$ is the preimage of $X'^{(n)}(X)$ under the canonical map $X'^n \to X'^{(n)}$. From this square we get a canonical isomorphism

$$E^{(n)} \mid_{X'^{(n)}(X)} \widetilde{\to} (\delta^* E)^{(n)} \mid_{X'^{(n)}(X)}$$

for any local system E on X. Denote by $\mathrm{Sh}_0^n(X')$ the stack of torsion sheaves of length n on X'. Denote by $\mathrm{Sh}_0^n(X,X') \subset \mathrm{Sh}_0^n(X')$ the preimage of $X'^{(n)}(X)$ under $\mathrm{div}: \mathrm{Sh}_0^n(X') \to X'^{(n)}$. We have a cartesian square

$$\begin{array}{ccc} \mathrm{Sh}_0^n(X,X') & \stackrel{\delta^n}{\to} & \mathrm{Sh}_0^n(X) \\ \downarrow \mathrm{div} & & \downarrow \mathrm{div} \\ X'^{(n)}(X) & \to & X^{(n)}, \end{array}$$

where δ^n is given by $T \in \mathrm{Sh}_0^n(X') \mapsto \delta_* T \in \mathrm{Sh}_0^n(X)$. (This notation agrees with [2]). Recall that for any local system E on X

$$(\delta^n)^* \mathcal{L}_E^n \widetilde{\to} \mathcal{L}_{\delta^* E}^n \mid_{\mathrm{Sh}_0^n(X,X')}$$

canonically. Chose a lifting $\mathrm{Spec}\, k \to X'^{(n)}(X) \to X^{(n)}$ of $D : \mathrm{Spec}\, k \to X^{(n)}$. It induces a map $X^{(n),D} \to X'^{(n)}(X)$ over $X^{(n)}$ and, hence, a map $X^{(n),D} \times_{X^{(n)}} \mathrm{Sh}_0^n \stackrel{g}{\to} \mathrm{Sh}_0^n(X,X')$. We see that the restriction of $\mathcal{L}_E^n$ to $X^{(n),D} \times_{X^{(n)}} \mathrm{Sh}_0^n$ is isomorphic to the inverse image of $\mathcal{L}_{\delta^* E}^n \mid_{\mathrm{Sh}_0^n(X,X')}$ under g. Since $\delta : X' \to X$ was an arbitrary Galois covering, our assertion follows.
□(Lemma 29)

Lemma 30. *Let E_1, E_2, E_1', E_2' be local systems on X of rank 2. Let $\mathrm{Spec}\, k \stackrel{D}{\to} X^{(n)}$ be a k-point and $X^{(n),D} \to X^{(n)}$ be the (strict) henselization of $X^{(n)}$ at D. Then the restrictions of $\mathcal{F}_{E_1,E_2}^n$ and of $\mathcal{F}_{E_1',E_2'}^n$ to $X^{(n),D}$ are isomorphic.*

Proof The morphism η from Lemma 20 can be written as the composition ${}^{00}\mathrm{Sh}_1''^{n} \stackrel{\eta'}{\to} \mathrm{Sh}_0^n \times_{X^{(n)}} \mathrm{Sh}_0^n \stackrel{\varkappa}{\to} X^{(n)}$, so that

$$\mathcal{F}_{E_1,E_2}^n \widetilde{\to} \varkappa_! (\mathcal{N}^n \otimes (\mathcal{L}_{E_1}^n \boxtimes \mathcal{L}_{E_2}^n)),$$

where $\mathcal{N}^n = \eta'_! ({}^0\nu)^* K^n[n]$. By Lemma 29, the restrictions of $\mathcal{N}^n \otimes (\mathcal{L}_{E_1}^n \boxtimes \mathcal{L}_{E_2}^n)$ and of $\mathcal{N}^n \otimes (\mathcal{L}_{E_1'}^n \boxtimes \mathcal{L}_{E_2'}^n)$ to $X^{(n),D} \times_{X^{(n)}} (\mathrm{Sh}_0^n \times_{X^{(n)}} \mathrm{Sh}_0^n)$ are isomorphic. Our assertion follows by the base change theorem. □

Lemma 31. *Let Y be a k-scheme of finite type. Let $\mathcal{F}_1, \mathcal{F}_2$ be constructable $\mathbb{Q}_\ell$-sheaves on Y. Suppose that for any k-point $\mathrm{Spec}\, k \stackrel{y}{\to} Y$ the restrictions of $\mathcal{F}_1$ and of $\mathcal{F}_2$ to Y^y are isomorphic, where Y^y is the (strict) henselization of Y at y. If for an open subscheme $j : Y' \to Y$ the natural map $\mathcal{F}_1 \to (\mathrm{R}^0 j_*) j^* \mathcal{F}_1$ is an isomorphism then the same holds for $\mathcal{F}_2$.*

Proof Consider the cartesian square

$$\begin{array}{ccc} Y' & \leftarrow & Y'^y \\ \downarrow j & & \downarrow \\ Y & \leftarrow & Y^y \end{array}$$

To say that $(\mathcal{F}_2)_y \to ((\mathrm{R}^0 j_*) j^* \mathcal{F}_2)_y$ is an isomorphism is equivalent to say that $\mathrm{H}^0(Y^y, \mathcal{F}_2) \to \mathrm{H}^0(Y'^y, \mathcal{F}_2)$ is an isomorphism. The latter property depends only on the restriction of $\mathcal{F}_2$ to Y^y. $\square$

According to our plan (cf. sect. 3.1.2), Theorem 3 is proved. So, Main Local Theorem is also proved.

4 Perverse sheaves $\mathcal{P}_E^n$

This section is, in essential, independent of all the rest of the paper. Here we introduce the sheaves $\mathcal{P}_E^n$ on $X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2}$. The purpose is to show that they are perverse. Besides, all the cohomology sheaves of $\mathcal{P}_E^n$ w.r.t. the usual t-structure are calculated (cf. Proposition 15).

Except to the proof of Proposition 7, the results of this section are not needed to prove Main Global Theorem.

4.1 Definition and first properties of $\mathcal{P}_E^n$

4.1.1

Let E be a local system on X of rank 2. Consider the perverse sheaf

$$\mathrm{pr}_1^* \pi_0'^* \mathcal{L}_E^n \otimes \mu^* \mathcal{L}_\psi [2n+1-g] \tag{9}$$

on ${}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2}$. Let $\phi_n : {}^0\,\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2}$ be the morphism defined in sect. 3.1.3.

Proposition 14. *Both*

$$(\phi_n)_!(\mathrm{pr}_1^* \pi_0'^* \mathcal{L}_E^n \otimes \mu^* \mathcal{L}_\psi)[2n+1-g]$$

and

$$(\phi_n)_*(\mathrm{pr}_1^* \pi_0'^* \mathcal{L}_E^n \otimes \mu^* \mathcal{L}_\psi)[2n+1-g]$$

are perverse sheaves on

$$X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2}$$

Definition 7. For any local system E on X of rank 2 put

$$\mathcal{P}_E^n = (\phi_n)_!(\mathrm{pr}_1^* \pi_0'^* \mathcal{L}_E^n \otimes \mu^* \mathcal{L}_\psi)[2n+1-g]$$

In fact, $\mathcal{P}_E^n$ depends also on ψ. We will write $\mathcal{P}_{E,\psi}^n$ if we want to express this dependence.

The proof of Proposition 14 will be given in sect. 4.1.5.

4.1.2

We stratify the stack $\overline{\mathrm{Flag}}_{n+2g-2}$ as follows. The strata are numbered by $m \geq 0$. The stratum indexed by m is $\overline{\mathrm{Flag}}^m_{n+2g-2} = \overline{\mathrm{Flag}}_{n+2g-2} \times_{\mathrm{Sh}^n_1} \mathrm{Sh}^{n;m}_1$. Recall (cf. sect. 1.4.3) that we have a natural isomorphism

$$\mathcal{W}^{n-2m} \times X^{(m)} \tilde{\to} \overline{\mathrm{Flag}}^m_{n+2g-2}$$

Denote by $\mu : X^{(k)} \times_{\mathrm{Pic}^k X} \mathcal{W}^k \to \mathbb{A}^1_k$ the natural pairing. Denote by

$$i' : (X^{(n-2m)} \times_{\mathrm{Pic}^{n-2m} X} \mathcal{W}^{n-2m}) \times X^{(m)} \to X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}^m_{n+2g-2}$$

the closed immersion that sends a collection $(D' \in X^{(n-2m)}, D \in X^{(m)}, 0 \to \Omega \to M \to \mathcal{O}(D') \to 0)$ to $(D' + 2D \in X^{(n)}, (0 \to \Omega \to M \to \mathcal{O}(D') \to 0, D) \in \mathcal{W}^{n-2m} \times X^{(m)})$.

Proposition 15. *For any $m \geq 0$ the restriction of $\mathcal{P}^n_E$ to $X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}^m_{n+2g-2}$ is supported at the closed substack*

$$i' : (X^{(n-2m)} \times_{\mathrm{Pic}^{n-2m} X} \mathcal{W}^{n-2m}) \times X^{(m)} \to X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}^m_{n+2g-2}$$

and is canonically isomorphic to

$$i'_*(\mathrm{pr}_1^* E^{(n-2m)} \otimes \mathrm{pr}_3^*(\det E)^{(m)} \otimes \mathrm{pr}_{12}^* \mu^* \mathcal{L}_\psi)(-m)[2n + 1 - g - 2m]$$

In particular, this restriction coincides with the $(2m - 2n - 1 + g)$-th cohomology sheaf of $\mathcal{P}^n_E$ (w.r.t. the usual t-structure). Besides, $\mathcal{P}^n_E$ has cohomology sheaves (w.r.t. the usual t-structure) in degrees $2m - 2n - 1 + g, 0 \leq m \leq [\frac{n}{2}]$.

Recall that $X^{(m)} \to \mathrm{Sh}^m_0$ is the map that sends a divisor D to $\Omega(D)/\Omega$. Denote by $\mu : \mathrm{Sh}'^m_0 \times_{\mathrm{Sh}^m_0} X^{(m)} \to \mathbb{A}^1_k$ the natural pairing. Let $\gamma'_{n,m} : X^{(n-m)} \times \mathrm{Sh}'^m_0 \to {}^0\mathrm{Sh}'^n_1$ be the morphism defined by the cartesian square

$$\begin{array}{ccc} {}^0\mathrm{Sh}'^n_1 & \stackrel{\gamma'_{n,m}}{\leftarrow} & X^{(n-m)} \times \mathrm{Sh}'^m_0 \\ \downarrow & & \downarrow \\ \mathrm{Sh}^n_1 & \stackrel{\gamma_{n,m}}{\leftarrow} & (\mathrm{Pic}^{n-m} X) \times \mathrm{Sh}^m_0 \end{array}$$

($\gamma_{n,m}$ was defined in sect. 3.1.1). Denote by $\delta : X^{(n-2m)} \times X^{(m)} \to X^{(n-m)} \times X^{(m)}$ the closed immersion that sends (D', D) to $(D' + D, D)$.

