MY NOTES OF EMBRYO GL SEMINAR

1. Commutative factorization algebras

1.0.1. Let X/k be a separated scheme of finite type. Fix one of our 4 sheaf theories. If the sheaf theory is \mathcal{D} -modules then assume X proper, otherwise it is arbitrary.

For $p: X \to \operatorname{Spec} k$ we have the adjoint pair $p_!: Shv(X) \leftrightarrows \operatorname{Vect} : p^!$. The functor $p^!: (\operatorname{Vect}, \otimes) \to (Shv(X), \otimes^!)$ is symmetric monoidal, hence yields a functor $p^!_{CAlg}: CAlg^{nu}(\operatorname{Vect}) \to CAlg^{nu}(Shv(X))$. This functor preseves limits, both categories are presentable, so $p^!_{CAlg}$ admits a left adjoint denoted $C_c^{\operatorname{Fact}}(X, \bullet)$. This will be our functor of factorization homology.

However, $C_c^{\text{Fact}}(X, \bullet)$ will not commute with obly : $CAlg^{nu}(\text{Vect}) \to \text{Vect}$, obly : $CAlg^{nu}(Shv(X)) \to Shv(X)$. The following diagram does commute

$$Shv(X) \stackrel{free}{\to} CAlg^{nu}(Shv(X))$$
$$\downarrow_{p_!} \qquad \qquad \downarrow_{C_c^{\text{Fact}}(X,\bullet)}$$
$$\text{Vect} \stackrel{free}{\to} CAlg^{nu}(\text{Vect})$$

For $V \in \text{Vect}$, $free(V) = \bigoplus_{d>0} Sym^d(V)$, where $\text{Sym}^d(V)$ is as in ([11], 3.0.40), similarly for $free : Shv(X) \to CAlg^{nu}(Shv(X))$.

1.0.2. Example (recheck). Let G be a simply-connected semisimple group. It is known that the reduced cohomology $C_{rd}(B(G), e) = \operatorname{Sym}(\mathfrak{a}) := \bigoplus_{d>0} \operatorname{Sym}^d(\mathfrak{a})$ for some $\mathfrak{a} \in$ Vect, the cohomologically graded vector space of Chern classes. Assume X is a proper curve. The Atyah-Bott formula says that

$$C^{\cdot}_{rd}(\operatorname{Bun}_{G}, e) \xrightarrow{\sim} C^{\operatorname{Fact}}_{c}(X, p^{!}C^{\cdot}_{rd}(B(G), e)) \xrightarrow{\sim} \operatorname{Sym}(\mathfrak{a} \otimes C^{\cdot}_{c}(X, \omega_{X}))$$

The map in one direction is defined as follows. We define a map

$$C_c^{\operatorname{Fact}}(X, p^! C_{rd}^{\cdot}(B(G), e)) \to C_{rd}^{\cdot}(\operatorname{Bun}_G, e),$$

equivalently, a map of nonunital commutative algebras $p^!C_{rd}^{\cdot}(B(G),e) \to p^!C_{rd}^{\cdot}(\operatorname{Bun}_G,e)$ in Shv(X). Namely, the diagram

$$\begin{array}{ccc} X \times \operatorname{Bun}_G & \xrightarrow{\gamma} & X \times B(G) \\ & \searrow h & & \downarrow q \\ & & X \end{array}$$

gives a map $q_* \to q_* \gamma_* \gamma^* = h_* \gamma^*$, which gives $q_*(e) \to h_* e$.

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1.0.3. Assume X connected. Then $C_c^{\text{Fact}}(X,\omega) \xrightarrow{\sim} e$ canonically as a commutative algebra in Vect. This follows from Example 2 below and ([4], 1.6.5) giving $C_c^{\cdot}(\text{Ran}_X,\omega) \xrightarrow{\sim} k$ canonically (here X is not necessarily complete).

Now if $\mathcal{A} \in CAlg^{nu}(Shv(X))$ is actually unital then the map $\omega \to \mathcal{A}$ provides a map $C_c^{\text{Fact}}(X,\omega) \to C_c^{\text{Fact}}(X,\mathcal{A})$, which is a unit of this algebra in Vect.

1.0.4. How to construct $C_c^{\text{Fact}}(X, \bullet)$? Consider an example of X a finite set. In this case the functor $p_{CAlg}^!$ is the diagonal map

$$\prod_{x \in X} CAlg^{nu}(\text{Vect}) \leftarrow CAlg^{nu}(\text{Vect})$$

Assume for a moment we are in the unital setting. Then its left adjoint sends $\{B_x\}_{x \in X}$ to $\bigotimes_{x \in X} B_x$ (in the nonunital setting this is not the left adjoint).

Remark 1.0.5. Let \mathcal{A} be a stable symmetric monoidal category. A unital augmented commutative algebra in \mathcal{A} is an object $A \in CAlg(\mathcal{A})$ together with a morphism $A \to 1$ in $CAlg(\mathcal{A})$.

1) One has the equivalence $CAlg(\mathcal{A})_{/1} \xrightarrow{\sim} CAlg^{nu}(\mathcal{A})$. Namely, if $A \to 1$ is an augmented commutative algebra in \mathcal{A} then 1 is a retract of A, so that we have canonically $A \xrightarrow{\sim} 1 \oplus \mathfrak{a}$ for some $\mathfrak{a} \in A$. This $\mathfrak{a} = \operatorname{Fib}(A \to 1)$ is naturally a non-unital commutative algebra. Conversely, if $\mathfrak{a} \in CAlg^{nu}(\mathcal{A})$ then $1 \oplus \mathfrak{a} \in CAlg(\mathcal{A})_{/1}$ naturally.

2) A finite coproduct in $CAlg^{nu}(\mathcal{A})$ is given as follows. Given B_x for $x \in X$, where X is a finite set, one has

(the desired copoduct)
$$\oplus e \xrightarrow{\sim} \underset{x \in X}{\otimes} (e \oplus B_x)$$

in $CAlg(\mathcal{A})_{/1}$. So, the desired copoduct) $\xrightarrow{\longrightarrow} \bigoplus_{X' \subset X} (\underset{x \in X'}{\otimes} B_x)$, the sum over non empty subsets of X.

The coproduct in $CAlg^{nu}$ (Vect) is given by the above remark.

1.0.6. Assume X proper. Then $\operatorname{Ran}_X \to \operatorname{Spec} k$ is pseudo-proper. So, for $F \in Shv(\operatorname{Ran}_X)$,

$$C_c^{\cdot}(\operatorname{Ran}_X, F) \xrightarrow{\sim} \operatorname{colim}_{I \in fSet^{op}} C_c^{\cdot}(X^I, (\Delta^I)^! F)$$

Here fSet is the category of finite nonempty sets and surjections. Indeed, in $Shv(Ran_X)$

$$F \xrightarrow{\sim} \underset{I \in fSet^{op}}{\operatorname{colim}} (\Delta^{I})_{*} (\Delta^{I})^{!} I$$

The map $u: \operatorname{Ran}_X \times \operatorname{Ran}_X \to \operatorname{Ran}_X$ is pseudo-proper. Indeed, for a finite nonempty set I one has

 $(\operatorname{Ran}_X \times \operatorname{Ran}_X) \times_{u, \operatorname{Ran}_X} X^I \xrightarrow{\sim} \operatorname{colim}_{(I_1, I_2) \in fSet^{op} \times fSet^{op}} (X^{I_1} \times X^{I_2}) \times_{\operatorname{Ran}_X} X^I,$

and $(X^{I_1 \sqcup I_2}) \times_{\operatorname{Ran}_X} X^I \to X^I$ is described in ([3], 8.1.2). So, $Shv(\operatorname{Ran}_X)$ is monoidal with the convolution monoidal structure given by $u_!$. We denote

$$F_1 \star F_2 = u_!(F_1 \boxtimes F_2)$$

for $F_i \in Shv(\operatorname{Ran}_X)$. Write $CAlg^{nu,\star}(Shv(\operatorname{Ran}_X))$ for the category of non-unital commutative algebras for the \star -monoidal structure.

Definition 1.0.7. Let $CAlq^{Fact}(\operatorname{Ran}_X) \subset CAlq^{nu,\star}(Shv(\operatorname{Ran}_X))$ be the full subcategory of $\mathcal{A} \in CAlg^{nu,\star}(Shv(\operatorname{Ran}_X))$ for which the corresponding map $\mathcal{A} \boxtimes \mathcal{A} \to u^! \mathcal{A}$ restricted to $(\operatorname{Ran}_X \times \operatorname{Ran}_X)_d$ becomes an isomorphism. (Since u is etale over $(\operatorname{Ran}_X \times \operatorname{Ran}_X)_d$, this is unambiguous).

1.0.8. For $\Delta: X \to \operatorname{Ran}_X$ the functor $\Delta^!: Shv(\operatorname{Ran}_X) \to Shv(X)$ is non-unital symmetric monoidal, so gives a functor $CAlg^{nu,\star}(Shv(\operatorname{Ran}_X)) \to CAlg^{nu,!}(Shv(X))$. Here ! means that we use the symmetric monoidal category $(Shv(X), \otimes^!)$.

Theorem 1.0.9. The restriction of the above functor

$$CAlg^{\operatorname{Fact}}(\operatorname{Ran}_X) \to CAlg^{nu,!}(Shv(X))$$

is an equivalence.

(This theorem is found in [7]). The inverse functor is denoted $A \mapsto Fact(A)$. The desired functor $C_c^{\text{Fact}}(X, \bullet)$ will be $C_c(\text{Ran}_X, \text{Fact}(\bullet))$.

1.0.10. Example 1. Let A = Sym(M) for some $M \in Shv(X)$, where Sym is understood in the non-unital sense for $(Shv(X), \otimes^!)$. Then $Fact(A) \xrightarrow{\sim} Sym^*(\Delta_! M)$, where Sym^{*} denotes the non-unital symmetric algebra in the non-unital symmetric monoidal category $(Shv(\operatorname{Ran}_X), \star)$.

Explicitly, $\operatorname{Sym}^{\star}(\Delta_{!} M) = \bigoplus_{d>0} \operatorname{Sym}^{\star,d}(\Delta_{!} M)$, where $\operatorname{Sym}^{\star,d}(\Delta_{!} M)$ is the S_{d} coinvariants in $(\Delta_n)_!(M^{\boxtimes n})$. Here $\Delta_n \colon X^n \to \operatorname{Ran}_X$ is the natural map. Recall that this means $\operatorname{colim}_{B(S_n)}(\Delta_n)_!(M^{\boxtimes n})$.

1.0.11. Example 2. Note that $\omega \in CAlg^{nu,\star}(Shv(\operatorname{Ran}_X))$ naturally and it is a factorization algebra. Besides, $\triangle^! \omega \xrightarrow{\sim} \omega$ on X, so Fact $(\omega) \xrightarrow{\sim} \omega$ by Theorem 1.0.9.

1.0.12. Recall that for $C \in 1$ – Cat one has $Tw(C) \in 1$ – Cat, see ([12], 1.0.1). For $J \in fSet \ \text{let} \ \Delta^J \colon X^J \to \operatorname{Ran}_X$ be the natural map. Now for $A \in CAlg^{nu,!}(Shv(X))$ one has

$$\operatorname{Fact}(A) \xrightarrow{\sim} \operatorname{colim}_{(I \xrightarrow{\phi} J) \in Tw(fSet)} (\Delta^J)_! A^{\otimes \phi}$$

taken in $Shv(\operatorname{Ran}_X)$. Here for $(\phi: I \to J) \in Tw(fSet)$ we set $A^{\otimes \phi} = \bigotimes_{j \in J} (A^{\otimes I_j})$, where

 $A^{\otimes I_j}$ is the tensor power in $(Shv(X), \otimes^!)$.

For a morphism in Tw(fSet)

(1)
$$\begin{array}{ccc} I & \stackrel{\phi}{\to} & J \\ \downarrow & \uparrow \\ I' & \stackrel{\phi'}{\to} & J' \end{array}$$

the transition map

(2)
$$(\Delta^J)_! A^{\otimes \phi} \to (\Delta^{J'})_! A^{\otimes \phi}$$

is obtained applying $\triangle_!^{J'}$ to the following map. For the closed immersion $\triangle^{(J'/J)}: X^J \rightarrow$ $X^{J'}$ one has

(3)
$$\Delta^{(J'/J)!} (\bigotimes_{j' \in J'} A^{\otimes I'_{j'}}) \xrightarrow{\sim} \bigotimes_{j \in J} A^{\otimes I'_{j}}$$

Now (2) over $X^{J'}$ is the composition

$$\Delta_!^{(J'/J)} \left(\operatornamewithlimits{\boxtimes}_{j \in J} (A^{\otimes I_j}) \right) \to \Delta_!^{(J'/J)} \left(\operatornamewithlimits{\boxtimes}_{j \in J} (A^{\otimes I'_j}) \right) \to \operatornamewithlimits{\boxtimes}_{j' \in J'} A^{\otimes I'_{j'}}$$

where the first map is the exterior product over $j \in J$ of the products in the algebra A along maps $I_j \to I'_j$, and the second one comes from (3). A rigorous definition as a functor $Tw(fSet) \to Shv(\text{Ran})$ follows from the explanation of Justin below.

1.0.13. We have an adjoint pair $l : fSet \cong Tw(fSet) : r$, where $l(I) = (I \to *)$ and $r(I \to J) = I$. Here *l* is fully faithful.

Let now $\mathbb{C} \in 1-\mathbb{C}$ at be cocomplete say and $e: fSet \to \mathbb{C}$ be a functor. Let $LKE(e): Tw(fSet) \to \mathbb{C}$ be the LKE of e along l. Then $LKE(e) = e \circ r$ by ([11], 2.2.39). So, $e: fSet \to \mathbb{C}$ identifies with the LKE of LKE(e) along $r: Tw(fSet) \to fSet$. For this reason, $\operatorname{colim}_{Tw(fSet)}(e \circ r) \xrightarrow{\sim} \operatorname{colim}_{fSet} e$. Now fSet has a final object *, so $\operatorname{colim}_{fSet} e \xrightarrow{\sim} e(*)$.

1.0.14. For $I \in fSet$ let $A_I = (\triangle^I)! \operatorname{Fact}(A)$. For I = * we get $A_I \xrightarrow{\sim} A$ canonically. Indeed, applying $\triangle^!$ for $\triangle: X \to \operatorname{Ran}_X$, we get

$$\triangle^! \operatorname{Fact}(A) \xrightarrow{\sim} \operatorname{colim}_{(I \xrightarrow{\phi} J) \in Tw(fSet)} A^{\otimes I}$$

where for a morphism (1) in Tw(fSet) the transition map is the product $A^{\otimes I} \to A^{\otimes I'}$ along $I \to I'$. By the previous subsection, the above colimit identifies with A.

1.0.15. If $I, J \in fSet$ then

$$X^I \times_{\operatorname{Ran}_X} X^J \xrightarrow{\sim} \operatorname{colim}_{I \twoheadrightarrow K \twoheadleftarrow J} X^K$$

naturally. More precisely, inside the colimit is over $(fSet_{I/} \times_{fSet} fSet_{J/})^{op}$. In particular, $Shv(X^I \times_{\operatorname{Ran}_X} X^J) \xrightarrow{\sim} \underset{I \to K \leftarrow J}{\operatorname{colim}} Shv(X^K)$ with respect to !-extensions, and for any $F \in Shv(X^I)$ we have

$$\Delta^{J!} \Delta^I_! \ F \xrightarrow{\sim} \underset{I \to K \not\leftarrow J}{\operatorname{colim}} \ \Delta^{(J/K)}_! \Delta^{K!} \ F$$

Here $\triangle^{(J/K)}: X^K \to X^J$.

Proposition 1.0.16. For $I \in fSet$ the object A_I identifies with

$$\operatorname{colim}_{I \to J \xrightarrow{\phi} K} \Delta^K_! (A^{\otimes \phi}).$$

where the colimit is over $Tw(fSet) \times_{fSet} fSet_{I/}$.

