MY NOTES OF EMBRYO GL SEMINAR

1. Commutative factorization algebras

1.0.1. Let X/k be a separated scheme of finite type. Fix one of our 4 sheaf theories. If the sheaf theory is $\mathcal{D}$-modules then assume X proper, otherwise it is arbitary.

For $p : X \to \operatorname{Spec} k$ we have the adjoint pair $p_! : Shv(X) \leftrightarrows \mathrm{Vect} : p^!$. The functor $p^! : (\mathrm{Vect}, \otimes) \to (Shv(X), \otimes^!)$ is symmetric monoidal, hence yields a functor $p^!_{CAlg} : CAlg^{nu}(\mathrm{Vect}) \to CAlg^{nu}(Shv(X))$. This functor preseves limits, both categories are presentable, so $p^!_{CAlg}$ admits a left adjoint denoted $C_c^{\mathrm{Fact}}(X, \bullet)$. This will be our functor of factorization homology.

However, $C_c^{\mathrm{Fact}}(X, \bullet)$ will not commute with $\mathrm{oblv} : CAlg^{nu}(\mathrm{Vect}) \to \mathrm{Vect}$, $\mathrm{oblv} : CAlg^{nu}(Shv(X)) \to Shv(X)$. The following diagram does commute

$$\begin{array}{ccc} Shv(X) & \overset{free}{\to} & CAlg^{nu}(Shv(X)) \\ \downarrow p_! & & \downarrow C_c^{\mathrm{Fact}}(X,\bullet) \\ \mathrm{Vect} & \overset{free}{\to} & CAlg^{nu}(\mathrm{Vect}) \end{array}$$

For $V \in \mathrm{Vect}$, $free(V) = \oplus_{d>0} Sym^d(V)$, where $\mathrm{Sym}^d(V)$ is as in ([9], 3.0.40), similarly for $free : Shv(X) \to CAlg^{nu}(Shv(X))$.

1.0.2. *Example (recheck).* Let G be a simply-connected semisimple group. It is known that the reduced cohomology $C^\cdot_{rd}(B(G), e) = \mathrm{Sym}(\mathfrak{a}) := \oplus_{d>0} \mathrm{Sym}^d(\mathfrak{a})$ for some $\mathfrak{a} \in \mathrm{Vect}$, the cohomologically graded vector space of Chern classes. Assume X is a proper curve. The Atyah-Bott formula says that

$$C^\cdot_{rd}(\mathrm{Bun}_G, e) \widetilde{\to} C_c^{\mathrm{Fact}}(X, p^! C^\cdot_{rd}(B(G), e)) \widetilde{\to} \mathrm{Sym}(\mathfrak{a} \otimes C^\cdot_c(X, \omega_X))$$

The map in one direction is defined as follows. We define a map

$$C_c^{\mathrm{Fact}}(X, p^! C^\cdot_{rd}(B(G), e)) \to C^\cdot_{rd}(\mathrm{Bun}_G, e),$$

equivalently, a map of nonunital commutative algebras $p^! C^\cdot_{rd}(B(G), e) \to p^! C^\cdot_{rd}(\mathrm{Bun}_G, e)$ in $Shv(X)$. Namely, the diagram

$$\begin{array}{ccc} X \times \mathrm{Bun}_G & \overset{\gamma}{\to} & X \times B(G) \\ & \searrow h & \downarrow q \\ & & X \end{array}$$

gives a map $q_* \to q_* \gamma_* \gamma^* = h_* \gamma^*$, which gives $q_*(e) \to h_* e$.

Date: 28 Sept 2021.

1.0.3. Assume X connected. Then $C_c^{\mathrm{Fact}}(X,\omega)\widetilde{\to} e$ canonically as a commutative algebra in Vect. This follows from Example 2 below and ([4], 1.6.5) giving $C^{\cdot}_c(\mathrm{Ran}_X,\omega)\widetilde{\to} k$ canonically (here X is not necessarily complete).

Now if $\mathcal{A}\in CAlg^{nu}(Shv(X))$ is actully unital then the map $\omega\to\mathcal{A}$ provides a map $C_c^{\mathrm{Fact}}(X,\omega)\to C_c^{\mathrm{Fact}}(X,\mathcal{A})$, which is a unit of this algebra in Vect.

1.0.4. *How to construct* $C_c^{\mathrm{Fact}}(X,\bullet)$*?* Consider an example of X a finite set. In this case the functor $p^!_{CAlg}$ is the diagonal map

$$\prod_{x\in X} CAlg^{nu}(\mathrm{Vect})\leftarrow CAlg^{nu}(\mathrm{Vect})$$

Assume for a moment we are in the unital setting. Then its left adjoint sends $\{B_x\}_{x\in X}$ to $\otimes_{x\in X}B_x$ (in the nonunital setting this is not the left adjoint).

Remark 1.0.5. *Let $\mathcal{A}$ be a stable symmetric monoidal category. A unital augmented algebra in $\mathcal{A}$ is an object $A\in CAlg(\mathcal{A})$ together with a morphism $A\to 1$ in $CAlg(\mathcal{A})$.*
1) One has the equivalence $CAlg(\mathcal{A})_{/1}\widetilde{\to} CAlg^{nu}(\mathcal{A})$. Namely, if $A\to 1$ is an augmented commutative algebra in $\mathcal{A}$ then 1 is a retract of A, so that we have canonically $A\widetilde{\to} 1\oplus\mathfrak{a}$ for some $\mathfrak{a}\in A$. This $\mathfrak{a}=\mathrm{Fib}(A\to 1)$ is naturally a non-unital commutative algebra. Conversely, if $\mathfrak{a}\in CAlg^{nu}(\mathcal{A})$ then $1\oplus\mathfrak{a}\in CAlg(\mathcal{A})_{/1}$ naturally.
2) A finite coproduct in $CAlg^{nu}(\mathcal{A})$ is given as follows. Given B_x for $x\in X$, where X is a finite set, one has

$$(\textit{the desired copoduct})\oplus e\widetilde{\to}\underset{x\in X}{\otimes}(e\oplus B_x)$$

in $CAlg(\mathcal{A})_{/1}$. So, the desired copoduct$)\widetilde{\to}\underset{X'\subset X}{\oplus}(\underset{x\in X'}{\otimes}B_x)$, the sum over non empty subsets of X.

The coproduct in $CAlg^{nu}(\mathrm{Vect})$ is given by the above remark.

1.0.6. Assume X proper. Then $\mathrm{Ran}_X\to\mathrm{Spec}\,k$ is pseudo-proper. So, for $F\in Shv(\mathrm{Ran}_X)$,

$$C^{\cdot}_c(\mathrm{Ran}_X,F)\widetilde{\to}\underset{I\in fSet^{op}}{\mathrm{colim}}\, C^{\cdot}_c(X^I,(\triangle^I)^!F)$$

Here $fSet$ is the category of finite nonempty sets and surjections. Indeed, in $Shv(Ran_X)$

$$F\widetilde{\to}\underset{I\in fSet^{op}}{\mathrm{colim}}(\triangle^I)_*(\triangle^I)^!F$$

The map $u:\mathrm{Ran}_X\times\mathrm{Ran}_X\to\mathrm{Ran}_X$ is pseudo-proper. Indeed, for a finite nonempty set I one has

$$(\mathrm{Ran}_X\times\mathrm{Ran}_X)\times_{u,\mathrm{Ran}_X}X^I\widetilde{\to}\mathrm{colim}_{(I_1,I_2)\in fSet^{op}\times fSet^{op}}(X^{I_1}\times X^{I_2})\times_{\mathrm{Ran}_X}X^I,$$

and $(X^{I_1\sqcup I_2})\times_{\mathrm{Ran}_X}X^I\to X^I$ is described in ([3], 8.1.2). So, $Shv(\mathrm{Ran}_X)$ is monoidal with the convolution monoidal structure given by $u_!$. We denote

$$F_1\star F_2=u_!(F_1\boxtimes F_2)$$

for $F_i\in Shv(\mathrm{Ran}_X)$. Write $CAlg^{nu,\star}(Shv(\mathrm{Ran}_X))$ for the category of non-unital commutative algebras for the $\star$-monoidal structure.

Definition 1.0.7. *Let* $CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X) \subset CAlg^{nu,\star}(Shv(\mathrm{Ran}_X))$ *be the full subcategory of* $\mathcal{A} \in CAlg^{nu,\star}(Shv(\mathrm{Ran}_X))$ *for which the corresponding map* $\mathcal{A} \boxtimes \mathcal{A} \to u^!\mathcal{A}$ *restricted to* $(\mathrm{Ran}_X \times \mathrm{Ran}_X)_d$ *becomes an isomorphism. (Since* u *is etale over* $(\mathrm{Ran}_X \times \mathrm{Ran}_X)_d$*, this is unambiguous).*

1.0.8. For $\vartriangle\colon X \to \mathrm{Ran}_X$ the functor $\vartriangle^!\colon Shv(\mathrm{Ran}_X) \to Shv(X)$ is non-unital symmetric monoidal, so gives a functor $CAlg^{nu,\star}(Shv(\mathrm{Ran}_X)) \to CAlg^{nu,!}(Shv(X))$. Here ! means that we use the symmetric monoidal category $(Shv(X), \otimes^!)$.

Theorem 1.0.9. *The restriction of the above functor*

$$CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X) \to CAlg^{nu,!}(Shv(X))$$

is an equivalence.

(This theorem is found in [5]). The inverse functor is denoted $A \mapsto \mathrm{Fact}(A)$.

The desired functor $C_c^{\mathrm{Fact}}(X, \bullet)$ will be $C_c^{\cdot}(\mathrm{Ran}_X, \mathrm{Fact}(\bullet))$.

1.0.10. *Example 1.* Let $A = \mathrm{Sym}(M)$ for some $M \in Shv(X)$, where Sym is understood in the non-unital sense for $(Shv(X), \otimes^!)$. Then $\mathrm{Fact}(A) \widetilde{\to} \mathrm{Sym}^\star(\vartriangle_! M)$, where $\mathrm{Sym}^\star$ denotes the non-unital symmetric algebra in the non-unital symmetric monoidal category $(Shv(\mathrm{Ran}_X), \star)$.

Explicitely, $\mathrm{Sym}^\star(\vartriangle_! M) = \oplus_{d>0} \mathrm{Sym}^{\star,d}(\vartriangle_! M)$, where $\mathrm{Sym}^{\star,d}(\vartriangle_! M)$ is the S_d-coinvariants in $(\vartriangle_n)_!(M^{\boxtimes n})$. Here $\vartriangle_n\colon X^n \to \mathrm{Ran}_X$ is the natural map.

Recall that this means $\underset{B(S_n)}{\mathrm{colim}} (\vartriangle_n)_!(M^{\boxtimes n})$.

