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Local geometrized Rankin–Selberg method for $\mathrm{GL}(n)$ and its application

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Abstract. We propose a geometric interpretation of the classical Rankin–Selberg method for $\mathrm{GL}(n)$ in the framework of the geometric Langlands program. Our main result is local. For $n = 2$ we apply it to prove a global result that gives a geometric version of the computation of the scalar product of two normalized cusp Hecke eigenforms as a residue of the Rankin–Selberg convolution. © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Méthode de Rankin–Selberg locale géométrisée pour $\mathrm{GL}(n)$ ***et son application***

Résumé. *Nous proposons une interprétation géométrique de la méthode classique de Rankin–Selberg pour* $\mathrm{GL}(n)$ *dans le cadre du programme de Langlands géométrique. Notre résultat principal est local. Pour* $n = 2$ *on en déduit un résultat global qui fournit une version géométrique du calcul du produit scalaire de deux formes automorphes cuspidales comme un résidu de la convolution de Rankin–Selberg.* © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Étant donné une courbe X (*voir* ci-dessous), considérons un $\overline{\mathbf{Q}}_\ell$-faisceau E_0 lisse irréductible de rang 2 sur X. On ne dispose pas d'espace de modules des systèmes locaux ℓ-adiques sur X. Cependant, on peut considérer les déformations de E_0 au sens suivant : si A est une $\overline{\mathbf{Q}}_\ell$-algèbre locale artinienne à corps résiduel $\overline{\mathbf{Q}}_\ell$ alors une *A-déformation de* E_0 est un A-faisceau lisse F sur X muni d'un isomorphisme $F \otimes_A \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} E_0$.

Le système local E_0 admet une déformation universelle (R, E), où R est isomorphe à un anneau des séries formelles sur $\overline{\mathbf{Q}}_\ell$ de dimension $8g - 6$, et E est un R-faisceau lisse de rang 2 sur X muni d'un isomorphisme $E \otimes_R \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} E_0$. (On pensera à un R-faisceau comme un $\overline{\mathbf{Q}}_\ell$-faisceau sur « $X \times \mathrm{Spf}\,(R)$ »).

Note présentée par Jena-Marc Fontaine.

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D'après V. Drinfeld (et.D. Gaitsgory, [3]), on peut associer à E_0 un $\overline{\mathbf{Q}}_\ell$-faisceau pervers irréductible Aut_{E_0} sur le champs Bun_2 de modules des fibrés vectoriels de rang 2 sur X, qui est un vecteur propre des opérateurs de Hecke par rapport à E_0. Notons $\mathrm{Aut}^d_{E_0}$ sa restriction à la composante connexe Bun^d_2 de Bun_2 qui correspond aux fibrés de degré d. La construction de $\mathrm{Aut}^d_{E_0}$ reste valable pour les déformations de E_0, ce qui permet à définir un R-faisceau pervers Aut^d_E sur Bun^d_2.

Les automorphismes scalaires des fibrés vectoriels définissent une action de $\mathbf{G}_m$ sur Bun^d_2 par 2-automorphismes de champs. Notons $\overline{\mathrm{Bun}}^d_2$ le quotient de Bun^d_2 par cette action, de sorte que le morphisme naturel $\mathrm{Bun}^d_2 \to \overline{\mathrm{Bun}}^d_2$ est une $\mathbf{G}_m$-gerbe. Du fait que Aut^d_E est pervers, il est l'image réciproque de $\overline{\mathrm{Aut}}^d_E[-1](-\frac{1}{2})$ sur $\overline{\mathrm{Bun}}^d_2$, où $\overline{\mathrm{Aut}}^d_E$ est un R-faisceau pervers.

THÉORÈME GLOBAL ([6]). – *Pour tout $d \in \mathbf{Z}$ il existe un isomorphisme canonique de $R\widehat{\otimes}R$-modules*

$$\mathrm{H}^i_c\Big(\overline{\mathrm{Bun}}^d_2,\ \mathrm{pr}_1^* \overline{\mathrm{Aut}}^d_{E^*} \otimes_{R\widehat{\otimes}R} \mathrm{pr}_2^* \overline{\mathrm{Aut}}^d_E\Big) \xrightarrow{\sim} \begin{cases} \triangle_* R \text{ si } i = 0, \\ 0 \text{ sinon,} \end{cases}$$

où $\triangle : \mathrm{Spf}(R) \to \mathrm{Spf}(R\widehat{\otimes}R)$ est le morphisme diagonal, et $\mathrm{pr}_1,\ \mathrm{pr}_2 : \mathrm{Spf}(R\widehat{\otimes}R) \to \mathrm{Spf}(R)$ sont les deux projections.