Proof. Consider the category \mathcal{E} , whose objects are diagrams $(I \to \overline{K} \leftarrow K \stackrel{\phi}{\leftarrow} J)$ in fSet, and a morphism from 1 to 2 is given by a diagram in fSet

From the description of $X^K \times_{\operatorname{Ran}_X} X^I$ we get

$$(4) \qquad A_{I} \xrightarrow{\sim} \underset{(J \xrightarrow{\phi} K) \in Tw(fSet)}{\operatorname{colim}} \Delta^{I!} \Delta^{K}_{!} (A^{\otimes \phi}) \xrightarrow{\sim} \underset{(I \to \bar{K} \leftarrow K \xleftarrow{\phi} J) \in \mathcal{E}}{\operatorname{colim}} \Delta^{(I/K)}_{!} \Delta^{(K/\bar{K})!} A^{\otimes \phi}$$

Write $\bar{\phi}$ for the composition $J \xrightarrow{\phi} K \to \bar{K}$ for an object of \mathcal{E} . Then $\triangle^{(K/\bar{K})!} A^{\otimes \phi} \xrightarrow{\sim} A^{\otimes \bar{\phi}}$.

Let $r_0 : \mathcal{E}_0 \subset \mathcal{E}$ be the full subcategory given by the property that $K \to \overline{K}$ is an isomorphism. We think of \mathcal{E}_0 as the category classifying diagrams $(I \to \overline{K} \leftarrow J)$, where $(\overline{K} \leftarrow J) \in Tw(fSet)$ and $I \to \overline{K}$ is a surjection of finite sets. We write $\mathcal{E}_{0,I}$ if we need to express the dependence on I.

Let $l_0: \mathcal{E} \to \mathcal{E}_0$ be the functor sending $(I \to \overline{K} \leftarrow K \leftarrow J)$ to $(I \to \overline{K} \leftarrow J)$. Then r_0 is left adjoint to l_0 . So, l_0 is cofinal. Therefore, (4) identifies with

$$\operatorname{colim}_{(I \to K \xleftarrow{\phi} J) \in \mathcal{E}_0} \Delta_!^{(I/K)} A^{\otimes \phi}$$

in $Shv(X^I)$. Let $\mathcal{F}: \mathcal{E}_0 \to Shv(X^I)$ be the functor whose colimit is the latter expession.

Consider now the category $\mathcal{C} = Tw(fSet) \times_{fSet} (fSet)_{I/}$. We will write \mathcal{C}_I if we need to express the dependence on I. This is the category of diagrams $(K \leftarrow J \leftarrow I)$ in fSetwith $(K \leftarrow J) \in Tw(fSet)$. We have the functor $q : \mathcal{C} \to \mathcal{E}_0$ sending $(K \xleftarrow{\phi} J \xleftarrow{a} I)$ to $(I \xrightarrow{\phi \circ a} K \xleftarrow{\phi} J)$.

We claim that the natural map $\operatorname{colim}_{\mathfrak{C}} \mathfrak{F} \circ q \to \operatorname{colim}_{\mathfrak{E}_0} \mathfrak{F}$ is an isomorphism in $Shv(X^I)$. Is the map q cofinal? Let $\eta := (I \xrightarrow{\alpha} K \xleftarrow{\phi} J) \in \mathfrak{E}_0$. We need to check that the category $\mathfrak{C} \times_{\mathfrak{E}_0} (\mathfrak{E}_0)_{\eta/}$ is contractible. An object of the latter category is a diagram

such that $\nu_1 \phi_1 \tau_1 = \alpha$. A morphism from the above object to another object

(5)
$$\begin{array}{cccc} K_2 & \stackrel{\phi_2}{\leftarrow} & J_2 & \stackrel{\tau_2}{\leftarrow} I \\ \downarrow \nu_2 & \uparrow \\ K & \stackrel{\phi}{\leftarrow} & J \end{array}$$

(satisfying $\nu_2 \phi_2 \tau_2 = \alpha$) is a commutative diagram

such that the vertical compositions are the corresponding maps from (5).

If K is a 1-element set then $\mathfrak{C} \times_{\mathcal{E}_0} (\mathcal{E}_0)_{\eta/}$ has an initial object given by the diagram

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For general $K, \mathfrak{C} \times_{\mathcal{E}_0} (\mathcal{E}_0)_{\eta/}$ will be a product of categories over the set K. Namely, denote for $k \in K$ by I_k, J_k the corresponding fibres, so that $\eta_k := (I_k \to \{k\} \leftarrow J_k) \in$ \mathcal{E}_{0,I_k} . For each $k \in K$ we have the category

$$\mathcal{C}_{I_k} \times_{\mathcal{E}_{0,I_k}} (\mathcal{E}_{0,I_k})_{\eta_k/},$$

and their product identifies with $\mathcal{C} \times_{\mathcal{E}_0} (\mathcal{E}_0)_{\eta/}$. Thus, $\mathcal{C} \times_{\mathcal{E}_0} (\mathcal{E}_0)_{\eta/}$ is contractible, and q is cofinal.

The formula from Proposition 1.0.16 has appeared for commutative factorization categories (instead of algebras) in (|8|, 8.1.6).

1.0.17. For I of order 2 we get $A_I \xrightarrow{\sim} A \underset{\Delta_! A^{\otimes^! 2}}{\sqcup} (A \boxtimes A)$ for $\Delta: X \to X^2$, where the map $A^{\otimes^! 2} \to A$ is the product in the algebra.

1.1. 2nd seminar.

1.1.1. We assume X a separated scheme of finite type. In the case of \mathcal{D} -modules we assume X proper. For p_{Ran} : $\text{Ran}_X \to \text{Spec } k$ the functor $p_{\text{Ran}}^!$: $\text{Vect} \to (Shv(\text{Ran}_X), \star)$ is right-lax non-unital symmetric monoidal, and the diagram (of right-lax non-unital symmetric monoidal functors) commutes

$$(Shv(X), \otimes^!) \stackrel{\triangleq^!}{\leftarrow} (Shv(\operatorname{Ran}_X), \star) \\ \stackrel{\swarrow}{\leftarrow} p_X^! \stackrel{\uparrow}{\to} p_{\operatorname{Ran}}^! \\ \operatorname{Vect}$$

(here $p_X^!, \Delta^!$ are symmetric monoidal). So, the diagram commutes

$$CAlg^{nu,!}(Shv(X)) \quad \stackrel{\triangleq^!}{\leftarrow} CAlg^{nu,\star}(Shv(\operatorname{Ran}_X)) \\ \stackrel{\bigwedge}{\leftarrow} p_X^! \stackrel{\uparrow}{\to} p_{\operatorname{Ran}}^! \\ CAlg^{nu}(\operatorname{Vect})$$

So, we have for the corresponding left adjoint functors between the categories of nonunital commutative algebras $(p_{\text{Ran}})_{!}^{CAlg} \circ \bigtriangleup_{!}^{CAlg} \xrightarrow{\sim} C_{c}^{\text{Fact}}(X, \bullet)$. The key is that the functor $(p_{\text{Ran}})_{!} : (Shv(\text{Ran}), \star) \to \text{Vect}$ is nonunital symmetric

monoidal! So, $(p_{\text{Ran}})_{l}^{CAlg}$ is just the usual direct image $(p_{\text{Ran}})_{l}$.

Theorem 1.1.2. The functor $\triangle^!$: $CAlg^{nu,\star}(Shv(\operatorname{Ran}_X)) \to CAlg^{nu,!}(Shv(X))$ admits a fully faithful left adjoint, whose essential image is precisely $CAlg^{Fact}(Ran_X)$.

Proof. The categories $CAlg^{nu,\star}(Shv(\operatorname{Ran}_X))$, $CAlg^{nu,!}(Shv(X))$ are presentable by (HA, 3.2.3.5), and \triangle ! preserves limits and commutes with obly, so its left adjoint Δ_1^{CAlg} exists. By Lemma 1.1.3 below, Δ_1^{CAlg} is fully faithful.

By the proof of Lemma 1.1.3 below, $CAlg^{nu,!}(Shv(X))$ is generated under geometric realizations by free algebras. So, to check that $\triangle_{!}^{CAlg}$ takes values in $CAlg^{\text{Fact}}(\text{Ran}_X)$, it suffices to show that for any $K \in Shv(X)$, $\triangle_{!}^{CAlg}(\text{Sym}(K))$ lies in $CAlg^{\text{Fact}}(\text{Ran}_X)$, where by Sym we mean the left adjoint to $oblv : CAlg^{nu,!}(Shv(X)) \to Shv(X)$. Indeed, obly $: CAlg^{nu,*}(Shv(\text{Ran}_X)) \to Shv(\text{Ran}_X)$ preserves sifted colimits, so that $CAlg^{\text{Fact}}(\text{Ran}_X) \subset CAlg^{nu,*}(Shv(\text{Ran}_X))$ is stable under sifted colimits.

We know that $\Delta_!^{CAlg}(\operatorname{Sym}(K)) \xrightarrow{\sim} \operatorname{Sym}^*(\Delta_! K)$ for $K \in Shv(X)$ by Section 1.0.10 of this file. One checks by hands that this object has the factorization property:

We have for n > 0,

(6)
$$u^! \operatorname{Sym}^{n,\star}(\Delta_! K) \mid_{(\operatorname{Ran}_X \times \operatorname{Ran}_X)_d} \xrightarrow{\sim} \bigoplus_{0 < s < n} \operatorname{Sym}^{s,\star}(\Delta_! K) \boxtimes \operatorname{Sym}^{n-s,\star}(\Delta_! K)$$

Indeed, if for a point $S \to X^n$ given by (z_k) we are given $(x_i), (y_j) \in (\operatorname{Ran}_X \times \operatorname{Ran}_X)_d$ with $(x_i) \cup (y_j) = (z_k)$, there is a unique decomposition $I_1 \sqcup I_2 = \{1, \ldots, n\}$ such that our point is a product of two points $S \to X^{I_1}, S \to X^{I_2}$ giving a point of $(X^{I_1} \times X^{I_2})_d$, which produces our point of $(\operatorname{Ran}_X \times \operatorname{Ran}_X)_d$. The group S_n acts transitively on such decompositions, and the stabilizor of a decomposition is $S(I_1) \times S(I_2)$, the product of symmetric groups. If $|I_1| = s$ this gives the contribution

$$\operatorname{Sym}^{s,\star}(\Delta_! K) \boxtimes \operatorname{Sym}^{n-s,\star}(\Delta_! K)$$

Now the RHS of (6) over $(\operatorname{Ran}_X \times \operatorname{Ran}_X)_d$ identifies with *n*-th graded component of $\operatorname{Sym}^*(\Delta_! K) \boxtimes \operatorname{Sym}^*(\Delta_! K)$.

It remains to show that $\triangle_!^{CAlg}$: $CAlg^{nu,!}(Shv(X)) \rightarrow CAlg^{Fact}(Ran_X)$ generates $CAlg^{Fact}(Ran_X)$ under geometric realizations. The idea is to use ([10], Pp. 4.7.3.14). We check first that

(7)
$$\Delta^!: CAlg^{\operatorname{Fact}}(\operatorname{Ran}_X) \to CAlg^{nu,!}(Shv(X))$$

is conservative. The latter fact follows from the factorization property, because we know it over X.

Indeed, let $f: A_1 \to A_2$ be a map in $CAlg^{\text{Fact}}(\text{Ran}_X)$ with $\triangle^! f$ an isomorphism. We want to show that the !-restriction of f under $X^I \to \text{Ran}_X$ is an isomorphism. It suffices to show that for any quotient set $I \to J$ the !-restriction of f under $X_d^J \hookrightarrow X^I \to \text{Ran}_X$ is an isomorphism, and this follows from the factorization property. Here $X_d^J \subset X^J$ is the complement to all the diagonals.

To finish, note that (7) preserves sifted colimits, because

$$CAlg^{\text{Fact}}(\text{Ran}_X) \subset CAlg^{nu,*}(Shv(\text{Ran}_X))$$

is stable under sifted colimits. Thus, ([10], 4.7.3.14) applies.

Lemma 1.1.3. Suppose $l: C \leftrightarrows D: r$ is an adjoint pair in $DGCat_{cont}$ with l fully faithful. Suppose that $C, D \in CAlg^{nu}(DGCat_{cont})$, and r is nonunital symmetric monoidal, so l is left-lax nonunital symmetric monoidal. The functor $r^{CAlg}: CAlg^{nu}(D) \rightarrow CAlg^{nu}(C)$ preserves limits, so admits a left adjoint l^{CAlg} , as both categories are presentable. Then l^{CAlg} is fully faithful.

Proof. We check that the natural map id $\rightarrow r^{CAlg} l^{CAlg}$ is an isomorphism. My understanding is that $CAlg^{nu}(C)$ is not a DG-category, just a presentable category. The

functor obly : $CAlg^{nu}(C) \to C$ is conservative. The essential image of its left adjoint $free: C \to CAlg^{nu}(C)$ generates $CAlg^{nu}(C)$ under colimits in the sense of ([9], 5.1.5.7), this follows from the (*) claim inside the proof of ([10], Corollary 3.2.3.3). Another way is to say that for any reduced operad \mathcal{P} in the sense of ([2], ch. IV.2, 1.1.2), obly : $\mathcal{P} - Alg(C) \to C$ is monadic, so we may apply ([10], 4.7.3.14). The functor obly : $CAlg^{nu}(C) \to C$ preserves sifted colimits (by [11], 9.4.12).

So, it suffices to show that for any $c \in C$,

$$free(c) \rightarrow r^{CAlg} l^{CAlg} (free(c))$$

is an isomorphism. Now $l^{CAlg} \circ free \xrightarrow{\sim} free \circ l$. Since r is nonunital symmetric monoidal, we have for n > 0 and $d \in D$, $r(\operatorname{Sym}^n(d)) \xrightarrow{\sim} \operatorname{Sym}^n(r(d))$, because

$$\operatorname{Sym}^n(d) \xrightarrow{\sim} \operatorname{colim}_{B(S_n)} d^{\otimes n}$$

the colimit being taken in D. So, $oblv \circ r^{CAlg} \circ free \xrightarrow{\sim} oblv \circ free \circ r$. The claim follows.

Passing to left adjoints, we see that the diagram commutes

$$\begin{array}{ccc} Shv(\operatorname{Ran}) & \stackrel{free}{\to} & CAlg^{nu,\star}(Shv(\operatorname{Ran})) \\ \uparrow & & \uparrow & \\ Shv(X) & \stackrel{free}{\to} & CAlg^{nu,!}(Shv(X)) \end{array}$$

Here is a dual version of Lemma 1.1.3.

Lemma 1.1.4. Let $C, D \in CAlg^{nu}(\text{DGCat}_{cont})$, let $L : C \to D$ be non-unital symmetric monoidal functor in DGCat_{cont} . Assume it has a fully faithful continuous right adjoint $R : D \to C$, so R is non-unital right-lax symmetric monoidal. Let

 $\mathcal{L}: ComCoAlg^{nu}(C) \to ComCoAlg^{nu}(D)$

be the functor induced by L. Then \mathcal{L} admits a right adjoint \mathcal{R} . (Is this \mathcal{R} fully faithful??? Not clear!!)

Proof. First, by ([11], 9.2.66), $ComCoAlg^{nu}(C)$, $ComCoAlg^{nu}(D)$ are presentable. The functor $\mathcal{L}^{op} : CAlg^{nu}(C^{op}) \to CAlg^{nu}(D^{op})$ preserves limits. So, \mathcal{L} admits a right adjoint \mathcal{R} . Is it true that $\mathcal{LR} \to \mathrm{id}$ is an isomorphism?

First, obly : $ComCoAlg^{nu}(C) \to C$ preserves colimits, so has a right adjoint $cofree : C \to ComCoAlg^{nu}(C)$.

Passing to the right adjoints in $oblv L \xrightarrow{\sim} \mathcal{L} oblv$, we get $cofree \circ R \xrightarrow{\sim} \mathcal{R} \circ cofree$. It is plausible that the essential image of $cofree : C \to ComCoAlg^{nu}(C)$ generates $ComCoAlg^{nu}(C)$ under colimits, but I don't see a proof!

