1. Comments to: Gaitsgory, Lysenko, Metaplectic Whittaker category and quantum groups: the "small" FLE (version April 21, 2020)

1.1. For 0.1.3. The normalization for the Kac-Moody extension not precised. Let take for the corresponding 2-cocycle the map $x \otimes f(t), y \otimes g(t) \mapsto -(x, y)_\kappa \operatorname{Res}_{t=0} f dg$ for $x, y \in \mathfrak{g}$, $f, g \in k((t))$ from [18].

1.1.1. For (0.20). The setting for existence of such functor could be as follows. Let $f : A \to B$ be a map in DGCat_{cont}, H an affine algebraic group, $T_H \subset H$ a closed subgroup. We get the objects in $CAlg(\mathrm{DGCat}_{cont})$, $\mathrm{Rep} H = \mathrm{QCoh}(B(H))$, same for T_H. The map $B(T_H) \to B(H)$ yields a symmetric monoidal functor $\mathrm{Rep} H \to \mathrm{Rep} T_H$. Assume A is a left $\mathrm{Rep}(H)$-module, B is a left $\mathrm{Rep}(T_H)$-module, and the functor f is a morphism of $\mathrm{Rep}(H)$-modules. Then the functor $\mathrm{Rep}(T_H) \times A \to B$, $(V, a) \mapsto V * f(a)$ extends to a functor $\mathrm{Rep}(T_H) \otimes_{\mathrm{Rep}(H)} A \to B$ by bilinearity. The latter functor is $\mathrm{Rep}(T_H)$-linear. My understanding is that in these terms the induction functor $A \to \mathrm{Rep}(T_H) \otimes_{\mathrm{Rep}(H)} A$ sends a to $e \boxtimes a$. Here e is the trivial T_H-module. We write e for the base field of coefficients.

Question: is the functor $\mathfrak{J}^{Quant}_{!*}$ lax braided monoidal?

1.1.2. For Rem. 0.3.11. Learn the definition of local finiteness in the definition of $\mathrm{Rep}_q(\check{G})$ and $\mathrm{Rep}_q^{mxd}(\check{G})$.

The relation between Ω_q^{small} and $\Omega_q^{Lus} \in \mathrm{Rep}_q(\check{T})$?

1.1.3. What is Lurie's equivalence between $\mathbb{E}_2$-algebras and factorization algebras mentioned in 0.5.5?

1.1.4. What is the sense of the objects N^λ defined in ([44], formula (29)) in terms of $\overset{\bullet}{u}_q(\check{G}) - mod$?

1.1.5. What is "the full force of the Drinfeld-Plucker formalism" mentioned in Sect. 0.7.8?

1.1.6. For 0.8.10. By $C^{\cdot}$ he means the functor $\mathrm{R}\Gamma(Y, \cdot)$. By $\mathcal{H}om$ he means the functor $\mathrm{R}\Gamma\, \mathcal{H}om$.

1.1.7. If G is a finite abelian group let $\alpha : B(G) \to pt$ be the projection. Then $\alpha_* : Shv(B(G)) \to \mathrm{Vect}$ admits a continuous right adjoint, namely, $= \alpha^!$ is this right adjoint.

Assume now G is a torsion abelian group. Write $G_n = \{g \in G \mid g^n = 1\}$. By definition, $\alpha_* : Shv(B(G)) \to \mathrm{Vect}$ comes from the compatible system of functors $(\alpha_n)_* : Shv(B(G_n)) \to \mathrm{Vect}$ by passing to the colimit over $n \in \mathbb{N}$ with the divisibility relation, here $\alpha_n : B(G_n) \to pt$. In particular α_* is continuous. Then $\alpha_* : Shv(B(G)) \to \mathrm{Vect}$ also admit the continuous right adjoint $\alpha^!$. Indeed, we have $B(G) \widetilde{\to} \operatorname{colim}_{n \in \mathbb{N}} B(G_n)$ with respect to the divisibility relation on $\mathbb{N}$, and

$$Shv(B(G)) \widetilde{\to} \lim_{n \in \mathbb{N}^{op}} Shv(G_n) \widetilde{\to} \operatorname{colim}_{n \in \mathbb{N}} Shv(B(G_n)),$$

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as we may pass to left adjoints in the limit system $\lim_{n\in\mathbb{N}^{op}} Shv(G_n)$. Now the functor α_* is obtained by passing to the colimit over $n\in\mathbb{N}$ in the functors $(\alpha_n)_* : Shv(B(G_n))\to$ Vect. Our claim follows now from ([43], end of Section 9.2.6). We may also note that α is pseudo-proper, so $\alpha_!$ is defined for all the 4 sheaf theories by [21].

We claim also that $\alpha^!$ is left adjoint to α_*????? Not clear.

1.1.8. In 0.8.11 It is essential that gerbes are of finite order! Indeed, for finite groups A, B we have $Shv(B_{et}(A))\otimes Shv(B_{et}(B))\widetilde{\to} Shv(B_{et}(A\times B))$. In fact, this also holds for torsion discrete groups.

Lemma 1.1.9. *Let H, G be torsion discrete groups. Then the natural map $Shv(B(H))\otimes Shv(B(G))\to Shv(B(H\times G))$ is an equivalence.*

Proof. If H, G are finite, this is easy. Now consider $\mathbb{N}$ with the divisibility relation. The diagonal map $\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ is cofinal. For H a torsion group write $H_n=\{h\in H\mid h^n=1\}$. Then $B(H)\widetilde{\to}\operatorname{colim}_{n\in\mathbb{N}} B(H_n)$ canonically, and one may pass to left adjoint in the presentation $Shv(B(H))\widetilde{\to}\lim\limits_{n\in\mathbb{N}^{op}} Shv(B(H_n))$. So,

$$Shv(B(H))\otimes Shv(B(G))\widetilde{\to}\operatorname*{colim}_{[n],[m]\in\boldsymbol{\Delta}^{op}} Shv(B(G_n\times H_n))\widetilde{\to} Shv(B(H\times G))$$

□

The assumptions that gerbes are of finite order is needed to show that the functor $(\mathrm{PreStk}_{lft} + Grb)^{op}\to \mathrm{DGCat}$, $(Y,\mathcal{G})\mapsto Shv_{\mathcal{G}}(Y)$ is right-lax symmetric monoidal I think.

Indeed, given $Y, Y'\in\mathrm{PreStk}_{lft}$ and $\mathcal{G}: Y\to B_{et}(e^{*,tors})$, $\mathcal{G}': Y'\to B_{et}(e^{*,tors})$ recall that

$$Shv_{\mathcal{G}}(Y)=e-comod(Shv(\tilde{Y})),\quad Shv_{\mathcal{G}}(Y')=e-comod(Shv(\tilde{Y}')),$$

where $\tilde{Y},\tilde{Y}'$ are the total spaces of these gerbes. Now $\mathcal{G},\mathcal{G}'$ give rise to $\mathcal{G}\boxtimes\mathcal{G}'$, which is the composition

$$Y\times Y'\stackrel{\mathcal{G}\times\mathcal{G}'}{\to} B_{et}(e^{*,tors}))\times Shv(B_{et}(e^{*,tors})\stackrel{m}{\to} B_{et}(e^{*,tors}))$$

Write $\widetilde{Y\times Y'}$ for the total space of $\mathcal{G}\boxtimes\mathcal{G}'$. We have

$$e\boxtimes e\in coAlg(Shv(B_{et}(e^{*,tors}\times e^{*,tors})))$$

We need a map

$$(e-comod(Shv(\tilde{Y})))\otimes(e-comod(Shv(\tilde{Y}')))\to e-comod(Shv(\widetilde{Y\times Y'}))$$

Now

$$Shv(B_{et}(e^{*,tors}\times e^{*,tors}))\widetilde{\to} Shv(B_{et}(e^{*,tors}))\otimes Shv(B_{et}(e^{*,tors}))$$

acts on $Shv(\tilde{Y}\times\tilde{Y}')$. We have the natural map

$$(e-comod(Shv(\tilde{Y})))\otimes(e-comod(Shv(\tilde{Y}')))\to(e\boxtimes e)-comod(Shv(\tilde{Y}\times\tilde{Y}'))$$

Write F for the composition

$$e-comod(Shv(\widetilde{Y\times Y'})\stackrel{oblv}{\to} Shv(\widetilde{Y\times Y'})\stackrel{\alpha^!}{\to} Shv(\tilde{Y}\times\tilde{Y}')$$

for $\alpha : \tilde{Y}\times\tilde{Y}' \to \widetilde{Y\times Y'}$. We claim that F is comonadic. Indeed, write coind for the right adjoint to oblv. Then $\mathrm{coind}\circ(\alpha^!)^R$ is the right adjoint to F. Both oblv and $\alpha^!$ are conservative, so $\alpha^!\circ\mathrm{oblv}$ is conservative. It suffices to show now that F admits a left adjoint. The functor $\alpha^!$ admits the left adjoint $\alpha_!$ by Section 1.1.7. It suffices to show that $\mathrm{oblv} : e-comod(Shv(\widetilde{Y\times Y'}))\to Shv(\widetilde{Y\times Y'})$ has a left adjoint. In fact, it suffices to show that the functor $Shv(\widetilde{Y\times Y'})\to Shv(\widetilde{Y\times Y'}), K\mapsto e*K$ preserves totalizations, where we view e as a coalgebra in $Shv(B(e^{*,tors}))$.

Write $\mathrm{act} : B(e^{*,tors})\times\widetilde{Y\times Y'}\to\widetilde{Y\times Y'}$ for the action map. We need that $K\mapsto \mathrm{act}_*(e\boxtimes K)$ preserves totalizations. For this we would need the existence of a left adjoint of act_*. To simplify, assume that $e^{*,tors}$ is replaced everywhere by a finite subgroup A. Then act_* admits a left adjoint act^*, and in turn the above functor $K\mapsto\mathrm{act}_*(e\boxtimes K)$ has a left adjoint. So, F is comonadic.

The comonad FF^R corresponding to F is what? Let $mB(A)\times B(A)\to B(A)$ be the product map. Then $e\boxtimes e\widetilde{\to} m^*e$ canonically. This is why for $K\in Shv(\tilde{Y}\times\tilde{Y}')$ one has $\alpha^*(e*(\alpha_*K))\widetilde{\to}(e\boxtimes e)*K$.

This gives the desired equivalence

$$(e\boxtimes e)-comod(Shv(\tilde{Y}\times\tilde{Y}'))\widetilde{\to} e-comod(Shv(\widetilde{Y\times Y'}))$$

Question The object $e\in Shv(e^{*,tors})$ is dualizable by ([21], Lemma 1.4.6). Indeed, its !-restriction to each $Shv(\mu_n(e))$ is dualizable for any n for the map $B(\mu_n(e))\to B(e^{*,tors})$. Moreover, the dual is e itself. Maybe this would allow to rewrite $e-comod(Shv(\tilde{Y}))$ as modules?

Let G be any torsion abelian group. Then $B^2(G)\widetilde{\to}\mathrm{colim}_{n\in\mathbb{N}} B^2(G_n)$ in PreStk, hence also $B^2_{et}(G)\widetilde{\to}\mathrm{colim}_{n\in\mathbb{N}} B^2_{et}(G_n)$ in Stk, where the colimit is calculated in Stk, as the sheafification preserves colimits.

For a scheme of finite type S, any map $q : S\to B^2_{et}(G)$ is of finite order. I see this as follows: there is an etale cover $f : S'\to S$ with $S'\in\mathrm{Sch}^{aff}_{ft}$ such that our gerbe trivializes over S'. So, it suffices to show that any etale G-torsor on $S'\times_S S'$ is of finite order, so we make a kind of induction. By induction on n, we want to show that any map $S\to B^n_{et}(G)$ is of finite order. The base of induction: the map $S\to G$ factors through G_m for some $m\in\mathbb{N}$.

1.1.10. For any sheaf theory $Shv : (\mathrm{Sch}^{aff}_{ft})^{op}\to\mathrm{DGCat}_{cont}$ the category $Shv(S)$ is dualizable, because it is compactly generated.