1.0.11. *Example 2.* Note that $\omega \in CAlg^{nu,\star}(Shv(\mathrm{Ran}_X))$ naturally and it is a factorization algebra. Besides, $\vartriangle^! \omega \widetilde{\to} \omega$ on X, so $\mathrm{Fact}(\omega) \widetilde{\to} \omega$ by Theorem 1.0.9.

1.0.12. Recall that for $C \in 1-\mathcal{C}at$ one has $Tw(C) \in 1-\mathcal{C}at$, see ([10], 1.0.1). For $J \in fSet$ let $\vartriangle^J\colon X^J \to \mathrm{Ran}_X$ be the natural map. Now for $A \in CAlg^{nu,!}(Shv(X))$ one has

$$\mathrm{Fact}(A) \widetilde{\to} \underset{(I \overset{\phi}{\to} J) \in Tw(fSet)}{\mathrm{colim}} (\vartriangle^J)_! A^{\otimes \phi}$$

taken in $Shv(\mathrm{Ran}_X)$. Here for $(\phi : I \to J) \in Tw(fSet)$ we set $A^{\otimes \phi} = \underset{j \in J}{\boxtimes} (A^{\otimes I_j})$, where $A^{\otimes I_j}$ is the tensor power in $(Shv(X), \otimes^!)$.

For a morphism in $Tw(fSet)$

$$\begin{array}{ccc} I & \overset{\phi}{\to} & J \\ \downarrow & & \uparrow \\ I' & \overset{\phi'}{\to} & J' \end{array} \tag{1}$$

the transition map

$$(\vartriangle^J)_! A^{\otimes \phi} \to (\vartriangle^{J'})_! A^{\otimes \phi'} \tag{2}$$

is obtained applying $\vartriangle_!^{J'}$ to the following map. For the closed immersion $\vartriangle^{(J'/J)}\colon X^J \to X^{J'}$ one has

$$\vartriangle^{(J'/J)!} (\underset{j' \in J'}{\boxtimes} A^{\otimes I'_{j'}}) \widetilde{\to} \underset{j \in J}{\boxtimes} A^{\otimes I'_j} \tag{3}$$

Now (2) over $X^{J'}$ is the composition

$$\vartriangle_!^{(J'/J)}(\underset{j\in J}{\boxtimes}(A^{\otimes I_j}))\to\vartriangle_!^{(J'/J)}(\underset{j\in J}{\boxtimes}(A^{\otimes I'_j}))\to\underset{j'\in J'}{\boxtimes}A^{\otimes I'_{j'}}$$

where the first map is the exterior product over $j\in J$ of the products in the algebra A along maps $I_j\to I'_j$, and the second one comes from (3). A rigorous definition as a functor $Tw(fSet)\to Shv(\mathrm{Ran})$ follows from the explanation of Justin below.

1.0.13. We have an adjoint pair $l: fSet\leftrightarrows Tw(fSet): r$, where $l(I)=(I\to *)$ and $r(I\to J)=I$. Here l is fully faithful.

Let now $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$ be cocomplete say and $e: fSet\to\mathcal{C}$ be a functor. Let $LKE(e): Tw(fSet)\to\mathcal{C}$ be the LKE of e along l. Then $LKE(e)=e\circ r$ by ([9], 2.2.39). So, $e: fSet\to\mathcal{C}$ identifies with the LKE of $LKE(e)$ along $r: Tw(fSet)\to fSet$. For this reason, $\mathrm{colim}_{Tw(fSet)}(e\circ r)\widetilde{\to}\mathrm{colim}_{fSet}\, e$. Now $fSet$ has a final object $*$, so $\mathrm{colim}_{fSet}\, e\widetilde{\to} e(*)$.

1.0.14. For $I\in fSet$ let $A_I=(\vartriangle^I)^!\,\mathrm{Fact}(A)$. For $I=*$ we get $A_I\widetilde{\to}A$ canonically. Indeed, applying $\vartriangle^!$ for $\vartriangle: X\to\mathrm{Ran}_X$, we get

$$\vartriangle^!\mathrm{Fact}(A)\widetilde{\to}\underset{(I\overset{\phi}{\to}J)\in Tw(fSet)}{\mathrm{colim}}A^{\otimes I}$$

where for a morphism (1) in $Tw(fSet)$ the transition map is the product $A^{\otimes I}\to A^{\otimes I'}$ along $I\to I'$. By the previous subsection, the above colimit identifies with A.

1.0.15. If $I,J\in fSet$ then

$$X^I\times_{\mathrm{Ran}_X}X^J\widetilde{\to}\underset{I\twoheadrightarrow K\twoheadleftarrow J}{\mathrm{colim}}X^K$$

naturally. More precisely, inside the colimit is over $(fSet_{I/}\times_{fSet}fSet_{J/})^{op}$. In particular, $Shv(X^I\times_{\mathrm{Ran}_X}X^J)\widetilde{\to}\underset{I\twoheadrightarrow K\twoheadleftarrow J}{\mathrm{colim}}Shv(X^K)$ with respect to !-extensions, and for any $F\in Shv(X^I)$ we have

$$\vartriangle^{J!}\vartriangle^I_!F\widetilde{\to}\underset{I\twoheadrightarrow K\twoheadleftarrow J}{\mathrm{colim}}\vartriangle_!^{(J/K)}\vartriangle^{K!}F$$

Here $\vartriangle^{(J/K)}: X^K\to X^J$.

Proposition 1.0.16. *For $I\in fSet$ the object A_I identifies with*

$$\underset{I\to J\overset{\phi}{\to}K}{\mathrm{colim}}\vartriangle_!^K(A^{\otimes\phi}),$$

where the colimit is over $Tw(fSet)\times_{fSet}fSet_{I/}$.

Proof. Consider the category $\mathcal{E}$, whose objects are diagrams $(I\to\bar{K}\leftarrow K\overset{\phi}{\leftarrow}J)$ in $fSet$, and a morphism from 1 to 2 is given by a diagram in $fSet$

$$\begin{array}{ccccccc} I & \to & \bar{K}_1 & \leftarrow & K_1 & \overset{\phi_1}{\leftarrow} & J_1\\ & \searrow & \uparrow & & \uparrow & & \downarrow\\ & & \bar{K}_1 & \leftarrow & K_2 & \overset{\phi_2}{\leftarrow} & J_2 \end{array}$$

From the description of $X^K \times_{\mathrm{Ran}_X} X^I$ we get

$$A_I \widetilde{\to} \underset{(J \overset{\phi}{\to} K)\in Tw(fSet)}{\mathrm{colim}} \vartriangle^{I!} \vartriangle_!^K (A^{\otimes \phi}) \widetilde{\to} \underset{(I\to \bar{K} \leftarrow K \overset{\phi}{\leftarrow} J)\in \mathcal{E}}{\mathrm{colim}} \vartriangle_!^{(I/K)} \vartriangle^{(K/\bar{K})!} A^{\otimes\phi} \tag{4}$$

Write $\bar{\phi}$ for the composition $J \overset{\phi}{\to} K \to \bar{K}$ for an object of $\mathcal{E}$. Then $\vartriangle^{(K/\bar{K})!} A^{\otimes\phi} \widetilde{\to} A^{\otimes\bar{\phi}}$.

Let $r_0 : \mathcal{E}_0 \subset \mathcal{E}$ be the full subcategory given by the property that $K \to \bar{K}$ is an isomorphism. We think of $\mathcal{E}_0$ as the category classifying diagrams $(I \to \bar{K} \leftarrow J)$, where $(\bar{K} \leftarrow J) \in Tw(fSet)$ and $I \to \bar{K}$ is a surjection of finite sets. We write $\mathcal{E}_{0,I}$ if we need to express the dependence on I.

Let $l_0 : \mathcal{E} \to \mathcal{E}_0$ be the functor sending $(I \to \bar{K} \leftarrow K \leftarrow J)$ to $(I \to \bar{K} \leftarrow J)$. Then r_0 is right adjoint to l_0. So, the LKE along l_0 is calculated as composition with r_0. Therefore, (4) identifies with

$$\underset{(I\to K \overset{\phi}{\leftarrow} J)\in \mathcal{E}_0}{\mathrm{colim}} \vartriangle_!^{(I/K)} A^{\otimes\phi}$$

in $Shv(X^I)$. Let $\mathcal{F} : \mathcal{E}_0 \to Shv(X^I)$ be the functor whose colimit is the latter expession.

Consider now the category $\mathcal{C} = Tw(fSet) \times_{fSet} (fSet)_{I/}$. We will write $\mathcal{C}_I$ if we need to express the dependence on I. This is the category of diagrams $(K \leftarrow J \leftarrow I)$ in $fSet$ with $(K \leftarrow J) \in Tw(fSet)$. We have the functor $q : \mathcal{C} \to \mathcal{E}_0$ sending $(K \overset{\phi}{\leftarrow} J \overset{a}{\leftarrow} I)$ to $(I \overset{\phi\circ a}{\to} K \overset{\phi}{\leftarrow} J)$.

We claim that the natural map $\mathrm{colim}_{\mathcal{C}}\, \mathcal{F} \circ q \to \mathrm{colim}_{\mathcal{E}_0} \mathcal{F}$ is an isomorphism in $Shv(X^I)$. Is the map q cofinal? Let $\eta := (I \overset{\alpha}{\to} K \overset{\phi}{\leftarrow} J) \in \mathcal{E}_0$. We need to check that the category $\mathcal{C} \times_{\mathcal{E}_0} (\mathcal{E}_0)_{\eta/}$ is contractible. An object of the latter category is a diagram

$$\begin{array}{ccccc} K_1 & \overset{\phi_1}{\leftarrow} & J_1 & \overset{\tau_1}{\leftarrow} & I \\ \downarrow \nu_1 & & \uparrow & & \\ K & \overset{\phi}{\leftarrow} & J & & \end{array}$$

such that $\nu_1\phi_1\tau_1 = \alpha$. A morphism from the above object to another object

$$\begin{array}{ccccc} K_2 & \overset{\phi_2}{\leftarrow} & J_2 & \overset{\tau_2}{\leftarrow} & I \\ \downarrow \nu_2 & & \uparrow & & \\ K & \overset{\phi}{\leftarrow} & J & & \end{array} \tag{5}$$

(satisfying $\nu_2\phi_2\tau_2 = \alpha$) is a commutative diagram

$$\begin{array}{ccccc} K & \overset{\phi}{\leftarrow} & J & & \\ \uparrow & & \downarrow & & \\ K_1 & \overset{\phi_1}{\leftarrow} & J_1 & \overset{\tau_1}{\leftarrow} & I \\ \uparrow & & \downarrow & & \downarrow \mathrm{id} \\ K_2 & \overset{\phi_2}{\leftarrow} & J_2 & \overset{\tau_2}{\leftarrow} & I \end{array}$$

such that the vertical compositions are the corresponding maps from (5).