Our Proposition 15 will be a consequence of the following result.

Proposition 16. *Let E be a local system on X of rank 2. Consider the sheaf*

$$\mathrm{pr}_{12}^* \gamma_{n,m}'^* \pi_0'^* \mathcal{L}_E^n \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi$$

on $X^{(n-m)} \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)})$. For the projection

$$\mathrm{pr}_{13} : X^{(n-m)} \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}) \to X^{(n-m)} \times X^{(m)}$$

the sheaf $(\mathrm{pr}_{13})_!(\mathrm{pr}_{12}^* \gamma_{n,m}'^* \pi_0'^* \mathcal{L}_E^n \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi)$ is supported at the closed subscheme $\delta : X^{(n-2m)} \times X^{(m)} \to X^{(n-m)} \times X^{(m)}$ and is canonically isomorphic to

$$\delta_*(E^{(n-2m)} \boxtimes (\det E)^{(m)})(-m)[-2m]$$

4.1.3

Following [3], for $0 \le m \le \frac{n}{2}$ denote by $i_{\underline{m}} : X^{(n-2m)} \times X^{(m)} \to \mathrm{Sh}_0^n$ the morphism that sends (D', D) to $\mathcal{O}(D'+D)/\mathcal{O} \oplus \Omega(D)/\Omega$. Our next result is a weakened version of Theorem (3.3.8), [3] in the special case corresponding to the partition $(n-m, m)$ of n and a rank 2 local system. It will be used to define the isomorphism of Proposition 16.

Lemma 32. *Let E be a local system on X of rank 2. For $0 \le m \le \frac{n}{2}$ there is a natural morphism*

$$i_{\underline{m}}^* \mathcal{L}_E^n \to E^{(n-2m)} \boxtimes (\det E)^{(m)}(-m)[-2m]$$

Proof Denote by $j : U \to X^{(n-2m)} \times X^{(m)}$ the open subscheme corresponding to pairs of divisors (D', D), where D' and D are reduced subschemes of X that do not intersect. The morphism $p_{0,\dots,0}^{1,\dots,1} : \mathcal{F}l_{0,\dots,0}^{1,\dots,1} \to \mathrm{Sh}_0^n$ is a locally trivial fibration over the image of U under $i_{\underline{m}}$. It follows that $\mathcal{H}^{2m}(i_{\underline{m}}^* \mathcal{L}_E^n)$ is locally constant over U. Using Theorem (3.3.8) of [3], one shows that

$$\mathcal{H}^{2m}(i_{\underline{m}}^* \mathcal{L}_E^n) \widetilde{\to} E^{(n-2m)} \boxtimes (\det E)^{(m)}(-m)$$

over U. Furthermore, $E^{(n-2m)} \boxtimes (\det E)^{(m)}(-m) \to (\mathrm{R}^0 j_*) j^* E^{(n-2m)} \boxtimes (\det E)^{(m)}(-m)$ is an isomorphism. This provides a morphism

$$\mathcal{H}^{2m}(i_{\underline{m}}^* \mathcal{L}_E^n) \to E^{(n-2m)} \boxtimes (\det E)^{(m)}(-m)$$

and we are done. □

Remark 14. It can be checked that the morphism of the above lemma induces an isomorphism on the highest cohomology sheaves (w.r.t. the usual t-structure)

$$\mathcal{H}^{2m}(i_{\underline{m}}^* \mathcal{L}_E^n) \widetilde{\to} E^{(n-2m)} \boxtimes (\det E)^{(m)}(-m)$$

We will not use this fact. For generalizations of this assertion see Appendix D.

4.1.4

Define the morphism δ' from the cartesian square

$$\begin{array}{ccc} X^{(n-m)} \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}) & \stackrel{\mathrm{pr}_{13}}{\to} & X^{(n-m)} \times X^{(m)} \\ \uparrow \delta' & & \uparrow \delta \\ X^{(n-2m)} \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}) & \to & X^{(n-2m)} \times X^{(m)} \end{array}$$

Define the morphism $p' : X^{(n-2m)} \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}) \to \mathrm{Sh}_0^n$ by $p' = \pi_0' \circ \gamma_{n,m}' \circ \mathrm{pr}_{12} \circ \delta'$.

Proof of Proposition 16

Step 1. The morphism $\gamma_{n,m}$ factors through the locally closed immersion $\mathrm{Sh}_1^{n;m} \to \mathrm{Sh}_1^n$, and the corresponding morphism $(\mathrm{Pic}^{n-m} X) \times \mathrm{Sh}_0^m \to \mathrm{Sh}_1^{n;m}$ is an affine bundle of rank m. Put ${}^0\mathrm{Sh}_1'^{n;m} = {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \mathrm{Sh}_1^{n;m}$ and define $\alpha : X^{(n-m)} \times \mathrm{Sh}_0'^m \to {}^0\mathrm{Sh}_1'^{n;m}$ from the cartesian square

$$\begin{array}{ccc} X^{(n-m)} \times \mathrm{Sh}_0'^m & \stackrel{\alpha}{\to} & {}^0\mathrm{Sh}_1'^{n;m} \\ \downarrow & & \downarrow \\ (\mathrm{Pic}^{n-m} X) \times \mathrm{Sh}_0^m & \to & \mathrm{Sh}_1^{n;m} \end{array}$$

Then pr_{13} can be written as the composition

$$X^{(n-m)} \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}) \stackrel{\alpha\times\mathrm{id}}{\to} {}^0\mathrm{Sh}_1'^{n;m} \times_{\mathrm{Sh}_0^m} X^{(m)} \to X^{(n-m)} \times X^{(m)}$$

The sheaf $\mathrm{pr}_{12}^* \gamma_{n,m}'^* \pi_0'^* \mathcal{L}_E^n$ comes from ${}^0\mathrm{Sh}_1'^{n;m} \times_{\mathrm{Sh}_0^m} X^{(m)}$, so we will use the projection formula for $\alpha \times \mathrm{id}$.

Define the stack $\tilde{\mathcal{Y}}^{n;m}$ from the cartesian square

$$\begin{array}{ccc} {}^0\mathrm{Sh}_1'^{n;m} \times_{\mathrm{Sh}_0^m} X^{(m)} & \to & X^{(n-m)} \times X^{(m)} \\ \uparrow & & \uparrow \delta \\ \tilde{\mathcal{Y}}^{n;m} & \to & X^{(n-2m)} \times X^{(m)} \end{array}$$

So, $\tilde{\mathcal{Y}}^{n;m}$ is the stack classifying the collections: $D \in X^{(m)}$, $D' \in X^{(n-2m)}$ and a diagram

$$\begin{array}{ccccccc} 0 \to & \Omega(D)/\Omega & \to & F_1 & \to & \mathcal{O}(D'+D) & \to 0, \\ & & & \uparrow s & \nearrow & & \\ & & & \mathcal{O} & & & \end{array} \qquad (10)$$

where $\mathcal{O} \to \mathcal{O}(D'+D)$ is the canonical section.

Given an exact sequence $0 \to \Omega(D)/\Omega \to F_1 \to \mathcal{O}(D'+D) \to 0$, there exists a canonical inclusion $\mathcal{O}(D') \to F_1$ making commute the diagram

$$\begin{array}{ccc} F_1 & \to & \mathcal{O}(D'+D) \\ \uparrow & \nearrow & \\ \mathcal{O}(D') & & \end{array}$$

Therefore, there is a distinguished section $s_0 : \mathcal{O} \to F_1$ defined as the composition $\mathcal{O} \to \mathcal{O}(D') \to F_1$. It follows that $\tilde{\mathcal{Y}}^{n;m}$ can be considered as the stack that classifies the collections: $D \in X^{(m)}, D' \in X^{(n-2m)}, s' : \mathcal{O} \to \Omega(D)/\Omega$ and an exact sequence $0 \to \Omega(D)/\Omega \to F_1 \to \mathcal{O}(D'+D) \to 0$. (We put $s = s' + s_0$).

In particular, there is a morphism $\tilde{\beta} : \tilde{\mathcal{Y}}^{n;m} \to \mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}$ that sends a collection as above to (s', D).

One easily checks that $(\alpha \times \mathrm{id})_! \, \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi$ is supported at $\tilde{\mathcal{Y}}^{n;m}$ and is naturally isomorphic to $\tilde{\beta}^* \mu^* \mathcal{L}_\psi(-m)[-2m]$ over $\tilde{\mathcal{Y}}^{n;m}$.

Step 2. The diagram commutes

$$\begin{array}{ccccc} \mathrm{Sh}_0^n & \stackrel{p'}{\leftarrow} & X^{(n-2m)} \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}) & \stackrel{\mathrm{pr}_{13}}{\to} & X^{(n-2m)} \times X^{(m)} \\ & \nwarrow i_{\underline{m}} & \uparrow i_0 & \nearrow \mathrm{id} & \\ & & X^{(n-2m)} \times X^{(m)} & & \end{array}$$

where i_0 is the zero section of the vector bundle pr_{13}. Let us define a morphism

$$(pr_{13})_!(p'^* \mathcal{L}_E^n \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi) \to E^{(n-2m)} \boxtimes (\det E)^{(m)}(-m)[-2m]$$

as follows. Using Lemma 32, we get the morphisms

$$p'^* \mathcal{L}_E^n \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi \to (i_0)_* i_0^* (p'^* \mathcal{L}_E^n \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi) \widetilde{\to} (i_0)_* i_{\underline{m}}^* \mathcal{L}_E^n \to$$
$$(i_0)_* (E^{(n-2m)} \boxtimes (\det E)^{(m)})(-m)[-2m]$$

Applying $(\mathrm{pr}_{13})_!$, we get the morphism

$$(pr_{13})_!(p'^* \mathcal{L}_E^n \otimes \mathrm{pr}_{23}^* \mu^* \mathcal{L}_\psi) \to E^{(n-2m)} \boxtimes (\det E)^{(m)}(-m)[-2m]$$

It remains to show that this is an isomorphism.