Dans [6] on a déduit ce théorème d'un résultat local pour GL(2). Le but de cette Note est de proposer une généralisation du résultat local correspondant pour $\mathrm{GL}(n)$ (théorème local ci-dessous).

1. Local result for $\mathrm{GL}(n)$

Fix an algebraically closed ground field k of characteristic $p > 0$, a prime $\ell \neq p$, an algebraic closure $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$, and a square root of p in $\overline{\mathbf{Q}}_\ell$.

Let X be a smooth complete connected curve of genus $g \geq 1$ over k. Fix $n > 0$, $d \geq 0$. Denote by $\mathcal{Q}_d$ the stack that classifies the collections

$$\big(0 = L_0 \subset L_1 \subset \cdots \subset L_n \subset L,\ \ (s_i)\big), \tag{1}$$

where $L_n \subset L$ is a modification of rank n vector bundles on X with $\deg(L/L_n) = d$, (L_i) is a complete flag of subbundles on L_n, and $s_i : \Omega^{n-i} \xrightarrow{\sim} L_i/L_{i-1}$ is an isomorphism (($i = 1, \dots, n$), Ω is the canonical invertible sheaf on X). We have a map $\mu : \mathcal{Q}_d \to \mathbf{A}^1_k$ which at the level of k-points sends the above collection to the sum of $n-1$ classes in

$$k \xrightarrow{\sim} \mathrm{Ext}^1\big(\Omega^{n-i-1}, \Omega^{n-i}\big) \xrightarrow{\sim} \mathrm{Ext}^1\big(L_{i+1}/L_i, L_i/L_{i-1}\big)$$

that correspond to the succesive extensions $0 \to L_i/L_{i-1} \to L_{i+1}/L_{i-1} \to L_{i+1}/L_i \to 0$.

Denote by $\mathcal{L}_\psi$ the Artin–Schreier sheaf on $\mathbf{A}^1_k$ corresponding to a nontrivial additive character $\psi : \mathbf{F}_p \to \overline{\mathbf{Q}}^*_\ell$. Let Sh^d_0 be the stack that classifies coherent torsion sheaves on X of degree d. To any smooth $\overline{\mathbf{Q}}_\ell$-sheaf E on X is associated Laumon's $\overline{\mathbf{Q}}_\ell$-perverse sheaf $\mathcal{L}^d_E$ on Sh^d_0 (*cf.* [4], [2], [3], [6]). We have a map $\beta : \mathcal{Q}_d \to \mathrm{Sh}^d_0$ that sends (1) to L/L_n. It is smooth of relative dimension $b = b(n,d) = nd + (1-g)\sum_{i=1}^{n-1} i^2$. So, if E is a smooth $\overline{\mathbf{Q}}_\ell$-sheaf of rank n on X then on $\mathcal{Q}_d$ we have a perverse sheaf

$$\mathcal{F}^d_{E,\psi} = \beta^* \mathcal{L}^d_E \otimes \mu^* \mathcal{L}_\psi[b]\left(\frac{b}{2}\right).$$

Denote by $\mathcal{X}_d$ the stack classifying the collections $(L,(t_i))$, where L is an n-bundle on X,

$$t_i\,:\,\Omega^{(n-1)+(n-2)+\cdots+(n-i)}\hookrightarrow\wedge^i L$$

is an inclusion of $\mathcal{O}_X$-modules ($i=1,\ldots,n$), and $\deg L-\deg\big(\Omega^{(n-1)+(n-2)+\cdots+(n-n)}\big)=d$.

