Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# COMPATIBILITY OF THE THETA CORRESPONDENCE WITH THE WHITTAKER FUNCTORS

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Tome 139
Fascicule 1

2011

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 75-88

*Bull. Soc. math. France*
**139** (1), 2011, p. 75–88

# COMPATIBILITY OF THE THETA CORRESPONDENCE WITH THE WHITTAKER FUNCTORS

BY VINCENT LAFFORGUE & SERGEY LYSENKO

ABSTRACT. — We prove that the global geometric theta-lifting functor for the dual pair $(H, G)$ is compatible with the Whittaker functors, where $(H, G)$ is one of the pairs $(\mathbb{SO}_{2n}, \mathbb{Sp}_{2n})$, $(\mathbb{Sp}_{2n}, \mathbb{SO}_{2n+2})$ or $(\mathbb{GL}_n, \mathbb{GL}_{n+1})$. That is, the composition of the theta-lifting functor from $H$ to $G$ with the Whittaker functor for $G$ is isomorphic to the Whittaker functor for $H$.

RÉSUMÉ (*Compatibilité de la thêta-correspondence avec les foncteurs de Whittaker*)
Nous démontrons que le foncteur géométrique de théta-lifting pour la paire duale $(H, G)$ est compatible avec la normalisation de Whittaker, où $(H, G)$ est l'une des paires $(\mathbb{SO}_{2n}, \mathbb{Sp}_{2n})$, $(\mathbb{Sp}_{2n}, \mathbb{SO}_{2n+2})$ ou $(\mathbb{GL}_n, \mathbb{GL}_{n+1})$. Plus précisément, le composé du foncteur de théta-lifting de $H$ vers $G$ et du foncteur de Whittaker pour $G$ est isomorphe au foncteur de Whittaker pour $H$.

We prove in this note that the global geometric theta lifting for the pair $(H, G)$ is compatible with the Whittaker normalization, where $(H, G) = (\mathbb{SO}_{2n}, \mathbb{Sp}_{2n})$, $(\mathbb{Sp}_{2n}, \mathbb{SO}_{2n+2})$, or $(\mathbb{GL}_n, \mathbb{GL}_{n+1})$. More precisely, let $k$ be an algebraically closed field of characteristic $p > 2$. Let $X$ be a smooth projective connected curve over $k$. For a stack $S$ write $\mathrm{D}(S)$ for the derived category of étale constructible $\bar{\mathbb{Q}}_\ell$-sheaves on $S$. For a reductive group $G$ over $k$ write

*Texte reçu le 3 février 2009, accepté le 26 novembre 2009*

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2000 Mathematics Subject Classification. — 11R39; 14H60.
Key words and phrases. — Geometric Langlands, Whittaker functor, theta-lifting.

$\mathrm{Bun}_G$ for the stack of $G$-torsors on $X$. The usual Whittaker distribution admits a natural geometrization $\mathrm{Whit}_G : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Spec}\, k)$.

We construct an isomorphism of functors between $\mathrm{Whit}_G \circ F$ and $\mathrm{Whit}_H$, where $F : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ is the theta lifting functor (cf. Theorems 1, 2 and 3).

This result at the level of functions (on $\mathrm{Bun}_H(k)$ and $\mathrm{Bun}_G(k)$ when $k$ is a finite field) is well known since a long time and the geometrization of the argument is straightforward. We wrote this note for the following reason.

Our proof holds also for $k = \mathbb{C}$ in the setting of $D$-modules. In this case for a reductive group $G$, Beilinson and Drinfeld proposed a conjecture, which (in a form that should be made more precise) says that there exists an equivalence $\alpha_G$ between the derived category of D-modules on $\mathrm{Bun}_G$ and the derived category of $\mathscr{O}$-modules on $\mathrm{Loc}_{\check{G}}$. Here $\mathrm{Loc}_{\check{G}}$ is the stack of $\check{G}$-local systems on $X$, and $\check{G}$ is the Langlands dual group to $G$. Moreover, $\mathrm{Whit}_G$ should be the composition $\mathrm{D}(D\text{-mod}(\mathrm{Bun}_G)) \stackrel{\alpha_G}{\to} \mathrm{D}(\mathrm{Loc}_{\check{G}}, \mathscr{O}) \stackrel{\mathrm{R}\Gamma}{\to} \mathrm{D}(\mathrm{Spec}\, \mathbb{C})$.

A morphism $\gamma : \check{H} \to \check{G}$ gives rise to the extension of scalars morphism $\bar{\gamma} : \mathrm{Loc}_{\check{H}} \to \mathrm{Loc}_{\check{G}}$. The functor $\bar{\gamma}_* : \mathrm{D}(\mathrm{Loc}_{\check{H}}, \mathscr{O}) \to \mathrm{D}(\mathrm{Loc}_{\check{G}}, \mathscr{O})$ should give rise to the Langlands functoriality functor

$$\gamma_L = \alpha_G^{-1} \circ \bar{\gamma}_* \circ \alpha_H : \mathrm{D}(D\text{-mod}(\mathrm{Bun}_H)) \to \mathrm{D}(D\text{-mod}(\mathrm{Bun}_G))$$

compatible with the action of Hecke functors.

In the cases $(H, G) = (\mathbb{SO}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathbb{SO}_{2n+2})$ or $(\mathbb{GL}_n, \mathbb{GL}_{n+1})$ the compatibility of the theta lifting functor $F : \mathrm{D}(D\text{-mod}(\mathrm{Bun}_H)) \to D(D\text{-mod}(\mathrm{Bun}_G))$ with the Hecke functors ([6]) and the compatibility of $F$ with the Whittaker functors (proved in this paper) indicate that $F$ should be the Langlands functoriality functor.

Notation. From now on $k$ denotes an algebraically closed field of characteristic $p > 2$, all the stacks we consider are defined over $k$. Let $X$ be a smooth projective curve of genus $g$. Fix a prime $\ell \neq p$ and a non-trivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$, and denote by $\mathscr{L}_\psi$ the corresponding Artin-Schreier sheaf on $\mathbb{A}^1$. Since $k$ is algebraically closed, we systematically ignore the Tate twists.

For a $k$-stack locally of finite type $S$ write simply $\mathrm{D}(S)$ for the category introduced in ([3], Remark 3.21) and denoted $\mathrm{D}_c(S, \bar{\mathbb{Q}}_\ell)$ in *loc.cit.* It should be thought of as the unbounded derived category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on $S$. For $* = +, -, b$ we have the full triangulated subcategory $\mathrm{D}^*(S) \subset \mathrm{D}(S)$ denoted $\mathrm{D}_c^*(S, \bar{\mathbb{Q}}_\ell)$ in *loc.cit.* Write $\mathrm{D}^*(S)_! \subset \mathrm{D}^*(S)$ for the full subcategory of objects which are extensions by zero from some open substack of finite type. Write $\mathrm{D}^{\prec}(S) \subset \mathrm{D}(S)$ for the full subcategory of complexes $K \in D(S)$ such that for any open substack $U \subset S$ of finite type we have $K \mid_U \in D^-(U)$.

For any vector space (or bundle) $E$, we define $\mathrm{Sym}^2(E)$ and $\Lambda^2(E)$ as quotients of $E\otimes E$ (and denote by $x.y$ and $x\wedge y$ the images of $x\otimes y$) and we will use in this article the embeddings

$$\begin{array}{lcl} \mathrm{Sym}^2(E) \to & E\otimes E \quad \text{and} \quad \Lambda^2(E) \to & E\otimes E \\ x.y \quad \mapsto & \frac{x\otimes y+y\otimes x}{2} \qquad\qquad x\wedge y \ \mapsto & \frac{x\otimes y-y\otimes x}{2} \end{array} \tag{1}$$

