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TOWARDS CANONICAL REPRESENTATIONS OF FINITE HEISENBERG GROUPS

by Sergey Lysenko

Abstract. — We consider a finite abelian group M of odd exponent n with a symplectic form $\omega : M \times M \to \mu_n$ and the Heisenberg extension $1 \to \mu_n \to H \to M \to 1$ with the commutator ω. According to the Stone–von Neumann theorem, H admits an irreducible representation with the tautological central character (defined up to a nonunique isomorphism). We construct such an irreducible representation of H defined up to a unique isomorphism, so canonical in this sense.

Résumé (*Vers les représentations canoniques des groupes de Heisenberg finis*). — On considère un groupe fini abélien M d'exposant impair n avec une forme symplectique $\omega : M \times M \to \mu_n$. Soit $1 \to \mu_n \to H \to M \to 1$ une extension de Heisenberg dont le commutateur est ω. D'après un théorème de Stone-von Neumann, H admet une représentation irréductible avec le caractère central tautologique, qui est définie à un isomorphisme non unique près. Nous construisons une telle représentation définie à un unique isomorphisme près, donc canonique dans ce sens.

1. Introduction

1.0.1. Consider a finite abelian group M of odd exponent n with a symplectic form $\omega : M \times M \to \mu_n$. It admits a unique symmetric Heisenberg extension $1 \to \mu_n \to H \to M \to 1$ with the commutator ω. According to the Stone–von Neumann theorem, H admits an irreducible representation with the tautolog-

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ical central character (defined up to a nonunique isomorphism). We construct such an irreducible representation of H defined up to a unique isomorphism, so canonical in this sense, over a suitable finite extension of $\mathbb{Q}$.

1.0.2. We are motivated by the following question of Dennis Gaitsgory about [7]. Let X be a smooth projective connected curve over an algebraically closed field k. Let T be a torus over k with a geometric metaplectic data $\mathcal{G}$ as in [5].[1] To fix ideas, consider the sheaf-theoretic context of ℓ-adic sheaves on finite type schemes over k. Let $(H, \mathcal{G}_{Z_H}, \epsilon)$ be the metaplectic Langlands dual datum associated to $(T, \mathcal{G})$ in ([5], Section 6), so H is a torus over $\bar{\mathbb{Q}}_\ell$ isogenous to the Langlands dual of T. Let σ be a twisted local system on X, for $(H, \mathcal{G}_{Z_H})$ in the sense of [5], Section 8.4. To this data in ([5], Section 9.5.3) we attached the DG-category of Hecke eigensheaves. The question is whether this category identifies canonically with the DG-category Vect of $\bar{\mathbb{Q}}_\ell$-vector spaces. In [7], we constructed such an irreducible Hecke eigensheaf for σ out of a given irreducible representation of a certain finite Heisenberg group (denoted by Γ given by formula (33) in [7], Section 5.2.4) with the tautological central character.

This is why we are interested in constructing a canonical irreducible representation of finite Heisenberg groups as in Section 1.0.1. We do this only assuming the order of M odd, the case of even order remains open.

2. Main result

2.0.1. Let e be an algebraically closed field of characteristic zero. Let M be a finite abelian group, $\omega : M \times M \to e^*$ a bilinear form, which is alternating, that is, $\omega(m, m) = 0$ for any $m \in M$. Assume the induced map $M \to \mathrm{Hom}(M, e^*)$ is an isomorphism, that is, the form is nondegenerate.

If $L \subset M$ is a subgroup, $L^\perp = \{m \in M \mid \omega(m, l) = 0 \text{ for all } l \in L\}$ is its orthogonal complement; this is a subgroup. The group L is isotropic if $L \subset L^\perp$. The subgroup L is Lagrangian if $L^\perp = L$. For a Lagrangian subgroup, we get an exact sequence

$$0 \to L \to M \to L^* \to 0, \tag{1}$$

where $L^* = \mathrm{Hom}(L, e^*)$. Namely, we send $m \in M$ to the character $l \mapsto \omega(m, l)$ of L. This exact sequence always admits a splitting $L^* \to M$, which is a homomorphism; see, for example, ([4], Lemma 5.2). For such a splitting after the obtained identification $M \widetilde{\to} L \times L^*$, the form ω becomes

$$\omega((l_1, \chi_1), (l_2, \chi_2)) = \frac{\chi_1(l_2)}{\chi_2(l_1)}, \tag{2}$$

for $l_i \in L, \chi_i \in L^*$.

