IMRN International Mathematics Research Notices
1997, No. 19

On the Moduli of $SL(2)$-bundles with Connections on $\mathbf{P}^1 \setminus \{x_1, \ldots, x_4\}$

D. Arinkin and S. Lysenko

Introduction

The moduli spaces of bundles with connections on algebraic curves have been studied from various points of view (see [6], [10]). Our interest in this subject was motivated by its relation with the Painlevé equations, and also by the important role of bundles with connections in the geometric Langlands program [4] (for more details see the remarks at the end of the introduction).

In this work, we consider $SL(2)$-bundles on $\mathbf{P}^1$ with connections. These connections are supposed to have poles of order 1 at fixed n points, and the eigenvalues $\pm\lambda_i$ of the residues are fixed. We call these bundles $(\lambda_1, \ldots, \lambda_n)$-bundles. Our aim is to find all invertible sheaves on the moduli space of $(\lambda_1, \ldots, \lambda_n)$-bundles and to compute the cohomology of these sheaves for $n = 4$.

In this work, the ground field is $\mathbf{C}$, that is, 'space' means '$\mathbf{C}$-space', $\mathbf{P}^1$ means $\mathbf{P}^1_{\mathbf{C}}$, and so on.

Let us formulate the main results of this work.

Fix $x_1, \ldots, x_n \in \mathbf{P}^1(\mathbf{C})$, $n \geq 4$, $x_i \neq x_j$ for $i \neq j$, and $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$.

Definition 1. A $(\lambda_1, \ldots, \lambda_n)$-*bundle* on $\mathbf{P}^1$ is a triple (L, ∇, φ) such that L is a rank 2 vector bundle on $\mathbf{P}^1$, $\nabla\colon L \to L \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n)$ is a connection, $\varphi\colon \Lambda^2 L \xrightarrow{\sim} O_{\mathbf{P}^1}$ is a horizontal isomorphism, and the residue R_i of the connection ∇ at x_i has eigenvalues $\pm\lambda_i$, $1 \leq i \leq n$.

Received 16 September 1997.
Communicated by Yu. I. Manin.

In the sequel, we suppose that

$$\sum_{i=1}^{n} \epsilon_i \lambda_i \notin \mathbf{Z} \tag{1}$$

for any (ϵ_i), $\epsilon_i \in \mu_2 := \{1, -1\}$.

Denote by $\mathcal{M}$ the moduli stack of $(\lambda_1, \dots, \lambda_n)$-bundles, and by M the corresponding coarse moduli space.

Theorem 1. Suppose that (1) holds and $\lambda_1, \dots, \lambda_n \neq 0$. Then

(i) M is a smooth irreducible separated scheme, $\dim M = 2n - 6$, and $\mathcal{M}$ is a μ_2-gerbe over M;

(ii) $H^i(M, \mathcal{F}) = 0$ for $i > n - 3$ for any quasicoherent O_M-module $\mathcal{F}$;

(iii) $\operatorname{Pic} \mathcal{M}$ is the free abelian group with generators $\delta, \xi_1, \dots, \xi_n$. Here δ (resp. ξ_i) is the invertible sheaf on $\mathcal{M}$ whose fiber over (L, ∇, φ) equals $\det R\Gamma(\mathbf{P}^1, L)$ (resp. $l_i := \operatorname{Ker}(R_i - \lambda_i) \subset L_{x_i}$, $R_i \colon L_{x_i} \to L_{x_i}$ is the residue of ∇ at x_i);

(iv) $\operatorname{Pic} M \subset \operatorname{Pic} \mathcal{M}$ is an index 2 subgroup, $\xi_1, \dots, \xi_n \notin \operatorname{Pic} M$, $\delta \in \operatorname{Pic} M$;

(v) the cohomology class $[\alpha] \in H^2_{ét}(M, \mu_2)$ corresponding to the μ_2-gerbe $\mathcal{M} \to M$ is the image of the nonzero element of $2 \operatorname{Pic} \mathcal{M} / 2 \operatorname{Pic} M$ via the canonical embedding $\operatorname{Pic} M / 2 \operatorname{Pic} M \to H^2_{ét}(M, \mu_2)$. In particular, $[\alpha] \neq 0$. □

Theorem 2. Let $n = 4$. Suppose that (1) holds and $2\lambda_i \notin \mathbf{Z}$, $1 \leq i \leq 4$. Define $\deg \colon \operatorname{Pic} M \to \mathbf{Z}$ by $\deg(a\delta + \sum_{i=1}^4 a_i \xi_i) := -a$. Let γ be an invertible sheaf on M.

(i) If $\deg \gamma > 0$, then $\dim H^0(M, \gamma) = \infty$, $H^i(M, \gamma) = 0$ for $i \neq 0$.

(ii) If $\deg \gamma < 0$, then $\dim H^1(M, \gamma) = \infty$, $H^i(M, \gamma) = 0$ for $i \neq 1$.

(iii) If $\gamma \simeq O_M$, then $\dim H^0(M, \gamma) = 1$, $H^i(M, \gamma) = 0$ for $i \neq 0$.

(iv) If $\deg \gamma = 0$ and $\gamma \not\simeq O_M$, then $\dim H^1(M, \gamma) = -[\langle \gamma, \gamma \rangle / 2] - 1$, $H^i(M, \gamma) = 0$ for $i \neq 1$. Here the bilinear form $\langle \cdot, \cdot \rangle$ is defined by

$$\left\langle \sum_{i=1}^{4} a_i \xi_i, \sum_{i=1}^{4} b_i \xi_i \right\rangle := -\frac{\sum_{i=1}^{4} a_i b_i}{2},$$

and $[a]$ is the integral part of a. □

Let us describe the general plan of this work.

In the first part (Sections 1–3), we study $(\lambda_1, \dots, \lambda_n)$-bundles for arbitrary n.

In Section 1, we prove the basic properties of $(\lambda_1, \dots, \lambda_n)$-bundles. We prove that M is a separated algebraic space. All the results of this section are still valid for any curve.

In Section 2, we construct an affine bundle $M \to N$, where N is the coarse moduli space of quasiparabolic bundles of a certain kind. We use this construction to prove that

M is a smooth scheme of dimension $2n-6$ and to show that the cohomological dimension of M is at most $n-3$.

Section 3 contains the calculation of the Picard group of M. This calculation uses the ideas of [3]. We also compute the cohomology class of the gerbe $\mathcal{M} \to M$.

In Sections 4 and 5, we assume that $n = 4$.

In Section 4, we give an explicit geometric description of M. This description goes back to Okamoto ([7], [9]) who studied M as the space of initial conditions of the Painlevé equation P_{VI} rather than the moduli space of bundles with connections.

In Section 5, we compute the cohomology of invertible sheaves on M.

Remarks. (1) The description of $\operatorname{Pic} M$ from Theorem 1 was used in [2] to describe all the isomorphisms between the varieties M for $n = 4$, and thereby to give a geometric explanation of the mysterious symmetries of the P_{VI} equation found by Okamoto [8].

(2) Theorem 2 was used by one of the authors (D. Arinkin) to prove the following orthogonality relations: if $n = 4$ and $x, y \in \mathbf{P}^1 \setminus \{x_1, \ldots, x_4\}$, then

$$H^i(\mathcal{M}, \xi_x \otimes \xi_y) = 0 \text{ unless } x = y, i = 0, \text{ and}$$

$$H^0(\mathcal{M}, \xi_x \otimes \xi_x) = \mathbf{C}$$

where ξ_x is the vector bundle on $\mathcal{M}$ whose fiber at (L, ∇, φ) equals L_x. These formulas can be interpreted in terms of the geometric Langlands program.

(3) The results of this paper were announced in [1].

1 $(\lambda_1, \ldots, \lambda_n)$-bundles

1.1 Basic properties of $(\lambda_1, \ldots, \lambda_n)$-bundles

Let (L, ∇, φ) be a $(\lambda_1, \ldots, \lambda_n)$-bundle.

Proposition 1. (L, ∇) is irreducible (i.e., there is no rank 1 ∇-invariant subbundle $L_1 \subset L$). □

Proof. Suppose there is an invariant rank 1 subbundle $L_1 \subset L$. Then $\nabla_1 := \nabla|_{L_1}$ is a connection on L_1. $(L_1)_{x_i} \subset L_{x_i}$ is an eigenspace of $R_i := \operatorname{res}_{x_i}(\nabla)$. Hence $\operatorname{res}_{x_i}(\nabla_1)$ is an eigenvalue of R_i, that is, $\operatorname{res}_{x_i}(\nabla_1) = \pm\lambda_i$. But $\sum_{i=1}^n \operatorname{res}_{x_i}(\nabla_1) = -\deg L_1 \in \mathbf{Z}$. This contradicts (1). ■

Remark 4. Denote by V the fiber of L over the generic point of $\mathbf{P}^1$. V is a 2-dimensional vector space over $\mathbf{C}(z)$ (here $\operatorname{Spec} \mathbf{C}(z) \in \mathbf{P}^1$ is the generic point); 1-dimensional subspaces of V correspond to rank 1 subbundles of L. ∇ induces a $\mathbf{C}$-linear morphism $V \to V \otimes_{\mathbf{C}(z)}$

$\Omega_{\mathrm{Spec}(\mathbf{C}(z))}$. So the proposition implies that V is irreducible (as a $\mathbf{C}(z)$-space) with respect to this morphism.