## 1. Whittaker functors

Let $G$ be a reductive group over $k$. We pick a maximal torus and a Borel subgroup $T\subset B\subset G$ and we denote by $\Delta_G$ the set of simple roots of $G$. The Whittaker functor

$$\mathrm{Whit}_G : \mathrm{D}^{\prec}(\mathrm{Bun}_G)\to \mathrm{D}^{-}(\mathrm{Spec}\, k)$$

is defined as follows. Write $\Omega$ for the canonical line bundle on $X$. Pick a $T$-torsor $\mathcal{F}_T$ on $X$ with a trivial conductor, that is, for each $\check\alpha\in\Delta_G$ it is equipped with an isomorphism $\delta_{\check\alpha} : \mathcal{L}^{\check\alpha}_{\mathcal{F}_T}\to\Omega$. Here $\mathcal{L}^{\check\alpha}_{\mathcal{F}_T}$ is the line bundle obtained from $\mathcal{F}_T$ via extension of scalars $T\stackrel{\check\alpha}{\to}\mathbb{G}_m$. Let $\mathrm{Bun}_N^{\mathcal{F}_T}$ be the stack classifying a $B$-torsor $\mathcal{F}_B$ together with an isomorphism

$$\zeta : \mathcal{F}_B\times_B T\widetilde{\to}\mathcal{F}_T$$

Let $\epsilon : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathbb{A}^1$ be the evaluation map (cf. [1], 4.3.1 where it is denoted $ev_{\tilde\omega}$). Just recall that for each $\check\alpha\in\Delta_G$ the class of the extension of $\mathcal{O}$ by $\Omega$ associated to $\mathcal{F}_B$, $\zeta$ and $\delta_{\check\alpha}$ gives $\epsilon_{\check\alpha} : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathbb{A}^1$ and that $\epsilon=\sum_{\check\alpha\in\Delta_G}\epsilon_{\check\alpha}$. Write $\pi : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathrm{Bun}_G$ for the extension of scalars $(\mathcal{F}_B,\zeta)\mapsto\mathcal{F}_B\times_B G$. Set $P^0_\psi=\epsilon^*\mathcal{L}_\psi[d_N]$, where $d_N=\dim\mathrm{Bun}_N^{\mathcal{F}_T}$. Let $d_G=\dim\mathrm{Bun}_G$. As in ([7], Definition 2) for $\mathcal{F}\in\mathrm{D}^{\prec}(\mathrm{Bun}_G)$ set

$$\mathrm{Whit}_G(\mathcal{F})=\mathrm{R}\Gamma_c(\mathrm{Bun}_N^{\mathcal{F}_T}, P^0_\psi\otimes\pi^*(\mathcal{F}))[-d_G] \tag{2}$$

Remark 1. — The collection $(\mathcal{F}_T,(\delta_{\check\alpha})_{\check\alpha\in\Delta_G})$ as above exists, because $k$ is algebraically closed, and one can take $\mathcal{F}_T=(\sqrt{\Omega})^{2\rho}$ for some square root $\sqrt{\Omega}$ of $\Omega$. One has an exact sequence of abelian group schemes $1\to Z\to T\stackrel{\prod\check\alpha}{\to}\mathbb{G}_m^{\Delta_G}\to 1$, where $Z$ denotes the center of $G$. So, two choices of the collection $(\mathcal{F}_T,(\delta_{\check\alpha})_{\check\alpha\in\Delta_G})$ are related by a point of $\mathrm{Bun}_Z(k)$ and the associated Whittaker functors are isomorphic up to the automophism of $\mathrm{Bun}_G$ given by tensoring with the corresponding $Z$-torsor.

Remark 2. — When $\mathcal{F}_T$ is fixed, the functor $\mathrm{Whit}_G : \mathrm{D}^{\prec}(\mathrm{Bun}_G)\to\mathrm{D}^{-}(\mathrm{Spec}\, k)$ does not depend, up to isomorphism, on the choice of the isomorphisms $(\delta_{\check\alpha})_{\check\alpha\in\Delta_G}$. That is, for any $(\lambda_{\check\alpha})_{\check\alpha\in\Delta_G}\in(k^*)^{\Delta_G}$, the functors associated to $(\mathcal{F}_T,(\delta_{\check\alpha})_{\check\alpha\in\Delta_G})$ and $(\mathcal{F}_T,(\lambda_{\check\alpha}\delta_{\check\alpha})_{\check\alpha\in\Delta_G})$ are isomorphic. Indeed, the two diagrams $\mathrm{Bun}_G \overset{\pi}{\leftarrow} \mathrm{Bun}_N^{\mathcal{F}_T} \overset{\epsilon}{\to} \mathbb{A}^1$ associated to $(\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ and $(\lambda_{\check{\alpha}}\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ are isomorphic for the following reason. Since $k$ is algebraically closed, $T(k)\to(k^*)^{\Delta_G}$ is surjective. We pick any preimage $\gamma\in T(k)$ of $(\lambda_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ and get the automorphism $(\mathcal{F}_B,\zeta)\mapsto(\mathcal{F}_B,\gamma\zeta)$ of $\mathrm{Bun}_N^{\mathcal{F}_T}$, which together with the idendity of $\mathrm{Bun}_G$ and $\mathbb{A}^1$ intertwines the two diagrams.

**1.1. Whittaker functor for $\mathbb{GL}_n$.** — For $i,j\in\mathbb{Z}$ with $i\le j$ we denote by $\mathcal{N}_{i,j}$ the stack classifying the extensions of $\Omega^i$ by $\Omega^{i+1}$ ... by $\Omega^j$, i.e. classifying a vector bundle $E_{j-i+1}$ on $X$ with a complete flag of vector subbundles $0=E_0\subset E_1\subset ...\subset E_{j-i+1}$ together with isomorphisms $E_{k+1}/E_k\simeq\Omega^{j-k}$ for $k=0,...,j-i$. Write $\epsilon_{i,j}:\mathcal{N}_{i,j}\to\mathbb{A}^1$ for the map given by the sum of the classes in $\mathrm{Ext}^1(\mathcal{O},\Omega)\widetilde{\to}\mathbb{A}^1$ of the extensions $0\to E_{k+1}/E_k\to E_{k+2}/E_k\to E_{k+2}/E_{k+1}\to 0$ for $k=0,\dots,j-i-1$.

For $G=\mathbb{GL}_n$, we consider the diagram $\mathrm{Bun}_n\overset{\pi_{0,n-1}}{\leftarrow}\mathcal{N}_{0,n-1}\overset{\epsilon_{0,n-1}}{\to}\mathbb{A}^1$, where $\pi_{0,n-1}:\mathcal{N}_{0,n-1}\to\mathrm{Bun}_n$ is $(0=E_0\subset\cdots\subset E_n)\mapsto E_n$. This diagram is isomorphic to the diagram $\mathrm{Bun}_G \overset{\pi}{\leftarrow} \mathrm{Bun}_N^{\mathcal{F}_T} \overset{\epsilon}{\to} \mathbb{A}^1$ associated to the choice of $\mathcal{F}_T$ whose image in $\mathrm{Bun}_n$ is $\Omega^{n-1}\oplus\Omega^{n-2}\oplus\cdots\oplus\mathcal{O}$.

Therefore the functor $\mathrm{Whit}_{\mathbb{GL}_n}:\mathrm{D}^{\prec}(\mathrm{Bun}_n)\to\mathrm{D}^-(\mathrm{Spec}\,k)$ associated to the above choice of $\mathcal{F}_T$ is given by

$$\mathrm{Whit}_{\mathbb{GL}_n}(\mathcal{F})=\mathrm{R}\Gamma_c(\mathcal{N}_{0,n-1},\epsilon_{0,n-1}^*(\mathcal{L}_\psi)\otimes\pi_{0,n-1}^*(\mathcal{F}))[\dim\mathcal{N}_{0,n-1}-\dim\mathrm{Bun}_n].$$

REMARK 3. — If $E$ is an irreducible rank $n$ local system on $X$ let $\mathrm{Aut}_E$ be the corresponding automorphic sheaf on $\mathrm{Bun}_n$ (cf. [2]) normalized to be perverse. Then $\mathrm{Aut}_E$ is equipped with a canonical isomorphism $\mathrm{Whit}_{\mathbb{GL}_n}(\mathrm{Aut}_E)\widetilde{\to}\bar{\mathbb{Q}}_\ell$. This is our motivation for the above shift normalization in (2).

**1.2. Whittaker functor for $\mathbb{Sp}_{2n}$.** — Write $G_n$ for the group scheme on $X$ of automorphisms of $\mathcal{O}^n\oplus\Omega^n$ preserving the natural symplectic form $\wedge^2(\mathcal{O}^n\oplus\Omega^n)\to\Omega$. The stack $\mathrm{Bun}_{G_n}$ of $G_n$-torsors on $X$ can be seen as the stack classifying vector bundles $M$ over $X$ of rank $2n$ equipped with a non-degenerate symplectic form $\Lambda^2M\to\Omega$.