1. There is a published version of this paper. However, the reader should use the version from the References, which corrects some mistakes of the published version.

We have $\mathrm{H}^2(M, e^*) \widetilde{\to} \mathrm{Hom}(\mathrm{H}_2(M, \mathbb{Z}), e^*)$ by the universal coefficient theorem [2]. So, up to an isomorphism, there is a unique central extension

$$1 \to e^* \to H_{e^*} \to M \to 1, \tag{3}$$

with the commutator ω. We are interested in understanding the category of representations of H_{e^*} with the tautological central character.

2.0.2. For a finite abelian group L, its exponent is the least common multiple of the orders of the elements of L. Let n be the exponent of M; this is a divisor of $\sqrt{|\, M\,|} \in \mathbb{N}$.

Let $\mu_n = \mu_n(e)$. Let us be given a central extension

$$1 \to \mu_n \to H \to M \to 1, \tag{4}$$

together with a symmetric structure σ in the sense of [1], Section 1.1,[2] and commutator ω. That is, σ is an automorphism of H such that $\sigma^2 = \mathrm{id}$, $\sigma|_{\mu_n} = \mathrm{id}$, and $\sigma \bmod \mu_n$ is the involution $m \mapsto -m$ of M.

From now, on assume n odd. Then by [1], Sections 1.1–1.3, there is a unique symmetric central extension (4) with commutator ω (up to a unique isomorphism). Besides, (3) is isomorphic to the push-out of (4) under the tautological character $\iota : \mu_n \hookrightarrow e^*$.

The extension H is constructed as follows. Let $\beta : M \times M \to \mu_n$ be the unique alternating bilinear form such that $\beta^2 = \omega$. We take $H = M \times \mu_n$ with the product

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 a_2 \beta(m_1, m_2)),$$

for $m_i \in M, a_i \in \mu_n$. Then $\sigma(m, a) = (-m, a)$ for $m \in M, a \in \mu_n$.

Let $G = \mathbb{S}\mathrm{p}(M)$, the group of automorphisms of M preserving ω. Let $g \in G$ act on H, sending (m, a) to (gm, a). This gives the semidirect product $H \rtimes G$.

2.0.3. The following version of the Stone–von Neumann theorem holds for H; the proof is left to the reader.

Proposition 2.1. — *Up to an isomorphism, there is a unique irreducible representation of H over e with the tautological central character $\iota : \mu_n \hookrightarrow e^*$.*

2.0.4. Write $\mathcal{L}(M)$ for the set of Lagrangian subgroups in M. For $L \in \mathcal{L}(M)$, let $\bar{L}$ be the preimage of L in H; this is a subgroup. If $\chi_L : \bar{L} \to e^*$ is a character extending the tautological character $\iota : \mu_n \hookrightarrow e^*$, set

$$\mathcal{H}_L = \{ f : H \to e \mid f(\bar{l}h) = \chi_L(\bar{l}) f(h), \text{ for } \bar{l} \in \bar{L}, h \in H \}.$$

This is a representation of H by right translations. It is irreducible with central character ι.

2. This paper is published as *Beilinson, A. (2006), Langlands parameters for Heisenberg modules. In Bernstein, J., Hinich, V., Melnikov, A. (eds.) Studies in Lie Theory. Progress in Mathematics, vol 243. Birkhäuser Boston.*. However, this definition seems absent in the published version.

