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Geometric Bessel Models for $\mathbb{G}\mathbb{S}p_4$ and Multiplicity One

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1 Introduction

1.1 Classical Bessel models

In this paper, which is a sequel to [6], we study Bessel models of representations of $\mathbb{G}\mathbb{S}p_4$ in the framework of the geometric Langlands program. These models introduced by Novodvorsky and Piatetski-Shapiro, satisfy the following multiplicity one property (see [8]).

Set $k = \mathbb{F}_q$ and $\mathcal{O} = k[[t]] \subset F = k((t))$. Let $\widetilde{F}$ be an étale F-algebra with $\dim_F(\widetilde{F}) = 2$ such that k is algebraically closed in $\widetilde{F}$. Write $\widetilde{\mathcal{O}}$ for the integral closure of $\mathcal{O}$ in $\widetilde{F}$. We have two cases:

(i) $\widetilde{F} \widetilde{\to} k((t^{1/2}))$ (nonsplit case),

(ii) $\widetilde{F} \widetilde{\to} F \oplus F$ (split case).

Write L for $\widetilde{\mathcal{O}}$ viewed as $\mathcal{O}$-module, it is equipped with a quadratic form $s : \mathrm{Sym}^2 L \to \mathcal{O}$ given by the determinant. Write $\Omega_{\mathcal{O}}$ for the completed module of relative differentials of $\mathcal{O}$ over k.

Set $\mathcal{M} = L \oplus (L^* \otimes \Omega_{\mathcal{O}}^{-1})$. This $\mathcal{O}$-module is equipped with a symplectic form $\wedge^2 \mathcal{M} \to L \otimes L^* \otimes \Omega_{\mathcal{O}}^{-1} \to \Omega_{\mathcal{O}}^{-1}$. Set $G = \mathbb{G}\mathbb{S}p(\mathcal{M})$, this is a group scheme over $\mathrm{Spec}\,\mathcal{O}$. Write $P \subset G$ for the Siegel parabolic subgroup preserving the Lagrangian submodule L. Its unipotent radical U has a distinguished character

$$\mathrm{ev} : U \widetilde{\longrightarrow} \Omega_{\mathcal{O}} \otimes \mathrm{Sym}^2 L \xrightarrow{s} \Omega_{\mathcal{O}} \qquad (1.1)$$

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(here we view $\Omega_{\mathcal{O}}$ as a commutative group scheme over Spec $\mathcal{O}$). Set

$$\widetilde{R} = \{p \in P \mid \mathrm{ev}(pup^{-1}) = \mathrm{ev}(u) \text{ for } u \in U\}. \tag{1.2}$$

View GL(L) as a group scheme over Spec $\mathcal{O}$ and $\widetilde{\mathcal{O}}^*$ as its closed subgroup. Write α for the composition $\widetilde{\mathcal{O}}^* \hookrightarrow \mathrm{GL}(L) \stackrel{\det}{\to} \mathcal{O}^*$. Fix a section $\widetilde{\mathcal{O}}^* \hookrightarrow \widetilde{R}$ given by $g \mapsto (g, \alpha(g)(g^*)^{-1})$. Then $R = \widetilde{\mathcal{O}}^* U \subset \widetilde{R}$ is a closed subgroup, and the map $R \stackrel{\xi}{\to} \Omega_{\mathcal{O}} \times \widetilde{\mathcal{O}}^*$ sending tu to $(\mathrm{ev}(u), t)$ is a homomorphism of group schemes over Spec $\mathcal{O}$.

Let ℓ be a prime invertible in k. Fix a character $\chi : \widetilde{F}^*/\widetilde{\mathcal{O}}^* \to \bar{\mathbb{Q}}_\ell^*$ and a nontrivial additive character $\psi : k \to \bar{\mathbb{Q}}_\ell^*$. Write τ for the composition

$$R(F) \xrightarrow{\xi} \Omega_F \times \widetilde{F}^* \xrightarrow{\mathrm{Res} \times \mathrm{pr}} k \times \widetilde{F}^*/\widetilde{\mathcal{O}}^* \xrightarrow{\psi \times \chi} \bar{\mathbb{Q}}_\ell^*. \tag{1.3}$$

The Bessel module is the vector space

$$\begin{aligned} \mathrm{BM}_\tau = \{ & f : G(F)/G(\mathcal{O}) \longrightarrow \bar{\mathbb{Q}}_\ell \mid f(rg) = \tau(r) f(g) \text{ for } r \in R(F), \\ & f \text{ is of compact support modulo} R(F)\}. \end{aligned} \tag{1.4}$$

Let $\chi_c : F^*/\mathcal{O}^* \to \bar{\mathbb{Q}}_\ell^*$ denote the restriction of χ. The Hecke algebra

$$\begin{aligned} H_{\chi_c} = \{ & h : G(\mathcal{O}) \backslash G(F)/G(\mathcal{O}) \longrightarrow \bar{\mathbb{Q}}_\ell \mid h(zg) = \chi_c(z) h(g) \text{ for } z \in F^*, \\ & h \text{ is of compact support modulo} F^*\} \end{aligned} \tag{1.5}$$

acts on BM_τ by convolutions. Then BM_τ is *a free module of rank one* over H_{χ_c}. In this paper we prove a geometric version of this result.

Recall that the affine Grassmannian $\mathrm{Gr}_G = G(F)/G(\mathcal{O})$ can be viewed as an ind-scheme over k. According to "fonctions-faisceaux" philosophy, the space BM_τ should have a geometric counterpart. A natural candidate for that would be the category of ℓ-adic perverse sheaves on Gr_G that change under the action of $R(F)$ by τ. However, the $R(F)$-orbits on Gr_G are infinite-dimensional, and this naive definition does not make sense.

The same difficulty appears when one tries to define Whittaker categories for any reductive group. In [3] Frenkel, Gaitsgory, and Vilonen have overcome this by replacing the corresponding local statement by its globalization, which admits a geometric counterpart leading to a definition of Whittaker categories with expected properties. We follow the strategy of [3] replacing the above local statement by a global one, which we further geometrize.

1.2 Geometrization

Fix a smooth projective absolutely irreducible curve X over k. Let $\pi : \widetilde{X} \to X$ be a two-sheeted covering ramified at some effective divisor D_π of X (we assume $\widetilde{X}$ smooth over k). The vector bundle $L = \pi_* \mathcal{O}_{\widetilde{X}}$ is equipped with a quadratic form $s : \mathrm{Sym}^2 L \to \mathcal{O}_X$.

Write Ω for the canonical line bundle on X. Set $\mathcal{M} = L \oplus (L^* \otimes \Omega^{-1})$, it is equipped with a symplectic form

$$\wedge^2 \mathcal{M} \longrightarrow L \otimes L^* \otimes \Omega^{-1} \longrightarrow \Omega^{-1}. \tag{1.6}$$

Let G be the group scheme (over X) of automorphisms of $\mathcal{M}$ preserving this symplectic form up to a multiple. Let $P \subset G$ denote the Siegel parabolic subgroup preserving L, $U \subset P$ its unipotent radical. Then U is equipped with a homomorphism of group schemes over X

$$\mathrm{ev} : U \widetilde{\longrightarrow} \Omega \otimes \mathrm{Sym}^2 L \xrightarrow{s} \Omega. \tag{1.7}$$

Let T be the functor sending a X-scheme S to the group $H^0(\widetilde{X} \times_X S, \mathcal{O}^*)$. Then T is a group scheme over X, a subgroup of $GL(L)$. Write α for the composition $T \hookrightarrow GL(L) \overset{\det}{\to} \mathbb{G}_m$. Set

$$\widetilde{R} = \left\{ p \in P \mid \mathrm{ev}\left(pup^{-1}\right) = \mathrm{ev}(u)\ \forall u \in U \right\}. \tag{1.8}$$

Fix a section $T \hookrightarrow \widetilde{R}$ given by $g \mapsto (g, \alpha(g)(g^*)^{-1})$. Then $R = TU \subset \widetilde{R}$ is a closed subgroup, and the map $R \overset{\xi}{\to} \Omega \times T$ sending tu to $(\mathrm{ev}(u), t)$ is a homomorphism of group schemes over X.

Let $F = k(X)$, let $\mathbb{A}$ be the adele ring of F, and $\mathcal{O} \subset \mathbb{A}$ the entire adeles. Write F_x for the completion of F at $x \in X$ and $\mathcal{O}_x \subset F_x$ for its ring of integers. Fix a nonramified character $\chi : T(F) \backslash T(\mathbb{A}) / T(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell^*$. Let τ be the composition

$$R(\mathbb{A}) \xrightarrow{\xi} \Omega(\mathbb{A}) \times T(\mathbb{A}) \xrightarrow{r \times \chi} \bar{\mathbb{Q}}_\ell^*, \tag{1.9}$$

where $r : \Omega(\mathbb{A}) \to \bar{\mathbb{Q}}_\ell^*$ is given by

$$r(\omega_x) = \psi\left(\sum_{x \in X} \mathrm{tr}_{k(x)/k} \mathrm{Res}\, \omega_x \right). \tag{1.10}$$

Fix $x \in X(k)$. Let Y denote the restricted product $G(F_x)/G(\mathcal{O}_x) \times \prod'_{y \neq x} R(F_y)/R(\mathcal{O}_y)$. Let $\mathcal{Y}(k)$ be the quotient of Y by the diagonal action of $R(F)$. Set

$$\begin{aligned} BM_{X,\tau} = \big\{ & f : Y \longrightarrow \bar{\mathbb{Q}}_\ell \mid f(rg) = \tau(r) f(g) \text{ for } r \in R(\mathbb{A}), \\ & f \text{ is of compact support modulo } R(\mathbb{A}) \big\}. \end{aligned} \tag{1.11}$$

View elements of $BM_{X,\tau}$ as functions on $\mathcal{Y}(k)$. Let $\chi_c : F_x^*/\mathcal{O}_x^* \to \bar{\mathbb{Q}}_\ell^*$ be the restriction of χ. As in Section 1.1, the Hecke algebra H_{χ_c} of the pair $(G(F_x), G(\mathcal{O}_x))$ acts on $BM_{X,\tau}$ by convolutions. The restriction under

$$G(F_x)/G(\mathcal{O}_x) \hookrightarrow Y \tag{1.12}$$

yields an isomorphism of H_{χ_c}-modules $BM_{X,\tau} \to BM_\tau$.

We introduce an ind-algebraic stack ${}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}$ whose set of k-points contains $\mathcal{Y}(k)$. We define the Bessel category $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$, a category of perverse sheaves on ${}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}$ with some equivariance property. This is a geometric version of $BM_{X,\tau}$.

Let $\mathrm{Sph}(\mathrm{Gr}_G)$ denote the category of $G(\mathcal{O}_x)$-equivariant perverse sheaves on the affine Grassmannian $G(F_x)/G(\mathcal{O}_x)$. By [7], this is a tensor category equivalent to the category of representations of the Langlands dual group $\check{G}\widetilde{\to}\mathbb{GS}p_4$. The category $\mathrm{Sph}(\mathrm{Gr}_G)$ acts on the derived category $D({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$ by Hecke functors.

Our main result is Theorem 3.10 describing the action of $\mathrm{Sph}(\mathrm{Gr}_G)$ on the irreducible objects of $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$. It implies the above multiplicity one. It also implies that the action of $\mathrm{Sph}(\mathrm{Gr}_G)$ on $D({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$ preserves $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$. The same phenomenon takes place for Whittaker and Waldspurger models.

Compared to the case of Whittaker categories, the Bessel category $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$ is not semisimple (cf. Section 3.12).

The explicit Casselman-Shalika formula for the Bessel models has been established in [2, Corollaries 1.8 and 1.9], where it is presented in the base of BM_τ consisting of functions supported at a single $R(F)$-orbit on Gr_G. Our Theorem 3.10 yields a geometric version of this formula. At the level of functions it yields another base $\{B^\lambda\}$ of BM_τ (cf. Section 3.14). In this new base, the Casselman-Shalika formula writes in an essentially uniform way for Bessel, Waldspurger, and Whittaker models.

In Section 2 we propose a general framework that gives a uniform way to define Whittaker, Waldspurger, and Bessel categories (the case of Waldspurger models was studied in [6]).

2 Compactifications and equivariant categories

2.1 Notation

We keep the following notation from [6]. Let k denote an algebraically closed field of characteristic $p \geq 0$. All the schemes (or stacks) we consider are defined over k. Let X be a smooth projective connected curve. Fix a prime $\ell \neq p$. For a scheme (or stack) S write

$D(S)$ for the bounded derived category of ℓ-adic étale sheaves on S, and $P(S) \subset D(S)$ for the category of perverse sheaves.

Write Ω for the canonical line bundle on X. For a group scheme G on X write $\mathcal{F}^0_G$ for the trivial G-torsor on X.

2.2 Generalized R-bundles

2.2.1. Let G' be a connected reductive group over k. Given a G'-torsor $\mathfrak{F}_{G'}$ on X let G be the group scheme (over X) of automorphisms of $\mathfrak{F}_{G'}$. Write Bun_G for the stack of G-bundles on X. Note that $\mathfrak{F}_{G'}$ can be viewed as a G-torsor as well as a G'-torsor on X. We identify Bun_G and $\mathrm{Bun}_{G'}$ via the isomorphism that sends a G-torsor $\mathcal{F}_G$ to the G'-torsor $\mathcal{F}_{G'} = \mathfrak{F}_{G'} \times^G \mathcal{F}_G$.

Let $R \subset G$ be a closed group subscheme over X. Say that G/R is *strongly quasi-affine over* X if for the projection $\mathrm{pr} : G/R \to X$ the $\mathcal{O}_X$-algebra $\mathrm{pr}_* \mathcal{O}_{G/R}$ is finitely generated (locally in Zarisky topology), and the natural map $G/R \to \overline{G/R}$ is an open immersion. Here $\overline{G/R} = \mathrm{Spec}(\mathrm{pr}_* \mathcal{O}_{G/R})$.

Let V be a vector bundle on X on which G acts, that is, we are given a homomorphism of group schemes $G \to \mathrm{Aut}(V)$ on X. Assume that R is obtained through the following procedure. There is a section $\mathcal{O}_X \stackrel{s}{\hookrightarrow} V$ such that $V/\mathcal{O}_X$ is locally free and $R = \{g \in G \mid gs = s\}$. Let Z be the closure of Gs in the total space of V, so $G/R \subset Z$. Let Z' be the complement of Gs in Z. The following is a consequence of [5, Theorem 2].

Lemma 2.1. Assume that any fibre of the projection $\mathrm{pr} : Z' \to X$ is of codimension ≥ 2 in the corresponding fibre of $\mathrm{pr} : Z \to X$. Then G/R is strongly quasi-affine over X, and Z is the affine closure $\overline{G/R}$ of G/R. □

Assume that R satisfies the conditions of Lemma 2.1 (this holds in our examples below).

Definition 2.2. Let $\overline{\mathrm{Bun}}_R$ be the following stack. For a scheme S, an S-point of $\overline{\mathrm{Bun}}_R$ is a pair $(\mathcal{F}_G, \beta)$, where $\mathcal{F}_G$ is an $(S \times X) \times_X G$-torsor on $S \times X$, and β is a G-equivariant map $\beta : \mathcal{F}_G \to S \times \overline{G/R}$ over $S \times X$ with the following property. For any geometric point $s \in S$ there is a nonempty open subset $U^s \subset s \times X$ such that

$$\beta : \mathcal{F}_G\big|_{U^s} \longrightarrow (s \times \overline{G/R})\big|_{U^s} \tag{2.1}$$

factors through $(s \times G/R)\mid_{U^s} \subset (s \times \overline{G/R})\mid_{U^s}$.

An S-point of $\overline{\mathrm{Bun}}_R$ can also be seen as a pair $(\mathcal{F}_G, \alpha)$, where $\mathcal{F}_G$ is an $(S \times X) \times_X$ G-torsor on $S \times X$, and $\alpha : \mathcal{O}_{S\times X} \to V_{\mathcal{F}_G}$ is a section with the following property. First, $\alpha(1)$ lies in $\overline{G/R} \times^G \mathcal{F}_G$. Secondly, for any geometric point $s \in S$ there is a nonempty open subset $U^s \subset s \times X$ such that $\alpha(1) \mid_{U^s}$ lies in $(G/R \times^G \mathcal{F}_G) \mid_{U^s}$. Here $V_{\mathcal{F}_G}$ is the vector bundle $(V \otimes \mathcal{O}_{S\times X}) \times^G \mathcal{F}_G$ on $S \times X$.

Let Bun_R denote the stack of R-bundles on X.

Lemma 2.3. The stack $\overline{\mathrm{Bun}}_R$ is algebraic, locally of finite type, and $\mathrm{Bun}_R \subset \overline{\mathrm{Bun}}_R$ is an open substack. □

Proof. Consider the stack $\mathcal{X}$ classifying pairs $(\mathcal{F}_G, \alpha)$, where $\mathcal{F}_G$ is a G-torsor on X, and $\alpha : \mathcal{O}_X \to V_{\mathcal{F}_G}$ is a section. It is well known that this stack is algebraic, locally of finite type. The condition that $\alpha(1)$ lies in $\overline{G/R} \times^G \mathcal{F}_G$ defines a closed substack $\mathcal{X}' \subset \mathcal{X}$. The condition that $\alpha(1)$ factors through $G/R \times^G \mathcal{F}_G$ at the generic point of X is open in $\mathcal{X}'$. Finally, the condition that $\alpha(1)$ lies in $G/R \times^G \mathcal{F}_G$ everywhere over X is also open. ■

2.2.2. Fix a closed point $x \in X$. Write $\mathcal{O}_x$ for the completed local ring of $\mathcal{O}_X$ at x, and F_x for its fractions field.

Let ${}_{x,\infty}\overline{\mathrm{Bun}}_R$ be the following stack. Its S-point is a pair $(\mathcal{F}_G, \alpha)$, where $\mathcal{F}_G$ is an $(S \times X) \times_X$ G-torsor on $S \times X$, and

$$\alpha : \mathcal{O}_{S\times X} \longrightarrow V_{\mathcal{F}_G}(\infty x) \tag{2.2}$$

is a section with the following property. First, $\alpha(1) \mid_{S\times(X-x)}$ lies in $\overline{G/R} \times^G \mathcal{F}_G \mid_{S\times(X-x)}$. Secondly, for any geometric point $s \in S$ there is a nonempty open subset $U^s \subset s \times (X - x)$ such that $\alpha(1) \mid_{U^s}$ lies in $(G/R \times^G \mathcal{F}_G) \mid_{U^s}$.