The diagram $\mathrm{Bun}_{G_n}\overset{\pi_{G_n}}{\leftarrow}\mathcal{N}_{G_n}\overset{\epsilon_{G_n}}{\to}\mathbb{A}^1$ constructed in the next definition is isomorphic to the diagram $\mathrm{Bun}_G \overset{\pi}{\leftarrow} \mathrm{Bun}_N^{\mathcal{F}_T} \overset{\epsilon}{\to} \mathbb{A}^1$ associated, for $G=G_n$, to the choice of $\mathcal{F}_T$ whose image in $\mathrm{Bun}_{G_n}$ is $L\oplus L^*\otimes\Omega$ with $L=\Omega^n\oplus\Omega^{n-1}\oplus\cdots\oplus\Omega$ (with the natural symplectic structure for which $L$ and $L^*\otimes\Omega$ are lagrangians).

DEFINITION 1. — Let $\mathcal{N}_{G_n}$ be the stack classifying $((L_1,...,L_n),E)$, where $(0=L_0\subset L_1\subset...\subset L_n)\in\mathcal{N}_{1,n}$, and $E$ is an extension of $\mathcal{O}_X$-modules

$$0\to\mathrm{Sym}^2L_n\to E\to\Omega\to 0 \quad (3)$$

We associate to (3) an extension

$$0 \to L_n \to M \to L_n^* \otimes \Omega \to 0 \tag{4}$$

with $M \in \mathrm{Bun}_{G_n}$ and $L_n$ lagrangian as follows. Equip $L_n \oplus L_n^* \otimes \Omega$ with the symplectic form $(l, l^*), (u, u^*) \mapsto \langle l, u^* \rangle - \langle u, l^* \rangle$ for $l, u \in L, l^*, u^* \in L^*$. Here $\langle ., . \rangle$ is the canonical paring between $L_n$ and $L_n^*$. Using (1), we consider (3) as a torsor on $X$ under the sheaf of symmetric morphisms $L_n^* \otimes \Omega \to L_n$. The latter sheaf acts naturally on $L_n \oplus L_n^* \otimes \Omega$ preserving the symplectic form. Then $M$ is the twisting of $L_n \oplus L_n^* \otimes \Omega$ by the above torsor. This defines a morphism $\pi_{G_n} : \mathscr{N}_{G_n} \to \mathrm{Bun}_{G_n}$.

Note that the extension of $\Omega$ by $L_n \otimes L_n$ obtained from (4) is the push-forward of (3) by the embedding $\mathrm{Sym}^2 L_n \to L_n \otimes L_n$ we have fixed in (1).

Let $\epsilon_{G_n} : \mathscr{N}_{G_n} \to \mathbb{A}^1$ denote the sum of $\epsilon_{1,n}(L_1, ..., L_n)$ with the class in $\mathrm{Ext}(\mathcal{O}, \Omega) = \mathbb{A}^1$ of the push-forward of (3) by $\mathrm{Sym}^2 L_n \to \mathrm{Sym}^2(L_n/L_{n-1}) = \Omega^2$.

The functor $\mathrm{Whit}_{G_n} : \mathrm{D}^{\prec}(\mathrm{Bun}_{G_n}) \to \mathrm{D}^-(\mathrm{Spec}\, k)$ associated to the above choice of $\mathscr{F}_T$ is given by

$$\mathrm{Whit}_{G_n}(\mathscr{F}) = \mathrm{R}\Gamma_c(\mathscr{N}_{G_n}, \epsilon_{G_n}^*(\mathscr{L}_\psi) \otimes \pi_{G_n}^*(\mathscr{F}))[d_{N(G_n)} - d_{G_n}]$$

with $d_{N(G_n)} = \dim \mathscr{N}_{G_n}$ and $d_{G_n} = \dim \mathrm{Bun}_{G_n}$.

**1.3. Whittaker functor for $\mathbb{SO}_{2n}$ (first form).** — Let $H_n = \mathbb{SO}_{2n}$.The stack $\mathrm{Bun}_{H_n}$ of $H_n$-torsors can be seen as the stack classifying vector bundles $V$ over $X$ equipped with a non-degenerate symmetric form $\mathrm{Sym}^2 V \to \mathcal{O}$ and a compatible trivialization $\det V \widetilde{\to} \mathcal{O}$.

The diagram $\mathrm{Bun}_{H_n} \overset{\pi_{H_n}}{\leftarrow} \mathscr{N}_{H_n} \overset{\epsilon_{H_n}}{\to} \mathbb{A}^1$ constructed in the next definition is isomorphic to the diagram $\mathrm{Bun}_G \overset{\pi}{\leftarrow} \mathrm{Bun}_N^{\mathscr{F}_T} \overset{\epsilon}{\to} \mathbb{A}^1$ associated, for $G = H_n$, to the choice of $\mathscr{F}_T$ whose image in $\mathrm{Bun}_{H_n}$ is $U \oplus U^*$ with $U = \Omega^{n-1} \oplus \Omega^{n-2} \oplus \cdots \oplus \mathcal{O}$ (with the natural symmetric structure for which $U$ and $U^*$ are isotropic).

Definition 2. — Let $\mathscr{N}_{H_n}$ be the stack classifying $((U_1, ..., U_n), E)$, where $(U_1, ..., U_n) \in \mathscr{N}_{0,n-1}$ (i.e. we have a filtration $0 = U_0 \subset U_1 \subset ... \subset U_n$ with $U_i/U_{i-1} \simeq \Omega^{n-i}$ for $i = 1, ..., n$), and $E$ is an extension of $\mathcal{O}_X$-modules

$$0 \to \Lambda^2 U_n \to E \to \mathcal{O} \to 0 \tag{5}$$

We associate to (5) an extension

$$0 \to U_n \to V \to U_n^* \to 0 \tag{6}$$

with $V \in \mathrm{Bun}_{H_n}$ and $U_n$ isotropic as follows. Equip $U_n \oplus U_n^*$ with the symmetric form given by $(u, u^*), (v, v^*) \mapsto \langle u, v^* \rangle + \langle v, u^* \rangle$ with $u, v \in U_n, u^*, v^* \in U_n^*$. Using (1), we consider (5) as a torsor under the sheaf of antisymmetric morphisms $U_n^* \to U_n$ of $\mathcal{O}_X$-modules. This sheaf acts naturally on $U_n \oplus U_n^*$ preserving the symmetric form and the trivialization of $\det(U_n \oplus U_n^*)$. Then (6) is the twisting of $U_n \oplus U_n^*$ by the above torsor. This defines a morphism $\pi_{H_n} : \mathcal{N}_{H_n} \to \mathrm{Bun}_{H_n}$.

Note that the extension of $\mathcal{O}_X$ by $U_n \otimes U_n$ obtained from (6) is the push-forward of (5) by the embedding $\Lambda^2 U_n \to U_n \otimes U_n$ fixed in (1).

For $\lambda \in k^*$ let $\epsilon_{H_n,\lambda} : \mathcal{N}_{H_n} \to \mathbb{A}^1$ be the sum of $\epsilon_{0,n-1}(U_1, ..., U_n)$ with $\lambda u$, where $u \in \mathrm{Ext}(\mathcal{O}, \Omega) = \mathbb{A}^1$ is the class of the push-forward of (5) by $\Lambda^2 U_n \to \Lambda^2(U_n/U_{n-2}) = \Omega$. Set $\epsilon_{H_n} = \epsilon_{H_n,1}$.

The functor $\mathrm{Whit}_{H_n} : \mathrm{D}^{\prec}(\mathrm{Bun}_{H_n}) \to \mathrm{D}^-(\mathrm{Spec}\, k)$ associated to the above choice of $\mathcal{F}_T$ sends $\mathcal{F} \in D^{\prec}(\mathrm{Bun}_{H_n})$ to

$$\mathrm{Whit}_{H_n}(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathcal{N}_{H_n}, \epsilon_{H_n}^*(\mathcal{L}_\psi) \otimes \pi_{H_n}^*(M))[d_{N(H_n)} - d_{H_n}] \tag{7}$$

with $d_{N(H_n)} = \dim \mathcal{N}_{H_n}$ and $d_{H_n} = \dim \mathrm{Bun}_{H_n}$. By Remark 2, if we replace in (7) $\epsilon_{H_n}$ by $\epsilon_{H_n,\lambda}$ then the functor $\mathrm{Whit}_{H_n}$ gets replaced by an isomorphic one.

### 1.4. Whittaker functor for $\mathbb{SO}_{2n}$ (second form)

Definition 3. — Let $\widetilde{\mathcal{N}}_{H_n}$ be the stack classifying $(V_1 \subset \cdots \subset V_n \subset V)$, where $V \in \mathrm{Bun}_{H_n}$, $V_n \subset V$ is a subbundle, $(V_1, ..., V_n) \in \mathcal{N}_{0,n-1}$ (i.e. we have a filtration $0 = V_0 \subset V_1 \subset ... \subset V_n$ with $V_i/V_{i-1} \simeq \Omega^{n-i}$ for $i = 1, ..., n$), and the composition

$$\mathrm{Sym}^2 V_n \to \mathrm{Sym}^2 V \to \mathcal{O}$$

coincides with $\mathrm{Sym}^2 V_n \to \mathrm{Sym}^2(V_n/V_{n-1}) = \mathcal{O}$ (in particular $V_{n-1}$ is isotropic).