Given an object of $\mathcal{Q}_d$, we get the morphisms

$$t_i\,:\,\Omega^{(n-1)+\cdots+(n-i)}\xrightarrow{\sim}\wedge^i L_i\hookrightarrow\wedge^i L.$$

This defines a map $\varphi:\mathcal{Q}_d\to\mathcal{X}_d$. Let $X^{(d)}$ be the d-th symmetric power of X. Denote by $\phi:\mathcal{X}_d\to X^{(d)}$ the map that sends $(L,(t_i))$ to the divisor $D\in X^{(d)}$ such that t_n induces an isomorphism $\Omega^{(n-1)+\cdots+(n-n)}(D)\xrightarrow{\sim}\wedge^n L$. Denote also by $f:\mathcal{Q}_d\times_{\mathcal{X}_d}\mathcal{Q}_d\to X^{(d)}$ the composition $\mathcal{Q}_d\times_{\mathcal{X}_d}\mathcal{Q}_d\to\mathcal{X}_d\xrightarrow{\phi}X^{(d)}$, where the first map is the natural projection.

Our main result is the following:

LOCAL THEOREM. – *For any smooth $\overline{\mathbf{Q}}_\ell$-sheaves E_1, E_2 of rank n on X there is a canonical isomorphism*

$$f_!\big(\mathcal{F}^d_{E_1,\psi}\boxtimes\mathcal{F}^d_{E_2,\psi^{-1}}\big)\xrightarrow{\sim}\big(E_1\otimes E_2\big)^{(d)}(d)[2d].\tag{2}$$

Remark 1. (i) Recall that the perverse sheaves $\mathcal{F}^d_{E,\psi}$ are the ingredients of the (conjectural) geometric construction of the automorphic sheaves Aut_E on the moduli stack Bun_n of rank n vector bundles on X (*cf.* [2], [5]). Namely, let $\mathcal{M}_d$ be the stack classifying the pairs $\big(\Omega^{n-1}\hookrightarrow L\big)$, where L is an n-bundle on X with $\deg L-\deg\big(\Omega^{(n-1)+(n-2)+\cdots+(n-n)}\big)=d$. We have the forgetful map $\nu:\mathcal{Q}_d\to\mathcal{M}_d$. Laumon's conjecture claims that if E is a smooth irreducible $\overline{\mathbf{Q}}_\ell$-sheaf of rank n on X and $d\geq 0$, then $\nu_!\mathcal{F}^d_{E,\psi}$ is the inverse image of a perverse sheaf Aut_E from Bun_n.

Notice also that ν factors through $\mathcal{Q}_d\xrightarrow{\varphi}\mathcal{X}_d\to\mathcal{M}_d$.

(ii) One can also impose the "Plücker type" relations on a point $(L,(t_i))$ of $\mathcal{X}_d$ as follows. For $k\geq i$ (with $n\geq k\geq i\geq 1$) denote by

$$\alpha_{k,i}\,:\,\wedge^k L\otimes\wedge^i L\longrightarrow\wedge^{k+1}L\otimes\wedge^{i-1}L$$

the morphism defined as in Appendix A. Let $\mathcal{Y}_d\hookrightarrow\mathcal{X}_d$ be the closed substack given by the conditions $\alpha_{k,i}(t_k\otimes t_i)=0$ for $n\geq k\geq i\geq 1$. The map φ factors through $\mathcal{Q}_d\to\mathcal{Y}_d\hookrightarrow\mathcal{X}_d$.

2. Ideas of the proof of Local theorem

Our proof of Local theorem consists of the following parts:

0) We notice that the sheaf $(E_1\otimes E_2)^{(d)}$ has the property: for any (nonempty) open subscheme $j:U\hookrightarrow X^{(d)}$ the natural map $(E_1\otimes E_2)^{(d)}\to(\mathrm{R}^0 j_*)j^*(E_1\otimes E_2)^{(d)}$ is an isomorphism.
1) We establish the isomorphism (2) over the open subscheme ${}^{\mathrm{rss}}X^{(d)}\subset X^{(d)}$ that parametrizes divisors consisting of pairwise different points ("rss" stands for "regular semisimple").
2) We show that $\mathcal{S}^d_{E_1,E_2}\overset{\mathrm{def}}{=}f_!\big(\mathcal{F}^d_{E_1,\psi}\boxtimes\mathcal{F}^d_{E_2,\psi^{-1}}\big)(-d)[-2d]$ is a $\overline{\mathbf{Q}}_\ell$-sheaf, and for any open subscheme $j:U\hookrightarrow X^{(d)}$ the map $\mathcal{S}^d_{E_1,E_2}\to(\mathrm{R}^0 j_*)j^*\mathcal{S}^d_{E_1,E_2}$ is an isomorphism.