Let $\mathcal{Y}_i \subset {}_{x,\infty}\overline{\mathrm{Bun}}_R$ be the closed substack given by the condition that (2.2) factors through $V_{\mathcal{F}_G}(ix) \subset V_{\mathcal{F}_G}(\infty x)$. In particular, $\mathcal{Y}_0 = \overline{\mathrm{Bun}}_R$. As in Lemma 2.3, one shows that $\mathcal{Y}_i$ is algebraic locally of finite type. Since ${}_{x,\infty}\overline{\mathrm{Bun}}_R$ is the direct limit of $\mathcal{Y}_i$, the stack ${}_{x,\infty}\overline{\mathrm{Bun}}_R$ is ind-algebraic.

Recall that if a stack $\mathcal{Y}$ admits a presentation as a direct limit of algebraic stacks, locally of finite type $\mathcal{Y}_i$, then we have the derived category $D(\mathcal{Y})$, which is an inductive 2-limit of $D(\mathcal{Y}_i)$. In particular, any $K \in D(\mathcal{Y})$ is the extension by zero from some closed algebraic substack of $\mathcal{Y}$, and similarly for the category $P(\mathcal{Y})$ of perverse sheaves on $\mathcal{Y}$ (cf. [4, Appendices A.1–A.2] and [1, Section 0.4.4] for details).

For a scheme S, one can also view an S-point of ${}_{x,\infty}\overline{\mathrm{Bun}}_R$ as a pair $(\mathcal{F}_G, \beta)$, where $\mathcal{F}_G$ is an $(S \times X) \times_X$ G-torsor on $S \times X$, and β is a G-equivariant map $\beta : \mathcal{F}_G \mid_{S\times(X-x)} \to S \times (\overline{G/R} \mid_{X-x})$ with the following property. For any geometric point $s \in S$, there is a nonempty open subset $U^s \subset s \times (X - x)$ such that

$$\beta : \mathcal{F}_G \mid_{U^s} \longrightarrow (s \times \overline{G/R}) \mid_{U^s} \tag{2.3}$$

factors through $(s \times G/R) \mid_{U^s} \subset (s \times \overline{G/R}) \mid_{U^s}$.

Let H be an abelian group scheme over X, and let $R \to H$ be a homomorphism of group schemes over X. Assume that the stack Bun_H of H-bundles on X is algebraic.

Fix a rank-one local system $\mathcal{L}$ on Bun_H trivialized at the trivial H-torsor $\mathcal{F}^0_H$. Assume that for the tensor product map $m : \mathrm{Bun}_H \times \mathrm{Bun}_H \to \mathrm{Bun}_H$ there exists an isomorphism $m^*\mathcal{L} \widetilde{\to} \mathcal{L} \boxtimes \mathcal{L}$ whose restriction to the k-point $(\mathcal{F}^0_H, \mathcal{F}^0_H)$ is the identity.

2.2.3. We would like to define a category $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ of *$\mathcal{L}$-equivariant perverse sheaves on* ${}_{x,\infty}\overline{\mathrm{Bun}}_R$, and similarly for $\overline{\mathrm{Bun}}_R$.

Let ${}_X\mathcal{Y} \subset (X - x) \times {}_{x,\infty}\overline{\mathrm{Bun}}_R$ be the open substack classifying collections $y \in X - x$, $(\mathcal{F}_G, \beta) \in {}_{x,\infty}\overline{\mathrm{Bun}}_R$ such that the map $\beta : \mathcal{F}_G \to \overline{G/R}$ factors through $G/R \subset \overline{G/R}$ in a neighbourhood of y.

Set $D_y = \mathrm{Spec}\, \mathcal{O}_y$. By definition, for a point of ${}_X\mathcal{Y}$, the G-torsor $\mathcal{F}_G \mid_{D_y}$ is equipped with a reduction to an R-torsor that we denote $\mathcal{F}_R$.

Let ${}_X\mathcal{X}$ be the stack classifying $(y, \mathcal{F}_G, \beta) \in {}_X\mathcal{Y}$, $(y, \mathcal{F}'_G, \beta') \in {}_X\mathcal{Y}$ and

$$\tau : \mathcal{F}_G \mid_{X-y} \;\widetilde{\longrightarrow}\; \mathcal{F}'_G \mid_{X-y} \tag{2.4}$$

such that the diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_G \mid_{X-y} & \xrightarrow{\;\beta\;} & \overline{G/R}\mid_{X-y} \\ \Big\downarrow{\scriptstyle\tau} & \nearrow{\scriptstyle\beta'} & \\ \mathcal{F}'_G\mid_{X-y} & & \end{array} \tag{2.5}$$

Let pr (resp., act) denote the projection ${}_X\mathcal{X} \to {}_X\mathcal{Y}$ sending the above collection to $(y, \mathcal{F}_G, \beta)$ (resp., to $(y, \mathcal{F}'_G, \beta')$). They provide ${}_X\mathcal{X}$ with a structure of a groupoid over ${}_X\mathcal{Y}$.

Set $D^*_y = \mathrm{Spec}\, F_y$. Let ${}_X\mathcal{G}r_R$ denote the stack classifying $(y \in X - x, \mathcal{F}_R, \mathcal{F}'_R, \tau)$, where $\mathcal{F}_R$ and $\mathcal{F}'_R$ are R-torsors on D_y and

$$\tau : \mathcal{F}_R \mid_{D^*_y} \;\widetilde{\longrightarrow}\; \mathcal{F}'_R \mid_{D^*_y} \tag{2.6}$$

is an isomorphism.

We have a map ${}_X\mathcal{X} \to {}_X\mathcal{G}r_R$ sending the above collection to $(y, \mathcal{F}_R, \mathcal{F}'_R, \tau)$, where $\mathcal{F}_R$ and $\mathcal{F}'_R$ are R-torsors on D_y obtained from $(\mathcal{F}_G, \beta)$ and $(\mathcal{F}'_G, \beta')$ and τ is the restriction of (2.4).

Let ${}_X\operatorname{Gr}_H$ denote the affine Grassmannian of H over $X - x$, namely the ind-scheme classifying $y \in X - x$ and an H-torsor on D_y trivialized over D_y^*. We have a map ${}_X\mathcal{G}r_R \to {}_X\operatorname{Gr}_H$ sending $(y, \mathcal{F}_R, \mathcal{F}'_R, \tau)$ to $(y, \mathcal{F}_H, \tau)$, where

$$\mathcal{F}_H = \operatorname{Isom}\left(\mathcal{F}_R \times_R H, \mathcal{F}'_R \times_R H\right), \tag{2.7}$$

and $\tau : \mathcal{F}_H \widetilde{\to} \mathcal{F}^0_H \mid_{D_y^*}$ is the induced trivialization.

We have a map ${}_X\operatorname{Gr}_H \to \operatorname{Bun}_H$ sending $(y, \mathcal{F}_H, \tau)$ to $\widetilde{\mathcal{F}}_H$, where $\widetilde{\mathcal{F}}_H$ is the gluing of $\mathcal{F}^0_H \mid_{X-y}$ and $\mathcal{F}_H \mid_{D_y}$ via the isomorphism $\tau : \mathcal{F}_H \widetilde{\to} \mathcal{F}^0_H \mid_{D_y^*}$.

Define the evaluation map $\operatorname{ev}_{\mathcal{X}} : {}_X\mathcal{X} \to \operatorname{Bun}_H$ as the composition

$${}_X\mathcal{X} \longrightarrow {}_X\mathcal{G}r_R \longrightarrow {}_X\operatorname{Gr}_H \longrightarrow \operatorname{Bun}_H . \tag{2.8}$$

We would like $P^{\mathcal{L}}({}_{x,\infty}\overline{\operatorname{Bun}}_R)$ to be the category of perverse sheaves K on ${}_{x,\infty}\overline{\operatorname{Bun}}_R$ equipped with an isomorphism

$$\operatorname{act}^* \widetilde{K} \widetilde{\longrightarrow} \operatorname{pr}^* \widetilde{K} \otimes \operatorname{ev}^*_{\mathcal{X}} \mathcal{L} \tag{2.9}$$

satisfying the usual associativity condition, and such that its restriction to the unit section of ${}_X\mathcal{X}$ is the identity. Here $\widetilde{K}$ is the restriction of K under ${}_X\mathcal{Y} \to {}_{x,\infty}\overline{\operatorname{Bun}}_R$. However, this naive definition does not apply directly, because $\operatorname{pr}, \operatorname{act} : {}_X\mathcal{X} \to {}_X\mathcal{Y}$ are not smooth in general. (One more source of difficulties is that the affine Grassmannian $\operatorname{Gr}_{R,y}$ may be highly nonreduced, this happens, e.g., for R a torus.)

We remedy the difficulty under an additional assumption satisfied in our examples. Suppose that R fits into an exact sequence of group schemes $1 \to U \to R \to T \to 1$ over X, where U is a unipotent group scheme, and T is as follows. There is an integer $b \geq 0$ and a (ramified) Galois covering $\pi : \widetilde{X} \to X$, where $\widetilde{X}$ is a smooth projective curve, such that for an X-scheme S we have

$$T(S) = \operatorname{Hom}\left(\widetilde{X} \times_X S, \mathbb{G}_m^b\right). \tag{2.10}$$

In this case Bun_T is nothing but the stack of $\mathbb{G}_m^b$-torsors on $\widetilde{X}$. For a divisor D on $\widetilde{X}$ with values in the coweight lattice of $\mathbb{G}_m^b$, and for a T-torsor $\mathcal{F}_T$ on X, we denote by $\mathcal{F}_T(D)$ the corresponding twisted T-torsor on X.

The stack ${}_X\mathcal{X}$ can be seen as the one classifying $(y, \mathcal{F}_G, \beta) \in {}_X\mathcal{Y}$, an R-torsor $\mathcal{F}'_R$ on D_y, and an isomorphism $\tau : \mathcal{F}_R \mid_{D_y^*} \widetilde{\to} \mathcal{F}'_R \mid_{D_y^*}$, where $\mathcal{F}_R$ is the R-torsor on D_y obtained from $(\mathcal{F}_G, \beta)$. From this point of view the projection $\mathrm{pr} : {}_X\mathcal{X} \to {}_X\mathcal{Y}$ is the map forgetting $\mathcal{F}'_R$.

Modify the definition of ${}_X\mathcal{X}$ and of ${}_X\mathcal{Y}$ as follows. Let

$$ {}_{\tilde{X}}\mathcal{Y} \subset \tilde{X} \times {}_{x,\infty}\overline{\mathrm{Bun}}_R \tag{2.11} $$

be the open substack classifying $\tilde{y} \in \tilde{X}$ with π nonramified at $\tilde{y}$ and $y := \pi(\tilde{y}) \neq x$, $(\mathcal{F}_G, \beta) \in {}_{x,\infty}\overline{\mathrm{Bun}}_R$ such that the map $\beta : \mathcal{F}_G \to \overline{G/R}$ factors through $G/R \subset \overline{G/R}$ in a neighbourhood of y.

Given for each $\sigma \in \Sigma = \mathrm{Gal}(\tilde{X}/X)$ a coweight $\gamma_\sigma : \mathbb{G}_m \to \mathbb{G}_m^b$, we set $\gamma = \{\gamma_\sigma\}$. Let

$$ \mathrm{pr} : {}_{\tilde{X}}\mathcal{X}_\gamma \longrightarrow {}_{\tilde{X}}\mathcal{Y} \tag{2.12} $$

be the stack whose fibre over $(\tilde{y}, \mathcal{F}_G, \beta) \in {}_{\tilde{X}}\mathcal{Y}$ is the ind-scheme classifying an R-torsor $\mathcal{F}'_R$ on D_y, an isomorphism $\mathcal{F}_R \widetilde{\to} \mathcal{F}'_R \mid_{D_y^*}$, and an extension of the induced isomorphism

$$ \mathcal{F}_R \times_R T \widetilde{\longrightarrow} \mathcal{F}'_R \times_R T \mid_{D_y^*} \tag{2.13} $$

to an isomorphism over D_y,

$$ \mathcal{F}_R \times_R T \widetilde{\longrightarrow} (\mathcal{F}'_R \times_R T)\left(\sum_{\sigma\in\Sigma} \gamma_\sigma \sigma(\tilde{y})\right). \tag{2.14} $$

Here $y = \pi(\tilde{y})$, and $\mathcal{F}_R$ is the R-torsor on D_y obtained from $(\mathcal{F}_G, \beta)$.

As above, we have an action map $\mathrm{act} : {}_{\tilde{X}}\mathcal{X}_\gamma \to {}_{\tilde{X}}\mathcal{Y}$. The advantage is that any fibre of each of the maps $\mathrm{pr}, \mathrm{act} : {}_{\tilde{X}}\mathcal{X}_\gamma \to {}_{\tilde{X}}\mathcal{Y}$ is reduced (it identifies with the affine Grassmannian at y of a unipotent group scheme over X).

Now proceed as in [3]. Recall that $U(F_y)$ is an ind-group scheme, it can be written as a direct limit of some group schemes U^{-m}, $m \geq 0$, such that $U^{-m} \hookrightarrow U^{-m-1}$ is a closed subgroup, $U^0 = U(\mathcal{O}_y)$, and U^{-m}/U^0 are smooth of finite type [3, Section 3.1].

For this reason, for $m \geq 0$ there exist closed substacks

$$ {}_{\tilde{X}}\mathcal{X}_{\gamma,m} \hookrightarrow {}_{\tilde{X}}\mathcal{X}_{\gamma,m+1} \hookrightarrow \cdots \hookrightarrow {}_{\tilde{X}}\mathcal{X}_\gamma \tag{2.15} $$

such that both maps $\mathrm{pr}, \mathrm{act} : {}_{\tilde{X}}\mathcal{X}_{\gamma,m} \to {}_{\tilde{X}}\mathcal{Y}$ are of finite type and smooth of the same relative dimension, and ${}_{\tilde{X}}\mathcal{X}_{\gamma}$ is a direct limit of the stacks ${}_{\tilde{X}}\mathcal{X}_{\gamma,m}$.

As above, we have a map ${}_{\tilde{X}}\mathcal{X}_{\gamma} \to {}_{X}\mathcal{G}r_R$, hence also the evaluation map $\mathrm{ev}_{\tilde{X},\gamma} : {}_{\tilde{X}}\mathcal{X}_{\gamma} \to \mathrm{Bun}_H$.

Definition 2.4. Let $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ denote the category of perverse sheaves on ${}_{x,\infty}\overline{\mathrm{Bun}}_R$ equipped for each γ and $m \geq 0$ with isomorphisms

$$\alpha_{\gamma,m} : \mathrm{act}^* \widetilde{K} \widetilde{\longrightarrow} \mathrm{pr}^* \widetilde{K} \otimes \mathrm{ev}^*_{\mathcal{X},\gamma} \mathcal{L} \tag{2.16}$$

over ${}_{\tilde{X}}\mathcal{X}_{\gamma,m}$. Here $\widetilde{K}$ denotes the restriction of K under ${}_{\tilde{X}}\mathcal{Y} \to {}_{x,\infty}\overline{\mathrm{Bun}}_R$. It is required that for $m_1 < m_2$ the restriction of α_{γ,m_2} to ${}_{\tilde{X}}\mathcal{X}_{\gamma,m_1}$ equals α_{γ,m_1}, the restriction of $\alpha_{0,m}$ to the unit section of ${}_{\tilde{X}}\mathcal{X}_{0,m}$ is the identity, and the usual associativity condition holds.

Denote by $P^{\mathcal{L}}(\overline{\mathrm{Bun}}_R)$ the full subcategory of $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ consisting of perverse sheaves, which are extensions by zero under $\overline{\mathrm{Bun}}_R \hookrightarrow {}_{x,\infty}\overline{\mathrm{Bun}}_R$.

2.3 Hecke functors

Let ${}_x\mathcal{H}_G$ denote the Hecke stack classifying G-torsors $\mathcal{F}_G, \mathcal{F}'_G$ on X together with an isomorphism $\tau : \mathcal{F}_G \widetilde{\to} \mathcal{F}'_G \mid_{X-x}$. Let $\mathfrak{q} : {}_x\mathcal{H}_G \to \mathrm{Bun}_G$ (resp., $\mathfrak{p} : {}_x\mathcal{H}_G \to \mathrm{Bun}_G$) denote the map forgetting $\mathcal{F}_G$ (resp., $\mathcal{F}'_G$). Consider the diagram

$${}_{x,\infty}\overline{\mathrm{Bun}}_R \xleftarrow{\mathfrak{p}_R} {}_{x,\infty}\overline{\mathrm{Bun}}_R \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G \xrightarrow{\mathfrak{q}_R} {}_{x,\infty}\overline{\mathrm{Bun}}_R, \tag{2.17}$$

where we used $\mathfrak{p}$ to define the fibred product, $\mathfrak{p}_R$ forgets $\mathcal{F}'_G$, and $\mathfrak{q}_R$ sends $(\mathcal{F}_G, \beta, \mathcal{F}'_G, \tau)$ to $(\mathcal{F}'_G, \beta')$, where β' is the composition

$$\mathcal{F}'_G \xrightarrow{\tau^{-1}} \mathcal{F}_G \xrightarrow{\beta} \overline{G/R}. \tag{2.18}$$

In the same way one gets the diagram

$${}_{\tilde{X}}\mathcal{Y} \xleftarrow{\mathfrak{p}_{\mathcal{Y}}} {}_{\tilde{X}}\mathcal{Y} \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G \xrightarrow{\mathfrak{q}_{\mathcal{Y}}} {}_{\tilde{X}}\mathcal{Y}. \tag{2.19}$$

The action of the groupoid ${}_{\tilde{X}}\mathcal{X}$ on ${}_{\tilde{X}}\mathcal{Y}$ lifts to an action on this diagram (in the sense of [6, Appendix A.1]). Namely, for each γ we have two diagrams, where the squares are cartesian:

$$
\begin{array}{ccc}
{}_{\tilde{x}}\mathcal{X}_\gamma & \xrightarrow{\ \mathrm{pr}\ } & {}_{\tilde{x}}\mathcal{Y} \\
\uparrow{\scriptstyle \mathfrak{p}_{\mathcal{X}}} & & \uparrow{\scriptstyle \mathfrak{p}_{\mathcal{Y}}} \\
{}_{\tilde{x}}\mathcal{X}_\gamma \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G & \xrightarrow{\ \mathrm{pr}\ } & {}_{\tilde{x}}\mathcal{Y} \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G \\
\downarrow{\scriptstyle \mathfrak{q}_{\mathcal{X}}} & & \downarrow{\scriptstyle \mathfrak{q}_{\mathcal{Y}}} \\
{}_{\tilde{x}}\mathcal{X}_\gamma & \xrightarrow{\ \mathrm{pr}\ } & {}_{\tilde{x}}\mathcal{Y} \\
 & & \\
{}_{\tilde{x}}\mathcal{X}_\gamma & \xrightarrow{\ \mathrm{act}\ } & {}_{\tilde{x}}\mathcal{Y} \\
\uparrow{\scriptstyle \mathfrak{p}_{\mathcal{X}}} & & \uparrow{\scriptstyle \mathfrak{p}_{\mathcal{Y}}} \\
{}_{\tilde{x}}\mathcal{X}_\gamma \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G & \xrightarrow{\ \mathrm{act}\ } & {}_{\tilde{x}}\mathcal{Y} \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G \\
\downarrow{\scriptstyle \mathfrak{q}_{\mathcal{X}}} & & \downarrow{\scriptstyle \mathfrak{q}_{\mathcal{Y}}} \\
{}_{\tilde{x}}\mathcal{X}_\gamma & \xrightarrow{\ \mathrm{act}\ } & {}_{\tilde{x}}\mathcal{Y}
\end{array}
\tag{2.20}
$$

Write $\mathrm{Sph}(\mathrm{Gr}_{G',x})$ for the category of $G'(\mathcal{O}_x)$-equivariant perverse sheaves on the affine Grassmannian $\mathrm{Gr}_{G',x} = G'(F_x)/G'(\mathcal{O}_x)$. This is a tensor category equivalent to the category of representations of the Langlands dual group $\check{G}'$ over $\bar{\mathbb{Q}}_\ell$ [7].