If K is a 1-element set then $\mathcal{C}\times_{\mathcal{E}_0}(\mathcal{E}_0)_{\eta/}$ has an initial object given by the diagram

$$\begin{array}{ccc} * & \leftarrow & I\sqcup J \quad \leftarrow I \\ \downarrow & & \uparrow \\ * & \leftarrow & J \end{array}$$

For general K, $\mathcal{C}\times_{\mathcal{E}_0}(\mathcal{E}_0)_{\eta/}$ will be a product of categories over the set K. Namely, denote for $k\in K$ by I_k, J_k the corresponding fibres, so that $\eta_k := (I_k\to\{k\}\leftarrow J_k)\in\mathcal{E}_{0,I_k}$. For each $k\in K$ we have the category

$$\mathcal{C}_{I_k}\times_{\mathcal{E}_{0,I_k}}(\mathcal{E}_{0,I_k})_{\eta_k/},$$

and their product identifies with $\mathcal{C}\times_{\mathcal{E}_0}(\mathcal{E}_0)_{\eta/}$. Thus, $\mathcal{C}\times_{\mathcal{E}_0}(\mathcal{E}_0)_{\eta/}$ is contractible, and q is cofinal. □

The formula from Proposition 1.0.16 has appeared for commutative factorization categories (instead of algebras) in ([6], 8.1.6).

1.0.17. For I of order 2 we get $A_I\tilde{\to}A\underset{\triangle_! A^{\otimes^! 2}}{\sqcup}(A\boxtimes A)$ for $\triangle\colon X\to X^2$, where the map $A^{\otimes^! 2}\to A$ is the product in the algebra.

1.1. **2nd seminar.**

1.1.1. for $p:\mathrm{Ran}_X\to\operatorname{Spec}k$ the functor $p^!:\mathrm{Vect}\to(Shv(\mathrm{Ran}_X),\star)$ is right-lax non-unital monoidal, and the diagram (of right-lax symmetric monoidal functors) commutes

$$\begin{array}{ccc} (Shv(X),\otimes^!) & \overset{\triangle^!}{\leftarrow} & (Shv(\mathrm{Ran}_X),\star) \\ & \nwarrow p_X^! & \uparrow p^!_{\mathrm{Ran}} \\ & & \mathrm{Vect} \end{array}$$

(here $p_X^!, \triangle^!$ are symmetric monoidal). So, the diagram commutes

$$\begin{array}{ccc} CAlg^{nu,!}(Shv(X)) & \overset{\triangle^!}{\leftarrow} & CAlg^{nu,\star}(Shv(\mathrm{Ran}_X)) \\ & \nwarrow p_X^! & \uparrow p^!_{\mathrm{Ran}} \\ & & CAlg^{nu}(\mathrm{Vect}) \end{array}$$

So, we have for the corresponding left adjoint functors between the categories of nonunital commutative algebras $(p_{\mathrm{Ran}})_!^{CAlg}\circ\triangle_!^{CAlg}\tilde{\to}C_c^{\mathrm{Fact}}(X,\bullet)$.

The key is that the functor $(p_{\mathrm{Ran}})_!:(Shv(\mathrm{Ran}),\star)\to\mathrm{Vect}$ is nonunital symmerric monoidal! So, $(p_{\mathrm{Ran}})_!^{CAlg}$ is just the usual direct image $(p_{\mathrm{Ran}})_!$.

Theorem 1.1.2. *The functor $\triangle^!\colon CAlg^{nu,\star}(Shv(\mathrm{Ran}_X))\to CAlg^{nu,!}(Shv(X))$ admits a fully faithful left adjoint, whose essential image is precisely $CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X)$.*

Proof. The categories $CAlg^{nu,\star}(Shv(\mathrm{Ran}_X))$, $CAlg^{nu,!}(Shv(X))$ are presentable, and $\triangle^!$ preserves limits and commutes with oblv, so its left adjoint $\triangle_!^{CAlg}$ exists. By Lemma 1.1.3 below, $\triangle_!^{CAlg}$ is fully faithful.

By the proof of Lemma 1.1.3 below, $CAlg^{nu,!}(Shv(X))$ is generated under geometric realizations by free algebras. So, to check that $\triangle_!^{CAlg}$ takes values in $CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X)$, it suffices to show that for any $K\in Shv(X)$, $\triangle_!^{CAlg}(\mathrm{Sym}(K))$ lies in $CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X)$, where by Sym we mean the left adjoint to $oblv : CAlg^{nu,!}(Shv(X)) \to Shv(X)$. Indeed, oblv : $CAlg^{nu,*}(Shv(\mathrm{Ran}_X)) \to Shv(\mathrm{Ran}_X)$ preserves sifted colimits, so that $CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X) \subset CAlg^{nu,*}(Shv(\mathrm{Ran}_X))$ is stable under sifted colimits.

We know that $\vartriangle_!^{CAlg}(\mathrm{Sym}(K)) \widetilde{\to} \mathrm{Sym}^\star(\vartriangle_! K)$ for $K \in Shv(X)$ by Section 1.0.10 of this file. One checks by hands that this object has the factorization property:

We have for $n > 0$,

$$u^! \,\mathrm{Sym}^{n,\star}(\vartriangle_! K) \mid_{(\mathrm{Ran}_X \times \mathrm{Ran}_X)_d} \widetilde{\to} \bigoplus_{0<s<n} \mathrm{Sym}^{s,\star}(\vartriangle_! K) \boxtimes \mathrm{Sym}^{n-s,\star}(\vartriangle_! K) \tag{6}$$

Indeed, if for a point $S \to X^n$ given by (z_k) we are given $(x_i), (y_j) \in (\mathrm{Ran}_X \times \mathrm{Ran}_X)_d$ with $(x_i) \cup (y_j) = (z_k)$, there is a unique decomposition $I_1 \sqcup I_2 = \{1, \dots, n\}$ such that our point is a product of two points $S \to X^{I_1}$, $S \to X^{I_2}$ giving a point of $(X^{I_1} \times X^{I_2})_d$, which produces our point of $(\mathrm{Ran}_X \times \mathrm{Ran}_X)_d$. The group S_n acts transitively on such decompositions, and the stabilizor of a decompostion is $S(I_1) \times S(I_2)$, the product of symmetric groups. If $\mid I_1 \mid = s$ this gives the contribution

$$\mathrm{Sym}^{s,\star}(\vartriangle_! K) \boxtimes \mathrm{Sym}^{n-s,\star}(\vartriangle_! K)$$

Now the RHS of (6) over $(\mathrm{Ran}_X \times \mathrm{Ran}_X)_d$ identifies with n-th graded component of $\mathrm{Sym}^\star(\vartriangle_! K) \boxtimes \mathrm{Sym}^\star(\vartriangle_! K)$.

It remains to show that $\vartriangle_!^{CAlg}\colon CAlg^{nu,!}(Shv(X)) \to CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X)$ generates $CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X)$ under geometric realizations. The idea is to use ([8], Pp. 4.7.3.14). We check first that

$$\vartriangle^!\colon CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X) \to CAlg^{nu,!}(Shv(X)) \tag{7}$$

is conservative. The latter fact follows from the factorization property, because we know it over X.

Indeed, let $f : A_1 \to A_2$ be a map in $CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X)$ with $\vartriangle^! f$ an isomorphism. We want to show that the !-restriction of f under $X^I \to \mathrm{Ran}_X$ is an isomorphism. It suffices to show that for any quotient set $I \to J$ the !-restriction of f under $X^J_d \hookrightarrow X^I \to \mathrm{Ran}_X$ is an isomorphism, and this follows from the factorization property. Here $X^J_d \subset X^J$ is the complement to all the diagonals.

To finish, note that (7) preserves sifted colimits, because

$$CAlg^{\mathrm{Fact}}(\mathrm{Ran}_X) \subset CAlg^{nu,*}(Shv(\mathrm{Ran}_X))$$

is stable under sifted colimits. Thus, ([8], 4.7.3.14) applies. □

Lemma 1.1.3. *Suppose $l : C \leftrightarrows D : r$ is an adjoint pair in DGCat_{cont} with l fully faithful. Suppose that $C, D \in CAlg^{nu}(\mathrm{DGCat}_{cont})$, and r is nonunital symmetric monoidal, so l is left-lax nonunital symmetric monoidal. The functor $r^{CAlg} : CAlg^{nu}(D) \to CAlg^{nu}(C)$ preserves limits, so admits a left adjoint l^{CAlg}, as both categories are presentable. Then l^{CAlg} is fully faithful.*

Proof. We check that the natural map $\mathrm{id} \to r^{CAlg} l^{CAlg}$ is an isomorphism. My understanding is that $CAlg^{nu}(C)$ is not a DG-category, just a presentable category. The functor oblv : $CAlg^{nu}(C) \to C$ is conservative. The essential image of its left adjoint $free : C \to CAlg^{nu}(C)$ generates $CAlg^{nu}(C)$ under colimits in the sense of ([7], 5.1.5.7), this follows from the (*) claim inside the proof of ([8], Corollary 3.2.3.3). Another way is to say that for any reduced operad $\mathcal{P}$ in the sense of ([2], ch. IV.2, 1.1.2), $\mathrm{oblv} : \mathcal{P} - Alg(C) \to C$ is monadic, so we may apply ([8], 4.7.3.14). The functor $\mathrm{oblv} : CAlg^{nu}(C) \to C$ preserves sifted colimits (by [9], 9.4.12).

So, it suffices to show that for any $c \in C$,

$$free(c) \to r^{CAlg} l^{CAlg}(free(c))$$

is an isomorphism. Now $l^{CAlg} \circ free \widetilde{\to} free \circ l$. Since r is nonunital symmetric monoidal, we have for $n > 0$ and $d \in D$, $r(\mathrm{Sym}^n(d)) \widetilde{\to} \mathrm{Sym}^n(r(d))$, because

$$\mathrm{Sym}^n(d) \widetilde{\to} \operatorname*{colim}_{B(S_n)} d^{\otimes n}$$

the colimit being taken in D. So, $\mathrm{oblv} \circ r^{CAlg} \circ free \widetilde{\to} \mathrm{oblv} \circ free \circ r$. The claim follows. □

1.2. Justin's explanation.

1.2.1. Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjoint pair in DGCat_{cont}, where both $\mathcal{C}, \mathcal{D} \in CAlg^{nu}(\mathrm{DGCat}_{cont})$, and G is nonunital symmetric monoidal. So, F is left-lax nonunital symmetric monoidal. We want to "force" in some universal way F to be strictly nonuinital symmetric monoidal.