1.2. Justin's explanation.

1.2.1. Let $F : \mathfrak{C} \hookrightarrow \mathfrak{D} : G$ be an adjoint pair in DGCat_{cont} , where both $\mathfrak{C}, \mathfrak{D} \in CAlg^{nu}(\mathrm{DGCat}_{cont})$, and G is nonunital symmetric monoidal. So, F is left-lax nonunital symmetric monoidal. We want to "force" in some universal way F to be strictly nonunital symmetric monoidal.

Recall that fSet is the category of finite nonempty sets and surjections. View fSet as nonunital symmetric monoidal with respect to the disjoint union. Note that $pt \in CAlg^{nu}(fSet)$, the product is the map $pt \sqcup pt \to pt$. It has the following universal property: for any nonunital symmetric monoidal category \mathcal{C} , one has a commutative diagram

(8)
$$\begin{aligned} \operatorname{Fun}^{\otimes}(fSet, \mathbb{C}) & \xrightarrow{\sim} & CAlg^{nu}(\mathbb{C}) \\ \downarrow & \swarrow & \swarrow & \mathsf{oblv} \\ \mathbb{C}. \end{aligned}$$

where the vertical arrow is the evaluation at $pt \in fSet$.

Indeed, recall the ∞ -operad Surj, which is a subcategory of $\mathcal{F}in_*$ with the same objects and where we keep the morphisms that are surjective. Its monoidal envelope in the sense of ([10], 2.2.4.1) evidently identifies with fSet. Namely, to $I \in fSet$ we associate $I \sqcup \{*\} \in Surj$. So, (8) is a particular case of ([10], 2.2.4.9).

Remark 1.2.2. What is the Fin_* -monoidal envelope of Fin_* ? By ([10], 2.2.4.3), it is as follows. Let $fSet_{\emptyset}$ be the category of finite (possibly empty) sets and any morphisms between them. The map $fSet_{\emptyset} \to \operatorname{Act}(\operatorname{Fin}_*)$, $I \mapsto I \sqcup \{*\}$ is an equivalence. By ([10], 2.2.4.4), $fSet_{\emptyset}$ has a symmetric monoidal structure given by the disjoint union, and this is the monoidal envelope of the ∞ -operad $\operatorname{Fin}_* \stackrel{\mathrm{id}}{\to} \operatorname{Fin}_*$. We may think of the ∞ -operad $\operatorname{Fin}_* \stackrel{\mathrm{id}}{\to} \operatorname{Fin}_*$ just as the category $\{*\}$ with the natural symmetric monoidal structure. So, ([10], 2.2.4.9) in this case says that for any symmetric monoidal ∞ -category D the restriction along $\{*\} \hookrightarrow fSet_{\emptyset}$ yields an equivalence

$$\operatorname{Fun}^{\otimes}(fSet_{\emptyset}, D) \xrightarrow{\sim} CAlg(D)$$

Here $\operatorname{Fun}^{\otimes}(fSet_{\emptyset}, D)$ is the category of symmetric monoidal functors $fSet_{\emptyset} \to D$.

1.2.3. For any $C \in CAlg(1 - Cat)$ consider the functor $h : C^{op} \times C \to \text{Spc}, (x, y) \mapsto \text{Map}_C(x, y)$. We equip C^{op} with the induced symmetric monoidal structure, and similarly for $C^{op} \times C$, and Spc with the cartesian symmetric monoidal structure. Then h is right-lax symmetric monoidal: we have natural maps $* \stackrel{\text{id}}{\to} \text{Map}_C(1, 1)$ and $\text{Map}_C(x_1, y_1) \times \text{Map}_C(x_2, y_2) \to \text{Map}_C(x_1 \otimes x_2, y_1 \otimes y_2)$ given by the tensor product of maps.

Let $h': \mathfrak{X} \to C^{op} \times C$ be the cocartesian fibration in spaces attached to h. By ([14], 5.15), \mathfrak{X} is naturally symmetric monoidal and h' is symmetric monoidal. So, \mathfrak{X}^{op} is also symmetric monoidal, and $h'^{op}: \mathfrak{X}^{op} \to C \times C^{op}$ is symmetric monoidal. The latter functor is a cartesian fibration attached to the functor $C^{op} \times C \to \operatorname{Spc}$, $(x, y) \mapsto \operatorname{Map}_C(x, y)^{op} \xrightarrow{\sim} \operatorname{Map}_C(x, y)$. So, $\mathfrak{X}^{op} \xrightarrow{\sim} Tw(C)$.

We similarly have a nonunital version for $C \in CAlg^{nu}(1 - Cat)$ giving a nonunital symmetric monoidal structure on Tw(C).

1.2.4. We apply this to C = fSet, and see that Tw(fSet) is nonunital symmetric monoidal. The projection $Tw(fSet) \rightarrow fSet \times fSet^{op}$ is nonunital symmetric monoidal. The tensor product of the objects $f : I \rightarrow J$ and $f' : I' \rightarrow J'$ of fSet is the map $f \sqcup f' : I \sqcup I' \rightarrow J \sqcup J'$.

1.2.5. Consider the functor $l: fSet \to Tw(fSet)$ sending I to $(I \to *)$. It is left-lax nonunital symmetric monoidal.

Lemma 1.2.6. For any $D \in CAlg^{nu}(1 - Cat)$ the composition with l yields an equivalence

$$\operatorname{Fun}^{\otimes}(Tw(fSet), D) \xrightarrow{\sim} \operatorname{Fun}^{llax}(fSet, D)$$

Here $\operatorname{Fun}^{\otimes}(Tw(fSet), D)$ is the category of nonunital symmetric monoidal functors.

Proof. A general claim first.

Step 1 Let $\operatorname{Surj} \to 1-\operatorname{Cat}$ be a Surj-monoid in $1-\operatorname{Cat}$ in the sense of ([10], 2.4.2.1), let \mathcal{A} be the value of this functor on $\langle 1 \rangle$. Let $\tilde{\mathcal{A}}^{\otimes} \to \operatorname{Surj}$ be the cocartesian fibration attached to it, so this is an ∞ -operad defining the nonunital symmetric monoidal category \mathcal{A} . Recall the subcategory $fSet \subset \operatorname{Surj}$, this functor sends I to $I \sqcup \{*\}$. Let $\mathcal{A}^{\otimes} \to fSet$ be obtained from $\tilde{\mathcal{A}}^{\otimes} \to \operatorname{Surj}$ by the base change $fSet \to \operatorname{Surj}$. Then \mathcal{A}^{\otimes} identifies with the Surj-monoidal envelope of the ∞ -operad $\tilde{\mathcal{A}}^{\otimes} \to \operatorname{Surj}$ in the sense of ([10], 2.2.4.3). So, \mathcal{A}^{\otimes} acquires a nonunital symmetric monoidal structure, and the natural right-lax nonunital symmetric monoidal map $\mathcal{A} \to \mathcal{A}^{\otimes}$ giving rise, by ([10], 2.2.4.9), to an equivalence

$$\operatorname{Fun}^{\otimes}(\mathcal{A}^{\otimes}, D) \xrightarrow{\sim} \operatorname{Fun}^{rlax}(\mathcal{A}, D)$$

Here $\operatorname{Fun}^{rlax}(\mathcal{A}, D)$ is what is called the category of $\tilde{\mathcal{A}}^{\otimes}$ -algebras in D^{\otimes} over Surj in [10], and $\operatorname{Fun}^{\otimes}(\mathcal{A}^{\otimes}, D)$ is the category of nonunital symmetric monoidal functors.

An object of \mathcal{A}^{\otimes} is a finite nonempty set I and a collection $a_i \in \mathcal{A}$ for $i \in I$. A map in \mathcal{A}^{\otimes} from $(I, \{a_i\})$ to $(J, \{b_j\})$ is a surjection $f: I \to J$ and for each $j \in J$ a map $\bigotimes_{i \in f^{-1}(j)} a_i \to b_j$ in \mathcal{A} . The tensor product in the symmetric monoidal category \mathcal{A}^{\otimes} of $(I, \{a_i\})$ and $(I, \{b_i\})$ is $(I \mapsto I, \{a_i, b_i\})$ by ([10], 2, 2, 4, 6). The functor $i \in \mathcal{A}$ and $(I, \{b_i\})$ is $(I \mapsto I, \{a_i\})$ by ([10], 2, 2, 4, 6).

 $(I, \{a_i\})$ and $(J, \{b_j\})$ is $(I \sqcup J, \{a_i, b_j\})$ by ([10], 2.2.4.6). The functor $i : \mathcal{A} \to \mathcal{A}^{\otimes}$ sends a to (*, a).

We also have a symmetric monoidal functor $\mu : \mathcal{A}^{\otimes} \to \mathcal{A}, (I, \{a_i\}) \mapsto \otimes_{i \in I} a_i$. Note that μ is left adjoint to i.

Step 2 Consider $fSet^{op}$ with the induced nonunital symmetric monoidal structure. Then the Surj-monoidal envelope $((fSet)^{op})^{\otimes}$ of $(fSet)^{op}$ is $Tw(fSet)^{op}$. Indeed, by the above, an object of $((fSet)^{op})^{\otimes}$ is a collection $(I, \{J_i\}_{i \in I})$, where I is a finite nonempty set, and for $i \in I$, J_i is a finite nonempty set. A map in $((fSet)^{op})^{\otimes}$ from $(I, \{J_i\}_{i \in I})$ to $(I', \{J'_{i'}\}_{i' \in I'})$ is a surjection $\phi : I \to I'$, and for each $i' \in I'$ a surjection of finite sets

$$J'_{i'} \to \bigsqcup_{i \in \phi^{-1}(i')} J_i$$

We may view $(I, \{J_i\}_{i \in I})$ as an object $\tau : J \to I$ of fSet. The above morphism in $((fSet)^{op})^{\otimes}$ from $(J \xrightarrow{\tau} I)$ to $(J' \xrightarrow{\tau'} I')$ becomes a commutative diagram

$$\begin{array}{cccc} J' & \frac{\tau'}{\rightarrow} & I' \\ \downarrow \phi' & \uparrow \phi \\ J & \stackrel{\tau}{\rightarrow} & I \end{array}$$

Now the right-lax functor $fSet^{op} \to ((fSet)^{op})^{\otimes} \xrightarrow{\sim} Tw(fSet)^{op}$ sends I to $(I \xrightarrow{\tau} *)$, this is the functor l^{op} from Section 1.2.5. By Step 1, we get an equivalence

$$\begin{split} \operatorname{Fun}^{\otimes}(Tw(fSet),D)^{op} & \xrightarrow{\sim} \operatorname{Fun}^{\otimes}(Tw(fSet)^{op},D^{op}) \xrightarrow{\sim} \operatorname{Fun}^{rlax}(fSet^{op},D^{op}) \\ & \xrightarrow{\sim} \operatorname{Fun}^{llax}(fSet,D)^{op} \end{split}$$

The functor $\mu: Tw(fSet)^{op} \xrightarrow{\sim} ((fSet)^{op})^{\otimes} \to (fSet)^{op}$ sends $(J \xrightarrow{\tau} I)$ to J.

Remark 1.2.7. If $A, B \in CAlg^{nu}(1 - Cat)$ then $\operatorname{Fun}^{\otimes}(A, B)^{op} \xrightarrow{\sim} \operatorname{Fun}^{\otimes}(A^{op}, B^{op})$ and $\operatorname{Fun}^{rlax}(A, B)^{op} \xrightarrow{\sim} \operatorname{Fun}^{llax}(A^{op}, B^{op})$.

1.2.8. Let $\mathfrak{I}, \mathfrak{C} \in CAlg(1 - \mathfrak{Cat})$ with \mathfrak{C} cocomplete such that the tensor product in \mathfrak{C} preserves colimits separately in each variable, and \mathfrak{I} is small. Then Fun($\mathfrak{I}, \mathfrak{C}$) has a symmetric monoidal structure given by Day convolution ([10], 2.2.6.17).

Let $F : \mathfrak{I} \to \mathfrak{J}$ be a map in $CAlg(1 - \mathbb{C}at)$, where $\mathfrak{I}, \mathfrak{J}$ are small. We have an adjoint pair l: Fun $(\mathfrak{I}, \mathbb{C}) \rightleftharpoons$ Fun $(\mathfrak{J}, \mathbb{C})$: r, where l is the left Kan extension along F, and r is the composition with F. Then l is symmetric monoidal. Indeed, for $i \in \mathfrak{I}$ and $f_1, f_2 \in \operatorname{Fun}(\mathfrak{I}, \mathbb{C})$ we have

$$(f_1 \otimes f_2)(i) \xrightarrow{\sim} \underset{(i_1,i_2) \in \mathfrak{I} \times \mathfrak{I}, \ i_1 \otimes i_2 \to i}{\operatorname{colim}} f_1(i_1) \otimes f_2(i_2)$$

Let $\bar{f}_i = l(f_i)$. Then the LKE of the composition $\mathfrak{I} \times \mathfrak{I} \xrightarrow{f_1 \times f_2} \mathfrak{C} \times \mathfrak{C} \xrightarrow{\otimes} C$ along $\mathfrak{I} \times \mathfrak{I} \to \mathfrak{J} \times \mathfrak{J}$ is the composition $\mathfrak{J} \times \mathfrak{J} \xrightarrow{\bar{f}_1 \times \bar{f}_2} \mathfrak{C} \times \mathfrak{C} \xrightarrow{\otimes} \mathfrak{C}$. Indeed, for $(j_1, j_2) \in \mathfrak{J} \times \mathfrak{J}$ the value of this LKE at (j_1, j_2) is

$$\underset{(i_1,i_2)\in\mathbb{J}\times\mathbb{J},F(i_1)\to j_1,F(i_2)\to j_2}{\operatorname{colim}} f_1(i_1)\otimes f_2(i_2) \xrightarrow{\sim} (\underset{i_1\in\mathbb{J},F(i_1)\to j_1}{\operatorname{colim}} f_1(i_1))\otimes (\underset{i_2\in\mathbb{J},F(i_2)\to j_2}{\operatorname{colim}} f_2(i_2))$$
$$\xrightarrow{\sim} \overline{f_1}(j_1)\otimes \overline{f_2}(j_2)$$

Now $l(f_1 \otimes f_2)$ is the LKE of the composition $\mathfrak{I} \times \mathfrak{I} \xrightarrow{f_1 \times f_2} \mathfrak{C} \times \mathfrak{C} \xrightarrow{\otimes} C$ along the composition $\mathfrak{I} \times \mathfrak{I} \xrightarrow{\otimes} \mathfrak{I} \xrightarrow{F} \mathfrak{J}$. So, our claim follows from the commutativity of the diagram

$$\begin{array}{cccc} \mathcal{J} \times \mathcal{J} & \stackrel{\otimes}{\to} & \mathcal{J} \\ \uparrow F \times F & \uparrow F \\ \mathcal{J} \times \mathcal{J} & \stackrel{\otimes}{\to} & \mathcal{J} \end{array}$$

and the transitivity of LKE.

More generally, for a (maybe empty) finite set S and a collection $f_s \in Fun(\mathcal{I}, \mathcal{C})$ for $s \in S$, we have

$$(\underset{s\in S}{\otimes} f_s)(i) \xrightarrow{\sim} \underset{\{i_s\}\in \prod \atop s\in S}{\operatorname{colim}} \underset{i\in S}{\operatorname{colim}} (\underset{i\in S}{\otimes} f_s(i_s))$$

In particular, for $S = \emptyset$ this gives the unit of Fun($\mathfrak{I}, \mathfrak{C}$) for the Day covolution: its value at i is $\operatorname{colim}_{\operatorname{Map}_{\mathcal{J}}(1,i)} 1$ taken in C. The space $\operatorname{Map}_{\mathcal{J}}(1,i)$ is not necessarily contractible, so in Lurie's notation this is $Map_1(1, i) \otimes 1$, as C is tensored over Spc. If 1 is initial in J then the unit of Fun($\mathfrak{I}, \mathfrak{C}$) for the Day convolution is the constant functor with value $1 \in \mathfrak{C}$.

