MEMOIRE

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Mémoire pour obtenir l'habilitation à diriger des recherches:

Certains aspects du programme de Langlands géométrique

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Résumé: Ce rapport présente mes travaux dans la direction du programme de Langlands géométrique. Ceux-ci abordent plusieurs aspects de ce thème: méthode de Rankin-Selberg locale et globale, les foncteurs de Whittaker et de Bessel pour $\mathbb{GS}p_4$, catégorification et la version géométrique de la multiplicité un pour les models de Bessel, les faisceaux Théta et programme de Langlands géométrique pour le groupe métaplectique, correspondance de Howe géométrique.

Abstract: This report presents my work on the geometric Langlands program. The following directions of this program are considered: local and global Rankin-Selberg method, Whittaker and Bessel functors for $\mathbb{GS}p_4$, categorification and geometric version of multiplicity one for Bessel models, Theta sheaves and geometric Langlands program for the metaplectic group, geometric Howe correspondence.

Codes MSC (2000):

Key words: geometric Langlands program, Rankin-Selberg method, Whittaker and Bessel models, metaplectic group and Theta sheaves, geometric theta-lifting (geometric Howe correspondence).

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1. Introduction: Geometric Langlands program

1.1 This memoire consists of several papers that all focus on different aspects of the geometric Langlands program. We start with a breif description of the subject, which is now refered to as the *geometric Langlands program.*

The classical Langlands program has emerged in the late 60's as a series of far-reaching conjectures tying together seemingly unrelated objects in number theory, algebraic geometry, and the theory of automorphic forms. It has two related parts, local and global. We restrict ourself to the totally unramified case.

In the function field case one starts with a smooth projective absolutely irreducible curve over a finite field $k = \mathbb{F}_q$ with q elements. Let $F = k(X)$ be the field of rational functions on X, $\mathbb{A}$ be the adèles ring of F, $\mathcal{O} \subset \mathbb{A}$ the entire adèles. Fix a separable closure $\bar{F}$ of F. For a closed point $x \in X$ let F_x denote the completion of F at x. Write $\mathcal{O}_x$ for the completed local ring of $\mathcal{O}_X$ at x, and k_x for its residue field. Let q_x be the number of elements of k_x. Let G be a connected split reductive group over F. Pick a prime ℓ that does not divide q.

1.2 Local picture Equip $G(F_x)$ with a Haar measure such that $G(\mathcal{O}_x)$ has volume one. Let H_x be the space of compactly supported $\bar{\mathbb{Q}}_\ell$-valued functions on $G(F_x)$, which are $G(\mathcal{O}_x)$-bi-invariant. This is an algebra with respect to the convolution product, which is called the *spherical Hecke algebra.* Let $\check{G}$ denote the Langlands dual group to G, this is a connected reductive group over $\bar{\mathbb{Q}}_\ell$, whose root datum is dual to that of G. The Satake isomorphism is a statement that the algebra H_x is canonically isomorphic to the Grothendieck ring $\operatorname{Rep}(\check{G})$ of the category of $\check{G}$-representations over $\bar{\mathbb{Q}}_\ell$.

On the automorphic side one is interested in smooth irreducible representations V of $G(F_x)$, which are unramified (that is, the space $V^{G(\mathcal{O}_x)}$ of $G(\mathcal{O}_x)$-invariants in nonzero). For such a representation the space $V^{G(\mathcal{O}_x)}$ is 1-dimensional, and H_x naturally acts on it by some character $\chi_V : H_x \to \bar{\mathbb{Q}}_\ell$.

On the Galois side one looks at continuous homomoprhisms $\operatorname{Gal}(\bar{F}_x/F_x) \to \check{G}(\bar{\mathbb{Q}}_\ell)$, which are unramified. This means that the inertia group I_x acts trivially. Remind the exact sequence $1 \to I_x \to \operatorname{Gal}(\bar{F}_x/F_x) \to \operatorname{Gal}(\bar{k}_x/k_x) \to 1$, where $\bar{k}_x$ (resp., $\bar{F}_x$) is a separable closure of k_x (resp., of F_x). The geometric Frobenius element $\operatorname{Fr}_x \in \operatorname{Gal}(\bar{k}_x/k_x)$ is defined to be the inverse of the automorphism $y \mapsto y^{q_x}$ of $\bar{k}_x$. Then $\operatorname{Gal}(\bar{k}_x/k_x)$ is a profinite completion of the free abelian group generated by Fr_x. So, a continuous homomorphism $\sigma : \operatorname{Gal}(\bar{k}_x/k_x) \to \check{G}(\bar{\mathbb{Q}}_\ell)$ is completely determined by $\sigma(\operatorname{Fr}_x)$. An isomorphism class of such σ yields a conjugacy class of $\sigma(\operatorname{Fr}_x)$ in $\check{G}(\bar{\mathbb{Q}}_\ell)$. Finally, a conjugacy class γ in $\check{G}(\bar{\mathbb{Q}}_\ell)$ yields a character $\chi_\gamma : \operatorname{Rep}(\check{G}) \to \bar{\mathbb{Q}}_\ell$ sending a representation W to $\operatorname{tr}(\gamma, W)$, hence a character of H_x. Let us say that a continuous nonramified homomoprhism $\sigma : \operatorname{Gal}(\bar{F}_x/F_x) \to \check{G}(\bar{\mathbb{Q}}_\ell)$ and a nonramified smooth irreducible representation V of $G(F_x)$ match if the characters $\chi_V : H_x \to \bar{\mathbb{Q}}_\ell$ and $\chi_{\sigma(\operatorname{Fr}_x)} : H_x \to \bar{\mathbb{Q}}_\ell$ coincide.

1.3 Global picture On the Galois side one now considers continuous ℓ-adic homomorphisms $\sigma : \operatorname{Gal}(\bar{F}/F) \to \check{G}(\bar{\mathbb{Q}}_\ell)$, which are everywhere nonramified. This means that for each closed point $x \in X$ the following holds. Pick a point $\bar{x}$ of $\bar{F}$ lying over x. The subgroup of $\operatorname{Gal}(\bar{F}/F)$

preserving $\bar{x}$ is called the decomposition group D_x. If we make a different choice of $\bar{x}$ then it gets conjugated in $\mathrm{Gal}(\bar{F}/F)$. We have $D_x \widetilde{\to} \mathrm{Gal}(\bar{F}_x/F_x)$, and it is required that σ is trivial on the corresponding inertia subgroup $I_x \subset D_x$. Under this assumption σ gives rise to a well-defined conjugacy class of $\sigma(\mathrm{Fr}_x)$ in $\check{G}(\bar{\mathbb{Q}}_\ell)$.

Equip $G(\mathbb{A})$ with a Haar measure such that $G(\mathcal{O})$ has measure one. The space $G(F)\backslash G(\mathbb{A})$ has the induced measure, so we can consider the space $L^2(G(F)\backslash G(\mathbb{A}))$ of square-integrable functions, this is a representation of $G(\mathbb{A})$ by right translations. On the automorphic side one looks at smooth unramified irreducible representations π of $G(\mathbb{A})$. One can show that such a reprentation decomposes into a (restricted) tensor product

$$\pi \widetilde{\to} \otimes'_{x\in|X|} \pi_x,$$

where π_x is a smooth irreducible unramified representation of $G(F_x)$. One is interested in such π satisfying an additional requiment to be an *automorphic representation*. Roughly speaking, this means that π 'appears' in $L^2(G(F)\backslash G(\mathbb{A}))$ (a precise notion is more complecated, as it may 'appear' not descretely).

The classical Langlands program proposes a series of conjectures relating such σ and π in a way compatible with the local correspondence. In particular, for σ and π to match it is required that for each closed point x of X the corresponding local data $\sigma_x : \mathrm{Gal}(\bar{F}_x/F_x) \to \check{G}(\bar{\mathbb{Q}}_\ell)$ and π_x match. Here σ_x is the restriction of σ to the decomposition subgroup at x.

1.4 Geometrization A starting point is Grothendieck's philosophy that many mathematical results for functions (for example, formulas for number of points of varieties over $\mathbb{F}_q$) are actually traces of deeper results that hold in the derived categories of ℓ-adic sheaves on the corresponding varieties (and maybe in other cohomology theories also). The geometric Langlands program is a trial to lift the classical theory of automorphic forms and Galois representations to this level.

