1. Weakly multiplicative quasi-coherent sheaves

In this note we give some examples of weakly-multiplicative objects in the multiplicative category QCoh(X) for monoids X in PreStk (in the sense of (Raskin, Chiral algebras, 5.25.1)). We follow the notations and conventions of *loc.cit.*

1.1. Work over a field k. For a monoid $X \in$ PreStk let $m : X \times X \to X$ be the product and $e : \operatorname{Spec} k \to X$ be the unit. A weakly multiplicative object $F \in \mathrm{QCoh}(X)$ is given by maps $p : F \boxtimes F \to m^*F$ and $u : k \to e^*F$ such that (X, F) is a commutative algebra (in the sense of Raskin, Chiral algebras, 5.25.1). That is, p is symmetric, and we have commutativity and associativity (with all coherent compatibilities).

The category $Mult^w(X)$ formed by such F is a symmetric monoidal ∞-category with respect to the tensor product. If $f : Y \to X$ is a map of commutative algebras in PreStk then $f^* : Mult^w(X) \to Mult^w(Y)$ is symmetric monoidal.

If $X = \operatorname{Spec} k$ then $Mult^w(\operatorname{Spec} k) = CAlg(\mathrm{Vect})$. So, for any $Y \in CAlg(\mathrm{PreStk})$ we have the examples $f^*C \in Mult^w(Y)$ for $C \in CAlg(\mathrm{Vect})$. These are constant weakly multiplicative sheaves.

On the other hand, given $F \in Mult^w(Y)$, we get a commutative algebra $e^*F \in CAlg(\mathrm{Vect})$, which is an invariant of F.

1.1.1. Consider the example $Y = \mathbb{A}^1$ with $m : Y \times Y \to Y$ given by the sum. Take F isomorphic to $\mathcal{O}_Y$, for the moment we do not fix this trivialization. What are the structures of a weakly multiplicative sheaf on it? As above, e^*F is a 1-dimensional k-algebra, so the unit map $u : k \to e^*F$ is an isomorphism trivializing this algebra. Since any invertible section in $\mathrm{H}^0(\mathbb{A}^1, \mathcal{O})$ is constant, we may assume given a trivialization $F \widetilde{\to} \mathcal{O}_Y$ such that $u : k \to e^*F$ is the identity.

Assume $p : F \boxtimes F \to m^*F$ is given by $f \in \mathrm{H}^0(\mathbb{A}^2, \mathcal{O}) = k[x, y]$. Then we must have the following properties:

1) $f(x, y) = f(y, x)$ and $f(0, x) = 1$
2) $f(x, y)f(x + y, z) = f(y, z)f(x, y + z)$ in $k[x, y, z]$

Write $f = f_0 + f_1 + \ldots + f_n$, where $f_i(tx, ty) = t^i f(x, y)$ is the i-th homogeneous component of f. Let

$$f_n = \sum_{i=0}^{n} a_i x^i y^{n-i}$$

be the top nonzero component. We have $f_0 = 1$ and $f_j(x, y) = f_j(y, x)$ for all j. In particular, $a_i = a_{n-i}$ for all i.

The property 2) gives

$$f_n(x, y)f_n(x + y, z) = f_n(y, z)f_n(x, y + z) \tag{1}$$

Assume $n > 0$. We have $f_n(0, y) = a_0 y^n = 0$ by 1), so $a_0 = a_n = 0$ and $n \geq 2$. So, (1) writes

$$(\sum_i a_i x^i y^{n-i})(\sum_i a_i (x + y)^i z^{n-i}) = (\sum_i a_i z^i y^{n-i})(\sum_i a_i x^{n-i} (y + z)^i)$$

The top degree in z on the RHS is $a_{n-1}^2 xyz^{2n-2}$, and on the left the biggest degree in z is $n - 1$. So, $a_1 = a_{n-1} = 0$. So, we may assume $n \geq 4$. Again compare the biggest degrees in z on both sides. On the LHS we get the biggest degree in z is at most $n-2$. On the RHS we get the biggest term $a_{n-2}^2 x^2 y^2 z^{2n-4}$. Since $n \geq 4$, this gives $a_2 = a_{n-2} = 0$.