For 1.3.2. Given a G-torsor $\mathcal{F}$ on X the induced map $\mathrm{Ran}\to\mathfrak{L}^+(B(G))$ sends $\mathcal{J}$ to the restriction of $\mathcal{F}$ under $\mathcal{D}_{\mathcal{J}}\to S\times X\to X$.

For 1.4.3. Note that G^{ω^ρ} is the group scheme of automorphisms of teh G-torsor ω^ρ. Now

$$\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}=\mathfrak{L}(G^{\omega^\rho})_{\mathrm{Ran}}/\mathfrak{L}^+(G^{\omega^\rho})_{\mathrm{Ran}}$$

For $S\in\mathrm{Sch}^{aff}$ its S-point is $\mathcal{J}\in\mathrm{Ran}(S)$ and a G^{ω^ρ}-torsor $\mathcal{P}$ on $\mathcal{D}_{\mathcal{J}}$ with a trivialization over $\overset{\circ}{\mathcal{D}}_{\mathcal{J}}$. We may equivalently think of its S-point over $\mathcal{J}$ as a G-torsor $\mathcal{P}^G$ on $\mathcal{D}_{\mathcal{J}}$ with an isomorphism of G-torsors $\mathcal{P}^G\widetilde{\to}\omega^\rho$ on $\overset{\circ}{\mathcal{D}}_{\mathcal{J}}$.

For 1.4.4. If G is a group scheme on a base S, $\mathcal{F}$ is a G-torsor on S then consider the group scheme $\mathcal{F}_G\times^G G$ with respect to the adjoint action of G. This is the group scheme of automorphisms of the G-torsor $\mathcal{F}$.

Let now $\mathcal{F}_G$ be a G-torsor on D_x, and $\operatorname{Aut}(\mathcal{F}_G)$ its group scheme of automorphisms. Then $\operatorname{Gr}_{\operatorname{Aut}(\mathcal{F}_G)}$ classifies a G-torsor $\mathcal{F}'_G$ on D_x and an isomorphism $\mathcal{F}'_G\widetilde{\to}\mathcal{F}_G\mid_{\overset{\circ}{D}_x}$.

1.1.11. For 1.5.3. Recall rigorous definition of a factorization prestack over Ran. This is a map $Z_{\operatorname{Ran}}\to\operatorname{Ran}$ in PreStk, which is lifted to a morphism of non-unital commutative algebras in $\operatorname{PreStk}_{corr}$ and such that for any J the induced morphism

$$Z^J_{\operatorname{Ran}}\times_{\operatorname{Ran}^J}\operatorname{Ran}^J_{disj}\to Z_{\operatorname{Ran}}\times_{\operatorname{Ran}}\operatorname{Ran}^J_{disj}$$

is an isomorphism.

Let now $Z_{\operatorname{Ran}_x}\to\operatorname{Ran}_x$ be a map in PreStk. A structure of a factorization module space over Z_{Ran} on it is a structure of a module in $\operatorname{PreStk}_{corr}$ over the non-unital commutative algebra Z_{Ran} such that the following hold: 1) this is a morphism of Z_{Ran}-modules in $\operatorname{PreStk}_{corr}$, where Z_{Ran} acts on Ran_x via $Z_{\operatorname{Ran}}\to\operatorname{Ran}$. So, for any J we have a commutative diagram

$$\begin{array}{ccccc}
Z^J_{\operatorname{Ran}}\times Z_{\operatorname{Ran}_x} & \leftarrow & \operatorname{mult}_{J,Z} & \to & Z_{\operatorname{Ran}_x}\\
\downarrow & & \downarrow & & \downarrow\\
\operatorname{Ran}^J\times\operatorname{Ran}_x & \leftarrow & (\operatorname{Ran}^J\times\operatorname{Ran}_x)_{disj} & \to & \operatorname{Ran}_x,
\end{array}$$

where the left square is cartesian (and the upper row defines the corresponding action map in $\operatorname{PreStk}_{corr}$). 2) It is required that the right square is also cartesian.

1.1.12. For 1.6.5. I would add that $(\operatorname{Ran}\times\operatorname{Ran})^{\subset}$ is a "category object" in PreStk acting on Ran.

By a category object in $C\in 1-\mathcal{C}at$ we mean a map $\mathcal{X}:\boldsymbol{\Delta}^{op}\to C$ such that for any $n\geq 0$ the morphisms $[1]\overset{i,i+1}{\to}[n]$ yield an isomorphism

$$\mathcal{X}([n])\widetilde{\to}\mathcal{X}[1]\times_{\mathcal{X}[0]}\mathcal{X}([1])\times_{\mathcal{X}[0]}\dots\mathcal{X}[1],$$

where [1] appears n times. Then we say that $\mathcal{X}[1]$ acts on $\mathcal{X}[0]$.

Recall that if, in addition, C. has finite limits then $\mathcal{X}[1]\in Alg(Corr(C))$ naturally by ([20], published version, Cor. 4.4.5, Chapter 9).

Now given a map $\tau:c\to\mathcal{X}[0]$ in C, we may define the notion that the $\mathcal{X}[1]$-action on $\mathcal{X}[0]$ is extended to a right $\mathcal{X}$-action on c. This means that we get a category object $\mathcal{X}':\boldsymbol{\Delta}^{op}\to C$ and a map $\mathcal{X}'\to\mathcal{X}$ of category objects in C such that $\mathcal{X}'[0]\to\mathcal{X}[0]$ is the map τ, and the square is cartesian

$$\begin{array}{ccc}
\mathcal{X}'[0] & \overset{s}{\leftarrow} & \mathcal{X}'[1]\\
\downarrow\tau & & \downarrow\\
\mathcal{X}[0] & \overset{s}{\leftarrow} & \mathcal{X}[1]
\end{array}$$

Here s is the source map attached to $[0] \xrightarrow{0} [1]$. The action map $t : \mathcal{X}'[1] \to c$ is then attached to $[0] \xrightarrow{1} [1]$. Note that in this situation the diagram

$$\begin{array}{ccc} \mathcal{X}'[0] & \stackrel{t}{\leftarrow} & \mathcal{X}'[1] \\ \downarrow \tau & & \downarrow \\ \mathcal{X}[0] & \stackrel{t}{\leftarrow} & \mathcal{X}[1] \end{array}$$

is not necessarily cartesian, as in the case of the action of $(\mathrm{Ran} \times \mathrm{Ran})^{\subset}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$.

Now given a map $Z_{\mathrm{Ran}} \to \mathrm{Ran}$ in PreStk, a unital structure on Z_{Ran} is a right $(\mathrm{Ran} \times \mathrm{Ran})^{\subset}$-action on Z such that the map $Z_{\mathrm{Ran}} \to \mathrm{Ran}$ is a equivariant with respect to the right actions of $(\mathrm{Ran} \times \mathrm{Ran})^{\subset}$. This is just a way to think. This is better as we are working with ∞-categories.

Note that Ran itself has a unital structure in the sense of Sect. 1.6.5. Make a precise relation with a factorization lax prestacks over Ran^{un} from [54].

Let $Z_{\mathrm{Ran}} \to \mathrm{Ran}$ be a factorization prestack, assume given a unital structure on Z_{Ran}. By definition, these structures are compatible if the map

$$\varphi_{big} : Z_{\mathrm{Ran}} \times_{\mathrm{Ran}, \varphi_{small}} (\mathrm{Ran} \times \mathrm{Ran})^{\subset} \to Z_{\mathrm{Ran}}$$

is a morphism of factorization prestacks over Ran.

Let $C \in 1 - \mathcal{C}\mathrm{at}$ and $\mathcal{X} : \boldsymbol{\Delta}^{op} \to C$ be a category object, $S = \mathcal{X}[0], H = \mathcal{X}[1]$. Consider the map $q : \boldsymbol{\Delta} \to \boldsymbol{\Delta}$, $[n] \mapsto [n+1]$. It sends a morphism $\beta : [n] \to [m]$ to the morphism $q(\beta) : [n+1] \to [m+1]$ given by $0 \mapsto 0$ and $k+1 \mapsto \beta(k)+1$ for all $n \geq k \geq 0$. Composing $\mathcal{X}$ with q^{op}, we get a new category objects, which realizes the right action of H on itself.

Consider the natural transformation of functors $\mathrm{id} \to q$ from $\boldsymbol{\Delta}$ to $\boldsymbol{\Delta}$ given on on $[n]$ by $\tau_n : [n] \to [n+1], i \mapsto i+1$ and naturally on morphisms. Applying $\mathcal{X}$ to this natural transformation, we see that $t : H \to S$ becomes a H-equivariant morphism with respect to the natural H-action on itself on the right.

Example: a monoid gives a category object acting on the final object $*$ of C.

1.1.13. For 1.6.5. As in [54], we have the lax prestack Ran^{un} (we supress X from the notation of [54]). To treat unital structures, one should more generally, I think, to consider a map $f : Z \to \mathrm{Ran}^{un}$ in PreStk^{lax}. Recall that Ran^{un} is a commutative algebra in $\mathrm{PreStk}^{lax}_{corr}$. Then the factorization structure on Z should be a lifting of f to a morphism of commutative algebras in $\mathrm{PreStk}^{lax}_{corr}$. This would mean in particular that we have for a finite set J a commutative diagram in PreStk^{lax}

$$\begin{array}{ccccc} Z^J & \leftarrow & mult_{J,Z} & \to & Z \\ \downarrow & & \downarrow & & \downarrow \\ (\mathrm{Ran}^{un})^J & \leftarrow & (\mathrm{Ran}^{un})^J_{disj} & \to & \mathrm{Ran}^{un}, \end{array}$$

where the left square is cartesian, and the top row defines the corresponding product map in PreStk_{corr}. Then we should similarly require that the right square is cartesian.

1.1.14. For 1.6.9. The unital and factorization structures on $\mathrm{Gr}_{G,\mathrm{Ran}}$ are compatible.

1.1.15. For 2.1.4. Let $\mathcal{G}$ be a factorization gerbe over $\mathrm{Gr}_{G,\mathrm{Ran}}$. Then its restriction to the unit section $i : \mathrm{Ran} \to \mathrm{Gr}_{G,\mathrm{Ran}}$ is canonically trivialized. Indeed, Ran is contractible, so $i^*\mathcal{G}$ is constant with value $\mathcal{G}_0$. Now given $x, y \in X$ with $x \neq y$ we have $(i^*\mathcal{G})_{x\cup y} \widetilde{\to} (i^*\mathcal{G})_x \otimes (i^*\mathcal{G})_y$, so $\mathcal{G}_0$ is trivialized.

This explains why $\mathcal{G}^{G,G,ratio}$ is trivialized after restriction to $\mathfrak{L}^+(G)_{\mathrm{Ran}}/\mathrm{Ran} \to \mathrm{Hecke}^{loc}_{G,\mathrm{Ran}}$.

In my comments below I use your definition of $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,x}$, not the one I propose. (Change this!!!)

1.1.16. For 2.1.6. The point $t^\lambda \in \mathrm{Gr}^{\omega^\rho}_{T,x}$ corresponds to $(\omega^\rho_T)(-\lambda x)$ over $D_{\mathcal{J}}$ together with an isomorphism $(\omega^\rho_T)(-\lambda x) \widetilde{\to} \omega^\rho_T$ over $\overset{\circ}{D}_{\mathcal{J}}$.

The fibre of the gerbe $\mathcal{G}^T$ on $\mathrm{Gr}^{\omega^\rho}_{T,x}$ at $t^0 \in \mathrm{Gr}^{\omega^\rho}_{T,x}$ is trivial, because for any T-torsor $\mathcal{F}_T$ on D_x, the fibre of $\mathcal{G}^T$ at $(\mathcal{F}_T, \mathcal{F}_T, \mathrm{id} : \mathcal{F}_T \widetilde{\to} \mathcal{F}_T) \in \mathrm{Hecke}^{loc}_{T,\mathrm{Ran}}$ is trivial.

The description of the fibres of $\mathcal{G}^T$ in 2.1.6 fixes a definition of the bilinear form on Λ associated to a factorization gerbe.