The morphism $\widetilde{\pi}_{H_n} : \widetilde{\mathcal{N}}_{H_n} \to \mathrm{Bun}_{H_n}$ sends $((V_1, ..., V_n), V)$ to $V$. The morphism $\widetilde{\epsilon}_{H_n} : \widetilde{\mathcal{N}}_{H_n} \to \mathbb{A}^1$ is given by $\widetilde{\epsilon}_{H_n}((V_1, ..., V_n), V) = \epsilon_{0,n-1}(V_1, ..., V_n)$.

Define a morphism $\kappa : \mathcal{N}_{H_n} \to \widetilde{\mathcal{N}}_{H_n}$ as follows. Let $(U_1, ..., U_n), E) \in \mathcal{N}_{H_n}$ and let $V$ be as in Definition 2. For $i = 1, ..., n-1$ define $V_i$ as the image of $U_i$ in $V$ and $V_{2n-i}$ as the orthogonal of $V_i$ in $V$. Then we have a filtration

$$0 = V_0 \subset V_1 \subset ... \subset V_{n-1} \subset V_{n+1} \subset ... \subset V_{2n-1} \subset V_{2n} = V.$$

Recall that we have an identification $U_n/U_{n-1} \simeq \mathcal{O}$. The exact sequence $0 \to U_n/U_{n-1} \to V_{n+1}/V_{n-1} \to V_{n+1}/U_n \to 0$ admits a unique splitting $s$ such that the image of $\mathcal{O} = V_{n+1}/U_n \overset{s}{\to} V_{n+1}/V_{n-1}$ is isotropic. Thus, $V_{n+1}/V_{n-1}$ is canonically identified with $\mathcal{O} \oplus \mathcal{O}$ in such a way that the symmetric bilinear form $\mathrm{Sym}^2(\mathcal{O} \oplus \mathcal{O}) \to \mathcal{O}$ becomes

$$(1,0).(1,0) \mapsto 0, \ (1,0).(0,1) \mapsto 1, \ (0,1).(0,1) \mapsto 0$$

Under this identification $\mathcal{O} = U_n/U_{n-1} \to V_{n+1}/V_{n-1} = \mathcal{O} \oplus \mathcal{O}$ sends 1 to $(1,0)$.

Define $V_n$, equipped with $\mathcal{O} \simeq V_n/V_{n-1}$ by the property that $\mathcal{O} \simeq V_n/V_{n-1} \hookrightarrow V_{n+1}/V_{n-1}$ sends 1 to $(1, \frac{1}{2}) \in \mathcal{O} \oplus \mathcal{O}$. The following is easy to check.

Lemma 1. — *The map* $\kappa : \mathcal{N}_{H_n} \to \widetilde{\mathcal{N}}_{H_n}$ *is an isomorphism. There exists* $\lambda \in k^*$ *such that* $\tilde{\epsilon}_{H_n} \circ \kappa = \epsilon_{H_n,\lambda}$ *and* $\tilde{\pi}_{H_n} \circ \kappa = \pi_{H_n}$.

By Remark 2, if we replace in (7) $\epsilon_{H_n}, \pi_{H_n}$ by $\tilde{\epsilon}_{H_n}, \tilde{\pi}_{H_n}$ then the functor $\mathrm{Whit}_{H_n}$ gets replaced by an isomorphic one.

## 2. Main statements

Write $\mathrm{Bun}_n$ for the stack of rank $n$ vector bundles on $X$. Let $\mathrm{Bun}_{P_n}$ be the stack classifying $L \in \mathrm{Bun}_n$ and an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$. Remind the complex $S_{P,\psi}$ on $\mathrm{Bun}_{P_n}$ introduced in ([4], 5.2). Let $\mathcal{V}$ be the stack over $\mathrm{Bun}_n$ whose fibre over $L$ is $\mathrm{Hom}(L, \Omega)$. For $\mathcal{X}_n = \mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_{P_n}$ let $p : \mathcal{X}_n \to \mathrm{Bun}_{P_n}$ be the projection. Write $q : \mathcal{X}_n \to \mathbb{A}^1$ for the map sending $s \in \mathrm{Hom}(L, \Omega)$ to the pairing of $s \otimes s \in \mathrm{Hom}(\mathrm{Sym}^2 L, \Omega^2)$ with the exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$. Let $d_{\mathcal{X}_n}$ be the "corrected" dimension of $\mathcal{X}_n$, i.e. the locally constant function $\dim \mathrm{Bun}_{P_n} - \chi(L)$. Set

$$S_{P,\psi} = p_! q^* \mathcal{L}_\psi [d_{\mathcal{X}_n}].$$

Let $\mathcal{A}$ be the line bundle on $\mathrm{Bun}_{G_n}$ whose fibre at $M$ is $\det \mathrm{R}\Gamma(X, M)$. Write $\widetilde{\mathrm{Bun}}_{G_n}$ for the gerb of square roots of $\mathcal{A}$ and Aut for the theta-sheaf on $\widetilde{\mathrm{Bun}}_{G_n}$ ([4], Definition 1). The projection $\nu_n : \mathrm{Bun}_{P_n} \to \mathrm{Bun}_{G_n}$ lifts naturally to a map $\tilde{\nu}_n : \mathrm{Bun}_{P_n} \to \widetilde{\mathrm{Bun}}_{G_n}$. In what follows, we pick an isomorphism[(1)]

$$S_{P,\psi} \widetilde{\to} \tilde{\nu}_n^* \mathrm{Aut}[\dim.\mathrm{rel}(\tilde{\nu}_n))] \tag{8}$$

provided by ([5], Proposition 1). Here $\dim.\mathrm{rel}(\tilde{\nu}_n)$ is the relative dimension of $\tilde{\nu}_n$. The isomorphisms we construct below may depend on this choice.

**2.1. From $\mathbb{S}\mathrm{p}_{2n}$ to $\mathbb{S}\mathbb{O}_{2n+2}$.** — Let $F : \mathrm{D}^-(\mathrm{Bun}_{G_n})_! \to \mathrm{D}^{\prec}(\mathrm{Bun}_{H_{n+1}})$ be the theta lifting functor introduced in ([6], Definition 2).

Theorem 1. — *The functors* $\mathrm{Whit}_{H_{n+1}} \circ F$ *and* $\mathrm{Whit}_{G_n}$ *from* $\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$ *to* $\mathrm{D}^-(\mathrm{Spec}\, k)$ *are isomorphic.*

Let $\mathcal{X}$ be the stack classifying $(M, (U_1, ..., U_{n+1}), E, s)$ with $M \in \mathrm{Bun}_{G_n}$, $(U_1, ..., U_{n+1}) \in \mathcal{N}_{0,n}$ (i.e. $U_{k+1}/U_k = \Omega^{n-k}$ for $k = 0, ..., n$), $E$ an extension $0 \to \Lambda^2 U_{n+1} \to E \to \mathcal{O} \to 0$, and $s : U_{n+1} \to M$ a morphism of $\mathcal{O}_X$-modules.

Let $\alpha_{\mathcal{X}} : \mathcal{X} \to \mathrm{Bun}_{G_n}$ be the morphism $(M, (U_1, ..., U_{n+1}), E, s) \mapsto M$. Let $\beta_{\mathcal{X}} : \mathcal{X} \to \mathbb{A}^1$ be defined as follows. For $(M, (U_1, ..., U_{n+1}), E, s) \in \mathcal{X}$,

$$\beta_{\mathcal{X}}(M, (U_1, ..., U_{n+1}), E, s) = \epsilon_{0,n}(U_1, ..., U_{n+1}) + \gamma(E) - \langle E, \Lambda^2 s\rangle$$

(1) Once $\sqrt{-1} \in k$ is chosen, this isomorphism is well defined up to a sign.

where $\gamma(E)$ is the pairing between the class of $E$ in $\mathrm{Ext}(\mathcal{O},\Lambda^2 U_{n+1})$ and the morphism $\Lambda^2 U_{n+1}\to\Lambda^2(U_{n+1}/U_{n-1})=\Omega$ and $\langle E,\Lambda^2 s\rangle$ is the pairing between the class of $E$ in $\mathrm{Ext}(\mathcal{O},\Lambda^2 U_{n+1})$ and $\Lambda^2 s:\Lambda^2 U_{n+1}\to\Lambda^2 M$ followed by $\Lambda^2 M\to\Omega$.