Now we may assume $n \geq 6$ and do the same. The biggest term in degree z in the RHS is $a_{n-3}^2 x^3 y^3 z^{2n-6}$, and in the LHS the degrees in z are at most $n-3$. So, $a_3 = a_{n-3} = 0$.

Continuing as above, we get that n is as large as we want, a contradiction. So, $n = 0$. We see that this weakly multiplicative sheaf is trivial.

1.1.2. Automorphisms of weakly multiplicative sheaves? Let A be a commutative k-algebra. For example, assume A finite-dimensional then $\mathrm{Aut}(A) \subset \mathrm{GL}(A)$ is a closed subgroup, so this is an algebraic group over k. Here $\mathrm{Aut}(A)$ is the group of automorphisms of the k-algebra A. Write the product map $A \otimes A \to A$ as $(a, b) \mapsto ab$.

Let X be a commutative monoid in schemes written additively. Consider the constant weakly multiplicative sheaf $A \otimes \mathcal{O}_X$ on X. What is the group of its automorphisms as a weakly multiplicative sheaf? Such an automorphism $\bar{g}$ is given by a regular map $g : X \to \mathrm{GL}(A)$ such that $\bar{g}(x, a) = (x, g(x)a)$ for $(x \in X, a \in A)$. It has to satisfy the properties:

- For $x_i \in X, a_i \in A$, $(g(x_1)a_1)(g(x_2)a_2) = g(x_1 + x_2)(a_1 a_2)$.
- $g(0) \in \mathrm{Aut}(A)$.

In particular, constant maps $g : X \to * \to \mathrm{Aut}(A)$ satisfy these properties, so of course $\mathrm{Aut}(A)$ acts on this weakly multiplicative sheaf.

1.1.3. Example: $A = k$, the trivial k-algebra. Then $g : X \to \mathbb{G}_m$ must be a homomorphism of monoids $g : X \to \mathbb{G}_m$. If X is an algebraic group with a nontrivial homomorphism $g : X \to \mathbb{G}_m$, we get nontrivial automorphisms of the weakly multiplicative sheaf $\mathcal{O}_X$ with $A = k$. For example we may take $X = T$ a split torus then $\mathrm{Hom}(T, \mathbb{G}_m)$ is the group of automorphisms of this weakly multiplicative sheaf.

1.2. Assume $\dim A$ finite, let $\mathcal{F} \to X$ be a H-torsor, where H is a commutative group with a homomorphism $H \to \mathrm{GL}(A)$. We underline that we do not require H to act by automorphisms of the algebra A. Let $A_{\mathcal{F}}$ be the quotient of $\mathcal{F} \times A$ by the diagonal action of H, so $A_{\mathcal{F}} \to X$ is a vector bundle on X with fibre A.

We want to understand when $A_{\mathcal{F}}$ has a natural structure of a weakly multiplicative sheaf. Write $p_A : A \otimes A \to A$ for the product.

The vector bundles $A_{\mathcal{F}} \boxtimes A_{\mathcal{F}}$ and $m^*(A \otimes A)_{\mathcal{F}}$ on $X \times X$ have the same fibre $A \otimes A$. It seems unreasonable to require that $A_{\mathcal{F}} \boxtimes A_{\mathcal{F}}$ descends under $m : X \times X \to X$ to the vector bundle $(A \otimes A)_{\mathcal{F}}$. This almost never happens.

There is the following however. Assume given an isomorphism

$$\mu : (\mathcal{F} \times \mathcal{F}) \times_{H \times H} H \widetilde{\to} m^*\mathcal{F}$$

of H-torsors over X^2. Denote by $\epsilon : \mathcal{F} \times_X \mathcal{F} \to m^*\mathcal{F}$ the corresponding map over X^2. Assume also the action of H on A satisfies

(P) for $a_i \in A, h_i \in H$ one has $(h_1 a_1)(h_2 a_2) = (h_1 + h_2)(a_1 a_2)$.

Then

$$\epsilon \times p_A : (\mathcal{F} \times_X \mathcal{F}) \times (A \otimes A) \to (\mathcal{F} \times_X X^2) \times A$$

is $H \times H$-equivariant, where $H \times H$ acts diagonally on the LHS, and on the RHS via the product map $H \times H \to H$. The map $\epsilon \times p$ then yields the desired map $p : A_{\mathcal{F}} \boxtimes A_{\mathcal{F}} \to m^* A_{\mathcal{F}}$.