The point 2) is local with respect to the étale topology. Its proof is divided into several steps:

2.1) If E_1, E_2, E_1', E_2' are smooth $\overline{\mathbf{Q}}_\ell$-sheaves of rank n on X, $X^{(d),D}\to X^{(d)}$ is the (strict) henselization of $X^{(d)}$ at a k-point $\mathrm{Spec}\,k\xrightarrow{D}X^{(d)}$ then we show that the restrictions of $\mathcal{S}^d_{E_1,E_2}$ and of $\mathcal{S}^d_{E_1',E_2'}$ to $X^{(d),D}$ are isomorphic.

2.2) We prove Local theorem under the additional assumption: $E_i = \sum_{j=1}^{n} E_{ij}$ $(i = 1, 2)$, where E_{ij} are smooth $\overline{\mathbf{Q}}_\ell$-sheaves of rank 1 on X. (In particular, combining 2.1) and 2.2) we learn that $\mathcal{S}^d_{E_1,E_2}$ is a sheaf, not a complex.)

2.3) We conclude with the following simple observation. Let Y be a k-scheme of finite type, $\mathcal{F}_1$, $\mathcal{F}_2$ be constructible $\overline{\mathbf{Q}}_\ell$-sheaves on Y. Suppose that for any k-point $\operatorname{Spec} k \xrightarrow{y} Y$ the restrictions of $\mathcal{F}_1$ and of $\mathcal{F}_2$ to Y^y are isomorphic where Y^y is the (strict) henselization of Y at y. If for an open subscheme $j : U \hookrightarrow Y$ the natural map $\mathcal{F}_1 \to (\mathrm{R}^0 j_*) j^* \mathcal{F}_1$ is an isomorphism then the same holds for $\mathcal{F}_2$.

3. Reduction to Proposition 1

Proof of 1). – Over the preimage of ${}^{\mathrm{rss}}X^{(d)}$ under $\operatorname{div} : \mathrm{Sh}^d_0 \to X^{(d)}$ we have $\mathcal{L}^d_E \xrightarrow{\sim} \operatorname{div}^* E^{(d)}$ canonically. Therefore, by the projection formulae, the complex $f_!(\mathcal{F}^d_{E_1,\psi} \boxtimes \mathcal{F}^d_{E_2,\psi^{-1}})$ is isomorphic to

$$E_1^{(d)} \otimes E_2^{(d)} \otimes \phi_!((\varphi_! \mu^* \mathcal{L}_\psi) \otimes (\varphi_! \mu^* \mathcal{L}_{\psi^{-1}}))(b)[2b]$$

over ${}^{\mathrm{rss}}X^{(d)}$. Denote by ${}^0\mathcal{Y}_d \subset \mathcal{Y}_d$ the open substack given by the conditions: the image of t_i is a *line subbundle* in $\bigwedge^i L$ $(i = 1, \dots, n-1)$. Then over ${}^0\mathcal{Y}_d$ the map $\varphi : \mathcal{Q}_d \to \mathcal{Y}_d$ is an isomorphism, and one checks that $\varphi_! \mu^* \mathcal{L}_\psi$ is the extension by zero of its restriction to ${}^0\mathcal{Y}_d$.

Since over ${}^{\mathrm{rss}}X^{(d)}$ the natural map $E_1^{(d)} \otimes E_2^{(d)} \to (E_1 \otimes E_2)^{(d)}$ is an isomorphism, our assertion follows from the fact that the restriction of $\phi : \mathcal{Y}_d \to X^{(d)}$ to ${}^0\mathcal{Y}_d$ is a composition of generalized vector fibrations whose sum of ranks equals $b - d$ [(1)]. □