Corollary 1. The only automorphisms of (L, ∇, φ) are 1 and -1 (in other words, the group of automorphisms of (L, ∇, φ) is μ_2). □

Proof. Let A be any automorphism of (L, ∇, φ). Clearly it has an eigenvalue $e \in \mathbf{C}$. Then $\mathrm{Ker}(A - e) \subset L$ is an invariant subbundle, $\mathrm{Ker}(A - e) \neq 0$, so $\mathrm{Ker}(A - e) = L$ and $A = e$. But $\det(A) = 1$, so $A = \pm 1$. ■

Corollary 2. Let $L_1 \subset L$ be a rank 1 subbundle. Then $\deg L_1 \leq (n - 2)/2$. □

Proof. By Proposition 1, the map $L_1 \to (L/L_1) \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n)$ induced by ∇ is not zero. So $\deg L_1 \leq \deg(L/L_1) + n - 2$. The corollary easily follows. ■

Remark. Let us consider $(\lambda_1, \ldots, \lambda_n)$-bundles on a curve of genus $g > 0$. Then Proposition 1 is still true, and Corollary 2 has the form

$$\deg L_1 \leq \frac{n + 2g - 2}{2}.$$

1.2 Moduli space of $(\lambda_1, \ldots, \lambda_n)$-bundles

The notion of a family of $(\lambda_1, \ldots, \lambda_n)$-bundles on $\mathbf{P}^1$ is defined in the usual way. $(\lambda_1, \ldots, \lambda_n)$-bundles on $\mathbf{P}^1$ form a stack $\mathcal{M}$. So $\mathcal{M}_S$ (the category of 1-morphisms from S to $\mathcal{M}$) is the category of families of $(\lambda_1, \ldots, \lambda_n)$-bundles parametrized by a scheme S.

Proposition 2. $\mathcal{M}$ is a separated algebraic stack. □

Proof. Denote by $\mathrm{Bun}_{SL(2)}\mathbf{P}^1$ the moduli stack of $SL(2)$-bundles on $\mathbf{P}^1$. It is well known ([5, Theorem 4.14.2.1]) that $\mathrm{Bun}_{SL(2)}\mathbf{P}^1$ is an algebraic stack. Clearly the natural map $\mathcal{M} \to \mathrm{Bun}_{SL(2)}\mathbf{P}^1$ is a representable (and even affine) 1-morphism of stacks. Hence $\mathcal{M}$ is algebraic.

Using the valuative criterion for algebraic stacks ([5, Theorem 3.19, Remark 3.20.2]), one can derive from Lemma 1 that $\mathcal{M}$ is separated. ■

Lemma 1. Let A be a discrete valuation ring, K the fraction field of A, $\eta := \mathrm{Spec}(K)$, $y_0 = (L_0, \nabla_0, \varphi_0) \in \mathrm{Ob}(\mathcal{M}_\eta)$ (i.e., y_0 is a family of $(\lambda_1, \ldots, \lambda_n)$-bundles parametrized by η). If an extension of y_0 to $y \in \mathrm{Ob}(\mathcal{M}_U)$, $U := \mathrm{Spec}(A)$ exists, it is unique. □

Proof. Let $y_i = (L_i, \nabla_i, \varphi_i) \in \mathrm{Ob}(\mathcal{M}_U)$, $i = 1, 2$ be two extensions of y_0. Denote by $\mathcal{F}_i$ the sheaf of sections of L_i, $i = 0, 1, 2$. Let $\widetilde{\mathcal{F}_0}$ be the direct image of $\mathcal{F}_0$ to $U \times \mathbf{P}^1$. Then ∇_0 (resp. φ_0) induces a connection $\nabla \colon \widetilde{\mathcal{F}_0} \to \widetilde{\mathcal{F}_0} \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n)$ (resp. a horizontal isomorphism

$\varphi\colon \Lambda^2\widetilde{\mathcal{F}_0}\widetilde{\rightarrow}O_{\eta\times\mathbf{P}^1}$). Since y_i is an extension of y_0, $\mathcal{F}_i$ is identified with a subsheaf of $\widetilde{\mathcal{F}_0}$; this identification agrees with ∇ and φ. Set $\mathcal{F} := \mathcal{F}_1 \cap \mathcal{F}_2$.

Denote by k the residue field of A (so $\operatorname{Spec} k \in U$ is the special point), and by $p \in \mathbf{P}^1_k \subset U \times \mathbf{P}^1$ the generic point of the special fiber $\mathbf{P}^1_k \subset U \times \mathbf{P}^1$.

There is $i \in \{1, 2\}$ such that $\mathcal{F}(\mathbf{P}^1_k) \not\subset \mathcal{F}_i$. We may assume that $i = 1$.

Denote by V_1 the fiber of L_1 over p, and by $V \subset V_1$ the image of $\mathcal{F} \subset \mathcal{F}_1$. Since $\mathcal{F} \not\subset \mathcal{F}_1(-\mathbf{P}^1_k)$, we have $V \neq 0$.

$\nabla(\mathcal{F}_i) \subset \mathcal{F}_i \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n)$, so $\nabla(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n)$. Therefore $V \subset V_1$ is ∇-invariant and, by Remark 4, $V = V_1$.

$\mathcal{F} \subset \mathcal{F}_1$ is locally free so $\mathcal{F} = \mathcal{F}_1$ and $\mathcal{F}_2 \supset \mathcal{F}_1$. But $\varphi(\Lambda^2\mathcal{F}_1) = \varphi(\Lambda^2\mathcal{F}_2)$, so $\mathcal{F}_2 = \mathcal{F}_1$. ■

For a scheme S, denote by $\underline{M}(S)$ the set of isomorphism classes of families of $(\lambda_1, \dots, \lambda_n)$-bundles parametrized by S. Denote by M the sheaf for the fppf-topology associated with the presheaf $\underline{M}$.

By Corollary 1, $\mathcal{M}$ is a μ_2-gerbe over M. In particular, the 1-morphism $\mathcal{M} \to M$ is smooth, surjective, and proper. This implies that M is a separated algebraic space (M is the coarse moduli space of $(\lambda_1, \dots, \lambda_n)$-bundles).

2 Structure of affine bundle on M

2.1 Quasiparabolic bundles

A quasiparabolic SL(2)-bundle on $\mathbf{P}^1$ is a collection $(L, \varphi, l_1, \dots, l_n)$ such that L is a rank 2 vector bundle on $\mathbf{P}^1$, $\varphi\colon \Lambda^2 L\widetilde{\rightarrow}O_{\mathbf{P}^1}$, and $l_i \subset L_{x_i}$ is a 1-dimensional subspace. Quasiparabolic SL(2)-bundles form a stack $\overline{\mathcal{N}}$. Using the same arguments as in Proposition 2, one can prove that $\overline{\mathcal{N}}$ is algebraic.

Suppose that $\lambda_1, \dots, \lambda_n \neq 0$. For a $(\lambda_1, \dots, \lambda_n)$-bundle (L, ∇, φ), we construct a quasiparabolic SL(2)-bundle $(L, \varphi, l_1, \dots, l_n)$ by setting $l_i := \operatorname{Ker}(R_i - \lambda_i)$, where $R_i\colon L_{x_i} \to L_{x_i}$ is the residue of ∇ at x_i. This yields a morphism $\overline{f}\colon \mathcal{M} \to \overline{\mathcal{N}}$. Let us give an explicit description of the image of $\overline{f}$.

Proposition 3. For a quasiparabolic SL(2)-bundle $(L, \varphi, l_1, \dots, l_n)$, the following conditions are equivalent:

(i) $(L, \varphi, l_1, \dots, l_n)$ belongs to the image of $\overline{f}\colon \mathcal{M} \to \overline{\mathcal{N}}$;

(ii) $\operatorname{Aut}(L, \varphi, l_1, \dots, l_n) = \mu_2$;

(ii′) $\operatorname{End}(L, l_1, \dots, l_n) = \mathbf{C}$;

(iii) $(L, \varphi, l_1, \dots, l_n)$ is indecomposable; that is, there are no $L_1, L_2 \neq 0$ such that $L = L_1 \oplus L_2$, and for any i, either $l_i = (L_1)_{x_i}$ or $l_i = (L_2)_{x_i}$. □

Proof (i) $\Rightarrow$ (iii). Suppose $(L, \varphi, l_1, \ldots, l_n)$ belongs to the image of $\bar{f}$; that is, there is a $\nabla\colon L \to L \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n)$ such that (L, ∇, φ) is a $(\lambda_1, \ldots, \lambda_n)$-bundle and $l_i = \mathrm{Ker}(R_i - \lambda_i)$. Suppose $L = L_1 \oplus L_2$ for $L_1, L_2 \neq 0$. The composition $\nabla_1\colon L_1 \to L \xrightarrow{\nabla} L \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n) \to L_1 \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n)$ is a connection on L_1. (1) implies that $\mathrm{res}_{x_i} \nabla_1 \neq \pm\lambda_i$ for some i. It is easy to prove that $l_i \neq (L_1)_{x_i}, (L_2)_{x_i}$ for this i.