2.0.5. We study the following.

Problem. — Describe the category $\mathrm{Rep}_\iota(H)$ of representations of H over e with central character $\iota:\mu_n\hookrightarrow e^*$. Is there an object of $\mathrm{Rep}_\iota(H)$, which is irreducible and defined up to a unique isomorphism? (If yes, it would provide an equivalence between $\mathrm{Rep}_\iota(H)$ and the category of e-vector spaces).

2.0.6. Let I be the set of primes appearing in the decomposition of n; write $n=\prod_{p\in I}p^{r(p)}$ with $r(p)>0$. Let $K\subset e$ be the subfield generated over $\mathbb{Q}$ by $\{\sqrt{p}\mid p\in I\}$ and μ_n.

Theorem 2.2. — *There is an irreducible representation π of H over K with central character $\iota:\mu_n\hookrightarrow K^*$ defined up to a unique isomorphism. The H-action on π extends naturally to an action of $H\rtimes G$.*

Remark 2.3. — Let $K'\subset e$ be the subfield generated over $\mathbb{Q}$ by μ_n. The field of definition of the character of π is K'. However, we do not expect that for any H with n odd Theorem 2.2 holds already with K replaced by K', but we have not checked that. Our choices of $\sqrt{p}$ for $p\in I$ are made to use the results of [6], and the formulas from [6] do not work without these choices. Note also the following. If L is an odd abelian group, and $b:L\times L\to e^*$ is a nondegenerate symmetric bilinear form, then the Gauss sum of b is defined as

$$G(L,b)=\sum_{l\in L}b(l,l).$$

Using the classification of such symmetric bilinear forms given in [8], one can check that $G(L,b)^4=\mid L\mid^2$. Since the construction of π in Theorem 2.2 is related to representing the corresponding 2-cocyle (given essentially by certain Gauss sums) as a coboundary (after some minimal additional choices), we expect that our choices of $\sqrt{p}$ for $p\in I$ are necessary.

Remark 2.4. — For $L\in\mathcal{L}(M)$, the H-representation $\mathcal{H}_L$ from Section 2.0.4 is defined over K. We sometimes view it as a representation over K; it is hoped that the precise meaning is clear from the context.

3. Proof of Theorem 2.2

3.0.1. *Reduction.* — For $p\in I$, let

$$H_p=\{h\in H\mid h^{(p^s)}=1\text{ for }s\text{ large enough}\}$$

and similarly for M_p. Then, $H_p\subset H$ is a subgroup that fits into an exact sequence $1\to\mu_{p^{r(p)}}\to H_p\to M_p\to 1$, and $H=\prod_{p\in I}H_p$, a product of groups. Indeed, $\omega(H_p,H_q)=1$, for $p,q\in I$, $p\neq q$. Besides, σ preserves H_p for each $p\in I$, so $(H_p,\sigma|_{H_p})$ is a symmetric Heisenberg extension of (M_p,ω_p) by $\mu_{p^{r(p)}}$. Here, $\omega_p:M_p\times M_p\to\mu_{p^{r(p)}}$ is the restriction of ω. So, Problem 2.0.5 reduces to the case of a prime n. If $M_p = 0$, then take π_p to be the one-dimensional representation given by the tautological character $\mu_{p^{r(p)}} \hookrightarrow e^*$.

For $p \in I$ odd, let $K_p \subset e$ be the subfield generated over $\mathbb{Q}$ by $\mu_{p^{r(p)}}$ and $\sqrt{p}$. We prove Theorem 2.2 in the case of an odd prime n, getting for $p \in I$ a representation π_p of H_p over K_p, hence over K also. Then, for any odd n, $\pi = \bigotimes_{p\in I} \pi_p$ is the desired representation.

3.0.2. From now on, we assume $n = p^r$ for an odd prime p.

3.1. Case $r = 1$. —

3.1.1. In this section, we assume M is a $\mathbb{F}_p$-vector space of dimension $2d$. To apply the results of [6], pick an isomorphism $\psi : \mathbb{F}_p \widetilde{\to} \mu_p$. It allows us to identify H with $M \times \mathbb{F}_p$. We then view $\mathcal{L}(M), H$ as algebraic varieties over $\mathbb{F}_p$. We also allow the case $d = 0$.