Let Bun_G^x be the stack classifying a G-bundle $\mathcal{F}_G$ on X with an isomorphism of G-torsors $\mathcal{F}_G \widetilde{\to} \mathfrak{F}_{G'}\mid_{D_x}$. In a way compatible with our identification $\mathrm{Bun}_G \widetilde{\to} \mathrm{Bun}_{G'}$ one can view Bun_G^x as the stack classifying a G'-torsor $\mathcal{F}_{G'}$ with a trivialization $\mathcal{F}_{G'} \widetilde{\to} \mathcal{F}^0_{G'}\mid_{D_x}$. So, the projection $\mathfrak{q} : {}_x\mathcal{H}_G \to \mathrm{Bun}_G$ can be written as a fibration

$$
\mathrm{Bun}_G^x \times_{G'(\mathcal{O}_x)} \mathrm{Gr}_{G',x} \longrightarrow \mathrm{Bun}_G\,. \tag{2.21}
$$

Now for $\mathcal{A} \in \mathrm{Sph}(\mathrm{Gr}_{G',x})$ and $K \in D({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ we can form their twisted exterior product

$$
K \tilde{\boxtimes} \mathcal{A} \in D({}_{x,\infty}\overline{\mathrm{Bun}}_R \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G). \tag{2.22}
$$

It is normalized so that it is perverse for K perverse and $\mathbb{D}(K \tilde{\boxtimes} \mathcal{A}) \widetilde{\to} \mathbb{D}(K) \tilde{\boxtimes} \mathbb{D}(\mathcal{A})$.

Define the Hecke functor $H(\mathcal{A}, \cdot) : D({}_{x,\infty}\overline{\mathrm{Bun}}_R) \to D({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ by

$$
H(\mathcal{A}, K) = (\mathfrak{p}_R)_!(K \tilde{\boxtimes} \mathcal{A}). \tag{2.23}
$$

These functors are compatible with the tensor structure on $\mathrm{Sph}(\mathrm{Gr}_{G',x})$. Namely, we have canonically

$$\mathrm{H}\left(\mathcal{A}_1, \mathrm{H}\left(\mathcal{A}_2, \mathrm{K}\right)\right) \widetilde{\longrightarrow} \mathrm{H}\left(\mathcal{A}_1 * \mathcal{A}_2, \mathrm{K}\right), \tag{2.24}$$

where $\mathcal{A}_1 * \mathcal{A}_2 \in \mathrm{Sph}(\mathrm{Gr}_{G',x})$ is the convolution [3, Section 5].

As in Section 2.2, one defines the category $\mathrm{P}^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G)$. If $K \in \mathrm{P}^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$, then

$$K \widetilde{\boxtimes} \mathcal{A} \in \mathrm{P}^{\mathcal{L}}\left({}_{x,\infty}\overline{\mathrm{Bun}}_R \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G\right), \tag{2.25}$$

so the complex $\mathrm{H}(\mathcal{A}, K)$ inherits a $\mathcal{L}$-equivariant structure. Each perverse cohomology sheaf of $\mathrm{H}(\mathcal{A}, K)$ lies in $\mathrm{P}^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$.

2.4 Substacks of ${}_{x,\infty}\overline{\mathrm{Bun}}_R$

Let $\Lambda_{\mathcal{Y}}$ be the set of $R(F_x)$-orbits on the affine Grassmannian $\mathrm{Gr}_{G,x} = G(F_x)/G(\mathcal{O}_x)$. We are interested in the situations where $\Lambda_{\mathcal{Y}}$ is *discrete*. Write $\mathrm{Orb}_\mu \subset \mathrm{Gr}_{G,x}$ for the $R(F_x)$-orbit corresponding to $\mu \in \Lambda_{\mathcal{Y}}$.

Let $\mathcal{Y}_{loc}$ be the stack classifying a G-torsor $\mathcal{F}_G$ on D_x, an R-torsor $\mathcal{F}_R$ on D_x^*, and an R-equivariant map $\mathcal{F}_R \to \mathcal{F}_G \mid_{D_x^*}$. Then $\mathcal{Y}_{loc}$ identifies with the stack quotient of $\mathrm{Gr}_{G,x}$ by $R(F_x)$.

For $\mu \in \Lambda_{\mathcal{Y}}$, let $\mathcal{Y}^\mu_{loc}$ (resp., $\mathcal{Y}^{\le\mu}_{loc}$) denote the stack quotient of Orb_μ (resp., of $\overline{\mathrm{Orb}}_\mu$) by $R(F_x)$. (We do not precise for the moment the scheme structure on $\overline{\mathrm{Orb}}_\mu$.) We have an order on $\Lambda_{\mathcal{Y}}$ given by $\mu' \le \mu$ if and only if $\mathrm{Orb}_{\mu'} \subset \overline{\mathrm{Orb}}_\mu$.

We have a map ${}_{x,\infty}\overline{\mathrm{Bun}}_R \to \mathcal{Y}_{loc}$ sending $(\mathcal{F}_G, \beta)$ to its restriction to D_x. For $\mu \in \Lambda_{\mathcal{Y}}$, set

$${}_{x,\mu}\overline{\mathrm{Bun}}_R = {}_{x,\infty}\overline{\mathrm{Bun}}_R \times_{\mathcal{Y}_{loc}} \mathcal{Y}^{\le\mu}_{loc}, \qquad {}_{x,\mu}\widetilde{\mathrm{Bun}}_R = {}_{x,\infty}\overline{\mathrm{Bun}}_R \times_{\mathcal{Y}_{loc}} \mathcal{Y}^{\mu}_{loc}. \tag{2.26}$$

Let ${}_{x,\mu}\mathrm{Bun}_R \subset {}_{x,\mu}\widetilde{\mathrm{Bun}}_R$ be the open substack given by the condition that

$$\beta : \mathcal{F}_G\big|_{X-x} \longrightarrow \overline{G/R}\big|_{X-x} \tag{2.27}$$

factors through $G/R \mid_{X-x} \subset \overline{G/R} \mid_{X-x}$.

To summarize, we have a sequence of embeddings,

$${}_{x,\mu}\mathrm{Bun}_R \hookrightarrow {}_{x,\mu}\widetilde{\mathrm{Bun}}_R \hookrightarrow {}_{x,\mu}\overline{\mathrm{Bun}}_R \hookrightarrow {}_{x,\infty}\overline{\mathrm{Bun}}_R, \tag{2.28}$$

where the first two arrows are open embeddings and the last arrow is a closed one.

2.5 $\mathcal{L}$-equivalent perverse sheaves

The stack ${}_{x,\mu}\operatorname{Bun}_R$ classifies a G-torsor $\mathcal{F}_G$ on X, a G-equivariant map $\beta : \mathcal{F}_G \to G/R\,|_{X-x}$ such that the restriction of $(\mathcal{F}_G, \beta)$ to D_x lies in $\mathcal{Y}^{\mu}_{loc}$. Set

$$ {}_{\mu}\mathcal{X} = {}_{x,\mu}\operatorname{Bun}_R \times_{\mathcal{Y}_{loc}} {}_{x,\mu}\operatorname{Bun}_R; \tag{2.29}$$

this is a groupoid over ${}_{x,\mu}\operatorname{Bun}_R$ for the two projections $\operatorname{pr}, \operatorname{act} : {}_{\mu}\mathcal{X} \to {}_{x,\mu}\operatorname{Bun}_R$.

View ${}_{\mu}\mathcal{X}$ as the stack classifying R-torsors $\mathcal{F}_R, \mathcal{F}'_R$ on $X - x$ with an isomorphism $\tau : \mathcal{F}_R \widetilde{\to} \mathcal{F}'_R\,|_{D^*_x}$, a G-torsor $\mathcal{F}_G$ on X, and an R-equivariant map $\mathcal{F}_R \to \mathcal{F}_G\,|_{X-x}$, whose restriction to D_x lies in $\mathcal{Y}^{\mu}_{loc}$. The projection $\operatorname{pr} : {}_{\mu}\mathcal{X} \to {}_{x,\mu}\operatorname{Bun}_R$ forgets $\mathcal{F}'_R$.

Let ${}_{\mu}\operatorname{ev}_{\mathcal{X}} : {}_{\mu}\mathcal{X} \to \operatorname{Bun}_H$ be the map sending the above collection to the H-torsor $\widetilde{\mathcal{F}}_H$ on X obtained by the following gluing procedure. Let $\mathcal{F}_H$ denote the H-torsor on $X - x$ of isomorphisms

$$ \operatorname{Isom}\left(\mathcal{F}_R \times_R H, \mathcal{F}'_R \times_R H\right). \tag{2.30}$$

Then $\widetilde{\mathcal{F}}_H$ is the gluing of $\mathcal{F}_H$ and of $\mathcal{F}^0_H\,|_{D_x}$ over D^*_x via $\tau : \mathcal{F}_H \widetilde{\to} \mathcal{F}^0_H\,|_{D^*_x}$.

We say that $\mu \in \Lambda_{\mathcal{Y}}$ is *relevant* if there exists a morphism $\operatorname{ev}^{\mu} : {}_{x,\mu}\operatorname{Bun}_R \to \operatorname{Bun}_H$ making the following diagram commutative:

$$\begin{array}{ccccc}
\operatorname{Bun}_H \times_{x,\mu} \operatorname{Bun}_R & \xrightarrow{\operatorname{id}\times \operatorname{ev}^{\mu}} & \operatorname{Bun}_H \times \operatorname{Bun}_H & \xrightarrow{m} & \operatorname{Bun}_H \\
\uparrow {\scriptstyle {}_{\mu}\operatorname{ev}_{\mathcal{X}} \times \operatorname{pr}} & & & & \uparrow {\scriptstyle \operatorname{ev}^{\mu}} \\
{}_{\mu}\mathcal{X} & & \xrightarrow{\operatorname{act}} & & {}_{x,\mu}\operatorname{Bun}_R
\end{array} \tag{2.31}$$

If such ev^{μ} exists, it is unique up to a tensoring by a fixed H-torsor on X. Write $\Lambda^+_{\mathcal{Y}}$ for the set of relevant $\mu \in \Lambda_{\mathcal{Y}}$.

Write $0 \in \Lambda_{\mathcal{Y}}$ for the $R(F_x)$-orbit on $\operatorname{Gr}_{G,x}$ passing by 1. Then ${}_{x,0}\operatorname{Bun}_R$ is nothing but the stack Bun_R of R-bundles on X. The homomorphism $R \to H$ yields a map $\operatorname{ev}^0 : {}_{x,0}\operatorname{Bun}_R \to \operatorname{Bun}_H$ such that (2.31) commutes, so $0 \in \Lambda^+_{\mathcal{Y}}$.

For $\mu \in \Lambda^+_{\mathcal{Y}}$ we denote by $\mathcal{B}^{\mu}$ the Goresky-MacPherson extension of

$$ \left(\operatorname{ev}^{\mu}\right)^* \mathcal{L} \otimes \bar{\mathbb{Q}}_{\ell}[1]\left(\frac{1}{2}\right)^{\otimes \dim {}_{x,\mu}\operatorname{Bun}_R} \tag{2.32}$$

under ${}_{x,\mu}\operatorname{Bun}_R \hookrightarrow {}_{x,\mu}\overline{\operatorname{Bun}}_R$. By construction, $\mathcal{B}^{\mu} \in \operatorname{P}^{\mathcal{L}}({}_{x,\infty}\overline{\operatorname{Bun}}_R)$.

The examples of the above situation include Whittaker models, Waldspurger models for GL_2, and Bessel models for $G\mathbb{S}p_4$ (the latter is studied in Section 3).

2.6 Whittaker models

Let G' be a connected reductive group over k, $B' \subset G'$ a Borel subgroup, $U' \subset B'$ its unipotent radical. Set $T' = B'/U'$. Assume that $[G', G']$ is simply connected. Let $\mathcal{J}$ denote the set of vertices of the Dynkin diagram, and $\{\check{\alpha}_i, i \in \mathcal{J}\}$ the simple roots corresponding to B'. Fix a B'-torsor $\mathfrak{F}_{B'}$ on X and a conductor for the induced T'-torsor $\mathfrak{F}_{T'}$. That is, for each $i \in \mathcal{J}$ we fix an inclusion of coherent sheaves

$$\widetilde{\omega}_i : \mathcal{L}^{\check{\alpha}_i}_{\mathfrak{F}_{T'}} \hookrightarrow \Omega. \tag{2.33}$$

Write $\mathfrak{F}_{G'}$ for the G'-torsor induced from $\mathfrak{F}_{B'}$. Now G is the group scheme of automorphisms of $\mathfrak{F}_{G'}$. Let $R \subset G$ denote the group scheme of automorphisms of $\mathfrak{F}_{B'}$ acting trivially on $\mathfrak{F}_{T'}$.

To satisfy the assumptions of Lemma 2.1, take

$$V = \oplus_i \mathcal{H}om\big(\mathcal{L}^{\check{\omega}_i}_{\mathfrak{F}_{T'}}, \mathcal{V}^{\check{\omega}_i}_{\mathfrak{F}_{G'}}\big), \tag{2.34}$$

the sum being taken over the set of fundamental weights $\check{\omega}_i$ of G'. Here $\mathcal{V}^{\check{\lambda}}$ is the Weil G'-module corresponding to $\check{\lambda}$. Then G acts on V, and V is equipped with a canonical section $\mathcal{O}_X \hookrightarrow V$. By [1, Theorem 1.1.2], G/R is strongly quasi-affine over X.

The group scheme of automorphisms of $\mathfrak{F}_{B'/[U',U']}$ acting trivially on $\mathfrak{F}_{T'}$ is canonically

$$\oplus_{i\in\mathcal{J}} \mathcal{L}^{\check{\alpha}_i}_{\mathfrak{F}_{T'}}. \tag{2.35}$$

Set $H = \oplus_{i\in\mathcal{J}}\Omega$. Define a homomorphism of group schemes $R \to H$ over X as the composition

$$R \longrightarrow \oplus_{i\in\mathcal{J}} \mathcal{L}^{\check{\alpha}_i}_{\mathfrak{F}_{T'}} \xrightarrow{\widetilde{\omega}} H. \tag{2.36}$$

The stack $\overline{\mathrm{Bun}}_R$ identifies with the one classifying pairs $(\mathcal{F}_{G'}, \kappa)$, where $\mathcal{F}_{G'}$ is a G'-torsor on X, and κ is a collection of maps

$$\kappa^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathfrak{F}_{T'}} \hookrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_{G'}} \tag{2.37}$$

for each dominant weight $\check{\lambda}$ of G', satisfying Plücker relations ([3], Section 2.2.2).

The set $\Lambda_{\mathcal{Y}}$ identifies in this case with the group $\mathrm{Hom}(\mathbb{G}_m, T')$ of coweights of T'.

For $\lambda \in \Lambda_{\mathcal{Y}}$ the stack ${}_{x,\lambda}\overline{\mathrm{Bun}}_R$ classifies a G'-torsor $\mathcal{F}_{G'}$ on X, a collection of maps

$$\kappa^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathfrak{F}_{T'}} \hookrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_{G'}}(\langle \lambda, \check{\lambda} \rangle x) \tag{2.38}$$

for each dominant weight $\check{\lambda}$ of G', satisfying Plücker relations.

Assume that the base field k is of characteristic $p > 0$, and fix a nontrivial additive character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$. Write $\mathcal{L}_\psi$ for the corresponding Artin-Shreier sheaf on $\mathbb{A}^1_k$. Take $\mathcal{L}$ to be the restriction of $\mathcal{L}_\psi$ under the map

$$\mathrm{Bun}_H \longrightarrow \prod_{i \in \mathcal{I}} H^1(X, \Omega) \xrightarrow{\text{sum}} \mathbb{A}^1_k. \tag{2.39}$$

The corresponding Whittaker category $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ has been described by Frenkel, Gaitsgory, and Vilonen in [3].

2.7 Waldspurger models

The ground field k is of characteristic $p \neq 2$. Let $\pi : \widetilde{X} \to X$ be a two-sheeted covering ramified over some divisor D_π on X, where $\widetilde{X}$ is a smooth projective curve. Set $L_\pi = \pi_* \mathcal{O}_{\widetilde{X}}$ and $G' = GL_2$. View L_π as a G'-torsor $\mathfrak{F}_{G'}$ on X. Let G be the group scheme of automorphisms of $\mathfrak{F}_{G'}$. Let R be the group scheme over X such that for an X-scheme S we have $R(S) = \mathrm{Hom}(\widetilde{X} \times_X S, \mathbb{G}_m)$, so R is a closed group subscheme of G over X.