In fact, the unit of Fun($\mathfrak{I}, \mathfrak{C}$) is the LKE of $* \xrightarrow{1} \mathfrak{C}$ along $* \xrightarrow{1} \mathfrak{I}$.

This shows that l sends unit to unit. Note that r is right-lax monoidal for the Day convolution structures. The functors r, l induce a diagram of adjoint functors $l^{CAlg}: CAlg(\operatorname{Fun}(\mathfrak{I}, \mathfrak{C})) \leftrightarrows CAlg(\operatorname{Fun}(\mathfrak{J}, \mathfrak{C})): r^{CAlg}$ given by composing with r and l respectively. This follows from remark below.

Remark 1.2.9. Let $C^{\otimes} \to O^{\otimes} \leftarrow D^{\otimes}$ be a diagram of cocartesian fibrations of ∞ operads. Let $A^{\otimes} \to O^{\otimes}$ be a map of ∞ -operads. Assume given an adjoint pair L: $C^{\otimes} \leftrightarrows D^{\otimes} : R \text{ in } 1 - \mathfrak{Cat}, \text{ where } L, R \text{ are maps of } \infty \text{-operads over } O^{\otimes}.$ Assume that L is a morphism of O^{\otimes} -monoidal categories, that is, sends a cocartesian arrow over O^{\otimes} to a cocartesian arrow over O^{\otimes} . Let $L' : Alg_{A/O}(C) \subseteq Alg_{A/O}(D) : R'$ be obtained by composing with L and R. Then (L', R') is an adjoint pair in 1 - Cat.

A special case of this has appeared as ([11], 3.0.20).

1.2.10. Now using $CAlg(\operatorname{Fun}(\mathfrak{I}, \mathfrak{C})) \xrightarrow{\sim} \operatorname{Fun}^{rlax}(\mathfrak{I}, \mathfrak{C})$ given by ([10], 2.2.6.8), we get an adjoint pair l^{CAlg} : $\operatorname{Fun}^{rlax}(\mathfrak{I}, \mathfrak{C}) \leftrightarrows \operatorname{Fun}^{rlax}(\mathfrak{I}, \mathfrak{C}) : r^{CAlg}$.

1.2.11. Let \Pr^L be the category of presentable categories and colimit preserving functors, it is equipped with the Lurie tensor product by ([10], 4.8.1.15). Consider 1 - Catwith the cartesian symmetric monoidal structure. Then the functor $1 - Cat \rightarrow Pr^L$, $I \mapsto \mathcal{P}(I) = \operatorname{Fun}(I^{op}, \operatorname{Spc})$ is symmetric monoidal (see [10], inside the proof of 4.8.1.15). For $\mathcal{C} \in \mathbb{P}r^L$ and I small there is an isomorphism in $\mathbb{P}r^L$

$$\operatorname{Fun}(I, \mathfrak{C}) \xrightarrow{\sim} \mathfrak{P}(I^{op}) \otimes \mathfrak{C}$$

Indeed, one has

 $\operatorname{Fun}(I, \mathfrak{C}) \xrightarrow{\sim} \operatorname{Fun}(I^{op}, \mathfrak{C}^{op})^{op} \xrightarrow{\sim} (\operatorname{Fun}^{L}(\mathfrak{P}(I^{op}), \mathfrak{C}^{op}))^{op} \xrightarrow{\sim} \operatorname{Fun}^{R}(\mathfrak{P}(I^{op})^{op}, \mathfrak{C}) \xrightarrow{\sim} \mathfrak{P}(I^{op}) \otimes \mathfrak{C}$ here the last isomorphism is by ([10], 4.8.1.17). Here $\operatorname{Fun}^{L}(\mathcal{P}(I^{op}), \mathcal{C}) \subset \operatorname{Fun}(\mathcal{P}(I^{op}), \mathcal{C})$ is the full subcategory of colimit preserving functors, and

$$\operatorname{Fun}^{R}(\mathfrak{P}(I^{op})^{op}, \mathfrak{C}^{op}) \subset \operatorname{Fun}(\mathfrak{P}(I^{op})^{op}, \mathfrak{C}^{op})$$

is the full subcategory of limit preserving functors.

1.2.12. The results of Section 1.2.8 have a nonunital version: in this case $\mathfrak{I}, \mathfrak{C} \in \mathfrak{I}$ $CAlg^{nu}(1-Cat)$ with C cocomplete such that the tensor product preserves colimits separately in each variable, and \mathcal{I} is small. Then $\operatorname{Fun}(\mathcal{I}, \mathcal{C})$ has a nonunital symmetric monoidal structure given by Day convolution. Namely, apply ([10], Construction 2.2.6.7) with the base ∞ -operad $\mathcal{O}^{\otimes} = \operatorname{Surj}^{\otimes}$.

Let now $F: \mathcal{I} \to \mathcal{J}$ be a map in $CAlg^{nu}(1-Cat)$, where \mathcal{I}, \mathcal{J} are small. We have the same adjoint pair l: Fun $(\mathfrak{I}, \mathfrak{C}) \leftrightarrows$ Fun $(\mathfrak{J}, \mathfrak{C})$: r, where l is the LKE along F. Then l is nonunital symmetric monoidal, and r is right-lax nonunital symmetric monoidal for the Dav convolution structures. So, we get an adjoint pair $l^{CAlg}: CAlg^{nu}(\operatorname{Fun}(\mathfrak{I}, \mathfrak{C})) \leftrightarrows$

 $CAlg^{nu}(\operatorname{Fun}(\mathcal{J}, \mathbb{C})): r^{CAlg}$, here the functors are given by composing with l and with r. In this case ([10], 2.2.6.8) gives $CAlg^{nu}(\operatorname{Fun}(\mathcal{J}, \mathbb{C})) \xrightarrow{\sim} \operatorname{Fun}^{rlax}(\mathcal{J}, \mathbb{C})$, where on the right hand side we mean nonunital right-lax symmetric monoidal functors.

1.2.13. Assume we are in the situation of Section 1.2.1, so we have an adjoint pair $F: \mathfrak{C} \leftrightarrows \mathfrak{D}: G$ in DGCat_{cont} , where $\mathfrak{C}, \mathfrak{D} \in CAlg^{nu}(\mathrm{DGCat}_{cont})$, and G is nonunital symmetric monoidal, so F is left-lax nonunital symmetric monoidal. We get an adjoint pair $G_{enh}^L: CAlg^{nu}(\mathfrak{C}) \leftrightarrows CAlg^{nu}(\mathfrak{D}): G_{enh}$, where G_{enh} commutes with obly, and G_{enh}^L is its left adjoint. Here G_{enh}^L is not a lifting of F.

Proposition 1.2.14. In the situation of Section 1.2.13 the following diagram commutes

Here the low horizontal isomorphism is given by Lemma 1.2.6.

In the above proposition we used the fact that $Tw(fSet) \to pt$ is nonunital symmetric monoidal, so we apply the result of Section 1.2.12 to this functor to conclude that colim : $\operatorname{Fun}(Tw(fSet), \mathcal{D}) \to \operatorname{Fun}(pt, \mathcal{D})$ is nonunital symmetric monoidal, so induces a functor $CAlq^{nu}(\operatorname{Fun}(Tw(fSet), \mathcal{D})) \to CAlq^{nu}(\operatorname{Fun}(pt, \mathcal{D}))$

$$\begin{array}{rcl} CAlg^{nu}(\operatorname{Fun}(Tw(fSet), \mathcal{D})) & \to & CAlg^{nu}(\operatorname{Fun}(pt, \mathcal{D})) \\ & & & || \\ & & & || \\ \operatorname{Fun}^{rlax}(Tw(fSet), \mathcal{D})) & & & \operatorname{Fun}^{rlax}(pt, \mathcal{D}) \end{array}$$

The long composition in the diagram (10) is as follows. Let $c \in CAlg^{nu}(\mathbb{C})$. The corresponding nonunital symmetric monoidal functor $fSet \to \mathbb{C}$ sends I to $\underset{i \in I}{\otimes} c$, and a

 $\begin{array}{l} \max \phi: I \to J \text{ in } fSet \text{ to } \underset{\phi}{\otimes} : \underset{i \in I}{\otimes} c \to \underset{j \in J}{\otimes} c, \text{ the tensor product along } \phi. \\ \text{This gives a right-lax functor } fSet^{op} \to \mathcal{D}^{op}, \ I \mapsto F(\underset{i \in I}{\otimes} c). \text{ It associates to a map} \\ \tau: J \to I \text{ in } fSet^{op} \text{ as above the morphism} \end{array}$

$$F(\bigotimes_{\tau}): F(\bigotimes_{j\in J} c) \to F(\bigotimes_{i\in I} c)$$

in \mathcal{D}^{op} . The corresponding nonunital symmetric monoidal functor $\theta: Tw(fSet)^{op} \to \mathcal{D}^{op}$ sends $(J \xrightarrow{\tau} I) \in Tw(fSet)$ to

$$\underset{i\in I}{\otimes} F(\underset{j\in J_i}{\otimes} c)$$

Now given a morphism from $(J' \xrightarrow{\tau'} I')$ to $(J \xrightarrow{\tau} I)$ in Tw(fSet) given by (9), θ sends it to the morphism in \mathcal{D}

$$\underset{i'\in I'}{\otimes} F(\underset{j'\in J'_{i'}}{\otimes} c) \to \underset{i\in I}{\otimes} F(\underset{j\in J_i}{\otimes} c)$$

obtained as the tensor product in \mathcal{D} over $i' \in I'$ of the morphisms

$$F(\underset{j'\in J'_{i'}}{\otimes} c) \to \underset{i\in I_{i'}}{\otimes} F(\underset{j\in J_i}{\otimes} c),$$

which are the compositions

(11)
$$F(\underset{j'\in J'_{i'}}{\otimes} c) \to F(\underset{i\in I_{i'}}{\otimes} c) \to \underset{i\in I_{i'}}{\otimes} F(\underset{j\in J_i}{\otimes} c)$$

Here we used surjections $\phi'_{i'}: J'_{i'} \to \bigsqcup_{i \in I_{i'}} J_i$. The first map in (11) is $F(\bigotimes)_{\phi'_{i'}}$, and the second one comes from the left-lax structure on F.

Proof of Proposition 1.2.14. From Remark 1.2.9 (by passing to the opposite categories) one derives an adjont pair \mathcal{L} : Fun^{llax}($fSet, \mathcal{C}$) \leftrightarrows Fun^{llax}($fSet, \mathcal{D}$) : \mathcal{R} , where \mathcal{L} and \mathcal{R} are compositions with F and G respectively. Note that Fun^{\otimes}($fSet, \mathcal{C}$) \subset Fun^{llax}($fSet, \mathcal{C}$) is a full subcategory.

Step 1 Let $d \in \mathcal{D}$ (resp., $c \in \mathcal{C}$) be a nonunital commutative algebra, and $\alpha_d : fSet \to \mathcal{D}$ (resp., $\alpha_c : fSet \to \mathcal{C}$) the corresponding nonunital symmetric monoidal functor. Then $G \circ \alpha_d$ is nonunital symmetric monoidal, so

 $\begin{aligned} \operatorname{Map}_{\operatorname{Fun}^{\otimes}(fSet,\mathbb{C})}(\alpha_{c},G\circ\alpha_{d}) & \xrightarrow{\sim} \operatorname{Map}_{\operatorname{Fun}^{llax}(fSet,\mathbb{C})}(\alpha_{c},G\circ\alpha_{d}) \xrightarrow{\sim} \operatorname{Map}_{\operatorname{Fun}^{llax}(fSet,\mathbb{D})}(F\circ\alpha_{c},\alpha_{d}) \\ \end{aligned}$ $\begin{aligned} \operatorname{Write} \ \bar{\alpha}_{c}, \bar{\alpha}_{d} \in \operatorname{Fun}^{\otimes}(Tw(fSet),\mathcal{D}) \text{ for the images of } F\circ\alpha_{c} \text{ and } \alpha_{d} \text{ respectively under} \end{aligned}$

$$\operatorname{Fun}^{llax}(fSet, \mathcal{D}) \xrightarrow{\sim} \operatorname{Fun}^{\otimes}(Tw(fSet), \mathcal{D})$$

The functor $\bar{\alpha}_d$ is the composition $Tw(fSet) \xrightarrow{r} fSet \xrightarrow{\alpha_d} \mathcal{D}$, where $r(J \to I) = J$. Let $\tilde{\alpha}_c : fSet \to \mathcal{D}$ be the LKE of $\bar{\alpha}_c : Tw(fSet) \to \mathcal{D}$ under $r : Tw(fSet) \to fSet$. We get

$$\operatorname{Map}_{\operatorname{Fun}^{\otimes}(Tw(fSet),\mathcal{D})}(\bar{\alpha}_{c},\bar{\alpha}_{d}) \xrightarrow{\sim} \operatorname{Map}_{\operatorname{Fun}^{rlax}(fSet,\mathcal{D})}(\tilde{\alpha}_{c},\alpha_{d})$$

We used the fact that r is nonunital symmetric monoidal, so by Section 1.2.12 gives rise an an adjunction

$$\operatorname{Fun}^{rlax}(Tw(fSet), \mathcal{D}) \leftrightarrows \operatorname{Fun}^{rlax}(fSet, \mathcal{D}).$$

where the left adjoint is given by the LKE, and the right adjoint is the composition with r. Besides, $\operatorname{Fun}^{\otimes}(Tw(fSet), \mathcal{D}) \subset \operatorname{Fun}^{rlax}(fSet, \mathcal{D})$ is a full subcategory.

Since pt is a final object of fSet, the composition

$$CAlg^{nu}(\mathcal{C}) \xrightarrow{\sim} \operatorname{Fun}^{\otimes}(fSet, \mathcal{C}) \xrightarrow{\operatorname{colim}} \mathcal{C}$$

is obly : $CAlg^{nu}(\mathcal{C}) \to \mathcal{C}$. By ([11], 9.2.13), Fun($fSet, \mathcal{C}$) \in DGCat_{cont}, and compositions with F and G yield an adjoint pair Fun($fSet, \mathcal{C}$) \leftrightarrows Fun($fSet, \mathcal{D}$) in DGCat_{cont}.

Decompose colim : $\operatorname{Fun}(Tw(fSet), \mathcal{D})) \to \operatorname{Fun}(pt, \mathcal{D})$ as

$$\operatorname{Fun}(Tw(fSet), \mathcal{D})) \xrightarrow{LKE} \operatorname{Fun}(fSet, \mathcal{D}) \xrightarrow{\operatorname{colim}} \operatorname{Fun}(pt, \mathcal{D})$$

where LKE is along $Tw(fSet) \xrightarrow{r} fSet$.

Step 2 To finish, it suffices to show that $\tilde{\alpha}_c : fSet \to \mathcal{D}$ is nonunital symmetric monoidal. This is a combination of the fact that $\bar{\alpha}_c$ is symmetric monoidal and of Lemma 1.2.15 below. Namely, let $J_i \in fSet$. We must show that the natural map $\tilde{\alpha}_c(J_1) \otimes \tilde{\alpha}_c(J_2) \to \tilde{\alpha}_c(J_1 \sqcup J_2)$ is an isomorphism. For i = 1, 2 we have

$$\tilde{\alpha}_c(J_i) \xrightarrow{\sim} \underset{(I_i \to K_i) \in Tw(fSet), I_i \to J_i}{\operatorname{colim}} \bar{\alpha}_c(I_i \to K_i)$$

in \mathcal{D} , and

(12)
$$\tilde{\alpha}_c(J_1 \sqcup J_2) \xrightarrow{\sim} \underset{(I \to K) \in Tw(fSet), I \to J_1 \sqcup J_2}{\operatorname{colim}} \bar{\alpha}_c(I \to K)$$

So,

$$\tilde{\alpha}_{c}(J_{1}) \otimes \tilde{\alpha}_{c}(J_{2}) \xrightarrow{\sim} \underset{(I_{1} \to K_{1}) \in Tw(fSet), I_{1} \to J_{1}}{\operatorname{colim}} \bar{\alpha}_{c}(I_{1} \sqcup I_{2} \to K_{1} \sqcup K_{2})$$

We used in the above the fact that $\bar{\alpha}_c(I_1 \to K_1) \otimes \bar{\alpha}_c(I_2 \to K_2) \xrightarrow{\sim} \bar{\alpha}_c(I_1 \sqcup I_2 \to K_1 \sqcup K_2)$. The latter colimit identifies with (12) by Lemma 1.2.15.