Proof of 2.1). – The map f is the composition $\mathcal{Q}_d \times_{\mathcal{Y}_d} \mathcal{Q}_d \xrightarrow{\beta \times \beta} \mathrm{Sh}^d_0 \times_{X^{(d)}} \mathrm{Sh}^d_0 \xrightarrow{\operatorname{div} \times \operatorname{div}} X^{(d)}$. So,

$$f_!(\mathcal{F}^d_{E_1,\psi} \boxtimes \mathcal{F}^d_{E_2,\psi^{-1}}) \xrightarrow{\sim} (\operatorname{div} \times \operatorname{div})_!((\mathcal{L}^d_{E_1} \boxtimes \mathcal{L}^d_{E_2}) \otimes \mathcal{N}^d),$$

where $\mathcal{N}^d = (\beta \times \beta)_!(\mu^* \mathcal{L}_\psi \boxtimes \mu^* \mathcal{L}_{\psi^{-1}})(b)[2b]$. Our assertion follows now from the fact that the restrictions of $\mathcal{L}^d_E$ and of $\mathcal{L}^d_{E'}$ to $X^{(d),D} \times_{X^{(d)}} \mathrm{Sh}^d_0$ are isomorphic for any smooth $\overline{\mathbf{Q}}_\ell$-sheaves E, E' on X of the same rank (*cf.* [6], Lemma 29, Section 3.2.7). □

Denote by I_d the set of n-tuples $\bar{c} = (c_1, \dots, c_n)$ with $c_i \in \mathbf{Z}_+$, $\sum c_i = d$. Let $\mathcal{F}\mathrm{l}^{\bar{c}}$ be the stack of collections $(0 = F^0 \subset F^1 \subset \dots \subset F^n)$, where F^i is a coherent torsion sheaf on X, and $\deg(F^i/F^{i-1}) = c_i$ $(i = 1, \dots, n)$. Put also $X^{(\bar{c})} = X^{(c_1)} \times \dots \times X^{(c_n)}$. By $\operatorname{div}^{\bar{c}} : \mathcal{F}\mathrm{l}^{\bar{c}} \to X^{(\bar{c})}$ we will denote the composition $\mathcal{F}\mathrm{l}^{\bar{c}} \to \mathrm{Sh}^{c_1}_0 \times \dots \times \mathrm{Sh}^{c_n}_0 \xrightarrow{\operatorname{div} \times \dots \times \operatorname{div}} X^{(\bar{c})}$.

Summation of divisors yields a map $X^{(\bar{c})} \to X^{(d)}$, and for a pair $c \stackrel{\mathrm{def}}{=} (\bar{c}, \bar{c}') \in I_d \times I_d$ we put $V^c = X^{(\bar{c})} \times_{X^{(d)}} X^{(\bar{c}')}$. We also set $\mathcal{Q}^{\bar{c}} = \mathcal{Q}_d \times_{\mathrm{Sh}^d_0} \mathcal{F}\mathrm{l}^{\bar{c}}$.

Let $f^c : \mathcal{Q}^{\bar{c}} \times_{\mathcal{Y}_d} \mathcal{Q}^{\bar{c}'} \to V^c$ be the composition $\mathcal{Q}^{\bar{c}} \times_{\mathcal{Y}_d} \mathcal{Q}^{\bar{c}'} \to \mathcal{F}\mathrm{l}^{\bar{c}} \times_{X^{(d)}} \mathcal{F}\mathrm{l}^{\bar{c}'} \xrightarrow{\operatorname{div}^{\bar{c}} \times \operatorname{div}^{\bar{c}'}} V^c$.

By abuse of notation, the composition $\mathcal{Q}^{\bar{c}} \to \mathcal{Q}_d \xrightarrow{\mu} \mathbf{A}^1$ will also be denoted by μ. Set

$$\mathcal{S}^c = (f^c)_!(\mu^* \mathcal{L}_\psi \boxtimes \mu^* \mathcal{L}_{\psi^{-1}})(b - d)[2b - 2d]. \tag{3}$$

The point 2.2) easily follows from:

PROPOSITION 1. – *There is a canonical isomorphism $\mathcal{S}^c[d] \xrightarrow{\sim} \mathrm{IC}$, where* IC *is the intersection cohomology sheaf on V^c.*

In the rest of the paper we sketch the proof of Proposition 1.