Let σ be the nontrivial automorphism of $\widetilde{X}$ over X, so $L_\pi \widetilde{\to} \mathcal{O} \oplus \mathcal{E}$, where $\mathcal{E}$ are σ-anti-invariants in L_π. It is equipped with $\mathcal{E}^2 \widetilde{\to} \mathcal{O}_X(-D_\pi)$. Take $V = \mathcal{E}nd_0(L_\pi) \otimes \mathcal{E}^{-1}$, where $\mathcal{E}nd_0(L_\pi)$ stands for the sheaf of traceless endomorphisms of L_π. The group scheme G acts on V via its action on L_π (the action of G on $\mathcal{E}$ is trivial).

We have

$$V \widetilde{\longrightarrow} \mathcal{O}(D_\pi) \oplus \mathcal{O} \oplus \mathcal{E}^{-1}. \tag{2.40}$$

Consider the section $\mathcal{O} \to V$ given by $(-1, 1, 0)$. The assumptions of Lemma 2.1 are satisfied.

Set $H = R$. The stack Bun_H classifies line bundles on $\widetilde{X}$. Pick a rank-one local system $\widetilde{E}$ on $\widetilde{X}$. Take $\mathcal{L}$ to be the automorphic local system on Bun_H corresponding to $\widetilde{E}$. The stack ${}_{x,\infty}\overline{\mathrm{Bun}}_R$ in this case is canonically isomorphic to the stack $\mathcal{W}ald^x_\pi$ introduced in [6, Section 8.2]. The corresponding Waldspurger category $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_R)$ has been studied in [6, Section 8.2].

3 Bessel categories

3.1 Notation

3.1.1 The group G. From now on, k is an algebraically closed field of characteristic $p > 2$. We change the notation compared to Section 2. From now on $G = G\mathbb{S}p_4$, so G is the quotient of $\mathbb{G}_m \times \mathbb{S}p_4$ by the diagonally embedded $\{\pm 1\}$. We realize G as the subgroup of $GL(k^4)$ preserving up to a scalar the bilinear form given by the matrix

$$\begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}, \tag{3.1}$$

where E_2 is the unit matrix of GL_2.

Let T be the maximal torus of G given by $\{(y_1, \dots, y_4) \mid y_i y_{2+i}$ does not depend on $i\}$. Let Λ (resp., $\check{\Lambda}$) denote the coweight (resp., weight) lattice of T. Let $\check{e}_i \in \check{\Lambda}$ be the character that sends a point of T to y_i. We have $\Lambda = \{(a_1, \dots, a_4) \in \mathbb{Z}^4 \mid a_i + a_{2+i}$ does not depend on $i\}$ and

$$\check{\Lambda} = \mathbb{Z}^4 / \{\check{e}_1 + \check{e}_3 - \check{e}_2 - \check{e}_4\}. \tag{3.2}$$

Fix the Borel subgroup of G preserving the flag $ke_1 \subset ke_1 \oplus ke_2$ of isotropic subspaces in the standard representation. The corresponding positive roots are

$$\{\check{\alpha}_{12}, \check{\beta}_{ij}, 1 \le i \le j \le 2\}, \tag{3.3}$$

where $\check{\alpha}_{12} = \check{e}_1 - \check{e}_2$ and $\check{\beta}_{ij} = \check{e}_i - \check{e}_{2+j}$. The simple roots are $\check{\alpha}_{12}$ and $\check{\beta}_{22}$. Write $V^{\check{\lambda}}$ for the irreducible representation of G of highest weight $\check{\lambda}$.

Fix fundamental weights $\check{\omega}_1 = (1, 0, 0, 0)$ and $\check{\omega}_2 = (1, 1, 0, 0)$ of G. So, $V^{\check{\omega}_1}$ is the standard representation of G. The orthogonal to the coroot lattice is $\mathbb{Z}\check{\omega}_0$ with $\check{\omega}_0 = (1, 0, 1, 0)$. The orthogonal to the root lattice is $\mathbb{Z}\omega$ with $\omega = (1, 1, 1, 1)$.

Let $P \subset G$ be the Siegel parabolic subgroup preserving the Lagrangian subspace $ke_1 \oplus ke_2 \subset k^4$. Write U for the unipotent radical of P, set $M = P/U$.

Let $\check{G}$ (resp., $\check{M}$) denote the Langlands dual group over $\bar{\mathbb{Q}}_\ell$. Write V^λ (resp., U^λ) for the irreducible representation of $\check{G}$ (resp., of $\check{M}$) with the highest weight λ.

Let w_0 be the longest element of the Weil group of G. Write Λ^+ for the set of dominant coweights of G. The half sum of positive roots of G is denoted by $\check{\rho}$. The corresponding objects for M are denoted by $\Lambda^+_M, w_0^M, \check{\rho}_M$.

Set $G_{ad} = G/Z$, where $Z \subset G$ is the center. Set $\check{\nu}_1 = \check{\omega}_2 - \check{\omega}_0$ and $\check{\nu}_2 = 2\check{\omega}_1 - \check{\omega}_0$. So, $V^{\check{\nu}_1}$ is the standard representation of G_{ad} and $\wedge^2 V^{\check{\nu}_1} \widetilde{\to} V^{\check{\nu}_2}$. Let $\Lambda_{G_{ad}}$ be the coweights lattice of G_{ad}. Write $\Lambda^{pos}_{G_{ad}}$ for the $\mathbb{Z}_+$-span of positive coroots in $\Lambda_{G_{ad}}$.

3.1.2. For $d \geq 0$ write $X^{(d)}$ for the dth symmetric power of X and view it as the scheme of effective divisors of degree d on X. Let ${}^{rss}X^{(d)} \subset X^{(d)}$ denote the open subscheme of divisors of the form $x_1 + \cdots + x_d$ with x_i pairwise distinct. Write Bun_i for the stack of rank-i vector bundles on X. Set

$$\mathrm{RCov}^d = \mathrm{Bun}_1 \times_{\mathrm{Bun}_1} {}^{rss}X^{(d)}, \tag{3.4}$$

where the map ${}^{rss}X^{(d)} \to \mathrm{Bun}_1$ sends D to $\mathcal{O}_X(-D)$, and the map $\mathrm{Bun}_1 \to \mathrm{Bun}_1$ takes a line bundle to its tensor square. It is understood that ${}^{rss}X^{(0)} = \mathrm{Spec}\, k$ and the point ${}^{rss}X^{(0)} \to \mathrm{Bun}_1$ is $\mathcal{O}_X$. Then RCov^d is the stack classifying two-sheeted coverings $\pi : \widetilde{X} \to X$ ramified exactly at $D \in {}^{rss}X^{(d)}$ with $\widetilde{X}$ smooth [6, Section 7.7.2].

Fix a character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$ and write $\mathcal{L}_\psi$ for the corresponding Artin-Shreier sheaf on $\mathbb{A}^1$.

3.2 Group schemes over X

3.2.1. Fix a k-point of RCov^d given by $D_\pi \in {}^{rss}X^{(d)}$ and $\pi : \widetilde{X} \to X$ ramified exactly at D_π. Let σ denote the nontrivial automorphism of $\widetilde{X}$ over X and let $\mathcal{E}$ be the σ-anti-invariants in $L_\pi := \pi_* \mathcal{O}_{\widetilde{X}}$. It is equipped with an isomorphism

$$\kappa : \mathcal{E}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}\big(-D_\pi\big). \tag{3.5}$$

Recall that L_π is equipped with a symmetric form $\mathrm{Sym}^2 L_\pi \xrightarrow{s} \mathcal{O}$ such that $\mathrm{div}(L_\pi^*/L_\pi) = D_\pi$ for the induced map $L_\pi \hookrightarrow L_\pi^*$ [6, Proposition 14]. Set $\mathcal{M}_\pi = L_\pi \oplus (L_\pi^* \otimes \Omega^{-1})$. It is equipped with a symplectic form

$$\wedge^2 \mathcal{M}_\pi \longrightarrow L_\pi \otimes \big(L_\pi^* \otimes \Omega^{-1}\big) \longrightarrow \Omega^{-1}. \tag{3.6}$$

Write $\mathfrak{F}_G$ for the G-torsor $(\mathcal{M}_\pi, \Omega^{-1})$ on X. Let G_π be the group scheme (over X) of automorphisms of $\mathfrak{F}_G$. Write $\mathcal{A}_\pi$ for the line bundle Ω^{-1} on X equipped with the corresponding action of G_π.

Let $P_\pi \subset G_\pi$ denote the Siegel parabolic subgroup preserving L_π, and $U_\pi \subset P_\pi$ its unipotent radical. Then U_π is equipped with a homomorphism of group schemes on X:

$$\mathrm{ev}_\pi : U_\pi \xrightarrow{\sim} \Omega \otimes \mathrm{Sym}^2 L_\pi \xrightarrow{s} \Omega. \tag{3.7}$$

Denote by $\widetilde{R}_\pi \subset P_\pi$ the subgroup stabilizing ev_π, that is,

$$\widetilde{R}_\pi = \big\{p \in P_\pi \mid \mathrm{ev}_\pi\left(pup^{-1}\right) = \mathrm{ev}_\pi(u)\ \forall u \in U_\pi\big\}. \tag{3.8}$$

Let $GL(L_\pi)$ denote the group scheme (over X) of automorphisms of the $\mathcal{O}_X$-module L_π. Let T_π denote the functor associating to an X-scheme V the group $H^0(\widetilde{X} \times_X V, \mathcal{O}^*)$. Then T_π is a group scheme over X, a subgroup of $GL(L_\pi)$.

Write Bun_{T_π} for the stack of T_π-bundles on X, that is, for a scheme S, the S-points of Bun_{T_π} constitute the category of $(S \times X) \times_X T_\pi$-torsors on $S \times X$. Given a $\mathbb{G}_m$-torsor on $S \times \widetilde{X}$, its direct image under $\mathrm{id} \times \pi : S \times \widetilde{X} \to S \times X$ is as $(S \times X) \times_X T_\pi$-torsor. In this way one identifies Bun_{T_π} with the Picard stack $\mathrm{Pic}\,\widetilde{X}$.

Let $\alpha : T_\pi \to \mathbb{G}_m$ be the character by which T_π acts on $\det(L_\pi)$. Fix an inclusion $T_\pi \hookrightarrow \widetilde{R}_\pi$ by making $t \in T_\pi$ act on $L_\pi \oplus (L_\pi^* \otimes \Omega^{-1})$ as $(t, \alpha(t)(t^*)^{-1})$, where $t^* \in \mathrm{Aut}(L_\pi^*)$ is the adjoint operator. Set $R_\pi = T_\pi U_\pi$, so $R_\pi \subset \widetilde{R}_\pi$ is a subgroup. Actually, $\widetilde{R}_\pi / U_\pi$ identifies with the group of those $g \in GL(L_\pi)$ for which there exists $\widetilde{\alpha}(g) \in \mathbb{G}_m$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Sym}^2 L_\pi & \xrightarrow{s} & \mathcal{O} \\ \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle \widetilde{\alpha}(g)} \\ \mathrm{Sym}^2 L_\pi & \xrightarrow{s} & \mathcal{O} \end{array} \tag{3.9}$$

So, $\widetilde{R}_\pi / U_\pi$ is equipped with a character $\widetilde{\alpha} : \widetilde{R}_\pi / U_\pi \to \mathbb{G}_m$ whose restriction to R_π equals α. For $g \in \widetilde{R}_\pi / U_\pi$ the following diagram commutes:

$$\begin{array}{ccc} L_\pi & \xrightarrow{s} & L_\pi^* \\ \uparrow{\scriptstyle g} & & \downarrow{\scriptstyle g^*} \\ L_\pi & \xrightarrow{\widetilde{\alpha}(g)s} & L_\pi^* \end{array} \tag{3.10}$$

so $(\det g)^2 = \widetilde{\alpha}(g)^2$. We see that R_π is the connected component of $\widetilde{R}_\pi$ given by the additional condition $\det g = \widetilde{\alpha}(g)$.

Lemma 3.1. The conditions of Lemma 2.1 are satisfied, so G_π / R_π is strongly quasi-affine over X. □

Proof. Define a G_π-module W_π by the exact sequence $0 \to W_\pi \to \mathcal{A}_\pi^{-1} \otimes \wedge^2 \mathcal{M}_\pi \to \mathcal{O}_X \to 0$ of $\mathcal{O}_X$-modules. So, W_π is equipped with a nondegenerate symmetric form $\mathrm{Sym}^2 W_\pi \to \mathcal{O}$, and the center of G_π acts trivially on W_π.

We have a subbundle $W_{\pi,1} := \mathcal{A}_\pi^{-1} \otimes \det L_\pi \widetilde{\to} \Omega \otimes \mathcal{E}$ in W_π. Let $W_{\pi,-1}$ denote the orthogonal complement to $W_{\pi,1}$ in W_π. Then $W_{\pi,-1} / W_{\pi,1} \widetilde{\to} \mathcal{E}nd_0(L_\pi)$. As in Section 2.7, we have a subbundle $\mathcal{E} \hookrightarrow \mathcal{E}nd_0(L_\pi)$. It gives rise to a subbundle

$$\Omega(-D_\pi) \hookrightarrow W_{\pi,1} \otimes \big(W_{\pi,-1} / W_{\pi,1}\big) \hookrightarrow \wedge^2 W_\pi. \tag{3.11}$$

Set

$$V = (\Omega^{-1} \otimes \mathcal{E}^{-1} \otimes W_\pi) \oplus (\Omega^{-1}(D_\pi) \otimes \wedge^2 W_\pi), \tag{3.12}$$

with the action of G_π coming from its action on W_π. We get a subbundle $\mathcal{O}_X \overset{s}{\hookrightarrow} V$, which is the sum of the above two sections. One checks that $R = \{g \in G \mid gs = s\}$, and the pair (V, s) satisfies the assumptions of Lemma 2.1. ■

3.2.2. Fix a k-point $x \in X$ and write $\mathcal{O}_x$ for the completed local ring of X at x and F_x for its fraction field. Set $D_x = \operatorname{Spec} \mathcal{O}_x$ and $D_x^* = \operatorname{Spec} F_x$.

Write $\widetilde{F}_x$ for the étale F_x-algebra of regular functions on $\widetilde{X} \times_X D_x^*$. If $x \in D_\pi$, then $\widetilde{F}_x$ is nonsplit; otherwise it splits over F_x. Denote by $\widetilde{\mathcal{O}}_x$ the ring of regular functions on $\widetilde{X} \times_X D_x$.

Write $\mathrm{Gr}_{G_\pi,x}$ for the affine Grassmannian $G_\pi(F_x)/G_\pi(\mathcal{O}_x)$. This is an ind-scheme over k that can be seen as the moduli scheme of pairs $(\mathcal{F}_{G_\pi}, \beta)$, where $\mathcal{F}_{G_\pi}$ is a G_π-torsor over D_x and $\beta : \mathcal{F}_{G_\pi} \widetilde{\to} \mathcal{F}^0_{G_\pi}$ is an isomorphism over D_x^*.

In concrete terms, $\mathrm{Gr}_{G_\pi,x}$ classifies the pairs $\mathcal{O}_x$-lattices $\mathcal{M} \subset \mathcal{M}_\pi \otimes F_x$ and $\mathcal{A} \subset \Omega^{-1} \otimes F_x$ such that the following diagram commutes:

$$\begin{array}{ccc} \wedge^2 \mathcal{M}_\pi \otimes F_x & \longrightarrow & \Omega^{-1} \otimes F_x \\ \cup & & \cup \\ \wedge^2 \mathcal{M} & \longrightarrow & \mathcal{A} \end{array} \tag{3.13}$$

and induces an isomorphism $\mathcal{M} \widetilde{\to} \mathcal{M}^* \otimes \mathcal{A}$ of $\mathcal{O}_x$-modules.

Definition 3.2. Let $\mathcal{Y}_{\mathrm{loc}}$ denote the stack classifying

- (i) a free $\widetilde{F}_x$-module $\mathcal{B}$ of rank one; then write L for $\mathcal{B}$ viewed as F_x-module; it is equipped with the nondegenerate form $\operatorname{Sym}^2 L \to \mathcal{C}$, where $\mathcal{C} = (\mathcal{E} \otimes F_x) \otimes \det L$ [**6**, Proposition 14];
- (ii) a G-bundle $(\mathcal{M}, \mathcal{A})$ on $\operatorname{Spec} \mathcal{O}_x$; here $\mathcal{M}$ is a free $\mathcal{O}_x$-module of rank 4 and $\mathcal{A}$ is a free $\mathcal{O}_x$-module of rank 1 with a symplectic form $\wedge^2 \mathcal{M} \to \mathcal{A}$ (it induces $\mathcal{M} \widetilde{\to} \mathcal{M}^* \otimes \mathcal{A}$);
- (iii) an inclusion $L \hookrightarrow \mathcal{M} \otimes_{\mathcal{O}_x} F_x$ of F_x-vector spaces, whose image is an isotropic subspace;
- (iv) an isomorphism $\Omega \otimes \mathcal{A} \otimes F_x \widetilde{\to} \mathcal{C}$ of F_x-vector spaces.

Lemma 3.3. The stack $\mathcal{Y}_{\mathrm{loc}}$ identifies with the stack quotient of $\mathrm{Gr}_{G_\pi,x}$ by $R_\pi(F_x)$. □

Proof. Given a point of $\mathcal{Y}_{\mathrm{loc}}$, it defines a P_π-torsor over $\operatorname{Spec} F_x$. Fix a splitting of the corresponding exact sequence $0 \to \operatorname{Sym}^2 L \otimes F_x \to ? \to \mathcal{A} \otimes F_x \to 0$. Fix also a trivialization

$\mathcal{B}\widetilde{\to}F_x$. Then our data becomes just a point of $\mathrm{Gr}_{G_\pi,x}$. Changing the two trivializations above corresponds to the action of $R_\pi(F_x)$ on $\mathrm{Gr}_{G_\pi,x}$. So, $\mathcal{Y}_{loc}$ classifies a G_π-torsor $\mathcal{F}_{G_\pi}$ on D_x equipped with an R_π-structure over D_x^*. ∎

The $R_\pi(F_x)$-orbits on $\mathrm{Gr}_{G_\pi,x}$ are described in [2, Section 1]. Set $\Lambda_{\mathcal{B}} = \{(a_1, a_2) \in \mathbb{Z}^2 \mid a_2 \geq 0\}$.