3.1.2. Recall the following construction from ([6], Theorem 1).[3]

Pick a prime $\ell \neq p$ and an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. We assume that $\bar{\mathbb{Q}}_\ell$ is chosen in such a way that $K \subset \bar{\mathbb{Q}}_\ell$ is a subfield. In particular, we get $\sqrt{p} \in K \subset \bar{\mathbb{Q}}_\ell$. This gives rise to the $\bar{\mathbb{Q}}_\ell$-sheaf $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$ over $\operatorname{Spec} \mathbb{F}_p$.

Pick a one-dimensional $\mathbb{F}_p$-vector space $\mathcal{J}$ of parity $d \bmod 2$ as $\mathbb{Z}/2\mathbb{Z}$-graded. Let $\mathcal{A}$ be the line bundle (of parity zero as $\mathbb{Z}/2\mathbb{Z}$-graded) on $\mathcal{L}(M)$ with fiber $\mathcal{J} \otimes \det L$ at $L \in \mathcal{L}(M)$. Write $\widetilde{\mathcal{L}}(M)$ for the gerbe of square roots of $\mathcal{A}$.

In *loc.cit*, we constructed an irreducible perverse sheaf F on $\widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$. Although in *loc.cit.*, we mostly worked over an algebraic closure $\overline{\mathbb{F}}_p$, F is defined over $\mathbb{F}_p$.

Lemma 3.1. — *For any* $i : \operatorname{Spec} \mathbb{F}_p \to \widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$, $\operatorname{tr}(\mathrm{Fr}, i^*F) \in K$. *Here,* Fr *is the geometric Frobenius endomorphism.*

Proof. — This follows from formula (10) in [6], Section 3.3. Namely, after a surjective smooth localization (the choice of an additional Lagrangian in M), there is an explicit formula for F as the convolution along H of two explicit rank one local systems. Their traces of Frobenius lie in K, as their definition involves only the Artin–Schreier sheaf and Tate twists. So, the same holds after the convolution along the finite group $H(\mathbb{F}_p)$. □

3.1.3. For an algebraic stack $S \to \operatorname{Spec} \mathbb{F}_p$, we write $S(\mathbb{F}_p)$ for the set of isomorphism classes of its $\mathbb{F}_p$-points. In view of the isomorphism $\psi : \mathbb{F}_p \widetilde{\to} \mu_p$ fixed above, for $L \in \mathcal{L}(M)(\mathbb{F}_p)$, we identify $\bar{L} = L \times \mu_p$ with $L \times \mathbb{F}_p$. Let

$$F^{cl} : \widetilde{\mathcal{L}}(M)(\mathbb{F}_p) \times \widetilde{\mathcal{L}}(M)(\mathbb{F}_p) \times H(\mathbb{F}_p) \to K$$

be the function trace of Frobenius of F.

3. For this construction, we adopt the conventions of *loc.cit* about $\mathbb{Z}/2\mathbb{Z}$-gradings and étale $\bar{\mathbb{Q}}_\ell$-sheaves on schemes over $\mathbb{F}_p$.

For $L \in \mathcal{L}(M)(\mathbb{F}_p)$, its preimage in $\widetilde{\mathcal{L}}(M)(\mathbb{F}_p)$ consists of two elements. We let μ_2 act on $\widetilde{\mathcal{L}}(M)(\mathbb{F}_p)$ over $\mathcal{L}(M)(\mathbb{F}_p)$ permuting the elements in the preimage of each $L \in \mathcal{L}(M)(\mathbb{F}_p)$. We call a function $h : \widetilde{\mathcal{L}}(M)(\mathbb{F}_p) \to K$ genuine, if it changes by the nontrivial character of μ_2 under this μ_2-action. Recall that F^{cl} is genuine with respect the first and second variables.