The property (P) shows that $h \in H$ has to act on $a \in A$ as $h(a) = a \cdot h(1)$, the product in A. In fact, (P) is equivalent to requring that for $h_i \in H$, $(h_1 + h_2)(1) = h_1(1) \cdot h_2(1)$, the product in A, and $h(1) \in A^*$. Here A^* is the group of invertible elements of A. In other words, we are given a homomorphism $\gamma : H \to A^*$, $h \mapsto h(1)$. Since $\dim A < \infty$, A^* is a commutative affine algebraic group of finite type.

What about associativity, commutativity and unit section $k \to e^* A_{\mathcal{F}}$? For associativity and commutativity we require the isomorphism μ to be associative and commutative. Even better, we simply require that $\mathcal{F} : X \to B(H)$ is a morphism of commutative algebras in PreStk, in particular is unital.

In this case we are reduced to describe the weakly multiplicative sheaves on $B(H)$, this is done in Proposition 1.3.1. Then given such a sheaf F on $B(H)$, the corresponding sheaf on X is the pull-back under $\mathcal{F} : X \to B(H)$.

1.2.1. *Example.* Take H a finite abelian group, $A = \mathrm{H}^0(H, \mathcal{O})$ with convolution algebra structure. Let H act on A so that $h \in H$ sends $f \in \mathrm{H}^0(H, \mathcal{O})$ to the function hf, where $(hf)(x) = f(x - h)$. Then (P) holds. Namely, this action of H on A comes from the homomorphism $\gamma : H \to A^*$, $\gamma(h) = h$.

1.3. Weakly multiplicative sheaves on $B(H)$. Let H be a commutative group scheme, affine of finite type. Recall that the category of (classical) quasi-coherent sheaves on H is equivalent to $\mathrm{Rep}(H)$. Let $A \in \mathrm{Rep}(H)$ be a weakly multiplicative sheaf on $B(H)$, write also A for its restriction to $\operatorname{Spec} k$, so A is an algebra. The map $A \otimes A \to A$ has to be $H \times H$-equivariant, where $H \times H$ acts on $A \otimes A$ naturally, and on A via the product $H \times H \to H$. This is precisely (P), so we are given a homomorphism $\gamma : H \to A^*$, $h \mapsto h(1)$ of algebraic groups over k. Once (P) holds, the unit section is defined automatically, namely this is the unit section $k \to A$ of the algebra A. The associativity is also automatic. We have proved the following.

Proposition 1.3.1. *Let H be a commutative group scheme, affine of finite type. Any weakly multiplicative sheaf on $B(H)$ is given by a (classical) algebra A in vector spaces and a homomorphism of algebraic groups $H \to A^*$ over k. (We denote it by $\underline{A}$).*

In particular, there is a canonical weakly multiplicative sheaf $\underline{A}$ on $B(A^*)$ given by the algebra A and the identity homomorphism $\mathrm{id} : A^* \to A^*$.

1.3.2. *Example as in Lizao.* Let H be a discrete commutative group, maybe infinite. Let A be the vector space of functions on H with a finite support. Then A is an algebra with respect to the convolution. We have a homomorphism $\gamma : H \to A^*$ sending $h \in H$ to the charactestic function of h. This gives a weakly multiplicative sheaf $\underline{A}$ on $B(H)$ as in Proposition 1.3.1.

1.4. Let C be a weakly multiplicative coherent sheaf on X and $f : Y \to X$ a homomorphism of commutative groups in Sch. Assume f^*C is identified with the constant multiplicative sheaf $Y \times A$. What are the descent data that appear on $Y \times A$? Assume

f surjective with kernel H. On $H \times Y$ we get an automorphism of the constant mutiplicative sheaf with fibre A. The latter is given by a map $g : H \times Y \to \mathrm{GL}(A)$ such that $H \times Y \times A \to H \times Y \times A$, $(h, y, a) \mapsto (h, y, g(h, y)a)$ is our automorphism. The fact that it is an automorphism of the weakly multiplicative sheaf means that

- for $h_i \in H, y_i \in Y, a_i \in A$, $g(h_1 + h_2, y_1 + y_2)(a_1 a_2) = (g(h_1, y_1)a_1)(g(h_2, y_2)a_2)$
- $g(0, 0) \in \mathrm{Aut}(A)$

The cocycle condition is as follows. Given $h_i \in H, y \in H$ one has

$$g(h_2, h_1 + y)g(h_1, y) = g(h_1 + h_2, y) \in \mathrm{GL}(A)$$

For example, if g is independent of $y \in Y$ then the above means that $H \to \mathrm{GL}(A)$ is a homomorphism, and (P) holds.