One more subtle thing here: the isomorphism (2.1) fixes a normalization of the map from factorizable gerbes on Gr_T to quadratic forms. There are two such possible normalizations, in (2.1) changing this normalization corresponds to replacing b by $-b$. I think the normalization chosen by this isomorphism is different from the one chosen in [GLys]. (If you agree then I propose to correct in this section as follows: replace the point $t^\lambda \in \mathrm{Gr}^{\omega^\rho}_{T,x}$ by the T-torsor $(\omega^\rho_T)(\lambda x)$, and the same for $\mathrm{Gr}_{T,x}$).

Verification: consider the case of $G = \mathrm{SL}_2$. Consider for $a \in e^{*,tors}(-1)$ the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}_{G,x}$ whose fibre at $(L, L \widetilde{\to} \mathcal{O}^2 \mid_{X-x})$ is $\det(L : \mathcal{O}^2)^a$. We know that the corresponding $q : \Lambda = \mathbb{Z} \to e^*(-1)$ sends 1 to a. So, $b(1,1) = 2a$. We identified $\mathbb{Z} \to \Lambda$ via $1 \mapsto \alpha$, where α is the simple coroot. Our $T \subset G$ is the standard maximal torus, and $\mathcal{G}^T$ is the restriction of $\mathcal{G}^G$ under $\mathrm{Gr}_{T,x} \to \mathrm{Gr}_{G,x}$. So, the fibre of $\mathcal{G}^T$ at $t^\alpha \in \mathrm{Gr}^{\omega^\rho}_{T,x}$ is

$$\det(\Omega^{\frac{1}{2}}(-x) \oplus \Omega^{-\frac{1}{2}}(x) : \mathcal{O}^2)^a$$

The gerbe $\mathcal{G}^T_{\alpha,x}$ from (2.1) becomes $\det(\mathcal{O}(-x) \oplus \mathcal{O}(x) : \mathcal{O}^2)^a \widetilde{\to} \omega_x^{-a}$, and

$$\det(\Omega^{\frac{1}{2}}(-x) \oplus \Omega^{-\frac{1}{2}}(x) : \Omega^{\frac{1}{2}} \oplus \Omega^{-\frac{1}{2}})^a \widetilde{\to} \omega_x^{-2a}$$

So, we see that the two normalizations are different!

1.1.17. To understand the gerbes in 2.1.4-2.1.8 consider an example when ti comes from a factorization line bundle. Namely, assume L is some representation of G, $a \in e^{*,tors}(-1)$, and the gerbe on Bun_G is with fibre

$$(\det \mathrm{R}\Gamma(X, L_{\mathcal{F}}) / \det \mathrm{R}\Gamma(X, L_{\mathcal{F}^0}))^a$$

at $\mathcal{F}$. Then the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ attaches to $(\mathcal{F}, \mathcal{J}, \beta : \mathcal{F} \widetilde{\to} \mathcal{F}^0 \mid_{\overset{\circ}{\mathcal{D}}_{\mathcal{J}}})$ the fibre

$$(\underset{i\in I}{\otimes} \det(L_{\mathcal{F},x_i} : L_{\mathcal{F}^0,x_i}))^a,$$

in the case $\mathcal{J}=\{x_i\mid i\in\mathcal{J}\}$, and the points x_i are disjoint. The gerbe $\mathcal{G}^G$ over $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ attaches to $(\mathcal{F},\mathcal{J},\alpha:\mathcal{F}\tilde{\to}\omega^\rho\mid_{\overset{\circ}{\mathcal{D}}_{\mathcal{J}}})$ the fibre

$$(\underset{i\in I}{\otimes}\det(L_{\mathcal{F},x_i}:L_{\omega^\rho,x_i}))^a,$$

in the case when $\mathcal{J}=\{x_i\mid i\in\mathcal{J}\}$, and the points x_i are disjoint.

So, in Remark 2.1.9 of the paper if $\tau:\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}\to\mathrm{Bun}_G$ is the natural maps then $(\mathcal{G}^G)\mid_{\omega^\rho}\otimes\tau^*\mathcal{G}^G\tilde{\to}\mathcal{G}^G$.

1.1.18. For 2.4.1-2.4.2. For the definition of the convolution, one uses the convolution diagram as in ([31], Sect 7.4) that I would denote $\widetilde{\mathrm{Gr}}_{G,x}$. It classifies G-torsors $\mathcal{P}^1_G,\mathcal{P}^2_G$ on D_x with isomorphisms $\alpha_1:\mathcal{P}^0_G\tilde{\to}\mathcal{P}^1_G\mid_{\overset{\circ}{D}_x}$, $\alpha_2:\mathcal{P}^1_G,\tilde{\to}\mathcal{P}^2_G\mid_{\overset{\circ}{D}_x}$. Let $conv:\widetilde{\mathrm{Gr}}_{G,x}\to\mathrm{Gr}_{G,x}$ be the map sending the above point to $(\mathcal{P}^2_G,\alpha_2\circ\alpha_1)$. Here $\mathcal{P}^0_G$ is the trivial G-torsor. As in ([31], Sect 7.4), since $\mathcal{G}^G$ is $\mathfrak{L}^+(G)_x$-equivariant, we have the twisted product $\mathcal{G}^G\tilde{\boxtimes}\mathcal{G}^G$, which according to *loc.cit.* identifies canonically with $conv^*(\mathcal{G}^G)$. The map *conv* is ind-proper. For this reason the functor $conv_!$ is defined for any of the 4 sheaf theories by ([21], 1.5.2).

Pick a presentation $\mathrm{Gr}_G=\mathrm{colim}_{i\in I}Y_i$, where Y_i are closed $\mathfrak{L}^+(G)$-invariant subschemes. Define $Shv(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)$ as $\lim_{i\in I^{op}}Shv(\mathfrak{L}^+(G)\backslash Y_i)$. For each i the action of $\mathfrak{L}^+(G)$ factors through certain finite-dimensional group G_i with prounipotent kernel, so we set $Shv(\mathfrak{L}^+(G)\backslash Y_i)=Shv(G_i\backslash Y_i)$. If $\mathfrak{L}^+(G)\to G'_i\to G_i$ are surjections (both kernels are unipotent) then we identify $Shv(G_i\backslash Y_i)$ and $Shv(G'_i\backslash Y_i)$ via the functor q^* for $q:G'_i\backslash Y_i\to G_i\backslash Y_i$. It is important that there is no cohomological shift in q^*.

Now for $\mathrm{pr}:\mathrm{Gr}_G\to\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G$ we get the functor $\mathrm{pr}^*:Shv(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)\to Shv(\mathrm{Gr}_G)$ and identify $\mathrm{oblv}:Shv(\mathrm{Gr}_G)^{\mathfrak{L}^+(G)}\to Shv(\mathrm{Gr}_G)$ with pr^*.

Namely, for each i pick G_i as above then for the projection $q:Y_i\to G_i\backslash Y_i$ we get the functor $q^*:Shv(G_i\backslash Y_i)\to Shv(Y_i)$. They are compatible and define the desired functor pr^*.

The category $Shv(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)$ is equpped with the following t-structure. For $K\in Shv(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)$ we say that it lies in nonpositive (resp. positive) degrees if pr^*K lies in nonpositive (resp. positive) perverse degrees. Thus, pr^* is t-exact.

For 2.4.3. We restrict the corresponding morphisms of sheaves of categories over Ran to the point x. Since we only want a monoidal functor, we may forget about ϵ, though it is not trivial. Namely, this $Sat_{q,G}:\mathrm{Rep}(H)\to\mathrm{Sph}_{q,x}(G)$ is not compatible with factorization in general. We also ignored the gerbe $\det\mathfrak{g}^{\frac{1}{2}}$ appearing in the definition of the metaplectic spherical category $Sph_{q,x}(G)$. The gerbe $\det^{\frac{1}{2}}_{\mathfrak{g}}$ on $\mathrm{Gr}_{G,x}$ is trivial, but this trivialization is not copmpatible with the factorization structure.

1.1.19. For 2.4.4. The fact that $\mathrm{Rep}(T_H)\tilde{\to}Shv(\mathrm{Gr}_{T^\sharp,x})$ should be the following general thing. First, $Shv(\mathrm{Gr}_{T^\sharp,x})\tilde{\to}\prod_{\lambda\in\Lambda^\sharp}\mathrm{Vect}$.

Recall that for any G, which is a group scheme of finite type,

$$\mathrm{Rep}(G)=\mathrm{QCoh}(B(G))\tilde{\to}\mathrm{R}\Gamma(G,\mathcal{O})-comod(\mathrm{Vect})$$

This is done as in ([43], 10.2.9). Now for a torus T_H over e we have

$$k[T_H] - comod(\mathrm{Vect}) \widetilde{\to} \prod_{\lambda\in\check{\Lambda}_H} \mathrm{Vect},$$

where $\check{\Lambda}_H$ is the weight lattice of T_H. The map in one direction is the evident one $\oplus_{\lambda\in\check{\Lambda}_H}\mathrm{Vect} \to k[T_H] - comod(\mathrm{Vect})$. Its λ-component sends $V \in \mathrm{Vect}$ to V on which T_H acts by λ. We use the fact that coproducts in DGCat_{cont} coincide with products ([43], 9.2.9).

1.1.20. For 2.4.5. The isomorphism (2.6) follows from [GLys, isomorphism (8.11) and Sect. 9.4.3]. By $\mathcal{F}\mid_{S^\gamma}$ we mean the !-restriction, and Γ denotes the $*$-direct image.

1.1.21. For 2.5.1. Let us study the question for ℓ-adic sheaves. Let G be a group scheme of finite type, $Shv(G)$ the DG-category of ℓ-adic sheaves of G equipped with the convolution monoidal structure defined as $Shv(G)\otimes Shv(G) \to Shv(G\times G) \stackrel{m_!}{\to} Shv(G)$, where $m : G\times G \to G$ is the product. The unit is $i_!\bar{\mathbb{Q}}_\ell$, where $i : \operatorname{Spec} k \to G$ is the unit. Let $r : G \to G$ be given by $r(g) = g^{-1}$. For $F \in Shv(G)^c$ we have a natural map $F * \mathbb{D}(r^*F) \to i_*\bar{\mathbb{Q}}_\ell$ coming by base change from

$$\mathrm{R}\Gamma_c(G, F\otimes\mathbb{D}(F)) \to \mathrm{R}\Gamma_c(G, \mathcal{K}_G) \to \bar{\mathbb{Q}}_\ell,$$

here $\mathcal{K}_G = p^!\bar{\mathbb{Q}}_\ell$ for $p : G \to \operatorname{Spec} k$. We want to check if the functor $(Shv(G)^c)^{op} \to Shv(G)^c, F \mapsto \mathbb{D}(r^*F)$ defines a monoidal dual for this convolution monoidal structure.

For $F, F' \in Shv(G)^c$, the inner hom $\mathcal{H}om(F, F')$ for this monoidal structure exists and identifies with $(p_1)_*\mathcal{H}om(p_2^*F, m^!F')$, where $p_i : G\times G \to G$ is the i-th projection. The existence of inner homs always holds for any algebra in DGCat_{cont}, so inner homs exist in $Shv(G)$. It is easy to see that for $F \in Shv(G)^c$ we get indeed $\mathcal{H}om(F, i_*\bar{\mathbb{Q}}_\ell) \widetilde{\to} \mathbb{D}(r^*F)$. The above candidate for the counit map is the morphism $\mathcal{H}om(F, i_*\bar{\mathbb{Q}}_\ell)\otimes F \to i_*\bar{\mathbb{Q}}_\ell$ given by the universal property of $\mathcal{H}om$. To check that the above map extends to a duality datum we have to establish for $D, A \in Shv(G)$ an isomorphism in Vect

$$\mathrm{Hom}(D, A * \mathbb{D}(r^*F)) \widetilde{\to} \mathrm{Hom}(D * F, A),$$

here by Hom we mean $\mathrm{R}\Gamma\,\mathcal{H}om$, where $\mathcal{H}om$ is the inner hom in $Shv(G)$ with the pointwise tensor product monoidal structure. By the above calculation of the inner hom, it suffices to establish the isomorphism for $A \in Shv(G)$

$$A * \mathbb{D}(r^*F) \widetilde{\to} (p_1)_*\mathcal{H}om(p_2^*F, m^!A) \tag{1}$$

Lemma 1.1.22. *For $A \in Shv(G), F \in Shv(G)^c$ there is a natural isomorphism in $Shv(G)$*

$$m_*(A\boxtimes\mathbb{D}(r^*F)) \widetilde{\to} (p_1)_*\mathcal{H}om(p_2^*F, m^!A)$$

Proof. It suffices to prove this for A compact. Indeed, we assume $m^! : Shv(G) \to Shv(G\times G)$ and $(p_1)_*$ continuous. Note also that for $A \in Shv(S)^c$, where S is a separated scheme of finite type, $\mathcal{H}om(A, \cdot)$ preserves filtered colimits. So, the RHS preserves filtered colimits as a functor of A. The LHS also preserves filtered colimits as a functor of A.