Let us write L^0 for a point of $\widetilde{\mathcal{L}}(M)(\mathbb{F}_p)$ over $L \in \mathcal{L}(M)(\mathbb{F}_p)$. As in [6], Section 2, for $L^0, N^0 \in \widetilde{\mathcal{L}}(M)(\mathbb{F}_p)$, viewing $\mathcal{H}_L, \mathcal{H}_N$ as H-representations over K, we define the canonical intertwining operator

$$F_{N^0,L^0} : \mathcal{H}_L \to \mathcal{H}_N$$

by

$$(F_{N^0,L^0} f)(h_1) = \int_{h_2 \in H} F^{cl}_{N^0,L^0}(h_1 h_2^{-1}) f(h_2) dh_2,$$

where our measure dh_2 is normalized by requiring that the volume of a point is 1. Here, $F^{cl}_{N^0,L^0}$ is the restriction of F^{cl}, where we fix the first two variables to be N^0, L^0.

Let $G = \mathbb{S}\mathrm{p}(M)$ viewed as an algebraic group over $\mathbb{F}_p$. It acts naturally on $\mathcal{L}(M), H$, and $\widetilde{\mathcal{L}}(M)$. By definition, for $g \in G, (m,a) \in H$, $g(m,a) = (gm, a)$ for $m \in M, a \in \mathbb{F}_p$, and this action preserves the symmetric structure σ on H. If $g \in G$, $f : H \to K$, then $gf : H \to K$ is given by $(gf)(h) = f(g^{-1}h)$. Then, $g \in G(\mathbb{F}_p)$ yields an isomorphism $\mathcal{H}_L \widetilde{\to} \mathcal{H}_{gL}$. We let G act diagonally on $\widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$.

The above intertwining operators satisfy the following properties.

- $F_{L^0,L^0} = \mathrm{id}$.
- $F_{R^0,N^0} \circ F_{N^0,L^0} = F_{R^0,L^0}$ for any $R^0, N^0, L^0 \in \widetilde{\mathcal{L}}(M)(\mathbb{F}_p)$.
- For any $g \in G(\mathbb{F}_p)$, we have $g \circ F_{N^0,L^0} \circ g^{-1} = F_{gN^0,gL^0}$.

Definition 3.2. — Let π be the K-vector space of collections $f_{L^0} \in \mathcal{H}_L$, for $L^0 \in \widetilde{\mathcal{L}}(M)(\mathbb{F}_p)$, satisfying the property: for $N^0, L^0 \in \widetilde{\mathcal{L}}(M)(\mathbb{F}_p)$, one has

$$F_{N^0,L^0}(f_{L^0}) = f_{N^0}.$$

This is our canonical H-representation over K.

We let $G(\mathbb{F}_p)$ act on $\widetilde{\mathcal{L}}(M)(\mathbb{F}_p) \times H(\mathbb{F}_p)$ diagonally. This yields a $G(\mathbb{F}_p)$-action on π, sending $\{f_{L^0}\} \in \pi$ to the collection $L^0 \mapsto g(f_{g^{-1}L^0})$.

3.2. Case $r \geq 1$. —

3.2.1. Let L be a finite abelian group and p be any prime number. For $k \geq 0$, let $L[p^k] = \{l \in L \mid p^k l = 0\}$ and

$$\rho_k(L) = L[p^k]/(L[p^{k-1}] + pL[p^{k+1}]).$$

Each $\rho_k(L)$ is a vector space over $\mathbb{F}_p$. Note that

$$\rho_k(\mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbb{Z}/p\mathbb{Z}, & m = k \\ 0, & \text{otherwise.} \end{cases}$$

For finite abelian groups L, L', one has canonically $\rho_k(L\times L') \xrightarrow{\sim} \rho_k(L)\times\rho_k(L')$.

3.2.2. *Canonical isotropic subgroup.* — Let p be any prime and M be a finite abelian p-group of exponent $n = p^r$ with an alternating nondegenerate bilinear form $\omega : M \times M \to \mu_n$. We first construct by induction on r a canonical isotropic subgroup $S \subset M$ such that $\mathrm{Aut}(M)$ fixes S, and $S^\perp/S$ is an $\mathbb{F}_p$-vector space.