Lemma 1.2.15. Let $J_1, J_2 \in fSet$. Consider the functor

$$\epsilon: \prod_{i=1,2} (Tw(fSet) \times_{fSet} fSet_{/J_i}) \to Tw(fSet) \times_{fSet} fSet_{/J_1 \sqcup J_2}$$

sending $(J_1 \leftarrow I_1 \rightarrow K_1), (J_2 \leftarrow I_2 \rightarrow K_2)$ to

$$(J_1 \sqcup J_2 \leftarrow I_1 \sqcup I_2 \to K_1 \sqcup K_2)$$

Here $(I_i \to K_i) \in Tw(Set)$. Then ϵ is cofinal.

Proof. We claim that ϵ has a left adjoint R given as follows. Let $(J_1 \sqcup J_2 \xleftarrow{\alpha} I \xrightarrow{\beta} K) \in Tw(fSet) \times_{fSet} fSet_{J_1 \sqcup J_2}$. Set $I_i = \alpha^{-1}(J_i)$. For i = 1, 2 consider the equivalence relation on I_i given by $x \sim y$ iff $\beta(x) = \beta(y)$. The quotient by this equivalence relation defines a surjection $\beta_i : I_i \to K_i$. In addition, we get a surjection $K_1 \sqcup K_2 \to K$. Consider a morphism in $Tw(fSet) \times_{fSet} fSet_{J_1 \sqcup J_2}$, it is given by the diagram

where I_i (resp. I'_i) is the preimage of J_i . We get natural morphisms $K'_i \to K_i$ for i = 1, 2. Indeed, fix $i \in \{1, 2\}$ and let $k', m' \in I'_i$ be such that their images in K' coincide. Pick $k, m \in I_i$ with $\gamma_i(k) = k', \gamma_i(m) = m'$. Then the images of k, m in K coincide, so k and m define the same element $x \in K_i$. We send k' and m' to x. The obtained map $I'_i \to K_i$ factors uniquely through $K'_i \to K_i$. Thus, we defined a functor R sending $(J_1 \sqcup J_2 \stackrel{\alpha}{\leftarrow} I \stackrel{\beta}{\to} K)$ to the pair $(J_1 \leftarrow I_1 \to K_1), (J_2 \leftarrow I_2 \to K_2)$. Then R is left adjoint to ϵ .

Remark 1.2.16. Trying to replace algebras by coalgebras in Proposition 1.2.14, one immediately gets the following. Let $F : \mathfrak{C} \leftrightarrows \mathfrak{D} : G$ be an adjoint pair in $1 - \mathfrak{Cat}$, where $\mathfrak{C}, \mathfrak{D} \in CAlg^{nu}(1 - \mathfrak{Cat})$, G is nonunital symmetric monoidal, so F is left-lax nonunital symmetric monoidal. Composing with F gives $F^{enh} : ComCoAlg^{nu}(\mathfrak{C}) \rightarrow$ $ComCoAlg^{nu}(\mathfrak{D})$. Composing with G gives $G^{enh} : ComCoAlg^{nu}(\mathfrak{D}) \rightarrow ComCoAlg^{nu}(\mathfrak{C})$. Then $F^{enh} : ComCoAlg^{nu}(\mathfrak{C}) \leftrightarrow ComCoAlg^{nu}(\mathfrak{D}) : G^{enh}$ is an adjoint pair.

1.3. Unital version of Justin's argument.

1.3.1. Let us try to work out a unital version of Proposition 1.2.14. Assume given an adjoint pair $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ in DGCat_{cont} , where $\mathcal{C}, \mathcal{D} \in CAlg(\mathrm{DGCat}_{cont})$, the functor G is symmetric monoidal, so that F is left-lax symmetric monoidal. The functor $G_{enh} : CAlg(\mathcal{D}) \to CAlg(\mathcal{C})$ preserves limits, so has a left adjoint G_{enh}^L , because $CAlg(\mathcal{D})$ is presentable.

Recall the equivalence $CAlg(\mathcal{C}) \xrightarrow{\sim} Fun^{\otimes}(fSet_{\emptyset}, \mathcal{C})$ given by Remark 1.2.2. As in the nonunital case, Remark 1.2.9 gives an adjoint pair

(13)
$$\mathcal{L}: \operatorname{Fun}^{llax}(fSet_{\emptyset}, \mathfrak{C}) \leftrightarrows \operatorname{Fun}^{llax}(fSet_{\emptyset}, \mathfrak{D}) : \mathfrak{R},$$

where the functors \mathcal{L}, \mathcal{R} are compositions with F and G.

1.3.2. Consider the functor $l: fSet_{\emptyset} \to Tw(fSet_{\emptyset})$ sending I to $(I \to *)$. This functor is left-lax symmetric monoidal. Here $Tw(fSet_{\emptyset})$ is equipped with the symmetric monoidal structure sending $(J \to I), (J' \to I')$ to $(J \sqcup J' \to I \sqcup I')$.

Lemma 1.3.3. For any $D \in CAlg(1-Cat)$ the composition with l yields an equivalence $\operatorname{Fun}^{\otimes}(Tw(fSet_{\mathbb{A}}), D) \xrightarrow{\sim} \operatorname{Fun}^{llax}(fSet_{\mathbb{A}}, D)$

Proof. We apply ([10], 2.2.4.9) to the symmetric monoidal category $fSet_{\emptyset}^{op}$. We claim that its symmetric monoidal envelope $Env(fSet_{\emptyset}^{op})$ identifies with $Tw(fSet_{\emptyset})^{op}$. Indeed, by ([10], 2.2.46), an object of $Env(fSet_{\emptyset}^{op})$ is a collection: $I \in fSet_{\emptyset}$ and $J_i \in fSet_{\emptyset}$ for each $i \in I$. We simply encode this as a map $J \xrightarrow{\tau} I$ in $fSet_{\emptyset}$.

A morphism from $(I, \{J_i\})$ to $(I', \{J'_{i'}\})$ in $Env(fSet_{\emptyset}^{op})$ is a collection: a map $\phi : I \to I'$ in $fSet_{\emptyset}$, and for each $i' \in I'$ a morphism $J'_{i'} \to \bigsqcup_{i \in \phi^{-1}(i')} J_i$ in $fSet_{\emptyset}$. This morphism is nothing but a diagram

$$egin{array}{ccc} J' &
ightarrow & I' \ \downarrow & & \uparrow \phi \ J &
ightarrow & I \end{array}$$

in $fSet_{\emptyset}$. Now the composition with $l: fSet_{\emptyset}^{op} \to Tw(fSet_{\emptyset})^{op}$ yields an equivalence

$$\operatorname{Fun}^{\otimes}(Tw(fSet_{\emptyset})^{op}, D^{op}) \xrightarrow{\sim} \operatorname{Fun}^{rlax}(fSet_{\emptyset}^{op}, D^{op})$$

Proposition 1.3.4. In the situation of Section 1.3.1 the diagram commutes

$$\begin{array}{ccc} CAlg(\mathfrak{C}) & \stackrel{G_{exh}^{L}}{\to} & CAlg(\mathfrak{D}) \\ & || & & || \\ \operatorname{Fun}^{\otimes}(fSet_{\emptyset}, \mathfrak{C}) & \operatorname{Fun}^{rlax}(pt, \mathfrak{D}) \\ & \downarrow^{F \circ \bullet} & \uparrow \operatorname{colim} \\ \operatorname{Fun}^{llax}(fSet_{\emptyset}, \mathfrak{D}) & \xrightarrow{\rightarrow} & \operatorname{Fun}^{\otimes}(Tw(fSet_{\emptyset}), \mathfrak{D}) \end{array}$$

Proof. Step 1 Let $c \in \mathcal{C}, d \in \mathcal{D}$ be commutative algebras in \mathcal{C} and \mathcal{D} . Write $\alpha_c : fSet_{\emptyset} \to \mathcal{C}, \alpha_d : fSet_{\emptyset} \to \mathcal{D}$ for the corresponding symmetric monoidal functors. Using (13), we get

$$\operatorname{Map}_{\operatorname{Fun}^{\otimes}(fSet_{\emptyset}, \mathcal{C})}(\alpha_{c}, G \circ \alpha_{d}) \xrightarrow{\sim} \operatorname{Map}_{\operatorname{Fun}^{llax}(fSet_{\emptyset}, \mathcal{D})}(F \circ \alpha_{c}, \alpha_{d})$$

Write $\bar{\alpha}_c, \bar{\alpha}_d$ for the images of $F \circ \alpha_c$ and α_d respectively under

$$\operatorname{Fun}^{llax}(fSet_{\emptyset}, \mathcal{D}) \xrightarrow{\sim} \operatorname{Fun}^{\otimes}(Tw(fSet_{\emptyset}), \mathcal{D})$$

Note that α_d sends $I \in fSet_{\emptyset}$ to $\underset{I}{\otimes} d$, and it send a map $I \xrightarrow{\phi} I'$ to $\underset{\phi}{\otimes} : \underset{I}{\otimes} d \xrightarrow{\phi} d$ given by the algebra structure on d.

One has $\bar{\alpha}_d = r \circ \alpha_d$ for $r: Tw(fSet_{\emptyset}) \to fSet_{\emptyset}$ sending $(J \to I)$ to J. We have an adjoint pair $l: fSet_{\emptyset} \leftrightarrows Tw(fSet_{\emptyset}) : r$, and r is symmetric monoidal. Note that the unit of $Tw(fSet_{\emptyset})$ is $(\emptyset \to \emptyset)$. Now we apply Section 1.2.10 and get an adjoint pair

$$\operatorname{Fun}^{rlax}(Tw(fSet_{\emptyset}), \mathcal{D}) \leftrightarrows \operatorname{Fun}^{rlax}(fSet_{\emptyset}, \mathcal{D})$$

where the left adjoint is the LKE along $r: Tw(fSet_{\emptyset}) \to fSet_{\emptyset}$ and the right adjoint is the restriction along r. So,

$$\operatorname{Map}_{\operatorname{Fun}^{rlax}(Tw(fSet_{\emptyset}), \mathcal{D})}(\bar{\alpha}_{c}, r \circ \alpha_{d}) \xrightarrow{\sim} \operatorname{Map}_{\operatorname{Fun}^{rlax}(fSet_{\emptyset}, \mathcal{D})}(\tilde{\alpha}_{c}, \alpha_{d}),$$

where $\tilde{\alpha}_c$ is the LKE of $\bar{\alpha}_c : Tw(fSet_{\emptyset}) \to \mathcal{D}$ along r.

The category $fSet_{\emptyset}$ has a final object pt. So, the value of $\tilde{\alpha}_c$ on pt is $\underset{(J \to I) \in Tw(fSet_{\emptyset})}{\operatorname{colim}} \bar{\alpha}_c$.

Step 2 To finish, it suffices to show that $\tilde{\alpha}_c$ is strict, that is, symmetric monoidal. Let $J_i \in fSet_{\emptyset}$. We must show that the natural map $\tilde{\alpha}_c(J_1) \otimes \tilde{\alpha}_c(J_2) \to \tilde{\alpha}_c(J_1 \sqcup J_2)$ is an isomorphism in \mathcal{D} . For i = 1, 2 we have

$$\tilde{\alpha}_c(J_i) \xrightarrow{\sim} \underset{(I_i \to K_i) \in Tw(fSet_{\emptyset}), \ I_i \to J_i}{\operatorname{colim}} \bar{\alpha}_c(I_i \to K_i)$$

in \mathcal{D} , and

$$\tilde{\alpha}_c(J_1 \sqcup J_2) \xrightarrow{\sim} \underset{(I \to K) \in Tw(fSet_{\emptyset}), \ I \to J_1 \sqcup J_2}{\operatorname{colim}} \bar{\alpha}_c(I \to K)$$

So,

$$\tilde{\alpha}_{c}(J_{1}) \otimes \tilde{\alpha}_{c}(J_{2}) \xrightarrow{\sim} \underset{\substack{(I_{1} \to K_{1}) \in Tw(fSet_{\emptyset}), I_{1} \to J_{1} \\ (I_{2} \to K_{2}) \in Tw(fSet_{\emptyset}), I_{2} \to J_{2}}}{\operatorname{colim}} \bar{\alpha}_{c}(I_{1} \sqcup I_{2} \to K_{1} \sqcup K_{2})$$

We used in the above the fact that $\bar{\alpha}_c(I_1 \to K_1) \otimes \bar{\alpha}_c(I_2 \to K_2) \xrightarrow{\sim} \bar{\alpha}_c(I_1 \sqcup I_2 \to K_1 \sqcup K_2)$. So, our claim follows from Lemma 1.3.5 below.

Lemma 1.3.5. The result of Lemma 1.2.15 remains true if we replace everywhere in its formulation fSet by $fSet_{\emptyset}$.

Proof. We construct a left adjoint R of ϵ as follows. Let $(J_1 \sqcup J_2 \stackrel{\alpha}{\leftarrow} I \stackrel{\beta}{\to} K) \in Tw(fSet_{\emptyset}) \times_{fSet_{\emptyset}} (fSet_{\emptyset})_{/J_1 \sqcup J_2}$. Set $I_i = \alpha^{-1}(J_i)$. Then R sends the above object to the pair $(J_1 \leftarrow I_1 \rightarrow K), (J_2 \leftarrow I_2 \rightarrow K)$. This is naturally a functor in the opposite direction. Then R is left adjoint to ϵ .

1.3.6. In the situation of Section 1.3.1 assume $d \in CAlg(\mathcal{D})$. Let us describe the counit of the adjunction map $G_{enh}^L(G(d)) \to d$ in \mathcal{D} . It is the map

$$\operatorname{colim}_{(J \to I) \in Tw(fSet_{\emptyset})} \underset{i \in I}{\otimes} F(G(d)^{\otimes J_i}) \to d$$

given by a compatible system of maps $\underset{i \in I}{\otimes} F(G(d)^{\otimes J_i}) \to d$ for $(J \to I) \in Tw(fSet_{\emptyset})$. The desired map is the composition

(14)
$$\underset{i \in I}{\otimes} F(G(d)^{\otimes J_i}) \to \underset{i \in I}{\otimes} F(G(d)) \to d,$$

where the first map is the tensor product over $i \in I$ of the maps $F(G(d)^{\otimes J_i}) \to F(G(d))$ obtained by applying F to the product map $G(d)^{\otimes J_i} \to G(d)$, here we use the unital commutative algebra structure on G(d). The second map in (14) comes from the counit of the adjunction $F(G(d)) \to d$ giving rise to the composition $F(G(d))^{\otimes I} \to d^{\otimes I} \to d$, where the second map is the product using the algebra structure of d.

1.4. Factorization homology functor.

1.4.1. Combining Theorem 1.1.2 and Section 1.1.1 we conclude that the composition

$$CAlg^{nu,!}(Shv(X)) \xrightarrow{\text{Fact}} CAlg^{\text{Fact}}(\text{Ran}_X) \xrightarrow{(p_{\text{Ran}})_!} CAlg^{nu}(\text{Vect})$$

is the desired functor $C_c^{\text{Fact}}(X, \bullet)$ of factorization homology.

1.4.2. Let $p: Y \to X$ be an affine scheme with a connection over X, here we are over the algebraically closed ground field of characteristic zero. This means that $p_* \mathcal{O} \in CAlg(\operatorname{QCoh}(X))$ is lifted to an object of $CAlg(\mathcal{D} - mod(X))$. Such schemes with a connection along X form a category $\mathcal{D} - \operatorname{Sch}_{/X}^{aff}$. More generally, there is a category $\mathcal{D} - \operatorname{Sch}_{/X}$ of \mathcal{D}_X -schemes which are not necessarily affine.