Lemma 3.4. The k-points of $\mathcal{Y}_{loc}$ are indexed by $\Lambda_{\mathcal{B}}$. □

Proof. Given a k-point of $\mathcal{Y}_{loc}$, set $L_2 = \mathcal{M} \cap L$. We get a P_π-torsor over D_x given by an exact sequence $0 \to \mathrm{Sym}^2 L_2 \to ? \to \mathcal{A} \to 0$ of $\mathcal{O}_x$-modules. There is a unique $a_1 \in \mathbb{Z}$ such that the isomorphism over F_x extends to an isomorphism $\Omega \otimes \mathcal{A}\widetilde{\to}(\mathcal{E} \otimes \det L_2)(D_\pi + a_1 x)$ of $\mathcal{O}_x$-modules.

Further, $(L_2, \mathcal{B}, L\widetilde{\to}L_2 \otimes F_x)$ is a k-point of $\mathcal{W}ald_\pi^{x,loc}$ given by some $a_2 \geq 0$. Namely, if $\mathcal{B}_{ex} \subset \mathcal{B}$ is the smallest $\widetilde{\mathcal{O}}_x$-lattice such that $L_2 \subset \mathcal{B}_{ex}$, then $a_2 = \dim(\mathcal{B}_{ex}/L_2)$ [6, Section 8.1]. ∎

We realize $\Lambda_{\mathcal{B}}$ as a subsemigroup of $\Lambda_{G_{ad}}$ via the map sending (a_1, a_2) to $\lambda \in \Lambda_{G_{ad}}$ given by $\langle \lambda, \check{\nu}_1 \rangle = a_1$ and $\langle \lambda, \check{\nu}_2 \rangle = a_1 + a_2$. Then $\Lambda_{\mathcal{B}} = \{\lambda \in \Lambda_{G_{ad}} \mid \langle \lambda, \check{\alpha}_{12} \rangle \geq 0\}$.

The image of α_{12} in $\Lambda_{G_{ad}}$ is divisible by two. Define the subsemigroup $\Lambda_{\mathcal{B}}^{pos} \subset \Lambda_{G_{ad}}$ as the $\mathbb{Z}_+$-span of $(1/2)\alpha_{12}, \beta_{22}$. Then

$$\Lambda_{\mathcal{B}}^{pos} = \left\{\lambda \in \Lambda_{G_{ad}} \mid \left\langle \lambda, \check{\nu}_i \right\rangle \geq 0 \text{ for } i = 1, 2\right\}. \tag{3.14}$$

We introduce an order on $\Lambda_{\mathcal{B}}$ as follows. For $\lambda, \mu \in \Lambda_{\mathcal{B}}$ write $\lambda \geq \mu$ if and only if $\lambda - \mu \in \Lambda_{\mathcal{B}}^{pos}$. The reader should be cautioned that this is *not* the order induced from $\Lambda_{G_{ad}}$ (the latter order is never used in this paper).

3.3 Generalized R_π-bundles

3.3.1. The stack Bun_{R_π} classifies the following collections: a line bundle $\mathcal{B}_{ex}$ on $\widetilde{X}$, for which we set $L_{ex} = \pi_* \mathcal{B}_{ex}$, and an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathrm{Sym}^2 L_{ex} \longrightarrow ? \longrightarrow \Omega^{-1} \otimes \mathcal{E}^{-1} \otimes \det L_{ex} \longrightarrow 0. \tag{3.15}$$

By [6, Proposition 14], L_{ex} is equipped with a symmetric form

$$\mathrm{Sym}^2 L_{ex} \longrightarrow \mathcal{E}^{-1} \otimes \det L_{ex}. \tag{3.16}$$

It admits a canonical section $\mathcal{E} \otimes \det L_{ex} \overset{s}{\hookrightarrow} \mathrm{Sym}^2 L_{ex}$.

Here is a Plücker-type description of Bun_{R_π}. It is the stack classifying

(i) a G-bundle $(\mathcal{M}, \mathcal{A})$ on X; here $\mathcal{M} \in \mathrm{Bun}_4$, $\mathcal{A} \in \mathrm{Bun}_1$ with a symplectic form $\wedge^2\mathcal{M} \to \mathcal{A}$, for which we set $W = \mathrm{Ker}(\mathcal{A}^{-1} \otimes \wedge^2\mathcal{M} \to \mathcal{O}_X)$;

(ii) two subbundles

$$\begin{aligned} &\kappa_1 : \Omega \otimes \mathcal{E} \hookrightarrow W, \\ &\kappa_2 : \Omega\big(-D_\pi\big) \hookrightarrow \wedge^2 W. \end{aligned} \tag{3.17}$$

It is required that there is a Lagrangian subbundle $L_{ex} \hookrightarrow \mathcal{M}$, a line bundle $\mathcal{B}_{ex}$ on $\widetilde{X}$, and an isomorphism $L_{ex} \widetilde{\to} \pi_*\mathcal{B}_{ex}$ with the following properties. Let W_{-1} denote the orthogonal complement to $W_1 = \mathcal{A}^{-1} \otimes \det L_{ex}$ in W, so that $W_{-1}/W_1 \widetilde{\to} \mathcal{E}nd_0(L_{ex})$ is equipped with $\mathcal{E} \overset{s}{\hookrightarrow} \mathcal{E}nd_0(L_{ex})$. Then

(a) κ_1 factors as $\Omega \otimes \mathcal{E} \widetilde{\to} W_1 \hookrightarrow W$;

(b) κ_2 factors as $\Omega(-D_\pi) \overset{s}{\hookrightarrow} W_1 \otimes W_{-1}/W_1 \hookrightarrow \wedge^2 W$.

3.3.2. As in Section 2.2, we have the stacks $\overline{\mathrm{Bun}}_{R_\pi} \hookrightarrow {}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}$. By definition, ${}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}$ classifies pairs $(\mathcal{F}_{G_\pi}, \beta)$, where $\mathcal{F}_{G_\pi}$ is a G_π-torsor on X, and $\beta : \mathcal{F}_{G_\pi} \to \overline{G_\pi/R_\pi} \mid_{X-x}$ is a G_π-equivariant map such that β factors through G_π/R_π over some nonempty open subset of $X - x$.

Here is a Plücker-type description. The stack ${}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}$ classifies

(i) a G-bundle $(\mathcal{M}, \mathcal{A})$ on X; here $\mathcal{M} \in \mathrm{Bun}_4$, $\mathcal{A} \in \mathrm{Bun}_1$ with a symplectic form $\wedge^2\mathcal{M} \to \mathcal{A}$, for which we set $W = \mathrm{Ker}(\mathcal{A}^{-1} \otimes \wedge^2\mathcal{M} \to \mathcal{O}_X)$;

(ii) nonzero sections

$$\begin{aligned} &\kappa_1 : \Omega \otimes \mathcal{E} \hookrightarrow W(\infty x), \\ &\kappa_2 : \Omega(-D_\pi) \hookrightarrow \wedge^2 W(\infty x). \end{aligned} \tag{3.18}$$

It is required that for some nonempty open subset $X^0 \subset X - x$ there be a Lagrangian subbundle $L \hookrightarrow \mathcal{M} \mid_{X^0}$, a line bundle $\mathcal{B}$ on $\pi^{-1}(X^0)$, and an isomorphism $L \widetilde{\to} \pi_*\mathcal{B} \mid_{X^0}$ with the following properties. Let W_{-1} denote the orthogonal complement to $W_1 = \mathcal{A}^{-1} \otimes \det L$ in $W \mid_{X^0}$, so $W_{-1}/W_1 \widetilde{\to} \mathcal{E}nd_0 L$ is equipped with $\mathcal{E} \overset{s}{\hookrightarrow} \mathcal{E}nd_0 L$. Then

(a) $\kappa_1 \mid_{X^0}$ factors as $\Omega \otimes \mathcal{E} \widetilde{\to} W_1 \hookrightarrow W \mid_{X^0}$;

(b) $\kappa_2 \mid_{X^0}$ factors as $\Omega(-D_\pi) \overset{s}{\hookrightarrow} W_1 \otimes W_{-1}/W_1 \hookrightarrow \wedge^2 W \mid_{X^0}$.

Definition 3.5. For $\lambda \in \Lambda_{\mathcal{B}}$ denote by ${}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi} \hookrightarrow {}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}$ the closed substack given by the condition that the maps

$$\begin{aligned} &\kappa_1 : \Omega \otimes \mathcal{E}\big(-\langle \lambda, \check{\nu}_1 \rangle x\big) \hookrightarrow W, \\ &\kappa_2 : \Omega\big(-D_\pi - \langle \lambda, \check{\nu}_2 \rangle x\big) \hookrightarrow \wedge^2 W \end{aligned} \tag{3.19}$$

initially defined over $X - x$ are regular over X.

For $\lambda, \mu \in \Lambda_{\mathcal{B}}$ we have ${}_{x,\mu}\overline{\mathrm{Bun}}_{R_\pi} \subset {}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi}$ if and only if $\mu \leq \lambda$.

As in Section 2.4, we have the open substacks

$${}_{x,\lambda}\mathrm{Bun}_{R_\pi} \subset {}_{x,\lambda}\widetilde{\mathrm{Bun}}_{R_\pi} \subset {}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi}, \tag{3.20}$$

given by requiring that κ_1, κ_2 are maximal everywhere on X (resp., in a neighbourhood of x).

3.4 Stratifications

The following lemma is straightforward.

Lemma 3.6. Let $\lambda \in \Lambda_{\mathcal{B}}$. For any k-point of ${}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi}$ there is a unique divisor D on X with values in $-\Lambda^{pos}_{\mathcal{B}}$ such that the maps

$$\begin{aligned} &\kappa_1 : \Omega \otimes \mathcal{E}\big(-\langle \lambda x + D, \check{\nu}_1 \rangle\big) \hookrightarrow W, \\ &\kappa_2 : \Omega\big(-D_\pi - \langle \lambda x + D, \check{\nu}_2 \rangle\big) \hookrightarrow \wedge^2 W \end{aligned} \tag{3.21}$$

are regular and maximal everywhere on X, and $D + \lambda x$ is a divisor with values in $\Lambda_{\mathcal{B}}$. □

Consider a $\Lambda_{\mathcal{B}}$-valued divisor D on X with $D = \lambda x + \sum_{y \neq x} \lambda_y y$ such that $\lambda_y \in -\Lambda^{pos}_{\mathcal{B}}$ for $y \neq x$. Denote by ${}_D\mathrm{Bun}_{R_\pi} \subset {}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi}$ the substack given by the condition that the maps

$$\begin{aligned} &\kappa_1 : \Omega \otimes \mathcal{E}(-\langle D, \check{\nu}_1 \rangle) \hookrightarrow W, \\ &\kappa_2 : \Omega(-D_\pi - \langle D, \check{\nu}_2 \rangle) \hookrightarrow \wedge^2 W \end{aligned} \tag{3.22}$$

are regular and maximal everywhere on X. In particular, for $D = \lambda x$ we get ${}_D\mathrm{Bun}_{R_\pi} \widetilde{\rightarrow} {}_{x,\lambda}\mathrm{Bun}_{R_\pi}$.

Actually, ${}_D\mathrm{Bun}_{R_\pi}$ is the stack classifying a line bundle $\mathcal{B}_{ex}$ on $\widetilde{X}$, for which we set $L_{ex} = \pi_* \mathcal{B}_{ex}$, a modification $L_2 \subset L_{ex}$ of rank-2 vector bundles on X such that the composition is surjective:

$$\mathrm{Sym}^2 L_2 \longrightarrow \mathrm{Sym}^2 L_{ex} \longrightarrow \mathcal{E}^{-1} \otimes \det L_{ex} \tag{3.23}$$

and $\mathrm{div}(L_{ex}/L_2) = \langle D, \check{\nu}_2 - \check{\nu}_1 \rangle$, and an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathrm{Sym}^2 L_2 \longrightarrow ? \longrightarrow \mathcal{A} \longrightarrow 0, \tag{3.24}$$

where $\mathcal{A} = (\Omega^{-1} \otimes \mathcal{E}^{-1} \otimes \det L_2)(\langle D, \check{\nu}_1 \rangle)$. We have used here the description of $\mathcal{W}\mathrm{ald}_\pi^{x,a}$ from [6, Section 8.2].

Remark 3.7. For $a_1 \in \mathbb{Z}$ denote by ${}^{a_1}_{x}\overline{\mathrm{Bun}}_{R_\pi} \subset {}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}$ the substack given by the condition that the map

$$\kappa_1 : \Omega \otimes \mathcal{E}\big(-a_1 x\big) \hookrightarrow W \tag{3.25}$$

is regular and maximal everywhere on X. This is the stack classifying the following collections: $L_2 \in \mathrm{Bun}_2$, an exact sequence $0 \to \mathrm{Sym}^2 L_2 \to ? \to \mathcal{A} \to 0$ on X with $\mathcal{A} = (\Omega^{-1} \otimes \mathcal{E}^{-1} \otimes \det L_2)(a_1 x)$, a line bundle $\mathcal{B}$ on $\pi^{-1}(X - x)$, and an isomorphism $\pi_* \mathcal{B} \widetilde{\to} L_2 \mid_{X-x}$. We have the projection

$$^{a_1}_{x}\overline{\mathrm{Bun}}_{R_\pi} \longrightarrow \mathcal{W}\mathrm{ald}_\pi^x \tag{3.26}$$

sending the above point to $(L_2, \mathcal{B}, \pi_* \mathcal{B} \widetilde{\to} L_2 \mid_{X-x})$ (cf. [6, Section 8.2]).

For $\lambda = (a_1, a_2) \in \Lambda_{\mathcal{B}}$ write ${}^{a_1}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi}$ for the preimage of $\mathcal{W}\mathrm{ald}_\pi^{x, \le a_2}$ under this map. The preimage of $\mathcal{W}\mathrm{ald}_\pi^{x,a_2}$ under the same map identifies with ${}_{x,\lambda}\mathrm{Bun}_{R_\pi}$. Note that

$$^{a_1}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi} \subset {}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi} \tag{3.27}$$

is an open substack. This will be used in Section 3.12.

3.5 Bessel category

Set $H = \Omega \times T_\pi$. Denote by $\chi_\pi : R_\pi \to H$ the homomorphism of group schemes over X given by $\chi_\pi(tu) = (\mathrm{ev}_\pi(u), t)$, $t \in T_\pi$, $u \in U_\pi$. Let

$$\mathrm{ev}^0 : \mathrm{Bun}_{R_\pi} \longrightarrow \mathbb{A}^1 \times \mathrm{Pic}\,\widetilde{X} \tag{3.28}$$

be the map sending a point of Bun_{R_π} to the pair $(\epsilon, \mathcal{B}_{ex})$, where ϵ is the class of the pushforward of (3.15) by (3.16).

Fix a rank-one local system $\widetilde{E}$ on $\widetilde{X}$. Write $A\widetilde{E}$ for the automorphic local system on $\operatorname{Pic}\widetilde{X}$ corresponding to $\widetilde{E}$. For $d \geq 0$ its inverse image under $\widetilde{X}^{(d)} \to \operatorname{Pic}^d \widetilde{X}$ identifies with the symmetric power $\widetilde{E}^{(d)}$ of $\widetilde{E}$.

Let $\mathcal{L}$ denote the restriction of $\mathcal{L}_\psi \boxtimes A\widetilde{E}$ under the natural map $\operatorname{Bun}_H \to \mathbb{A}^1 \times \operatorname{Pic}\widetilde{X}$. As in Section 2.2, our data give rise to *the Bessel category* $P^{\mathcal{L}}({}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi})$.

One checks that $\lambda = (a_1, a_2) \in \Lambda_{\mathcal{B}}$ is *relevant* (in the sense of Section 2.4) if and only if $a_1 \geq a_2$. Write $\Lambda^+_{\mathcal{B}}$ for the set of relevant $\lambda \in \Lambda_{\mathcal{B}}$.

3.6 Perverse sheaves $\mathcal{B}^\lambda$

Consider a stratum ${}_D\operatorname{Bun}_{R_\pi}$ of ${}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi}$ as in Section 3.4, so D is a $\Lambda_{\mathcal{B}}$-valued divisor on X. Arguing as in Section 2.2.3 (with the difference that now $\widetilde{y} \in \widetilde{X}$ satisfies an additional assumption: $\pi(\widetilde{y})$ does not lie in the support of D), one defines the category $P^{\mathcal{L}}({}_D\operatorname{Bun}_{R_\pi})$.

We say that the stratum ${}_D\operatorname{Bun}_{R_\pi}$ is *relevant* if $P^{\mathcal{L}}({}_D\operatorname{Bun}_{R_\pi})$ contains a nonzero object. As in [3, Lemma 6.2.8], one shows that the stratum ${}_D\operatorname{Bun}_{R_\pi}$ is relevant if and only if $D = \lambda x$ with $\lambda \in \Lambda^+_{\mathcal{B}}$.

For $\lambda \in \Lambda^+_{\mathcal{B}}$ denote by

$$\mathrm{ev}^\lambda : {}_{x,\lambda}\operatorname{Bun}_{R_\pi} \longrightarrow \mathbb{A}^1 \times \operatorname{Pic}\widetilde{X} \tag{3.29}$$

the following map. Given a point of ${}_{x,\lambda}\operatorname{Bun}_{R_\pi}$ as in Section 3.4, ev^λ sends it to the pair $(\epsilon, \mathcal{B}_{ex})$, where ϵ is the class of the push-forward of (3.24) under the map $\operatorname{Sym}^2 L_2 \to \mathcal{A} \otimes \Omega$, obtained from the symmetric form on L_{ex}.

For $\lambda \in \Lambda^+_{\mathcal{B}}$ let $\mathcal{B}^\lambda$ be the Goresky-MacPherson extension of

$$(\mathrm{ev}^\lambda)^*(\mathcal{L}_\psi \boxtimes A\widetilde{E}) \otimes \bar{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right)^{\otimes \dim {}_{x,\lambda}\operatorname{Bun}_{R_\pi}} \tag{3.30}$$

under ${}_{x,\lambda}\operatorname{Bun}_{R_\pi} \hookrightarrow {}_{x,\lambda}\overline{\operatorname{Bun}}_{R_\pi}$. The irreducible objects of $P^{\mathcal{L}}({}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi})$ are (up to isomorphism) exactly $\mathcal{B}^\lambda, \lambda \in \Lambda^+_{\mathcal{B}}$.

Let us underline that for $0 \in \Lambda^+_{\mathcal{B}}$ the only relevant stratum of ${}_{x,0}\overline{\operatorname{Bun}}_{R_\pi} = \overline{\operatorname{Bun}}_{R_\pi}$ is ${}_{x,0}\operatorname{Bun}_{R_\pi}$. So, $\mathcal{B}^0$ is the extension by zero from ${}_{x,0}\operatorname{Bun}_{R_\pi}$. As in [3], we say that $\mathcal{B}^0$ is *clean* with respect to the open immersion ${}_{x,0}\operatorname{Bun}_{R_\pi} \hookrightarrow \overline{\operatorname{Bun}}_{R_\pi}$. The same argument proves the following.