Step 3. Denote by $\mathcal{Y}^{n;m}$ the stack that classifies the collections: $D \in X^{(m)}, D' \in X^{(n-2m)}$ and an exact sequence $0 \to \Omega(D)/\Omega \to F \to \mathcal{O}(D'+D)/\mathcal{O} \to 0$. The natural projection $q : \mathcal{Y}^{n;m} \to X^{(n-2m)} \times X^{(m)}$ is a generalized vector bundle of rank 0. We have an isomorphism $\tilde{\mathcal{Y}}^{n;m} \widetilde{\to} \mathcal{Y}^{n;m}$ over $X^{(n-2m)} \times X^{(m)}$ that sends a diagram (10) to

$$(0 \to \Omega(D)/\Omega \to F_1/\operatorname{Im} s \to \mathcal{O}(D'+D)/\mathcal{O} \to 0)$$

Denote by $\beta : \mathcal{Y}^{n;m} \to \mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}$ the composition

$$\mathcal{Y}^{n;m} \widetilde{\to} \tilde{\mathcal{Y}}^{n;m} \stackrel{\tilde{\beta}}{\to} \mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}$$

Denote by $p : \mathcal{Y}^{n;m} \to \mathrm{Sh}_0^n$ the morphism that sends $(D, D', 0 \to \Omega(D)/\Omega \to F \to \mathcal{O}(D'+D)/\mathcal{O} \to 0)$ to F.

We have to show that

$$q_!(p^*\mathcal{L}_E^n \otimes \beta^*\mu^*\mathcal{L}_\psi) \to E^{(n-2m)} \boxtimes (\det E)^{(m)}$$

is an isomorphism. Since the morphism is already constructed, the problem is local w.r.t. X, and one can suppose E to be trivial. However, we will only assume $E = E_1 \oplus E_2$, where E_1, E_2 are rank 1 local systems on X. Recall that on the one hand, by ((3) of Proposition 2, [2]) we have $\mathcal{L}_E^n \widetilde{\to} \oplus_{0 \le c \le n} \mathcal{L}_{E_1,E_2}^{c,n-c}$ and on the other hand, $(E_1 \oplus E_2)^{(n)} \widetilde{\to} \oplus_{0 \le c \le n} (\mathrm{sum})_*(E_1^{(c)} \boxtimes E_2^{(n-c)})$, where $\mathrm{sum} : X^{(c)} \times X^{(n-c)} \to X^{(n)}$ is the summation of divisors.

We will show that $q_!(p^*\mathcal{L}_{E_1,E_2}^{c,n-c} \otimes \beta^*\mu^*\mathcal{L}_\psi)$ vanishes unless $m \le c \le n-m$, and for $m \le c \le n-m$ there is an isomorphism

$$q_!(p^*\mathcal{L}_{E_1,E_2}^{c,n-c} \otimes \beta^*\mu^*\mathcal{L}_\psi) \widetilde{\to} ((\mathrm{sum})_!(E_1^{(c-m)} \boxtimes E_2^{(n-m-c)})) \boxtimes (E_1 \otimes E_2)^{(m)}$$

Recall that

$$p^*\mathcal{L}_{E_1,E_2}^{c,n-c} \widetilde{\to} (\mathrm{pr}_1)!\, \mathrm{pr}_2^*(q_{0,0}^{c,n-c})^*(\mathrm{div} \times \mathrm{div})^*(E_1^{(c)} \boxtimes E_2^{(n-c)}),$$

where the involved morphisms are illustrated by the diagram

$$\begin{array}{ccccccc} \mathcal{Y}^{n;m} \times_{\mathrm{Sh}_0^n} \mathcal{F}l_{0,0}^{c,n-c} & \stackrel{\mathrm{pr}_2}{\to} & \mathcal{F}l_{0,0}^{c,n-c} & \stackrel{q_{0,0}^{c,n-c}}{\to} & \mathrm{Sh}_0^c \times \mathrm{Sh}_0^{n-c} & \stackrel{\mathrm{div}\times\mathrm{div}}{\to} & X^{(c)} \times X^{(n-c)} \\ \downarrow \mathrm{pr}_1 & & & & & & \\ \mathcal{Y}^{n;m} & & & & & & \end{array}$$

So, the problem is to calculate the direct image of

$$\mathrm{pr}_2^*(q_{0,0}^{c,n-c})^*(\mathrm{div} \times \mathrm{div})^*(E_1^{(c)} \boxtimes E_2^{(n-c)}) \otimes \mathrm{pr}_1^*\beta^*\mu^*\mathcal{L}_\psi \tag{11}$$

under the composition

$$\mathcal{Y}^{n;m} \times_{\mathrm{Sh}_0^n} \mathcal{F}l_{0,0}^{c,n-c} \to \mathcal{Y}^{n;m} \stackrel{q}{\to} X^{(n-2m)} \times X^{(m)}$$

To do so we stratify the stack $\mathcal{Y}^{n;m} \times_{\mathrm{Sh}_0^n} \mathcal{F}l_{0,0}^{c,n-c}$ by locally closed substacks U_{c_1,c_2} parametrized by pairs (c_1, c_2), where $0 \le c_1 \le m, 0 \le c_2 \le n-m$ and $c_1 + c_2 = c$. We say that a point

$$(D, D', 0 \to \Omega(D)/\Omega \to F \to \mathcal{O}(D' + D)/\mathcal{O} \to 0, F' \subset F) \tag{12}$$

is a point of U_{c_1,c_2} if

$$\mathrm{length}(F_0) = c_1 \text{ and } \mathrm{length}(G_0) = c_2,$$

where $F_0 = (\Omega(D)/\Omega) \cap F'$ and G_0 is the image of F' in $\mathcal{O}(D' + D)/\mathcal{O}$. Put $D_0 = \mathrm{div}\, F_0, D_1 = \mathrm{div}\, G_0$. We have a map

$$g_{c_1,c_2} : U_{c_1,c_2} \to (X^{(c_2)} \times X^{(n-m-c_2)}) \times (X^{(c_1)} \times X^{(m-c_1)})$$

that sends (12) to $(D_1, D' + D - D_1, D_0, D - D_0)$. It factors through the closed immersion δ_{c_1,c_2} defined by the cartesian square

$$\begin{array}{ccc} X^{n,m}_{c_1,c_2} & \stackrel{\delta_{c_1,c_2}}{\to} & (X^{(c_2)} \times X^{(n-m-c_2)}) \times (X^{(c_1)} \times X^{(m-c_1)}) \\ \downarrow & & \downarrow \mathrm{sum} \times \mathrm{sum} \\ X^{(n-2m)} \times X^{(m)} & \stackrel{\delta}{\to} & X^{(n-m)} \times X^{(m)} \end{array}$$

Notice that $U_{c_1,c_2} \to X^{n,m}_{c_1,c_2}$ is a generalized (non-representable) vector fibration of rank 0. The diagram commutes

$$\begin{array}{ccccc} \mathcal{Y}^{n;m} \times_{\mathrm{Sh}_0^n} \mathcal{F}l_{0,0}^{c,n-c} & \to \mathcal{Y}^{n;m} & \stackrel{q}{\to} & X^{(n-2m)} \times X^{(m)} \\ \uparrow & & & \uparrow \\ U_{c_1,c_2} & & \longrightarrow & X^{n,m}_{c_1,c_2} \end{array}$$

The restriction of $\mathrm{pr}_2^*(q_{0,0}^{c,n-c})^*(\mathrm{div} \times \mathrm{div})^*(E_1^{(c)} \boxtimes E_2^{(n-c)})$ to U_{c_1,c_2} is isomorphic to the inverse image of $E_1^{(c)} \boxtimes E_2^{(n-c)}$ under the composition

$$U_{c_1,c_2} \stackrel{g_{c_1,c_2}}{\to} (X^{(c_2)} \times X^{(n-m-c_2)}) \times (X^{(c_1)} \times X^{(m-c_1)}) \to X^{(c)} \times X^{(n-c)}$$

Denote by $\mathcal{T}_{c_1,c_2}$ the restriction of $\mathrm{pr}_1^* \beta^* \mu^* \mathcal{L}_\psi$ to U_{c_1,c_2}. For $m \le c \le n-m$ the closed subscheme of $X^{n,m}_{0,c}$ defined by the condition $D_1 \ge D$ is given by the obvious closed immersion $(X^{(c-m)} \times X^{(n-m-c)}) \times X^{(m)} \to X^{n,m}_{0,c}$.

Lemma 33. *We have $(g_{c_1,c_2})_! \mathcal{T}_{c_1,c_2} = 0$ unless $c_1 = 0, c_2 \ge m$. If $m \le c \le n-m$ then $(g_{0,c})_! \mathcal{T}_{0,c}$ is supported at the closed subscheme $(X^{(c-m)} \times X^{(n-m-c)}) \times X^{(m)} \to X^{n,m}_{0,c}$ and is naturally isomorphic to the constant sheaf $\bar{\mathbb{Q}}_\ell$ over it.*

Proof of the lemma We have the morphisms

$$U_{c_1,c_2} \to \mathcal{Y}^{n;m} \times_{(X^{(n-2m)}\times X^{(m)})} X^{n,m}_{c_1,c_2} \stackrel{(\mu\circ\beta)\times \mathrm{id}}{\to} \mathbb{A}^1 \times X^{n,m}_{c_1,c_2}$$

of generalized vector bundles over $X^{n,m}_{c_1,c_2}$. Denote by ξ their composition. Fix a point $\operatorname{Spec} k \to X^{n,m}_{c_1,c_2}$ given by a collection of divisors (D, D', D_0, D_1) as above. Consider the morphism $\xi \times \mathrm{id} : U_{c_1,c_2} \times_{X^{n,m}_{c_1,c_2}} \operatorname{Spec} k \to \mathbb{A}^1_k$ (of generalized vector bundles over $\operatorname{Spec} k$) obtained from ξ by the base change $\operatorname{Spec} k \to X^{n,m}_{c_1,c_2}$.

1) Let us show that $\xi \times \mathrm{id}$ vanishes if and only if $D_0 = 0$ and $D_1 \geq D$.