The first example is the geometric Satake equivalence (cf. [Gi],[MV]). The set $G(F_x)/G(\mathcal{O}_x)$ is naturally the set of k-points of an ind-scheme over k called the affine grassmanian $\mathrm{Gr}_{G,x}$ of G at x. Similarly, $G(\mathcal{O}_x)$ is the set of k-points of a group scheme over k (denoted by the same symbol). The geometric counterpart of H_x is the category $\mathrm{Sph}(\mathrm{Gr}_{G,x})$ of $G(\mathcal{O}_x)$-equivariant ℓ-adic perverse sheaves on $\mathrm{Gr}_{G,x}$. One may equip $\mathrm{Sph}(\mathrm{Gr}_{G,x})$ with a convolution product (as well as a commutativity and associativity constraints) making it a tensor category. There is a canonical equivalence of tensor categories $\mathrm{Sph}(\mathrm{Gr}_{G,x}) \widetilde{\to} \mathcal{R}\mathrm{ep}(\check{G})$, where $\mathcal{R}\mathrm{ep}(\check{G})$ is the tensor category of $\bar{\mathbb{Q}}_\ell$-representations of $\check{G}$ (the Satake equivalence).

Write Bun_G for the stack of G-bundles on X, it is known to be algebraic. A key observaion is that (assuming for simplicity that any G-torsor over $\mathrm{Spec}\, F$ is trivial) there is a canonical bijection

$$G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \longleftrightarrow \{\text{isomorphism classes of } G-\text{torsors on } X\} \qquad (1)$$

The Hecke algebra H_x acts naturally on the space of functions on $G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$. This phenomenon also can be geometrized as follows. For each object $\mathcal{A} \in \mathrm{Sph}(\mathrm{Gr}_{G,x})$ there is a natural Hecke functor $\mathrm{H}(\mathcal{A},\cdot) : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)$. These functors are compatible with the tensor structure on $\mathrm{Sph}(\mathrm{Gr}_{G,x})$. Letting x move around X, one similarly introduces Hecke functors $\mathrm{H}(\mathcal{A},\cdot) : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_G)$.

If π is a smooth unramified irreducible automorphic representation of $G(\mathbb{A})$, write

$$\pi \widetilde{\to} \otimes'_{x\in|X|} \pi_x,$$

the restricted product with respect to the vectors $v_x \in (\pi_x)^{G(\mathcal{O}_x)}$. Since π appears in the space of functions on $G(F)\backslash G(\mathbb{A})$, the vector $\otimes_x v_x$ becomes a function

$$\phi_\pi : G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell$$

By construction, we have the following *Hecke property* of ϕ_π. For any closed point $x \in X$ and $h \in H_x$ we have an equality of functions on $G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$

$$h(\phi_\pi) = \chi_{\pi_x}(h)\phi_\pi \tag{2}$$

The geometric analog of π would be an object K_π of the derived category of ℓ-agic sheaves $\mathrm{D}(\mathrm{Bun}_G)$ on Bun_G, whose function 'trace of Frobenius' equals ϕ_π. Moreover, one wants it to satisfy the *Hecke property* in the following sense.

A geometric analog of a nonramified continuous Galois representation $\sigma : \mathrm{Gal}(\bar{F}/F) \to \check{G}(\bar{\mathbb{Q}}_\ell)$ is an ℓ-adic $\check{G}$-local system $E = E_\sigma$ on X. For each representation $V \in \mathcal{R}\mathrm{ep}(\check{G})$ write V_E for the local system on X obtained from E by extension of scalars $\check{G} \to \mathrm{GL}(V)$. Then for a closed point $x \in X$ we have

$$\mathrm{tr}(\mathrm{Fr}_x, V_E) = \mathrm{tr}(\sigma(\mathrm{Fr}_x), V) = \chi_{\sigma(\mathrm{Fr}_x)}(V)$$

Let $\mathcal{A}_V \in \mathrm{Sph}(\mathrm{Gr}_{G,x})$ correspond to $V \in \mathrm{Rep}(\check{G})$ and let $h \in H_x$ be the trace of Frobenius function of $\mathcal{A}_V$. Then (2) can be written as

$$\mathrm{tr}(\mathrm{Fr}_g, \mathrm{H}(\mathcal{A}_V, K_\pi)) = \chi_{\pi_x}(V)\,\mathrm{tr}(\mathrm{Fr}_x, K_\pi) = \mathrm{tr}(\mathrm{Fr}_x, V_E)\,\mathrm{tr}(\mathrm{Fr}_g, K_\pi)$$

for $g \in G(\mathbb{A})$. We have assumed here that π and σ match in the sense of 1.3.

The latter formula admits a geometric version, namely, an isomorphism in $\mathrm{D}(X \times \mathrm{Bun}_G)$

$$\mathrm{H}(\mathcal{A}_V, K_\pi) \widetilde{\to} V_E \boxtimes K_\pi \tag{3}$$

A Hecke eigensheaf corresponding to a $\check{G}$-local system E on X is a complex K_π equipped with isomorphisms (3) for each $V \in \mathcal{R}\mathrm{ep}(\check{G})$ satisfying some compatibility conditions (that we do not precise).

The geometric Langlands program is an attempt to analyze the spectrum of the tensor categories $\mathrm{Sph}(\mathrm{Gr}_{G,x})$ acting on $\mathrm{D}(\mathrm{Bun}_G)$, to look for automorphic sheaves or, more generally, for a kind of spectral decomposition of $\mathrm{D}(\mathrm{Bun}_G)$.

One of the major achievements in this field at present is a proof of the classical Langlands correspondence over function fields for $G = \mathrm{GL}_n$ by Lafforgue [Laf] (the case $n = 1$ was earlier known as the class fields theory, and the case $n = 2$ was proved by Drinfeld [D]). In the nonramified geometric setting the corresponding result for GL_n has beed recently proved by Frenkel, Gaitsgory and Vilonen ([FGV], [G]). They attach to each irreducible rank n ℓ-adic local system E on X a perverse sheaf Aut_E on $\mathrm{Bun}_{\mathrm{GL}_n}$ that satisfies the Hecke property for E.

2. Rankin-Selberg convolutions for GL_n

2.1 The papers [1] and [2] form a series, in which we propose a geometric version of the classical Rankin-Selberg method for computation of the scalar product of two cuspidal automorphic forms on GL_n over a function field.

Let us describe briefly the main results. Let X be a smooth, projectice, geometrically connected curve over $k = \mathbb{F}_q$. Write Bun_n for the stack of rank n vector bundles on X. Let Bun_n^d denote its connected component corresponding to vector bundles of degree d.

For an irreducible rank n local system E on X write Aut_E for the automorphic sheaf on Bun_n corresponding to E (normalized as in [FGV], in particular, $\mathbb{D}(\mathrm{Aut}_E) \widetilde{\to} \mathrm{Aut}_{E^*}$, here $\mathbb{D}$ is the Verdier duality functor). Write $\phi_E : \mathrm{Bun}_n(k) \to \bar{\mathbb{Q}}_\ell$ for the function 'trace of Frobenius' of Aut_E. Write $\#(A)$ for the number of elements of a set A.

The classical Rankin-Selberg result for GL_n claims that the sum

$$\sum_{L \in \mathrm{Bun}_n^d(k)} \frac{1}{\# \mathrm{Aut}\, L} \phi_{E_1^*}(L) \phi_{E_2}(L) \tag{4}$$

vanishes unless $E_1 \widetilde{\to} E_2 \widetilde{\to} E$, in the latter case the answer is expressed in terms of the action of Fr on $\mathrm{H}^1(X \otimes \bar{k}, \mathrm{End}\, E)$. Here Fr denotes the geometric Frobenius automophism for k.

The computation of (4) is based on the equality of formal series

$$\sum_{d \ge 0} \sum_{(\Omega^{n-1} \hookrightarrow L) \in {}_n\mathcal{M}_d(\mathbb{F}_q)} \frac{1}{\# \mathrm{Aut}(\Omega^{n-1} \hookrightarrow L)} \varphi_{E_1^*}(L) \varphi_{E_2}(L) t^d = L(E_1^* \otimes E_2,\, q^{-1} t) \tag{5}$$

Here ${}_n\mathcal{M}_d(\mathbb{F}_q)$ is the set of isomorphism classes of pairs $(\Omega^{n-1} \hookrightarrow L)$, where L is a vector bundle on X of rank n and degree $d + n(n-1)(g-1)$, and Ω is the canonical invertible sheaf on X (Ω^{n-1} is embedded in L as a subsheaf, i.e., the quotient is allowed to have torsion). We have denoted by $L(E_1^* \otimes E_2, t)$ the L-function attached to the local system $E_1^* \otimes E_2$ on X.