Then we may consider the scheme $Sect_{\nabla}(X,Y) \in Sch^{aff}$ of horizontal sections of p. It is defined by an isomorphism of functors $Sch^{aff} \to Spc$: for $S \in Sch^{aff}$,

$$\operatorname{Map}_{\operatorname{Sch}^{aff}}(S, \operatorname{Sect}_{\nabla}(X, Y)) \xrightarrow{\sim} \operatorname{Map}_{\mathcal{D}-\operatorname{Sch}^{aff}_{/X}}(S \times X, Y)$$

One checks that

$$C^{\cdot}(Sect_{\nabla}(X,Y),\mathbb{O}) \xrightarrow{\sim} C_{c}^{\operatorname{Fact}}(X,\operatorname{Fact}(p_{*}\mathbb{O}_{Y}))$$

1.4.3. *Example.* Let $p : Z \to X$ be an affine morphism. Define Sect(X, Z) by an isomorphism of functors $(Sch^{aff})^{op} \to Spc$, for $S \in Sch^{aff}$,

$$\operatorname{Map}_{\operatorname{Sch}}(S, \operatorname{Sect}(X, Z)) \xrightarrow{\sim} \operatorname{Map}_{\operatorname{Sch}_{/X}}(S \times X, Z)$$

Then we may describe $C^{\cdot}(Sect(X, Z), \mathbb{O})$.

Namely, the functor obly : $\mathcal{D} - Sch_{/X} \to Sch_{/X}$ has a right adjoint *Jets* given by the scheme of jets ([1], 2.3.2). One checks that for $x \in X$, $Jets(Z)_x \to Sect(\hat{D}_x, Z)$, here \hat{D}_x is the formal neighbourhood of x in X. Besides, *Jets* takes values in $\mathcal{D} - Sch_{/X}^{aff}$.

One has $Sect(X, Z) \xrightarrow{\sim} Sect_{\nabla}(X, Jets(Z))$ immediately. So,

$$C^{\cdot}(Sect(X,Z),\mathbb{O}) \xrightarrow{\sim} C_{c}^{Fact}(X,q_*\mathbb{O}_{Jets(Z)})$$

for the projection $q: Jets(Z) \to X$. This is an example of a local-to-global principle.

1.5. Graded version.

1.5.1. Assume X is a smooth projective curve, \mathcal{A} is a $\mathbb{Z}^{>0}$ -graded object in $CAlg^{nu,!}(Shv(X))$, that is, the product is compatible with gradings.

Let $\operatorname{Div}^{eff} = \bigsqcup_{d>0} X^{(d)}$. Then Div^{eff} is naturally a semi-group with respect to the sum of divisors. Now $Shv(\operatorname{Div}^{eff}) \xrightarrow{\sim} \prod_{d>0} Shv(X^{(d)})$ is equipped with the convolution monoidal structure. For the sum $u : \operatorname{Div}^{eff} \times \operatorname{Div}^{eff} \to \operatorname{Div}^{eff}$ we let

$$F_1 \star F_2 = u_!(F_1 \boxtimes F_2)$$

Let $\Delta: \bigsqcup_{d>0} X \to \operatorname{Div}^{eff}$ be the inclusion given by the diagonal $\Delta^d: X \hookrightarrow X^{(d)}$ for each d > 0. We view $Shv(\bigsqcup_{d>0} X) = \prod_{d>0} Shv(X) = Shv(X)^{\mathbb{Z}^{>0}}$ as the category of $\mathbb{Z}^{>0}$ -graded sheaves on X.

The functor $\Delta^!$: $(Shv(\operatorname{Div}^{eff}), \star) \to (Shv(X)^{\mathbb{Z}^{>0}}, \otimes^!)$ is nonunital symmetric monoidal. It has a fully faithful left adjoint $\Delta_!$: $Shv(X)^{\mathbb{Z}^{>0}} \to Shv(\operatorname{Div}^{eff})$, which is so left-lax nonunital symmetric monoidal. By Lemma 1.1.3, the functor

$$\Delta^{!}: CAlg^{nu,\star}(Shv(\operatorname{Div}^{eff})) \to CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}})$$

has a fully faithful left adjoint denoted Fact. Let

$$(\operatorname{Div}^{eff} \times \operatorname{Div}^{eff})_d \subset (\operatorname{Div}^{eff} \times \operatorname{Div}^{eff})$$

be the open subscheme of disjoint divisors.

Definition 1.5.2. Define $CAlg^{Fact}(Shv(\text{Div}^{eff})) \subset CAlg^{nu,\star}(Shv(\text{Div}^{eff}))$ as the full subcategory of those commutative algebras \mathcal{A} for which the induced map $\mathcal{A} \boxtimes \mathcal{A} \to u^! \mathcal{A}$ restricted to $(\text{Div}^{eff} \times \text{Div}^{eff})_d$ is an isomorphism.

Theorem 1.5.3. The functor

Fact :
$$CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}}) \to CAlg^{nu,\star}(Shv(\operatorname{Div}^{eff}))$$

is fully faithful with the essential image $CAlg^{Fact}(Shv(\text{Div}^{eff}))$.

Proof. Similar to Theorem 1.1.2.

The explicit formula for Fact is again given by Proposition 1.2.14. Namely, for $\mathcal{A} \in CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}})$ given as $\mathcal{A} = \bigoplus_{d>0} \mathcal{A}_d$, we get a functor $\theta : Tw(fSet) \to Shv(\operatorname{Div}^{eff})$ sending $(J \xrightarrow{\tau} I)$ to $\underset{i \in I}{\star} \bigtriangleup_{i \in I} (\underset{j \in J_i}{\otimes} \mathcal{A})$. Here $\underset{j \in J_i}{\otimes}$ denotes the product in the symmetric monoidal category $(Shv(X)^{\mathbb{Z}^{>0}}, \otimes^!)$. The *d*-th component of the latter is

$$\bigoplus_{\{d_j \in \mathbb{Z}^{>0}\}_{j \in J_i}, \ \sum_j d_j = d} \ \left(\underset{i \in I}{\star} \Delta_! \ \left(\underset{j \in J_i}{\otimes} \mathcal{A}_{d_j} \right) \right)$$

Given a map from $(J' \xrightarrow{\tau'} I')$ to $(J \xrightarrow{\tau} I)$ in Tw(fSet) given by (9), θ sends it to the morphism in $Shv(\text{Div}^{eff})$

$$\underset{i'\in I'}{\star} \Delta_! (\underset{j'\in J'_{i'}}{\otimes} \mathcal{A}) \to \underset{i\in I}{\star} \Delta_! (\underset{j\in J_i}{\otimes} \mathcal{A}),$$

which is the composition

$$\underset{i' \in I'}{\star} \vartriangle_! (\underset{j' \in J'_{i'}}{\otimes} \mathcal{A}) \to \underset{i' \in I'}{\star} \vartriangle_! (\underset{j \in J_{i'}}{\otimes} \mathcal{A}) \to \underset{i \in I}{\star} \vartriangle_! (\underset{j \in J_i}{\otimes} \mathcal{A})$$

where the first functor is obtained by applying $\Delta_!$ to the product maps along $J'_{i'} \to J_{i'}$, and the second one comes from the left-lax symmetric monoidal structure of $\Delta_!$.

1.5.4. Let $\operatorname{Vect}^{\mathbb{Z}^{>0}} = \prod_{d>0} \operatorname{Vect}$ be the category of $\mathbb{Z}^{>0}$ -graded objects of Vect. For the projection $p: X \to \operatorname{Spec} k$ the functors $p^!: \operatorname{Vect}^{\mathbb{Z}^{>0}} \to (\operatorname{Shv}(X)^{\mathbb{Z}^{>0}}, !)$ is nonunital symmetric monoidal, so gives rise to a functor

$$p_{CAlg}^!: CAlg^{nu}(\operatorname{Vect}^{\mathbb{Z}^{>0}}) \to CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}})$$

Its left adjoint is also called the functor $C_c^{\text{Fact}}(X, \bullet)$ of factorization homology (in the graded context).

One shows similarly that $C_c^{\text{Fact}}(X, \bullet)$ is the composition

$$CAlg^{nu,!}(Shv(X)^{\mathbb{Z}^{>0}}) \xrightarrow{\text{Fact}} CAlg^{nu,\star}(Shv(\text{Div}^{eff})) \xrightarrow{(p_{\text{Div}^{eff}})_!} CAlg^{nu}(\text{Vect}^{\mathbb{Z}^{>0}})$$

Here $p_{\text{Div}^{eff}}$: $\text{Div}^{eff} \to \text{Spec } k$ is the projection, and

$$(p_{\operatorname{Div}^{eff}})_! : (Shv(\operatorname{Div}^{eff}), \star) \to \operatorname{Vect}^{\mathbb{Z}^{>0}}$$

is nonunital symmetric monoidal.

1.6. Λ^{pos} -graded version of commutative factorization algebras.

1.6.1. Let Λ^{pos} be a semigroup isomorphic to $(\mathbb{Z}_+)^m$ for some $m \geq 1$. For $\lambda \in \Lambda$ let X^{λ} be the moduli scheme of $\Lambda^{pos} - \{0\}$ -valued divisors of degree λ . Let Conf = $\sqcup_{\lambda \in \Lambda^{pos} - 0} X^{\lambda}.$ Then Conf is a non-unital semigroup, and as above $Shv(Conf) \xrightarrow{\sim} \prod_{\lambda \in \Lambda^{pos} - 0} Shv(X^{\lambda})$

is equipped with the convolution symmetric monoidal structure denoted \star . We view $Shv(X)^{\Lambda^{pos}-0} \xrightarrow{\sim} Shv(\underset{\lambda \in \Lambda^{pos}-0}{\sqcup} X)$ as the category of $\Lambda^{pos}-0$ -graded sheaves on X. We equip it with the symmetric monoidal structure so that for $F = \bigoplus_{\lambda} F^{\lambda}, K =$

 $\oplus_{\lambda} K^{\lambda}$, the ν -component of the tensor product $F \otimes^! K$ is $\oplus_{\lambda+\mu=\nu} F^{\lambda} \otimes^! K^{\mu}$.

We have the embedding

$$\Delta \colon \bigsqcup_{\lambda \in \Lambda^{pos} - 0} X \to \bigsqcup_{\lambda \in \Lambda^{pos} - 0} X^{\lambda}$$

The functor $\triangle^!$: $Shv(Conf, \star) \to (Shv(X)^{\Lambda^{pos}-0}, \otimes^!)$ is non-unital symmetric monoidal, its left adjoint \triangle_1 is fully faithful. The corresponding functor

$$\Delta^{!}: CAlg^{nu}(Shv(Conf, \star)) \to CAlg^{nu,!}(Shv(X)^{\Lambda^{pos}-0})$$

has a fully faithful left adjoint denoted Fact. Let $u: \operatorname{Conf} \times \operatorname{Conf} \to \operatorname{Conf}$ be the sum.

Definition 1.6.2. Define $CAlg^{Fact}(Shv(Conf)) \subset CAlg^{nu}(Shv(Conf,\star))$ as the full subcategory of those commutative algebras for which the induced map $\mathcal{A} \boxtimes \mathcal{A} \to u^! \mathcal{A}$ restricted to $(\operatorname{Conf} \times \operatorname{Conf})_d$ is an isomorphism.

Theorem 1.6.3 ([5], Prop. 2.3.3). *The functor*

Fact :
$$CAlg^{nu,!}(Shv(X)^{\Lambda^{pos}-0}) \to CAlg^{nu}(Shv(\text{Conf},\star))$$

is fully faithful and its essential image is $CAlg^{Fact}(Shv(Conf))$.

Let $A = \bigoplus_{\lambda} A_{\lambda} \in CAlg^{nu,!}(Shv(X)^{\Lambda^{pos}-0})$. The explicit construction of the corresponding commutative factorization algebra is as follows. We get a functor θ : $Tw(fSet) \to Shv(\text{Conf})$ sending $(J \xrightarrow{\tau} I)$ to $\underset{i \in I}{\star} \bigtriangleup_{l} (\underset{j \in J_{i}}{\otimes} A)$, here \otimes denotes the product in the nonunital symmetric monoidal category $CAlg^{nu,!}(Shv(X)^{\Lambda^{pos}-0})$. The restriction of $\underset{i \in I}{\star} \bigtriangleup_{l} (\underset{j \in J_{i}}{\otimes} A)$ to X^{λ} is

$$\bigoplus_{\underline{\lambda}: J \to \Lambda^{pos} = 0, \sum_{j} \underline{\lambda}(j) = \lambda} \left(\underset{i \in I}{\star} \Delta_{\underline{1}} \left(\bigotimes_{j \in J_{i}}^{!} A_{\underline{\lambda}(j)} \right) \right)$$

Here we used the map

$$X^I \to X^{\lambda}, \ (x_i) \mapsto \sum_i (\sum_{j \in J_i} \underline{\lambda}(j)) x_i$$

and $\underset{i \in I}{\star} \Delta_! (\underset{j \in J_i}{\otimes} A_{\underline{\lambda}(j)})$ is the direct image of $\underset{i \in I}{\boxtimes} (\underset{j \in J_i}{\otimes} A_{\underline{\lambda}(j)})$ under the latter map.

Given a map from $(J' \xrightarrow{\tau'} I')$ to $(J \xrightarrow{\tau} I)$ in Tw(fSet) by (9), θ sends it to the morphism in Shv(Conf)

$$\underset{i'\in I'}{\star} \vartriangle_! (\underset{j'\in J'_{i'}}{\otimes} A) \to \underset{i\in I}{\star} \vartriangle_! (\underset{j\in J_i}{\otimes} A),$$

which is the composition

$$\underset{i'\in I'}{\star} \bigtriangleup_{!} (\underset{j'\in J'_{i'}}{\otimes} A) \to \underset{i'\in I'}{\star} \bigtriangleup_{!} (\underset{j\in J_{i'}}{\otimes} A) \to \underset{i\in I}{\star} \bigtriangleup_{!} (\underset{j\in J_{i}}{\otimes} A)$$

where the first functor is obtained by applying $\Delta_!$ to the product maps along $J'_{i'} \to J_{i'}$, and the second one comes from the left-lax symmetric monoidal structure of $\Delta_!$.