The stack $U_{c_1,c_2} \times_{X^{n,m}_{c_1,c_2}} \operatorname{Spec} k$ classifies the diagrams

$$\begin{array}{ccccccc} 0 \to & \Omega(D)/\Omega & \to & F & \to & \mathcal{O}(D'+D)/\mathcal{O} & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & \Omega(D_0)/\Omega & \to & F' & \to & \mathcal{O}(D_1)/\mathcal{O} & \to 0 \end{array}$$

Define an object K of the derived category of k-vector spaces from the exact triangle $K \to \mathrm{RHom}(\mathcal{O}(D'+D)/\mathcal{O}, \Omega(D)/\Omega) \to \mathrm{RHom}(\mathcal{O}(D_1)/\mathcal{O}, \Omega(D)/\Omega(D_0))$. The coarse moduli space of $U_{c_1,c_2} \times_{X^{n,m}_{c_1,c_2}} \operatorname{Spec} k$ coincides with $\mathrm{H}^1(K)$ (if needed, we identify a k-vector space and the corresponding k-scheme, whose set of k-points is this space). The morphism $\xi \times \mathrm{id}$ is the compostion of linear maps

$$U_{c_1,c_2} \times_{X^{n,m}_{c_1,c_2}} \operatorname{Spec} k \to \mathrm{H}^1(K) \to \mathbb{A}^1_k$$

The corresponding linear functional $\mathrm{H}^1(K) \to k$ is the composition

$$\mathrm{H}^1(K) \to \mathrm{Ext}^1(\mathcal{O}(D'+D)/\mathcal{O}, \Omega(D)/\Omega) \stackrel{f_0}{\to} k,$$

where $f_0 \in (\mathrm{Ext}^1(\mathcal{O}(D'+D)/\mathcal{O}, \Omega(D)/\Omega))^* \widetilde{\to} \mathrm{Hom}(\Omega(D)/\Omega, \Omega(D'+D)/\Omega)$ is the canonical inclusion $f_0 : \Omega(D)/\Omega \hookrightarrow \Omega(D'+D)/\Omega$. The sequence is exact

$$\mathrm{H}^1(K) \to \mathrm{Ext}^1(\mathcal{O}(D'+D)/\mathcal{O}, \Omega(D)/\Omega) \stackrel{g}{\to} \mathrm{Ext}^1(\mathcal{O}(D_1)/\mathcal{O}, \Omega(D)/\Omega(D_0))$$

Besides, f_0 vanishes on $\mathrm{Ker}(g)$ if and only if $D_0 = 0$ and $D_1 \geq D$. Our assertion follows.

2) Now let $m \leq c \leq n-m$. It is easy to see that the restriction of $\mathcal{T}_{0,c}$ to $((X^{(c-m)} \times X^{(n-m-c)}) \times X^{(m)}) \times_{X^{n,m}_{0,c}} U_{0,c}$ is the constant sheaf $\bar{\mathbb{Q}}_\ell$. Our lemma follows. $\square$ (Lemma 33)

For $m \leq c \leq n-m$ the diagram commutes

$$\begin{array}{ccc} (X^{(c-m)} \times X^{(n-m-c)}) \times X^{(m)} & \stackrel{\text{sum} \times \text{id}}{\rightarrow} & X^{(n-2m)} \times X^{(m)} \\ & \searrow & \uparrow \\ & & X^{n,m}_{0,c} \end{array}$$

Lemma 33 implies the degeneration of the spectral sequence that computes the direct image $(q \circ \text{pr}_1)_!$ of the sheaf (11) w.r.t. the stratification by the U_{c_1,c_2}'s. Thus,

$$(q \circ \text{pr}_1)_!(\text{pr}_2^*(q^{c,n-c}_{0,0})^*(\text{div} \times \text{div})^*(E_1^{(c)} \boxtimes E_2^{(n-c)}) \otimes \text{pr}_1^* \beta^* \mu^* \mathcal{L}_\psi) \qquad (13)$$

vanishes unless $m \leq c \leq n-m$. If $m \leq c \leq n-m$ then the only stratum that contributes to this direct image is the open one, and (13) is isomorphic to the direct image of $(E_1^{(c-m)} \boxtimes E_2^{(n-m-c)}) \boxtimes (E_1 \otimes E_2)^{(m)}$ under

$$\text{sum} \times \text{id} : (X^{(c-m)} \times X^{(n-m-c)}) \times X^{(m)} \to X^{(n-2m)} \times X^{(m)}$$

This concludes the proof.
□ (Proposition 16)

4.1.5

Proof of Proposition 15
Step 1. We have a closed immersion

$$i'' : X^{(n-m)} \times_{\text{Pic}^{n-m} X} \overline{\text{Flag}}^m_{n+2g-2} \to X^{(n)} \times_{\text{Pic}^n X} \overline{\text{Flag}}^m_{n+2g-2}$$

that sends $(D \in X^{(m)}, D_1 \in X^{(n-m)}, (0 \to \Omega \to F \to \mathcal{O}(D_1 - D) \to 0) \in \mathcal{W}^{n-2m})$ to $(D + D_1 \in X^{(n)}, D \in X^{(m)}, (0 \to \Omega \to F \to \mathcal{O}(D_1 - D) \to 0) \in \mathcal{W}^{n-2m})$. The morphism

$$\phi_n \times \text{id} : {}^0\text{Sh}'^n_1 \times_{Sh^n_1} \overline{\text{Flag}}^m_{n+2g-2} \to X^{(n)} \times_{\text{Pic}^n X} \overline{\text{Flag}}^m_{n+2g-2} \qquad (14)$$

obtained from ϕ_n by the base change $\overline{\text{Flag}}^m_{n+2g-2} \to \overline{\text{Flag}}_{n+2g-2}$ factors through i''. After the base change $(\text{Pic}^{n-m} X) \times \text{Sh}^m_0 \to \text{Sh}^{n;m}_1$ the morphism

$${}^0\text{Sh}'^n_1 \times_{\text{Sh}^n_1} \overline{\text{Flag}}^m_{n+2g-2} \to X^{(n-m)} \times_{\text{Pic}^{n-m} X} \overline{\text{Flag}}^m_{n+2g-2}$$

becomes the projection

$$(X^{(n-m)} \times_{(\text{Pic}^{n-m} X)} \mathcal{W}^{n-m}) \times (\text{Sh}'^m_0 \times_{\text{Sh}^m_0} X^{(m)}) \to (X^{(n-m)} \times_{(\text{Pic}^{n-m} X)} \mathcal{W}^{n-m}) \times X^{(m)}$$

The restriction of the sheaf (9) to

$$(X^{(n-m)} \times_{(\mathrm{Pic}^{n-m} X)} \mathcal{W}^{n-m}) \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)})$$

is

$$\mathrm{pr}_{13}^* \gamma_{n,m}'^* \pi_0'^* \mathcal{L}_E^n \otimes \mathrm{pr}_{12}^* \mu^* \mathcal{L}_\psi \otimes \mathrm{pr}_{34}^* \mu^* \mathcal{L}_\psi [2n+1-g]$$

So, by Proposition 16, the restriction of $\mathcal{P}_E^n$ to

$$(X^{(n-m)} \times_{(\mathrm{Pic}^{n-m} X)} \mathcal{W}^{n-m}) \times X^{(m)}$$

is isomorphic to

$$\mathrm{pr}_{12}^* \mu^* \mathcal{L}_\psi \otimes (\mathrm{pr}_{13}^* \delta_* (E^{(n-2m)} \boxtimes (\det E)^{(m)}))(-m)[2n+1-g-2m]$$

In particular, it follows that the restriction of $\mathcal{P}_E^n$ to $X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2}^m$ is supported at

$$i' : (X^{(n-2m)} \times_{\mathrm{Pic}^{n-2m} X} \mathcal{W}^{n-2m}) \times X^{(m)} \to X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2}^m$$

Step 2. After the base change under i' the morphism (14) becomes the projection

$$(X^{(n-2m)} \times_{(\mathrm{Pic}^{n-2m} X)} \mathcal{W}^{n-2m}) \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)}) \to$$
$$(X^{(n-2m)} \times_{(\mathrm{Pic}^{n-2m} X)} \mathcal{W}^{n-2m}) \times X^{(m)}$$

The restriction of our perverse sheaf (9) to

$$(X^{(n-2m)} \times_{(\mathrm{Pic}^{n-2m} X)} \mathcal{W}^{n-2m}) \times (\mathrm{Sh}_0'^m \times_{\mathrm{Sh}_0^m} X^{(m)})$$

is isomorphic to

$$\mathrm{pr}_{12}^* \mu^* \mathcal{L}_\psi \otimes \mathrm{pr}_{134}^* p'^* \mathcal{L}_E^n \otimes \mathrm{pr}_{34}^* \mu^* \mathcal{L}_\psi [2n+1-g]$$

Applying again Proposition 16 we get the desired result. □

Lemma 34. *1) The natural morphism* $\mathrm{Sh}_1'^n \to \mathrm{Pic}'^n X \times_{\mathrm{Pic}^n X} \mathrm{Sh}_1^n$ *is affine.*
2) The square

$$\begin{array}{ccc} \mathrm{Sh}_1'^n & \to & \mathrm{Pic}'^n X \times_{\mathrm{Pic}^n X} \mathrm{Sh}_1^n \\ \uparrow & & \uparrow \\ {}^0\mathrm{Sh}_1'^n & \to & X^{(n)} \times_{\mathrm{Pic}^n X} \mathrm{Sh}_1^n \end{array}$$

is cartesian. In particular,

$$\phi_n : {}^0\mathrm{Sh}_1'^n \times_{\mathrm{Sh}_1^n} \overline{\mathrm{Flag}}_{n+2g-2} \to X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2}$$

is an affine morphism.

Proof 1) Recall that the map $\mathrm{Sh}'^n_1 \to \mathrm{Pic}'^n X$ was defined in sect. 1.2.3. It yields the morphism $\mathrm{Sh}'^n_1 \to \mathrm{Pic}'^n X \times_{\mathrm{Pic}^n X} \mathrm{Sh}^n_1$.

Suppose that $S \in (\mathrm{Aff}/k)$ and we are given a morphism $S \to \mathrm{Pic}'^n X \times_{\mathrm{Pic}^n X} \mathrm{Sh}^n_1$ defined by a sheaf $F_1 \in (\mathrm{Sh}^n_1)_S$ and a section $\mathcal{O} \stackrel{s}{\to} \det F_1$. Let Z be the S-scheme of sections of F_1, Z' be the S-scheme of sections of $\det F_1$. The mapping $F_1 \to \det F_1$ provides an affine morphism $Z \to Z'$ over S. The section $\mathcal{O} \stackrel{s}{\to} \det F_1$ defines a morphism $S \to Z'$ over S. Put $\tilde{Z} = S \times_{Z'} Z$. Then the square

$$\begin{array}{ccc} \mathrm{Sh}'^n_1 & \to & \mathrm{Pic}'^n X \times_{\mathrm{Pic}^n X} \mathrm{Sh}^n_1 \\ \uparrow & & \uparrow \\ \tilde{Z} & \to & S \end{array}$$

is cartesian.
2) is straightforward. □

Proof of Proposition 14

By 2) of Lemma 34, $\mathcal{P}^n_E$ lives in non-negative perverse dimensions. On the other hand, by Proposition 15, $\mathcal{P}^n_E$ has (usual) cohomology sheaves in degrees $2m-(2n+1-g)$ for $0 \le m \le [\frac{n}{2}]$, and the support of the cohomology sheaf with number $2m-(2n+1-g)$ has dimension $(2n+1-g)-3m$. So, the support condition is satisfied, $\mathcal{P}^n_E$ lives in non-positive perverse dimensions. Thus, $\mathcal{P}^n_E$ is perverse.