(iii) $\Rightarrow$ (ii′). Suppose $A \in \mathrm{End}(L, l_1, \ldots, l_n)$. Denote by $e_1, e_2 \in \mathbf{C}$ the eigenvalues of A. If $e_1 \neq e_2$, L can be decomposed to the direct sum of the eigenspaces of A.

Assume that $e_1 = e_2$. Replacing A by $A - e_1$, we can assume that $e_1 = e_2 = 0$. Let us prove that $A = 0$. Assume the converse. Then $L_1 := \mathrm{Ker}(A) \subset L$ is a rank 1 subbundle. Set $T := \{i | l_i \neq (L_1)_{x_i}\}$. Locally on $\mathbf{P}^1$ we can construct L_2 such that $L_1 \oplus L_2 = L$, $(L_2)_{x_i} = l_i$ for $i \in T$. Obstructions to global existence of L_2 lie in $H^1(\mathbf{P}^1, \mathcal{H}om(L/L_1, L_1)(-\sum_{i\in T} x_i))$. Since $(L, l_1, \ldots, l_n)$ is indecomposable, this space is not zero. So $\deg(\mathcal{H}om(L/L_1, L_1)(-\sum_{i\in T} x_i)) < -1$, and $\{A_1 \in \mathrm{Hom}(L/L_1, L_1) | A_1(x_i) = 0 \text{ for } i \in T\} = H^0(\mathbf{P}^1, \mathcal{H}om(L/L_1, L_1)(-\sum_{i\in T} x_i)) = 0$. Clearly A induces a map $A_1\colon L/L_1 \to L_1$ such that $A_1(x_i) = 0$ for $i \in T$. Hence $A_1 = 0$ and $A = 0$.

(ii′) $\Rightarrow$ (i). Let us construct a connection $\nabla\colon L \to L \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n)$ such that (L, ∇, φ) is a $(\lambda_1, \ldots, \lambda_n)$-bundle, and $l_i = \mathrm{Ker}(R_i - \lambda_i)$. This can be done locally on $\mathbf{P}^1$. The obstructions to global construction lie in $H^1(\mathbf{P}^1, \mathcal{E})$, where $\mathcal{E} := \{A \in \mathcal{E}nd_0(L) \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_n) | (\mathrm{res}_{x_i} A)(l_i) = 0\}$. Here $\mathcal{E}nd_0(L) := \{A \in \mathcal{E}nd(L) | \operatorname{tr} A = 0\}$. By Serre's duality theorem, $H^1(\mathbf{P}^1, \mathcal{E})$ is dual to $H^0(\mathbf{P}^1, \{A \in \mathcal{E}nd_0(L) | A(x_i)(l_i) \subset l_i\}) = \{A \in \mathrm{End}(L, l_1, \ldots, l_n) | \operatorname{tr}(A) = 0\} = 0$. So there is a global ∇ with such properties.

(ii′) $\Rightarrow$ (ii). This implication is obvious since $\mathrm{Aut}(L, \varphi, l_1, \ldots, l_n) = \{A \in \mathrm{End}(L, l_1, \ldots, l_n) | \det(A) = 1\}$.

(not (iii)) $\Rightarrow$ (not (ii)). Let $L = L_1 \oplus L_2$ be a decomposition of L. Then $a \oplus a^{-1} \in \mathrm{Aut}(L, \varphi, l_1, \ldots, l_n)$ for $a \in \mathbf{C}^*$. ∎

Remark. The proof of the implication (iii) $\Rightarrow$ (ii′) does not work for curves of genus $g > 0$, because it uses the following property of $\mathbf{P}^1$: for every line bundle $\mathcal{F}$ on $\mathbf{P}^1$, either $H^0(\mathbf{P}^1, \mathcal{F}) = 0$ or $H^1(\mathbf{P}^1, \mathcal{F}) = 0$.

If $(L, \varphi, l_1, \ldots, l_n)$ satisfies the equivalent conditions of Proposition 3, the fiber of $\bar{f}$ over $(L, \varphi, l_1, \ldots, l_n)$ consists of all $(\lambda_1, \ldots, \lambda_n)$-bundles (L, ∇, φ) such that $l_i = \mathrm{Ker}(R_i - \lambda_i)$. Such ∇ form an affine space of dimension $n - 3$ because the corresponding vector space is dual to $H^1(\mathbf{P}^1, \mathcal{E}nd_0(L, l_1, \ldots, l_n))$, and the Euler characteristic of $\mathcal{E}nd_0(L, l_1, \ldots, l_n)$ equals $\chi(\mathcal{E}nd_0 L) - n = 3 - n$.

Denote by $\mathcal{N} \subset \overline{\mathcal{N}}$ the open substack defined by condition (ii′) from Proposition 3. $\bar{f}$ induces the morphism $f\colon \mathcal{M} \to \mathcal{N}$, which is a locally trivial affine bundle with fibers of dimension $n - 3$.

Denote by N the coarse moduli space of indecomposable quasiparabolic $SL(2)$-bundles on $\mathbf{P}^1$. The construction of the algebraic space N is similar to that of M (see Section 1.2). $\mathcal{N}$ is a μ_2-gerbe over N.

2.2 Modifications

Suppose L is a rank 2 bundle on $\mathbf{P}^1$, $x \in \mathbf{P}^1$, and $l \subset L_x$ is a dimension 1 subspace. Denote by $\mathcal{L}$ the sheaf of sections of L. The *lower (resp. upper)* (x, l)-*modification of* L is the rank 2 bundle on $\mathbf{P}^1$ whose sheaf of sections is $\widetilde{\mathcal{L}} := \{s \in \mathcal{L} | s(x) \in l\}$ (resp. $\widetilde{\mathcal{L}}(x)$). If $\widetilde{L}$ is the lower (x, l)-modification of L, the image of the natural map $\widetilde{L}_x \to L_x$ is l. Denote by $\widetilde{l} \subset \widetilde{L}_x$ the kernel of this map. Then L is the upper $(x, \widetilde{l})$-modification of $\widetilde{L}$.

Suppose $(L, l_1, \dots, l_n)$ is a *quasiparabolic bundle on* $\mathbf{P}^1$ (i.e., L is a rank 2 bundle on $\mathbf{P}^1$, and $l_i \subset L_{x_i}$ is a dimension 1 subspace). Then the lower (x_i, l_i)-modification $\widetilde{L}$ of L has a natural structure of a quasiparabolic bundle, namely, $(\widetilde{L}, l_1, \dots, \widetilde{l}_i, \dots, l_n)$, where $\widetilde{l}_i := \mathrm{Ker}(\widetilde{L}_{x_i} \to L_{x_i})$. Similarly, the upper (x_i, l_i)-modification of $(L, l_1, \dots, l_n)$ is a quasiparabolic bundle.

Clearly $(\widetilde{L}, l_1, \dots, \widetilde{l}_i, \dots, l_n)$ is indecomposable if and only if $(L, l_1, \dots, l_n)$ is indecomposable.

Lemma 2. Suppose $(L, l_1, \dots, l_n)$ is an indecomposable quasiparabolic bundle on $\mathbf{P}^1$. Then making (x_i, l_i)-modifications in some of the points x_i, one can transform $(L, l_1, \dots, l_n)$ to $(L', l'_1, \dots, l'_n)$ such that $L' \simeq O_{\mathbf{P}^1}(k')^2$ for some k'. □

Proof. Since L is a rank 2 bundle on $\mathbf{P}^1$, $L \simeq O_{\mathbf{P}^1}(k) \oplus O_{\mathbf{P}^1}(l)$ for some $k, l \in \mathbf{Z}$, $k \geq l$. The proof is given by induction on $k - l$.

For $k - l = 0$, there is nothing to prove.

Suppose $k - l > 0$. Denote by $L_1 \subset L$ the rank 1 subbundle of degree k. Since $(L, l_1, \dots, l_n)$ is indecomposable, $l_i \neq (L_1)_{x_i}$ for some i. Let $\widetilde{L}$ be the lower (x_i, l_i)-modification of L. Then L_1 defines a rank 1 subbundle $\widetilde{L}_1 \subset \widetilde{L}$ of degree $k-1$. Clearly $\widetilde{L}/\widetilde{L}_1 = L/L_1$, so $\deg(\widetilde{L}/\widetilde{L}_1) = l$. Hence $\widetilde{L} \simeq O_{\mathbf{P}^1}(k-1) \oplus O_{\mathbf{P}^1}(l)$. By the induction hypothesis, $\widetilde{L}$ can be modified to $(L', l'_1, \dots, l'_n)$ such that $L' \simeq O_{\mathbf{P}^1}(k')^2$ for some $k' \in \mathbf{Z}$. ■

Let us return to the case of $SL(2)$-bundles.