Finally, Fact(A) is defined as

$$\operatorname{colim}_{(J \xrightarrow{\tau} I) \in Tw(fSet)} \star_{i \in I} \Delta_! (\underset{j \in J_i}{\otimes} A)$$

The diagram commutes

$$\begin{array}{ccc} Shv(\operatorname{Conf}) & \stackrel{free}{\to} & CAlg^{nu}(Shv(\operatorname{Conf}),\star) \\ \uparrow \vartriangle_{l} & & \uparrow \operatorname{Fact} \\ Shv(X)^{\Lambda^{pos}-0} & \stackrel{free}{\to} & CAlg^{nu,!}(Shv(X)^{\Lambda^{pos}-0}) \end{array}$$

If $A = \bigoplus_{\lambda} A_{\lambda} \in Shv(X)^{\Lambda^{pos}-0}$ then free $\Delta_{!} A$ restricted to X^{λ} is the direct sum over all ways to write $\lambda = \sum_{k \in K} n_k \lambda_k$ with K finite nonempty, $\lambda_k \in \Lambda^{pos} - 0$, $n_k \geq 1$ of

$$\underset{k \in K}{\star} \operatorname{Sym}^{n_k, \star}(\Delta_! A_{\lambda_k})$$

Here $\operatorname{Sym}^{n_k,\star}(\Delta_! A_{\lambda_k}) \in Shv(X^{n_k\lambda_k}).$

1.6.4. For $\Delta: X \to X^{\lambda}$ we get $\Delta^! (\operatorname{Fact}(A)_{X^{\lambda}}) \xrightarrow{\sim} A_{\lambda}$ canonically by definition of Fact. Let Fact $(A)_{X^{\lambda}}$ be the restriction of Fact(A) to X^{λ} . For a point $D = \sum_{k} \lambda_k x_k \in X^{\lambda}$ with $x_k \neq x_{k'}$ for $k \neq k'$ and $\lambda_k \in \Lambda^{pos} - 0$ the !-fibre of Fact $(A)_{X^{\lambda}}$ at D is

 $\otimes_k A_{\lambda_k,x_k}$

Here $A_{\lambda,x}$ denotes the !-fibre of $A_{\lambda} \in Shv(X)$ at $x \in X$. More precisely, for $K \in fSet$ and $\lambda : K \to \Lambda^{pos} - 0$ with $\lambda = \sum_{k \in K} \lambda_k$ consider $h: X^K \to X^{\lambda}, (x_k) \mapsto \sum_k \lambda_k x_k$. Let $\overset{\circ}{X}^K$ be the complement to all the diagonals. Let

$$(\prod_k X^{\lambda_k})_d \subset \prod_k X^{\lambda_k}$$

be the open subscheme of those (D_k) such that for $k \neq k'$, the divisors D_k and $D_{k'}$ are disjoint. We get a diagram

with s etale. By definition, $s^!(\operatorname{Fact}(A)_{X^{\lambda}}) \xrightarrow{\sim} \boxtimes_k \operatorname{Fact}(A)_{X^{\lambda_k}}$ over $(\prod_k X^{\lambda_k})_d$, and over $\overset{\circ}{X}{}^{K}$ we have

$$f^!(\bigotimes_{k\in K} \operatorname{Fact}(A)_{X^{\lambda_k}}) \xrightarrow{\sim} \bigotimes_{k\in K} A_{\lambda_k}$$

1.6.5. Let $A, B \in CAlg^{nu,!}(Shv(X)^{\Lambda^{pos}-0})$. Then $A \otimes B \in CAlg^{nu,!}(Shv(X)^{\Lambda^{pos}-0})$, here \otimes denotes the tensor product in the latter non-unital symmetric monoidal category. Note that for $\lambda \in \Lambda^{pos} - 0$, $(A \otimes B)_{\lambda} = \bigoplus_{\lambda_1 + \lambda_2 = \lambda} A_{\lambda_1} \otimes B_{\lambda_2}$.

Recall a general fact that for a pair of not necessarily commutative factorization algebras $\mathcal{F}, \mathcal{F}' \in Alg^{\text{Fact}}(\text{Conf}), \mathcal{F} \star \mathcal{F}'$ is naturally an object of $Alg^{\text{Fact}}(\text{Conf})$. Here $Alg^{\text{Fact}}(\text{Conf})$ is the category of factorization algebras on Conf.

It is easy to see that $\operatorname{Fact}(A \otimes B) \xrightarrow{\sim} \operatorname{Fact}(A) \star \operatorname{Fact}(B)$ on Conf. That is, for $\lambda \in$ $\Lambda^{pos} - 0,$

$$\operatorname{Fact}(A \otimes B)_{\lambda} \xrightarrow{\sim} \bigoplus_{\lambda_1 + \lambda_2 = \lambda} \operatorname{Fact}(A)_{X^{\lambda_1}} \star \operatorname{Fact}(B)_{X^{\lambda_2}}$$

Here $\operatorname{Fact}(A)_{X^{\lambda_1}}$ denotes the restriction of $\operatorname{Fact}(A)$ to X^{λ_1} . So, the functor Fact of Theorem 1.6.3 is non-unital symmetric monoidal.

1.6.6. If A is a non-unital coalgebra in $CAlg^{nu}(Shv(X)^{\Lambda^{pos}-0}, \otimes^!)$ then the comultiplication $A \to A \otimes A$ is a map in $CAlg^{nu,!}(Shv(X)^{\Lambda^{pos}-0})$, so gives a morphism $\operatorname{Fact}(A) \to \operatorname{Fact}(A \otimes A) \xrightarrow{\sim} \operatorname{Fact}(A) \star \operatorname{Fact}(A) \text{ in } CAlg^{\operatorname{Fact}}(Shv(\operatorname{Conf})).$

More precisely, Fact induces a functor

$$CoAlg^{nu}(CAlg^{nu}(Shv(X)^{\Lambda^{pos}-0},\otimes^!)) \to CoAlg^{nu}(CAlg^{nu}(Shv(Conf,\star)))$$

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1.6.7. Let us try the following calculation (in the constructible context). Let $B = \bigoplus B_{\lambda}$ be a $(\Lambda^{pos} - 0)$ -graded non-unital commutative algebra in Vect. Set $A_{\lambda} = B_{\lambda} \otimes \omega$ on X for each λ , and $A = \bigoplus_{\lambda} A_{\lambda}$.

Pick $\lambda \in \Lambda^{pos} - 0$. Let $\Delta: X \to X^{\lambda}$ be the natural map.

Question: calculate the *-restriction \triangle^* (Fact $(A)_{X^{\lambda}}$) to X.

The functor $\triangle^* \theta : Tw(fSet) \to Shv(X)$ sends $(J \xrightarrow{\tau} I) \in Tw(fSet)$ to

$$\bigoplus_{\underline{\lambda}: J \to \Lambda^{pos} - 0, \sum_{j} \underline{\lambda}(j) = \lambda} \left(\bigotimes_{j \in J} B_{\underline{\lambda}(j)} \right) [2 \mid I \mid] = (\bigotimes_{j \in J} B)_{\lambda} [2 \mid I \mid]$$

here $e[2 \mid I \mid] = (\triangle^I)^* \omega_{X^I}$ for $\triangle^I \colon X \to X^I$. It sends the map (9) to the composition

$$(\underset{j\in J'}{\otimes} B)_{\lambda}[2\mid I'\mid] \to (\underset{j\in J}{\otimes} B)_{\lambda}[2\mid I'\mid] \stackrel{\mathrm{id}\,\otimes\epsilon}{\to} (\underset{j\in J}{\otimes} B)_{\lambda}[2\mid I\mid],$$

where the first map comes from the product in the algebra B, and the second map is obtained from $\epsilon : e[2 \mid I' \mid] \to e[2 \mid I \mid]$. We have denoted by ϵ here the morphism obtained from $i_{I}\omega_{X^{I'}} \to \omega_{X^{I}}$ by applying the functor $(\triangle^{I})^*$. Here $i: X^{I'} \to X^{I}$ is the diagonal attached to $\phi: I \to I'$. The answer is not clear in general.

1.7. Generalization of Proposition 1.2.14.

1.7.1. Let $\mathcal{C}, \mathcal{D} \in CAlq^{nu}(1-\operatorname{Cat})$, we assume C, D admit all small limits and colimits. Let us be given an adjoint pair $F: \mathfrak{C} \hookrightarrow \mathfrak{D}: G$ in $1-\mathfrak{C}$ at such that G is nonunital symmetric monoidal, so F is left-lax nonunital symmetric monoidal. Let G_{enh} : $CAlg^{nu}(\mathcal{D}) \to CAlg^{nu}(\mathcal{C})$ be the functor obtained from G, so $oblv \circ G_{enh} \xrightarrow{\sim} G \circ oblv$ for obly : $CAlg^{nu}(\mathcal{D}) \to \mathcal{D}$, obly : $CAlg^{nu}(\mathcal{C}) \to \mathcal{C}$.

We difference with Proposition 1.2.14 is that we do not assume that the tensor product in \mathcal{D} preserves colimits separately in each variable!

Conjecture 1.7.2. In the situation of Section 1.7.1 the following holds.

i) The functor colim : $\operatorname{Fun}^{rlax}(Tw(fSet), \mathcal{D}) \to \mathcal{D}$ upgrades naturally to a functor colim : Fun^{*rlax*}($Tw(fSet), \mathcal{D}$) \rightarrow Fun^{*rlax*}(pt, \mathcal{D}). *ii)* There is a left adjoint G_{enh}^L : $CAlg^{nu}(\mathcal{C}) \rightarrow CAlg^{nu}(\mathcal{D})$, which fits into the commu-

tative diagram (10).

1.7.3. To extend the proof of our Proposition 1.2.14 to this case, we should, I think, solve the following question. Let $\mathfrak{I}, \mathfrak{J} \in 1-\mathfrak{C}$ at be small nonuinital symmetric monioidal categories, $f : \mathcal{I} \to \mathcal{J}$ be a nonunital symmetric monoidal functor. Take $\mathcal{O}^{\otimes} =$ Suri, the ∞ -operad defined in Section 1.2.1. Then we have the ∞ -operads Fun^O $(\mathfrak{I}, \mathcal{D})^{\otimes}$, $\operatorname{Fun}^{\mathcal{O}}(\mathcal{J}, \mathcal{D})^{\otimes}$ defined in ([10], Construction 2.2.6.7). The functor f yields by functoriality a natural map

$$e_f: \operatorname{Fun}^{\mathcal{O}}(\mathcal{J}, \mathcal{D})^{\otimes} \to \operatorname{Fun}^{\mathcal{O}}(\mathcal{I}, \mathcal{D})^{\otimes}$$

of ∞ -operads. Its fibre over $\langle 1 \rangle \in \text{Surj}$ is $\overline{f} : \text{Fun}(\mathcal{J}, \mathcal{D}) \to \text{Fun}(\mathcal{I}, \mathcal{D})$, the composition with f. The functor \overline{f} has a left adjoint \overline{f}^L : Fun $(\mathfrak{I}, \mathcal{D}) \to$ Fun $(\mathfrak{J}, \mathcal{D})$ given by the LKE along f. We need to extend \bar{f}^L to a map of ∞ -operads $\operatorname{Fun}^{\mathcal{O}}(\mathcal{I}, \mathcal{D})^{\otimes} \to \operatorname{Fun}^{\mathcal{O}}(\mathcal{J}, \mathcal{D})^{\otimes}$ I think.

I think unfortunately, this does not work, then so obtained functor \bar{f}^L is only nonunital left-lax symmetric monoidal (it is not strict). So, I don't believe in the above Conjecture.

1.8. Λ^{pos} -graded version of cocommutative factorization coalgebras.

1.8.1. For this section we work in the constructible context.¹ Let Λ^{pos} , X^{λ} , Conf, Δ, u be as in Section 1.6.1. We consider the symmetric monoidal category $(Shv(Conf), \star)$ as in Section 1.6.1.

View now $Sh\nu(X)^{\Lambda^{pos}-0}$ as equipped with the symmetric monoidal structure so that for $F = \bigoplus_{\lambda} F^{\lambda}, K = \bigoplus_{\lambda} K^{\lambda}$, the ν -component of $F \otimes K$ is $\bigoplus_{\lambda+\mu=\nu} F^{\lambda} \otimes K^{\mu}$. So, we replaced \otimes ! by \otimes .

The functor Δ^* : $Shv(Conf, \star) \to Shv(X)^{\Lambda^{pos}-0}$ is non-unital symmetric monoidal, its right adjoint Δ_* is fully faithful and non-unital right-lax symmetric monoidal. The functor Δ^* induces a functor

(15)
$$\triangle^*: ComCoAlg^{nu}(Shv(Conf, \star)) \rightarrow ComCoAlg^{nu,*}(Shv(X)^{\Lambda^{pos}-0})$$

In the RHS the * refers to the fact that we use the \otimes -monoidal structure. By Lemma 1.1.4, (15) admits a right adjoint.

Definition 1.8.2. Let $ComCoAlg^{Fact}(Shv(Conf)) \subset ComCoAlg(Shv(Conf, \star))$ be the full subcategory of those \mathcal{A} for which the induced map $u^*\mathcal{A} \to \mathcal{A} \boxtimes \mathcal{A}$ is an isomorphism over $(Conf \times Conf)_d$. We used that u is proper and of finite type.

Remark 1.8.3. The map $u : (Conf \times Conf)_d \to Conf$ is etale, so that u^* over this locus exists even if we are not in the constructible context.

Question. Is it true that the right adjoint to (15) is fully faithful?

1.8.4. Let $A = \bigoplus_{\lambda} A_{\lambda} \in ComCoAlg^{nu,*}(Shv(X)^{\Lambda^{pos}-0})$. The construction of Fact(A) should go as follows. We get a functor $\theta : Tw(fSet)^{op} \to Shv(Conf)$ sending $(J \xrightarrow{\tau} I)$ to $\underset{i \in I}{\star} \Delta_* (\underset{j \in J_i}{\otimes} A)$, here \otimes denotes the product in the non-unital symmetric monoidal category $(Shv(X)^{\Lambda^{pos}-0}, \otimes)$. The restriction of the latter object to X^{λ} is

$$\underset{\underline{\lambda}:J\to\Lambda^{pos}-0,\ \sum_{j}\underline{\lambda}(j)=\lambda}{\oplus} \left(\underset{i\in I}{\star}\ \Delta_{!}\ (\underset{j\in J_{i}}{\otimes}\ A_{\underline{\lambda}(j)})\right)$$

Here we used the map

$$X^I \to X^{\lambda}, \ (x_i) \mapsto \sum_i (\sum_{j \in J_i} \underline{\lambda}(j)) x_i$$

and $\underset{i \in I}{\star} \bigtriangleup_! (\underset{j \in J_i}{\otimes} A_{\underline{\lambda}(j)}) \text{ is the direct image of } \underset{i \in I}{\boxtimes} (\underset{j \in J_i}{\otimes} A_{\underline{\lambda}(j)}) \text{ under the latter map.}$

Given a map from $(J' \xrightarrow{\tau'} I')$ to $(J \xrightarrow{\tau} I)$ in Tw(fSet) by (9), θ sends it to the morphism in Shv(Conf)

$$\underset{i \in I}{\star} \Delta_! \left(\underset{j \in J_i}{\otimes} A \right) \to \underset{i' \in I'}{\star} \Delta_! \left(\underset{j' \in J'_{i'}}{\otimes} A \right)$$

¹There should be a version for \mathcal{D} -modules but with imposing strong assumptions like all the sheaves are compact, I have not thought about this case.

which is the composition

$$\underset{i\in I}{\star} \vartriangle_! (\underset{j\in J_i}{\otimes} A) \to \underset{i'\in I'}{\star} \bigtriangleup_! (\underset{j\in J_{i'}}{\otimes} A) \to \underset{i'\in I'}{\star} \bigtriangleup_! (\underset{j'\in J_{i'}'}{\otimes} A)$$

Here the second map is the (*-product over I') of the coproduct maps $\underset{j\in J_{i'}}{\otimes} A \to \underset{j'\in J'_{i'}}{\otimes} A$ along surjections $J'_{i'} \to J_{i'}$, and the first map comes from the right-lax structure on Δ_* . Here $\Delta^{(I/I')}: X^{I'} \to X^I$, and we used that $(\Delta^{(I/I')})^*(\underset{i\in I}{\otimes} A^{\otimes J_i}) \xrightarrow{\cong} \underset{i'\in I'}{\otimes} A^{\otimes J_{i'}}$.

The right-lax structure above was used to get maps

$$\underset{i \in I_{i'}}{\star} \vartriangle_! (\underset{j \in J_i}{\otimes} A) \to \vartriangle_! (\underset{i \in I_{i'}}{\otimes} \underset{j \in J_i}{\otimes} A)$$

for each $i' \in I'$, and further take $\underset{i' \in I'}{\star}$ of these maps.

Now define

$$\operatorname{Fact}(A) = \lim_{Tw(fSet)^{op}} \theta$$

in Shv(Conf).

to

1.9. Ran-version of commutative factorization categories attached to coalgebras.

1.9.1. For this subsection we work with any sheaf theory. Let $A \in ComCoAlg^{nu}(\mathrm{DGCat}_{cont})$. In ([6], 2.5.5) Dennis proposed the following construction. Consider the functor Tw(A): $Tw(fSet)^{op} \to \mathrm{DGCat}_{cont}$ sending $(J \xrightarrow{\phi} I)$ to $A^{\otimes J} \otimes Shv(X^{I})$. It sends the map (9)

$$A^{\otimes J} \otimes Shv(X^{I}) \to A^{\otimes J'} \otimes Shv(X^{I'})$$

which is the tensor product of $\triangle^!$: $Shv(X^I) \to Shv(X^{I'})$ for $\triangle: X^{I'} \to X^I$ with the product map $A^{\otimes J} \to A^{\otimes J'}$. Then $\lim_{Tw(fSet)^{op}} Tw(A)$ in DGCat_{cont}. Note that this limit can also be calculated in Shv(Ran) - mod naturally.