Remind that for a local system E on X we have

$$L(E,t) = \sum_{d \ge 0} \sum_{D \in X^{(d)}(k)} \mathrm{tr}(\mathrm{Fr}, E_D^{(d)}) t^d = \prod_{r=0}^{2} \det(1 - \mathrm{Fr}\, t, \mathrm{H}^r(X \otimes \bar{k}, E))^{(-1)^{r+1}} \tag{6}$$

Here $X^{(d)}$ denotes the d-th symmetric power of X, and $E^{(d)}$ is the d-th symmetric power of E.

Let ${}_n\mathcal{M}_d$ denote the moduli stack of pairs $(\Omega^{n-1} \overset{s}{\hookrightarrow} L)$, where L is a vector bundle of rank n and degree $d + n(n-1)(g-1)$ on X, and s is an inclusion of $\mathcal{O}_X$-modules. Geometrizing a construction due to Shalika and Piatetski-Shapiro, Laumon has defined a complex of $\bar{\mathbb{Q}}_\ell$-sheaves ${}_n\mathcal{K}_E^d$ on ${}_n\mathcal{M}_d$, which is a part of the construction of the sheaf Aut_E. Namely, a theorem of Frenkel, Gaitsgory and Vilonen ([FGV]) says that, when E is irreducible of rank n, ${}_n\mathcal{K}_E^d$ descends with respect to the projection ${}_n\mathcal{M}_d \to \mathrm{Bun}_n$ to a perverse sheaf Aut_E.

Denote by

$$\tilde{\varphi}_E : {}_n\mathcal{M}_d(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$$

the function trace of Frobenius of ${}_n\mathcal{K}^d_E$. So, the restriction of ϕ_E to ${}_n\mathcal{M}_d(\mathbb{F}_q)$ equals $\tilde\varphi_E$.

We prove a geometric version of the equality

$$\sum_{(\Omega^{n-1}\hookrightarrow L)\in {}_n\mathcal{M}_d(\mathbb{F}_q)} \frac{1}{\#\operatorname{Aut}(\Omega^{n-1}\hookrightarrow L)}\tilde\varphi_{E_1^*}(L)\tilde\varphi_{E_2}(L) = q^{-d}\sum_{D\in X^{(d)}(\mathbb{F}_q)} \operatorname{tr}(\mathrm{Fr}, (E_1^*\otimes E_2)^{(d)}_D) \qquad (7)$$

of coefficients in (5). The point is that this equality holds for any local systems E_1, E_2 of rank n on X and is local in this sense.

We establish for any smooth $\bar{\mathbb{Q}}_\ell$-sheaves E_1, E_2 of rank n on X and any $d\ge 0$ a canonical isomorphism

$$\mathrm{R}\Gamma_c({}_n\mathcal{M}_d\,,\,{}_n\mathcal{K}^d_{E_1^*}\otimes{}_n\mathcal{K}^d_{E_2})\widetilde{\to}\mathrm{R}\Gamma(X^{(d)},\,(E_1^*\otimes E_2)^{(d)})(d)[2d],$$

Actually more general results of local nature are proved (cf. Theorems A,B and C in [1]).

2.2 The global part of the Rankin Selberg method appeared first as a calculation of (4) starting from (5). Namely, rewrite (5) as

$$\sum_{d\ge 0}\sum_{L\in \mathrm{Bun}^d_n(\mathbb{F}_q)} \frac{1}{\#\operatorname{Aut} L}(q^{\dim \operatorname{Hom}(\Omega^{n-1},L)}-1)\varphi_{E_1^*}(L)\varphi_{E_2}(L)t^d = L(E_1^*\otimes E_2, q^{-1}t) \qquad (8)$$

The cuspidality of φ_E implies that if $\varphi_E(L)\ne 0$ and $\deg L$ is large enough then $\operatorname{Ext}^1(\Omega^{n-1},L)=0$, and $\dim\operatorname{Hom}(\Omega^{n-1},L) = d-n^2(g-1)$. To conclude, it remains to study the asymptotic behaviour of the above series when t goes to 1, using the cohomological interpretation of the L-function (6).

Let us explain another source of motivation for the same calculation coming from the geometric Langlands program.

Remind the following result of Laumon and Rothstein ([Ro]) in the case of GL(1). Assume for a moment that the ground field is $\mathbb{C}$. Let M' be the Picard scheme classifying invertible $\mathcal{O}_X$-modules L of degree zero. Denote by M the coarse moduli space of invertible $\mathcal{O}_X$-modules L with connection $\nabla : L\to L\otimes_{\mathcal{O}_X}\Omega_X$. This is an abelian group scheme over $\mathbb{C}$ (for the tensor product), which has a natural structure of a $\mathrm{H}^0(X,\Omega_X)$-torsor over M'. In [Ro] a certain invertible $\mathcal{O}_{M\times M'}$-module Aut with connection (relative to M) is considered as a kernel of two integral functors

$$\mathcal{F} : \mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{D}_{M'})\to \mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{O}_M)$$

and

$$\mathcal{F}' : \mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{O}_M)\to \mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{D}_{M'})$$

Here $\mathcal{D}_S$ denotes the sheaf of differential operators on a smooth scheme S over $\mathbb{C}$. The theorem of Laumon and Rothstein claims that these functors are quasi-inverse to each other.

This result can be obtained as a consequence of two orthogonality relations. One of them states that *the complex*

$$\mathrm{R}(\mathrm{pr}_{12})_*(\mathrm{pr}_{13}^*\operatorname{Aut}\otimes\mathrm{pr}_{23}^*\operatorname{Aut})$$

is canonically isomorphic to $\triangle_* \mathcal{O}_M$ *in* $\mathrm{D}^b_{\mathrm{qcoh}}(\mathcal{O}_{M\times M})$ *(up to a shift and a sign)*, where

$$\mathrm{pr}_{13}, \mathrm{pr}_{23} : M \times M \times M' \to M \times M'$$

and $\mathrm{pr}_{12} : M \times M \times M' \to M \times M$ are the projections, $\triangle\colon M \to M \times M$ is the diagonal, and the functor $\mathrm{R}(\mathrm{pr}_{12})_*$ is understood in the $\mathcal{D}$-modules sense.

An ℓ-adic analogue of this orthogonality relation is the following. Assume again that $k = \mathbb{F}_q$ (and the genus of X is at least 1). Let E_0 be a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on X of rank 1.

The moduli space of ℓ-adic local systems on X is not known to exist. However, E_0 admits a unversal deformation E over $\bar{\mathbb{Q}}_\ell$. Let $\mathrm{Spec}(R)$ be the base of this deformation. In fact, R is isomorphic to the ring of formal power series over $\bar{\mathbb{Q}}_\ell$ in $2g$ variables.

Let $\underline{\mathrm{Pic}}^d X$ denote the Picard scheme of X parametrizing isomorphism classes of invertible $\mathcal{O}_X$-modules of degree d. The construction of the automorphic local system AE_0 of the geometric abelian class field theory makes sense for deformations also. So, one has a smooth R-sheaf AE on $\underline{\mathrm{Pic}}^d X$. Denote by AE_1, AE_2 the two liftings of AE to $\mathrm{Spec}(R \hat{\otimes}_{\bar{\mathbb{Q}}_\ell} R)$. we show that *there is a canonical isomorphism of* $R \hat{\otimes} R$*-modules*

$$\mathrm{H}^{2g}(\underline{\mathrm{Pic}}^d X, \mathcal{H}om(E_1, E_2)) \widetilde{\to} R(-g),$$

where the $R \hat{\otimes} R$*-module structure on* R *is given by the diagonal map* $R \hat{\otimes} R \to R$*. Besides, for* $i \neq 2g$ *we have* $\mathrm{H}^i(\underline{\mathrm{Pic}}^d X, \mathcal{H}om(E_1, E_2)) = 0$.

Applying the base change theorem for the above result, we get the 'scalar square' of E_0, namely

$$\mathrm{R}\Gamma(\underline{\mathrm{Pic}}^d X, \mathcal{E}nd(E_0^d, E_0^d)) \widetilde{\to} R \overset{L}{\otimes}_{R \hat{\otimes} R} \bar{\mathbb{Q}}_\ell(-g)[-2g]$$

As is easy to see, this complex has cohomology groups in all degrees $0, 1, \dots, 2g$. The trace of Frobenius Fr acting on this complex yields a formula for (4) in the case $n = 1$ (that agrees with the one obtained from (5)).