Let $(L, \varphi, l_1, \dots, l_n)$ be a quasiparabolic $SL(2)$-bundle, $T \subset \{1, \dots, n\}$. Denote by $(L', l'_1, \dots, l'_n)$ the lower modification of $(L, l_1, \dots, l_n)$ at (x_i, l_i) for all $i \in T$ (clearly, modifications at different points commute). Then φ induces an isomorphism $\varphi' \colon \Lambda^2 L' \xrightarrow{\sim} O_{\mathbf{P}^1}(-\sum_{i \in T} x_i)$. Suppose that $\mathrm{Card}\, T = 2k$, where $\mathrm{Card}\, T$ is the number of elements of the set T. We choose an isomorphism $s\colon\ O_{\mathbf{P}^1}(2kx_1 - \sum_{i\in T} x_i) \xrightarrow{\sim} O_{\mathbf{P}^1}$. $s \circ \varphi'$ gives a structure of quasiparabolic SL(2)-bundle on $L'(kx_1)$. This defines an automorphism $\overline{f}_T\colon \overline{\mathcal{N}} \xrightarrow{\sim} \overline{\mathcal{N}}$. Since $\overline{f}_T(\mathcal{N}) = \mathcal{N}$, this gives $f_T\colon N \xrightarrow{\sim} N$. Obviously, f_T does not depend on s.

Denote by Γ the set of all $T \subset \{1, \dots, n\}$ such that $\operatorname{Card} T$ is even. Γ is an abelian group with respect to the product $T_1 \triangle T_2 := (T_1 \cup T_2) \setminus (T_1 \cap T_2)$.

Proposition 4. (i) $f_{T_1} \circ f_{T_2} = f_{T_1 \triangle T_2}$ $(T_1, T_2 \in \Gamma)$.

(ii) Denote by $N_0 \subset N$ the open subspace formed by trivial SL(2)-bundles (i.e., $(L, \varphi, l_1, \dots, l_n) \in N_0$ if and only if $L \simeq O^2_{\mathbf{P}^1}$). Then $\bigcup_{T \in \Gamma} f_T(N_0) = N$. □

Proof. Statement (i) is obvious. Statement (ii) follows from Lemma 2. ■

2.3 Geometry of N

Let N_0 have the same meaning as in Proposition 4(ii).

Lemma 3. N_0 is a smooth irreducible nonseparated scheme of dimension $n - 3$. □

Proof. Denote by U the set of $(l_1, \dots, l_n) \in (\mathbf{P}^1)^n$ such that there are at least three different points among $l_1, \dots, l_n$. Then $N_0 = PGL(2) \setminus U$. Set $U_{ijk} := \{(l_1, \dots, l_n) \in (\mathbf{P}^1)^n | l_i \neq l_j, l_j \neq l_k, l_i \neq l_k\} \subset U$, where $1 \leq i < j < k \leq n$. Then $U_{ijk} \subset U$ is open, $\bigcup_{i,j,k} U_{ijk} = U$, and $\bigcap_{i,j,k} U_{ijk} = \{(l_1, \dots, l_n) \in (\mathbf{P}^1)^n | l_i \neq l_j \text{ for } i \neq j\} \neq \emptyset$. So N_0 is covered by pairwise intersecting open subsets $PGL(2) \setminus U_{ijk}$. Finally, $PGL(2) \setminus U_{ijk} \simeq (\mathbf{P}^1)^{n-3}$. ■

Proposition 5. N is a smooth irreducible nonseparated scheme of dimension $n - 3$. □

Proof. Since N is covered by $f_T(N_0)$, $T \in \Gamma$ (Proposition 4), and N_0 is a smooth irreducible nonseparated scheme (Lemma 3), it is enough to prove that $f_T(N_0) \cap N_0 \neq \emptyset$.

Any T can be represented as a product of $T_{ij} = \{i, j\} \in \Gamma$, $i \neq j$. Since N_0 is irreducible, it is enough to prove that $N_0 \cap f_{T_{ij}}(N_0) \neq \emptyset$. Clearly, $N_0 \cap f_{T_{ij}}(N_0) = PGL(2) \setminus \{(l_1, \dots, l_n) \in U | l_i \neq l_j\} \neq \emptyset$. ■

Using the affine bundle $f\colon M \to N$, one derives statements (i) and (ii) of Theorem 1 from Proposition 5.

Remark. In the special case $n = 4$, one can prove the following explicit description of N:

There is a map $N \to \mathbf{P}^1$ that identifies N and 'the projective line with doubled points $x_1, \dots, x_4$.' In other words, N can be obtained by glueing two copies of $\mathbf{P}^1$ outside $x_1, \dots, x_4$.

3 Invertible sheaves on M

3.1 Calculation of Pic $\overline{\mathcal{N}}$

Denote by ξ_i (resp. δ) the invertible sheaf on $\overline{\mathcal{N}}$ whose fiber over $(L, \varphi, l_1, \dots, l_n)$ is l_i (resp. $\det R\Gamma(\mathbf{P}^1, L)$).

Notation. For the sake of simplicity, we write ξ_i (resp. δ) for the inverse image of ξ_i (resp. δ) to $\mathcal{M}$.

The following proposition is an easy, special case of the general theorem due to Y. Laszlo and C. Sorger in [3, Theorem 1.1].

Proposition 6. Pic $\overline{\mathcal{N}}$ is the free abelian group with basis δ, ξ_i $(i = 1, \dots, n)$. □

Remark. The proof by Y. Laszlo and C. Sorger is based on the techniques of affine Grassmanianns. In our situation, Proposition 6 for $n = 0$ follows from the well-known description of the isomorphism classes of SL(2)-bundles on $\mathbf{P}^1$, and the case of an arbitrary n is easily reduced to $n = 0$.

3.2 Calculation of Pic M

Lemma 4. $\operatorname{codim}(\overline{\mathcal{N}} \setminus \mathcal{N}) \geq 2$. □

Proof. Denote by $\mathcal{N}_d$ the moduli stack of decompositions. In other words, $\mathcal{N}_d$ parametrizes $(L = L_1 \oplus L_2, \varphi; l_1, \dots, l_n)$ such that $(L, \varphi, l_1, \dots, l_n)$ is a quasiparabolic SL(2)-bundle, $\operatorname{rk} L_1 = \operatorname{rk} L_2 = 1$, and for any $i = 1, \dots, n$, either $l_i = (L_1)_{x_i}$ or $l_i = (L_2)_{x_i}$. Connected components of $\mathcal{N}_d$ are parametrized by $(\deg L_1, \{i | l_i = (L_1)_{x_i}\})$; hence the set of these components is countable. Besides, each component is of dimension -1.

Consider the natural map $\mathcal{N}_d \to \overline{\mathcal{N}}$. Its image is $\overline{\mathcal{N}} \setminus \mathcal{N}$, so $\dim \overline{\mathcal{N}} \setminus \mathcal{N} \leq -1$. On the other hand, $\dim \overline{\mathcal{N}} = n - 3 \geq 1$. ■

Corollary 3. Pic $\mathcal{M}$ = Pic $\mathcal{N}$ = Pic $\overline{\mathcal{N}}$ is the free abelian group with basis $\xi_1, \dots, \xi_n, \delta$. □

Proof. Since $\mathcal{M} \to \mathcal{N}$ is an affine bundle, Pic $\mathcal{M}$ = Pic $\mathcal{N}$. Since $\overline{\mathcal{N}}$ is a smooth stack, Lemma 4 implies Pic $\mathcal{N}$ = Pic $\overline{\mathcal{N}}$. Now the corollary follows from Proposition 6. ■

Proposition 7. Pic $M \subset$ Pic $\mathcal{M}$ is the subgroup of index 2 such that $\delta \in$ Pic M, $\xi_i \notin$ Pic M. □

Proof. Since $\mathcal{M}$ is a μ_2-gerbe over M, any $O_{\mathcal{M}}$-module has a natural action of μ_2. An $O_{\mathcal{M}}$-module is an O_M-module if and only if this action is trivial. It follows from the definitions that $-1 \in \mu_2$ acts as -1 on ξ_i and acts as 1 on δ. ■

We have proved statements (iii) and (iv) of Theorem 1. Statement (v) is a particular case of the following lemma.

Lemma 5. Let X be an algebraic space, $i\colon \mathcal{X} \to X$ a μ_2-gerbe, $[\alpha] \in H^2_{\text{ét}}(X, \mu_2)$ the corresponding cohomology class, and $\gamma \in \operatorname{Pic}\mathcal{X}$ the isomorphism class of a sheaf $\mathcal{E}$ such that $-1 \in \mu_2$ acts on $\mathcal{E}$ as -1. Then $[\alpha] = c_1(\gamma^{\otimes 2})$, where $c_1\colon \operatorname{Pic} X \to H^2_{\text{ét}}(X, \mu_2)$ is the Chern class. □

Proof. Fix a sheaf $\mathcal{F}$ in the class $\gamma^{\otimes 2} \in \operatorname{Pic} X$. Denote by $\mathcal{S}qr\,\mathcal{F}$ the μ_2-gerbe of square roots of $\mathcal{F}$ defined by $(\mathcal{S}qr\,\mathcal{F})_S := \{(f\colon S \to X, \mathcal{E}', \psi) | \mathcal{E}' \text{ is an invertible sheaf on } S, \psi\colon (\mathcal{E}')^{\otimes 2} \xrightarrow{\sim} f^*(\mathcal{F})\}$.

An isomorphism $\mathcal{E}^{\otimes 2} \xrightarrow{\sim} i^*\mathcal{F}$ yields a 1-morphism $\mathcal{X} \to \mathcal{S}qr\,\mathcal{F}$. Since $-1 \in \mu_2$ acts on $\mathcal{E}$ as -1, this is a μ_2-gerbe morphism. So μ_2-gerbes $\mathcal{X}$ and $\mathcal{S}qr\,\mathcal{F}$ are isomorphic.