So, we assume A compact. Consider the isomorphism $\alpha : G\times G\to G\times G$, $\alpha(z,v)=(zv,v^{-1})$. We have $\alpha_*\mathcal{H}om(p_2^*F, m^!A)\widetilde{\to}\mathcal{H}om(p_2^*r^*F, p_1^!A)$, and $m\circ\alpha=p_1$. This yields an isomorphism

$$m_*\mathcal{H}om(p_2^*r^*F, p_1^!A)\widetilde{\to}(p_1)_*\mathcal{H}om(p_2^*F, m^!A)$$

on G.

Recall that for $F_1, F_2\in Shv(S)^c$, where S is a scheme of finite type, one has $\mathbb{D}(F\otimes G)\widetilde{\to}\mathcal{H}om(F,\mathbb{D}G)$. This gives

$$\mathcal{H}om(p_2^*r^*F, p_1^!A)\widetilde{\to}\mathbb{D}(p_2^*r^*F\otimes\mathbb{D}p_1^!A)\widetilde{\to}\mathbb{D}(\mathbb{D}(A)\boxtimes r^*F)\widetilde{\to}A\boxtimes\mathbb{D}(r^*F)$$

This gives finally $m_*(A\boxtimes\mathbb{D}(r^*F))\widetilde{\to}(p_1)_*\mathcal{H}om(p_2^*F, m^!A)$. □

So, (1) is equivalent to the natural map $m_!(A\boxtimes\mathbb{D}(r^*F))\to m_*(A\boxtimes\mathbb{D}(r^*F))$ to be an isomorphism! We see that this is indeed the monoidal dual if G is proper.

1.1.23. Sam confirms that for any of our 4 sheaf theories, one has the following. Let $G\in\mathcal{G}\mathrm{rp}(\mathrm{Stk})$ be an ind-scheme of ind-finite type. Then we define the convolution monoidal structure on $Shv(G)$ using m_* for $m: G\times G\to G$ the product map. We have the Verdier duality equivalence $\mathbb{D}: (Shv(G)^c)^{op}\to Shv(G)^c$. Let $r: G\to G$, $r(g)=g^{-1}$. Similarly, if we assume G ind-proper then the functor $\mathbb{D}\circ r: (Shv(G)^c)^{op}\to Shv(G)^c$ is the monoidal dual.

Now consider the following situation in the constructible context. Let G be a group scheme of finite type, $H\subset G$ a closed smooth subgroup with G/H proper. Then $Shv(H\backslash G/H)$ is monoidal with the monoidal product $K_1 * K_2=\bar{m}_*q^!(K_1\boxtimes K_2)$ for

$$(H\backslash G/H)\times(H\backslash G/H)\stackrel{q}{\leftarrow} H\backslash G\times^H G/H\stackrel{\bar{m}}{\to} H\backslash G/H$$

This monoidal product preserves the full subcategory $Shv(H\backslash G/H)^{constr}$, because $\bar{m}$ is proper.

Given $F, F'\in Shv(H\backslash G/H)$, the inner hom $\mathcal{H}om^*(F,F')$ for this monoidal structure exists, it given for the i-th projection $p_i: (H\backslash G/H)\times(H\backslash G/H)\to(H\backslash G/H)$ by

$$\mathcal{H}om^*(F,F')=(p_1)_*\mathcal{H}om(p_2^*F, q_*\bar{m}^!F')[-2\dim H])$$

Here we used ([46], 0.2.2). Let $i: B(H)\hookrightarrow H\backslash G/H$ be the closed immersion. The unit of $Shv(H\backslash G/H)$ is $i_*\omega$. Let r be the involution of $H\backslash G/H$ coming from $G\to G, g\mapsto g^{-1}$. Verdier duality gives an equivalence

$$\mathbb{D}: (Shv(H\backslash G/H)^{constr})^{op}\widetilde{\to} Shv(H\backslash G/H)^{constr}$$

Let now $F\in Shv(H\backslash G/H)^{constr}$. We claim that F has the monoidal right dual $\mathbb{D}(r^!F)[-2\dim H]$.

Proof. We have $\mathcal{H}om^*(F, i_*\omega)\widetilde{\to}\mathbb{D}(r^!F)[-2\dim H]$ naturally, which gives a canonical map

$$F*\mathbb{D}(r^!F)[-2\dim H]\to i_*\omega$$

by the universal property of $\mathcal{H}om^*$. We. want to check this is a counit of a duality. For this we need to show that the for any $A\in Shv(H\backslash G/H), K\in Shv(H\backslash G/H)^{constr}$ one has canonically

$$\mathcal{H}om(K, A*F)\widetilde{\to}\mathcal{H}om(K*\mathbb{D}(r^!F)[-2\dim H], A)$$

We get by ([46], 0.2.2),

$$\mathcal{H}om(K * \mathbb{D}(r^! F)[-2\dim H], A) \widetilde{\to} \mathcal{H}om(K, (p_1)_* \mathcal{H}om(p_2^* \mathbb{D}(r^! F), q_* \bar{m}^! A))$$

The latter object, using $p_2^* \mathbb{D}(r^! F) \widetilde{\to} \mathbb{D}(p_2^! r^! F)$, identifies with

$$\mathcal{H}om(K, (p_1)_*(p_2^! r^! F \otimes^! q_* \bar{m}^! A)) \widetilde{\to} \mathcal{H}om(K, (p_1)_* q_*(\bar{m}^! A \otimes^! q^! p_2^! r^! F))$$

So, it suffices to establish an isomorphism

$$(p_1)_* q_*(\bar{m}^! A \otimes^! q^! p_2^! r^! F) \widetilde{\to} A * F \tag{2}$$

Consider the automorphism $\gamma : G \times G \widetilde{\to} G \times G$, $(g_1, g_2) \mapsto (g_1 g_2, g_2^{-1})$. It induces the isomorphism

$$\bar{\gamma} : H\backslash G \times^H G/H \widetilde{\to} H\backslash G \times^H G/H$$

Then $p_1 q \bar{\gamma} = \bar{m}$ and $p_2 q \bar{\gamma} = r p_2 q$. So, $\bar{m}^! A \widetilde{\to} \bar{\gamma}^! q^! p_1^! A$ and $\bar{\gamma}^! q^! p_2^! F \widetilde{\to} q^! p_2^! r^! F$. Now the LHS of (2) identifies with

$$(p_1)_* q_* \bar{\gamma}^! q^! (A \boxtimes F)$$

Now $(p_1) q \bar{\gamma}^{-1} = \bar{m}$, and we get the isomorphism (2). □

Consider now a more more case, where G is a placid group ind-scheme, and $H \subset G$ is a closed placid subgroup. Assume H prosmooth, and G/H is an ind-scheme of ind-finite type, which is ind-proper. We define the monoidal structure on $Shv(H\backslash G/H)$ by

$$K_1 * K_2 = \bar{m}_* q^*(K_1 \boxtimes K_2)$$

for the same diagram

$$(H\backslash G/H) \times (H\backslash G/H) \stackrel{q}{\leftarrow} H\backslash G \times^H G/H \stackrel{\bar{m}}{\to} H\backslash G/H$$

Define $Shv(H\backslash G/H)^{constr} \subset Shv(H\backslash G/H)$ as the full subactegory of those K for which $\mathrm{oblv}(K) \in Shv(G/H)^c$. This monoidal product preserves the full subcategory $Shv(H\backslash G/H)^{constr}$, because $\bar{m}$ is proper.

Let now $F \in Shv(H\backslash G/H)^{constr}$. We claim that F has the monoidal right dual $\mathbb{D}(r^! F)$.

Proof. Let $A \in Shv(H\backslash G/H), K \in Shv(H\backslash G/H)^{constr}$. It suffices to show that one has canonically

$$\mathcal{H}om(K, A * F) \widetilde{\to} \mathcal{H}om(K * \mathbb{D}(r^! F), A)$$

in Vect. We have

$$\mathcal{H}om(K * \mathbb{D}(r^! F), A) \widetilde{\to} \mathcal{H}om(K \boxtimes \mathbb{D}(r^! F), q_* m^! A)$$

Step 1. We claim that the latter complex identifies with

$$\mathcal{H}om(K, (p_1)_*(p_2^! r^! F \otimes^! q_* \bar{m}^! A))$$

with the same notations as above. Here the functor $p_2^!$ sends L to $\omega \boxtimes L$. The difficulty is that p_i^* do not make sense. Pick $Y_1, Y_2 \subset G/H$ such that $\mathrm{oblv}(K)$ is the extension by zero from Y_1, $\mathrm{oblv}(r^! F)$ is the extension by zero from Y_2. Let $p_{Y,i} : (H\backslash Y_1) \times (H\backslash Y_1) \to$

$(H\backslash Y_i)$ be the i-th projection. Then $p^*_{Y,i}$ make sense, and $K\boxtimes \mathbb{D}(r^!F)\widetilde{\to} p^*_{Y,1}K\otimes p^*_{Y,2}\mathbb{D}(r^!F)$. So,

$$\mathcal{H}om(K\boxtimes \mathbb{D}(r^!F), q_*m^!A)\widetilde{\to}\mathcal{H}om(p^*_{Y,1}K, \mathcal{H}om(p^*_{Y,2}\mathbb{D}(r^!F), q_*m^!A)$$

Then $\mathcal{H}om(p^*_{Y,2}\mathbb{D}(r^!F), q_*m^!A)\widetilde{\to} p^!_{Y,2}r^!F\otimes^! q_*m^!A$. So,

$$(p_{Y,1})_*\mathcal{H}om(K\boxtimes \mathbb{D}(r^!F), q_*m^!A)\widetilde{\to}\mathcal{H}om(K, (p_{Y,1})_*(p^!_{Y,2}r^!F\otimes^! q_*m^!A)$$

Our claim follows now from

$$(p_{Y,1})_*(p^!_{Y,2}r^!F\otimes^! q_*m^!A\widetilde{\to} p^!_{Y,1}(p_1)_*(p^!_2r^!F\otimes^! q_*\bar{m}^!A)$$

Step 2. The rest of the proof is as in the finite-dimensional case. □

1.1.24. For 2.5.5. In (2.12) over the low horizontal arrow there should be $Sat_{q^{-1},G}$. Note that $\mathbb{D}^{lin}\tau_H\widetilde{\to}\tau_H\mathbb{D}^{lin}$, so the order in the left vertical arrow in (2.12) does not matter. Besides, $\mathrm{Sat}_{q,G}$ does not preserve compact objects, it sends compact objects to constructible ones, correct!

1.1.25. For 3.1. The notion of a chiral category makes sense for ℓ-adic sheaves (and for all the 4 sheaf theories). It could be defined as in [54]. In particular, if $S\in \mathrm{PreStk}$ has a structure of a commutative algebra in PreStk_{corr} then we have the notion of a multiplicative sheaf of categories on S as in ([54], 5.4). The fact that for $S_i\in \mathrm{Sch}_{ft}$ separated, $Shv(S_1)\otimes Shv(S_2)\to Shv(S_1\times S_2)$ is maybe not an equivalence, is not a problem. The reason is that the exteriour tensor product of sheaves of categories is $Shv_{/S_1}\boxtimes Shv_{/S_2}\widetilde{\to} Shv(S_1\times S_2)$.