Further, we generalize these orthogonality relations for GL_n. Namely, let E_0 be a smooth irreducible $\bar{\mathbb{Q}}_\ell$-sheaf on X of rank n. Let (E, R) be a universal deformation of E_0 over $\bar{\mathbb{Q}}_\ell$. In fact, R is isomorphic to the ring of formal power series over $\bar{\mathbb{Q}}_\ell$ in $2 + (2g-2)n^2$ variables. The construction of Aut_{E_0} also makes sense for deformations, so one has a perverse R-sheaf Aut_E on Bun_n.

Scalar automorphisms of vector bundles provide an action of $\mathbb{G}_m$ on Bun_n by 2-automorphisms of the identity. We introduce a stack $\underline{\mathrm{Bun}}_n$ (cf. [2], Sect. 3.5), the quotient of Bun_n under this action. There exists a perverse R-sheaf $\underline{\mathrm{Aut}}_E$ on $\underline{\mathrm{Bun}}_n$ such that the inverse image of $\underline{\mathrm{Aut}}_E[-1]$ under the projection $\mathrm{Bun}_n \to \underline{\mathrm{Bun}}_n$ is naturally identified with Aut_E.

Our main global result claims that for any integers i and d there is a canonical isomorphism of $R \hat{\otimes} R$-modules

$$\mathrm{H}^i_c(\underline{\mathrm{Bun}}^d_n, \mathrm{pr}_1^* \underline{\mathrm{Aut}}^d_{E^*} \otimes_{R \hat{\otimes} R} \mathrm{pr}_2^* \underline{\mathrm{Aut}}^d_E) \widetilde{\to} \begin{cases} R, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0, \end{cases}$$

where the $R\hat{\otimes}R$-module structure on R is given by the diagonal map $R\hat{\otimes}R \to R$, and pr_i : $\mathrm{Spec}(R\hat{\otimes}R) \to R$ are the two projections. Again, the classical Rankin-Selberg convolution can be derived from this by the base change $R \to \bar{\mathbb{Q}}_\ell$.

3. Whittaker and Bessel functors for $\mathrm{G}\mathbb{S}\mathrm{p}_4$

3.1 The papers [3]-[7] are all written in the direction of the geometric Langlands program for groups different from GL_n.

The fundamental tools in the classical theory of automorphic forms and representations are various models of representations, which satisfy multiplicity one property (such as Whittaker models for arbitrary group G, Waldspurger models for GL_2, Bessel models for $\mathrm{S}\mathbb{O}_{2n+1}$). In geometric setting the theory of Whittaker functors (proposed in [G]) has played an important role in the proof of the geometric Langlands conjecture ([FGV], [G]).

In our paper [4] we introduce Whittaker and Bessel functors for $\mathrm{G}\mathbb{S}\mathrm{p}_4$ and study their properties. Our motivation is a further developpement of the geometric Langlands program for $\mathrm{G}\mathbb{S}\mathrm{p}_4$ (and maybe other groups).

Remind the following facts about automorphic forms on $G = \mathbb{S}\mathrm{p}_4$. (Now X is a smooth curve over $k = \mathbb{F}_q$ as above). Let B be a Borel subgroup of G and $U \subset B$ its unipotent radical. For a character $\psi : U(F)\backslash U(\mathbb{A}) \to \mathbb{C}^*$ one has a global Whittaker module over $G(\mathbb{A})$

$$WM_\psi = \{f : U(F)\backslash G(\mathbb{A}) \to \mathbb{C} \mid \; f(ug) = \psi(u)f(g) \text{ for } u \in U(\mathbb{A}),\; f \text{ is smooth}\}$$

Let $\mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A}))$ be the space of cusp forms on $G(F)\backslash G(\mathbb{A})$. The usual Whittaker operator $W_\psi : \mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A})) \to WM_\psi$ is given by

$$W_\psi(f)(g) = \int_{U(F)\backslash U(\mathbb{A})} f(ug)\psi(u^{-1})du,$$

where du is induced from a Haar measure on $U(\mathbb{A})$. Whence for GL_n (and generic ψ) the operator W_ψ is an injection, this is not always the case for more general groups. There are cuspidal automorphic representations of $\mathbb{S}\mathrm{p}_4$ that don't admit a ψ-Whittaker model for any ψ.

The $G(\mathbb{A})$-module $\mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A}))$ decomposes as a direct sum

$$\mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A})) = I \oplus I_{hcusp} \tag{9}$$

The decomposition is orthogonal with respect to the scalar product

$$\langle f, h\rangle = \int_{G(F)\backslash G(\mathbb{A})} f(x)\overline{h(x)}dx, \tag{10}$$

where dx is induced from a Haar measure on $G(\mathbb{A})$.

By definition, I_{hcusp} are those cuspidal forms on G whose θ-lifting to the groups $\mathbb{O}(2)(\mathbb{A})$, $\mathbb{O}_4(\mathbb{A})$ and $\mathbb{O}_6(\mathbb{A})$ vanishes. Here $\mathbb{O}_{2r}$ is the orthogonal group defined by the hyperbolic quadratic form in a $2r$-dimensional space.

The space I_{hcusp} is also the intersection of kernels of W_ψ for all ψ. It is known as the space of hyper-cuspidal forms on $G(F)\backslash G(\mathbb{A})$. Another description is as follows. Let $P_1 \subset G$ be the parabolic preserving a 1-dimensional isotropic subspace in the standard representation V of G, $U_1 \subset P_1$ be its unipotent radical, U_0 the center of U_1. Then $f \in \mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A}))$ lies in I_{hcusp} if and only if

$$\int_{U_0(F)\backslash U_0(\mathbb{A})} f(ug)du = 0$$

for all $g \in G(\mathbb{A})$.

Let now $G = \mathrm{G}\mathbb{S}\mathrm{p}_4$. Similarly to the above, we have full triangulated subcategories

$$\mathrm{D}_{hcusp}(\mathrm{Bun}_G) \subset \mathrm{D}_{cusp}(\mathrm{Bun}_G) \subset \mathrm{D}(\mathrm{Bun}_G)$$

of hyper-cuspidal (resp., cuspidal) objects of $\mathrm{D}(\mathrm{Bun}_G)$. Both they are preserved by Hecke functors. So, a step in the geometric Langlands program for G would be to analize the action of $\mathrm{Sph}(\mathrm{Gr}_G)$ on the triangulated categories $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$ and $\mathrm{D}_{cusp}(\mathrm{Bun}_G)/\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$. It is for the latter category that the machinery of Whittaker functors seems to be appropriate to apply. The category $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$ is a geometric analog of the space I_{hcusp}.

A G-bundle on X is a triple: a rank 4 vector bundle M on X, a line bundle $\mathcal{A}$ on X, and a symplectic form $\wedge^2 M \to \mathcal{A}$. Let $\alpha : \bar{\mathcal{Q}}_1 \to \mathrm{Bun}_G$ be the stack over Bun_G whose fibre over $(M, \mathcal{A})$ consists of all nonzero maps of coherent sheaves $\Omega \hookrightarrow M$, where Ω is the canonical line bundle on X.

We introduce a notion of cuspidality and hyper-cuspidality on $\bar{\mathcal{Q}}_1$, leading to full triangulated subcategories

$$\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \subset \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1) \subset \mathrm{D}(\bar{\mathcal{Q}}_1)$$

The theory of Whittaker functors gives a description of $\mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1)$. Namely, let $\bar{\mathcal{Q}}$ be the Drinfeld partial compactification of the stack of (twisted) U-bundles on X (cf. [4], Sect. 4). (Here B is a Borel subgroup of G, and $U \subset B$ is its unipotent radical. By 'twisted' we mean that one rather considers B-bundles with fixed induced T-bundle. The corresponding T-bundle is picked together with a trivial conductor).

There is a natural full triangulated subcategory $\mathrm{D}^W(\bar{\mathcal{Q}}) \subset \mathrm{D}(\bar{\mathcal{Q}})$, which is a geometric analog of the space WM_ψ. We construct Whittaker functors that give rise to an equivalence of triangulated categories

$$W : \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \widetilde{\to} \mathrm{D}^W(\bar{\mathcal{Q}})$$

The Hecke functor H corresponding to the standard representation of the Langlands dual group $\check{G} \widetilde{\to} \mathrm{G}\mathbb{S}\mathrm{p}_4$ acts on all these categories, and the above equivalence commutes with H. The restriction functor

$$\alpha^* : \mathrm{D}_{cusp}(\mathrm{Bun}_G)/\mathrm{D}_{hcusp}(\mathrm{Bun}_G) \to \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \qquad (11)$$

also commutes with H. As in the case of GL_n ([G], Theorem 7.9), the advantage of $\bar{\mathcal{Q}}$ over Bun_G is that the functor $\mathrm{H} : \mathrm{D}(\bar{\mathcal{Q}}) \to \mathrm{D}(X \times \bar{\mathcal{Q}})$ is *right-exact* for the perverse t-structures. The essential difference with the case of GL_n is that the Whittaker functor $W : \mathrm{D}(\bar{\mathcal{Q}}_1) \to \mathrm{D}^W(\bar{\mathcal{Q}})$ is not exact for the perverse t-structures.