Lemma 3.8. For $\lambda \in \Lambda^+_{\mathcal{B}}$ the $*$-restriction of $\mathcal{B}^\lambda$ to ${}_{x,\lambda}\widetilde{\operatorname{Bun}}_{R_\pi} - {}_{x,\lambda}\operatorname{Bun}_{R_\pi}$ vanishes. $\square$

3.7 Semigroups

The natural projection $\Lambda \to \Lambda_{G_{ad}}$ induces a map $i : \Lambda^+ \to \Lambda^+_{\mathcal{B}}$. Actually, we get an isomorphism of semigroups

$$\Lambda^+/\mathbb{Z}\omega \widetilde{\longrightarrow} \Lambda^+_{\mathcal{B}}. \tag{3.31}$$

The map i preserves the order, that is, if $\lambda \le \mu$ for $\lambda, \mu \in \Lambda^+$, then $i(\lambda) \le i(\mu)$. Besides, $i(-w_0(\lambda)) = i(\lambda)$. For $\mu \in \Lambda^+_{\mathcal{B}}$ an easy calculation shows that

$$\dim{}_{x,\mu}\operatorname{Bun}_{R_\pi} = \langle \mu, 2\check{\rho}\rangle + \dim \operatorname{Bun}_{R_\pi}. \tag{3.32}$$

Remark 3.9. Let $\lambda \in \Lambda^+$. The map $\lambda' \mapsto i(\lambda')$ provides a bijection between $\{\lambda' \in \Lambda^+ \mid \lambda' \le \lambda\}$ and $\{\mu \in \Lambda^+_{\mathcal{B}} \mid \mu \le i(\lambda);\ i(\lambda) - \mu = 0 \text{ in } \pi_1(G_{ad})\}$.

3.8 Main result

Recall that $G = G\mathbb{S}p_4$ and for each $\mathcal{A} \in \operatorname{Sph}(\operatorname{Gr}_{G,x})$ we have the Hecke functor $H(\mathcal{A}, \cdot) : D({}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi}) \to D({}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi})$ introduced in Section 2.3.

Here is our main result.

Theorem 3.10. (1) Set $\check{\nu} = (1/2)w_0(\check{\omega}_0 - \check{\beta}_{22})$, so $\check{\nu} \in \check{\Lambda}$. For $\lambda \in \Lambda^+$ there is a canonical isomorphism

$$H\left(\mathcal{A}_\lambda, \mathcal{B}^0\right) \widetilde{\longrightarrow} \begin{cases} \mathcal{B}^{i(\lambda)} \otimes \left(\widetilde{E}_{\widetilde{x}}\right)^{\otimes\langle\lambda, 2\check{\nu}\rangle}, & \text{the nonsplit case, } \pi(\widetilde{x}) = x, \\ \mathcal{B}^{i(\lambda)} \otimes \left(\widetilde{E}_{\widetilde{x}_1} \otimes \widetilde{E}_{\widetilde{x}_2}\right)^{\otimes\langle\lambda, \check{\nu}\rangle}, & \text{the split case, } \pi^{-1}(x) = \left\{\widetilde{x}_1, \widetilde{x}_2\right\}. \end{cases} \tag{3.33}$$

(2) For $\omega = (1,1,1,1) \in \Lambda^+$ and $\mu \in \Lambda^+_{\mathcal{B}}$ there is a canonical isomorphism

$$H(\mathcal{A}_\omega, \mathcal{B}^\mu) \widetilde{\longrightarrow} \begin{cases} \mathcal{B}^\mu \otimes \widetilde{E}_{\widetilde{x}}^{\otimes 2}, & \text{the nonsplit case, } \pi(\widetilde{x}) = x, \\ \mathcal{B}^\mu \otimes \widetilde{E}_{\widetilde{x}_1} \otimes \widetilde{E}_{\widetilde{x}_2}, & \text{the split case, } \pi^{-1}(x) = \{\widetilde{x}_1, \widetilde{x}_2\}. \end{cases} \tag{3.34}$$

□

3.9 Dimensions estimates

Given a G-torsor $\mathcal{F}_G$ over D_x, denote by $\operatorname{Gr}_{G,x}(\mathcal{F}_G)$ the affine Grassmannian classifying pairs $(\mathcal{F}'_G, \beta)$, where $\mathcal{F}'_G$ is a G-torsor over D_x and $\beta : \mathcal{F}'_G \widetilde{\to} \mathcal{F}_G \mid_{D^*_x}$ an isomorphism.

For $\lambda \in \Lambda^+$ we have the subschemes (cf. [1, Section 3.2.1])

$$\operatorname{Gr}^\lambda_{G,x}(\mathcal{F}_G) \subset \overline{\operatorname{Gr}}^\lambda_{G,x}(\mathcal{F}_G) \subset \operatorname{Gr}_{G,x}(\mathcal{F}_G). \tag{3.35}$$

A point $(\mathcal{F}'_G, \beta) \in \mathrm{Gr}_{G,x}(\mathcal{F}_G)$ lies in $\overline{\mathrm{Gr}}^{\lambda}_{G,x}(\mathcal{F}_G)$ if for any G-module V, whose weights are $\leq \check{\lambda}$, we have

$$V_{\mathcal{F}_G}\big(-\langle\lambda,\check{\lambda}\rangle x\big) \subset V_{\mathcal{F}'_G}. \tag{3.36}$$

Recall that we identify $\mathrm{Gr}_{G_\pi,x}$ with the ind-scheme $\mathrm{Gr}_{G,x}(\mathfrak{F}_G)$ classifying pairs $(\mathcal{F}_G, \widetilde{\beta})$, where $\mathcal{F}_G$ is a G-torsor on D_x and

$$\widetilde{\beta} : \mathcal{F}_G \widetilde{\longrightarrow} \mathfrak{F}_G \mid_{D^*_x} \tag{3.37}$$

is an isomorphism of G-torsors. A k-point $(\mathcal{F}_G, \widetilde{\beta})$ of $\mathrm{Gr}_{G_\pi,x}$ yields an inclusion $\overline{\mathrm{Gr}}^{\lambda}_{G,x}(\mathcal{F}_G) \hookrightarrow \mathrm{Gr}_{G_\pi,x}$ sending $(\mathcal{F}'_G, \beta)$ to $(\mathcal{F}'_G, \widetilde{\beta}\circ\beta)$. For $\mu \in \Lambda_{\mathcal{B}}$ we denote by $S^{\mu}_{R_\pi} \subset \mathrm{Gr}_{G_\pi,x}$ the $R_\pi(F_x)$-orbit on $\mathrm{Gr}_{G_\pi,x}$ corresponding to μ.

As in [3] and [6, Proposition 17], the following is a key point of our proof of Theorem 3.10.

Proposition 3.11. Let $\mu \in \Lambda^+_{\mathcal{B}}$. Let $(\mathcal{F}_G, \widetilde{\beta})$ be a k-point of $S^{\mu}_{R_\pi}$, where $\mathcal{F}_G$ is a G-torsor on D_x and $\widetilde{\beta} : \mathcal{F}_G \widetilde{\rightarrow} \mathfrak{F}_G \mid_{D^*_x}$ is an isomorphism of G-torsors. For any $\lambda \in \Lambda^+$ the scheme

$$\overline{\mathrm{Gr}}^{\lambda}_{G,x}(\mathcal{F}_G) \cap S^{0}_{R_\pi} \tag{3.38}$$

is empty unless $\mu \leq i(\lambda)$ in the sense of the order on $\Lambda^+_{\mathcal{B}}$. If $\mu \leq i(\lambda)$, then

$$\mathrm{Gr}^{\lambda}_{G,x}\,(\mathcal{F}_G) \cap S^{0}_{R_\pi} \tag{3.39}$$

is of dimension $\leq \langle\lambda,\check{\rho}\rangle - \langle\mu,\check{\rho}\rangle$. The equality holds if and only if there exists $\lambda' \in \Lambda^+$, $\lambda' \leq \lambda$, such that $\mu = i(\lambda')$, and in this case the irreducible components of (3.39) of maximal dimension form a base of

$$\mathrm{Hom}_{\check{M}}\,\big(U^{w_0^M w_0(\lambda')}, V^{\lambda}\big). \tag{3.40}$$

If $\mu = i(\lambda)$, then (3.39) is a point scheme. □

Remark 3.12. Consider the scheme (3.39) in the case $\lambda, \lambda' \in \Lambda^+$ with $\lambda' < \lambda$ and $\mu = i(\lambda')$. Our proof of Proposition 3.11 will also show that for such λ and μ in the nonsplit case, *all* the irreducible components of (3.39) are of the same dimension. In the split case, (3.39) may have irreducible components of different dimensions (e.g., this happens for $\lambda = (a, a, 0, 0) \in \Lambda^+$ and $\mu = 0$).

3.10 Proofs

For a P-torsor $\mathcal{F}_P$ over D_x let $\mathcal{F}_G = \mathcal{F}_P \times_P G$. For a coweight $\nu \in \Lambda^+_M$ denote by $S^\nu_P(\mathcal{F}_P)$ the ind-scheme classifying pairs $(\mathcal{F}'_P, \beta)$, where $\mathcal{F}'_P$ is a P-torsor on D_x and

$$\beta : \mathcal{F}'_P \widetilde{\longrightarrow} \mathcal{F}_P \mid_{D^*_x} \tag{3.41}$$

is an isomorphism such that the pair $(\mathcal{F}'_M, \beta)$ lies in $\mathrm{Gr}^\nu_{M,x}(\mathcal{F}_M)$. Here $\mathcal{F}_M$ and $\mathcal{F}'_M$ are the M-torsors induced from $\mathcal{F}_P$ and $\mathcal{F}'_P$, respectively. For $\lambda \in \Lambda^+$ denote by

$$\mathfrak{t}^\nu_P : S^\nu_P(\mathcal{F}_P) \cap \mathrm{Gr}^\lambda_{G,x}(\mathcal{F}_G) \longrightarrow \mathrm{Gr}^\nu_{M,x}(\mathcal{F}_M) \tag{3.42}$$

the natural projection. Our Proposition 3.11 is based on the following result established in [1, Proposition 4.3.3 and Section 5.3.7].

Proposition 3.13. All the irreducible components of any fibre of $\mathfrak{t}^\nu_P$ are of dimension $\langle \nu + \lambda, \check{\rho} \rangle - \langle \nu, 2\check{\rho}_M \rangle$. These components form a base of

$$\mathrm{Hom}_{\check{M}}(U^\nu, V^\lambda). \tag{3.43}$$

For $\nu = w^M_0 w_0(\lambda)$ the map (3.42) is an isomorphism. □

Proof of Proposition 3.11. Write $\mu = (a_1, a_2)$. The pair $(\mathcal{F}_G, \widetilde{\beta})$ is given by $\mathcal{O}_x$-lattices $\mathcal{M} \subset \mathcal{M}_\pi \otimes F_x$ and $\mathcal{A} \subset \Omega^{-1} \otimes F_x$ such that $(\mathcal{M}, \mathcal{A})$ is a G-bundle over Spec $\mathcal{O}_x$. Note that

$$\langle \mu, \check{\rho} \rangle = \frac{1}{2}(3a_1 + a_2). \tag{3.44}$$

(1) The nonsplit case.

Step 1. Acting by $R_\pi(F_x)$, we may assume that $(\mathcal{M}, \mathcal{A})$ has the standard form $\mathcal{M} = L_2 \oplus (L^*_2 \otimes \mathcal{A})$, where $\mathcal{A} = \Omega^{-1}((a_1 - a_2)x) \otimes \mathcal{O}_x$ and $L_2 = \mathcal{O}_x \oplus \mathcal{O}_x t^{a_2+1/2} \subset \widetilde{F}_x$; here $t \in \mathcal{O}_x$ is a local parameter [6, Section 8.1].

Any k-point of $S^0_{R_\pi}$ is given by a collection $(a \in \mathbb{Z},\ L'_2 \subset \mathcal{M}', \mathcal{A}')$, where $\mathcal{M}' \subset \mathcal{M}_\pi \otimes F_x$ is an $\mathcal{O}_x$-lattice, $\mathcal{A}' = \Omega^{-1}(-ax) \otimes \mathcal{O}_x$, and $L'_2 = \widetilde{\mathcal{O}}_x(-a\widetilde{x}) = \mathcal{M}' \cap (L_\pi \otimes F_x)$. Here $\pi(\widetilde{x}) = x$ and L'_2 is viewed as an $\mathcal{O}_x$-module, so

$$L'_2 = t^{a/2}\mathcal{O}_x \oplus t^{(a+1)/2}\mathcal{O}_x. \tag{3.45}$$

Set $\mathcal{W} = \mathrm{Ker}(\wedge^2\mathcal{M} \to \mathcal{A})$ and $\mathcal{W}' = \mathrm{Ker}(\wedge^2\mathcal{M}' \to \mathcal{A}')$.

The condition that $(\mathcal{F}'_G, \beta) = (\mathcal{M}', \mathcal{A}')$ lies in $\overline{\mathrm{Gr}}^{\lambda}_{G,x}(\mathcal{F}_G)$ implies that $\mathcal{A}' \widetilde{\to} \mathcal{A}(-\langle\lambda, \check{\omega}_0\rangle x)$, hence

$$a = \langle\lambda, \check{\omega}_0\rangle - (a_1 - a_2). \tag{3.46}$$

It also implies that

$$\mathcal{M}(-\langle\lambda, \check{\omega}_1\rangle x) \subset \mathcal{M}', \tag{3.47}$$

$$\mathcal{W}(-\langle\lambda, \check{\omega}_2\rangle x) \subset \mathcal{W}'. \tag{3.48}$$

The inclusion (3.47) fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L'_2 & \longrightarrow & \mathcal{M}' & \longrightarrow & L'^*_2 \otimes \mathcal{A}' & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & L_2(-\langle\lambda, \check{\omega}_1\rangle x) & \longrightarrow & \mathcal{M}(-\langle\lambda, \check{\omega}_1\rangle x) & \longrightarrow & L^*_2 \otimes \mathcal{A}(-\langle\lambda, \check{\omega}_1\rangle x) & \longrightarrow & 0 \end{array} \tag{3.49}$$

This yields an inclusion $L^*_2 \subset L'^*_2(\langle\lambda, \check{\omega}_1 - \check{\omega}_0\rangle)$, which implies $\langle\lambda, 2\check{\omega}_1 - \check{\omega}_0\rangle \geq a_1 + a_2$. Note that $2\check{\omega}_1 - \check{\omega}_0 = \check{\beta}_{12} + \check{\alpha}_{12}$.

Further, the inclusion (3.48) shows that $(\wedge^2 L^*_2) \otimes \mathcal{A}^2(-\langle\lambda, \check{\omega}_2\rangle x) \subset (\wedge^2 L'^*_2) \otimes \mathcal{A}'^2$, that is,

$$\langle\lambda, \check{\omega}_2 - \check{\omega}_0\rangle \geq a_1. \tag{3.50}$$

Since $\check{\omega}_2 - \check{\omega}_0 = \check{\beta}_{12}$, we get $\mu \leq i(\lambda)$.

Step 2. The above M-torsor $(L'_2, \mathcal{A}')$ is in a position ν with respect to $(L_2, \mathcal{A})$, where $\nu \in \Lambda^+_M$ is a dominant coweight for M that we are going to determine.

Clearly, $\langle\nu - \lambda, \check{\omega}_0\rangle = 0$. Further, $(\wedge^2 L_2)(-\langle\nu, \check{\omega}_2\rangle x) \widetilde{\to} \wedge^2 L'_2$, so $a_1 = \langle\nu, \check{\omega}_0 - \check{\omega}_2\rangle$. From $L_2(-\langle\nu, \check{\omega}_1\rangle x) \subset L'_2$ we get

$$\langle\nu, \check{\omega}_1\rangle = \begin{cases} \dfrac{a}{2}, & a \text{ is even}, \\ \dfrac{a+1}{2}, & a \text{ is odd}. \end{cases} \tag{3.51}$$

Now (3.39) identifies with the fibre of (3.42) over $(L'_2, \mathcal{A}') \in \mathrm{Gr}^{\nu}_{M,x}(\mathcal{F}_M)$. Here the M-torsor $\mathcal{F}_M$ is given by $(L_2, \mathcal{A})$.

By Remark 3.9, for a even, there exists a unique $\lambda' \in \Lambda^+$ with $\lambda' \leq \lambda$ such that $\mu = i(\lambda')$. In this case the above formulas imply $\nu = w_0^M w_0(\lambda')$.

If $\mu = i(\lambda)$, then $a = \langle \lambda, \check{\omega}_0 - \check{\beta}_{22} \rangle$ is even, because $\check{\omega}_0 - \check{\beta}_{22}$ is divisible by 2 in $\check{\Lambda}$. For $\mu = i(\lambda)$ we get $\nu = w_0^M w_0(\lambda)$.

Let us show that $\langle \mu, \check{\rho} \rangle + \langle \nu, \check{\rho} - 2\check{\rho}_M \rangle \leq 0$. Indeed, since $2\check{\omega}_1 - \check{\omega}_2 = \check{\alpha}_{12}$, we get

$$\langle \nu, \check{\alpha}_{12} \rangle = \begin{cases} a_2, & a \text{ is even}, \\ a_2 + 1, & a \text{ is odd}, \end{cases} \tag{3.52}$$

and $\langle \nu, \check{\alpha}_{12} + \check{\beta}_{22} \rangle = -a_1$. We have $\check{\rho} - 2\check{\rho}_M = \check{\alpha}_{12} + (3/2)\check{\beta}_{22}$ and $\check{\rho} = 2\check{\alpha}_{12} + (3/2)\check{\beta}_{22}$. So,

$$\langle \nu, \check{\rho} - 2\check{\rho}_M \rangle = \begin{cases} \frac{1}{2}\big(-3a_1 - a_2 \big), & a \text{ is even}, \\ \frac{1}{2}\big(-3a_1 - a_2 - 1 \big), & a \text{ is odd}. \end{cases} \tag{3.53}$$

The desired inequality follows now from (3.44), and it is an equality if and only if a is even, that is, $i(\lambda) - \mu$ vanishes in $\pi_1(G_{ad})$. Our assertion follows now from Proposition 3.13.

(2) The split case.