1.4.1. *Example of Alain.* Let $f : Y \to X$ be a surjective morphism of commutative groups, whose kernel is a finite subgroup $H \subset Y$. This is the same as a homomorphism of commutative groups $\mathcal{F} : X \to B(H)$ in PreStk.

Assume H finite and take $A = \mathrm{H}^0(H, \mathcal{O})$ with the convolution algebra structure. Let $\gamma : H \to A^*$ be the map sending h to the characteristic function of H. This is a homomorphism of groups, so by Proposition 1.3.1, we get a weakly multiplicative sheaf $\underline{A}$ on $B(H)$, and example of Alain is the pull-back of $\underline{A}$ by $\mathcal{F} : X \to B(H)$.

Old description: Take $C = f_*\mathcal{O}_Y$. Let $Z = X^2 \times_X Y$. The map $f \times f$ is the composition $Y \times Y \stackrel{\alpha}{\to} Z \stackrel{\beta}{\to} X \times X$. Note that Z is the quotient of $Y \times Y$ under the action of H, where $h \in H$ acts on (y_1, y_2) as $(h + y_1, y_2 - h)$. We assume characteristic of k prime to the order of H. Then $\alpha_*\mathcal{O} = \oplus \mathcal{E}_\chi$, the sum over the characters χ of H. Here $\mathcal{E}_\chi$ is a line bundle, and $\mathcal{E}_{triv} = \mathcal{O}$. So, we have the projection $\alpha_*\mathcal{O} \to \mathcal{O}$, and applying β_*, we get the desired map $C \boxtimes C \to m^*C$. We also have the unit section $u : k \to e^*C$ given by the constant function $H \to k$ with value 1. Then C is a weakly multiplicative sheaf on X.

We have

$$f^*C \widetilde{\to} \mathrm{H}^0(X, H) \otimes \mathcal{O}_Y \tag{2}$$

as $\mathcal{O}_Y$-modules. This is an isomorphism of weakly multiplicative sheaves, where $A = \mathrm{H}^0(H, \mathcal{O})$ is equipped with the convolution algebra structure.

1.5. Example. Consider a smooth projective connected curve over an algebraically closed field k, and the stack Bun_1 of line bundles on X. Let $\mathcal{L}$ be the universal line bundle on $X \times \mathrm{Bun}_1$, $q : X \times \mathrm{Bun}_1 \to \mathrm{Bun}_1$ the projection. Set $C = q_*\mathcal{L}$ in the derived sense, so $C \in \mathrm{QCoh}(\mathrm{Bun}_1)$.

We equipe C with a structure of a weakly multiplicative sheaf as follows. Write $m : \mathrm{Bun}_1 \times \mathrm{Bun}_1 \to \mathrm{Bun}_1$ for the product in this group stack. For the map

$$\mathrm{id} \times m : X \times \mathrm{Bun}_1 \times \mathrm{Bun}_1 \to X \times \mathrm{Bun}_1$$

we get

$$(\mathrm{id} \times m)^*\mathcal{L} \widetilde{\to} \mathrm{pr}_{13}^* \mathcal{L} \otimes \mathrm{pr}_{12}^* \mathcal{L} \tag{3}$$