We also study similar functors for G corresponding to Bessel models. They are closely related with the Waldspurger models of representations for GL_2. In ([4], Sect. 8), we consider a problem analogous to the main result of ([FGV2]) in the case of Waldspurger models for GL_2.

We define the Waldspurger category, which is a geometric counterpart of the Waldspurger module over the Hecke algebra of GL_2. We prove a geometric version of the multiplicity one result for the Waldspurger models. This kind of problems is further studied in [6].

4. Geometric Bessel models for $\mathbb{GS}\mathrm{p}_4$ and multiplicity one

4.1 In [6] we study Bessel models of representations of $\mathrm{G}\mathbb{S}\mathrm{p}_4$. These models introduced by Novodvorsky and Piatetski-Shapiro, satisfy the following multiplicity one property ([NP]).

Set $k = \mathbb{F}_q$ and $\mathcal{O} = k[[t]] \subset F = k((t))$. Let $\tilde{F}$ be an étale F-algebra with $\dim_F(\tilde{F}) = 2$ such that k is algebraically closed in $\tilde{F}$. Write $\tilde{\mathcal{O}}$ for the integral closure of $\mathcal{O}$ in $\tilde{F}$. We have two cases:

- $\tilde{F} \widetilde{\to} k((t^{\frac{1}{2}}))$ (nonsplit case)
- $\tilde{F} \widetilde{\to} F \oplus F$ (split case)

Write L for $\tilde{\mathcal{O}}$ viewed as $\mathcal{O}$-module, it is equipped with a quadratic form $s : \mathrm{Sym}^2 L \to \mathcal{O}$ given by the determinant. Write $\Omega_{\mathcal{O}}$ for the completed module of relative differentials of $\mathcal{O}$ over k.

Set $\mathcal{M} = L \oplus (L^* \otimes \Omega_{\mathcal{O}}^{-1})$. This $\mathcal{O}$-module is equipped with a symplectic form $\wedge^2\mathcal{M} \to L \otimes L^* \otimes \Omega_{\mathcal{O}}^{-1} \to \Omega_{\mathcal{O}}^{-1}$. Set $G = GSp(\mathcal{M})$, this is a group scheme over $\mathrm{Spec}\,\mathcal{O}$. Write $P \subset G$ for the Siegel parabolic subgroup preserving the lagrangian submodule L. Its unipotent radical U has a distinguished character

$$ev : U \widetilde{\to} \Omega_{\mathcal{O}} \otimes \mathrm{Sym}^2 L \stackrel{s}{\to} \Omega_{\mathcal{O}}$$

(here we view $\Omega_{\mathcal{O}}$ as a commutative group scheme over $\mathrm{Spec}\,\mathcal{O}$). Set

$$\tilde{R} = \{p \in P \mid ev(pup^{-1}) = ev(u) \text{ for } u \in U\}$$

View $\mathrm{GL}(L)$ as a group scheme over $\mathrm{Spec}\,\mathcal{O}$ and $\tilde{\mathcal{O}}^*$ as its closed subgroup. Write α for the composition $\tilde{\mathcal{O}}^* \hookrightarrow \mathrm{GL}(L) \stackrel{\det}{\to} \mathcal{O}^*$. Fix a section $\tilde{\mathcal{O}}^* \hookrightarrow \tilde{R}$ given by $g \mapsto (g, \alpha(g)(g^*)^{-1})$. Then $R = \tilde{\mathcal{O}}^* U \subset \tilde{R}$ is a closed subgroup, and the map $R \stackrel{\xi}{\to} \Omega_{\mathcal{O}} \times \tilde{\mathcal{O}}^*$ sending tu to $(ev(u), t)$ is a homomorphism of group schemes over $\mathrm{Spec}\,\mathcal{O}$.

Let ℓ be a prime invertible in k. Fix a character $\chi : \tilde{F}^*/\tilde{\mathcal{O}}^* \to \bar{\mathbb{Q}}_\ell^*$ and a nontrivial additive character $\psi : k \to \bar{\mathbb{Q}}_\ell^*$. Write τ for the composition

$$R(F) \stackrel{\xi}{\to} \Omega_F \times \tilde{F}^* \stackrel{\mathrm{Res} \times \mathrm{pr}}{\to} k \times \tilde{F}^*/\tilde{\mathcal{O}}^* \stackrel{\psi \times \chi}{\to} \bar{\mathbb{Q}}_\ell^*$$

The Bessel module is the vector space

$$\mathrm{BM}_\tau = \{f : G(F)/G(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell \mid f(rg) = \tau(r)f(g) \text{ for } r \in R(F),$$
$$f \text{ is of compact support modulo } R(F)\}$$

Let $\chi_c : F^*/\mathcal{O}^* \to \bar{\mathbb{Q}}_\ell^*$ denote the restriction of χ. The Hecke algebra

$$\mathrm{H}_{\chi_c} = \{h : G(\mathcal{O})\backslash G(F)/G(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell \mid h(zg) = \chi_c(z)h(g) \text{ for } z \in F^*,$$
$$h \text{ is of compact support modulo } F^*\}$$

acts on BM_τ by convolutions. Then BM_τ is *a free module of rank one* over H_{χ_c}. In [6] we prove a geometric version of this result.

Remind that the affine grassmanian $\mathrm{Gr}_G = G(F)/G(\mathcal{O})$ can be viewed as an ind-scheme over k. According to 'fonctions-faisceaux' philosophy, the space BM_τ should have a geometric counterpart. A natural candidate for that would be the category of ℓ-adic perverse sheaves on Gr_G that change under the action of $R(F)$ by τ. However, the $R(F)$-orbits on Gr_G are infinite-dimensional, and this naive definition does not make sense.

We overcome this diffucculty in a way similar to the one used by Frenkel, Gaitsgory and Vilonen in [FGV2]. The idea is to replace a local statement by an appropriate global one, which admits a geometric counterpart leading to a definition of Bessel categories with expected properties.

4.2 Fix a smooth projective absolutely irreducible curve X over k. Let $\pi : \tilde{X} \to X$ be a two-sheeted covering ramified at some effective divisor D_π of X (we assume $\tilde{X}$ smooth over k). The vector bundle $L = \pi_*\mathcal{O}_{\tilde{X}}$ is equipped with a quadratic form $s : \mathrm{Sym}^2 L \to \mathcal{O}_X$.

Write Ω for the canonical line bundle on X. Set $\mathcal{M} = L \oplus (L^* \otimes \Omega^{-1})$, it is equipped with a symplectic form

$$\wedge^2\mathcal{M} \to L \otimes L^* \otimes \Omega^{-1} \to \Omega^{-1}$$

Let G be the group scheme (over X) of automorphisms of $\mathcal{M}$ preserving this symplectic form up to a multiple. Let $P \subset G$ denote the Siegel parabolic subgroup preserving L, $U \subset P$ its unipotent radical. Then U is equipped with a homomorphism of group schemes over X

$$ev : U \,\tilde{\to}\, \Omega \otimes \mathrm{Sym}^2 L \stackrel{s}{\to} \Omega$$

Let T be the functor sending a X-scheme S to the group $\mathrm{H}^0(\tilde{X} \times_X S, \mathcal{O}^*)$. Then T is a group scheme over X, a subgroup of $\mathrm{GL}(L)$. Write α for the composition $T \hookrightarrow \mathrm{GL}(L) \stackrel{\det}{\to} \mathbb{G}_m$. Set

$$\tilde{R} = \{p \in P \mid ev(pup^{-1}) = ev(u) \text{ for all } u \in U\}$$

Fix a section $T \hookrightarrow \tilde{R}$ given by $g \mapsto (g, \alpha(g)(g^*)^{-1})$. Then $R = TU \subset \tilde{R}$ is a closed subgroup, and the map $R \stackrel{\xi}{\to} \Omega \times T$ sending tu to $(ev(u), t)$ is a homomorphism of group schemes over X.