Step 1. Acting by $R_\pi(F_x)$, we may assume that $(\mathcal{M}, \mathcal{A})$ has the following standard form $\mathcal{M} = L_2 \oplus L_2^* \otimes \mathcal{A}$, where

$$L_2 = \mathcal{O}_x t^{a_2} e_1 \oplus \mathcal{O}_x \big(e_1 + e_2 \big) \tag{3.54}$$

and $\mathcal{A} = \Omega^{-1}((a_1 - a_2)x) \otimes \mathcal{O}_x$. Here $\{e_i\}$ is a base of $\widetilde{\mathcal{O}}_x$ over $\mathcal{O}_x$ consisting of isotropic vectors [6, Section 8.1].

Any k-point of $S^0_{R_\pi}$ is given by a collection $(b_1, b_2 \in \mathbb{Z},\ L'_2 \subset \mathcal{M}', \mathcal{A}')$, where $\mathcal{M}' \subset \mathcal{M}_\pi \otimes F_x$ is an $\mathcal{O}_x$-lattice, $\mathcal{A}' = \Omega^{-1}(-(b_1 + b_2)x) \otimes \mathcal{O}_x$, and

$$L'_2 = \widetilde{\mathcal{O}}_x \big(-b_1 \tilde{x}_1 - b_2 \tilde{x}_2 \big) = \mathcal{M}' \cap \big(L_\pi \otimes F_x \big). \tag{3.55}$$

Here $\pi^{-1}(x) = \{\tilde{x}_1, \tilde{x}_2\}$ and L'_2 is viewed as an $\mathcal{O}_x$-module, so

$$L'_2 = \mathcal{O}_x t^{b_1} e_1 \oplus \mathcal{O}_x t^{b_2} e_2. \tag{3.56}$$

If $(\mathcal{F}'_G, \beta) = (\mathcal{M}', \mathcal{A}')$ lies in $\overline{\mathrm{Gr}}^{\lambda}_{G,x}(\mathcal{F}_G)$, then $\mathcal{A}' \widetilde{\to} \mathcal{A}(-\langle\lambda, \check{\omega}_0\rangle x)$, so

$$b_1 + b_2 = \langle \lambda, \check{\omega}_0 \rangle - a_1 + a_2. \tag{3.57}$$

As in the nonsplit case, the inclusion $L'_2(-\langle\lambda, \check{\omega}_1 - \check{\omega}_0\rangle x) \subset L_2$ yields

$$b_i + \langle \lambda, \check{\omega}_1 - \check{\omega}_0 \rangle \geq a_2 \tag{3.58}$$

for $i = 1, 2$. This implies $\langle\lambda, 2\check{\omega}_1 - \check{\omega}_0\rangle \geq a_1 + a_2$. As in the nonsplit case, $(\wedge^2 L'_2)(\langle\lambda, 2\check{\omega}_0 - \check{\omega}_2\rangle x) \subset \wedge^2 L_2$ implies $\langle\lambda, \check{\omega}_2 - \check{\omega}_0\rangle \geq a_1$. We have shown that $\mu \leq i(\lambda)$.

Step 2. Let us determine $\nu \in \Lambda^+_M$ such that $(L'_2, \mathcal{A}') \in \mathrm{Gr}^{\nu}_{M,x}(\mathcal{F}_M)$. Here $\mathcal{F}_M$ is given by $(L_2, \mathcal{A})$.

As in the nonsplit case, $\langle\nu - \lambda, \check{\omega}_0\rangle = 0$ and $(\wedge^2 L_2)(-\langle\nu, \check{\omega}_2\rangle x) \widetilde{\to} \wedge^2 L'_2$. So, $a_1 = \langle\nu, \check{\omega}_0 - \check{\omega}_2\rangle$. From $L_2(-\langle\nu, \check{\omega}_1\rangle x) \subset L'_2$ we get

$$\langle \nu, \check{\omega}_1 \rangle = \max\{b_1, b_2\}. \tag{3.59}$$

In particular, for $\mu = i(\lambda)$ we get from (3.57) and (3.58)

$$\begin{aligned} b_1 + b_2 &= \langle \lambda, \check{\omega}_0 - \check{\beta}_{22} \rangle, \\ b_i &\geq \langle \lambda, \check{\alpha}_{12} - \check{\omega}_1 + \check{\omega}_0 \rangle. \end{aligned} \tag{3.60}$$

But $2(\check{\alpha}_{12} - \check{\omega}_1 + \check{\omega}_0) = \check{\omega}_0 - \check{\beta}_{22}$, so in this case $b_i = \langle\lambda, \check{\alpha}_{12} - \check{\omega}_1 + \check{\omega}_0\rangle$ for $i = 1, 2$. It easily follows that for $\mu = i(\lambda)$ we get $\nu = w_0^M w_0(\lambda)$.

As in the nonsplit case, it remains to show that $\langle\mu, \check{\rho}\rangle + \langle\nu, \check{\rho} - 2\check{\rho}_M\rangle \leq 0$. We have $\langle\nu, \check{\alpha}_{12} + \check{\beta}_{22}\rangle = -a_1$ and $\langle\nu, \check{\alpha}_{12}\rangle = 2\max\{b_i\} - \langle\lambda, \check{\omega}_0\rangle + a_1$. So,

$$\langle \nu, \check{\rho} - 2\check{\rho}_M \rangle = -2a_1 - \max\{b_i\} + \frac{1}{2}\langle \lambda, \check{\omega}_0 \rangle. \tag{3.61}$$

The desired inequality follows now from (3.44), because $\max\{b_i\} \geq (1/2)(a_2 - a_1 + \langle\lambda, \check{\omega}_0\rangle) = (1/2)(b_1 + b_2)$. It is an equality if and only if $b_1 = b_2$, and this implies that $2b_i = \langle\lambda, \check{\omega}_0\rangle - (a_1 - a_2)$ is even.

If $b_1 = b_2$, then, as in the nonsplit case, we get $\langle\nu, \check{\alpha}_{12}\rangle = a_2$, so that $\nu = w_0^M w_0(\lambda')$ for $\lambda' \in \Lambda^+$ such that $\lambda' \leq \lambda$ and $i(\lambda') = \mu$. ■

Remark 3.14. Write $\check{B} \subset \check{G}$ for the dual Borel subgroup in $\check{G}$. The set of double-cosets $\check{M}\backslash\check{G}/\check{B}$ is finite, that is, $\check{M} \subset \check{G}$ is a Gelfand pair. So, for any character $\nu \in \Lambda$ with $\langle \nu, \check{\alpha}_{12}\rangle = 0$ and any $\lambda \in \Lambda^+$, the space $\mathrm{Hom}_{\check{M}}(U^\nu, V^\lambda)$ is at most 1-dimensional [9, Theorem 1]. This implies that for $\lambda', \lambda \in \Lambda^+$ with $\lambda' \leq \lambda$ and $\langle \lambda', \check{\alpha}_{12}\rangle = 0$ for $\mu = i(\lambda')$, the scheme (3.39) is irreducible.

Remark 3.15. Let $\mathcal{F}_G$ be a G-torsor on D_x. For a k-point $(\mathcal{F}'_G, \beta)$ of $\mathrm{Gr}_{G,x}(\mathcal{F}_G)$ we have $(\mathcal{F}'_G, \beta) \in \overline{\mathrm{Gr}}^\lambda_{G,x}(\mathcal{F}_G)$ if and only if

$$V^{\check{\omega}_i}_{\mathcal{F}'_G} \subset V^{\check{\omega}_i}_{\mathcal{F}_G}(\langle \lambda, -w_0(\check{\omega}_i)\rangle x) \tag{3.62}$$

for $i = 0, 1, 2$, and for $i = 0$, this is an isomorphism.

3.11 End of the proof

Recall the map $\chi_\pi : R_\pi \to \Omega \times T_\pi$ (cf. Section 3.5). Write $\chi_{\pi,x} : R_\pi(F_x) \to \mathbb{A}^1 \times \mathrm{Pic}\,\widetilde{X}$ for the composition

$$R_\pi(F_x) \xrightarrow{\chi_\pi} \Omega(F_x) \times T_\pi(F_x) \xrightarrow{\sim} \Omega(F_x) \times \widetilde{F}^*_x \xrightarrow{\mathrm{Res}\times\tau_x} \mathbb{A}^1 \times \mathrm{Pic}\,\widetilde{X}, \tag{3.63}$$

where τ_x is the natural map $\widetilde{F}^*_x \to \widetilde{F}^*_x/\widetilde{\mathcal{O}}^*_x \to \mathrm{Pic}\,\widetilde{X}$. It is easy to see that for $\mu \in \Lambda^+_{\mathcal{B}}$ there exists an $(R_\pi(F_x), \chi_{\pi,x})$-equivariant map $\chi^\mu : S^\mu_{R_\pi} \to \mathbb{A}^1 \times \mathrm{Pic}\,\widetilde{X}$, and such a map is unique up to an additive constant (with respect to the structure of an abelian group on $\mathbb{A}^1 \times \mathrm{Pic}\,\widetilde{X}$).

We need the following analog of [3, Proposition 7.1.7].

Lemma 3.16. Let $\lambda, \lambda' \in \Lambda^+$ with $\lambda' < \lambda$. Set $\mu = i(\lambda')$. Let $(\mathcal{F}_G, \widetilde{\beta})$ be a k-point of $S^\mu_{R_\pi}$. Let $\chi^0 : S^0_{R_\pi} \to \mathbb{A}^1 \times \mathrm{Pic}\,\widetilde{X}$ be an $(R_\pi(F_x), \chi_{\pi,x})$-equivariant map. Then the composition

$$\mathrm{Gr}^\lambda_{G,x}(\mathcal{F}_G) \cap S^0_{R_\pi} \xrightarrow{\chi^0} \mathbb{A}^1 \times \mathrm{Pic}\,\widetilde{X} \xrightarrow{\mathrm{pr}_1} \mathbb{A}^1 \tag{3.64}$$

maps each irreducible component of (3.39) of dimension $\langle \lambda, \check{\rho}\rangle - \langle \mu, \check{\rho}\rangle$ dominantly to $\mathbb{A}^1$. □

Proof. We may assume that $(\mathcal{F}_G, \widetilde{\beta})$ is given by the pair $(\mathcal{M}, \mathcal{A})$ in its standard form as in the proof of Proposition 3.11; in particular, it is reduced to a M-torsor. Write $\mu = (a_1, a_2)$. Set $\nu = w_0^M w_0(\lambda')$.

Let $z \in \mathbb{G}_m$ act on L_π as a multiplication by z and trivially on Ω^{-1}. The corresponding action of $\mathbb{G}_m$ on $\mathcal{M}_\pi = L_\pi \oplus L^*_\pi \otimes \Omega^{-1}$ defines a map $\mathbb{G}_m \to G_\pi$ whose image lies in the center of P_π/U_π. The corresponding action of $\mathbb{G}_m(\mathcal{O}_x) = \mathcal{O}_x^*$ on $\mathrm{Gr}_{G_\pi,x}$ fixes $(\mathcal{F}_G, \widetilde{\beta})$ and preserves the scheme (3.39).

The dimension estimates in Proposition 3.11 also show that the irreducible components of dimension $\langle\lambda, \check{\rho}\rangle - \langle\mu, \check{\rho}\rangle$ of the schemes $\mathrm{Gr}^\lambda_{G,x}(\mathcal{F}_G) \cap S^0_{R_\pi}$ and $\overline{\mathrm{Gr}}^\lambda_{G,x}(\mathcal{F}_G) \cap S^0_{R_\pi}$ are the same. We are going to describe the latter scheme explicitly.

(1) The split case. We have $\mathcal{M} = L_2 \oplus L_2^* \otimes \mathcal{A}$ with $L_2 = \mathcal{O}_x t^{a_2} e_1 \oplus \mathcal{O}_x(e_1 + e_2)$ and $\mathcal{A} = \Omega^{-1}((a_1 - a_2)x) \otimes \mathcal{O}_x$, where $\{e_i\}$ is a base of $\widetilde{\mathcal{O}}_x$ over $\mathcal{O}_x$ consisting of isotropic vectors, and $t \in \mathcal{O}_x$ is a local parameter. Let $\mathcal{F}_M$ be the M-torsor on $\mathrm{Spec}\, \mathcal{O}_x$ given by $(L_2, \mathcal{A})$.

Set $b = (1/2)(a_2 - a_1 + \langle\lambda, \check{\omega}_0\rangle)$. Consider the k-point of $\mathrm{Gr}_{M,x}(\mathcal{F}_M)$ given by $(L_2', \mathcal{A}')$ with $\mathcal{A}' = \Omega^{-1}(-2bx) \otimes \mathcal{O}_x$ and $L_2' = \widetilde{\mathcal{O}}_x(-b\widetilde{x}_1 - b\widetilde{x}_2)$, where $\pi^{-1}(x) = \{\widetilde{x}_1, \widetilde{x}_2\}$. Under our assumptions the scheme (3.38) identifies with the fibre, say Y, of

$$t_P^\nu : S^\nu(\mathcal{F}_P) \cap \overline{\mathrm{Gr}}^\lambda_{G,x}(\mathcal{F}_G) \longrightarrow \mathrm{Gr}^\nu_{M,x}(\mathcal{F}_M) \tag{3.65}$$

over $(L_2', \mathcal{A}')$. In matrix terms, Y is the scheme of those $u \in \mathrm{Gr}_{U,x}$ for which $gu \in \overline{\mathrm{Gr}}^\lambda_{G,x}$. Here

$$g = \begin{pmatrix} t^{b-a_2} & -t^{b-a_2} & 0 & 0 \\ 0 & t^b & 0 & 0 \\ 0 & 0 & t^{a_1+b} & 0 \\ 0 & 0 & t^{a_1-a_2+b} & t^{a_1-a_2+b} \end{pmatrix}. \tag{3.66}$$

Write

$$u = \begin{pmatrix} 1 & 0 & u_1 & u_2 \\ 0 & 1 & u_2 & u_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.67}$$

with $u_i \in \Omega(F_x)/\Omega(\mathcal{O}_x)$. By Remark 3.15, Y inside of $\mathrm{Gr}_{U,x}$ is given by the equations

$$\begin{aligned} &u_i \in t^{-b+\langle\lambda, w_0(\check{\omega}_1)\rangle}\Omega(\mathcal{O}_x), \\ &u_i - u_j \in t^\alpha \Omega(\mathcal{O}_x), \\ &u_1u_3 - u_2^2 \in t^\delta \Omega^{\otimes 2}(\mathcal{O}_x), \\ &u_i \in t^\delta \Omega(\mathcal{O}_x), \end{aligned} \tag{3.68}$$

where we have set for brevity $\delta = -2b + a_2 + \langle\lambda, w_0(\check{\omega}_2)\rangle$ and $\alpha = -b + a_2 + \langle\lambda, w_0(\check{\omega}_1)\rangle$.

We may assume that (3.64) sends (3.67) to $\mathrm{Res}\, u_2$. Let $Y' \subset Y$ be the closed subscheme given by $u_2 = 0$. The above $\mathcal{O}_x^*$-action on Y multiplies each u_i in (3.67) by the same scalar. So, it suffices to show that $\dim Y' < \langle \lambda, \check{\rho} \rangle - \langle \mu, \check{\rho} \rangle$.

The scheme Y' is contained in the scheme of pairs

$$\{u_1, u_3 \in t^{\delta}\Omega(\mathcal{O}_x)/\Omega(\mathcal{O}_x) \mid u_1 u_3 \in t^{\delta}\Omega(\mathcal{O}_x)/\Omega(\mathcal{O}_x)\}. \tag{3.69}$$

The dimension of the latter scheme is at most $-\delta$. We have $-\delta \leq \langle \lambda, \check{\rho} \rangle - \langle \mu, \check{\rho} \rangle$, and the equality holds if and only if $\alpha = 0$. But if $\alpha = 0$, then Y' is a point scheme. Since $\langle \lambda, \check{\rho} \rangle - \langle \mu, \check{\rho} \rangle$ is strictly positive, we are done.

(2) The nonsplit case. We have $\mathcal{M} = L_2 \oplus L_2^* \otimes \mathcal{A}$ with $L_2 \widetilde{\to} \mathcal{O}_x \oplus \mathcal{O}_x t^{a_2+(1/2)}$ and $\mathcal{A} \widetilde{\to} \Omega^{-1}((a_1 - a_2)x) \otimes \mathcal{O}_x$, where $t \in \mathcal{O}_x$ is a local parameter. Let $\mathcal{F}_M$ be the M-torsor on $\mathrm{Spec}\, \mathcal{O}_x$ given by $(L_2, \mathcal{A})$.

Set $L_2' = t^{a/2}\mathcal{O}_x \oplus t^{(a+1)/2}\mathcal{O}_x$ and $\mathcal{A}' = \Omega^{-1}(-ax) \otimes \mathcal{O}_x$ with $a = \langle \lambda, \check{\omega}_0 \rangle - a_1 + a_2$ and recall that a is even. The scheme (3.38) identifies with the fibre, say Y, of (3.65) over $(L_2', \mathcal{A}')$.

Consider the base $\{1, t^{1/2}\}$ in $L_\pi \otimes \mathcal{O}_x$ and the dual base in $L_\pi^* \otimes \mathcal{O}_x$. Then in matrix terms, Y becomes the scheme of those $u \in \mathrm{Gr}_{U,x}$ for which $gu \in \overline{\mathrm{Gr}}^{\lambda}_{G,x}$. Here $g = t^{a/2}\, \mathrm{diag}(1, t^{-a_2}, t^{a_1 - a_2}, t^{a_1})$. For $u \in \mathrm{Gr}_{U,x}$ written as in (3.67), the scheme Y is given by the equations

$$\begin{aligned} & u_1 \in t^{-(a/2) + \langle \lambda, w_0(\check{\omega}_1) \rangle}\Omega(\mathcal{O}_x), \\ & u_2, u_3 \in t^{\alpha}\Omega(\mathcal{O}_x), \\ & u_1 u_3 - u_2^2 \in t^{\delta}\Omega^{\otimes 2}(\mathcal{O}_x), \\ & u_i \in t^{\delta}\Omega(\mathcal{O}_x), \end{aligned} \tag{3.70}$$

where we have set $\alpha = a_2 - (a/2) + \langle \lambda, w_0(\check{\omega}_1) \rangle$ and $\delta = a_2 - a + \langle \lambda, w_0(\check{\omega}_2) \rangle$.

We may assume that (3.64) sends (3.67) to $\mathrm{Res}(u_1 - t u_3)$. Let $Y' \subset Y$ be the closed subscheme given by $u_1 = t u_3$. Since we have an action of $\mathcal{O}_x^*$, it suffices to show that $\dim Y' < \langle \lambda, \check{\rho} \rangle - \langle \mu, \check{\rho} \rangle$.