Let $a_n=n(n+1)(1-g)(n-\frac{1}{2})$, this is the dimension of the stack classifying extension $0\to\wedge^2 U_{n+1}\to ?\to\mathcal{O}\to 0$ of $\mathcal{O}_X$-modules for any fixed $(U_1,\dots,U_{n+1})\in\mathcal{N}_{0,n}$.

Let $d_{\alpha_{\mathcal{X}}}$ denote the "corrected" relative dimension of $\alpha_{\mathcal{X}}$, that is, $d_{\alpha_{\mathcal{X}}}=a_n+\dim\mathcal{N}_{0,n}+\chi(U_{n+1}^*\otimes M)$ for any $k$-points $M\in\mathrm{Bun}_{G_n}$ and $(U_1,\dots,U_{n+1})\in\mathcal{N}_{0,n}$. One checks that (8) yields for $\mathcal{F}\in\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$ an isomorphism in $\mathrm{D}^-(\mathrm{Spec}\,k)$

$$\mathrm{Whit}_{H_{n+1}}\circ F(\mathcal{F})\widetilde{\to}\mathrm{R}\Gamma_c(\mathcal{X},\alpha_{\mathcal{X}}^*(\mathcal{F})\otimes\beta_{\mathcal{X}}^*(\mathcal{L}_\psi)[d_{\alpha_{\mathcal{X}}}])$$

We will show later that Theorem 1 is reduced to the following proposition.

Proposition 1. — *There is an isomorphism* $\alpha_{\mathcal{X}!}(\beta_{\mathcal{X}}^*(\mathcal{L}_\psi)[2a_n])\widetilde{\to}\pi_{G_n!}\epsilon_{G_n}^*(\mathcal{L}_\psi)$ *in* $\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$.

The proposition is a consequence of the following lemmas. Let $\mathcal{Y}$ be the stack classifying $(M,(U_1,...,U_{n+1}),s)$ with $M\in\mathrm{Bun}_{G_n}$, $(U_1,...,U_{n+1})\in\mathcal{N}_{0,n}$ (i.e. $U_{k+1}/U_k=\Omega^{n-k}$ for $k=0,...,n$), and $s:U_{n+1}\to M$ a morphism such that the composition $\Lambda^2 U_{n+1}\overset{\Lambda^2 s}{\to}\Lambda^2 M\to\Omega$ coincides with $\Lambda^2 U_{n+1}\to\Lambda^2(U_{n+1}/U_{n-1})=\Omega$.

Let $\alpha_{\mathcal{Y}}:\mathcal{Y}\to\mathrm{Bun}_{G_n}$ be the morphism $(M,(U_1,...,U_{n+1}),s)\mapsto M$. Let $\beta_{\mathcal{Y}}:\mathcal{Y}\to\mathbb{A}^1$ be the map sending $(M,(U_1,...,U_{n+1}),s)\in\mathcal{Y}$ to $\epsilon_{0,n}(U_1,...,U_{n+1})$.

Lemma 2. — *There is an isomorphism* $\alpha_{\mathcal{X},!}\beta_{\mathcal{X}}^*(\mathcal{L}_\psi)=\alpha_{\mathcal{Y},!}\beta_{\mathcal{Y}}^*(\mathcal{L}_\psi)[-2a_n]$ *in* $\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$.

For $i\in\{1,...,n+1\}$ let $\mathcal{Y}_i$ denote the open subset of $\mathcal{Y}$ given by the condition that the image of $U_i$ by $s$ is a subbundle of $M$. One has open immersions $\mathcal{Y}_{n+1}\subset\mathcal{Y}_n\subset...\subset\mathcal{Y}_1\subset\mathcal{Y}$. Denote by $\alpha_{\mathcal{Y}_i}:\mathcal{Y}_i\to\mathrm{Bun}_{G_n}$ and $\beta_{\mathcal{Y}_i}:\mathcal{Y}_i\to\mathbb{A}^1$ the restrictions of $\alpha_{\mathcal{Y}}$ and $\beta_{\mathcal{Y}}$ to $\mathcal{Y}_i$.

Lemma 3. — *The natural maps* $\alpha_{\mathcal{Y}_{n+1},!}\beta_{\mathcal{Y}_{n+1}}^*(\mathcal{L}_\psi)\to\alpha_{\mathcal{Y}_n,!}\beta_{\mathcal{Y}_n}^*(\mathcal{L}_\psi)\to...\to\alpha_{\mathcal{Y}_1,!}\beta_{\mathcal{Y}_1}^*(\mathcal{L}_\psi)\to\alpha_{\mathcal{Y},!}\beta_{\mathcal{Y}}^*(\mathcal{L}_\psi)$ *are isomorphisms in* $\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$.

*Proof.* — First, one has $\mathcal{Y}_{n+1}=\mathcal{Y}_{n-1}$ thanks to the condition that the composition $\Lambda^2 U_{n+1}\overset{\Lambda^2 s}{\to}\Lambda^2 M\to\Omega$ coincides with $\Lambda^2 U_{n+1}\to\Lambda^2(U_{n+1}/U_{n-1})=\Omega$.

Write $\mathcal{Y}_0=\mathcal{Y}$. Let $i\in\{1,...,n-1\}$. We are going to prove that the natural map

$$\alpha_{\mathcal{Y}_i,!}\beta_{\mathcal{Y}_i}^*(\mathcal{L}_\psi)\to\alpha_{\mathcal{Y}_{i-1},!}\beta_{\mathcal{Y}_{i-1}}^*(\mathcal{L}_\psi)$$

is an isomorphism. Set $\mathcal{Z}_i = \mathcal{Y}_{i-1} \backslash \mathcal{Y}_i$, let $\alpha_{\mathcal{Z}_i}$ and $\beta_{\mathcal{Z}_i}$ be the restrictions of $\alpha_{\mathcal{Y}_{i-1}}$ and $\beta_{\mathcal{Y}_{i-1}}$ to $\mathcal{Z}_i$. We must prove that $\alpha_{\mathcal{Z}_i,!}\beta^*_{\mathcal{Z}_i}(\mathcal{L}_\psi) = 0$.

Let $\mathcal{T}_i$ be stack classifying $(M, (U_1, U_2, ..., U_i), s_i)$ with $M \in \mathrm{Bun}_{G_n}$, $(U_1, U_2, ..., U_i) \in \mathcal{N}_{n-i+1,n}$, $s_i : U_i \to M$ such that the restriction of $s_i$ to $U_{i-1}$ is injective and its image is a subbundle of $M$, but the image of $s_i$ is not a subbundle of $M$ of the same rank as $U_i$. The map $\alpha_{\mathcal{Z}_i}$ decomposes naturally as $\mathcal{Z}_i \stackrel{\gamma_{\mathcal{Z}_i}}{\to} \mathcal{T}_i \stackrel{\alpha_{\mathcal{T}_i}}{\to} \mathrm{Bun}_{G_n}$. It suffices to show that the $*$-fibre of $\gamma_{\mathcal{Z}_i,!}\beta^*_{\mathcal{Z}_i}(\mathcal{L}_\psi)$ at any closed point $(M, (U_1, U_2, ..., U_i), s_i) \in \mathcal{T}_i$ vanishes.

The fiber $\mathcal{Q}$ of $\gamma_{\mathcal{Z}_i}$ over this point is the stack classifying $((U_1, ..., U_{n+1}), s)$, where $(U_1, ..., U_{n+1}) \in \mathcal{N}_{0,n}$ extends $(U_1, U_2, ..., U_i)$, $s : U_{n+1} \to M$ extends $s_i$, and the composition $\Lambda^2 U_{n+1} \stackrel{\Lambda^2 s}{\to} \Lambda^2 M \to \Omega$ coincides with $\Lambda^2 U_{n+1} \to \Lambda^2(U_{n+1}/U_{n-1}) = \Omega$.

Let $F$ denote the smallest subbundle of $M$ containing $s(U_i)$, its rank is $i$ or $i-1$. Let $\mathcal{R}$ be stack classifying $((W_1, ..., W_{n+1-i}), t)$ with $(W_1, ..., W_{n+1-i}) \in \mathcal{N}_{0,n-i}$ and $t \in \mathrm{Hom}(W_{n+1-i}, M/F)$. There is a morphism $\rho : \mathcal{Q} \to \mathcal{R}$ which sends $((U_1, ..., U_{n+1}), s)$ to $((U_{i+1}/U_i, ..., U_{n+1}/U_i), \bar{s})$ where $\bar{s} : U_{n+1}/U_i \to M/F$ is the reduction of $s$. Let $\beta_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{A}^1$ be the restriction of $\beta_{\mathcal{Z}_i}$ to $\mathcal{Q}$. It suffices to show that $\rho_! \beta^*_{\mathcal{Q}}(\mathcal{L}_\psi) = 0$.