Recall that $fSet$ is the category of finite nonempty sets and surjections. View $fSet$ as nonunital symmetric monoidal with respect to the disjoint union. Note that $pt \in CAlg^{nu}(fSet)$, the product is the map $pt \sqcup pt \to pt$. It has the following universal property: for any nonunital symmetric monoidal category $\mathcal{C}$, one has a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}(fSet, \mathcal{C}) & \widetilde{\to} & CAlg^{nu}(\mathcal{C}) \\ \downarrow & \swarrow \mathrm{oblv} & \\ \mathcal{C}, & & \end{array} \tag{8}$$

where the vertical arrow is the evaluation at $pt \in fSet$.

Indeed, recall the ∞-operad Surj, which is a subcategory of $\mathcal{F}\mathrm{in}_*$ with the same objects and where we keep the morphisms that are surjective. Its monoidal envelope in the sense of ([8], 2.2.4.1) evidently identifies with $fSet$. Namely, to $I \in fSet$ we associate $I \sqcup \{*\} \in \mathrm{Surj}$. So, (8) is a particular case of ([8], 2.2.4.9).

Remark 1.2.2. *What is the $\mathcal{F}\mathrm{in}_*$-monoidal envelope of $\mathcal{F}\mathrm{in}_*$? By ([8], 2.2.4.3), it is as follows. Let $fSet_\emptyset$ be the category of finite (possibly empty) sets and any morphisms between them. The map $fSet_\emptyset \to Act(\mathcal{F}\mathrm{in}_*)$, $I \mapsto I \sqcup \{*\}$ is an equivalence. By ([8], 2.2.4.4), $fSet_\emptyset$ has a symmetric monoidal structure given by the disjoint union, and this is the monoidal envelope of the ∞-operad $\mathcal{F}\mathrm{in}_* \xrightarrow{\mathrm{id}} \mathcal{F}\mathrm{in}_*$. We may think of the ∞-operad $\mathcal{F}\mathrm{in}_* \xrightarrow{\mathrm{id}} \mathcal{F}\mathrm{in}_*$ just as the category $\{*\}$ with the natural symmetric monoidal structure. So, ([8], 2.2.4.9) in this case says that for any symmetric monoidal ∞-category D the restriction along $\{*\} \hookrightarrow fSet_\emptyset$ yields an equivalence*

$$\mathrm{Fun}^{\otimes}(fSet_\emptyset, D) \widetilde{\to} CAlg(D)$$

Here $\mathrm{Fun}^{\otimes}(fSet_\emptyset, D)$ is the category of symmetric monoidal functors $fSet_\emptyset \to D$.

1.2.3. For any $C \in CAlg(1-\mathcal{C}\mathrm{at})$ consider the functor $h : C^{op} \times C \to \mathrm{Spc}$, $(x, y) \mapsto \mathrm{Map}_C(x, y)$. We equip C^{op} with the induced symmetric monoidal structure, and similarly for $C^{op} \times C$, and Spc with the cartesian symmetric monoidal structure. Then h is right-lax symmetric monoidal: we have natural maps $* \stackrel{\mathrm{id}}{\to} \mathrm{Map}_C(1, 1)$ and $\mathrm{Map}_C(x_1, y_1) \times \mathrm{Map}_C(x_2, y_2) \to \mathrm{Map}_C(x_1 \otimes x_2, y_1 \otimes y_2)$ given by the tensor product of maps.

Let $h' : \mathcal{X} \to C^{op} \times C$ be the cocartesian fibration in spaces attached to h. By ([12], 5.15), $\mathcal{X}$ is naturally symmetric monoidal and h' is symmetric monoidal. So, $\mathcal{X}^{op}$ is also symmetric monoidal, and $h'^{op} : \mathcal{X}^{op} \to C \times C^{op}$ is symmetric monoidal. The latter functor is a cartesian fibration attached to the functor $C^{op} \times C \to \mathrm{Spc}$, $(x, y) \mapsto \mathrm{Map}_C(x, y)^{op} \widetilde{\to} \mathrm{Map}_C(x, y)$. So, $\mathcal{X}^{op} \widetilde{\to} Tw(C)$.

We similarly have a nonunital version for $C \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ giving a nonunital symmetric monoidal structure on $Tw(C)$.

1.2.4. We apply this to $C = fSet$, and see that $Tw(fSet)$ is nonunital symmetric monoidal. The projection $Tw(fSet) \to fSet \times fSet^{op}$ is nonunital symmetric monoidal. The tensor product of the objects $f : I \to J$ and $f' : I' \to J'$ of $fSet$ is the map $f \sqcup f' : I \sqcup I' \to J \sqcup J'$.

1.2.5. Consider the functor $l : fSet \to Tw(fSet)$ sending I to $(I \to *)$. It is left-lax nonunital symmetric monoidal.

Lemma 1.2.6. *For any $D \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ the composition with l yields an equivalence*

$$\mathrm{Fun}^{\otimes}(Tw(fSet), D) \widetilde{\to} \mathrm{Fun}^{llax}(fSet, D)$$

Here $\mathrm{Fun}^{\otimes}(Tw(fSet), D)$ is the category of nonunital symmetric monoidal functors.

Proof. A general claim first.

Step 1 Let $\mathrm{Surj} \to 1-\mathcal{C}\mathrm{at}$ be a Surj-monoid in $1-\mathcal{C}\mathrm{at}$ in the sense of ([8], 2.4.2.1), let $\mathcal{A}$ be the value of this functor on $\langle 1 \rangle$. Let $\tilde{\mathcal{A}}^{\otimes} \to \mathrm{Surj}$ be the cocartesian fibration attached to it, so this is an ∞-operad defining the nonunital symmetric monoidal category $\mathcal{A}$. Recall the subcategory $fSet \subset \mathrm{Surj}$, this functor sends I to $I \sqcup \{*\}$. Let $\mathcal{A}^{\otimes} \to \mathrm{Surj}$ be obtained from $\tilde{\mathcal{A}}^{\otimes} \to \mathrm{Surj}$ by the base change $fSet \to \mathrm{Surj}$. Then $\mathcal{A}^{\otimes}$ identifies with the Surj-monoidal envelope of the ∞-operad $\tilde{\mathcal{A}}^{\otimes} \to \mathrm{Surj}$ in the sense of ([8], 2.2.4.3). So, $\mathcal{A}^{\otimes}$ acquires a nonunital symmetric monoidal structure, and the natural right-lax nonunital symmetric monoidal map $\mathcal{A} \to \mathcal{A}^{\otimes}$ giving rise, by ([8], 2.2.4.9), to an equivalence

$$\mathrm{Fun}^{\otimes}(\mathcal{A}^{\otimes}, D) \widetilde{\to} \mathrm{Fun}^{rlax}(\mathcal{A}, D)$$

Here $\mathrm{Fun}^{rlax}(\mathcal{A}, D)$ is what is called the category of $\tilde{\mathcal{A}}^{\otimes}$-algebras in $D^{\otimes}$ over Surj in [8], and $\mathrm{Fun}^{\otimes}(\mathcal{A}^{\otimes}, D)$ is the category of nonunital symmetric monoidal functors.

An object of $\mathcal{A}^{\otimes}$ is a finite nonempty set I and a collection $a_i \in \mathcal{A}$ for $i \in I$. A map in $\mathcal{A}^{\otimes}$ from $(I, \{a_i\})$ to $(J, \{b_j\})$ is a surjection $f : I \to J$ and for each $j \in J$ a map $\bigotimes_{i \in f^{-1}(j)} a_i \to b_j$ in $\mathcal{A}$. The tensor product in the symmetric monoidal category $\mathcal{A}^{\otimes}$ of $(I, \{a_i\})$ and $(J, \{b_j\})$ is $(I \sqcup J, \{a_i, b_j\})$ by ([8], 2.2.4.6). The functor $i : \mathcal{A} \to \mathcal{A}^{\otimes}$ sends a to $(*, a)$.

We also have a symmetric monoidal functor $\mu : \mathcal{A}^{\otimes} \to \mathcal{A}$, $(I, \{a_i\}) \mapsto \otimes_{i\in I} a_i$. Note that μ is left adjoint to i.

Step 2 Consider $fSet^{op}$ with the induced nonunital symmetric monoidal structure. Then the Surj-monoidal envelope $((fSet)^{op})^{\otimes}$ of $(fSet)^{op}$ is $Tw(fSet)^{op}$. Indeed, by the above, an object of $((fSet)^{op})^{\otimes}$ is a collection $(I, \{J_i\}_{i\in I})$, where I is a finite nonempty set, and for $i \in I$, J_i is a finite nonempty set. A map in $((fSet)^{op})^{\otimes}$ from $(I, \{J_i\}_{i\in I})$ to $(I', \{J'_{i'}\}_{i'\in I'})$ is a surjection $\phi : I \to I'$, and for each $i' \in I'$ a surjection of finite sets

$$J'_{i'} \to \bigsqcup_{i\in\phi^{-1}(i')} J_i$$

We may view $(I, \{J_i\}_{i\in I})$ as an object $\tau : J \to I$ of $fSet$. The above morphism in $((fSet)^{op})^{\otimes}$ from $(J \xrightarrow{\tau} I)$ to $(J' \xrightarrow{\tau'} I')$ becomes a commutative diagram

$$\begin{array}{ccc} J' & \xrightarrow{\tau'} & I' \\ \downarrow \phi' & & \uparrow \phi \\ J & \xrightarrow{\tau} & I \end{array} \tag{9}$$

Now the right-lax functor $fSet^{op} \to ((fSet)^{op})^{\otimes} \widetilde{\to} Tw(fSet)^{op}$ sends I to $(I \xrightarrow{\tau} *)$, this is the functor l^{op} from Section 1.2.5. By Step 1, we get an equivalence

$$\begin{aligned} \mathrm{Fun}^{\otimes}(Tw(fSet), D)^{op} \widetilde{\to} \mathrm{Fun}^{\otimes}(Tw(fSet)^{op}, D^{op}) \widetilde{\to} \mathrm{Fun}^{rlax}(fSet^{op}, D^{op}) \\ \widetilde{\to} \mathrm{Fun}^{llax}(fSet, D)^{op} \end{aligned}$$

The functor $\mu : Tw(fSet)^{op} \widetilde{\to} ((fSet)^{op})^{\otimes} \to (fSet)^{op}$ sends $(J \xrightarrow{\tau} I)$ to J. □

Remark 1.2.7. *If* $A, B \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ *then* $\mathrm{Fun}^{\otimes}(A, B)^{op} \widetilde{\to} \mathrm{Fun}^{\otimes}(A^{op}, B^{op})$ *and* $\mathrm{Fun}^{rlax}(A, B)^{op} \widetilde{\to} \mathrm{Fun}^{llax}(A^{op}, B^{op})$.

1.2.8. Let $\mathcal{I}, \mathcal{C} \in CAlg(1-\mathcal{C}\mathrm{at})$ with $\mathcal{C}$ cocomplete such that the tensor product in $\mathcal{C}$ preserves colimits separately in each variable, and $\mathcal{I}$ is small. Then $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ has a symmetric monoidal structure given by Day convolution ([8], 2.2.6.17).