1.1.26. For 3.1.2. The sense of $fSet^{surj}$ is that it is the 1-full subcategory of the operad controlling the non-unital commutative algebras, where we keep only sets $\langle n\rangle$ for $n>0$ and active morphisms.

As in ([54], Appendix B), write $\mathcal{C}\mathrm{at}_{dir}$ for the ∞-category of categories with directions. Recall that we have an adjoint pair $\mathrm{Tw}: 1-\mathcal{C}\mathrm{at}\leftrightarrows \mathcal{C}\mathrm{at}_{dir}: corr$, where the right adjoint sends $C\mapsto C_{corr;hor,vert}$. View $1-\mathcal{C}\mathrm{at}$ as symmetric monoidal with the cartesian symmetric monoidal structure. View $\mathcal{C}\mathrm{at}_{dir}$ as symmetric monoidal, where for $(C, hor_C, vert_C), (D, hor_D, vert_D)$, on $C\times D$ we get a structure of a category with directions: $hor_{C\times D}=hor_C\times hor_D$, and $vert_{C\times D}=vert_C\times vect_D$. Then both Tw and $corr$ are symmetric monoidal, so yield an adjoint pair

$$\mathrm{Tw}: CAlg^{nu}(1-\mathcal{C}\mathrm{at})\leftrightarrows CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir}): corr$$

Now for $D, D'\in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$, the space $\mathrm{Map}_{CAlg^{nu}(1-\mathcal{C}\mathrm{at})}(D, D')$ is the space $\mathrm{Fun}^{\otimes}(D, D')^{\mathrm{Spc}}$ of symmetric monoidal functors.

In fact, $1-\mathcal{C}\mathrm{at}$ and $\mathcal{C}\mathrm{at}_{dir}$ are naturally 2-categories, and so are $CAlg^{nu}(1-\mathcal{C}\mathrm{at})$, $CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$. For $D, D'\in \mathcal{C}\mathrm{at}_{dir}$ the category of maps between them in $\mathcal{C}\mathrm{at}_{dir}$ is the full subcategory $\mathrm{Fun}^{dir}(D, D')\subset \mathrm{Fun}(D, D')$ classifying functors preserving horizontal (resp., vertical) morphisms, and sending cartesian products of $x\stackrel{a}{\to} y\stackrel{b}{\leftarrow} z$ with a horizontal and b vertical to cartesian squares.

Given $E, E'\in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$, the mapping category in $CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ from E to E' is the category $\mathrm{Fun}^{\otimes}(E, E')$ of non-unital symmetric monoidal functiors.

Namely, if $E^{\otimes} \to \mathrm{Surj}, E'^{\otimes} \to \mathrm{Surj}$ are the cocartesian fibrations corresponding to E, E' then $\mathrm{Fun}^{\otimes}(E, E') \subset \mathrm{Fun}_{\mathrm{Surj}}(E^{\otimes}, E'^{\otimes})$ is the full subcategory of functors sending Surj-cocartesian arrows to cocartesian arrows.

Let us assume that $Tw : 1-\mathcal{C}\mathrm{at} \leftrightarrows \mathcal{C}\mathrm{at}_{dir} : corr$ is an adjoint pair of 2-categories, that is, we have

$$\mathrm{Fun}(E, C_{corr;hor,vert}) \widetilde{\to} \mathrm{Fun}^{dir}(Tw(E), C)$$

for $C \in \mathcal{C}\mathrm{at}_{dir}$, $E \in 1-\mathcal{C}\mathrm{at}$ naturally.

Moreover, we assume the induced adjoint pair

$$Tw : CAlg^{nu}(1-\mathcal{C}\mathrm{at}) \leftrightarrows CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir}) : corr$$

is also an adjoint pair of 2-categories. For $D, D' \in CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$ the mapping category $\mathbf{Map}_{CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})}(D, D')$ is the category $\mathrm{Fun}^{\otimes,dir}(D, D')$ of those non-unital symmetric monoidal functors $f : D \to \mathrm{D}'$ whose image in $\mathrm{Fun}(D, D')$ lies in $\mathrm{Fun}^{dir}(D, D')$. I hope for $E \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ and $D \in CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$ one has a naturall equivalence

$$\mathrm{Fun}^{\otimes}(E, C_{corr,hor,vert}) \widetilde{\to} \mathrm{Fun}^{\otimes,dir}(Tw(E), C)$$

in $1-\mathcal{C}\mathrm{at}$.

1.1.27. For 3.1.2 more. Let now $C \in 1-\mathcal{C}\mathrm{at}$ admitting fibred products. We view it as a category with directions taking $hor = vert$ to be all morphisms. Assume moreover $C \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ with the cartesian symmetric monoidal structure. Then $C \in CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$ naturally, because the product map $C \times C \to C$ preserves the cartesian squares automatically. So, $C_{corr} \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ naturally.

The product map $C_{corr} \times C_{corr} \to C_{corr}$ sends (c_1, c_2) to $c_1 \times c_2$. Now by ([50], Lm. 1.2.6),

$$\mathrm{Fun}^{llax}(fSet^{surj}, C) \widetilde{\to} \mathrm{Fun}^{\otimes}(Tw(fSet^{surj}), C),$$

here the RHS is the category of non-unital symmetric monoidal functors, and the non-unital symmetric monoidal structure on $Tw(fSet^{surj})$ sends a pair $f : I \to J$, $g : I' \to J'$ to $f \sqcup g : I \sqcup I' \to J \sqcup J'$.

By the above,

$$\mathrm{Fun}^{\otimes,dir}(Tw(fSet^{surj}), C) \widetilde{\to} \mathrm{Fun}^{\otimes}(fSet^{surj}, C_{corr})$$

Finally, by ([50], 1.2.1),

$$\mathrm{Fun}^{\otimes}(fSet^{surj}, C_{corr}) \widetilde{\to} CAlg^{nu}(C_{corr})$$

Combining, we get a full embedding

$$CAlg^{nu}(C_{corr}) \hookrightarrow \mathrm{Fun}^{llax}(fSet^{surj}, C)$$

Explicitly, given $c \in CAlg^{nu}(C_{corr})$, we get a non-unital symmetric monoidal functor $f : fSet^{surj} \to C_{corr}$, $f(I) = c^I$, here f sends a surjection of finite non-empty sets $\alpha : I \to J$ to the map $c^I \leftarrow mult_{\alpha} \to c^J$ in C_{corr} corresponding to the product along α. Note that

$$mult_{\alpha} = \prod_{j \in J} mult_{I_j},$$

where for $K \in fSet^{surj}$ we denote by $c^K \leftarrow mult_K \to c$ the product map in C_{corr} along $K \to *$. The corresponding functor $\bar{f} : Tw(fSet^{surj}) \to C$ sends $(I \overset{\alpha}{\to} J) \in$

$Tw(fSet^{surj})$ to $mult_\alpha$. Finally, the resulting left-lax symmetric monoidal functor $\tilde{f} : fSet^{surj} \to C$ sends I to $mult_I$. The functor $\tilde{f}$ sends $\beta : I \to J$ to the map $\tilde{f}(\beta) : mult_I \to mult_J$, which fits into the diagram

$$\begin{array}{ccccc} mult_I & \stackrel{\tilde{f}(\beta)}{\to} & mult_J & \to * \\ \downarrow & & \downarrow & \\ mult_\beta & \to & c^J & \\ \downarrow & & & \\ c^I & & & \end{array}$$

The left-lax symmetric monoidal structure on $\tilde{f}$ is as follows. Given $I, J \in fSet^{surj}$, we construct the map $mult_{I\sqcup J} \to mult_I \times mult_J$ as follows. Consider the diagram $I \sqcup J \stackrel{\beta}{\to} * \sqcup * \to *$. The product diagram for β is

$$c^I \times c^J \leftarrow mult_I \times mult_J \to c \times c$$

Composing further in C_{corr} with $c \otimes c \leftarrow mult_{*\sqcup *} \to c$, we get a diagram

$$\begin{array}{ccccc} mult_{I\sqcup J} & \to & mult_{*\sqcup *} & \to & c \\ \downarrow & & \downarrow & & \\ mult_I \times mult_J & \to & c \times c & & \\ \downarrow & & & & \\ c^I \times c^J, & & & & \end{array}$$

which gives the desired map $mult_{I\sqcup J} \to mult_I \times mult_J$.

Conversely, let $h : fSet^{surj} \to C$ be a left lax symmetric monoidal functor. It gives rise to a symmetric monoidal functor $\bar{h} : Tw(fSet^{surj}) \to C$ sending $(I \stackrel{\alpha}{\to} J)$ to $\prod_{j\in J} h(I_j)$. It sends a diagram

$$\begin{array}{ccc} I & \stackrel{\alpha}{\to} & J \\ \downarrow & & \uparrow \\ I' & \stackrel{\alpha'}{\to} & J' \end{array}$$

to the morphism $\prod_{j\in J} h(I_j) \to \prod_{j'\in J'} h(I'_{j'})$, which is obtained as the product over $j \in J$ of the compositions

$$h(I_j) \to \prod_{j'\in J'_j} h(I_{j'}) \to \prod_{j'\in J'_j} h(I'_{j'})$$

Here the first map comes from tey left-lax structure, and the second is the functoriality of h on morphisms. The so obtained functor $\bar{h}$ lies in $\mathrm{Fun}^{\otimes,dir}(Tw(fSet^{surj}, C)$ iff for any maps $L \stackrel{\gamma}{\to} I \stackrel{\alpha}{\to} J \stackrel{\beta}{\to} K$ in $fSet^{surj}$ the square is cartesian

$$\begin{array}{ccc} \bar{h}(L \to K) & \to & \bar{h}(L \to J) \\ \downarrow & & \downarrow \\ \bar{h}(I \to K) & \to & \bar{h}(I \to J) \end{array}$$

This is not automatic! This is equivalent to the property that the square (which is commutative by definition of a left-lax functor)

$$\begin{array}{ccc} h(L) & \to & \prod_{j\in J} h(L_j) \\ \downarrow & & \downarrow \\ h(I) & \to & \prod_{j\in J} h(I_j) \end{array}$$

is cartesian.

In our case given a factorization space in the sense of Section 3.1.2, for $J \in fSet^{surj}$ the map $Z_J \to \mathrm{Ran}_d^J \times_{\mathrm{Ran}^J} (Z_*)^J$ induced by the left-lax structure is an isomorphism. This implies that the functor $J \mapsto Z_J$ comes from an object of $CAlg^{nu}(\mathrm{PreStk}_{corr})$ (the above squares are cartesian). So, indeed the two definitions are equivalent and give rise to equivalent categories of factorization prestacks over Ran.

1.1.28. For 3.1.4. First, I think it is important to explain to which notion from [54] your definition of factorization algebra corresponds. It corresponds to multiplicative object of a multiplicative sheaf of categories, as far as I understand.

One more thing here. If $f : Y \to C$ is a cartesian fibration corresponding to a functor $p : C^{op} \to 1-\mathcal{C}at$ then $\lim p \widetilde{\to} \mathrm{Fun}_C^{cart}(C, Y)$ canonically. Here $\mathrm{Fun}_C^{cart}(C, Y) \subset \mathrm{Fun}_C(C, Y)$ is the full subcategory of those functors that send any arrow to a cartesian arrow. This is something people (who tried to read your book) know. So, the category of factorization algebras in $Shv_{\mathcal{G}}(Z_{\mathrm{Ran}})$ in your sense maps naturally to

$$\lim_{I\in (fSet)^{surj})^{op}} Shv_{\mathcal{G}_I}(Z_I) \widetilde{\to} Shv_{\mathcal{G}}(Z_*)$$

We used that $fSet^{surj}$ has a final object.