Let $T := \operatorname{Isom}(O_X, \mathcal{F})$ be the $\mathbf{G_m}$-torsor corresponding to $\mathcal{F}$. Consider the exact sequence $0 \to \mu_2 \to \mathbf{G_m} \xrightarrow{x \mapsto x^2} \mathbf{G_m} \to 0$. The corresponding map $H^1_{\text{ét}}(X, \mathbf{G_m}) = \operatorname{Pic} X \to H^2_{\text{ét}}(X, \mu_2)$ is c_1. Now it is enough to notice that $\mathcal{S}qr\,\mathcal{F}$ is the gerbe of liftings of T with respect to

$$\mathbf{G_m} \xrightarrow{x \mapsto x^2} \mathbf{G_m}. \qquad ■$$

This completes the proof of Theorem 1.

4 Geometric description of M

Suppose that $n = 4$, $\lambda_i \neq 0$ ($i = 1, \dots, 4$), and $\lambda_1 \neq 1/2$. Recall that M is the coarse moduli space of $(\lambda_1, \dots, \lambda_4)$-bundles. The aim of this section is to prove the following statement:

Set $K := \mathbf{V}((\Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4))^*)$ (i.e., K is the vector bundle whose sheaf of sections is $\Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4)$). Denote by $b_i \subset K$ the fiber over $x_i \subset \mathbf{P}^1$. Since $(\Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4))_{x_i} = \mathbf{C}$, there is a natural isomorphism $r_i\colon b_i \xrightarrow{\sim} \mathbf{A}^1$. Set $\lambda_i^{\pm} := \pm\lambda_i$ for $i \neq 1$, $\lambda_1^+ := \lambda_1$, $\lambda_1^- := 1 - \lambda_1$, $c_i^{\pm} := r_i^{-1}(\lambda_i^{\pm}) \in b_i$. For every i, one has $\lambda_i^+ \neq \lambda_i^-$, so $c_i^+ \neq c_i^-$.

Theorem 3. Denote by $\widetilde{M}$ the blow-up of K in $c_i^{\pm}$. Then there is an open embedding $M \hookrightarrow \widetilde{M}$ such that $\widetilde{M} \setminus M$ is the union of the proper preimages of $b_i \subset K$, $i = 1, \dots, 4$. □

4.1 Construction of $M \to K$

Denote by M_1 the coarse moduli space of triples $(\widetilde{L}, \nabla, \varphi)$ such that $\widetilde{L}$ is a rank 2 vector bundle on $\mathbf{P}^1$, $\nabla\colon \widetilde{L} \to \widetilde{L} \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4)$ is a connection, $\varphi\colon \Lambda^2\widetilde{L} \xrightarrow{\sim} O_{\mathbf{P}^1}(-x_1)$ is a horizontal isomorphism, and the residue $\widetilde{R}_i$ of ∇ at x_i has eigenvalues $\lambda_i^{\pm}$. For any $(\lambda_1, \dots, \lambda_n)$-bundle (L, ∇, φ), consider the lower (x_1, l_1)-modification $\widetilde{L}$ of L. Here $l_1 := \operatorname{Ker}(R_1 - \lambda_1) \subset L_{x_1}$. The triple $(\widetilde{L}, \nabla|_{\widetilde{L}}, \varphi|_{\widetilde{L}})$ corresponds to a point of M_1. This gives us a map $M \to M_1$. The upper modification of $(\widetilde{L}, \nabla, \varphi)$ defines the inverse map, so $M \simeq M_1$.

Since $(\widetilde{L}, \nabla)$ is irreducible, $\widetilde{L} \simeq O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-1)$ (see Corollary 2). So there is a unique subsheaf $\widetilde{L}_0 \subset \widetilde{L}$ such that $\widetilde{L}_0 \simeq O_{\mathbf{P}^1}$. There is a unique connection d: $\widetilde{L}_0 \to \widetilde{L}_0 \otimes \Omega_{\mathbf{P}^1}$. The correspondence $(\widetilde{L}, \nabla, \varphi) \mapsto (\widetilde{L}_0 \subset \widetilde{L}, \nabla|_{\widetilde{L}_0} - d, \varphi)$ gives a map $M_1 \to K_1$, where K_1 is the coarse moduli space of collections $(\widetilde{L}_0 \subset \widetilde{L}, A, \varphi)$ such that $(\widetilde{L}_0 \subset \widetilde{L}) \simeq (O_{\mathbf{P}^1} \subset O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-1))$, $\varphi\colon \Lambda^2\widetilde{L} \tilde{\to} O_{\mathbf{P}^1}(-x_1)$, $A \in \mathrm{Hom}(\widetilde{L}_0, \widetilde{L} \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4))$, and $\mathrm{Im}\, A \not\subset \widetilde{L}_0 \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4)$.

Proposition 8. K_1 is isomorphic to K. □

Proof. Set $\Omega' := \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4)$. Denote by K_2 the moduli space of $(O_{\mathbf{P}^1} \subset \widetilde{L}, B)$ such that $\widetilde{L}/O_{\mathbf{P}^1} \simeq O_{\mathbf{P}^1}(-1)$, $B\colon (\Omega')^{-1} \to \widetilde{L}$, and $\mathrm{Im}\, B \not\subset O_{\mathbf{P}^1}$. Suppose $(\widetilde{L}_0 \subset \widetilde{L}, A, \varphi)$ corresponds to a point of K_1. A induces a morphism $B\colon (\Omega')^{-1} = (\Omega' \otimes \widetilde{L}_0)^{-1} \otimes \widetilde{L}_0 \to (\Omega' \otimes \widetilde{L}_0)^{-1} \otimes (\widetilde{L} \otimes \Omega') = \widetilde{L}_0^{-1} \otimes \widetilde{L}$. Clearly $(O_{\mathbf{P}^1} = \widetilde{L}_0^{-1} \otimes \widetilde{L}_0 \subset \widetilde{L}_0^{-1} \otimes \widetilde{L}, B)$ corresponds to a point of K_2. This yields a morphism $K_1 \to K_2$. It is not hard to check that this is an isomorphism. Using B, we consider $O_{\mathbf{P}^1} \oplus (\Omega')^{-1}$ as a subsheaf of $\widetilde{L}$. So K_2 is isomorphic to the moduli space of locally free sheaves $\widetilde{L} \supset O_{\mathbf{P}^1} \oplus (\Omega')^{-1}$ such that $O_{\mathbf{P}^1}$ is a subbundle (not only a subsheaf) of $\widetilde{L}$, and $\widetilde{L}/(O_{\mathbf{P}^1} \oplus (\Omega')^{-1})$ is a sky-scraper sheaf with 1-dimensional space of sections. Such $\widetilde{L}$ are the upper (x, l)-modifications of $O_{\mathbf{P}^1} \oplus (\Omega')^{-1}$ for $x \in \mathbf{P}^1$, $l \subset \mathbf{C} \oplus ((\Omega')^{-1})_x$, $l \neq \mathbf{C}$. The space of such pairs (x, l) is identified with K. Hence $K_1 = K_2 = K$. ■

This yields a map $M \to K_1 = K$.

4.2 Local calculations

Lemma 6. Suppose $(\widetilde{L}_0 \subset \widetilde{L}, A, \varphi)$ corresponds to a point of K_1, $\widetilde{R}_i$ is an operator $\widetilde{L}_{x_i} \to \widetilde{L}_{x_i}$ such that the eigenvalues of $\widetilde{R}_i$ are $\lambda_i^{\pm}$, and $\widetilde{R}_i|_{(\widetilde{L}_0)_{x_i}}$ coincides with the residue of A at x_i. Then there is a unique connection ∇ such that the following conditions hold:

(i) $\nabla|_{\widetilde{L}_0} = A + d$, where d: $\widetilde{L}_0 \to \widetilde{L}_0 \otimes \Omega_{\mathbf{P}^1}$ is the unique connection;

(ii) $\widetilde{R}_i = \mathrm{res}_{x_i} \nabla$;

(iii) $(\widetilde{L}, \nabla, \varphi)$ corresponds to a point of M_1. □

Proof. It is easy to see that such a ∇ exists locally on $\mathbf{P}^1$. Let ∇_1, ∇_2 be two connections defined on some open set $U \subset \mathbf{P}^1$ such that (i)–(iii) are satisfied. Set $E := \nabla_1 - \nabla_2$. Then we have:

(i′) $E \in H^0(U, \mathcal{H}om(L, L \otimes \Omega_{\mathbf{P}^1}))$;

(ii′) $E|_{L_0} = 0$;

(iii′) $\mathrm{tr}\, E = 0$.