To show the perversity of $(\phi_n)_*(\mathrm{pr}_1^* \pi_0'^* \mathcal{L}^n_E \otimes \mu^* \mathcal{L}_\psi)[2n+1-g]$ notice that the dual to the perverse sheaf (9) is

$$\mathrm{pr}_1^* \pi_0'^* \mathcal{L}^n_{E^*} \otimes \mu^* \mathcal{L}_{\psi^{-1}}(2n+1-g)[2n+1-g]$$

and use the Poincaré duality.
□ (Proposition 14)

4.1.6

In this subsection we explain how to express the complex $\mathcal{F}^n_{E_1,E_2}$ in terms of $\mathcal{P}^n_E$. (As a corolary of Proposition 15 we will get another proof of the fact that $\mathcal{F}^n_{E_1,E_2}[-n]$ is a $\bar{\mathbb{Q}}_\ell$-sheaf and another proof of Proposition 12).

Lemma 35. *Let E_1, E_2 be local systems on X of rank 2. Then there is a natural isomorphism*

$$(\mathrm{pr}_1)_!(\mathcal{P}^n_{E_1,\psi} \otimes \mathcal{P}^n_{E_2,\psi^{-1}})[-n] \widetilde{\to} \mathcal{F}^n_{E_1,E_2},$$

where

$$\mathrm{pr}_1 : X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2} \to X^{(n)}$$

is the projection.

Proof
1) Consider the morphism $[-1] : {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} \overline{\operatorname{Flag}}_{n+2g-2} \to {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} \overline{\operatorname{Flag}}_{n+2g-2}$ that sends $\mathcal{O} \stackrel{s}{\hookrightarrow} F_1$ to $\mathcal{O} \stackrel{-s}{\hookrightarrow} F_1$ and does not change an extension of F_1. The inverse image of $\mu^*\mathcal{L}_\psi$ under $[-1]$ is $\mu^*\mathcal{L}_{\psi^{-1}}$. We have also the morphism

$$[-1] : X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2}$$

that sends $\mathcal{O} \stackrel{s}{\hookrightarrow} \det F_1$ to $\mathcal{O} \stackrel{-s}{\hookrightarrow} \det F_1$ and preserves an extension of F_1. Clearly, the inverse image $[-1]^*\mathcal{P}^n_{E,\psi}$ is $\mathcal{P}^n_{E,\psi^{-1}}$.
2) One easily checks that the direct image (with compact support) of the sheaf

$$\operatorname{pr}_1^* \pi_0'^* \mathcal{L}^n_{E_1} \otimes \operatorname{pr}_2^* \pi_0'^* \mathcal{L}^n_{E_2} \otimes \operatorname{pr}_{13}^* \mu^*\mathcal{L}_\psi \otimes \operatorname{pr}_{23}^* \mu^*\mathcal{L}_\psi[3n+2-2g]$$

under the morphism

$${}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} {}^0\operatorname{Sh}_1'^n \times_{\operatorname{Sh}_1^n} \overline{\operatorname{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2}$$

is naturally isomorphic to

$$\operatorname{pr}_{13}^* \mathcal{P}^n_{E_1} \otimes \operatorname{pr}_{23}^* \mathcal{P}^n_{E_2}[-n] \tag{15}$$

The direct image (with compact suport) of the latter sheaf under the projection pr_{12} is supported at $i : X^{(n)} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)}$ and is naturally isomorphic to $i_*\mathcal{F}^n_{E_1,E_2}$. To complete the proof it remains to notice that, by 1), the restriction of (15) under

$$i \times \operatorname{id} : X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2} \to X^{(n)} \times_{\operatorname{Pic}^n X} X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2}$$

is $\mathcal{P}^n_{E_1,\psi} \otimes \mathcal{P}^n_{E_2,\psi^{-1}}$.
□

Denote by $\sigma_m : X^{(n-2m)} \times X^{(m)} \to X^{(n)}$ the morphism that sends (D_1, D_2) to $D_1 + 2D_2$.

Another proof of the fact that $\mathcal{F}^n_{E_1,E_2}[-n]$ is a $\bar{\mathbb{Q}}_\ell$-sheaf.

We calculate the direct image $(\operatorname{pr}_1)_!(\mathcal{P}^n_{E_1,\psi} \otimes \mathcal{P}^n_{E_2,\psi^{-1}})[-n]$ using the stratification of $X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}_{n+2g-2}$ by locally closed substacks $X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}^m_{n+2g-2}$, $(m \geq 0)$.

Denote by $\mathcal{Q}^m$ the restriction of $\mathcal{P}^n_{E_1,\psi} \otimes \mathcal{P}^n_{E_2,\psi^{-1}}[-n]$ to $X^{(n)} \times_{\operatorname{Pic}^n X} \overline{\operatorname{Flag}}^m_{n+2g-2}$. Our assertion is an immediate consequence of Lemma 36. □

Lemma 36. *For $0 \leq m \leq \frac{n}{2}$ we have a natural isomorphism*

$$(\mathrm{pr}_1)_! \mathcal{Q}^m \widetilde{\to} (\sigma_m)_* ((E_1^{(n-2m)} \otimes E_2^{(n-2m)}) \boxtimes ((\det E_1)^{(m)} \otimes (\det E_2)^{(m)}))(-n-1+g)[n],$$

where

$$\mathrm{pr}_1 : X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}^m_{n+2g-2} \to X^{(n)}$$

is the projection. If $m > \frac{n}{2}$ then $(\mathrm{pr}_1)_! \mathcal{Q}^m = 0$.

Proof The diagram commutes

$$\begin{array}{ccc} i' : (X^{(n-2m)} \times_{\mathrm{Pic}^{n-2m} X} \mathcal{W}^{n-2m}) \times X^{(m)} & \stackrel{i'}{\to} & X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}^m_{n+2g-2} \\ \downarrow \mathrm{pr}_{13} & & \downarrow \mathrm{pr}_1 \\ X^{(n-2m)} \times X^{(m)} & \stackrel{\sigma_m}{\to} & X^{(n)} \end{array}$$

By Proposition 15, $\mathcal{Q}^m$ is naturally isomorphic to

$$i'_* \mathrm{pr}_{13}^* ((E_1^{(n-2m)} \otimes E_2^{(n-2m)}) \boxtimes ((\det E_1)^{(m)} \otimes (\det E_2)^{(m)}))(-2m)[3n+2-2g-4m]$$

So, our assertion follows from the fact that for any k the morphism $\mathcal{W}^k \to \mathrm{Pic}^k X$ is a generalized vector fibration of rank $k+1-g$.
□ (Lemma 36)

Remark 15. The above argument also shows that on $\mathcal{F}^n_{E_1,E_2}(n+1-g)[-n]$ there is a canonical filtration

$$0 \subset F_0 \subset F_1 \subset \cdots \subset F_{[\frac{n}{2}]} = \mathcal{F}^n_{E_1,E_2}(n+1-g)[-n]$$

by constructable $\bar{\mathbb{Q}}_\ell$-subsheaves such that

$$F_m / F_{m-1} \widetilde{\to} (\sigma_m)_* ((E_1^{(n-2m)} \otimes E_2^{(n-2m)}) \boxtimes ((\det E_1)^{(m)} \otimes (\det E_2)^{(m)}))$$

for $0 \leq m \leq \frac{n}{2}$. (Here $F_{-1} = 0$).

Another proof of Proposition 12
Over $X^{(n)} - \triangle$ the filtration from Remark 15 is reduced to the isomorphism

$$F_0 \widetilde{\to} \mathcal{F}^n_{E_1,E_2}(n+1-g)[-n]$$

Since $F_0 \widetilde{\to} E_1^{(n)} \otimes E_2^{(n)}$ and the natural inclusion $E_1^{(n)} \otimes E_2^{(n)} \to (E_1 \otimes E_2)^{(n)}$ is an isomorphism over $X^{(n)} - \triangle$, our assertion follows. □

Remark 16. Let E_1, E_2 be local sytems on X of rank 2. One can construct a canonical filtration $0 \subset F'_0 \subset F'_1 \subset \cdots \subset F'_{[\frac{n}{2}]} = (E_1 \otimes E_2)^{(n)}$ on $(E_1 \otimes E_2)^{(n)}$ by constructable $\mathbb{Q}_\ell$-subsheaves with successive quotients

$$F'_m / F'_{m-1} \widetilde{\to} F_m / F_{m-1},$$

where (F_i) is the filtration from Remark 15. So, at the level of functions 'trace of Frobenius' (in other words, in the classical theory of automorphic forms) this is the end of the proof of Main Local Theorem . However, in the geometrical theory one still has to show that not only the corresponding graded objects but the filtered objects itself are isomorphic. This was one of the main difficulties of the geometrization of the classical Rankin-Selberg relations.

4.2 An application

In this subsection we prove Proposition 7.

Proof of Proposition 7
For $n \geq 0$ for the projection $\mathrm{pr}_2 : X^{(n)} \times_{\mathrm{Pic}^n X} \overline{\mathrm{Flag}}_{n+2g-2} \to \overline{\mathrm{Flag}}_{n+2g-2}$ we have $(\mathrm{pr}_2)_! \mathcal{P}^n_E(\frac{n+1-g}{2}) \widetilde{\to} \mathcal{K}^{n+2g-2}_E$. Our assertion follows immediately from Proposition 15. $\square$

A Cohomology with compact support of a stack

Let Λ be a noetherian ring such that $\operatorname{char}\Lambda$ is invertible in k.

Definition 8 ([7],(18.7.1)). A representable morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ of algebraic stacks is called universally n-acyclic (w.r.t. Λ) if for any morphism $U \to \mathcal{Y}_2$ with $U \in (\mathrm{Aff}/k)$, any étale sheaf $\mathcal{F}$ of Λ-modules on U the natural map $\mathcal{F} \to \mathrm{R}^0(f_U)_* f_U^* \mathcal{F}$ is an isomorphism and $\mathrm{R}^i(f_U)_*\mathcal{F} = 0$ for $1 \le i \le n$, where $f_U : U \times_{\mathcal{Y}_2} \mathcal{Y}_1 \to U$ is the projection.

Notice that the composition of two representable and universally n-acyclic morphisms is also universally n-acyclic. A morphism obtained by a base change from a representable and universally n-acyclic morphism is also universally n-acyclic.

Definition 9 ([7],(18.7.4)). An algebraic stack $\mathcal{X}$ is a Bernstein-Lunts stack (w.r.t. Λ) if for any $n \ge 0$ there exists a presentation of finite type $\pi : S \to \mathcal{X}$ such that π is universally n-acyclic (w.r.t. Λ), and S is separated.