4. A normalization of V^c

Denote by J_d the set of $n \times n$-matrices $e = (e_i^j)$ with $e_i^j \in \mathbf{Z}_+$, $\sum_{i,j} e_i^j = d$. We have a map $h : J_d \to I_d \times I_d$ that sends e to $c = (\bar{c}, \bar{c}')$, where $c_i = \sum_j e_i^j$ and $c'_j = \sum_i e_i^j$. For $e \in J_d$ put $Y^e = \prod_{i,j} X^{(e_i^j)}$. So, Y^e classifies the matrices of effective divisors (D_i^j) on X such that $\deg D_i^j = e_i^j$. We have a map

$$\mathrm{norm} : \bigsqcup_{e \in h^{-1}(c)} Y^e \longrightarrow V^c$$

that sends a matrix (D_i^j) to the collection $((D_i), (D'_i))$, where $D_i = \sum_j D_i^j$ and $D'_j = \sum_i D_i^j$.

LEMMA 1. – 1) *The scheme V^c is of pure dimension d, its irreducible components are numbered by the set $h^{-1}(c)$. Namely, to $e \in h^{-1}(c)$ there corresponds the component* $\mathrm{norm}(Y^e)$.

2) *The map* norm *is the normalization of V^c (more precisely, it is a finite morphism, an isomorphism over an open dense subscheme of V^c, and the scheme $\bigsqcup_{e \in h^{-1}(c)} Y^e$ is smooth). In particular,* $\mathrm{norm}_* \overline{\mathbf{Q}}_\ell[d] \xrightarrow{\sim} \mathrm{IC}$.

5. Highest direct image

LEMMA 2. – *The fibres of f^c are of pure dimension $b - d$, and the complex $\mathcal{S}^c$ is, in fact, a $\overline{\mathbf{Q}}_\ell$-sheaf. So, $\mathcal{S}^c$ is the* highest *direct image of $(\mu^* \mathcal{L}_\psi \boxtimes \mu^* \mathcal{L}_{\psi^{-1}})(b - d)$ with respect to f^c.*

Idea of proof. – We stratify $\mathcal{Q}^{\bar{c}}$ by fixing all the degrees $\deg(L_i^j)$, where L_i^j is the subbundle of L_n^j generated by L_i. (Here we have denoted by L_n^j the preimage of F^j under $L \to L/L_n = F^n$.) On the stack $\mathcal{Q}^{\bar{c}} \times_{\mathcal{Y}_d} \mathcal{Q}^{\bar{c}'}$ we get the stratification induced by both stratifications of $\mathcal{Q}^{\bar{c}}$ and of $\mathcal{Q}^{\bar{c}'}$.

We calculate the direct image (3) with respect to this stratification. The contributions of all strata are sheaves, placed in the same degree. Therefore, $\mathcal{S}^c$ is also a sheaf. □

6. End of the proof

Denote by g^c the restriction of f^c to $\mathcal{Q}^{\bar{c}} \times_{\mathcal{Q}_d} \mathcal{Q}^{\bar{c}'} \subset \mathcal{Q}^{\bar{c}} \times_{\mathcal{Y}_d} \mathcal{Q}^{\bar{c}'}$.

LEMMA 3. – *The natural map $\mathrm{R}^{2(b-d)}(f^c)_!(\mu^* \mathcal{L}_\psi \boxtimes \mu^* \mathcal{L}_{\psi^{-1}}) \to \mathrm{R}^{2(b-d)}(g^c)_! \overline{\mathbf{Q}}_\ell$ is an isomorphism.*

Put $W^c = \mathcal{F}l^{\bar{c}} \times_{\mathrm{Sh}_0^d} \mathcal{F}l^{\bar{c}'}$ and denote by $\mathrm{div}^c : W^c \to V^c$ the map $\mathrm{div}^{\bar{c}} \times \mathrm{div}^{\bar{c}'}$. Then $g^c = \mathrm{div}^c \circ \pi$, where π is the projection $\mathcal{Q}^{\bar{c}} \times_{\mathcal{Q}_d} \mathcal{Q}^{\bar{c}'} = \mathcal{Q}_d \times_{\mathrm{Sh}_0^d} W^c \to W^c$.