Let $F : \mathcal{I} \to \mathcal{J}$ be a map in $CAlg(1-\mathcal{C}\mathrm{at})$, where $\mathcal{I}, \mathcal{J}$ are small. We have an adjoint pair $l : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \rightleftarrows \mathrm{Fun}(\mathcal{J}, \mathcal{C}) : r$, where l is the left Kan extension along F, and r is the composition with F. Then l is symmetric monoidal. Indeed, for $i \in \mathcal{I}$ and $f_1, f_2 \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ we have

$$(f_1 \otimes f_2)(i) \widetilde{\to} \underset{(i_1,i_2)\in\mathcal{I}\times\mathcal{I},\ i_1\otimes i_2\to i}{\mathrm{colim}} f_1(i_1) \otimes f_2(i_2)$$

Let $\bar{f}_i = l(f_i)$. Then the LKE of the composition $\mathcal{I}\times\mathcal{I} \xrightarrow{f_1\times f_2} \mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} C$ along $\mathcal{I}\times\mathcal{I} \to \mathcal{J}\times\mathcal{J}$ is the composition $\mathcal{J}\times\mathcal{J} \xrightarrow{\bar{f}_1\times\bar{f}_2} \mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C}$. Indeed, for $(j_1, j_2) \in \mathcal{J}\times\mathcal{J}$ the value of this LKE at (j_1, j_2) is

$$\begin{aligned} \underset{(i_1,i_2)\in\mathcal{I}\times\mathcal{I}, F(i_1)\to j_1, F(i_2)\to j_2}{\mathrm{colim}} f_1(i_1)\otimes f_2(i_2) \widetilde{\to} (\underset{i_1\in\mathcal{I}, F(i_1)\to j_1}{\mathrm{colim}} f_1(i_1)) \otimes (\underset{i_2\in\mathcal{I}, F(i_2)\to j_2}{\mathrm{colim}} f_2(i_2)) \\ \widetilde{\to} \bar{f}_1(j_1)\otimes\bar{f}_2(j_2) \end{aligned}$$

Now $l(f_1 \otimes f_2)$ is the LKE of the composition $\mathcal{I} \times \mathcal{I} \stackrel{f_1 \times f_2}{\to} \mathcal{C} \times \mathcal{C} \stackrel{\otimes}{\to} C$ along the composition $\mathcal{I} \times \mathcal{I} \stackrel{\otimes}{\to} \mathcal{I} \stackrel{F}{\to} \mathcal{J}$. So, our claim follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{J} \times \mathcal{J} & \stackrel{\otimes}{\to} & \mathcal{J} \\ \uparrow F \times F & & \uparrow F \\ \mathcal{I} \times \mathcal{I} & \stackrel{\otimes}{\to} & \mathcal{I} \end{array}$$

and the transitivity of LKE.

More generally, for a (maybe empty) finite set S and a collection $f_s \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ for $s \in S$, we have

$$(\underset{s \in S}{\otimes} f_s)(i) \widetilde{\to} \underset{\{i_s\} \in \prod_{s \in S} \mathcal{I},\ \underset{i \in S}{\otimes} i_s \to i}{\mathrm{colim}} (\underset{i \in S}{\otimes} f_s(i_s))$$

In particular, for $S = \emptyset$ this gives the unit of $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ for the Day covolution: its value at i is $\underset{\mathrm{Map}_{\mathcal{I}}(1,i)}{\mathrm{colim}}\ 1$ taken in $\mathcal{C}$. The space $\mathrm{Map}_{\mathcal{I}}(1, i)$ is not necessarily contractible, so in Lurie's notation this is $\mathrm{Map}_{\mathcal{I}}(1, i) \otimes 1$, as $\mathcal{C}$ is tensored over Spc. If 1 is initial in $\mathcal{I}$ then the unit of $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ for the Day convolution is the constant functor with value $1 \in \mathcal{C}$. In fact, the unit of $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ is the LKE of $* \stackrel{1}{\to} \mathcal{C}$ along $* \stackrel{1}{\to} \mathcal{I}$.

This shows that l sends unit to unit. Note that r is right-lax monoidal for the Day convolution structures. The functors r, l induce a diagram of adjoint functors $l^{CAlg} : CAlg(\mathrm{Fun}(\mathcal{I}, \mathcal{C})) \leftrightarrows CAlg(\mathrm{Fun}(\mathcal{J}, \mathcal{C})) : r^{CAlg}$ given by composing with r and l respectively. This follows from remark below.

Remark 1.2.9. *Let $C^{\otimes} \to O^{\otimes} \leftarrow D^{\otimes}$ be a diagram of cocartesian fibrations of ∞-operads. Let $A^{\otimes} \to O^{\otimes}$ be a map of ∞-operads. Assume given an adjoint pair $L : C^{\otimes} \leftrightarrows D^{\otimes} : R$ in $1 - \mathcal{C}\mathrm{at}$, where L, R are maps of ∞-operads over $O^{\otimes}$. Assume that L is a morphism of $O^{\otimes}$-monoidal categories, that is, sends a cocartesian arrow over $O^{\otimes}$ to a cocartesian arrow over $O^{\otimes}$. Let $L' : Alg_{A/O}(C) \leftrightarrows Alg_{A/O}(D) : R'$ be obtained by composing with L and R. Then (L', R') is an adjoint pair in $1 - \mathcal{C}\mathrm{at}$.*

A special case of this has appeared as ([9], 3.0.20).

1.2.10. Now using $CAlg(\mathrm{Fun}(\mathcal{I}, \mathcal{C})) \widetilde{\to} \mathrm{Fun}^{rlax}(\mathcal{I}, \mathcal{C})$ given by ([8], 2.2.6.8), we get an adjoint pair $l^{CAlg} : \mathrm{Fun}^{rlax}(\mathcal{I}, \mathcal{C}) \leftrightarrows \mathrm{Fun}^{rlax}(\mathcal{J}, \mathcal{C}) : r^{CAlg}$.

1.2.11. Let $\mathcal{P}\mathrm{r}^L$ be the category of presentable categories and colimit preserving functors, it is equipped with the Lurie tensor product by ([8], 4.8.1.15). Consider $1 - \mathcal{C}\mathrm{at}$ with the cartesian symmetric monoidal structure. Then the functor $1 - \mathcal{C}\mathrm{at} \to \mathcal{P}\mathrm{r}^L$, $I \mapsto \mathcal{P}(I) = \mathrm{Fun}(I^{op}, \mathrm{Spc})$ is symmetric monoidal (see [8], inside the proof of 4.8.1.15).

For $\mathcal{C} \in \mathcal{P}\mathrm{r}^L$ and I small there is an isomorphism in $\mathcal{P}\mathrm{r}^L$

$$\mathrm{Fun}(I, \mathcal{C}) \widetilde{\to} \mathcal{P}(I^{op}) \otimes \mathcal{C}$$

Indeed, one has

$$\mathrm{Fun}(I, \mathcal{C}) \widetilde{\to} \mathrm{Fun}(I^{op}, \mathcal{C}^{op})^{op} \widetilde{\to} (\mathrm{Fun}^L(\mathcal{P}(I^{op}), \mathcal{C}^{op}))^{op} \widetilde{\to} \mathrm{Fun}^R(\mathcal{P}(I^{op})^{op}, \mathcal{C}) \widetilde{\to} \mathcal{P}(I^{op}) \otimes \mathcal{C}$$

here the last isomorphism is by ([8], 4.8.1.17). Here $\mathrm{Fun}^L(\mathcal{P}(I^{op}),\mathcal{C})\subset\mathrm{Fun}(\mathcal{P}(I^{op}),\mathcal{C})$ is the full subcategory of colimit preserving functors, and

$$\mathrm{Fun}^R(\mathcal{P}(I^{op})^{op},\mathcal{C}^{op})\subset\mathrm{Fun}(\mathcal{P}(I^{op})^{op},\mathcal{C}^{op})$$

is the full subcategory of limit preserving functors.

1.2.12. The results of Section 1.2.8 have a nonunital version: in this case $\mathcal{I},\mathcal{C}\in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ with $\mathcal{C}$ cocomplete such that the tensor product preserves colimits separately in each variable, and $\mathcal{I}$ is small. Then $\mathrm{Fun}(\mathcal{I},\mathcal{C})$ has a nonunital symmetric monoidal structure given by Day convolution. Namely, apply ([8], Construction 2.2.6.7) with the base ∞-operad $\mathcal{O}^{\otimes}=\mathrm{Surj}^{\otimes}$.

Let now $F:\mathcal{I}\to\mathcal{J}$ be a map in $CAlg^{nu}(1-\mathcal{C}\mathrm{at})$, where $\mathcal{I},\mathcal{J}$ are small. We have the same adjoint pair $l:\mathrm{Fun}(\mathcal{I},\mathcal{C})\leftrightarrows\mathrm{Fun}(\mathcal{J},\mathcal{C}):r$, where l is the LKE along F. Then l is nonunital symmetric monoidal, and r is right-lax nonunital symmetric monoidal for the Day convolution structures. So, we get an adjoint pair $l^{CAlg}:CAlg^{nu}(\mathrm{Fun}(\mathcal{I},\mathcal{C}))\leftrightarrows CAlg^{nu}(\mathrm{Fun}(\mathcal{J},\mathcal{C})):r^{CAlg}$, here the functors are given by composing with l and with r. In this case ([8], 2.2.6.8) gives $CAlg^{nu}(\mathrm{Fun}(\mathcal{I},\mathcal{C}))\widetilde{\rightarrow}\mathrm{Fun}^{rlax}(\mathcal{I},\mathcal{C})$, where on the right hand side we mean nonuital right-lax symmetric monoidal functors.

1.2.13. Assume we are in the situation of Section 1.2.1, so we have an adjoint pair $F:\mathcal{C}\leftrightarrows\mathcal{D}:G$ in DGCat_{cont}, where $\mathcal{C},\mathcal{D}\in CAlg^{nu}(\mathrm{DGCat}_{cont})$, and G is nonunital symmetric monoidal, so F is left-lax nonunital symmetric monoidal. We get an adjoint pair $G^L_{enh}:CAlg^{nu}(\mathcal{C})\leftrightarrows CAlg^{nu}(\mathcal{D}):G_{enh}$, where G_{enh} commutes with oblv, and G^L_{enh} is its left adjoint. Here G^L_{enh} is not a lifting of F.