Pick $((W_1, ..., W_{n+1-i}), t) \in \mathcal{R}$, let $\mathcal{S}$ be the fiber of $\rho$ over

$$((W_1, ..., W_{n+1-i}), t).$$

Write $\beta_{\mathcal{S}}$ for the restriction of $\beta_{\mathcal{Q}}$ to $\mathcal{S}$. We will show that $R\Gamma_c(\mathcal{S}, \beta^*_{\mathcal{S}}(\mathcal{L}_\psi)) = 0$.

If $F$ is of rank $i-1$ then $\mathcal{S}$ identifies with the stack classifying extensions $0 \to U_i/U_{i-1} \to ? \to U_{n+1}/U_i \to 0$ of $\mathcal{O}_X$-modules. Since $\beta_{\mathcal{S}}$ is a nontrivial character, we are done in this case.

If $F$ is of rank $i$ then $\mathcal{S}$ is a scheme with a free transitive action of $\mathrm{Hom}(U_{n+1}/U_i, F/s(U_i))$. Under the action of $\mathrm{Hom}(U_{n+1}/U_i, F/s(U_i))$, $\beta_{\mathcal{S}}$ changes by some character

$$\mathrm{Hom}(U_{n+1}/U_i, F/s(U_i)) \to \mathrm{Hom}(U_{i+1}/U_i, F/s(U_i)) \stackrel{\delta}{\to} \mathbb{A}^1.$$

If $D = \mathrm{div}(F/s(U_i))$ then $F/s(U_i) \widetilde{\to} \Omega^{n-i+1}(D)/\Omega^{n-i+1}$ naturally, and $\delta : \mathrm{H}^0(X, \Omega(D)/\Omega) \to \mathrm{H}^1(X, \Omega)$ is the map induced by the short exact sequence $0 \to \Omega \to \Omega(D) \to \Omega(D)/\Omega \to 0$, i.e. it is the sum of the residues. Since $D > 0$, $\delta$ is nontrivial, and we are done. □

Lemma 4. — *There is an isomorphism $\mu : \mathcal{Y}_{n+1} \to \mathcal{N}_{G_n}$ such that $\pi_{G_n} \circ \mu = \alpha_{\mathcal{Y}_{n+1}}$ and $\epsilon_{G_n} \circ \mu = \beta_{\mathcal{Y}_{n+1}}$.*

It remains to show that Proposition 1 implies Theorem 1 . By the base change theorem we have

$$\mathrm{Whit}_{G_n}(\mathcal{F}) \widetilde{\to} R\Gamma_c(\mathcal{N}_{G_n}, \epsilon^*_{G_n}(\mathcal{L}_\psi) \otimes \pi^*_{G_n}(\mathcal{F}))[d_{N(G_n)} - d_{G_n}]$$

$$\widetilde{\to} R\Gamma_c(\mathrm{Bun}_{G_n}, \pi_{G_n,!}\epsilon^*_{G_n}(\mathcal{L}_\psi)\otimes \mathcal{F})[d_{N(G_n)} - d_{G_n}]$$

and

$$R\Gamma_c(\mathcal{X}, \alpha^*_{\mathcal{X}}(\mathcal{F})\otimes \beta_{\mathcal{X}}(\mathcal{L}_\psi)[d_{\alpha_\mathcal{X}}])\widetilde{\to} R\Gamma_c(\mathrm{Bun}_{G_n}, \alpha_{\mathcal{X},!}(\beta^*_{\mathcal{X}}(\mathcal{L}_\psi)\otimes \mathcal{F}[d_{\alpha_\mathcal{X}}])).$$

It remains to prove $d_{\alpha_\mathcal{X}} - 2a_n = d_{N(G_n)} - d_{G_n}$. This follows from $d_{G_n} = -(1-g)n(2n+1)$, $d_{N(G_n)} - \dim \mathcal{N}_{0,n} = (1-g)(-n^2 + n(n+1)(n-\frac{1}{2}))$, and $\chi(U^*_{n+1}\otimes M) = (1-g)2n^2(n+1)$ where $(U_1,\dots,U_{n+1})$ and $M$ are closed points in $\mathcal{N}_{0,n}$ and $\mathrm{Bun}_{G_n}$.

**2.2. From $\mathbb{SO}_{2n}$ to $\mathbb{Sp}_{2n}$.** — Let $F : \mathrm{D}^-(\mathrm{Bun}_{H_n})_! \to \mathrm{D}^{\prec}(\mathrm{Bun}_{G_n})$ be the Theta functor introduced in ([6], Definition 2).

THEOREM 2. — *The functors* $\mathrm{Whit}_{G_n}\circ F$ *and* $\mathrm{Whit}_{H_n}$ *from* $\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$ *to* $\mathrm{D}^-(\mathrm{Spec}\, k)$ *are isomorphic.*

We use the same letters as in the last paragraph (with a different meaning), as the proof is very similar.

Let $\mathcal{X}$ be the stack classifying $(V,(L_1,...,L_n),E,s)$ with $V\in \mathrm{Bun}_{H_n}$, $(L_1,...,L_n)\in\mathcal{N}_{1,n}$ (i.e. $L_{k+1}/L_k = \Omega^{n-k}$ for $k=0,...,n-1$), an extension $0\to \mathrm{Sym}^2 L_n \to E\to\Omega\to 0$ of $\mathcal{O}_X$-modules, and a section $s: L_n\to V\otimes\Omega$.

Let $\alpha_\mathcal{X} : \mathcal{X}\to \mathrm{Bun}_{H_n}$ be the morphism $(V,(L_1,...,L_n),E,s)\mapsto V$. Let $\beta_\mathcal{X}:\mathcal{X}\to\mathbb{A}^1$ be the map sending $(V,(L_1,...,L_n),E,s)\in\mathcal{X}$ to

$$\epsilon_{1,n}(L_1,...,L_n) + \gamma(E) - \langle E, \mathrm{Sym}^2 s\rangle,$$

where $\gamma(E)$ is the pairing between the class of $E$ in $\mathrm{Ext}^1(\Omega,\mathrm{Sym}^2 L_n)$ and the map $\mathrm{Sym}^2 L_n\to \mathrm{Sym}^2(L_n/L_{n-1}) = \Omega^2$; $\langle E,\mathrm{Sym}^2 s\rangle$ is the pairing between the class of $E$ in $\mathrm{Ext}^1(\Omega,\mathrm{Sym}^2 L_n)$ and $\mathrm{Sym}^2 s : \mathrm{Sym}^2 L_n\to\mathrm{Sym}^2 V\otimes\Omega^2$ followed by $\mathrm{Sym}^2 V\to\mathcal{O}$.

Let $b_n = -\chi(\Omega^{-1}\otimes\mathrm{Sym}^2 L_n)$ for any $k$-point $(L_1,\dots,L_n)\in\mathcal{N}_{1,n}$. Write $d_{\alpha_\mathcal{X}}$ for the "corrected" relative dimension of $\alpha_\mathcal{X}$, that is,

$$d_{\alpha_\mathcal{X}} = \dim\mathcal{N}_{1,n} + b_n + \chi(L_n^*\otimes V\otimes\Omega)$$

for any $k$-points $(L_1,\dots,L_n)\in\mathcal{N}_{1,n}$ and $V\in\mathrm{Bun}_{H_n}$. One checks that (8) yields for $\mathcal{F}\in \mathrm{D}^-(\mathrm{Bun}_{H_n})_!$ an isomorphism in $\mathrm{D}^-(\mathrm{Spec}\, k)$

$$\mathrm{Whit}_{G_n}\circ F(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathcal{X},\alpha^*_\mathcal{X}(\mathcal{F})\otimes\beta^*_\mathcal{X}(\mathcal{L}_\psi))[d_{\alpha_\mathcal{X}}]$$

We will derive Theorem 2 from the following proposition.

PROPOSITION 2. — *There is an isomorphism* $\alpha_{\mathcal{X},!}\beta^*_\mathcal{X}(\mathcal{L}_\psi)[2b_n]\simeq\widetilde{\pi}_{H_n,!}\widetilde{\epsilon}^*_{H_n}(\mathcal{L}_\psi)$ *in* $\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$.