Let $i : X \to X \times X$ be the diagonal. For the map $i \times \mathrm{id} : X \times \mathrm{Bun}_1 \times \mathrm{Bun}_1 \to X \times X \times \mathrm{Bun}_1 \times \mathrm{Bun}_1$ the line bundle $(i \times \mathrm{id})^*(\mathrm{pr}_{14}^* \mathcal{L} \otimes \mathrm{pr}_{23}^* \mathcal{L})$ identifies canonically with (3). So, the natural map

$$\mathrm{pr}_{14}^* \mathcal{L} \otimes \mathrm{pr}_{23}^* \mathcal{L} \to (i \times \mathrm{id})_*(i \times \mathrm{id})^*(\mathrm{pr}_{14}^* \mathcal{L} \otimes \mathrm{pr}_{23}^* \mathcal{L})$$

yields a morphism $p : C \boxtimes C \to m^*C$. For the unit section $e : \mathrm{Spec}\, k \to \mathrm{Bun}_1$ we get $e^*C \widetilde{\to} \mathrm{R}\Gamma(X, \mathcal{O})$, it has the natural map $u : k \to e^*C$ given by $\mathrm{id} : k \widetilde{\to} \mathrm{H}^0(X, \mathcal{O})$. It is commutative and associative. (Probably, we may take here X any proper variety and understand by Bun_1 the corresponding derived stack).

1.5.1. A version of the previous construction for coherent sheaves. Let Coh be the stack of coherent sheaves on the curve X. Let $\mathcal{L}$ be the universal coherent sheaf on $X \times \mathrm{Coh}$. For the projection $q : X \times \mathrm{Coh} \to \mathrm{Coh}$ let $C = q_*\mathcal{O} \in \mathrm{QCoh}(\mathrm{Coh})$ in the derived sense. Then Coh has a structure of a commutative algebra in PreStk, $m : \mathrm{Coh} \times \mathrm{Coh} \to \mathrm{Coh}$ is given by the tensor product. For the map $\mathrm{id} \times m : X \times \mathrm{Coh} \times \mathrm{Coh} \to X \times \mathrm{Coh}$ the isomorphism (3) still holds. As in the previous section, one defines a map $p : C \boxtimes C \to m^*C$ and a unit section $u : k \to e^*C$. This is a weakly multiplicative object of QCoh(Coh).

1.5.2. Sam has proposed the following generalization of these two examples. First, if $S \in \mathrm{PreStk}$ and X is a commutative algebra in $\mathrm{PreStk}_{/S}$ then one has a notion of a weakly multiplicative object of the mutliplicative sheaf QCoh_X. (In Raskin, Chiral categories, only the absolute version was defined).

Assume now $\pi : S' \to S$ is a map in PreStk and X is a commutative algebra in $\mathrm{PreStk}_{/S}$. Set $X' = X \times_S S'$. Then X' is a commutative algebra in $\mathrm{PreStk}_{/S'}$. Let now $F \in \mathrm{QCoh}(X')$ be equipped with a structure of a weakly multiplicative object in $\mathrm{QCoh}_{X'}$. We may always consider the object $\pi'_*F \in \mathrm{QCoh}(X_2)$ for the map $\pi' : X' \to X$ obtained by the base change from π. We would like to affirm that it inherits a structure of a weakly multiplicative object in QCoh_X.

Proposition 1.5.3. *Assume the maps $\pi : S' \to S$ is quasi-compact quasi-separated schematic. Then π'_*F inherits a structure of a weakly multiplicative object of* QCoh_{X_2}.

Proof. The square is cartesian

$$\begin{array}{ccc} X' \times_{S'} X' & \stackrel{m}{\to} & X' \\ \downarrow \pi' \times \pi' & & \downarrow \pi' \\ X \times_S X & \stackrel{m}{\to} & X \end{array}$$

Apply ([1], ch. I.3, Pp. 2.2.2 and Lm. 3.2.4) to garantee that we have the base change for this square and the projection formula for pr_*. We get the morphism $\mathcal{G} \boxtimes \mathcal{G} \to m^*\mathcal{G}$ for $\mathcal{G} = \pi'_*F$. The map $e : S' \to X'$ is obtained from $e : S \to X$ by the base change $\pi : S' \to S$. So, the desired map $u_\mathcal{G} : \mathcal{O} \to e^*\mathcal{G}$ is the composition

$$\mathcal{O} \to \pi'_*\mathcal{O} \stackrel{\pi'_* u_F}{\to} \pi'_* e^* \mathcal{F}$$

Here $u_F : \mathcal{O} \to e^*F$ is the corresponding map for F. □

References

[1] D. Gaitsgory, N. Rozenblyum, A Study in Derived Algebraic Geometry: Volumes I and II, Mathematical Surveys and Monographs Volume: 221; 2017, version on Dennis' homepage