Write the set $\{r > 0 \mid \rho_r(M) \neq 0\}$ as $\{r_1, \dots, r_s\}$, with $0 < r_1 < r_2 < \dots < r_s$. There is an orthogonal direct sum $(M, \omega) \xrightarrow{\sim} \oplus_{i=1}^s (M_i, \omega_i)$, where $\omega_i : M_i \times M_i \to \mu_n$ is an alternating nondegenerate bilinear form, and M_i is a free $\mathbb{Z}/p^{r_i}$-module of finite rank.

Let

$$r' = \begin{cases} \frac{r_s}{2}, & r_s \text{ is even} \\ \frac{r_s+1}{2}, & r_s \text{ is odd.} \end{cases}$$

Set $S_1 = p^{r'}M$. Since ω takes values in $\mu_{p^{r_s}}$, S_1 is isotropic and fixed by $\mathrm{Aut}(M)$. By induction hypothesis, we have a canonical isotropic subgroup $S' \subset M_1 := S_1^\perp/S_1$ such that $S'^\perp/S'$ is an $\mathbb{F}_p$-vector space, where $S'^\perp$ denotes the orthogonal complement of S' in M_1. Let S be the preimage of S' under $S_1^\perp \to M_1$. This is our canonical isotropic subgroup in M.

Set $M_c = S^\perp/S$; it is equipped with the induced alternating nondegenerate bilinear form $\omega_c : M_c \times M_c \to \mu_p$; the subscript c stands for 'canonical'.

3.2.3. We keep the assumptions of Theorem 2.2, so p is odd. View S as a subgroup of H via $s \mapsto (s, 0) \in H$, for $s \in S$. Let $H^S = S^\perp \times \mu_n$; this is a subgroup of H. Since S lies in the kernel of $\beta : S^\perp \times S^\perp \to \mu_n$, we get the alternating nondegenerate bilinear form $\beta_c : M_c \times M_c \to \mu_p$ given by $\beta_c(m_1, m_2) = \beta(\tilde{m}_1, \tilde{m}_2)$, for $\tilde{m}_i \in S^\perp$ over m_i.

Set $H_c = M_c \times \mu_p$ with the product

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 a_2 \beta_c(m_1, m_2)).$$

This is a central extension $1 \to \mu_p \to H_c \to M_c \to 1$ with the commutator $\omega_c = \beta_c^2$ and the symmetric structure $\sigma_c(m, a) = (-m, a)$ for $(m, a) \in H_c$.

Let $\alpha_S : H^S \to H_c$ be the homomorphism sending (m, a) to $(m \bmod S, a)$, for $m \in S^\perp$; its kernel is S.

As in Section 3.1, we get the algebraic stacks $\widetilde{\mathcal{L}}(M_c), \mathcal{L}(M_c), H_c$ over $\mathbb{F}_p$. Let $G = \mathbb{S}\mathrm{p}(M, \omega)$ be the group of automorphisms of M preserving ω; this is a finite group. We let $g \in G$ act on H sending (m, a) to (gm, a). Let $g \in G$ act on functions $f : H \to K$ by $(gf)(h) = f(g^{-1}h)$ for $h \in H$. For $L \in \mathcal{L}(M)$, this yields an isomorphism $g : \mathcal{H}_L \widetilde{\to} \mathcal{H}_{gL}$ of K-vector spaces.

Since G preserves $S^{\perp}$, we have the homomorphism $G \to G_c := \mathbb{S}\mathrm{p}(M_c)(\mathbb{F}_p)$. Via this map, G acts on $\mathcal{L}(M_c)(\mathbb{F}_p)$, $\widetilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$, and H_c.