Remark 17. i) If $\mathcal{X}$ is a Bernstein-Lunts stack then for any $n \ge 0$ there exists a presentation $\pi : S \to \mathcal{X}$ as in the above definition such that π is, in addition, smooth of constant relative dimension.
ii) If $\mathcal{X}_1 \to \mathcal{X}_2$ is a representable and separated morphism of algebraic stacks, and $\mathcal{X}_2$ is a Bernstein-Lunts stack then $\mathcal{X}_1$ is also a Bernstein-Lunts stack.

Lemma 37. *If M is a separated algebraic space with an action of an affine algebraic group G then M/G is a Bernstein-Lunts stack (w.r.t. Λ).*

Proof (cf. [7], Lemma (18.7.5)). To construct a presentation $\pi : S \to M/G$ as in definition 9, it is enough to construct a universally n-acyclic smooth algebraic space U of finite type with a free action of G on U such that U/G is a separated algebraic space. Then we can set $S = (M \times U)/G$. Since $G \subset \mathrm{GL}(V)$ for some vector space V, it is enough to construct U in the case $G = \mathrm{GL}(V)$. Then we can take the Stieffel variety $U = \mathrm{Inj}(V, W) \subset \mathrm{Hom}(V, W)$ that parametrizes injective linear operators $V \to W$, where W is a vector space such that $\dim W \gg \dim V$ (if $\dim W \ge \dim V$ then the codimension in $\mathrm{Hom}(V, W)$ of the set of non-injective operators equals $\dim W - \dim V + 1$). $\Box$

Lemma 38. *The stack $\overline{\mathrm{Bun}}_i^n$ is a Bernstein-Lunts stack (w.r.t. Λ).*

Proof Consider the morphism $\tilde{\pi} : \overline{\mathrm{Bun}}_i^n \to \mathrm{Bun}_{\mathrm{PGL}_i}$ (cf. Remark 6, sect.1.2.1). It is representable and separated. By Lemma 37, $\mathrm{Bun}_{\mathrm{PGL}_i}$ is a Bernstein-Lunts stack. So, we are done. $\Box$

Let Y be an algebraic space of finite type, $\mathcal{X}$ be a Bernstein-Lunts stack, and $f : \mathcal{X} \to Y$ be a morphism of finite type. Let F be a bounded complex on $\mathcal{X}$. Pick an integer b such that F has nontrivial cohomology sheaves only in the usual degrees $\leq b$. Let d be an integer such that all the geometric fibres of f are of dimension $\leq d$. Following ([7], (18.8)) we will define for any $n \geq 0$ the complex $\tau_{\geq 2d+b-n}(f_! F)$ on Y. (Here $\tau_{\leq c}, \tau_{\geq c}$ denote the troncation functors corresponding to the usual t-structure).

Let $\pi : S \to \mathcal{X}$ be a presentation of finite type and universally n-acyclic, with S being separated. Then $f \circ \pi$ is also separated. We can suppose that π is smooth of constant relative dimension, say e.

First, consider the case, where f is representable and separated. Then $\mathcal{X}$ is an algebraic space, and π is separated. Using the Poincaré duality we see that the natural map $\pi_! \Lambda \to \Lambda(-e)[-2e]$ induces an isomorphism $\tau_{\geq 2e-n} \pi_! \Lambda \widetilde{\to} \Lambda(-e)[-2e]$. It follows that the natural map $f_! \pi_! \pi^* F(e)[2e] \to f_! F$ induces an isomorphism

$$\tau_{\geq 2d+b-n}((f \circ \pi)_! \pi^* F(e)[2e]) \widetilde{\to} \tau_{\geq 2d+b-n}(f_! F)$$

So, we return to the general case and put

$$\tau_{\geq 2d+b-n}(f_! F)_S = \tau_{\geq 2d+b-n}((f \circ \pi)_! \pi^* F(e)[2e])$$

(The subscript S means that this complex depends on the presentation $\pi : S \to \mathcal{X}$).

If, in addition, $\pi' : S' \to S$ is a smooth and surjective morphism of relative dimension e', and π' is universally n-acyclic then we have a natural isomorphism

$$\tau_{\geq 2d+b-n}((f \circ \pi \circ \pi')_! \pi'^* \pi^* F(e+e')[2e+2e']) \widetilde{\to} \tau_{\geq 2d+b-n}((f \circ \pi)_! \pi^* F(e)[2e])$$

Given two presentations $\pi_1 : S_1 \to \mathcal{X}, \pi_2 : S_2 \to \mathcal{X}$ as above, we define the isomorphism

$$\tau_{\geq 2d+b-n}(f_! F)_{S_1} \widetilde{\to} \tau_{\geq 2d+b-n}(f_! F)_{S_2}$$

to be the composition

$$\tau_{\geq 2d+b-n}(f_! F)_{S_1} \widetilde{\leftarrow} \tau_{\geq 2d+b-n}(f_! F)_{S_{12}} \widetilde{\to} \tau_{\geq 2d+b-n}(f_! F)_{S_2},$$

where $S_{12} = S_1 \times_{\mathcal{X}} S_2$. If $\pi_1 = \pi_2$ then this is the identical isomorphism. Given three presentations $\pi_i : S_i \to \mathcal{X}$ $(i = 1, 2, 3)$ as above, one checks the commutativity of the diagram

$$\begin{array}{ccc} \tau_{\geq 2d+b-n}(f_! F)_{S_1} & \to & \tau_{\geq 2d+b-n}(f_! F)_{S_2} \\ \downarrow & \swarrow & \\ \tau_{\geq 2d+b-n}(f_! F)_{S_3} & & \end{array}$$

using $S_{123} = S_1 \times_{\mathcal{X}} S_2 \times_{\mathcal{X}} S_3$. So, $\tau_{\geq 2d+b-n}(f_! F)_S$ does not depend on a presentation $\pi : S \to \mathcal{X}$ and we can skip the subscript S.

Example 1. Consider the classifying stack $B(\mathbb{G}_m)$ of the multiplicative group. We have $\mathrm{H}^i_c(B(\mathbb{G}_m), \Lambda) = \Lambda(n)$ if $i = -2n, n \in \mathbb{N}$, and $\mathrm{H}^i_c(B(\mathbb{G}_m), \Lambda) = 0$ for all other i.

Example 2. Let G be a finite group such that the order of G is invertible in Λ. Then $\mathrm{H}^0_c(B(G), \Lambda) = \Lambda$ and $\mathrm{H}^i_c(B(G), \Lambda) = 0$ for $i \neq 0$. (When working with ℓ-adic sheaves instead of torsion sheaves the assumption on the order of G is not neccesary).

B Some linear algebra

Let A be a commutative ring of characteristic 0. Let M be a bounded complex of A-modules. Then S_n acts on the complex $\overset{n}{\underset{i=1}{\otimes}} M$, and we put $\mathrm{Sym}^n(M) = (\overset{n}{\underset{i=1}{\otimes}} M)^{S_n}$. Clearly, $\mathrm{Sym}^{k+l}(M)$ is a direct summand of $\mathrm{Sym}^k(M) \otimes_A \mathrm{Sym}^l(M)$, so that we have both natural morphisms $\mathrm{Sym}^k(M) \otimes_A \mathrm{Sym}^l(M) \to \mathrm{Sym}^{k+l}(M)$ and $\mathrm{Sym}^{k+l}(M) \to \mathrm{Sym}^k(M) \otimes_A \mathrm{Sym}^l(M)$.

We will be interested in complexes M of the form $(\cdots \to M^{-2} \to M^{-1} \to A \to 0)$, i.e., we suppose that $M^0 = A$ and $M^i = 0$ for $i > 0$. Denote by $\sigma_{\leq k}, \sigma_{\geq k}$ the 'foolish' troncation functors. Put $V = \sigma_{\leq -1} M$, so the sequence of complexes

$$0 \to A \to M \to V \to 0$$

is exact. Define a morphism $f_n : \mathrm{Sym}^n(M) \to \mathrm{Sym}^{n+1}(M)$ as the composition $A \otimes \mathrm{Sym}^n(M) \to M \otimes \mathrm{Sym}^n(M) = \mathrm{Sym}^1(M) \otimes \mathrm{Sym}^n(M) \to \mathrm{Sym}^{n+1}(M)$. We get an inductive system of complexes $(\mathrm{Sym}^n(M), f_n)_{n \in \mathbb{N}}$. Put $\mathrm{Sym}^\infty(M) = \varinjlim \mathrm{Sym}^n(M)$.

Lemma 39. *For any $n \geq 0$ the sequence of complexes*

$$0 \to \mathrm{Sym}^n(M) \xrightarrow{f_n} \mathrm{Sym}^{n+1}(M) \to \mathrm{Sym}^{n+1}(V) \to 0$$

is exact, where the second arrow is defined by functoriality from the natural morphism $M \to V$.

Proof If $k \leq 0$ then $(\overset{n+1}{\underset{i=1}{\otimes}} M)^k$ is the direct sum

$$\Big[\bigotimes_{i_1+\cdots+i_{n+1}=k i_j=0 \text{ for some } j} M^{i_1} \otimes \cdots \otimes M^{i_{n+1}}\Big] \oplus \Big[(\overset{n+1}{\underset{i=1}{\otimes}} V)^k\Big]$$

The group S_{n+1} acts on every summand in square brackets. It is easy to understand that the invariants

$$\Big[\bigotimes_{i_1+\cdots+i_{n+1}=k i_j=0 \text{ for some } j} M^{i_1} \otimes \cdots \otimes M^{i_{n+1}}\Big]^{S_{n+1}}$$

are identified with $(\mathrm{Sym}^n(M))^k$. $\square$

From the above lemma it follows that the natural morphism $\mathrm{Sym}^n(M) \to \mathrm{Sym}^\infty(M)$ is injective and $\sigma_{\geq -n} \mathrm{Sym}^n(M) \to \sigma_{\geq -n} \mathrm{Sym}^\infty(M)$ is an isomorphism for any $n \geq 0$. So, $\mathrm{Sym}^\infty(M)$ is a filtered complex with the filtration $(\mathrm{Sym}^n(M))_{n \in \mathbb{N}}$. Since the morphisms $\mathrm{Sym}^k(M) \otimes \mathrm{Sym}^l(M) \to \mathrm{Sym}^{k+l}(M)$ are compatible with f_n, by passing to the limit we get the morphism of multiplication $\mathrm{Sym}^\infty(M) \otimes \mathrm{Sym}^\infty(M) \to \mathrm{Sym}^\infty(M)$ (compatible with the above filtration).