Proposition 1.2.14. *In the situation of Section 1.2.13 the following diagram commutes*

$$\begin{array}{ccc} CAlg^{nu}(\mathcal{C}) & \stackrel{G^L_{enh}}{\rightarrow} & CAlg^{nu}(\mathcal{D})\\ || & & ||\\ \mathrm{Fun}^{\otimes}(fSet,\mathcal{C}) & & \mathrm{Fun}^{rlax}(pt,\mathcal{D})\\ \downarrow F\circ\bullet & & \uparrow \mathrm{colim}\\ \mathrm{Fun}^{llax}(fSet,\mathcal{D}) & \widetilde{\rightarrow} & \mathrm{Fun}^{\otimes}(Tw(fSet),\mathcal{D}) \end{array} \tag{10}$$

Here the low horizontal isomorphism is given by Lemma 1.2.6.

In the above proposition we used the fact that $Tw(fSet)\to pt$ is nonunital symmetric monoidal, so we apply the result of Section 1.2.12 to this functor to conclude that $\mathrm{colim}:\mathrm{Fun}(Tw(fSet),\mathcal{D})\to\mathrm{Fun}(pt,\mathcal{D})$ is nonunital symmetric monoidal, so induces a functor

$$\begin{array}{ccc} CAlg^{nu}(\mathrm{Fun}(Tw(fSet),\mathcal{D})) & \rightarrow & CAlg^{nu}(\mathrm{Fun}(pt,\mathcal{D}))\\ || & & ||\\ \mathrm{Fun}^{rlax}(Tw(fSet),\mathcal{D})) & & \mathrm{Fun}^{rlax}(pt,\mathcal{D}) \end{array}$$

The long composition in the diagram (10) is as follows. Let $c\in CAlg^{nu}(\mathcal{C})$. The corresponding nonunital symmetric monoidal functor $fSet\to\mathcal{C}$ sends I to $\underset{i\in I}{\otimes}c$, and a map $\phi:I\to J$ in $fSet$ to $\underset{\phi}{\otimes}:\underset{i\in I}{\otimes}c\to\underset{j\in J}{\otimes}c$, the tensor product along ϕ.

This gives a right-lax functor $fSet^{op} \to \mathcal{D}^{op}$, $I \mapsto F(\underset{i\in I}{\otimes} c)$. It associates to a map $\tau : J \to I$ in $fSet^{op}$ as above the morphism

$$F(\underset{\tau}{\otimes}) : F(\underset{j\in J}{\otimes} c) \to F(\underset{i\in I}{\otimes} c)$$

in $\mathcal{D}^{op}$. The corresponding nonunital symmetric monoidal functor $\theta : Tw(fSet)^{op} \to \mathcal{D}^{op}$ sends $(J \xrightarrow{\tau} I) \in Tw(fSet)$ to

$$\underset{i\in I}{\otimes} F(\underset{j\in J_i}{\otimes} c)$$

Now given a morphism from $(J' \xrightarrow{\tau'} I')$ to $(J \xrightarrow{\tau} I)$ in $Tw(fSet)$ given by (9), θ sends it to the morphism in $\mathcal{D}$

$$\underset{i'\in I'}{\otimes} F(\underset{j'\in J'_{i'}}{\otimes} c) \to \underset{i\in I}{\otimes} F(\underset{j\in J_i}{\otimes} c)$$

obtained as the tensor product in $\mathcal{D}$ over $i' \in I'$ of the morphisms

$$F(\underset{j'\in J'_{i'}}{\otimes} c) \to \underset{i\in I_{i'}}{\otimes} F(\underset{j\in J_i}{\otimes} c),$$

which are the compositions

$$F(\underset{j'\in J'_{i'}}{\otimes} c) \to F(\underset{\underset{i\in I_{i'}}{\sqcup} J_i}{\otimes} c) \to \underset{i\in I_{i'}}{\otimes} F(\underset{j\in J_i}{\otimes} c) \tag{11}$$

Here we used surjections $\phi'_{i'} : J'_{i'} \to \underset{i\in I_{i'}}{\sqcup} J_i$. The first map in (11) is $F(\underset{\phi'_{i'}}{\otimes})$, and the second one comes from the left-lax structure on F.

Proof of Proposition 1.2.14. From Remark 1.2.9 (by passing to the opposite categories) one derives an adjont pair $\mathcal{L} : \mathrm{Fun}^{llax}(fSet, \mathcal{C}) \leftrightarrows \mathrm{Fun}^{llax}(fSet, \mathcal{D}) : \mathcal{R}$, where $\mathcal{L}$ and $\mathcal{R}$ are compositions with F and G respectively. Note that $\mathrm{Fun}^{\otimes}(fSet, \mathcal{C}) \subset \mathrm{Fun}^{llax}(fSet, \mathcal{C})$ is a full subcategory.

Step 1 Let $d \in \mathcal{D}$ (resp., $c \in \mathcal{C}$) be a nonunital commutative algebra, and $\alpha_d : fSet \to \mathcal{D}$ (resp., $\alpha_c : fSet \to \mathcal{C}$) the corresponding nonunital symmetric monoidal functor. Then $G \circ \alpha_d$ is nonunital symmetric monoidal, so

$$\mathrm{Map}_{\mathrm{Fun}^{\otimes}(fSet,\mathcal{C})}(\alpha_c, G\circ\alpha_d) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}^{llax}(fSet,\mathcal{C})}(\alpha_c, G\circ\alpha_d) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}^{llax}(fSet,\mathcal{D})}(F\circ\alpha_c, \alpha_d)$$

Write $\bar{\alpha}_c, \bar{\alpha}_d \in \mathrm{Fun}^{\otimes}(Tw(fSet), \mathcal{D})$ for the images of $F \circ \alpha_c$ and α_d respectively under

$$\mathrm{Fun}^{llax}(fSet, \mathcal{D}) \widetilde{\to} \mathrm{Fun}^{\otimes}(Tw(fSet), \mathcal{D})$$

The functor $\bar{\alpha}_d$ is the composition $Tw(fSet) \xrightarrow{r} fSet \xrightarrow{\alpha_d} \mathcal{D}$, where $r(J \to I) = J$. Let $\tilde{\alpha}_c : fSet \to \mathcal{D}$ be the LKE of $\bar{\alpha}_c : Tw(fSet) \to \mathcal{D}$ under $r : Tw(fSet) \to fSet$. We get

$$\mathrm{Map}_{\mathrm{Fun}^{\otimes}(Tw(fSet),\mathcal{D})}(\bar{\alpha}_c, \bar{\alpha}_d) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}^{rlax}(fSet,\mathcal{D})}(\tilde{\alpha}_c, \alpha_d)$$

We used the fact that r is nonunital symmetric monoidal, so by Section 1.2.12 gives rise an an adjunction

$$\mathrm{Fun}^{rlax}(Tw(fSet), \mathcal{D}) \leftrightarrows \mathrm{Fun}^{rlax}(fSet, \mathcal{D}),$$

where the left adoint is given by the LKE, and the right adjoint is the composition with r. Besides, $\mathrm{Fun}^{\otimes}(Tw(fSet), \mathcal{D}) \subset \mathrm{Fun}^{rlax}(fSet, \mathcal{D})$ is a full subcategory.

Since pt is a final object of $fSet$, the composition

$$CAlg^{nu}(\mathcal{C}) \tilde{\to} \mathrm{Fun}^{\otimes}(fSet, \mathcal{C}) \stackrel{\mathrm{colim}}{\to} \mathcal{C}$$

is oblv : $CAlg^{nu}(\mathcal{C}) \to \mathcal{C}$. By ([9], 9.2.13), $\mathrm{Fun}(fSet, \mathcal{C}) \in \mathrm{DGCat}_{cont}$, and compositions with F and G yield an adjoint pair $\mathrm{Fun}(fSet, \mathcal{C}) \leftrightarrows \mathrm{Fun}(fSet, \mathcal{D})$ in DGCat_{cont}.

Decompose colim : $\mathrm{Fun}(Tw(fSet), \mathcal{D})) \to \mathrm{Fun}(pt, \mathcal{D})$ as

$$\mathrm{Fun}(Tw(fSet), \mathcal{D})) \stackrel{LKE}{\to} \mathrm{Fun}(fSet, \mathcal{D}) \stackrel{\mathrm{colim}}{\to} \mathrm{Fun}(pt, \mathcal{D}),$$

where LKE is along $Tw(fSet) \stackrel{r}{\to} fSet$.

Step 2 To finish, it suffices to show that $\tilde{\alpha}_c : fSet \to \mathcal{D}$ is nonunital symmetric monoidal. This is a combination of the fact that $\bar{\alpha}_c$ is symmetric monoidal and of Lemma 1.2.15 below. Namely, let $J_i \in fSet$. We must show that the natural map $\tilde{\alpha}_c(J_1) \otimes \tilde{\alpha}_c(J_2) \to \tilde{\alpha}_c(J_1 \sqcup J_2)$ is an isomorphism. For $i = 1, 2$ we have

$$\tilde{\alpha}_c(J_i) \tilde{\to} \operatorname*{colim}_{(I_i \to K_i) \in Tw(fSet), I_i \to J_i} \bar{\alpha}_c(I_i \to K_i)$$

in $\mathcal{D}$, and

$$\tilde{\alpha}_c(J_1 \sqcup J_2) \tilde{\to} \operatorname*{colim}_{(I \to K) \in Tw(fSet), I \to J_1 \sqcup J_2} \bar{\alpha}_c(I \to K) \tag{12}$$

So,

$$\tilde{\alpha}_c(J_1) \otimes \tilde{\alpha}_c(J_2) \tilde{\to} \operatorname*{colim}_{\substack{(I_1 \to K_1) \in Tw(fSet), I_1 \to J_1 \\ (I_2 \to K_2) \in Tw(fSet), I_2 \to J_2}} \bar{\alpha}_c(I_1 \sqcup I_2 \to K_1 \sqcup K_2)$$

We used in the above the fact that $\bar{\alpha}_c(I_1 \to K_1) \otimes \bar{\alpha}_c(I_2 \to K_2) \tilde{\to} \bar{\alpha}_c(I_1 \sqcup I_2 \to K_1 \sqcup K_2)$. The latter colimit identifies with (12) by Lemma 1.2.15. □

Lemma 1.2.15. *Let $J_1, J_2 \in fSet$. Consider the functor*

$$\epsilon : \prod_{i=1,2} (Tw(fSet) \times_{fSet} fSet_{/J_i}) \to Tw(fSet) \times_{fSet} fSet_{/J_1 \sqcup J_2}$$

sending $(J_1 \leftarrow I_1 \to K_1), (J_2 \leftarrow I_2 \to K_2)$ to

$$(J_1 \sqcup J_2 \leftarrow I_1 \sqcup I_2 \to K_1 \sqcup K_2)$$

Here $(I_i \to K_i) \in Tw(Set)$. Then ϵ is cofinal.