3.2.4. We denote elements of $\mathcal{L}(M_c)$ by a capital letter with a subscript c. For $L_c \in \mathcal{L}(M_c)$, let $L \in \mathcal{L}(M)$ denote the preimage of L_c under $S^{\perp} \to M_c$.

For $L_c \in \mathcal{L}(M_c)$, we have the representation $\mathcal{H}_{L_c}$ of H_c over K defined in Section 3.1.3, and the H-representation $\mathcal{H}_L$ over K defined in Section 2.0.4.

For $L_c \in \mathcal{L}(M_c)$, any f in the space of invariants $\mathcal{H}_L^S$ is the extension by zero under $H^S \hookrightarrow H$. The space $\mathcal{H}_L^S$ is naturally a H_c-module. We get an isomorphism of H_c-modules $\tau_{L_c} : \mathcal{H}_{L_c} \widetilde{\to} \mathcal{H}_L^S$, sending f to the composition $H^S \xrightarrow{\alpha_S} H_c \xrightarrow{f} K$ extended by zero to H.

For $g \in G$, $L_c \in \mathcal{L}(M_c)$, the isomorphism $g : \mathcal{H}_L \widetilde{\to} \mathcal{H}_{gL}$ yields an isomorphism $g : \mathcal{H}_L^S \widetilde{\to} \mathcal{H}_{gL}^S$ of S-invariants.

3.2.5. Given $L_c^0, N_c^0 \in \widetilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$, we define a canonical intertwining operator

$$\mathcal{F}_{N_c^0, L_c^0} : \mathcal{H}_L \widetilde{\to} \mathcal{H}_N \tag{5}$$

as the unique isomorphism of H-modules such that the diagram commutes

$$\begin{array}{ccc} \mathcal{H}_L^S & \xrightarrow{\mathcal{F}_{N_c^0, L_c^0}} & \mathcal{H}_N^S \\ {\scriptstyle \tau_{L_c}} \uparrow & & \uparrow {\scriptstyle \tau_{N_c}} \\ \mathcal{H}_{L_c} & \xrightarrow[F_{N_c^0, L_c^0}]{} & \mathcal{H}_{N_c} \end{array}$$

Here, $F_{N_c^0, L_c^0}$ are the canonical intertwining operators from Section 3.1.3. The properties of the canonical intertwining operators of Section 3.1.3 imply the following properties of (5):

- $\mathcal{F}_{L_c^0, L_c^0} = \mathrm{id}$ for $L_c^0 \in \widetilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$.
- For $R_c^0, N_c^0, L_c^0 \in \widetilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$, one has
$$\mathcal{F}_{R_c^0, N_c^0} \circ \mathcal{F}_{N_c^0, L_c^0} = \mathcal{F}_{R_c^0, L_c^0}.$$
- For $g \in G, N_c^0, L_c^0 \in \widetilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$, we have $g \circ \mathcal{F}_{N_c^0, L_c^0} \circ g^{-1} = \mathcal{F}_{gN_c^0, gL_c^0}$.

Definition 3.3. — Let π be the K-vector space of collections $f_{L_c^0} \in \mathcal{H}_L$, for $L_c^0 \in \widetilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ satisfying the property: for $N_c^0, L_c^0 \in \widetilde{\mathcal{L}}(M)(\mathbb{F}_p)$, one has

$$\mathcal{F}_{N_c^0, L_c^0}(f_{L_c^0}) = f_{N_c^0}.$$

The element $h \in H$ sends $\{f_{L_c^0}\} \in \pi$ to the collection $\{h(f_{L_c^0})\} \in \pi$. This is our canonical H-representation over K.

The group G acts on π sending $\{f_{L_c^0}\} \in \pi$ to the collection $L_c^0 \mapsto g(f_{g^{-1}L_c^0})$. This is a version of the Weil representation of G. (This G-representation was also obtained in [3] for the case when the field of coefficients is $\mathbb{C}$; however, a canonical representation of H was not constructed in therein).

The above actions of H and G on π combine to an action of the semidirect product $H \rtimes G$ on π. Theorem 2.2 is proved.

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