Conversely, if a connection ∇ on U satisfies (i)–(iii), and E satisfies (i′)–(iii′), then the connection $\nabla + E$ on U satisfies (i)–(iii). Denote by $\mathcal{C}(U)$ the set of all connections on

U satisfying (i)–(iii), and denote by $\mathcal{E}(U)$ the set of all E satisfying (i′)–(iii′). $\mathcal{C}(U)$ form a sheaf of sets $\mathcal{C}$, and $\mathcal{E}(U)$ form a sheaf of abelian groups $\mathcal{E}$. Clearly, $\mathcal{C}$ is an $\mathcal{E}$-torsor and $\mathcal{E} = \{E \in \mathcal{H}om(L, L \otimes \Omega_{\mathbf{P}^1})\colon E|_{L_0} = 0; \operatorname{tr} E = 0\} = \mathcal{H}om(L/L_0, L_0 \otimes \Omega_{\mathbf{P}^1})$. $\deg \mathcal{E} = -1$, so any $\mathcal{E}$-torsor is trivial and has a unique global section. Hence there is a unique $\nabla \in \mathcal{C}(\mathbf{P}^1)$ that satisfies (i)–(iii) on $\mathbf{P}^1$. ■

We need the following simple lemma from linear algebra.

Lemma 7. Suppose V is a vector space, $\dim_{\mathbf{C}} V = 2$, $V_0 \subset V$, $\dim_{\mathbf{C}} V_0 = 1$, $R_0 \in \operatorname{Hom}_{\mathbf{C}}(V_0, V)$, $\lambda^{\pm} \in \mathbf{C}$, $\lambda^+ \neq \lambda^-$. Set $\mathcal{R} := \{R \in \operatorname{End}_{\mathbf{C}}(V)\colon R|_{V_0} = R_0$, the eigenvalues of R are $\lambda^+, \lambda^-\}$ and $\mathcal{L} := \{(l^+, l^-) | l^{\pm} \subset V, \dim_{\mathbf{C}} l^{\pm} = 1, l^{\pm} \supset (R_0 - \lambda^{\mp})V_0, l^+ \neq l^-\}$.

The map $F\colon \mathcal{R} \to \mathcal{L}\colon R \mapsto (\operatorname{Ker}(R - \lambda^+) = \operatorname{Im}(R - \lambda^-), \operatorname{Ker}(R - \lambda^-))$ is bijective. □

Proof. F is clearly injective. Let us prove surjectivity.

For $(l^+, l^-) \in \mathcal{L}$, denote by $P^{\pm}$ the projector $V \to V/l^{\mp} \xrightarrow{\sim} l^{\pm}$ (so $P^+ + P^- = \mathrm{Id}$). The condition $l^{\pm} \supset (R_0 - \lambda^{\mp})V_0$ implies $P^{\mp}(R_0 - \lambda^{\mp})V_0 = 0$. So $(P^-(R_0 - \lambda^-) + P^+(R_0 - \lambda^+))V_0 = 0$, or equivalently, $R_0 = (\lambda^+ P^+ + \lambda^- P^-)|_{V_0}$. Hence $R := (\lambda^+ P^+ + \lambda^- P^-) \in \mathcal{R}$ and $F(R) = (l^+, l^-)$. ■

Lemmas 6 and 7 imply the following corollary.

Corollary 4. M_1 is identified with the coarse moduli space of $((\widetilde{L}_0 \subset \widetilde{L}, A, \varphi); \widetilde{l}_1^+, \widetilde{l}_1^-, \dots, \widetilde{l}_4^+, \widetilde{l}_4^-)$ such that:

(i) $(\widetilde{L}_0 \subset \widetilde{L}, A, \varphi)$ corresponds to a point of K_1;
(ii) $\widetilde{l}_i^{\pm} \subset \widetilde{L}_{x_i}$ is a subspace such that $\dim \widetilde{l}_i^{\pm} = 1$, $(\operatorname{res}_{x_i} A - \lambda_i^{\mp})(\widetilde{L}_0)_{x_i} \subset \widetilde{l}_i^{\pm}$;
(iii) $\widetilde{l}_i^+ \neq \widetilde{l}_i^-$. □

Denote by $\widetilde{M}_1$ the coarse moduli space of $((\widetilde{L}_0 \subset \widetilde{L}, A, \varphi); \widetilde{l}_1^+, \widetilde{l}_1^-, \dots, \widetilde{l}_4^+, \widetilde{l}_4^-)$ such that conditions (i)–(ii) of Corollary 4 are satisfied. Then M_1 is identified with the open subset of $\widetilde{M}_1$ defined by (iii).

Denote by $\widetilde{\xi}_{x_i}$ (resp. $\widetilde{\delta}$) the bundle on K_1 whose fiber over $(\widetilde{L}, A, \varphi)$ is $\widetilde{L}_{x_i}$ (resp. $(\widetilde{L}_0)_{x_i} = \det R\Gamma(\mathbf{P}^1, \widetilde{L})$). The map $(\operatorname{res}_{x_i} A - \lambda_i^{\mp})\colon (\widetilde{L}_0)_{x_i} \to \widetilde{L}_{x_i}$ for variable $(\widetilde{L}, A, \varphi)$ defines a morphism $\widetilde{\delta} \to \widetilde{\xi}_{x_i}$. This morphism $\widetilde{\delta} \to \widetilde{\xi}_{x_i}$ has a unique simple zero in $c_i^{\mp}$. This proves that the natural map $\widetilde{M}_1 \to K_1$ is the blow-up at $c_i^{\pm}$, $i = 1, \dots, 4$. It is easy to see that the closed subset of $\widetilde{M}_1$ defined by the equation $\widetilde{l}_i^+ = \widetilde{l}_i^-$ is the proper preimage of b_i, so $\widetilde{M}_1 \setminus M_1$ is the union of these proper preimages.

This completes the proof of Theorem 3. ■

4.3 Description of invertible sheaves on M

Denote by $b_i^{\pm} \subset M_1$ the preimages of $c_i^{\pm} \subset K$.

Proposition 9. The group Pic M_1 is the abelian group generated by the classes $[b_i^{\pm}]$ with the defining relations

$$[b_1^+] + [b_1^-] = [b_2^+] + [b_2^-] = [b_3^+] + [b_3^-] = [b_4^+] + [b_4^-]. \qquad \square$$

Proof. Consider the composition $\pi_1\colon M_1 \to K_1 = K \to \mathbf{P}^1$. Set $U := \mathbf{P}^1 \setminus \{x_1, \dots, x_4\}$, $U' := \pi_1^{-1}(U)$. Denote by Γ the group of divisors D on M_1 such that $\operatorname{supp} D \cap U' = \emptyset$. By Theorem 3, $U' \simeq U \times \mathbf{A}^1$, so $\operatorname{Pic} U' = 0$, and the map $H^0(U, O_U^*) \to H^0(U', O_{U'}^*)$ is an isomorphism. Therefore, the morphism $\Gamma \to \operatorname{Pic} M_1$ is surjective and its kernel Γ_0 consists of the inverse images of principal divisors Δ on $\mathbf{P}^1$ such that $\operatorname{supp} \Delta \cap U = \emptyset$. Γ is the free abelian group generated by $b_i^{\pm}$, and Γ_0 is generated by $\pi_1^*(x_i - x_j) = (b_i^+ + b_i^-) - (b_j^+ + b_j^-)$. ■

Proposition 10. Let δ, $\xi_i^{\otimes 2}$ be the line bundles on M defined in Section 3. Then

$$\delta \simeq O_M(-b_1^-),$$

$$\xi_i^{\otimes 2} \simeq O_M(b_i^- - b_i^+).$$

$\square$

Proof. Denote by $\widetilde{\xi}_{x_i}$ (resp. $\widetilde{\xi}_i^{\pm}, \widetilde{\delta}$) the locally free sheaf on $\mathcal{M}_1$ (the moduli stack of $(\widetilde{L}, \nabla, \varphi)$) whose fiber over $(\widetilde{L}, \nabla, \varphi)$ is $\widetilde{L}_{x_i}$ (resp. $\widetilde{l}_i^{\pm} = \operatorname{Ker}(\widetilde{R}_i - \lambda_i^{\pm})$, $\det R\Gamma(\mathbf{P}^1, \widetilde{L}) = H^0(\mathbf{P}^1, \widetilde{L}) = (\widetilde{L}_0)_{x_i}$). Then $\widetilde{\xi}_i^{\pm}$ and $\widetilde{\delta}$ are subsheaves of $\widetilde{\xi}_{x_i}$.

Let $(\widetilde{L}, \nabla, \varphi)$ be a point of $\mathcal{M}_1$. Consider the map $(\widetilde{R}_i - \lambda_i^{\mp})\colon (\widetilde{L}_0)_{x_i} \to \widetilde{l}_i^{\pm}$. As $(\widetilde{L}, \nabla, \varphi)$ varies, it yields a morphism of $O_{\mathcal{M}_1}$-modules $\widetilde{\delta} \to \widetilde{\xi}_i^{\pm}$. It follows from the results of the previous subsection that this morphism identifies $\widetilde{\xi}_i^{\pm}$ with $\widetilde{\delta}(b_i^{\mp})$. Since $\widetilde{\xi}_{x_i} = \widetilde{\xi}_i^+ \oplus \widetilde{\xi}_i^-$ and $\Lambda^2 \widetilde{\xi}_{x_i} \simeq O_{\mathcal{M}_1}$, we have $\widetilde{\xi}_i^- \simeq (\widetilde{\xi}_i^+)^*$. Hence $(\widetilde{\delta})^{\otimes 2} \simeq O_{M_1}(-b_i^+ - b_i^-)$ and $(\widetilde{\xi}_i^{\pm})^{\otimes 2} \simeq O_{M_1}(b_i^{\mp} - b_i^{\pm})$. But $\widetilde{\xi}_i^+$ (resp. $\widetilde{\delta}$) corresponds to ξ_i (resp. $\delta \otimes \xi_1$) via the identification $\mathcal{M}_1 = \mathcal{M}$. The statement follows immediately. ■

5 Cohomology of invertible sheaves on M

In this section, we prove Theorem 2.