Let now $fSet^{surj} \to \mathrm{PreStk}_{lft}, I \mapsto Z_I$ be a factorization space in the sense of Section 3.1.2 of the paper, so a left lax nonunital symmetric monoidal functor with some properties. For $I \in fSet^{surj}$ the map $I \to *$ gives the morphism $g_I : Z_I \to Z_*$, which is the product map $Z_*^I \times_{\mathrm{Ran}^I} \mathrm{Ran}_d^I \to Z_*$. An object $\mathcal{F} \in Shv(Z_*)$ defines a cartesian section $fSet^{surj} \to Shv(Z_{fSet^{surj}})$ sending I to $g_I^! \mathcal{F}$. The left-lax monoidal structure on the functor Z is given for a pair $I, J \in fSet^{surj}$ by the open immersion

$$Z^{I\sqcup J} \times_{\mathrm{Ran}^{I\sqcup J}} \mathrm{Ran}_d^{I\sqcup J} \hookrightarrow (Z^I \times_{\mathrm{Ran}^I} \mathrm{Ran}_d^I) \times (Z^J \times_{\mathrm{Ran}^J} \mathrm{Ran}_d^J)$$

The fact that the above cartesian section is multiplicative means the following now. For $I, J \in fSet^{surj}$ the !-restriction of $(g_I^! \mathcal{F}) \boxtimes (g_J^! \mathcal{F})$ under the above open immersion is identified with $g_{I\sqcup J}^! \mathcal{F}$.

The comparison with the notion of multiplicative object from [54] is not evident, and deserves an explanation.

Before thinking about multiplicatove objects, one tries in your way to understand the multiplicative sheaves of categories. Namely, let

$$\mathrm{PreStk}_{lft}^{op} \to 1-\mathcal{C}at,\ Y \mapsto ShvCat_{/Y} \tag{3}$$

be the functor sending Y to the category of sheaves of categories $ShvCat_{/Y}$ over Y. This functor is the RKE of its restriction to $(\mathrm{Sch}_{ft}^{aff})^{op}$, and it sends $S \in (\mathrm{Sch}_{ft}^{aff})^{op}$ to $Shv(S) - mod$. The functor (3) is right-lax symmetric monoidal, where PreStk_{lft}

is equipped with the cartesian monoidal structure. Let now $Z_{\mathrm{Ran}} \to \mathrm{Ran}$ be a factorization prestack over Ran. So, we are given a left-lax symmetric monoidal functor $fSet^{surj} \to \mathrm{PreStk}_{lft}$, $J \mapsto Z_J$ as in Section 3.1.2 of the paper. Composing with (3), one gets a right-lax symmetric monoidal functor

$$(fSet^{surj})^{op} \to 1-\mathfrak{Cat},\ J \mapsto ShvCat_{/Z_J},$$

which gives rise to a cartesian fibration

$$ShvCat(Z_{fSet^{surj}}) \to fSet^{surj} \tag{4}$$

As in the paper, $ShvCat(Z_{fSet^{surj}})$ is equipped with a symmetric monoidal structure and (4) is symmetric monoidal.

So, we may consider the category of symmetric monoidal sections of (4) which are also cartesian. Does this category identify with $MultCat(Z_{\mathrm{Ran}})$? We have denoted as in ([54], 5.21.1) the category of multiplicative sheaves of categories on Z_{Ran}.

To answer the above, it is natural to consider the general situation in the next subsection.

1.1.29. Let C be a symmetric monoidal category admitting fibre products and final object, equip C with the cartesian non-unital symmetric monoidal structure. Let $F : C^{op} \to 1-\mathfrak{Cat}$ be a right lax symmetric monoidal functor. In the cases of interest, it factors through $1-\mathfrak{Cat}_{cocmpl} \to 1-\mathfrak{Cat}$, the latter is the 1-full subcategory, where we restrict categories to cocomplete ones, and functors to those preserving small colimits.

Consider the category $\mathrm{Groth}_{corr}(F)$ defined in ([54], 5.14). It is equipped with a functor $q : \mathrm{Groth}_{corr}(F) \to C_{corr}$, and the base change of the latter by $C^{op} \to C_{corr}$ identifies with $\mathrm{Groth}(F) \to C^{op}$, the cocartesian fibration attached to F. We know from ([54], 5.16) that $\mathrm{Groth}_{corr}(F)$ is symmetric monoidal, and q is symmetric monoidal.

Let us use the following notation for the monoidal structure on $\mathrm{Groth}_{corr}(F)$. Given $(c, x), (c', x') \in \mathrm{Groth}_{corr}(F)$, one has

$$(c, x) \otimes (c', x') = (c \otimes c', x \boxtimes x'),$$

where $x \boxtimes x'$ is the image of (x, x') under $F(c) \times F(c') \to F(c \otimes c')$.

Let $c \in CAlg^{nu}(C_{corr})$ be a non-unital commutative algebra in C_{corr}. Write $Mult^w_F(c)$ be the category of non-unital commutative algebras in $\mathrm{Groth}_{corr}(F)$ over $c \in CAlg^{nu}(C)$. For a non-empty finite set I let

$$c^{\otimes I} \overset{a_1}{\leftarrow} c_I \overset{a_2}{\to} c \tag{5}$$

be the action diagram in C for $I \to *$. We write $a_{2,I}, a_{1,I}$ to express the dependence on I if needed. For an object $(c, x \in F(c)) \in Mult^w_F(c)$ we get the product map $(c, x)^{\otimes I} \to (c, x)$ in $\mathrm{Groth}_{corr}(F)$ over (5). It is given by a morphism

$$\eta_I : a_1(x^{\boxtimes I}) \to a_2(x)$$

Let $Mult_F(c) \subset Mult^w_F(c)$ be the full subcategory of those (c, x) for which the map η_I is an isomorphism for any nonempty finite set I.

Consider the left lax symmetric monoidal functor $fSet^{surj} \to C$, $I \mapsto c_I$ attached to $c \in CAlg^{nu}(C_{corr})$ as above. Composing with F, one gets a right-lax symmetric monoidal functor $\bar{F} : (fSet^{surj})^{op} \to 1 - \mathcal{C}at$ giving rise to a cartesian fibration

$$\text{coGroth}(\bar{F}) \to fSet^{surj} \tag{6}$$

As in ([54], 5.15.2), coGroth$(\bar{F})$ is equipped with a symmetric monoidal structure, and (6) is symmetric monoidal. Write $Sect^{\otimes,cart}(\bar{F})$ for the category of symmetric monoidal sections of (6), which are cartesian.

Question. Do we have an equivalence $Sect^{\otimes,cart}(\bar{F}) \widetilde{\to} Mult_F(c)$? This would certainly help a reader.

To answer this question, I propose first to answer

Question'. Prove that the base change $Groth_{corr}(F) \times_{C_{corr}} C \to C$ of q is the cartesian fibration $coGroth(F) \to C$ attached to F.

Assume the answer to the latter question is positive. Then $coGroth(F)$ becomes a symmetric monoidal subcategory in $Groth_{corr}(F)$. Now given $(c, x \in F(c)) \in Mult_F(c)$, we get a functor $fSet^{surj} \to coGroth(\bar{F})$ sending I to $(I, a_{2,I}(x) \in F(c_I))$. To see that this is indeed a functor, recall that for a map $\gamma : I \to J$ in $fSet^{surj}$ we have a diagram

$$\begin{array}{ccccc} c_I & \stackrel{\bar{\gamma}}{\to} & c_J & \stackrel{a_{2,J}}{\to} & c \\ \downarrow & & \downarrow & & \\ c_\gamma & \to & c^{\otimes J} & & \\ \downarrow & & & & \\ c^{\otimes I} & & & & \end{array}$$

in C corresponding to products for the diagram $I \to J \to *$. The functor $F(c_J) \to F(c_I)$ sending an object to the end of a cartesian arrow over γ is the functor $\bar{\gamma} : F(c_J) \to F(c_I)$. To see that $(c, a_{2,I}(x)) \in coGroth(F)$ depends functorially on I, we need an isomorphism $\bar{\gamma}(a_{2,J}) \widetilde{\to} a_{2,I}$. It takes place because $a_{2,J} \circ \bar{\gamma} = a_{2,I}$. We also see this is a cartesian section.

Let now I, J be nonempty finite sets. To get a symmetric monoidal structure on this section, we need to establish the isomorphism

$$(I \sqcup J,\ a_{2,I\sqcup J}(x)) \widetilde{\to} (I, a_{2,I}(x)) \otimes (J, a_{2,J}(x)) = (I \sqcup J,\ \tau(a_{2,I}(x) \boxtimes a_{2,J}(x))) \in \text{coGroth}(\bar{F}),$$

where τ is the map from the diagram below

$$\begin{array}{ccccc} c_{I\sqcup J} & \to & c_{*\sqcup *} & \to & c \\ \downarrow \tau & & \downarrow & & \\ c_I \otimes c_J & \stackrel{a_{2,I} \otimes a_{2,J}}{\to} & c \otimes c & & \\ \downarrow a_{1,I} \otimes a_{1,J} & & & & \\ c^{\otimes I} \otimes c^{\otimes J}, & & & & \end{array}$$

Since $a_{2,I}(x) \widetilde{\to} a_{1,I}(x^{\boxtimes I})$ and $a_{2,J}(x) \widetilde{\to} a_{1,J}(x^{\boxtimes J})$, the above diagram gives isomorphisms

$$a_{2,I}(x) \boxtimes a_{2,J}(x) \widetilde{\to} (a_{2,I} \otimes a_{2,J})(x \boxtimes x) \widetilde{\to} (a_{1,I} \otimes a_{1,J})(x^{\boxtimes I \sqcup J})$$

This gives

$$\tau(a_{2,I}(x) \boxtimes a_{2,J}(x)) \widetilde{\to} a_{1,I\sqcup J}(x^{\boxtimes I\sqcup J}) \widetilde{\to} a_{2,I\sqcup J}(x)$$

as desired.

So, I hope the answer to both questions is yes.

1.1.30. For 3.2.1. First, a general observation. Let $\mathcal{F}, \mathcal{F}' : (fSet^{surj})^{op} \to 1-\mathcal{C}at$ be right lax symmetric monoidal functors, and $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of right-lax symmetric monoidal functors (recall that right-lax symmetric monoidal functors form a category). Let $\mathcal{X} \to fSet^{surj}, \mathcal{X}' \to fSet^{surj}$ be the corresponding cartesian fibratioins, and $\bar\alpha : \mathcal{X} \to \mathcal{X}'$ the induced functor over $fSet^{surj}$. Then $\bar\alpha$ sends cartesian arrows to cartesian arrows over $fSet^{surj}$. Besides, $\bar\alpha$ is non-unital symmetric monoidal. Now $\bar\alpha$ induces a morphism

$$\mathrm{Fun}^{\otimes, cart}_{fSet^{surj}}(fSet^{surj}, \mathcal{X}) \to \mathrm{Fun}^{\otimes, cart}_{fSet^{surj}}(fSet^{surj}, \mathcal{X}')$$

For this reason a morphism $f : Z^1 \to Z^2$ of factorization prestacks over Ran with gerbes $f^*\mathcal{G}_2 \widetilde{\to} \mathcal{G}_1$ induces a functor

$$f^! : \mathrm{FactAlg}(Shv_{\mathcal{G}_2}(Z^2)) \to \mathrm{FactAlg}(Shv_{\mathcal{G}_1}(Z^1))$$

At the levet of "main objects" it sends for $f : Z^1_* \to Z^2_*$ the corresponding object of $K \in Shv_{\mathcal{G}_2}(Z^2_*)$ to $f^!K \in Shv_{\mathcal{G}_1}(Z^1_*)$.