Proposition 2 is reduced to the following lemmas. Let $\mathcal{Y}$ be the stack classifying $(V,(L_1,\dots,L_n),s)$ with $V\in \mathrm{Bun}_{\mathbb{SO}_{2n}}$, $(L_1,\dots,L_n)\in \mathcal{N}_{1,n}$ (i.e., $L_{k+1}/L_k=\Omega^{n-k}$ for $k=0,\dots,n-1$) and $s: L_n\to V\otimes\Omega$ a morphism such that the composition $\mathrm{Sym}^2 L_n \stackrel{\mathrm{Sym}^2 s}{\to} (\mathrm{Sym}^2 V)\otimes\Omega^2\to\Omega^2$ coincides with

$$\mathrm{Sym}^2 L_n\to \mathrm{Sym}^2(L_n/L_{n-1})=\Omega^2$$

Let $\alpha_{\mathcal{Y}}:\mathcal{Y}\to \mathrm{Bun}_{H_n}$ be the map $(V,(L_1,\dots,L_n),s)\mapsto V$. Let $\beta_{\mathcal{Y}}:\mathcal{Y}\to\mathbb{A}^1$ be the map sending $(V,(L_1,\dots,L_n),s)\in\mathcal{Y}$ to $\epsilon_{1,n}(L_1,\dots,L_n)$.

Lemma 5. — *There is an isomorphism* $\alpha_{\mathcal{X},!}\beta^*_{\mathcal{X}}(\mathcal{L}_\psi)\widetilde{\to}\alpha_{\mathcal{Y},!}\beta^*_{\mathcal{Y}}(\mathcal{L}_\psi)[-2b_n]$ *in* $\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$.

For $i\in\{1,\dots,n\}$ let $\mathcal{Y}_i\subset\mathcal{Y}$ be the open substack given by the condition that $s(L_i)\subset V\otimes\Omega$ is a subbundle of rank $i$. We have inclusions $\mathcal{Y}_n\subset\mathcal{Y}_{n-1}\subset\dots\subset\mathcal{Y}_1\subset\mathcal{Y}$. Denote by $\alpha_{\mathcal{Y}_i}:\mathcal{Y}_i\to\mathrm{Bun}_{H_n}$ and $\beta_{\mathcal{Y}_i}:\mathcal{Y}_i\to\mathbb{A}^1$ the restrictions of $\alpha_{\mathcal{Y}}$ and $\beta_{\mathcal{Y}}$ to $\mathcal{Y}_i$.

As in Lemma 3, one proves

Lemma 6. — *The natural maps* $\alpha_{\mathcal{Y}_n,!}\beta^*_{\mathcal{Y}_n}(\mathcal{L}_\psi)\to\alpha_{\mathcal{Y}_{n-1},!}\beta^*_{\mathcal{Y}_{n-1}}(\mathcal{L}_\psi)\to\dots\to \alpha_{\mathcal{Y}_1,!}\beta^*_{\mathcal{Y}_1}(\mathcal{L}_\psi)\to\alpha_{\mathcal{Y},!}\beta^*_{\mathcal{Y}}(\mathcal{L}_\psi)$ *are isomorphisms in* $\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$.

Lemma 7. — *There is an isomorphism* $\mu:\mathcal{Y}_n\to\widetilde{\mathcal{N}}_{\mathbb{SO}_{2n}}$ *such that* $\widetilde{\pi}_{\mathbb{SO}_{2n}}\circ\mu=\alpha_{\mathcal{Y}_n}$ *and* $\widetilde{\epsilon}_{\mathbb{SO}_{2n}}\circ\mu=\beta_{\mathcal{Y}_n}$.

Theorem 2 follows from Proposition 2 because $d_{\alpha_{\mathcal{X}}}-2b_n=d_{N(H_n)}-d_{H_n}$. Let us just indicate that $d_{N(H_n)}-\dim\mathcal{N}_{1,n}=(1-g)n(n-1)(n-\frac{3}{2})$, $\chi(L_n^*\otimes V\otimes\Omega)=(1-g)2n^3$, $b_n=(1-g)n(n+1)(n-\frac{1}{2})$ and $d_{H_n}=-(1-g)n(2n-1)$, where $(L_1,\dots,L_n)$ and $V$ are closed points in $\mathcal{N}_{1,n}$ and $\mathrm{Bun}_{H_n}$.

**2.3. From $\mathbb{GL}_n$ to $\mathbb{GL}_{n+1}$.** — Let $F:\mathrm{D}^-(\mathrm{Bun}_n)_!\to\mathrm{D}^{\prec}(\mathrm{Bun}_{n+1})$ be the composition of the direct image by $\mathrm{Bun}_n\to\mathrm{Bun}_n$, $L\mapsto L^*$ and the theta functor $F_{n,n+1}:\mathrm{D}^-(\mathrm{Bun}_n)_!\to\mathrm{D}^{\prec}(\mathrm{Bun}_{n+1})$ introduced in ([6], Definition 3). It is a consequence of Theorem 5 in [6] that $F$ is compatible with Hecke functors according to the morphism of dual groups $\mathbb{GL}_n\to\mathbb{GL}_{n+1}$, $A\mapsto\begin{pmatrix}A&0\\0&1\end{pmatrix}$.

Let us recall the definition of $F$. Denote $\mathcal{W}$ be the classifying stack of $(L,U,s)$ with $L\in\mathrm{Bun}_n$, $U\in\mathrm{Bun}_{n+1}$ and $s:L\to U$ a morphism. We have $(h_n,h_{n+1}):\mathcal{W}\to\mathrm{Bun}_n\times\mathrm{Bun}_{n+1},(L,U,s)\mapsto(L,U)$. Then for $\mathcal{F}\in\mathrm{D}^-(\mathrm{Bun}_n)_!$,

$$F(\mathcal{F})=h_{n+1,!}((h_n^*\mathcal{F})[\dim\mathrm{Bun}_{n+1}+\chi(L^*\otimes U)]),$$

where $\chi(L^*\otimes U)$ is considered as a locally constant function on $\mathrm{Bun}_n\times\mathrm{Bun}_{n+1}$.

THEOREM 3. — *The functors* $\mathrm{Whit}_{\mathbb{GL}_{n+1}} \circ F$ *and* $\mathrm{Whit}_{\mathbb{GL}_n}$ *from* $\mathrm{D}^-(\mathrm{Bun}_n)_!$ *to* $\mathrm{D}^-(\mathrm{Spec}\, k)$ *are isomorphic.*

Let $\mathcal{X}$ be the stack classifying $L \in \mathrm{Bun}_n$, $(U_1, \dots, U_{n+1}) \in \mathcal{N}_{0,n}$, and $s : L \to U_{n+1}$ a morphism. We have $\alpha_{\mathcal{X}} : \mathcal{X} \to \mathrm{Bun}_n$ and $\beta_{\mathcal{X}} : \mathcal{X} \to \mathbb{A}^1$ which send $(L, (U_1, \dots U_{n+1}), s)$ to $L$ and $\epsilon_{0,n}(U_1, \dots U_{n+1})$.

We have

$$\mathrm{Whit}_{\mathbb{GL}_{n+1}} \circ F(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathrm{Bun}_n, \mathcal{F} \otimes \alpha_{\mathcal{X},!}\beta_{\mathcal{X}}^*(\mathcal{L}_\psi)[\dim \mathcal{N}_{0,n} + \chi(L^* \otimes U_{n+1})]) \text{ and}$$

$$\mathrm{Whit}_{\mathbb{GL}_n}(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathrm{Bun}_n, \mathcal{F} \otimes (\pi_{0,n-1})_!\epsilon_{0,n-1}^*(\mathcal{L}_\psi)[\dim \mathcal{N}_{0,n-1} - \dim \mathrm{Bun}_n]).$$

For $i \in \{0, \dots, n\}$ denote by $\mathcal{X}_i$ the open substack of $\mathcal{X}$ classifying $(L, (U_1, \dots U_{n+1}), s)$ such that the composition $L \xrightarrow{s} U_{n+1} \to U_{n+1}/U_{n+1-i}$ is surjective. We have $\mathcal{X} = \mathcal{X}_0 \supset \mathcal{X}_1 \supset \cdots \supset \mathcal{X}_n$ and we have an isomorphism $\mathcal{N}_{0,n-1} \to \mathcal{X}_n$ which sends $(E_1, \dots, E_n)$ to $(E_n, (\Omega^n, \Omega^n \oplus E_1, \dots, \Omega^n \oplus E_n), (0, \mathrm{Id}))$ with $(0, \mathrm{Id}) : E_n \to \Omega^n \oplus E_n$ the obvious inclusion.