Denote by ${}^{\leq n}\mathrm{Sh}_0^d \subset \mathrm{Sh}_0^d$ the open substack classifying sheaves $F \in \mathrm{Sh}_0^d$ such that all geometric fibres of F have dimension $\leq n$. Put ${}^{\leq n}W^c = W^c \times_{\mathrm{Sh}_0^d} {}^{\leq n}\mathrm{Sh}_0^d$.

LEMMA 4. – 1) *π is smooth, and its image equals ${}^{\leq n}W^c$. The fibres of π over points of ${}^{\leq n}W^c$ are connected and have dimension b.*

2) *div^c is of relative dimension $\leq -d$. The restriction of div^c to $W^c \setminus {}^{\leq n}W^c$ has relative dimension $< -d$.*

So, $\mathrm{R}^{2(b-d)}(g^c)_!\overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathrm{R}^{-2d}(\mathrm{div}^c)_!\overline{\mathbf{Q}}_\ell(-b)$ canonically. Now using Lemma 2 we reduce Proposition 1 to the following:

LEMMA 5. – *There is a canonical isomorphism* $\mathrm{R}^{-2d}(\mathrm{div}^c)_!\,\overline{\mathbf{Q}}_\ell(-d) \xrightarrow{\sim} \mathrm{norm}_*\overline{\mathbf{Q}}_\ell$.

Idea of the proof of Lemma 5. – We stratify the stack W^c by locally closed substacks $\mathcal{U}^e \hookrightarrow W^c$ indexed by $e \in h^{-1}(c)$. A point $\big(F^1 \subset \cdots \subset F^n = F,\ (F^1)' \subset \cdots \subset (F^n)' = F\big)$ of W^c lies in $\mathcal{U}^e$ if $\deg\big(F^i \cap (F^j)'\big) = \sum_{k\le i,\, m\le j} e_k^m$ for $i, j = 1, \ldots, n$.

Calculating the direct image $\mathrm{R}^{-2d}(\mathrm{div}^c)_!\,\overline{\mathbf{Q}}_\ell(-d)$ with respect to this stratification, we show that it is a sheaf with a filtration parametrized by the set $h^{-1}(c)$ with successive quotients being $(\mathrm{norm}^e)_*\overline{\mathbf{Q}}_\ell$, where $\mathrm{norm}^e : Y^e \to V^c$ is the restriction of norm. Further, any filtration with these successive quotients degenerates canonically into a direct sum. Indeed:

(i) the different successive quotients are supported on different irreducible components of V^c, so that our filtration degenerates into a direct sum over some open dense subscheme of V^c;

(ii) the sheaf $(\mathrm{norm}^e)_*\overline{\mathbf{Q}}_\ell[d]$ is perverse, it is the Goresky–MacPherson extension of its restriction to any open dense subscheme of V^c;

(iii) the property "perverse and the Goresky–MacPherson extension of its restriction to a given open subscheme of V^c" is preserved for extensions. □

Appendix A. Linear algebra

Let V be a vector space of dimension n. For $n \ge k \ge i \ge 1$ let $\alpha_{k,i} : \wedge^k V \otimes \wedge^i V \longrightarrow \wedge^{k+1} V \otimes \wedge^{i-1} V$ be the "contraction" that send $u\otimes(v_1\wedge v_2\wedge\cdots\wedge v_i)$ to $\sum_{j=1}^{i}(-1)^j(u\wedge v_j)\otimes\big(v_1\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_i\big)$.

LEMMA 6. – *Given nonzero elements $t_i \in \wedge^i V$ for $1 \le i \le n$, the following are equivalent:*

1) *there exists a complete flag of vector subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $t_i \in \wedge^i V_i \subset \wedge^i V$;*

2) *for $n \ge k \ge i \ge 1$ we have $\alpha_{k,i}(t_k \otimes t_i) = 0$.*

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(1) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We call f a *generalized vector fibration of rank m* if locally in smooth topology on $\mathcal{Y}$ there exists a homomorphism $L \to L'$ of locally free coherent sheaves on $\mathcal{Y}$ such that f is identified with $L'/L \to \mathcal{Y}$, the quotient L'/L being taken in stack sense, and $\mathrm{rk}\, L' - \mathrm{rk}\, L = m$.

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