5.1 The least smooth compactification $\overline{M} \supset M$

Set $\overline{K} := \mathbf{P}(O_{\mathbf{P}^1} \oplus \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4))$. K is the open subscheme $\overline{K} \setminus s_\infty$, where s_∞ is 'the infinite section.' Blowing up $c_i^{\pm} \subset \overline{K}$, we obtain a variety $\overline{M}$, which is a smooth compactification of $\widetilde{M}_1 \supset M_1 = M$. $\overline{M} \setminus M$ consists of the five irreducible components $s'_\infty, b'_1, \dots, b'_4$ (the proper preimages of $s_\infty, b_1, \dots, b_4 \subset \overline{K}$). Clearly on $\overline{K}$ we have $(s_\infty, \bar{b}_i) = 1$, $(\bar{b}_i, \bar{b}_j) = 0$, and $(s_\infty, s_\infty) = -2$. This implies

$$(s'_\infty, s'_\infty) = (b'_i, b'_i) = -2, \qquad (s'_\infty, b'_i) = 1. \tag{2}$$

Corollary 5. $\overline{M}$ is the least smooth compactification of M (i.e., any smooth compactification of M dominates $\overline{M}$). □

Proof. Let $\overline{M}'$ be another smooth compactification of M. Then there is a smooth compactification $\overline{M}''$ that dominates $\overline{M}$ and $\overline{M}'$. The morphisms $f\colon \overline{M}'' \to \overline{M}$ and $f'\colon \overline{M}'' \to \overline{M}'$ are compositions of σ-processes, and we may assume that the number of these σ-processes is minimal. Let us prove that f' is an isomorphism.

Assume the converse. Then there is an exceptional curve $C' \subset \overline{M}''$ of the first kind such that $\dim f'(C') = 0$. Clearly $C' \cap M = \emptyset$.

$\overline{M}'' \setminus M$ has the following irreducible components: b''_i, s''_∞ (the proper preimages of b'_i, s'_∞), and curves C such that $\dim f(C) = 0$. $(b''_i)^2 \le (b'_i)^2 < -1$ and $(s''_\infty)^2 \le (s'_\infty)^2 < -1$, so $\dim f(C') = 0$. But this contradicts the hypothesis that the number of σ-processes is minimal. ■

Remark. Let us interpret $\overline{K}$ and $\overline{M}$ as moduli spaces. Denote by $\overline{K}_1$ the coarse moduli space of $(\widetilde{L}_0 \subset \widetilde{L}, A, \varphi)$ such that $\widetilde{L}_0$ is an invertible sheaf of degree 0 on $\mathbf{P}^1$, $\widetilde{L}$ is a rank 2 locally free sheaf of degree -1 on $\mathbf{P}^1$, $A\colon \widetilde{L}_0 \to \widetilde{L} \otimes \Omega_{\mathbf{P}^1}(x_1 + \cdots + x_4)$, $\operatorname{Im} A \cap \widetilde{L}_0 = 0$, and $\varphi\colon \Lambda^2\widetilde{L} \tilde{\to} O_{\mathbf{P}^1}(-x_1)$. The isomorphism $K_1 \tilde{\to} K$ from Proposition 8 can be extended to $\overline{K}_1 \tilde{\to} \overline{K}$.

Denote by $\overline{M}_1$ the coarse moduli space of $((\widetilde{L}_0 \subset \widetilde{L}, A, \varphi); \widetilde{l}_1^+, \widetilde{l}_1^-, \dots, \widetilde{l}_4^+, \widetilde{l}_4^-)$ such that $(\widetilde{L}_0 \subset \widetilde{L}, A, \varphi)$ corresponds to a point of $\overline{K}_1$, $\widetilde{l}_i^\pm \subset \widetilde{L}_{x_i}$ is a 1-dimensional subspace, and $\widetilde{l}_i^\pm \supset (\operatorname{res} A - \lambda_i^\mp)(\widetilde{L}_0)_{x_i}$. Then there is an isomorphism $\overline{M}_1 \tilde{\to} \overline{M}$ such that the two compositions $\overline{M}_1 \tilde{\to} \overline{M} \to \overline{K}$ and $\overline{M}_1 \to \overline{K}_1 \tilde{\to} \overline{K}$ coincide.

5.2 The geometry of $\overline{M} \setminus M$

Set $D := 2s'_\infty + b'_1 + \cdots + b'_4$. Then

$$(D, D) = (D, s'_\infty) = (D, b'_i) = 0. \tag{3}$$

Since $\Omega^2_{\overline{K}} \simeq O_{\overline{K}}(-4\bar{b}_i - 2s_\infty)$, we have $\Omega^2_{\overline{M}} \simeq O_{\overline{M}}(-D)$.

Notation. For a positive divisor C, we denote the corresponding subscheme by the same letter C.

Consider $D \subset \overline{M}$ as a reducible nonreduced subscheme. Then b'_i, s'_∞, and $2s'_\infty$ are closed subschemes of D.

By the Riemann-Roch theorem, $\chi(O_D) = -D(D+K)/2$, where $K = -D$ is the canonical class of $\overline{M}$. So $\chi(O_D) = 0$. This implies the following statement.

Proposition 11. Let $\mathcal{E}$ be a locally free sheaf on D. Then $\chi(\mathcal{E}) = 2\deg(\mathcal{E}|_{s'_\infty}) + \sum_{i=1}^4 \deg(\mathcal{E}|_{b'_i})$. □

Lemma 8. Let $\mathcal{E}$ be a nontrivial invertible sheaf on D such that $\deg \mathcal{E}|_{s'_\infty} = 0$, and either $\deg \mathcal{E}|_{b'_i} = 0$ for all i, or one of the numbers $\deg \mathcal{E}|_{b'_i}$ is -1, another one is 1, and the remaining two equal zero. Then $H^k(D, \mathcal{E}) = 0$ for all k. □

Proof. By Proposition 11, $\chi(\mathcal{E}) = 0$. So it is enough to prove that $H^0(D, \mathcal{E}) = 0$.

Assume the converse. Let $f \in H^0(D, \mathcal{E})$, $f \neq 0$. $\chi(\mathcal{E}) = \chi(O_D)$, $\mathcal{E} \not\simeq O_D$, so f is zero on one of the irreducible components of D.

We may assume that $\deg \mathcal{E}|_{b'_i} \leq 0$ for $i \neq 1$. The closed subscheme $D_1 := s'_\infty + \sum_{i\neq 1} b_i \subset D$ is reduced and connected. Besides, $\mathcal{E}|_{D_1}$ has nonpositive degree on any irreducible component of D_1. So either $f|_{D_1} = 0$, or $f|_{D_1}$ has no zero. In the second case, $f|_C \neq 0$, where $C \subset D$ is any irreducible component. Therefore $f \in \operatorname{Ker}(H^0(D, \mathcal{E}) \to H^0(D_1, \mathcal{E}))$. In other words, $f \in H^0(D, \mathcal{E} \otimes I_{D_1})$, where $I_{D_1} := \{\tilde{f} \in O_D \colon \tilde{f}|_{D_1} = 0\}$ is the sheaf of ideals of $D_1 \subset D$.

We have $I_{D_1} = O_{\overline{M}}(-D_1)/O_{\overline{M}}(-D)$, $\operatorname{supp} I_{D_1} = s'_\infty + b'_1$. So $\deg I_{D_1}|_{b'_1} = \deg O(-D_1)|_{b'_1} = -1$. Therefore $\deg(\mathcal{E} \otimes I_{D_1})|_{b'_1} = \deg \mathcal{E}|_{b'_1} - 1 \leq 0$. In the same way, $\deg(\mathcal{E} \otimes I_{D_1})|_{s'_\infty} = \deg \mathcal{E}|_{s'_\infty} - 1 = -1$. Since $\mathcal{E} \otimes I_{D_1}$ is an invertible sheaf on the connected reduced scheme $s'_\infty + b'_1$, this implies $f \in H^0(D, \mathcal{E} \otimes I_{D_1}) = 0$. ■

Set $\operatorname{Pic}^0 D := \{\mathcal{E} \in \operatorname{Pic} D \mid \deg(\mathcal{E}|_{s'_\infty}) = 0, \deg(\mathcal{E}|_{b'_i}) = 0 \text{ for all } i\}$.

Proposition 12. $\operatorname{Pic}^0 D \simeq \mathbf{A}^1$. □

Proof. Set $D_{red} := s'_\infty + \sum_{i=1}^4 b'_i \subset D$. Then $\operatorname{Pic}^0 D = \operatorname{Ker}(\operatorname{Pic} D \to \operatorname{Pic} D_{red})$.