Let $F = k(X)$, $\mathbb{A}$ be the adele ring of F and $\mathcal{O} \subset \mathbb{A}$ the entire adeles. Write F_x for the completion of F at $x \in X$ and $\mathcal{O}_x \subset F_x$ for its ring of integers. Fix a nonramified character $\chi : T(F)\backslash T(\mathbb{A})/T(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell^*$. Let τ be the composition

$$R(\mathbb{A}) \stackrel{\xi}{\to} \Omega(\mathbb{A}) \times T(\mathbb{A}) \stackrel{r\times\chi}{\to} \bar{\mathbb{Q}}_\ell^*,$$

where $r : \Omega(\mathbb{A}) \to \bar{\mathbb{Q}}_\ell^*$ is given by

$$r(\omega_x) = \psi(\sum_{x\in X} \operatorname{tr}_{k(x)/k} \operatorname{Res} \omega_x)$$

Fix $x \in X(k)$. Let Y denote the restricted product $G(F_x)/G(\mathcal{O}_x) \times \prod'_{y\neq x} R(F_y)/R(\mathcal{O}_y)$. Let $\mathcal{Y}(k)$ be the quotient of Y by the diagonal action of $R(F)$. Set

$$\begin{aligned}\mathrm{BM}_{X,\tau} = \{f : Y \to \bar{\mathbb{Q}}_\ell \mid f(rg) = \tau(r)f(g) \text{ for } r \in R(\mathbb{A}),\\ f \text{ is of compact support modulo } R(\mathbb{A})\}\end{aligned}$$

View elements of $\mathrm{BM}_{X,\tau}$ as functions on $\mathcal{Y}(k)$. Let $\chi_c : F_x^*/\mathcal{O}_x^* \to \bar{\mathbb{Q}}_\ell^*$ be the restriction of χ. The Hecke algebra H_{χ_c} of the pair $(G(F_x), G(\mathcal{O}_x))$ acts on $\mathrm{BM}_{X,\tau}$ by convolutions. The restriction under

$$G(F_x)/G(\mathcal{O}_x) \hookrightarrow Y$$

yields an isomorphism of H_{χ_c}-modules $\mathrm{BM}_{X,\tau} \to \mathrm{BM}_\tau$.

We introduce an ind-algebraic stack ${}_{x,\infty}\overline{\mathrm{Bun}}_R$ whose set of k-points contains $\mathcal{Y}(k)$. We define the Bessel category $\mathrm{P}^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$, a category of perverse sheaves on ${}_{x,\infty}\overline{\mathrm{Bun}}_R$ with some equivariance property. This is a geometric version of $\mathrm{BM}_{X,\tau}$ (cf. [6] for details).

Let $\mathrm{Sph}(\mathrm{Gr}_G)$ denote the category of $G(\mathcal{O}_x)$-equivariant perverse sheaves on the affine grassmanian $G(F_x)/G(\mathcal{O}_x)$. By [MV], this is a tensor category equivalent to the category of representations of the Langlands dual group $\check{G} \widetilde{\to} \mathbb{GS}\mathrm{p}_4$. The category $\mathrm{Sph}(\mathrm{Gr}_G)$ acts on the derived category $\mathrm{D}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ by Hecke functors.

The main result of [6] describes the action of $\mathrm{Sph}(\mathrm{Gr}_G)$ on the irreducible objects of the category $\mathrm{P}^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$. It implies the above multiplicity one. It also implies that the action of $\mathrm{Sph}(\mathrm{Gr}_G)$ on $\mathrm{D}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ preserves $\mathrm{P}^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$. The same phenomenon takes place for Whittaker and Waldspurger models.

To the difference with the case of Whittaker categories, the Bessel category $\mathrm{P}^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ is not semi-simple.

5. Moduli of metaplectic bundles on curves and Theta-sheaves

5.1 The paper [5], which we consider the most important in this memoire, gives a geometric interpretation of the Weil representation of the metaplectic group, placing it in the framework of the geometric Langlands program. The discovery of this representation by A. Weil in his celebrated paper [We] has opened a new representation-theoretic approach to the classical theory of θ-series (such as, in one variable, $\sum q^{n^2}$). This is also one of the sources to construct automorphic forms.

Let $k = \mathbb{F}_q$ be a finite field with q odd. Set $K = k((t))$ and $\mathcal{O} = k[[t]]$. Let Ω denote the completed module of relative differentials of $\mathcal{O}$ over k. Let M be a free $\mathcal{O}$-module of rank $2n$ given with a nondegenerate symplectic form $\wedge^2 M \to \Omega$. It is known that the continuous $\mathrm{H}^2(\mathbb{S}\mathrm{p}(M)(K), \{\pm 1\}) \widetilde{\to} \mathbb{Z}/2\mathbb{Z}$ ([Mo], 10.4). As $\mathbb{S}\mathrm{p}(M)(K)$ is a perfect group, the corresponding metaplectic extension

$$1 \to \{\pm 1\} \xrightarrow{i} \widehat{\mathbb{S}\mathrm{p}}(M)(K) \to \mathbb{S}\mathrm{p}(M)(K) \to 1 \tag{12}$$

is unique up to unique isomorphism. It can be constructed in two essentially different ways.

Recall the classical construction of A. Weil ([We]). The Heisenberg group is $H(M) = M \oplus \Omega$ with operation

$$(m_1, \omega_1)(m_2, \omega_2) = (m_1 + m_2, \omega_1 + \omega_2 + \frac{1}{2}\langle m_1, m_2\rangle)$$

Fix a prime ℓ that does not divide q. Let $\psi : k \to \bar{\mathbb{Q}}_\ell^*$ be a nontrivial additive character. Let $\chi : \Omega(K) \to \bar{\mathbb{Q}}_\ell$ be given by $\chi(\omega) = \psi(\mathrm{Res}\,\omega)$. By the Stone and Von Neumann theorem ([MVW]), there is a unique (up to isomorphism) smooth irreducible representation $(\rho, \mathcal{S}_\psi)$ of $H(M)(K)$ over $\bar{\mathbb{Q}}_\ell$ with central character χ. The group $\mathbb{S}\mathrm{p}(M)$ acts on $H(M)$ by group automorphisms $(m, \omega) \xrightarrow{g} (gm, \omega)$ This gives rise to the group

$$\begin{aligned}\widetilde{\mathbb{S}\mathrm{p}}(M)(K) = \{(g, M[g]) \mid g \in \mathbb{S}\mathrm{p}(M)(K), M[g] \in \mathrm{Aut}\,\mathcal{S}_\psi \\ \rho(gm, \omega) \circ M[g] = M[g] \circ \rho(m, \omega) \;\text{ for } (m, \omega) \in H(M)(K)\}\end{aligned}$$

The group $\widetilde{\mathbb{S}\mathrm{p}}(M)(K)$ is an extension of $\mathbb{S}\mathrm{p}(M)(K)$ by $\bar{\mathbb{Q}}_\ell^*$. Its commutator subgroup is an extension of $\mathbb{S}\mathrm{p}(M)(K)$ by $\{\pm 1\} \hookrightarrow \bar{\mathbb{Q}}_\ell^*$, uniquely isomorphic to (12).

Another way is via Kac-Moody groups. Namely, view $\mathbb{S}\mathrm{p}(M)(K)$ as an ind-scheme over k. Let

$$1 \to \mathbb{G}_m \to \overline{\mathbb{S}\mathrm{p}}(M)(K) \to \mathbb{S}\mathrm{p}(M)(K) \to 1 \tag{13}$$

denote the canonical extension, here $\overline{\mathbb{S}\mathrm{p}}(M)(K)$ is an ind-scheme over k (cf. [F]). Passing to k-points we get an extension of abstract groups $1 \to k^* \to \overline{\mathbb{S}\mathrm{p}}(M)(K) \to \mathbb{S}\mathrm{p}(M)(K) \to 1$. Then (12) is the push-forward of this extension under $k^* \to k^*/(k^*)^2$.

The second construction underlies one of the main results of [5], the tannakian description of the Langlands dual to the metaplectic group. Namely, the canonical splitting of (13) over $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$ yields a splitting of (12) over $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$. Consider the Hecke algebra

$$\begin{aligned}\mathcal{H} = \{f : \mathbb{S}\mathrm{p}(M)(\mathcal{O})\backslash\widehat{\mathbb{S}\mathrm{p}}(M)(K)/\,\mathbb{S}\mathrm{p}(M)(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell \mid f(i(-1)g) = -f(g), \;\; g \in \widehat{\mathbb{S}\mathrm{p}}(M)(K); \\ f \text{ is of compact support}\}\end{aligned}$$

The product is convolution, defined using the Haar measure on $\widehat{\mathbb{S}\mathrm{p}}(M)(K)$ for which the inverse image of $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$ has volume 1.