1.10. Ran-version for commutative factorization categories attached to algebras.

1.10.1. For this subsection we work with any sheaf theory. Let $A \in CAlg^{nu}(\text{DGCat}_{cont})$. In ([6], 2.5.1) Dennis proposed the following construction. Let $Tw(A) : Tw(fSet) \to$ DGCat_{cont} be the functor sending $(J \xrightarrow{\phi} I)$ to $A^{\otimes J} \otimes Shv(X^{I})$. It sends the map (9) to

$$A^{\otimes J'} \otimes Shv(X^{I'}) \to A^{\otimes J} \otimes Shv(X^{I})$$

which is the tensor product of Δ_* : $Shv(X^{I'}) \to Shv(X^I)$ for Δ : $X^{I'} \to X^I$ with the product map $A^{\otimes J'} \to A^{\otimes J}$ along $J' \to J$. Then we may consider $\underset{Tw(fSet)}{\operatorname{colim}} Tw(A)$, this is the category of global sections over Ran of a suitable sheaf of categories attached to A as in ([8], 8.1), see my file ([13], Section 3.3).

1.11. Factorizable sheaves.

1.11.1. Let be X is a smooth curve. We may define $Shv(\operatorname{Ran}_X)^{\operatorname{Fact}}$ similarly to [13].

2. Non-commutative version

2.0.1. Let $D \in Alg^{nu}(\mathrm{DGCat}_{cont})$. Then $Alg^{nu}(D)$ is presentable. This is obtained from the following general fact: let C be a presentable ∞ -category, \mathcal{A} a monad on C such that the undelying functor $C \to C$ is accessible. Then $\mathcal{A} - mod(M)$ is presentable. In our case $Alg^{nu}(D)$ admits a sifted colimits by (HA, 3.2.3.1), and obly : $Alg^{nu}(D) \to D$ preserves sifted colimits and is conservative by (HA, 3.2.2.6). Thus, obly : $Alg^{nu}(D) \to D$ D is monadic, and the corresponding monad is continuous.

2.0.2. Let $C, D \in Alg^{nu}(\text{DGCat}_{cont})$. Let $l : C \leftrightarrows D : r$ be an adjoint pair in DGCat_{cont} , where r is non-unital monoidal, so l is left-lax non-unital monoidal. Write $r^{Alg} : Alg^{nu}(D) \to Alg^{nu}(C)$ for the induced functor, it preserves limits. Besides, $Alg^{nu}(C), Alg^{nu}(D)$ are presentable, so r^{Alg} admits a left adjoint l^{Alg} .

Write $free: C \to Alg^{nu}(C)$ for the left adjoint to obly $: Alg^{nu}(C) \to C$. Note that $l^{Alg} \circ free \xrightarrow{\sim} free \circ l$.

Lemma 2.0.3. Assume $l: C \to D$ fully faithful. Then l^{Alg} is also fully faithful.

Proof. The proof of Lemma 1.1.3 immediately generalizes to this case. Note that $\operatorname{Alg}^{nu}(C)$ is generated under geometric realization by free algebras. By a free algebra we mean an object of $\operatorname{Alg}^{nu}(C)$ lying in the essential image of $free: C \to \operatorname{Alg}^{nu}(C)$, see (HA, 3.2.3.3).

2.1. Example of the affine grassmanian.

2.1.1. Let G be split reductive, consider the affine grassmanian $\operatorname{Gr}_{G,\operatorname{Ran}}$. This is a factorization prestack over Ran. For $S \in \operatorname{Sch}_{ft}^{aff}$ given an S-point \mathcal{I} of Ran let $\hat{\mathcal{D}}_I$ denote the formal scheme obtained as the formal completion of $S \times X$ along $\Gamma_{\mathcal{I}}$, here $\Gamma_{\mathcal{I}}$ is the union of the graphs Γ_i for $i \in \mathcal{I}$. Let $\mathcal{D}_{\mathcal{I}}$ be the affine scheme attached $\hat{\mathcal{D}}_{\mathcal{I}}$ (by passing to the colimit inside affine schemes instead of prestacks). Write also $\hat{\mathcal{D}}_{\mathcal{I}} = \mathcal{D}_{\mathcal{I}} - \Gamma_{\mathcal{I}}$.

Let $\mathfrak{L}^+(G)_{\operatorname{Ran}}$ be the group scheme over Ran defined as follows. Given an S-point \mathfrak{I} of Ran, its lifting to an S-point of $\mathfrak{L}^+(G)_{\operatorname{Ran}}$ is a section $\mathcal{D}_{\mathfrak{I}} \to G$.

Let $\operatorname{Hecke}_{G}^{loc}$ be the stack over Ran classifying collections: $\mathfrak{I} \in \operatorname{Ran}$, G-torsors $\mathfrak{F}_{G}, \mathfrak{F}_{G}'$ on $\mathcal{D}_{\mathfrak{I}}$, and an isomorphism $\beta : \mathfrak{F} \xrightarrow{\sim} \mathfrak{F}_{G}'$ over $\overset{\circ}{\mathcal{D}}_{\mathfrak{I}}$. The stack quotient $\operatorname{Gr}_{G,\operatorname{Ran}}/\mathcal{L}^{+}(G)_{\operatorname{Ran}}$ (in the étale topology) identifies with $\operatorname{Hecke}_{G}^{loc}$.

2.1.2. Now $\operatorname{Hecke}_{G}^{loc}$ is naturally a factorization prestack over Ran. It is also an associative algebra in $\operatorname{PreStk}_{corr}$. Namely, for a linearly ordered set $I = \{1, \ldots, n\}$ the product is given by the diagram

$$(\operatorname{Hecke}_{G}^{loc})^{n} \stackrel{a}{\leftarrow} \operatorname{Conv}_{I} \stackrel{b}{\to} \operatorname{Hecke}_{G}^{loc}$$

Here Conv_I is the stack classifying $\mathfrak{I}_1, \ldots, \mathfrak{I}_n \in \operatorname{Ran}$, G-torsors $\mathfrak{F}_G^1, \ldots, \mathfrak{F}_G^{n+1}$ on $\mathfrak{D}_{\cup_i \mathfrak{I}_i}$ together with isomorphisms

$$\beta_1: \mathfrak{F}^1_G \widetilde{\to} \mathfrak{F}^2_G \mid_{\mathfrak{D}_{\cup_i \mathfrak{I}_i} - \Gamma_{\mathfrak{I}_1}}, \ \dots, \ \beta_n: \mathfrak{F}^n_G \widetilde{\to} \mathfrak{F}^{n+1}_G \mid_{\mathfrak{D}_{\cup_i \mathfrak{I}_i} - \Gamma_{\mathfrak{I}_n}}$$

The map a sends this collection to

$$(\mathfrak{F}^1_G,\mathfrak{F}^2_G,eta_1)\mid_{\mathfrak{D}_{\mathfrak{I}_1}},\ldots,(\mathfrak{F}^n_G,\mathfrak{F}^{n+1}_G,\mid_{\mathfrak{D}_{\mathfrak{I}_n}})\mid_{\mathfrak{D}_{\mathfrak{I}_n}}$$

The map b sends this point to $(\bigcup_{i=1}^{n} \mathfrak{I}_i, \beta, \mathfrak{F}_G^1, \mathfrak{F}_G^n)$, where β is the composition $\beta_n \circ \ldots \circ \beta_1$.

Though $\operatorname{Hecke}_{G}^{loc}$ is not locally of finite type, we may still define the DG-category $Shv(\operatorname{Hecke}_{G}^{loc})$ in our constructible context via the usual way. Moreover, $Shv(\operatorname{Hecke}_{G}^{loc})$ becomes a non-unital monoidal category via the convolution \star . Namely, given $K_i \in Shv(\operatorname{Hecke}_{G}^{loc})$, their convolution $K_1 \star \ldots \star K_n$ is defined as

$$K_1 \star \ldots \star K_n = b_* a^! (K_1 \boxtimes \ldots \boxtimes K_n)$$

Here the map b is pseudo-proper, so we should have $b_* \xrightarrow{\sim} b_!$ naturally. Write $Shv(\operatorname{Hecke}_G^{loc}, *) \in Alg^{nu}(\operatorname{DGCat}_{cont})$ for this monoidal category.

2.1.3. We view Ran as an object of Alg^{nu} (PreStk). The inclusion PreStk \subset PreStk_{corr} allows to view it also as an object of Alg^{nu} (PreStk_{corr}). Then the projection Hecke^{loc}_G \rightarrow Ran is a map in Alg^{nu} (PreStk_{corr}). In particular, for the linearly ordered set $I = \{1, \ldots, n\}$ the above diagram fits into a commutative diagram

$$\begin{array}{ccccccc} (\operatorname{Hecke}_{G}^{loc})^{n} & \stackrel{a}{\leftarrow} & \operatorname{Conv}_{I} & \stackrel{b}{\rightarrow} & \operatorname{Hecke}_{G}^{loc} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Ran}^{I} & \stackrel{\operatorname{id}}{\leftarrow} & \operatorname{Ran}^{I} & \stackrel{u}{\rightarrow} & \operatorname{Ran}, \end{array}$$

where u is the union operation on Ran.

2.1.4. Let $A^{\otimes} : \mathbf{\Delta}^{op} \to 1 - \mathbb{C}$ at be a monoidal ∞ -category, $\mathcal{A} = A^{\otimes}([1])$. Let $\tilde{\mathcal{A}} \to \mathbf{\Delta}^{op}$ be the corresponding cocartesian fibration. Let $\mathbf{\Delta}_s \subset \mathbf{\Delta}$ be the subcategory with the same objects, where we keep only injective morphisms $[n] \to [m]$. The category $Alg^{nu}(\mathcal{A})$ of non-unital associative algebras in a monoidal category \mathcal{A} is controled according to ([10], 5.4.3.3) as the full subcategory $Alg^{nu}(\mathcal{A}) \subset \operatorname{Funct}_{\mathbf{\Delta}^{op}}((\mathbf{\Delta}_s)^{op}, \tilde{\mathcal{A}})$ spanned by functors F that send morphisms of the form $[1] \to [n], 0 \mapsto i, 1 \mapsto i+1$ to a cocartesian arrow.

2.1.5. Let $\operatorname{Hecke}_{G,X}^{loc} = \operatorname{Hecke}_{G}^{loc} \times_{\operatorname{Ran}} X$. We may view $Shv(\operatorname{Hecke}_{G,X}^{loc})$ also as an object of $Alg^{nu}(Shv(X, \otimes^!) - mod)$ via convolution.

Namely, the non-unital monoidal structure on $\operatorname{Hecke}_{G}^{loc}$ is given by a functor $f_{\mathcal{H}}$: $\boldsymbol{\Delta}_{s}^{op} \to \operatorname{PreStk}_{corr}, [n] \mapsto (\operatorname{Hecke}_{G}^{loc})^{n}$, and $\phi : [n] \to [m]$ in $\boldsymbol{\Delta}$ goes to some diagram

$$(\operatorname{Hecke}_{G}^{loc})^{m} \leftarrow \operatorname{Conv}_{\phi} \rightarrow (\operatorname{Hecke}_{G}^{loc})^{n}$$

We get the functor $f_{\mathcal{H},X}: \mathbf{\Delta}^{op} \to (\operatorname{PreStk}_{/X})_{corr}$ sending [n] to

$$(\operatorname{Hecke}_{G}^{loc})^{m} \times_{\operatorname{Ran}^{m}} X = \prod_{\substack{/X\\1 \le i \le m}} \operatorname{Hecke}_{G,X}^{loc},$$

and $\phi: [n] \to [m]$ in $\boldsymbol{\Delta}$ to

 $(\operatorname{Hecke}_{G}^{loc})^m \times_{\operatorname{Ran}^m} X \leftarrow \operatorname{Conv}_{\phi} \times_{\operatorname{Ran}^I} X \to (\operatorname{Hecke}_{G}^{loc})^n \times_{\operatorname{Ran}^n} X$

Now $f_{\mathcal{H},X}$ is an object of $Alg^{nu}((\operatorname{PreStk}_{/X})_{corr})$. Applying the functor Shv, this gives on $Shv(\operatorname{Hecke}_{G,X}^{loc})$ the structure of an object of $Alg^{nu}(Shv(X, \otimes^!) - mod)$.

2.1.6. The functor obly : $Shv(X, \otimes^!) - mod \to DGCat_{cont}$ is right-lax symmetric monoidal, so on $Shv(\operatorname{Hecke}_{G,X}^{loc})$ we get a structure of an object of $Alg^{nu}(DGCat_{cont})$. We denote this convolution by *. For the linearly ordered set $I = \{1, \ldots, n\}$ consider the diagram

$$(\operatorname{Hecke}_{G,X}^{loc})^n \xleftarrow{q} (\operatorname{Hecke}_{G}^{loc})^n \times_{\operatorname{Ran}^n} X \xleftarrow{a_X} \operatorname{Conv}_{I,X} \xrightarrow{b_X} \operatorname{Hecke}_{G,X}^{loc},$$

where we denoted $\operatorname{Conv}_{I,X} := \operatorname{Conv}_I \times_{\operatorname{Ran}^I} X$, and a_X, b_X are obtained from a, b by the base change. The convolution product of $K_i \in Shv(\operatorname{Hecke}_{G,X}^{loc})$ is given as

$$K_1 * \ldots * K_n = (b_X)_* a_X^! q^! (K_1 \boxtimes \ldots \boxtimes K_n) \in Shv(\operatorname{Hecke}_{G,X}^{loc})$$

2.1.7. Let $i: \operatorname{Hecke}_{G,X}^{loc} \to \operatorname{Hecke}_{G}^{loc}$ be the natural map. Then $r = i^!: Shv(\operatorname{Hecke}_{G}^{loc}, \star) \to Shv(\operatorname{Hecke}_{G,X}^{loc}, \star)$ is non-unital monoidal. So, induces a functor

$$r^{Alg}: Alg^{nu}(Shv(\operatorname{Hecke}_{G}^{loc}, \star)) \to Alg^{nu}(Shv(\operatorname{Hecke}_{G,X}^{loc}, \star))$$

APPENDIX A. SOME REMARKS

A.0.1. Recall the adjoint pair $l : fSet \leftrightarrows Tw(fSet) : r$ from Section 1.0.13. Since $r : Tw(fSet) \rightarrow fSet$ is cofinal, r induces an isomorphism $| Tw(fSet) | \cong | fSet |$. Since fSet has a final object, $| fSet | \cong *$, so Tw(fSet) is contractible.

Consider the functor

$$q: Tw(fSet) \to fSet^{op}, \ (J \to K) \mapsto K$$

We claim that q is cofinal. Indeed, let $K \in fSet$. We show that the category $Tw(fSet) \times_{fSet^{op}} (fSet^{op})_{K/}$ is contractible. Its objects are collections $(J_1 \to K_1 \to K)$ with $(J_1 \to K_1) \in Tw(fSet)$. Taking the fibres $(J_1)_k \to (K_1)_k$ for each $k \in K$, one gets an isomorphism $Tw(fSet) \times_{fSet^{op}} (fSet^{op})_{K/} \xrightarrow{\sim} \prod_{k \in K} Tw(fSet)$. By the above, the latter category is contractible.

A.0.2. As an application, for $A = \omega_X \in Shv(X)$ the corresponding commutative factorization algebra on Ran is

$$\operatorname{colim}_{(J \to K) \in Tw(fSet)} \Delta^K_! \, \omega_{X^K}$$

for $\triangle^K \colon X^K \to \text{Ran}$. This is the colimit of the composition $Tw(fSet) \xrightarrow{q} fSet^{op} \to Shv(\text{Ran})$, so rewrites as

$$\operatorname{colim}_{K \in fSet^{op}} \omega_{X^K} \xrightarrow{\sim} \omega_{\operatorname{Ran}},$$

because $Shv(\operatorname{Ran}) \xrightarrow{\sim} \operatorname{colim}_{fSet^{op}} Shv(X^{I})$ in DGCat_{cont}.

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