Set $O' := \operatorname{Ker}(O^*_D \to O^*_{D_{red}})$. Then the exact sequence $0 \to O' \to O^*_D \to O^*_{D_{red}} \to 1$ defines an isomorphism $H^1(D, O') \to \operatorname{Pic}^0 D$. But O' is a locally free $O_{s'_\infty}$-module of degree $-(s'_\infty, D_{red}) = -2$. Hence $\operatorname{Pic}^0 D$ is a 1-dimensional $\mathbf{C}$-space. ■

Lemma 9. If $2\lambda_i \notin \mathbf{Z}$ for any i, then M contains no projective curve. □

Proof. Fix a point $x \in \mathbf{P}^1 \setminus \{x_1, \dots, x_4\}$. Consider the fundamental group $G := \pi_1(x, \mathbf{P}^1 \setminus \{x_1, \dots, x_4\})$. G is generated by the loops γ_i around x_i with the relation $\gamma_1 \times \cdots \times \gamma_4 = e$. Denote by W the moduli space of representations $\rho \colon G \to SL(2)$ such that $\rho(\gamma_i)$ has eigenvalues $\exp(\pm 2\pi\sqrt{-1}\lambda_i)$. Clearly W is an affine scheme.

The Riemann-Hilbert correspondence gives an analytic isomorphism $M_{an} \tilde{\to} W_{an}$. But W_{an} contains no compact curve, so M contains no projective curve. ■

Remark. Consider the case of n points on any curve for any n. Then one can prove in the same way that the only projective subvarieties in M are finite sets.

Lemma 10. The sheaf $\mathcal{N}_D := \mathcal{O}_{\overline{M}}(D)|_D$ is not trivial. □

Proof. Assume the converse. Let σ be a global section of $\mathcal{N}_D$ with no zeros. $\overline{M}$ is a smooth rational projective variety, $H^1(\overline{M}, \mathcal{O}_{\overline{M}}) = 0$, and therefore $\sigma \in H^0(D, \mathcal{N}_D) = H^0(\overline{M}, \mathcal{O}_{\overline{M}}(D)/\mathcal{O}_{\overline{M}})$ can be lifted to $s \in H^0(\overline{M}, \mathcal{O}_{\overline{M}}(D))$. Then (s) is an effective divisor equivalent to D, and $\operatorname{supp}(s) \subset M$. This contradicts Lemma 9. ■

Remark. One can give a direct (but more complicated) proof of this lemma.

Corollary 6. $H^i(D, (\mathcal{N}_D)^{\otimes k}) = 0$ for $k \neq 0$. □

Proof. By (3), $\mathcal{N}_D \in \operatorname{Pic}^0 D$. Lemma 10 and Proposition 12 imply $(\mathcal{N}_D)^{\otimes k} \not\simeq \mathcal{O}_D$ for $k \neq 0$. Lemma 8 completes the proof. ■

5.3 Calculation of cohomology

Let $\mathcal{E}$ be an invertible sheaf on M. We set $\deg \mathcal{E} := (\overline{\mathcal{E}}, D)$, where $\overline{\mathcal{E}}$ is an extension of $\mathcal{E}$ to an invertible sheaf on $\overline{M}$. (3) implies that $\deg \mathcal{E}$ is well defined. Besides, it follows from Proposition 10 that deg: $\operatorname{Pic} M \to \mathbf{Z}$ coincides with deg from Theorem 2.

If $\overline{\mathcal{E}}$ is an invertible sheaf on $\overline{M}$, $\mathcal{E} = \overline{\mathcal{E}}|_M$, then $H^j(M, \mathcal{E}) = \varinjlim H^j(\overline{M}, \mathcal{E}(kD))$. But $H^*(\overline{M}, \mathcal{O}_{\overline{M}}(kD)/\mathcal{O}_{\overline{M}}((k-1)D)) = 0$ for $k \neq 0$ (see Corollary 6). Hence $H^j(M, \mathcal{O}_M) = H^j(\overline{M}, \mathcal{O}_{\overline{M}})$, and the statement (iii) of Theorem 2 follows from the rationality of $\overline{M}$.

If $\deg \mathcal{E} = 0$, one can choose an extension $\overline{\mathcal{E}}$ such that $(\overline{\mathcal{E}}, s'_\infty) = 0$ and either $(\overline{\mathcal{E}}, b'_i) = 0$ for all i, or one of the numbers $(\overline{\mathcal{E}}, b'_i)$ is 1, another one is -1, and the remaining two are zero. Then Lemmas 8 and 10 and Proposition 12 imply that for all $k \in \mathbf{Z}$, maybe except for one value, $H^*(\overline{M}, \overline{\mathcal{E}}(kD)/\overline{\mathcal{E}}((k-1)D)) = 0$. Hence, $\dim H^j(M, \mathcal{E}) < \infty$ and

$$\chi(\mathcal{E}) = \chi(\overline{\mathcal{E}}) = 1 + \frac{(\overline{\mathcal{E}}, \overline{\mathcal{E}}(D))}{2} = 1 + \frac{(\overline{\mathcal{E}}, \overline{\mathcal{E}})}{2}.$$

One can check that $(\overline{\mathcal{E}}, \overline{\mathcal{E}})/2 = \left[\langle \mathcal{E}, \mathcal{E}\rangle/2\right]$, where $\langle , \rangle$ is the bilinear form from Theorem 2. So statement (iv) of Theorem 2 follows from Lemma 11.

Lemma 11. If $\deg \mathcal{E} \leq 0$, $\mathcal{E} \not\simeq \mathcal{O}_M$, then $H^0(M, \mathcal{E}) = 0$. □

Proof. Suppose $H^0(M, \mathcal{E}) \neq 0$, $\mathcal{E} \not\simeq \mathcal{O}_M$. Then $\mathcal{E} \simeq \mathcal{O}_M(C)$, $C > 0$. So $\deg \mathcal{E} = (\overline{C}, D)$, where $\overline{C}$ is the closure of C in $\overline{M}$. Hence by Lemma 9, $\deg \mathcal{E} > 0$. ■

Now we prove statement (i) of Theorem 2. Suppose $\deg \mathcal{E} > 0$, $\overline{\mathcal{E}}$ is an extension of $\mathcal{E}$ to $\overline{M}$. Then $\chi(\overline{\mathcal{E}}(kD)) \to \infty$ as $k \to \infty$. Since $H^2(\overline{M}, \overline{\mathcal{E}}(kD)) = 0$ for $k \gg 0$, we have $\dim H^0(\overline{M}, \overline{\mathcal{E}}(kD)) \to \infty$ as $k \to \infty$, that is, $\dim H^0(M, \mathcal{E}) = \infty$. Since $H^0(M, \mathcal{E}) \neq 0$, $\mathcal{E} \simeq O_M(C)$ for some $C > 0$. But $H^1(M, O_M) = 0$, and C is affine (see Lemma 9), so $H^1(M, \mathcal{E}) = 0$.

To complete the proof of Theorem 2, we should check that if $\deg \mathcal{E} < 0$, then $\dim H^1(M, \mathcal{E}) = \infty$. Since $H^0(M, \mathcal{E}^{-1}) \neq 0$, $\mathcal{E} \simeq O_M(-C)$ for some $C > 0$. Since C is affine and $H^0(M, O_M)$ is finite-dimensional, it is enough to use the exact sequence $0 \to O_M(-C) \to O_M \to O_M/O_M(-C) \to 0$.

Acknowledgments

We are very grateful to V. Drinfeld for stating problems and for numerous stimulating discussions.

The authors were partially supported by INTAS grant 94-4720.

References

[1] D. Arinkin and S. Lysenko, *Invertible sheaves on moduli spaces of* SL(2)*-bundles with connections on* $\mathbf{P}^1$, Dokl. Akad. Nauk. Ukraine **6** (1997), 7–11 (in Russian).

[2] ———, *Isomorphisms between moduli spaces of* SL(2)*-bundles with connections on* $\mathbf{P}^1 \setminus \{x_1, \ldots, x_4\}$, Math. Res. Lett. **4** (1997), 181–190.

[3] Y. Laszlo and C. Sorger, *The line bundles on the moduli of parabolic* G*-bundles over curves and their sections*, Ann. Sci. École Norm. Sup. (4) **30** (1997), 499–525.

[4] G. Laumon, *Correspondance de Langlands géométrique pour les corps de fonctions*, Duke Math. J. **54** (1987), 309–359.

[5] G. Laumon and L. Moret-Bailly, *Champs algébriques*, preprint, Université Paris 11 (Orsay), 1992.

[6] N. Nitsure, *Moduli of semistable logarithmic connections*, J. Amer. Math. Soc. **6** (1993), 597–609.

[7] K. Okamoto, *Isomonodromic deformation and Painlevé equations*, *and the Garnier system*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 575–618.

[8] ———, *Studies on the Painlevé equations*, *I: Sixth Painlevé equation PVI*, Ann. Mat. Pura Appl. **146** (1987), 337–381.

[9] ———, *Sur les feuilletages associés aux équation du second ordre à points critiques fixes de P. Painlevé. Espaces de conditions initiales*, Japan J. Math. **5** (1979), 1–79.

[10] C. Simpson, "Nonabelian Hodge theory" in *Proc. Internat. Cong. Math.*, *Vol. I*, *II (Kyoto*, *1990)*, Math. Soc. Japan, Tokyo, 1991, 747–756.

Physico-Technical Institute of Lower Temperatures, Mathematical Division, Lenin Avenue 47, Kharkov-164, 310164, Ukraine.

Current address of D. Arinkin: Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138, USA; arinkin@math.harvard.edu