Set $G = \mathbb{S}\mathrm{p}(M)$. Let $\check{G}$ denote $\mathbb{S}\mathrm{p}_{2n}$ viewed as an algebraic group over $\bar{\mathbb{Q}}_\ell$. Let $\mathrm{Rep}(\check{G})$ denote the category of finite-dimensional representations of $\check{G}$. Write $K(\mathrm{Rep}(\check{G}))$ for the Grothendieck ring of $\mathrm{Rep}(\check{G})$ over $\bar{\mathbb{Q}}_\ell$. There is a canonical isomorphism of $\bar{\mathbb{Q}}_\ell$-algebras

$$\mathcal{H} \widetilde{\to} K(\mathrm{Rep}(\check{G}))$$

Actually, we prove a categorical version of this isomorphism. Consider the affine grassmanian $\mathrm{Gr}_G = G(K)/G(\mathcal{O})$, viewed as an ind-scheme over k. Let W denote the nontrivial ℓ-adic local system of rank one on $\mathbb{G}_m$ corresponding to the covering $\mathbb{G}_m \to \mathbb{G}_m$, $x \mapsto x^2$. Denote by $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ the category of $G(\mathcal{O})$-equivariant perverse sheaves on $\overline{G}(K)/G(\mathcal{O})$, which are also $(\mathbb{G}_m, W)$-equivariant. Here $\widetilde{\mathrm{Gr}}_G$ denotes the stack quotient of $\overline{G}(K)/G(\mathcal{O})$ by $\mathbb{G}_m$ with respect to the action $g \stackrel{x}{\to} x^2 g$, $x \in \mathbb{G}_m, g \in \overline{G}(K)$. Actually, $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ is a full subcategory of the category of perverse sheaves on $\widetilde{\mathrm{Gr}}_G$.

Assuming for simplicity k algebraically closed, we equip $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ with the structure of a rigid tensor category. We establish a canonical equivalence of tensor categories

$$\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G) \widetilde{\to} \mathrm{Rep}(\check{G})$$

5.2 In the global setting let X be a smooth projective geometrically connected curve over k. Let G denote the sheaf of automorphisms of $\mathcal{O}_X^n \oplus \Omega^n$ (now Ω is the canonical line bundle on X) preserving the symplectic form $\wedge^2(\mathcal{O}_X^n \oplus \Omega^n) \to \Omega$.

The stack Bun_G of G-bundles (=G-torsors) on X classifies vector bundles M of rank $2n$ on X, given with a nondegenerate symplectic form $\wedge^2 M \to \Omega$. We introduce an algebraic stack $\widetilde{\mathrm{Bun}}_G$ of metaplectic bundles on X. The stack $\widetilde{\mathrm{Gr}}_G$ is a local version of $\widetilde{\mathrm{Bun}}_G$. The category $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ acts on $\mathrm{D}(\widetilde{\mathrm{Bun}}_G)$ by Hecke operators.

We construct a perverse sheaf Aut on $\widetilde{\mathrm{Bun}}_G$, a geometric analog of the Weil representation. We calculate the fibres of Aut and its constant terms for maximal parabolic subgroups of G. Finally, we argue that Aut is a Hecke eigensheaf on $\widetilde{\mathrm{Bun}}_G$ with eigenvalue

$$\mathrm{St} = \mathrm{R}\Gamma(\mathbb{P}^{2n-1}, \bar{\mathbb{Q}}_\ell) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes 2n-1}$$

viewed as a constant complex on X. Note that St is equipped with an action of SL_2 of Arthur, the corresponding representation of SL_2 is irreducible of dimension $2n$ and admits a unique, up to a multiple, symplectic form. One may imagine that Aut corresponds to a group homomorphism $\pi_1(X) \times \mathrm{SL}_2 \to \check{G}$ trivial on $\pi_1(X)$. This agrees with Arthur's conjectures.

6. Towards geometric theta-lifting

6.1 The Weil representation has been used by R. Howe who proposed a series of conjectures ([H]) establishing a correspondence between automorphic representations for dual reductive pairs (now referred to as theta-lifting or Howe correspondence) as well as their local analogs. These conjectures are now theorems (if the characteristic of the base field is not 2), which are among major achievements in the classical theory of automorphic forms (cf. [MVW], [R], [Pr], [Wa, Wa1, Wa2, Wa3]).

The paper [7], which is a sequel to [5], is a step towards a version of the Howe correspondence (or geometric theta-lifting) in the framework of the geometric Langlands program.

As above, let X be a smooth projective geometrically connected curve over k. Set $G = \mathrm{G}\mathbb{S}\mathrm{p}_{2n}$. Let $\pi : \tilde{X} \to X$ be an étale two-sheeted covering of X. Let $\mathrm{G}\mathbb{O}^0_{2m}$ denote the connected component of unity of the split orthogonal similitude group over $\operatorname{Spec} k$ (the corresponding orthogonal space is $2m$-dimensional). Pick an involution $\sigma \in \mathbb{O}_{2m}(k)$ with $\sigma \notin \mathrm{S}\mathbb{O}_{2m}(k)$, consider the corresponding $\mathbb{Z}/2\mathbb{Z}$-action on $\mathrm{G}\mathbb{O}^0_{2m}$ by conjugation. Let H denote the group scheme over X, the twisting of $\mathrm{G}\mathbb{O}^0_{2m}$ by the $\mathbb{Z}/2\mathbb{Z}$-torsor $\pi : \tilde{X} \to X$.

Then (H, G) is an (unramified) dual reductive pair over X. Let Bun_G (resp., Bun_H) denote the stack of G-torsors (resp., H-torsors) on X.

For $m \leq n$ (resp., $m > n$) we have a natural morphism of the Langlands L-groups $H^L \to G^L$ (resp., $G^L \to H^L$). According to the Langlands principle of functoriality, there should exist some correspondences between the automorphic representations of $G(\mathbb{A})$ and $H(\mathbb{A})$, whose local Langlands parameters are related by the corresponding maps of L-groups. (In everywhere nonramified geometric setting we formalize this problem in [7, appendix B]).

Using the theta-sheaf introduced in [5], we define functors $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ and $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ between the corresponding derived categories, which are geometric analogs of the theta-lifting operators. These are approximations to the solution of the corresponding functoriality problem.

It turns out that, in general, F_G and F_H do not commute with Hecke operators. The reason is that the automorphic representation of G (resp., of H) attached to a given automorphic representation of H (resp., of G) via the Howe correspondence is not always irreducible (and its components may correspond to non isomorphic local system eigen-values).

On the positive side, here is a geometric version of a 'folklore conjecture' that I found in a paper of Prasad ([Pr]).

Folklore Conjecture . *i) For $m \leq n$ let $K \in \mathrm{D}(\mathrm{Bun}_H)$ be a cuspidal automorphic sheaf such that $F_G(K) \in \mathrm{D}(\mathrm{Bun}_G)$ is cuspidal. Then $F_G(K)$ is an automorphic sheaf on Bun_G and $F_H(F_G(K))$ does not vanish. If, in addition, $F_H(F_G(K))$ is cuspidal then it is isomorphic to K.*
ii) For $m > n$ let $K \in \mathrm{D}(\mathrm{Bun}_G)$ be a cuspidal automorphic sheaf such that $F_H(K) \in \mathrm{D}(\mathrm{Bun}_H)$ is cuspidal. Then $F_H(K)$ is an automorphic sheaf on Bun_H and $F_G(F_H(K))$ does not vanish. If, in addition, $F_G(F_H(K))$ is cuspidal then it is isomorphic to K.

One of the main results of [7] is the geometric Langlands functoriality for the dual pair $(\mathrm{G}\mathbb{O}_2, \mathrm{GL}_2)$, where $\mathrm{G}\mathbb{O}_2 = \pi_*\mathbb{G}_m$ is a group scheme over X, here $\pi : \tilde{X} \to X$ is a nontrivial étale two-sheeted covering. In this case the functor F_G does commute with Hecke operators (cf. [7], Corolary 4). If $\tilde{E}$ is a rank one local system on $\tilde{X}$, let $K_{\tilde{E}}$ denote the automorphic sheaf on $\operatorname{Pic} \tilde{X}$ corresponding to $\tilde{E}$ (constructed via the geometric class field theory). Then $F_G(K_{\tilde{E}})$ is an automorphic sheaf on Bun_G corresponding to the local system $E = (\pi_*\tilde{E})^*$. This provides *a new proof of the geometric Laglands conjecture for E* independent of the existing proof due to Frenkel, Gaitsgory and Vilonen ([FGV], [G]).