Lemma 40. *Suppose that* $M = (M^{-1} \overset{d^{-1}}{\to} A)$, *i.e.,* $M^i = 0$ *for* $i < -1$. *Then* $\mathrm{Sym}^\infty(M)$ *is the Koszul complex for* (M^{-1}, d^{-1}). *In addition,*

$$\mathrm{Sym}^n(M) \to \sigma_{\geq -n} \mathrm{Sym}^\infty(M)$$

is an isomorphism for any $n \geq 0$. □

Now we impose on M the additional condition: $M^i = 0$ for $i < -2$ and $M^{-2} = A$, so $M = (A \to M^{-1} \to A)$. Then we have

Lemma 41. *Let* $n \geq 0$.
1) For any k *we have* $(\mathrm{Sym}^n(M))^k = (\mathrm{Sym}^n(M))^{-2n-k}$ *canonically.*
2) If $0 \leq k \leq n$ *then* $(\mathrm{Sym}^n(M))^{-k} = \wedge^k(M^{-1}) \oplus \wedge^{k-2}(M^{-1}) \oplus \wedge^{k-4}(M^{-1}) \oplus \dots$ □

Set $W = \sigma_{\geq -1} M$. Pick $c_1 \in A^*$ and consider the exact sequence $0 \to W \to M \to A[2] \to 0$, where $M \to A[2]$ is given by $M^{-2} \overset{c_1}{\to} A$. Let us define a morphism $g_n : \mathrm{Sym}^{n+1}(M) \to \mathrm{Sym}^n(M)[2]$ as the composition $\mathrm{Sym}^{n+1}(M) \to \mathrm{Sym}^1(M) \otimes \mathrm{Sym}^n(M) = M \otimes \mathrm{Sym}^n(M) \to (A[2]) \otimes \mathrm{Sym}^n(M)$. The proof of the next result is analogous to that of Lemma 39.

Lemma 42. *For any* $n \geq 0$ *the sequence*

$$0 \to \mathrm{Sym}^{n+1}(W) \to \mathrm{Sym}^{n+1}(M) \overset{g_n}{\to} \mathrm{Sym}^n(M)[2] \to 0$$

is exact, where the first arrow is defined by functoriality from the natural morphism $W \to M$. □

Remark 18. i) The morphism $(\mathrm{Sym}^{n+1}(M))^k \overset{g_n}{\to} (\mathrm{Sym}^n(M))^{k+2}$ can be also described as follows. For $-n-1 \leq k \leq 0$ this is the morphism $\wedge^{-k}(M^{-1}) \oplus \wedge^{-k-2}(M^{-1}) \oplus \dots \to \wedge^{-k-2}(M^{-1}) \oplus \dots$ that sends $\wedge^{-k}(M^{-1})$ to zero and induces isomorphisms on the others direct summands $\wedge^{-k-2}(M^{-1}) \widetilde{\to} \wedge^{-k-2}(M^{-1})$, … . For $k < -n-1$ this is an isomorphism (that preserves the corresponding direct summands).
ii) Passing to the limit we get an exact sequence

$$0 \to \mathrm{Sym}^\infty(W) \to \mathrm{Sym}^\infty(M) \overset{g}{\to} \mathrm{Sym}^\infty(M)[2] \to 0$$

Denote by φ_n the composition $\mathrm{Sym}^n(M) \overset{f_n}{\hookrightarrow} \mathrm{Sym}^{n+1}(M) \overset{g_n}{\to} \mathrm{Sym}^n(M)[2]$. In particular, φ_1 is given by the diagram

$$\begin{array}{ccccc} & & A & \to M^{-1} \to & A \\ & & \downarrow c_1 & & \\ A \to M^{-1} \to & & A & & \end{array}$$

Define the morphism $\varphi : \bigotimes_{i=1}^{n} M \to \bigotimes_{i=1}^{n} M[2]$ as $\varphi_1 \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \ldots + \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \varphi_1$. Then the diagram commutes

$$\begin{array}{ccc} \bigotimes_{i=1}^{n} M & \stackrel{\varphi}{\to} & \bigotimes_{i=1}^{n} M[2] \\ \cup & & \cup \\ \mathrm{Sym}^n(M) & \stackrel{\varphi_n}{\to} & \mathrm{Sym}^n(M)[2] \end{array}$$

Proof of Lemma B.1
$\mathrm{Sym}^\infty(W)$ is the Koszul complex for $d(e_1), \ldots, d(e_r) \in A$, so that the natural map $\mathrm{Sym}^\infty(W) \to A/I$ is a quasi-isomorphism. Our assertion follows now from Lemmas 42 and 39. $\square$

C Chern classes

Let S be a smooth scheme, $\mathcal{A} \in \operatorname{Pic} S, f : Y \to S$ be the complement of the zero section to the total space of $\mathcal{A}$. Fix $n > 0$ invertible as a function on S.

Lemma 43. *The complex $f_*\mu_n$ is included into a distinguished triangle*

$$f_*\mu_n \to \mathbb{Z}/n\mathbb{Z}[-1] \stackrel{c(\mathcal{A})}{\to} \mu_n[1],$$

where $c(\mathcal{A}) \in \mathrm{H}^2(S, \mu_n)$ is the Chern class of $\mathcal{A}$.

To prove this lemma we need the following straightforward result.

Lemma 44. *Let $\mathcal{A}$ be an abelian category, $\mathrm{D}(\mathcal{A})$ be its derived category. Let $K' \stackrel{\alpha}{\to} K \to K''$ be a distinguished triangle in $\mathrm{D}(\mathcal{A})$. Suppose that the morphism $\mathrm{H}^i(K'') \to \mathrm{H}^{i+1}(K')$ is surjective. Then there is a unique morphism $\tau_{\leq i+1}K' \to \tau_{\leq i}K$ such that the composition $\tau_{\leq i+1}K' \to \tau_{\leq i}K \to \tau_{\leq i+1}K$ is obtained from α by applying the functor $\tau_{\leq i+1}$, and the triangle $\tau_{\leq i+1}K' \to \tau_{\leq i}K \to \tau_{\leq i}K''$ is distinguished.* □

Proof of Lemma 43

Recall that the Kummer exact sequence $1 \to \mu_n \to \mathbb{G}_m \stackrel{x \mapsto x^n}{\to} \mathbb{G}_m \to 1$ on S defines a distinguished triangle $\mathbb{G}_m[1] \to \mathbb{G}_m[1] \stackrel{\delta}{\to} \mu_n[2]$, and the Chern class of $\mathcal{A} \in \operatorname{Pic} S = \operatorname{Hom}_{\mathrm{D}}(\mathbb{Z}, \mathbb{G}_m[1])$ is the composition $\delta \circ \mathcal{A} \in \operatorname{Hom}_{\mathrm{D}}(\mathbb{Z}, \mu_n[2])$ of morphisms in the derived category.

Consider now the Kummer exact sequence on Y. It provides a distinguished triangle $f_*\mu_n \to f_*\mathbb{G}_m \to f_*\mathbb{G}_m$ on S. As is easily seen, the moprhism $(\mathrm{R}^0 f_*)\mathbb{G}_m \to (\mathrm{R}^1 f_*)\mu_n$ is surjective (the question is local for the étale topology on S, and one can assume f to be the projection $S \times \mathbb{G}_m \to S$). Since $\tau_{\leq 1} f_*\mu_n \widetilde{\to} f_*\mu_n$, by Lemma 44 we get a distinguished triangle

$$f_*\mu_n \to (\mathrm{R}^0 f_*)\mathbb{G}_m \stackrel{\hat{n}}{\to} (\mathrm{R}^0 f_*)\mathbb{G}_m$$

Let us now construct a morphism $(\mathrm{R}^0 f_*)\mathbb{G}_m \to \mathbb{Z}$. If U is a smooth scheme then

$$\mathrm{H}^0(U \times \mathbb{G}_m, \mathbb{G}_m) \widetilde{\to} \mathrm{H}^0(U, \mathbb{G}_m) \times \mathrm{H}^0(U, \mathbb{Z})$$

canonically. Let $S' \to S$ be an étale morphism. Put $Y' = S' \times_S Y$ and denote by $g : \mathbb{G}_m \times Y' \to Y'$ the action of $\mathbb{G}_m$ on Y'. Let $s \in \mathrm{H}^0(S', (\mathrm{R}^0 f_*)\mathbb{G}_m)$. Since Y' is smooth, to $s \circ g$ there corresponds an element of $\mathrm{H}^0(Y', \mathbb{Z}) = \mathrm{H}^0(S', \mathbb{Z})$. This provides a morphism $(\mathrm{R}^0 f_*)\mathbb{G}_m \to \mathbb{Z}$. It is included into an exact sequence

$$0 \to \mathbb{G}_m \to (\mathrm{R}^0 f_*)\mathbb{G}_m \to \mathbb{Z} \to 0,$$

where the first arrow comes from the natural morphism $\mathbb{G}_m \to f_* f^* \mathbb{G}_m$. Using Chech coverings one proves that the corresponding element of $\mathrm{Ext}^1_S(\mathbb{Z}, \mathbb{G}_m) = \mathrm{Pic}\, S$ is $\mathcal{A}$. In other words, we get a distinguished triangle $(\mathrm{R}^0 f_*)\mathbb{G}_m \to \mathbb{Z} \stackrel{\mathcal{A}}{\to} \mathbb{G}_m[1]$ on S.

The morphism $\varkappa$ yeilds a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{G}_m & \to & (\mathrm{R}^0 f_*)\mathbb{G}_m & \to & \mathbb{Z} & \to & 0 \\ & & \downarrow x \mapsto x^n & & \downarrow \varkappa & & \downarrow n & & \\ 0 & \to & \mathbb{G}_m & \to & (\mathrm{R}^0 f_*)\mathbb{G}_m & \to & \mathbb{Z} & \to & 0 \end{array}$$

The latter provides a commutative diagram, where the rows and columns are distinguished triangles

$$\begin{array}{ccccc} (\mathrm{R}^0 f_*)\mathbb{G}_m & \to & \mathbb{Z} & \stackrel{\mathcal{A}}{\to} & \mathbb{G}_m[1] \\ \downarrow \varkappa & & \downarrow n & & \downarrow \\ (\mathrm{R}^0 f_*)\mathbb{G}_m & \to & \mathbb{Z} & \stackrel{\mathcal{A}}{\to} & \mathbb{G}_m[1] \\ \downarrow & & \downarrow & & \downarrow \delta \\ f_*\mu_n[1] & \to & \mathbb{Z}/n\mathbb{Z} & \to & \mu_n[2] \end{array}$$

So, the morphism $\mathbb{Z}/n\mathbb{Z} \to \mu_n[2]$ in the lowest row is $c(\mathcal{A})$. $\square$

The next assertion follows immediately from Lemma 43 by Verdier duality.