Let $f : Z^1 \to Z^2$ be a map of factorization spaces over Ran for which f^* exists. Assume a gerbe $\mathcal{G}$ restricts to $\mathcal{G}$ along f. Do we have a functor f^* between the corresponding factorization algebras? It exists in the constructible context if f is schematic locally of finite type. Indeed, for any $\alpha : I \to J$ in $fSet^{surj}$ from the diagram (7), since $\bar\alpha^! = \bar\alpha^*$, we see that we get a natural transformation μ of functors $(fSet^{surj})^{op} \to 1-\mathcal{C}at$ sending I to $Shv_{\mathcal{G}^2_I}(Z^2_I) \stackrel{f^*}{\to} Shv_{\mathcal{G}^1_I}(Z^1_I)$. Then μ is compatible with the right-lax symmetric monoidal structrures on these functors, as we see from (8). Hence, gives the desired morphism

$$f^* : \mathrm{FactAlg}(Shv_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}})) \to \mathrm{FactAlg}(Shv_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}}))$$

1.1.31. For 3.2.1. Let $f : Z^1 \to Z^2$ be a morphism of factorization prestacks over Ran. Assume $f : Z^1_* \to Z^2_*$ is ind-schematic. Then for any $I \in fSet^{surj}$, $f : Z^1_I \to Z^2_I$ is ind-schematic, as this is the map $\mathrm{Ran}^I_d \times_{\mathrm{Ran}} Z^1_* \to \mathrm{Ran}^I_d \times_{\mathrm{Ran}} Z^2_*$ obtained by base change from $f : Z^1_* \to Z^2_*$. Then we get a natural transformation η of functors $(fSet^{surj})^{op} \to 1-\mathcal{C}at$ sending I to $Shv_{\mathcal{G}^1_I}(Z^1_I) \stackrel{f_*}{\to} Shv_{\mathcal{G}^2_I}(Z^2_I)$, because for any $\alpha : I \to J$ in $fSet^{surj}$ the square is cartesian

$$\begin{array}{ccc} Z^1_I & \stackrel{f}{\to} & Z^2_I \\ \downarrow \bar\alpha & & \downarrow \bar\alpha \\ Z^1_J & \stackrel{f}{\to} & Z^2_J, \end{array} \tag{7}$$

and $\bar\alpha$ is etale. Moreover, η is compatible with the right-lax symmetric monoidal structures on the corresponding functors $(fSet^{surj})^{op} \to 1-\mathcal{C}at$, because for any $I, J \in fSet^{surj}$ the square is cartesian

$$\begin{array}{ccc} Z^1_{I\sqcup J} & \stackrel{r}{\to} & Z^1_I \times Z^1_J \\ \downarrow f & & \downarrow f\times f \\ Z^2_{I\sqcup J} & \stackrel{r}{\to} & Z^2_I \times Z^2_J, \end{array} \tag{8}$$

here r is an open immersion. For this reason we get as above the functor

$$f_* : \mathrm{FactAlg}(Shv_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}})) \to \mathrm{FactAlg}(Shv_{\mathcal{G}^1}(Z^2_{\mathrm{Ran}}))$$

1.1.32. For 3.3.2. The map $(\mathrm{Ran}^J)_{disj}\to \mathrm{Ran}$ used in the definition of Ran^J_* is the projection on the factor corresponding to $*\in J$.

The op-lax compatibility of (3.7) with the module structure is given by natural maps

$$(\mathrm{Ran}^{I\sqcup J})_{disj}\times_{\mathrm{Ran}}\mathrm{Ran}_x\to (\mathrm{Ran}^I)_{disj}\times(\mathrm{Ran}^J_*)_{disj}$$

for $I\in fSet^{surj}, J\in fSet^{surj}_*$. In the LHS the map $(\mathrm{Ran}^{I\sqcup J})_{disj}\to\mathrm{Ran}$ is the projection on the factor corresponding to $*\in I\sqcup J$.

1.1.33. For 3.3.3. I think it is necessary to write explicitly that op-lax compatibility with actions is given by maps $\tilde Z_{I\sqcup J}\to Z_I\times\tilde Z_J$ for $I\in fSet^{surj}, J\in fSet^{surj}_*$, which are open immersions.

1.1.34. *Digression: factorization modules.* Let C be a symmetric monoidal category admitting fibre products, we equip it with a non-unital cartesian symmetric monoidal structure. So, $C\in CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$. Equip C_{corr} with the induced non-unital symmetric monoidal structure, so $C_{corr}\in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$.

Assume $F: C^{op}\to 1-\mathcal{C}\mathrm{at}$ be a right-lax symmetric monoidal functor, we assume it factors through $1-\mathcal{C}\mathrm{at}_{cocmpl}$ as in Section 1.1.29. Consider the category $\mathrm{Groth}_{corr}(F)$ as above, it is symmetric monoidal, and the projection $q:\mathrm{Groth}_{corr}(F)\to C_{corr}$ is symmetric monoidal.

Write $CAlgMod^{nu}$ for the operad of a non-unital commutative algebra plus module (cf. [43], 3.4.5), we also denote it by $\mathcal{C}^{\otimes}$.

Let $(c,m)\in CAlgMod^{nu}(C_{corr})$, here $c\in CAlg^{nu}(C_{corr})$, so we have the categories $Mult_F(c)\subset Mult^w_F(c)$ as in Section 1.1.29.

Let $x\in Mult_F(c)$. Assume that x is extended to an object

$$(x,y)\in CAlgMod^{nu}(\mathrm{Groth}_{corr}(F))$$

over $(c,m)\in CAlgMod^{nu}(C_{corr})$. Here $x\in F(c), y\in F(m)$.

For a non-empty finite set I we have the action diagram $c^I\times m\stackrel{b_1}{\leftarrow} m_I\stackrel{b_2}{\to} m$. We say that the object (x,y) is multiplicative iff for any finite nonempty set I the natural map

$$b_1(x^{\boxtimes I}\boxtimes y)\to b_2(y)$$

in $F(m_I)$ is an isomorphism.

Let now $\mathrm{Env}_{Surj}(\mathcal{C})$ be the category considered in ([43], 3.4.6). Its objects are pairs $(I, S\subset I)$, where I is a non-empty finite set, here S could be empty. A morphism $(I,S)\to(I,S')$ is a surjective map $\alpha: I\to I'$ with $\alpha(S)\subset S'$ such that the induced map $S\to S'$ is bijective. The category $\mathrm{Env}_{Surj}(\mathcal{C})$ is non-unital symmetric monoidal: the product of (I,S) and (I',S') is $(I\sqcup I', S\sqcup S')$. Recall that we have canonically

$$\mathrm{Fun}^{\otimes}(\mathrm{Env}_{Surj}(\mathcal{C}), C_{corr})\widetilde{\to} CAlgMod^{nu}(C_{corr}),$$

where the LHS denotes the category of non-unital symmetric monoidal functors. The object (c,m) gives rise to a non-unital symmetric monoidal functor $f:\mathrm{Env}_{Surj}(\mathcal{C})\to C_{corr}$, $(I,S)\mapsto c^{I-S}\times m^S$. It sends a map $\alpha:(I,S)\to(I',S')$ to the corresponding action diagram $c^{I-S}\times m^S\leftarrow mult_\alpha\to c^{I'-S'}\times m^{S'}$ in C, which is a morphism in C_{corr}.

For $i' \in I'$ write $\alpha_{i'} : (I_{i'}, S_{i'}) \to (i', S'_{i'})$ denote the restriction of α, here $S'_{i'} = S' \cap \{i'\}$. Note that

$$mult_\alpha \widetilde{\to} \prod_{i' \in I'} mult_{\alpha_{i'}}.$$

By Section 1.1.26,

$$\mathrm{Fun}^{\otimes, dir}(Tw(\mathrm{Env}_{Surj}(\mathcal{C})), C) \widetilde{\to} \mathrm{Fun}^{\otimes}(\mathrm{Env}_{Surj}(\mathcal{C}), C_{corr})$$

The corresponding functor $\bar{f} : Tw(\mathrm{Env}_{Surj}(\mathcal{C})) \to C$ sends $(I, S) \stackrel{\alpha}{\to} (I', S')$ to $mult_\alpha$.

We will see that $Tw(\mathrm{Env}_{Surj}(\mathcal{C}))^{op}$ is a monoidal envelope of something.

Let $fSets$ be the category of nonempty finite sets and surjections. Let $fSets_*$ be the category of pointed finite sets and surjective morphisms (preserving the distinguished point). View $fSets_*$ as a module over $fSets$ via the disjoint union.

Now view $fSets_*^{op}$ as a module category over $fSets^{op}$. This pair defines a $\mathcal{C}^{\otimes}$-monoid in $1-\mathcal{C}\mathrm{at}$, hence (by [36], 2.4.2.4) a cocartesian fibration $\mathcal{X}^{\otimes} \to \mathcal{C}^{\otimes}$, which is a $\mathcal{C}^{\otimes}$-monoindal category. The composition $\mathcal{X}^{\otimes} \to \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is an ∞-operad.

Lemma 1.1.35. *There is a canonical equivalence of non-unital symmetric monoidal categories*

$$\mathrm{Env}_{Surj}(\mathcal{X}) \widetilde{\to} Tw(\mathrm{Env}_{Surj}(\mathcal{C}))^{op},$$

where $\mathrm{Env}_{Surj}(\mathcal{X})$ *is the* $Surj$*-monoidal envelope of* $\mathcal{X}^{\otimes}$*.*

Proof. The objects of $\mathrm{Env}_{Surj}(\mathcal{X})$ are collections: a non-empty finite set I, and a collection $\{J_i, S_i \subset J_i\}_{i \in I}$, where J_i is a finite nonempty set, and S_i has at most one element. We think of the above datum as an object $x = \oplus_{i \in I}(J_i, S_i)$ of $\mathcal{X}^{\otimes}$ over $(*, I \sqcup *) \in Surj$.

A morphism from $x = (I, \{J_i, S_i\}_{i \in I})$ to

$$x' = (I', \{J'_{i'}, S'_{i'}\}_{i' \in I'})$$

is a surjection $\phi : I \to I'$ defining an active morphism $\bar{\phi} : I \sqcup * \to I' \sqcup *$ in $Surj$ and an (automatically active) morphism $x \to x'$ in $\mathcal{X}^{\otimes}$ over $\bar{\phi}$.

We write x as a surjection of finite sets $\tau : J \to I$ together with a subset $S \subset J$ such that $S_i = S \cap \tau^{-1}(i)$. So, $\tau\mid_S : S \to I$ is injective, and we get a bijection $\tau\mid_S : S \to \tau(S)$. So, $\tau : (J, S) \to (I, \tau(S))$ is a map in $\mathrm{Env}_{Surj}(\mathcal{C})$, and $(* \sqcup I, \tau(S))$ is the underlying object of $\mathcal{C}^{\otimes}$.

Write similarly x' as a surjection of finite sets $\tau' : J' \to I'$ togerther with a subset $S' \subset J'$ defined as above, so $\tau'\mid_{S'} : S' \to I'$ is injective, and $\tau' : (J', S') \to (I', \tau'(S'))$ is a map in $\mathrm{Env}_{Surj}(\mathcal{C})$, and $(* \sqcup I', \tau'(S'))$ is the object of $\mathcal{C}^{\otimes}$ underlying x'.

So, the map $\bar{\phi}$ must be a morphism $\bar{\phi} : (* \sqcup I, \tau(S)) \to (* \sqcup I', \tau(S'))$ in $\mathcal{C}^{\otimes}$, in particular, $\phi : \tau(S) \to \tau'(S')$ is a bijection. Finally, it remains to provide for each $i' \in I'$ a surjection

$$(J'_{i'}, S'_{i'}) \to (\bigsqcup_{i \in \phi^{-1}(i')} J_i, \quad \bigsqcup_{i \in \phi^{-1}(i')} S_i)$$

preserving the corresponding distinguished subsets (mentioned after the comma). These data is the same as a morphism

$$\begin{array}{ccc} (J, S) & \stackrel{\tau}{\to} & (I, \tau(S)) \\ \uparrow & & \downarrow \phi \\ (J', S') & \stackrel{\tau'}{\to} & (I', \tau'(S')) \end{array}$$

from τ' to τ in $Tw(\mathrm{Env}_{Surj}(\mathcal{C}))$. $\square$

We get

$$\mathrm{Fun}^{\otimes}(Tw(\mathrm{Env}_{Surj}(\mathcal{C})), C)^{op} \tilde{\to} \mathrm{Fun}^{\otimes}(Tw(\mathrm{Env}_{Surj}(\mathcal{C}))^{op}, C^{op}) \tilde{\to} Alg_{\mathcal{X}/\,\mathrm{Surj}}(C^{op}),$$

by ([36], 2.2.4.9), where the RHS stands for the category operad maps $\mathcal{X}^{\otimes} \to (C^{op})^{\otimes}$ over $Surj$. Here we have denoted by $(C^{op})^{\otimes} \to Surj$ the cocartesian fibration attached to to the non-unital symmetric monoidal category C. We may think of $Alg_{\mathcal{X}/\,\mathrm{Surj}}(C^{op})$ as the category of pairs of functors $h_* : fSets_*^{op} \to C^{op}$, $h : fSets^{op} \to C^{op}$, where h is right lax symmetric monoidal, and h_* is right lax compatible with h in the sense of ([20], ch. I.1, 3.5.1).