The scheme Y' is contained in the scheme

$$\{u_2, u_3 \in t^{\delta}\Omega(\mathcal{O}_x)/\Omega(\mathcal{O}_x) \mid t u_3^2 - u_2^2 \in t^{\delta}\Omega^{\otimes 2}(\mathcal{O}_x)\}. \tag{3.71}$$

The latter scheme is included into Y'' given by

$$Y'' = \begin{cases} \{u_2, u_3 \in t^{\delta/2}\Omega(\mathcal{O}_x)/\Omega(\mathcal{O}_x)\}, & \text{for } \delta \text{ even}, \\ \{u_2 \in t^{(1+\delta)/2}\Omega(\mathcal{O}_x)/\Omega(\mathcal{O}_x), u_3 \in t^{(\delta-1)/2}\Omega(\mathcal{O}_x)/\Omega(\mathcal{O}_x)\}, & \text{for } \delta \text{ odd}. \end{cases} \tag{3.72}$$

This implies $\dim Y' \leq \dim Y'' \leq -\delta$. As in the split case, $-\delta \leq \langle\lambda, \check{\rho}\rangle - \langle\mu, \check{\rho}\rangle$ and the equality implies $\alpha = 0$. But for $\alpha = 0$ we get $Y' \widetilde{\to} \operatorname{Spec} k$. This concludes the proof. ■

Proof of Theorem 3.10. (2) Let $q_\omega : {}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi} \widetilde{\to} {}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi}$ denote the isomorphism sending $(\mathcal{M}, \mathcal{A}, \kappa_1, \kappa_2)$ to

$$(\mathcal{M}(x), \mathcal{A}(2x), \kappa_1, \kappa_2). \tag{3.73}$$

It preserves the stratification of ${}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi}$ by ${}_D\operatorname{Bun}_{R_\pi}$ introduced in Section 3.4, and we have a commutative diagram

$$\begin{array}{ccc} {}_{x,\mu}\operatorname{Bun}_{R_\pi} & \xrightarrow{q_\omega} & {}_{x,\mu}\operatorname{Bun}_{R_\pi} \\ \downarrow{\scriptstyle \mathrm{ev}^\mu} & & \downarrow{\scriptstyle \mathrm{ev}^\mu} \\ \mathbb{A}^1 \times \operatorname{Pic}\widetilde{X} & \xrightarrow{\mathrm{id}\times\widetilde{q}_\omega} & \mathbb{A}^1 \times \operatorname{Pic}\widetilde{X} \end{array} \tag{3.74}$$

where $\widetilde{q}_\omega$ sends $\mathcal{B}_{\mathrm{ex}}$ to $\mathcal{B}_{\mathrm{ex}}(2\widetilde{x})$ (resp., to $\mathcal{B}_{\mathrm{ex}}(\widetilde{x}_1 + \widetilde{x}_2)$) in the nonsplit (resp., split) case. Our assertion follows from the automorphic property of $A\widetilde{E}$.

(1) We change the notation replacing λ by $-w_0(\lambda)$. In other words, we will establish a canonical isomorphism $H(\mathcal{A}_{-w_0(\lambda)}, \mathcal{B}^0) \widetilde{\to} \mathcal{B}^{i(\lambda)} \otimes \mathcal{N}$ with

$$\mathcal{N} \widetilde{\longrightarrow} \begin{cases} (\widetilde{E}_{\widetilde{x}})^{\otimes\langle\lambda, 2\check{\nu}\rangle}, & \text{the nonsplit case}, \pi(\widetilde{x}) = x, \\ (\widetilde{E}_{\widetilde{x}_1} \otimes \widetilde{E}_{\widetilde{x}_2})^{\otimes\langle\lambda, \check{\nu}\rangle}, & \text{the split case}, \pi^{-1}(x) = \{\widetilde{x}_1, \widetilde{x}_2\}. \end{cases} \tag{3.75}$$

Denote by $\widetilde{K}_\mu$ (resp., by K_μ, ${}_D K$) the $*$-restriction of $H(\mathcal{A}_{-w_0(\lambda)}, \mathcal{B}^0)$ to ${}_{x,\mu}\widetilde{\operatorname{Bun}}_{R_\pi}$ (resp., to ${}_{x,\mu}\operatorname{Bun}_{R_\pi}$, ${}_D\operatorname{Bun}_{R_\pi}$). Here D is $\Lambda_{\mathcal{B}}$-valued divisor on X as in Section 3.4.

By Section 2.3, we know that each perverse cohomology sheaf of ${}_D K$ is $\mathcal{L}$-equivariant. So, ${}_D K = 0$ unless $D = \mu x$ with μ relevant. In particular, $\widetilde{K}_\mu$ is the extension by zero under ${}_{x,\mu}\operatorname{Bun}_{R_\pi} \hookrightarrow {}_{x,\mu}\widetilde{\operatorname{Bun}}_{R_\pi}$.

Since $\mathcal{B}^0$ is self-dual (up to replacing $\widetilde{E}$ by $\widetilde{E}^*$ and ψ by ψ^{-1}), our assertion is reduced to the following lemma. ■

Lemma 3.17. One has $\widetilde{K}_\mu = 0$ unless $\mu \le i(\lambda)$. The complex $\widetilde{K}_\mu$ lives in nonpositive (resp., strictly negative) perverse degrees for $\mu = i(\lambda)$ (resp., for $\mu < i(\lambda)$). One has canonically

$$K_{i(\lambda)} \xrightarrow{\sim} \left(\mathrm{ev}^{i(\lambda)}\right)^* \left(\mathcal{L}_\psi \boxtimes A\widetilde{E}\right) \otimes \mathcal{N} \otimes \bar{\mathbb{Q}}_\ell[1]\left(\frac{1}{2}\right)^{\otimes \dim\, _{x,i(\lambda)}\mathrm{Bun}_{R_\pi}}. \tag{3.76}$$

□

Proof. Write $_x\overline{\mathcal{H}}^\lambda_G$ for the substack of $_x\mathcal{H}_G$ that under the projection $q_G : {_x\mathcal{H}_G} \to \mathrm{Bun}_G$ identifies with

$$\mathrm{Bun}^x_G \times_{G(\mathcal{O}_x)} \overline{\mathrm{Gr}}^\lambda_{G,x} \longrightarrow \mathrm{Bun}_G\,. \tag{3.77}$$

For the diagram

$$_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi} \xleftarrow{p_R} {_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}} \times_{\mathrm{Bun}_G} {_x\overline{\mathcal{H}}^{-w_0(\lambda)}_G} \xrightarrow{q_R} {_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}}, \tag{3.78}$$

we have

$$\mathrm{H}\left(\mathcal{A}_{-w_0(\lambda)}, \cdot\right) = \left(p_R\right)_!\left(q_R^*(\cdot)\widetilde{\boxtimes}\mathcal{A}_{-w_0(\lambda)}\right). \tag{3.79}$$

Let $\mu = (a_1, a_2) \in \Lambda^+_{\mathcal{B}}$. Pick a k-point $\eta \in {_{x,\mu}\mathrm{Bun}_{R_\pi}}$ given by the following collection: a line bundle $\mathcal{B}_{ex}$ on $\widetilde{X}$, for which we set $L_{ex} = \pi_*\mathcal{B}_{ex}$, a modification $L_2 \subset L_{ex}$ of rank-2 vector bundles on X such that the composition is surjective:

$$\mathrm{Sym}^2 L_2 \longrightarrow \mathrm{Sym}^2 L_{ex} \longrightarrow \left(\mathcal{E} \otimes \det L_{ex}\right)\left(D_\pi\right), \tag{3.80}$$

and $a_2 x = \mathrm{div}(L_{ex}/L_2)$, and an exact sequence

$$0 \longrightarrow \mathrm{Sym}^2 L_2 \longrightarrow ? \longrightarrow \mathcal{A} \longrightarrow 0 \tag{3.81}$$

on X, where we have set $\mathcal{A} = (\Omega^{-1} \otimes \mathcal{E} \otimes \det L_2)(D_\pi + a_1 x)$.

The fibre of

$$p_R : {_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}} \times_{\mathrm{Bun}_G} {_x\overline{\mathcal{H}}^{-w_0(\lambda)}_G} \longrightarrow {_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}} \tag{3.82}$$

over η identifies with $\overline{\mathrm{Gr}}^\lambda_{G,x}(\mathcal{F}_G)$, where $\mathcal{F}_G = (\mathcal{M}, \mathcal{A}) \in \mathrm{Bun}_G$ is given by the P-torsor (3.81).

Fix a trivialization $\mathcal{B}_{ex}\otimes\widetilde{\mathcal{O}}_x\widetilde{\to}\widetilde{\mathcal{O}}_x$ and a splitting of (3.81) over Spec $\mathcal{O}_x$. They yield isomorphisms $\mathcal{M}\widetilde{\to}(L_2\oplus L_2^*\otimes\mathcal{A})\mid_{\operatorname{Spec}\mathcal{O}_x}$ and $\mathcal{A}\widetilde{\to}\Omega^{-1}((a_1-a_2)x)\mid_{\operatorname{Spec}\mathcal{O}_x}$. So, the pair

$$\begin{aligned}&\mathcal{M}\otimes\mathcal{O}_x\subset\mathcal{M}_\pi\otimes F_x,\\&\mathcal{A}\otimes\mathcal{O}_x\subset\Omega^{-1}\otimes F_x\end{aligned}\tag{3.83}$$

becomes a point of $\operatorname{Gr}_{G_\pi,x}$ lying in $S^\mu_{R_\pi}$.

Recall that $\mathcal{B}^0$ is clean with respect to the open immersion ${}_{x,0}\operatorname{Bun}_{R_\pi}\subset{}_{x,0}\overline{\operatorname{Bun}}_{R_\pi}$. So, only the stratum (3.38) contributes to K_μ. By **Proposition 3.11**, $K_\mu=0$ unless $\mu\le i(\lambda)$.

Assume that $\mu\le i(\lambda)$. Stratify (3.38) by locally closed subschemes $\operatorname{Gr}^{\lambda'}_{G,x}(\mathcal{F}_G)\cap S^0_{R_\pi}$ with $\lambda'\le\lambda$, where $\lambda'\in\Lambda^+$. The $*$-restriction of $\mathcal{A}_{-w_0(\lambda)}$ under

$$\operatorname{Gr}^{\lambda'}_{G,x}(\mathcal{F}_G)\hookrightarrow\overline{\operatorname{Gr}}^{\lambda}_{G,x}(\mathcal{F}_G)\tag{3.84}$$

is a constant complex placed in usual degree $\le-\dim\operatorname{Gr}^{\lambda'}_{G,x}(\mathcal{F}_G)=-\langle\lambda',2\check\rho\rangle$, the inequality is strict unless $\lambda'=\lambda$. From (3.32) and **Proposition 3.11**, we get

$$-\dim{}_{x,0}\operatorname{Bun}_{R_\pi}-\langle\lambda',2\check\rho\rangle+2\dim\left(\operatorname{Gr}^{\lambda'}_{G,x}(\mathcal{F}_G)\cap S^0_{R_\pi}\right)\le-\dim{}_{x,\mu}\operatorname{Bun}_{R_\pi}.\tag{3.85}$$

So, K_μ is placed in perverse degrees ≤ 0. If $\mu-i(\lambda)$ does not vanish in $\pi_1(G_{ad})$, then, by **Proposition 3.11**, K_μ is placed in strictly negative perverse degrees.

If $i(\lambda)-\mu$ vanishes in $\pi_1(G_{ad})$, let $\lambda'\in\Lambda^+$ be such that $\lambda'\le\lambda$ and $\mu=i(\lambda')$. Then only the stratum (3.39) could contribute to the 0th perverse cohomology sheaf of K_μ. For $\mu<i(\lambda)$ it does not contribute, because the restriction of $q_R^*(\mathcal{B}^0)\widetilde{\boxtimes}\mathcal{A}_{-w_0(\lambda)}$ to (3.39) is a nonconstant local system by **Lemma 3.16**.

If $\mu=i(\lambda)$, then (3.39) is a point scheme by **Proposition 3.11**, and the description of $K_{i(\lambda)}$ follows from the automorphic property of $A\widetilde{E}$. ■

3.12 Properties of the Bessel category

For $\lambda\in\Lambda^+_{\mathcal{B}}$ the perverse sheaf $\mathcal{B}^\lambda$ is not always the extension by zero from ${}_{x,\lambda}\operatorname{Bun}_{R_\pi}$. For example, take $\lambda=(1,1)$ and $\mu=(1,0)$. An easy calculation shows that, over ${}_{x,\lambda}\operatorname{Bun}_{R_\pi}\cup{}_{x,\mu}\operatorname{Bun}_{R_\pi}$, $\mathcal{B}^\lambda$ is a usual sheaf placed in cohomological degree $-\dim{}_{x,\lambda}\operatorname{Bun}_{R_\pi}$.

Now we can show that the category $P^{\mathcal{L}}({}_{x,\infty}\overline{\operatorname{Bun}}_{R_\pi})$ is not semisimple. Recall the stack ${}^{a_1}_{x}\overline{\operatorname{Bun}}_{R_\pi}$ (cf. **Remark 3.7**). Let $\lambda=(1,1)$ and $\mu=(1,0)$. We have a sequence of open embeddings

$${}_{x,\lambda}\operatorname{Bun}_{R_\pi}\overset{j}{\hookrightarrow}{}^{1}_{x,\lambda}\overline{\operatorname{Bun}}_{R_\pi}\overset{\tilde{j}}{\hookrightarrow}{}_{x,\lambda}\overline{\operatorname{Bun}}_{R_\pi},\tag{3.86}$$

where j is obtained from the affine open embedding $\mathcal{W}ald^{x,1}_{\pi} \hookrightarrow \mathcal{W}ald^{x,\le 1}_{\pi}$ by the base change

$$ {}^{1}_{x,\lambda}\overline{\mathrm{Bun}}_{R_\pi} \longrightarrow \mathcal{W}ald^{x,\le 1}_{\pi}. \tag{3.87}$$

Set $\mathcal{B}^{\lambda,\mu} = \widetilde{j}_{!*} j_! (\mathcal{B}^{\lambda} \mid_{x,\lambda\, \mathrm{Bun}_{R_\pi}})$. We get an exact sequence in $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$:

$$ 0 \longrightarrow K \longrightarrow \mathcal{B}^{\lambda,\mu} \longrightarrow \mathcal{B}^{\lambda} \longrightarrow 0. \tag{3.88}$$

If $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$ was semisimple, it would split; this contradicts the fact that the $*$-restriction of $\mathcal{B}^{\lambda}$ to ${}_{x,\mu}\mathrm{Bun}_{R_\pi}$ is not zero.

3.13 Geometric Casselman-Shalika formula

Recall that we write V^{μ} for the irreducible representation of $\check{G}$ of highest weight μ. Let E be a $\check{G}$-local system on $\operatorname{Spec} k$ equipped with an isomorphism

$$ V^{\omega}_{E} \widetilde{\longrightarrow} \begin{cases} \widetilde{E}^{\otimes 2}_{\widetilde{x}}, & \text{the nonsplit case, } \pi(\widetilde{x}) = x, \\ \widetilde{E}_{\widetilde{x}_1} \otimes \widetilde{E}_{\widetilde{x}_2}, & \text{the split case, } \pi^{-1}(x) = \{\widetilde{x}_1, \widetilde{x}_2\}. \end{cases} \tag{3.89}$$

We assign to E the ind-object K_E of $P^{\mathcal{L}}({}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi})$ given by

$$ K_E = \oplus_{\substack{\lambda\in\Lambda^+ \\ \langle\lambda,\check{\nu}\rangle = 0}} \mathcal{B}^{i(\lambda)} \otimes \left(V^{\lambda}\right)^{*}_{E}, \tag{3.90}$$

where $\check{\nu} \in \check{\Lambda}$ is that of Theorem 3.10. For a representation V of $\check{G}$ write $\mathcal{A}_V$ for the object of $\mathrm{Sph}(\mathrm{Gr}_{G,x})$ corresponding to V via the Satake equivalence $\mathrm{Rep}(\check{G}) \widetilde{\to} \mathrm{Sph}(\mathrm{Gr}_{G,x})$.

One formally derives from Theorem 3.10 the following.

Corollary 3.18. For any $V \in \mathrm{Rep}(\check{G})$, there is an isomorphism $\alpha_V : H(\mathcal{A}_V, K_E) \widetilde{\to} K_E \otimes V_E$. For $V, V' \in \mathrm{Rep}(\check{G})$ the following diagram commutes:

$$\begin{array}{ccc} H\left(\mathcal{A}_{V'}, H\left(\mathcal{A}_V, K_E\right)\right) & \xrightarrow{\alpha_V} & H\left(\mathcal{A}_{V'}, K_E \otimes V_E\right) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \alpha_{V'}\otimes \mathrm{id}} \\ H\left(\mathcal{A}_{V\otimes V'}, K_E\right) & \xrightarrow{\alpha_{V\otimes V'}} & K_E \otimes (V\otimes V')_E \end{array} \tag{3.91}$$

where η is the isomorphism (2.24). □

3.14 Multiplicity one

One may view $\mathrm{Gr}_{G_\pi,x}$ as the ind-scheme classifying a G_π-bundle $\mathcal{F}_{G_\pi}$ on X together with a trivialization $\mathcal{F}_{G_\pi} \tilde{\to} \mathcal{F}^0_{G_\pi} \mid_{X-x}$. This yields a map $\mathrm{Gr}_{G_\pi,x} \to {}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}$.

Theorem 3.10 holds also in the case of a finite base field $k = \mathbb{F}_q$. In this case we have the Bessel module BM_τ introduced in Section 1.1, which we now view as the space of functions on $G_\pi(F_x)/G_\pi(\mathcal{O}_x)$ that change by τ under the action of $R_\pi(F_x)$. Let B^λ denote the restriction under

$$G_\pi(F_x)/G_\pi(\mathcal{O}_x) \longrightarrow {}_{x,\infty}\overline{\mathrm{Bun}}_{R_\pi}(k) \tag{3.92}$$

of the trace of Frobenius function of $\mathcal{B}^\lambda$. Then $\{B^\lambda, \lambda \in \Lambda^+_{\mathcal{B}}\}$ is a base of BM_τ. From Theorem 1 it follows that BM_τ is a free module of rank one over the Hecke algebra H_{χ_c}.

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