Proposition C.1. *Suppose that S is a smooth separated scheme of finite type. Then $f_!\mu_n$ is included into a distinguished triangle $f_!\mu_n \to \mathbb{Z}/n\mathbb{Z}[-2] \stackrel{c(\mathcal{A})}{\to} \mu_n$ on S, where $c(\mathcal{A}) \in H^2(S, \mu_n)$ is the Chern class of $\mathcal{A}$.*

$\square$

D On the Laumon's perverse sheaf $\mathcal{L}_E^n$.

In this appendix we formulate a conjectural strengthened version of Theorem (3.3.8),[3] by Laumon. (It is nowhere used in the paper, but clarifies the place of our Lemma 32 in general theory). An analog of the corresponding constructible $\mathbb{Q}_\ell$-sheaf is also defined for an arbitrary reductive group.

D.1

Let E be a local system on X. Let $\underline{m} = (m_1, \dots, m_s)$ be a partition of m, i.e., $m_1 \geq m_2 \geq \dots \geq m_s > 0$ and $\sum m_i = m$. Put $d_i = m_i - m_{i+1}, m_{s+1} = 0$. We have a morphism

$$i_{\underline{m}} : X^{(d_1)} \times X^{(d_2)} \times \dots \times X^{(d_s)} \to \mathrm{Sh}_0^m$$

that sends $(D_1, \dots, D_s)$ to $\mathcal{O}_{D_1+\dots+D_s} \oplus \mathcal{O}_{D_2+\dots+D_s} \oplus \dots \oplus \mathcal{O}_{D_s}$.

Following [3], define the polynomial functor $R^{(d_1,\dots,d_s)}V$ of a $\bar{\mathbb{Q}}_\ell$-vector space V as follows. If $s > \dim V$ then $R^{(d_1,\dots,d_s)}V = 0$. If $s \leq \dim V$ then

$$R^{(d_1,\dots,d_s)}V \subset \mathrm{Sym}^{d_1} V \otimes \mathrm{Sym}^{d_2}(\wedge^2 V) \otimes \dots \otimes \mathrm{Sym}^{d_s}(\wedge^s V)$$

is the irreducible subrepresentation of $\mathrm{GL}(V)$ of the highest weight $d_1\omega_1 + \dots + d_s\omega_s$ (with ω_i being the highest weight of $\wedge^i V$). Consider the sheaf

$$E^{(d_1)} \boxtimes (\wedge^2 E)^{(d_2)} \boxtimes \dots \boxtimes (\wedge^s E)^{(d_s)} \tag{16}$$

on

$$X^{(d_1)} \times \dots \times X^{(d_s)} \tag{17}$$

If $(D_1, \dots, D_s)$ is a k-point of this scheme then we put $D_i = \sum_{x\in X} d_{i,x}[x]$ $(i = 1, \dots, s)$. The fibre of (16) at $(D_1, \dots, D_s)$ is the tensor product over $x \in X$ of the vector spaces

$$\bigotimes_{i=1}^{s} \mathrm{Sym}^{d_{i,x}}(\wedge^i E_x)$$

In particular, there is a canonical inclusion

$$\bigotimes_{x\in X} R^{(d_{1,x},\dots,d_{s,x})} E_x \to \bigotimes_{x\in X} (\bigotimes_{i=1}^{s} \mathrm{Sym}^{d_{i,x}}(\wedge^i E_x)) \tag{18}$$

Our next assertion is a particular case of Proposition D.2.

Proposition D.1. *There is a (unique) constructable $\mathbb{Q}_\ell$-subsheaf $\mathcal{S}_E^{\underline{m}}$ of (16) such that for any k-point $(D_1, \dots, D_s)$ of $X^{(d_1)} \times \cdots \times X^{(d_s)}$ the fibre*

$$(\mathcal{S}_E^{\underline{m}})_{(D_1,\dots,D_s)}$$

is the image of the canonical inclusion (18).

Conjecture 1. *There is a canonical isomorphism*

$$\mathcal{H}^{2k}(i_{\underline{m}}^* \mathcal{L}_E^n) \widetilde{\to} \mathcal{S}_E^{\underline{m}}(-k),$$

where $k = \sum_{i=1}^s d_i \frac{i^2-i}{2}$.

D.2

D.2.1

Let G be a connected reductive linear algebraic group over $\bar{\mathbb{Q}}_\ell$. Fix a Cartan subgroup $H \subset G$ and an ordering on the set of roots. Denote by Λ^+ the semigroup of dominant weights. Let Repr_G be the category of finite-dimensional representations of G (over $\bar{\mathbb{Q}}_\ell$). For $\alpha \in \Lambda^+$ denote by $V^\alpha \in \mathrm{Repr}_G$ the irreducible representation of G with highest weight α. We also denote by $W^\alpha \subset V^\alpha$ the 1-dimensional subspace generated by a highest weight vector.

For a finite set I denote by $E(I)$ the category of surjective morphisms $I \to I'$. (In other words, this is the category associated to the partially ordered set of equivalence relations on I). For any morphism $\alpha : I \to \Lambda^+$ we define a functor

$$f_\alpha : E(I)^\circ \to \mathrm{Repr}_G,$$

as follows.

Let $I \stackrel{h}{\to} I'$ be a surjection. Consider the representation $\bigotimes_{i \in I} V^{\alpha(i)}$ of G^I and restrict it to $G^{I'}$ via the diagonal mapping $G^{I'} \to G^I$. Generate a subrepresentation of $G^{I'}$ by the 1-dimensional subspace $\bigotimes_{i \in I} W^{\alpha(i)} \subset \bigotimes_{i \in I} V^{\alpha(i)}$ and restrict it to G via the diagonal mapping $G \to G^{I'}$. The obtained representation is, by definition, $f_\alpha(h)$. For a morphism

$$\begin{array}{ccc} I' & \stackrel{h}{\leftarrow} & I \\ \downarrow & \swarrow h' & \\ I'' & & \end{array} \qquad (19)$$

in $E(I)$ we have an obvious inclusion $f_\alpha(h') \subset f_\alpha(h)$, so f_α becomes a functor. Clearly,

$$f_\alpha(h) \widetilde{\to} \bigotimes_{i' \in I'} V^{\alpha'(i')},$$

where $\alpha' : I' \to \Lambda^+$ is given by $\alpha'(i') = \sum_{i \in h^{-1}(i')} \alpha(i)$.

D.2.2

Fix $s \geq 0$ and a collection $d = (d_1, \ldots, d_s) \in (\mathbb{Z}_+)^s$. The scheme (17) can be thought of as the moduli scheme of $(\mathbb{Z}_+)^s$-valued divisors on X of degree d. Namely, a point $(D_1, \ldots, D_s)$ of this scheme is the divisor that associates to $x \in X$ the collection

$$d_x = (d_{1,x}, \ldots, d_{s,x})$$

Put $I = \bigsqcup_{i=1}^{s} I_i$, where $I_i = \{1, 2, \ldots, d_i\}$. Define a map $\beta : I \to (\mathbb{Z}_+)^s$ constant on every I_i by $\beta(I_i) = (0, \ldots, 1, \ldots, 0)$. (1 occurs in the i-th place). For a surjection $I \overset{h}{\to} I'$ denote by $\beta' : I' \to (\mathbb{Z}_+)^s$ the map given by $\beta'(i') = \sum_{i \in h^{-1}(i')} \beta(i)$.

The scheme (17) admits a stratification by locally closed subschemes U_h indexed by the (isomorphism classes of) objects $h \in \operatorname{Ob} E(I)$. The stratum U_h parametrizes the divisors

$$\sum_{i \in I'} \beta'(i)[x_i],$$

where $\{x_i\}_{i \in I'}$ are distinct points of X. Clearly, given two surjections $I \overset{h}{\to} I', I \overset{h'}{\to} I''$, we have $U_{h'} \subset \overline{U_h}$ if and only if h' factors through h.

Now suppose that we are given a morphism $\gamma : (\mathbb{Z}_+)^s \to \Lambda^+$ of semigroups. Define $\alpha : I \to \Lambda^+$ by $\alpha = \gamma \circ \beta$. There exists a unique constructable $\mathbb{Q}_\ell$-sheaf $\mathcal{S}^{d,\gamma}$ on (17) with the following properties. The restriction of $\mathcal{S}^{d,\gamma}$ to U_h is the constant sheaf with fibre $f_\alpha(h)$. For a morphism (19) in $E(I)$ the corresponding cospecialization map is the inclusion $f_\alpha(h') \to f_\alpha(h)$.

Notice that for a sufficiently small open subscheme $j : U \to X^{(d_1)} \times \cdots \times X^{(d_s)}$ $\mathcal{S}^{d,\gamma} \to (\mathrm{R}^0 j_*) j^* \mathcal{S}^{d,\gamma}$ is an isomorphism.

D.2.3

Now we present a twisted version of the above construction.

Fix a geometric point $\bar{\eta} \to X$ over the generic point $\eta \to X$. Recall that G-local systems on X form a groupoïd whose objects are continuous homomorphisms

$F : \pi_1(X,\bar\eta) \to G(\bar{\mathbb{Q}}_\ell)$; morphisms are inner automorphisms of $G(\bar{\mathbb{Q}}_\ell)$ which make the natural diagram commutative. If $V \in \mathrm{Repr}_G$ and F is a G-local system on X then we denote by V_F the corresponding smooth $\bar{\mathbb{Q}}_\ell$-sheaf on X (its fibre at $\bar\eta$ is the representation $\pi_1(X,\bar\eta) \to G(\bar{\mathbb{Q}}_\ell) \to \mathrm{Aut}\, V$ of $\pi_1(X,\bar\eta)$).

Let $e_1,\dots,e_s$ be the standard basis of $(\mathbb{Z}_+)^s$.

Proposition D.2. *Given a G-local system F on X, $d \in (\mathbb{Z}_+)^s$, and $\gamma : (\mathbb{Z}_+)^s \to \Lambda^+$ as above, consider the constructable $\bar{\mathbb{Q}}_\ell$-sheaf*

$$\mathop{\boxtimes}_{i=1}^{s} (V_F^{\gamma(e_i)})^{(d_i)} \tag{20}$$

on (17). There is a (unique) constructable $\bar{\mathbb{Q}}_\ell$-subsheaf $\mathcal{S}_F^{d,\gamma}$ of (20) whose fibre at a k-point $(D_1,\dots,D_s)$ is

$$\bigotimes_{x\in X} (V_F^{\gamma(d_x)})_x$$

□

Notice that for the trivial G-local system F on X we get $\mathcal{S}_F^{d,\gamma} \tilde{\to} \mathcal{S}^{d,\gamma}$. Again for a sufficiently small open subscheme $j : U \to X^{(d_1)} \times \cdots \times X^{(d_s)}$ the natural morphism $\mathcal{S}_F^{d,\gamma} \to (\mathrm{R}^0 j_*) j^* \mathcal{S}_F^{d,\gamma}$ is an isomorphism.

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