It is easy to compute that for $L = E_n$ with $(E_1, \dots, E_n) \in \mathcal{N}_{0,n-1}$ and $(U_1, \dots, U_{n+1}) \in \mathcal{N}_{0,n}$ we have $\dim \mathcal{N}_{0,n} + \chi(L^* \otimes U_{n+1}) = \dim \mathcal{N}_{0,n-1} - \dim \mathrm{Bun}_n$.

Therefore we are reduced to the following lemma. We denote by $\alpha_{\mathcal{X}_i} : \mathcal{X}_i \to \mathrm{Bun}_n$ and $\beta_{\mathcal{X}_i} : \mathcal{X}_i \to \mathbb{A}^1$ the restrictions of $\alpha_{\mathcal{X}}$ and $\beta_{\mathcal{X}}$ to $\mathcal{X}_i$.

LEMMA 8. — *The natural maps* $\alpha_{\mathcal{X}_n,!}\beta_{\mathcal{X}_n}^*(\mathcal{L}_\psi) \to \alpha_{\mathcal{X}_{n-1},!}\beta_{\mathcal{X}_{n-1}}^*(\mathcal{L}_\psi) \to \dots \to \alpha_{\mathcal{X}_1,!}\beta_{\mathcal{X}_1}^*(\mathcal{L}_\psi) \to \alpha_{\mathcal{X},!}\beta_{\mathcal{X}}^*(\mathcal{L}_\psi)$ *are isomorphisms in* $\mathrm{D}^-(\mathrm{Bun}_n)_!$.

*Proof.* — We recall that $\mathcal{X} = \mathcal{X}_0$. Let $i \in \{1, \dots, n\}$. We are going to prove that the natural map

$$\alpha_{\mathcal{X}_i,!}\beta_{\mathcal{X}_i}^*(\mathcal{L}_\psi) \to \alpha_{\mathcal{X}_{i-1},!}\beta_{\mathcal{X}_{i-1}}^*(\mathcal{L}_\psi)$$

is an isomorphism. Set $\mathcal{Z}_i = \mathcal{X}_{i-1} \backslash \mathcal{X}_i$, let $\alpha_{\mathcal{Z}_i}$ and $\beta_{\mathcal{Z}_i}$ be the restrictions of $\alpha_{\mathcal{X}_{i-1}}$ and $\beta_{\mathcal{X}_{i-1}}$ to $\mathcal{Z}_i$. We must prove that $\alpha_{\mathcal{Z}_i,!}\beta_{\mathcal{Z}_i}^*(\mathcal{L}_\psi) = 0$.

Let $\mathcal{T}_i$ be stack classifying $(L, (V_1, V_2, \dots, V_i), t)$ with $L \in \mathrm{Bun}_n$, $(V_1, V_2, \dots, V_i) \in \mathcal{N}_{0,i-1}$, $t : L \to V_i$ such that the composition $L \xrightarrow{t} V_i \to V_i/V_1$ is surjective but $t$ is not surjective. The map $\alpha_{\mathcal{Z}_i}$ decomposes naturally as $\mathcal{Z}_i \xrightarrow{\gamma_{\mathcal{Z}_i}} \mathcal{T}_i \xrightarrow{\alpha_{\mathcal{T}_i}} \mathrm{Bun}_n$ where $\gamma_{\mathcal{Z}_i}(L, (U_1, \dots U_{n+1}), s) = (L, (U_{n+2-i}/U_{n+1-i}, \dots, U_{n+1}/U_{n+1-i}), \bar{s})$ and $\alpha_{\mathcal{T}_i}(L, (U_1, \dots U_{n+1}), s) = L$. It suffices to show that the $*$-fibre of $\gamma_{\mathcal{Z}_i,!}\beta_{\mathcal{Z}_i}^*(\mathcal{L}_\psi)$ at any closed point $(L, (V_1, V_2, \dots, V_i), t) \in \mathcal{T}_i$ vanishes.

Let us choose a closed point $(L, (V_1, V_2, \dots, V_i), t) \in \mathcal{T}_i$ and define $L'' = \mathrm{Ker}\, t$ and $L'$ the kernel of the composition $L \xrightarrow{t} V_i \to V_i/V_1$. Then $L'$ is a subbundle of $L$ of rank $n+1-i$ and $L''$ is a subbundle of $L$ of rank $n+1-i$ or $n-i$.

The fiber $\mathcal{Q}$ of $\gamma_{Z_i}$ over this closed point is the stack classifying $((U_1,...,U_{n+1}),s)$, with an isomorphism between $U_{n+1}/U_{n+1-i}$ and $V_i$ sending $U_{j+n+1-i}/U_{n+1-i}$ to $V_j$ for any $j\in\{0,\dots,i\}$ and $s:L\to U_{n+1}$ such that the composition $L\xrightarrow{s}U_{n+1}\to U_{n+1}/U_{n+1-i}\simeq V_i$ is $t$. Let $\mathcal{R}$ be stack classifying $((U_1,\dots,U_{n+1-i}),s_i)$ with $(U_1,\dots,U_{n+1-i})\in\mathcal{N}_{i,n}$ and $s_i\in\mathrm{Hom}(L'',U_{n+1-i})$. There is a morphism $\rho:\mathcal{Q}\to\mathcal{R}$ which sends $((U_1,...,U_{n+1}),s)$ to $((U_1,\dots,U_{n+1-i}),s_i)$ where $s_i$ is the restriction of $s$ to $L''$. Let $\beta_{\mathcal{Q}}:\mathcal{Q}\to\mathbb{A}^1$ be the restriction of $\beta_{Z_i}$ to $\mathcal{Q}$. It suffices to show that $\rho_!\beta_{\mathcal{Q}}^*(\mathcal{L}_\psi)=0$.

Pick $((U_1,\dots,U_{n+1-i}),s_i)\in\mathcal{R}$, let $\mathcal{S}$ be the fiber of $\rho$ over $((U_1,\dots,U_{n+1-i}),s_i)$. Write $\beta_{\mathcal{S}}$ for the restriction of $\beta_{\mathcal{Q}}$ to $\mathcal{S}$. We will show that $R\Gamma_c(\mathcal{S},\beta_{\mathcal{S}}^*(\mathcal{L}_\psi))=0$.

If $L'=L''$ we have an exact sequence $0\to L/L''\to U_{n+1}/U_{n+1-i}\to U_{n+2-i}/U_{n+1-i}\to 0$, and $\mathcal{S}$ identifies with the stack classifying extensions $0\to U_{n+1-i}\to ?\to U_{n+2-i}/U_{n+1-i}\to 0$ of $\mathcal{O}_X$-modules. Since $\beta_{\mathcal{S}}$ is a nontrivial character, we are done in this case.

If $L'/L''$ is a line bundle then $\mathcal{S}$ is a scheme with a free transitive action of the $H^0$ of the cone of the morphism of complexes of $k$-vector spaces

$$\mathrm{RHom}(U_{n+1}/U_{n+1-i},U_{n+1-i})\to\mathrm{RHom}(L/L'',U_{n+1-i})$$

which is also the cone of the morphism of complexes

$$\mathrm{RHom}(U_{n+2-i}/U_{n+1-i},U_{n+1-i})\to\mathrm{RHom}(L'/L'',U_{n+1-i})$$

and whose cohomology is concentrated in degree 0. The last morphism of complexes comes from the non zero morphism $L'/L''\to U_{n+2-i}/U_{n+1-i}=\Omega^{i-1}$ which identifies $L'/L''$ to $\Omega^{i-1}(-D)$ for some effective non zero divisor $D$. Therefore the $H^0$ of this cone is equal to

$$H^0(X,U_{n+1-i}\otimes\Omega^{1-i}(D)/U_{n+1-i}\otimes\Omega^{1-i})$$

and $\beta_{\mathcal{S}}^*(\mathcal{L}_\psi)$ transforms under this action through the character

$$H^0(X,U_{n+1-i}\otimes\Omega^{1-i}(D)/U_{n+1-i}\otimes\Omega^{1-i})\to$$

$$H^0(X,(U_{n+1-i}/U_{n-i})\otimes\Omega^{1-i}(D)/(U_{n+1-i}/U_{n-i})\otimes\Omega^{1-i})=H^0(X,\Omega(D)/\Omega)\xrightarrow{\sigma}\mathbb{A}^1$$

where $\sigma$ is the sum of the residues. Since $D$ is non zero, $\sigma$ is a non zero character and we are done. □

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