Proof. We claim that ϵ has a left adjoint R given as follows. Let $(J_1 \sqcup J_2 \stackrel{\alpha}{\leftarrow} I \stackrel{\beta}{\to} K) \in Tw(fSet) \times_{fSet} fSet_{/J_1 \sqcup J_2}$. Set $I_i = \alpha^{-1}(J_i)$. For $i = 1, 2$ consider the equivalence relation on I_i given by $x \sim y$ iff $\beta(x) = \beta(y)$. The quotient by this equivalence relation defines a surjection $\beta_i : I_i \to K_i$. In addition, we get a surjection $K_1 \sqcup K_2 \to K$. Consider a morphism in $Tw(fSet) \times_{fSet} fSet_{/J_1 \sqcup J_2}$, it is given by the diagram

$$\begin{array}{ccccc} & & I_1 \sqcup I_2 & \to & K \\ & \swarrow & \downarrow \gamma_1 \sqcup \gamma_2 & & \uparrow \\ J_1 \sqcup J_2 & \leftarrow & I'_1 \sqcup I'_2 & \to & K' \end{array}$$

where I_i (resp. I'_i) is the preimage of J_i. We get natural morphisms $K'_i \to K_i$ for $i = 1, 2$. Indeed, fix $i \in \{1, 2\}$ and let $k', m' \in I'_i$ be such that their images in K' coincide. Pick $k, m \in I_i$ with $\gamma_i(k) = k', \gamma_i(m) = m'$. Then the images of k, m in K coincide, so k and m define the same element $x \in K_i$. We send k' and m' to x. The obtained map $I'_i \to K_i$ factors uniquely through $K'_i \to K_i$. Thus, we defined a functor R sending $(J_1 \sqcup J_2 \stackrel{\alpha}{\leftarrow} I \stackrel{\beta}{\to} K)$ to the pair $(J_1 \leftarrow I_1 \to K_1), (J_2 \leftarrow I_2 \to K_2)$. Then R is left adjoint to ϵ. □

1.3. **Unital version of Justin's argument.**

1.3.1. Let us try to wotk out a unital version of Proposition 1.2.14. Assume given an adjoint pair $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ in DGCat_{cont}, where $\mathcal{C}, \mathcal{D} \in CAlg(\mathrm{DGCat}_{cont})$, the functor G is symmetric monoidal, so that F is left-lax symmetric monoidal. The functor $G_{enh} : CAlg(\mathcal{D}) \to CAlg(\mathcal{C})$ preserves limits, so has a left adjont G^L_{enh}, because $CAlg(\mathcal{D})$ is presentable.

Recall the equivalence $CAlg(\mathcal{C}) \widetilde{\to} \mathrm{Fun}^{\otimes}(fSet_{\emptyset}, \mathcal{C})$ given by Remark 1.2.2. As in the nonunital case, Remark 1.2.9 gives an adjoint pair

$$\mathcal{L} : \mathrm{Fun}^{llax}(fSet_{\emptyset}, \mathcal{C}) \leftrightarrows \mathrm{Fun}^{llax}(fSet_{\emptyset}, \mathcal{D}) : \mathcal{R}, \tag{13}$$

where the functors $\mathcal{L}, \mathcal{R}$ are compositions with F and G.

1.3.2. Consider the functor $l : fSet_{\emptyset} \to Tw(fSet_{\emptyset})$ sending I to $(I \to *)$. This functor is left-lax symmetric monoidal. Here $Tw(fSet_{\emptyset})$ is equipped with the symmetric monoidal structure sending $(J \to I), (J' \to I')$ to $(J \sqcup J' \to I \sqcup I')$.

Lemma 1.3.3. *For any $D \in CAlg(1-\mathcal{C}\mathrm{at})$ the composition with l yields an equivalence*

$$\mathrm{Fun}^{\otimes}(Tw(fSet_{\emptyset}), D) \widetilde{\to} \mathrm{Fun}^{llax}(fSet_{\emptyset}, D)$$

Proof. We apply ([8], 2.2.4.9) to the symmetric monoidal category $fSet^{op}_{\emptyset}$. We claim that its symmetric monoidal envelope $Env(fSet^{op}_{\emptyset})$ identifies with $Tw(fSet_{\emptyset})^{op}$. Indeed, by ([8], 2.2.46), an object of $Env(fSet^{op}_{\emptyset})$ is a collection: $I \in fSet_{\emptyset}$ and $J_i \in fSet_{\emptyset}$ for each $i \in I$. We simply encode this as a map $J \stackrel{\tau}{\to} I$ in $fSet_{\emptyset}$.

A morphism from $(I, \{J_i\})$ to $(I', \{J'_{i'}\})$ in $Env(fSet^{op}_{\emptyset})$ is a collection: a map $\phi : I \to I'$ in $fSet_{\emptyset}$, and for each $i' \in I'$ a morphism $J'_{i'} \to \underset{i \in \phi^{-1}(i')}{\sqcup} J_i$ in $fSet_{\emptyset}$. This morphism is nothing but a diagram

$$\begin{array}{ccc} J' & \to & I' \\ \downarrow & & \uparrow \phi \\ J & \to & I \end{array}$$

in $fSet_{\emptyset}$. Now the composition with $l : fSet^{op}_{\emptyset} \to Tw(fSet_{\emptyset})^{op}$ yields an equivalence

$$\mathrm{Fun}^{\otimes}(Tw(fSet_{\emptyset})^{op}, D^{op}) \widetilde{\to} \mathrm{Fun}^{rlax}(fSet^{op}_{\emptyset}, D^{op})$$

□

Proposition 1.3.4. *In the situation of Section 1.3.1 the diagram commutes*

$$\begin{array}{ccc} CAlg(\mathcal{C}) & \overset{G^L_{enh}}{\to} & CAlg(\mathcal{D}) \\ || & & || \\ \mathrm{Fun}^{\otimes}(fSet_{\emptyset}, \mathcal{C}) & & \mathrm{Fun}^{rlax}(pt, \mathcal{D}) \\ \downarrow F\circ\bullet & & \uparrow \mathrm{colim} \\ \mathrm{Fun}^{llax}(fSet_{\emptyset}, \mathcal{D}) & \widetilde{\to} & \mathrm{Fun}^{\otimes}(Tw(fSet_{\emptyset}), \mathcal{D}) \end{array}$$

Proof. **Step 1** Let $c \in \mathcal{C}, d \in \mathcal{D}$ be commutative algebras in $\mathcal{C}$ and $\mathcal{D}$. Write $\alpha_c : fSet_{\emptyset} \to \mathcal{C}$, $\alpha_d : fSet_{\emptyset} \to \mathcal{D}$ for the corresponding symmetric monoidal functors. Using (13), we get

$$\mathrm{Map}_{\mathrm{Fun}^{\otimes}(fSet_{\emptyset}, \mathcal{C})}(\alpha_c, G \circ \alpha_d) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}^{llax}(fSet_{\emptyset}, \mathcal{D})}(F \circ \alpha_c, \alpha_d)$$

Write $\bar{\alpha}_c, \bar{\alpha}_d$ for the images of $F \circ \alpha_c$ and α_d respedtively under

$$\mathrm{Fun}^{llax}(fSet_{\emptyset}, \mathcal{D}) \widetilde{\to} \mathrm{Fun}^{\otimes}(Tw(fSet_{\emptyset}), \mathcal{D})$$

Note that α_d sends $I \in fSet_{\emptyset}$ to $\underset{I}{\otimes} d$, and it send a map $I \overset{\phi}{\to} I'$ to $\underset{\phi}{\otimes} : \underset{I}{\otimes} d \to \underset{I'}{\otimes} d$ given by the algebra structure on d.

One has $\bar{\alpha}_d = r \circ \alpha_d$ for $r : Tw(fSet_{\emptyset}) \to fSet_{\emptyset}$ sending $(J \to I)$ to J. We have an adjoint pair $l : fSet_{\emptyset} \leftrightarrows Tw(fSet_{\emptyset}) : r$, and r is symmetric monoidal. Note that the unit of $Tw(fSet_{\emptyset})$ is $(\emptyset \to \emptyset)$. Now we apply Section 1.2.10 and get an adjoint pair

$$\mathrm{Fun}^{rlax}(Tw(fSet_{\emptyset}), \mathcal{D}) \leftrightarrows \mathrm{Fun}^{rlax}(fSet_{\emptyset}, \mathcal{D})$$

where the left adjoint is the LKE along $r : Tw(fSet_{\emptyset}) \to fSet_{\emptyset}$ and the right adjoint is the restriction along r. So,

$$\mathrm{Map}_{\mathrm{Fun}^{rlax}(Tw(fSet_{\emptyset}), \mathcal{D})}(\bar{\alpha}_c, r \circ \alpha_d) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}^{rlax}(fSet_{\emptyset}, \mathcal{D})}(\tilde{\alpha}_c, \alpha_d),$$

where $\tilde{\alpha}_c$ is the LKE of $\bar{\alpha}_c : Tw(fSet_{\emptyset}) \to \mathcal{D}$ along r.

The category $fSet_{\emptyset}$ has a final object pt. So, the value of $\tilde{\alpha}_c$ on pt is $\underset{(J \to I) \in Tw(fSet_{\emptyset})}{\mathrm{colim}} \bar{\alpha}_c$.

Step 2 To finish, it suffices to show that $\tilde{\alpha}_c$ is strict, that is, symmetric monoidal. Let $J_i \in fSet_{\emptyset}$. We must show that the natural map $\tilde{\alpha}_c(J_1) \otimes \tilde{\alpha}_c(J_2) \to \tilde{\alpha}_c(J_1 \sqcup J_2)$ is an isomorphism in $\mathcal{D}$. For $i = 1, 2$ we have

$$\tilde{\alpha}_c(J_i) \widetilde{\to} \underset{(I_i \to K_i) \in Tw(fSet_{\emptyset}),\ I_i \to J_i}{\mathrm{colim}} \bar{\alpha}_c(I_i \to K_i)$$

in $\mathcal{D}$, and

$$\tilde{\alpha}_c(J_1 \sqcup J_2) \widetilde{\to} \underset{(I \to K) \in Tw(fSet_{\emptyset}),\ I \to J_1 \sqcup J_2}{\mathrm{colim}} \bar{\alpha}_c(I \to K)$$

So,

$$\tilde{\alpha}_c(J_1) \otimes \tilde{\alpha}_c(J_2) \widetilde{\to} \underset{\substack{(I_1 \to K_1) \in Tw(fSet_{\emptyset}), I_1 \to J_1 \\ (I_2 \to K_2) \in Tw(fSet_{\emptyset}), I_2 \to J_2}}{\mathrm{colim}} \bar{\alpha}_c(I_1 \sqcup I_2 \to K_1 \sqcup K_2)$$

We used in the above the fact that $\bar{\alpha}_c(I_1 \to K_1) \otimes \bar{\alpha}_c(I_2 \to K_2) \widetilde{\to} \bar{\alpha}_c(I_1 \sqcup I_2 \to K_1 \sqcup K_2)$. So, our claim follows from Lemma 1.3.5 below. □

Lemma 1.3.5. *The result of Lemma 1.2.15 remains true if we replace everywhere in its formulation* $fSet$ *by* $fSet_{\emptyset}$.