We also check that (up to a tensoring by a 1-dimensional vector space) the sheaf $F_G(K_{\tilde{E}})$ coincides with the perverse sheaf Aut_E constructed via Whittaker models in *loc.cit.* (this is our Proposition 6, [7]).

The above results also allow us to calculate the following (global) Rankin-Selberg type convolution. Let E be an irreducible rank 2 local system on X, E_1 be a rank one local system on X. We denote by $\mathrm{Aut}_{E_1\oplus\bar{\mathbb{Q}}_\ell}$ the corresponding Eisenstein series (cf. [BG]). Theorem 2 from [7] provides an explicit calculation of $F_{\mathrm{H}}(\mathrm{Aut}_{E_1\oplus\bar{\mathbb{Q}}_\ell}\otimes\mathrm{Aut}_E)$. The method of its proof is inspired by [1]. Namely, we first note that this global statement can be reduced to some local statement (which is the corresponding local Rankin-Selberg convolution, a calculation that makes sense for any, non necessairily irreducible, local system E on X).

6.2 Let us formulate just one consequence of the results of [7], which we find striking.

Assume the ground field $k=\mathbb{F}_q$ finite of q elements (with q odd). Set $G=\mathrm{GL}_2$. Let E be a rank 2 irreducible ℓ-adic local system on X. Remind that Aut_E denotes the corresponding automorphic sheaf on Bun_G (cf. [FGV]). Let $f_E:\mathrm{Bun}_G(k)\to\bar{\mathbb{Q}}_\ell$ denote the function 'trace of Frobenius' of Aut_E.

Let $\phi:Y\to X$ be a nontrivial étale two-sheeted covering. Write $\operatorname{Pic}Y$ for the Picard stack of Y. Let $\mathcal{J}$ be a rank one local system on Y equipped with an isomorphism $N(\mathcal{J})\,\widetilde{\to}\,\det E$, where $N(\mathcal{J})$ is the norm of $\mathcal{J}$ (cf. [7], appendix A). Write $f_{\mathcal{J}}:(\operatorname{Pic}Y)(k)\to\bar{\mathbb{Q}}_\ell$ for the corresponding character (the trace of Frobenius of the automorphic local system $A\mathcal{J}$ corresponding to $\mathcal{J}$). The Waldspurger period of f_E is

$$\int_{\mathcal{B}\in(\operatorname{Pic}Y)(k)/(\operatorname{Pic}X)(k)} f_E(\phi_*\mathcal{B})f_{\mathcal{J}}^{-1}(\mathcal{B})d\mathcal{B}$$

(the function that we integrate do not change when $\mathcal{B}$ is tensored by ϕ^*L, $L\in\operatorname{Pic}X$), here $d\mathcal{B}$ is a Haar measure. A beautiful theorem of Waldspurger says that *the square* of this period is equal (up to an explicit harmless coefficient) to the value of the L-function $L(\phi^*E\otimes\mathcal{J}^{-1},\frac{1}{2})$ (cf. [Wa]).

Theorem 4 from [7] is a geometric version of this result. The role of the L-function in geometric setting is played by the complex

$$\oplus_{d\geq 0}\mathrm{R}\Gamma(Y^{(d)},(\phi^*E\otimes\mathcal{J}^*)^{(d)})[d] \tag{14}$$

Here $Y^{(d)}$ is the d-th symmetric power of Y, and $V^{(d)}$ denotes the d-th symmetric power of a local system V on X. If ϕ^*E is still irreducible then (14) is bounded.

The geometric Waldspurger period is

$$\mathrm{R}\Gamma_c(\operatorname{Pic}Y/\operatorname{Pic}X,\phi_1^*\mathrm{Aut}_E\otimes A\mathcal{J}^{-1}), \tag{15}$$

where $\phi_1:\operatorname{Pic}Y\to\mathrm{Bun}_G$ sends $\mathcal{B}$ to $\phi_*\mathcal{B}$. The sense of the quotient $\operatorname{Pic}Y/\operatorname{Pic}X$ is precised in ([7], 6.3.3), this stack has two connected components (the degree of $\mathcal{B}$ modulo two), so that (15) and its tensor square is naturally $\mathbb{Z}/2\mathbb{Z}$-graded.

We show that the tensor square of (15) is isomorphic (up to a shift and a tensoring by an explicit 1-dimensional space) to (14). Moreover, this isomrphism is $\mathbb{Z}/2\mathbb{Z}$-graded, where the $\mathbb{Z}/2\mathbb{Z}$-grading of (14) is given by the parity of d.

6.3 In [7] we also consider the cases $m = 2$ and $n = 2$ (that is, $H = \mathrm{G}\mathbb{O}_4$ and $G = \mathrm{G}\mathbb{S}\mathrm{p}_4$), but our results in these cases are only partial.

Assume $m = 2$. Write $\mathrm{Bun}_{2,\tilde{X}}$ for the stack of rank 2 vector bundles on $\tilde{X}$. We have an exact sequence $1 \to U_\pi \to \pi_* \mathrm{GL}_2 \to H \to 1$ of group schemes on X, where U_π is the twisting of $\mathbb{G}_m$ by the $\mathbb{Z}/2\mathbb{Z}$-torsor $\pi : \tilde{X} \to X$ (the corresponding involution of $\mathbb{G}_m$ sends z to z^{-1}). This yields a natural map $\rho : \mathrm{Bun}_{2,\tilde{X}} \to \mathrm{Bun}_{\tilde{H}}$.

Let $\tilde{E}$ be an irreducible rank 2 local system on $\tilde{X}$, χ be a rank one local system on X equipped with $\det \tilde{E} \widetilde{\to} \pi^*\chi$. The corresponding automorphic sheaf $\mathrm{Aut}_{\tilde{E}}$ descends with respect to ρ to a perverse sheaf $K_{\tilde{E},\chi,H}$ on Bun_H.

Assume $n = 2$, so $G = \mathrm{G}\mathbb{S}\mathrm{p}_4$. If $\tilde{E}$ does not descend to X then we conjecture that

$$F_G(K_{\tilde{E},\chi,H}) \tag{16}$$

is a cuspidal automorphic sheaf on Bun_G ([7], Conjecture 1). This agrees with the classical Howe correspondence.

Assume that E is an irreducible rank 2 local system on X equipped with $\pi^* E \widetilde{\to} \tilde{E}$ and $\det E \widetilde{\to} \chi$. For an object $K \in \mathrm{D}(\mathrm{Bun}_G)$ there is a notion of Bessel periods of K (cf. [BFF]). Theorem 3 from [7] is essentially a calculation of the Bessel periods of (16). Its proof follows closely a similar calculation of Waldspurger at the level of functions ([Wa], cf. also [BFF]).

Now assume $n = 1$. If $\tilde{X}$ splits then we show that $F_H(\mathrm{Aut}_{E^*})$ is isomorphic to $K_{\pi^*E,\det E,H}$ up to a tensoring by a 1-dimensional space ([7], Proposition 8). This is a geometric version of a theorem of Shimizu (cf. [Wa]).

Papers included in this memoire

1. S. Lysenko, Local geometrized Rankin-Selberg method for GL(n), Duke Math. J., 111 (2002), no. 3, 451–493.
2. S. Lysenko, Global geometric Rankin-Selberg method for GL_n, math.AG/0108208 (38 pages)
3. S. Lysenko, On the automorphic sheaves on Bun_G, math.RT/0211067 (30 pages)
4. S. Lysenko, Whittaker and Bessel functors for $\mathrm{G}\mathbb{S}\mathrm{p}_4$, math.AG/0310361, to appear in Annales de l'Institut Fourier (52 pages)
5. S. Lysenko, Moduli of metaplectic bundles on curves and Theta-sheaves, math.AG/0405021, to appear in Annales Sci. Ecole Norm. Sup (54 pages)
6. S. Lysenko, Geometric Bessel models for $\mathrm{G}\mathbb{S}\mathrm{p}_4$ and multiplicity one, IMRN, vol. 43 (2005), p. 2657-2694
7. S. Lysenko, Geometric Waldspurger periods, math.AG/0510110 submitted (54 pages)

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