Proof. We construct a left adjoint R of ϵ as follows. Let $(J_1 \sqcup J_2 \overset{\alpha}{\leftarrow} I \overset{\beta}{\rightarrow} K) \in Tw(fSet_\emptyset) \times_{fSet_\emptyset} (fSet_\emptyset)_{/J_1 \sqcup J_2}$. Set $I_i = \alpha^{-1}(J_i)$. Then R sends the above object to the pair $(J_1 \leftarrow I_1 \rightarrow K), (J_2 \leftarrow I_2 \rightarrow K)$. This is naturally a functor in the opposite direction. Then R is left adjoint to ϵ. $\square$

1.4. **Factorization homology functor.**

1.4.1. Combining Theorem 1.1.2 and Section 1.1.1 we conclude that the composition

$$CAlg^{nu,!}(Shv(X)) \overset{\text{Fact}}{\rightarrow} CAlg^{\text{Fact}}(\text{Ran}_X) \overset{(p_{\text{Ran}})_!^{CAlg}}{\rightarrow} CAlg^{nu}(\text{Vect})$$

is the desired functor $C_c^{\text{Fact}}(X, \bullet)$ of factorization homology.

1.4.2. Let $p : Y \rightarrow X$ be an affine scheme with a connection over X, here we are over the algebraically closed ground field of characteristic zero. This means that $p_*\mathcal{O} \in CAlg(\text{QCoh}(X))$ is lifted to an object of $CAlg(\mathcal{D} - mod(X))$. Such schemes with a connection along X form a category $\mathcal{D} - \text{Sch}_{/X}^{aff}$. More generally, there is a category $\mathcal{D} - \text{Sch}_{/X}$ of $\mathcal{D}_X$-schemes which are not necessarily affine.

Then we may consider the scheme $Sect_\nabla(X, Y) \in Sch^{aff}$ of horizontal sections of p. It is defined by an isomorphism of functors $\text{Sch}^{aff} \rightarrow \text{Spc}$: for $S \in Sch^{aff}$,

$$\text{Map}_{\text{Sch}^{aff}}(S, Sect_\nabla(X, Y)) \widetilde{\rightarrow} \text{Map}_{\mathcal{D}-\text{Sch}_{/X}^{aff}}(S \times X, Y)$$

One checks that

$$C^\cdot(Sect_\nabla(X, Y), \mathcal{O}) \widetilde{\rightarrow} C_c^{\text{Fact}}(X, \text{Fact}(p_*\mathcal{O}_Y))$$

1.4.3. *Example.* Let $p : Z \rightarrow X$ be an affine morphism. Define $Sect(X, Z)$ by an isomorphism of functors $\text{Sch}^{aff} \rightarrow \text{Spc}$, for $S \in \text{Sch}^{aff}$,

$$\text{Map}_{\text{Sch}}(S, Sect(X, Z)) \widetilde{\rightarrow} \text{Map}_{\text{Sch}_{/X}}(S \times X, Z)$$

Then we may describe $C^\cdot(Sect(X, Z), \mathcal{O})$.

Namely, the functor $\text{oblv} : \mathcal{D} - Sch_{/X} \rightarrow \text{Sch}_{/X}$ has a right adjoint $Jets$ given by the scheme of jets ([1], 2.3.2). One checks that for $x \in X$, $Jets(Z)_x \widetilde{\rightarrow} Sect(\hat{D}_x, Z)$, here $\hat{D}_x$ is the formal neighbourhood of x in X. Besides, $Jets$ takes values in $\mathcal{D} - \text{Sch}_{/X}^{aff}$.

One has $Sect(X, Z) \widetilde{\rightarrow} Sect_\nabla(X, Jets(Z))$ immediately. So,

$$C^\cdot(Sect(X, Z), \mathcal{O}) \widetilde{\rightarrow} C_c^{\text{Fact}}(X, q_*\mathcal{O}_{Jets(Z)})$$

for the projection $q : Jets(Z) \rightarrow X$. This is an example of a local-to-global principle.

1.5. **Graded version.**

1.5.1. Assume X is a smooth projective curve, $\mathcal{A}$ is a $\mathbb{Z}^{>0}$-graded object in $CAlg^{nu,!}(Shv(X))$, that is, the product is compatible with gradings.

Let $\mathrm{Div}^{eff} = \underset{d>0}{\sqcup} X^{(d)}$. Then Div^{eff} is naturally a semi-group with respect to the sum of divisors. Now $Shv(\mathrm{Div}^{eff}) \widetilde{\to} \prod_{d>0} Shv(X^{(d)})$ is equipped with the convolution monoidal structure. For the sum $u : \mathrm{Div}^{eff} \times \mathrm{Div}^{eff} \to \mathrm{Div}^{eff}$ we let

$$F_1 \star F_2 = u_!(F_1 \boxtimes F_2)$$

Let $\vartriangle\colon \underset{d>0}{\sqcup} X \to \mathrm{Div}^{eff}$ be the inclusion given by the diagonal $\vartriangle^d\colon X \hookrightarrow X^{(d)}$ for each $d > 0$. We view $Shv(\underset{d>0}{\sqcup} X) = \prod_{d>0} Shv(X) = Shv(X)^{\mathbb{Z}^{>0}}$ as the category of $\mathbb{Z}^{>0}$-graded sheaves on X.

The functor $\vartriangle^!\colon (Shv(\mathrm{Div}^{eff}), \star) \to (Shv(X)^{\mathbb{Z}^{>0}}, !)$ is nonunital symmetric monoidal. It has a fully faithful left adjoint $\vartriangle_!\colon Shv(X)^{\mathbb{Z}^{>0}} \to Shv(\mathrm{Div}^{eff})$, which is so left-lax nonunital symmetric monoidal. By Lemma 1.1.3, the functor

$$\vartriangle^!\colon CAlg^{nu,\star}(Shv(\mathrm{Div}^{eff})) \to CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}})$$

has a fully faithful left adjoint denoted Fact. Let

$$(\mathrm{Div}^{eff} \times \mathrm{Div}^{eff})_d \subset (\mathrm{Div}^{eff} \times \mathrm{Div}^{eff})$$

be the open subscheme of disjoint divisors.

Definition 1.5.2. *Define $CAlg^{\mathrm{Fact}}(Shv(\mathrm{Div}^{eff})) \subset CAlg^{nu,\star}(Shv(\mathrm{Div}^{eff}))$ as the full subcategory of those commutative algebras $\mathcal{A}$ for which the induced map $\mathcal{A} \boxtimes \mathcal{A} \to u^!\mathcal{A}$ restricted to $(\mathrm{Div}^{eff} \times \mathrm{Div}^{eff})_d$ is an isomorphism.*

Theorem 1.5.3. *The functor*

$$\mathrm{Fact} : CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}}) \to CAlg^{nu,\star}(Shv(\mathrm{Div}^{eff}))$$

is fully faithful with the essential image $CAlg^{\mathrm{Fact}}(Shv(\mathrm{Div}^{eff}))$.

Proof. Similar to Theorem 1.1.2. □

The explicit formula for Fact is again given by Proposition 1.2.14. Namely, for $\mathcal{A} \in CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}})$ given as $\mathcal{A} = \oplus_{d>0}\mathcal{A}_d$, we get a functor $\theta : Tw(fSet) \to Shv(\mathrm{Div}^{eff})$ sending $(J \xrightarrow{\tau} I)$ to $\underset{i\in I}{\star} \vartriangle_! (\underset{j\in J_i}{\otimes} \mathcal{A})$. The d-th component of the latter is

$$\underset{\{d_j \in \mathbb{Z}^{>0}\}_{j\in J_i},\ \sum_j d_j = d}{\oplus} (\underset{i\in I}{\star} \vartriangle_! (\underset{j\in J_i}{\otimes} \mathcal{A})_{d_j})$$

Given a map from $(J' \xrightarrow{\tau'} I')$ to $(J \xrightarrow{\tau} I)$ in $Tw(fSet)$ given by (9), θ sends it to the morphism in $Shv(\mathrm{Div}^{eff})$

$$\underset{i'\in I'}{\star} \vartriangle_! (\underset{j'\in J'_{i'}}{\otimes} \mathcal{A}) \to \underset{i\in I}{\star} \vartriangle_! (\underset{j\in J_i}{\otimes} \mathcal{A}),$$

which is the composition

$$\underset{i'\in I'}{\star} \vartriangle_! (\underset{j'\in J'_{i'}}{\otimes} \mathcal{A}) \to \underset{i'\in I'}{\star} \vartriangle_! (\underset{j\in J_{i'}}{\otimes} \mathcal{A}) \to \underset{i\in I}{\star} \vartriangle_! (\underset{j\in J_i}{\otimes} \mathcal{A})$$

where the first functor is obtained by applying $\triangle_!$ to the product maps along $J'_{i'} \to J_{i'}$, and the second one comes from the left-lax symmetric monoidal structure of $\triangle$.

1.5.4. Let $\mathrm{Vect}^{\mathbb{Z}^{>0}} = \prod_{d>0} \mathrm{Vect}$ be the category of $\mathbb{Z}^{>0}$-graded objects of Vect. For the projection $p : X \to \operatorname{Spec} k$ the functors $p^! : \mathrm{Vect}^{\mathbb{Z}^{>0}} \to (Shv(X)^{\mathbb{Z}^{>0}}, !)$ is nonunital symmetric monoidal, so gives rise to a functor

$$p^!_{CAlg} : CAlg^{nu}(\mathrm{Vect}^{\mathbb{Z}^{>0}}) \to CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}})$$

Its left adjoint is also called the functor $C_c^{\mathrm{Fact}}(X, \bullet)$ of factorization homology (in the graded context).

One shows similarly that $C_c^{\mathrm{Fact}}(X, \bullet)$ is the composition

$$CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}}) \overset{\mathrm{Fact}}{\to} CAlg^{nu,\star}(Shv(\mathrm{Div}^{eff})) \overset{(p_{\mathrm{Div}^{eff}})_!}{\to} CAlg^{nu}(\mathrm{Vect}^{\mathbb{Z}^{>0}})$$

Here $p_{\mathrm{Div}^{eff}} : \mathrm{Div}^{eff} \to \operatorname{Spec} k$ is the projection, and

$$(p_{\mathrm{Div}^{eff}})_! : (Shv(\mathrm{Div}^{eff}), \star) \to \mathrm{Vect}^{\mathbb{Z}^{>0}}$$

is nonunital symmetric monoidal.

1.6. **Factorizable sheaves.**

1.6.1. Let be X is a smooth curve. We may define $Shv(\mathrm{Ran}_X)^{\mathrm{Fact}}$ similarly to [11]. Namely,

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