The conclusion is that we get a full embedding of $CAlgMod^{nu}(C_{corr})$ into

$$Alg_{\mathcal{X}/\,\mathrm{Surj}}(C^{op})^{op},$$

the latter category can be reformulated in terms of left lax functors.

Explicitly, consider the natural map

$$l : \mathcal{X} \to \mathrm{Env}_{Surj}(\mathcal{X}) \tilde{\to} Tw(\mathrm{Env}_{Surj}(\mathcal{C}))^{op},$$

here $\mathcal{X} = \mathcal{X}^{\otimes}_{\langle 1\rangle}$. The left lax functor $h : fSets \to C$ attached to c was described in Section 1.1.27. For $I \in fSets$ write $c^I \leftarrow mult_I \to c$ for the product map in C_{corr}, write $c^I \times m \leftarrow mult_I^m \to m$ for the action map in C_{corr}. Then $h(I) = mult_I$, and $h_*(* \sqcup I) = mult_I^m$. The left lax structure on h_* is as follows: given $I \in fSets, (* \sqcup J) \in fSets_*$ we have a canonical map

$$mult^m_{I\sqcup J} \to mult_I \times mult^m_J$$

It comes from a morphism in $Tw(\mathrm{Env}_{Surj}(\mathcal{C}))$ given by the diagram

$$\begin{array}{ccc} I \sqcup J \sqcup * & \stackrel{(q,\gamma)}{\to} & * \sqcup * \\ \uparrow \mathrm{id} & & \downarrow \\ I \sqcup J \sqcup * & \to & * \end{array}$$

Here $q : I \to *$ and $\gamma : J \sqcup * \to *$.

1.1.36. For 3.4.1. If $Z^i \to \mathrm{Ran}_x$ are factorization module spaces with respect to a factorization space $Z \to \mathrm{Ran}$, let $f : Z^1 \to Z^2$ be a map of Z-factorization spaces over Ran_x. Assume that $(\mathcal{G}, \mathcal{G}^2)$ is a pair of compatible factorization gerbes over Z and Z^2, and $\mathcal{G}^1 = f^*\mathcal{G}^2$. For $I \in fSet_*^{surj}, J \in fSet^{surj}$ the square is cartesian

$$\begin{array}{ccc} Z^1_{I\sqcup J} & \hookrightarrow & Z_J \times Z^1_I \\ \downarrow & & \downarrow \\ Z^2_{I\sqcup J} & \hookrightarrow & Z_J \times Z^2_I, \end{array} \tag{9}$$

and the horizontal arrows are open immersions. Let now $\mathcal{A} \in \mathrm{FactAlg}(Shv_{\mathcal{G}}(Z))$. We claim that in the constructible context we get the morphism

$$f_! : \mathcal{A} - \mathrm{FactMod}(Shv_{\mathcal{G}^1}(Z^1)) \to \mathcal{A} - \mathrm{FactMod}(Shv_{\mathcal{G}^2}(Z^2))$$

Indeed, consider the corresponding functors $(fSet_*^{surj})\to 1-\mathcal{C}\mathrm{at},\ I\mapsto Shv_{\mathcal{G}^i}(Z^i_I)$. Then

$$Shv_{\mathcal{G}^1}(Z^i_1)\stackrel{f_!}{\to} Shv_{\mathcal{G}^2}(Z^2_I)$$

is a natural transformation of these functors right-lax compatible with the actions. Indeed, first for $\alpha: I\to J$ in $fSet_*^{surj}$ the square is cartesian

$$\begin{array}{ccc} Z^1_I & \stackrel{f}{\to} & Z^2_I\\ \downarrow \bar\alpha && \downarrow \bar\alpha\\ Z^1_J & \stackrel{f}{\to} & Z^2_J, \end{array}$$

and the maps $\bar\alpha$ are etale. This gives $\bar\alpha^! f_!\widetilde{\to} f_!\bar\alpha^!$, so we get a morphism of functors.

Now (9) gives the commutativity of the diagram

$$\begin{array}{ccc} Shv_{\mathcal{G}}(Z_J)\otimes Shv_{\mathcal{G}^1}(Z^1_I) & \to & Shv_{\mathcal{G}^1}(Z^1_{I\sqcup J})\\ \downarrow \mathrm{id}\otimes f_! && \downarrow f_!\\ Shv_{\mathcal{G}}(Z_J)\otimes Shv_{\mathcal{G}^2}(Z^2_I) & \to & Shv_{\mathcal{G}^2}(Z^2_{I\sqcup J}) \end{array}$$

where the top horizontal arrow sends (L,M) to $(L\boxtimes M)\mid_{Z^1_{I\sqcup J}}$, and similarly for the low horizontal arrow. This gives our claim.

1.1.37. For 4.3.1. The scheme Conf is naturally an object of $CAlg^{nu}(\mathrm{PreStk}_{lft})$, so we have the notion of a factorization gerbe on it as for any object of $CAlg^{nu}(\mathrm{PreStk}_{lft})$. Namely,

$$\mathrm{Map}_{CAlg^{nu}(\mathrm{PreStk}_{corr})}(Conf, B^2_{et}(A))\times_{\mathrm{Map}_{\mathrm{PreStk}_{corr}}(Conf,B^2_{et}(A))}\mathrm{Map}_{\mathrm{PreStk}}(Conf,B^2_{et}(A))$$

is the space of factorization A-gerbes on Conf.

1.1.38. For 4.3.3. We use here ([43], 9.2.28).

Let us check the following. Let Y be a scheme of finite type, $f: Y'\to Y$ be etale surjective, $\mathcal{G}$ be a $e^{*,tors}$-gerbe on Y, which becomes trivial on Y'. We want to check that $Shv_{\mathcal{G}}(Y)$ is compactly generated. We have the adjoint pair $f^*: Shv_{\mathcal{G}}(Y)\rightleftarrows Shv_{\mathcal{G}}(Y'): f_*$, and $f^*=f^!$. Here we denoted also by $\mathcal{G}$ the restriction of $\mathcal{G}$ to Y'. The category $Shv_{\mathcal{G}}(Y')=Shv(Y')$ is compactly generated, because Y' is of finite type. Besides, $f^*: Shv_{\mathcal{G}}(Y)^c\to Shv(Y')^c$, because its right adjoint is continuous.

We may assume that A is a finite abelian group and f is a $B_{et}(A)$-torsor on Y. In this case $f_!\widetilde{\to} f_*$ canonically. Indeed, for $F\in Shv(Y')$ is suffices to show that $f_!F\to f_*F$ is an isomorphism after an etale localization, which reduces the question to the case of the trivial $B_{et}(A)$-torsor on Y. In this case both $f_!F$ and f_*F is the the direct summand of F on which A acts trivially. Further, $f_!$ is left adjoint to $f^!=f^*$. Since f^* is conservative, the essential image of $f_!$ generates $Shv_{\mathcal{G}}(Y)$. Why $Shv(Y')$ is compactly generated? In the constructible context this is automatic.

It is better maybe to argue as follows. Assume $f: Y'\to Y$ etale and schematic, surjective. So, f is an etale cover for Y. Let $Y'^{\bullet}/Y$ be the Cech nerve of f. Since Shv satisfies the etale descent, $Shv_{\mathcal{G}}(Y)=Tot(Shv(Y'^{\bullet}/Y))$. Moreover, for each transition map $a: Y'^m/Y\to Y'^n/Y$ the functor $a^!=a^*$ admits a left adjoint $a_!$. Passing to left adjoints, we get $Shv_{\mathcal{G}}(Y)\widetilde{\to}\mathrm{colim}_{\boldsymbol{\Delta}^{op}} Shv(Y'^{\bullet}/Y)$.

Now for any injective map $\alpha : [n] \to [m]$ and the corresponding map $\bar{\alpha} : Y'^m/Y \to Y'^n/Y$, $\bar{\alpha}_!$ preserves compact object, because $\bar{\alpha}^!$ is continuous. Since each $Shv(Y'^m/Y)$ is compactly generated, we may apply ([20], ch. I.1, 7.2.7) with my impovement ([43], 4.2.8). Thus, $Shv_{\mathcal{G}}(Y)$ is compactly generated.

Consider now an ind-scheme of ind-finite type Y with a gerbe $\mathcal{G}$. Write $Y = \operatorname{colim}_i Y_i$, where the transition maps $f_{ij} : Y_i \to Y_j$ are closed immersions. Write also $\mathcal{G}$ for the restriction of $\mathcal{G}$ to Y_i for each i. We get $Shv_{\mathcal{G}}(Y) \widetilde{\to} \operatorname{colim}_i Shv_{\mathcal{G}}(Y_i)$ in DGCat_{cont}. Each $Shv_{\mathcal{G}}(Y_i)$ is compactly generated by the above, and the functor $(f_{ij})_! : Shv_{\mathcal{G}}(Y_i) \to Shv_{\mathcal{G}}(Y_j)$ preserves compact objects, because $f_{ij}^!$ is continuous. So, as above, $Shv_{\mathcal{G}}(Y)$ is compactly generated. We may assume actually that Y is pseudo-proper here.

Lemma 1.1.39. *1) Let $Y = \operatorname{colim}_{i \in I} Y_i$ be an ind-scheme of ind-finite type, here I is filtered, Y_i is of finite type. If $i \to j$ in I then $Y_i \to Y_j$ is a closed immersion. Then any $K \in Shv(Y)^c$ is of the form $(i_i)_! K'$ for $i_i : Y_i \to Y$, $K' \in Shv(Y_i)^c$ for some i.*

2) If $Y = \operatorname{colim}_{i \in I} Y_i$ be an ind-algebraic stack, I filtered, Y_i is an algebraic stack locally of finite type. If $i \to j$ in I then $Y_i \to Y_j$ is a closed immersion. Then any $K \in Shv(Y)^c$ is of the form $(i_i)_! K'$ for $i_i : Y_i \to Y$, $K' \in Shv(Y_i)^c$ for some i.

Proof. 1) Write $Y \widetilde{\to} \operatorname{colim}_{i \in I} Y_i$, where Y_i is a scheme of finite type (algebraic stack of finite type), and I is filtered. Then $Shv(Y) \widetilde{\to} \operatorname{colim}_i Shv(Y_i)$. Now we apply ([20], ch. I.1, 7.2.6) to describe $Shv(Y)^c$. By (HA, 1.1.4.6), $1 - \mathcal{C}\mathrm{at}^{St}$ admits filtered colimits, and the inclusion $1 - \mathcal{C}\mathrm{at}^{St} \to 1 - \mathcal{C}\mathrm{at}$ preserves filtered colimits. Recall that

$$Shv(Y) \widetilde{\to} \operatorname{Ind}(\operatorname{colim}_{i \in I} Shv(Y_i)^c),$$

where the colimit inside is calculated in $\mathrm{DGCat}^{non-cocmpl}$ (notation from [20], ch. I.1, 10.3.1). By ([20], ch. I.1, 7.2.4), any compact object of $Shv(Y)$ is a direct summand in $Shv(Y)$ of an object $F \in \operatorname{colim}_{i \in I} Shv(Y_i)^c$. By ([43], 13.1.14), F comes from an object of $Shv(Y_i)^c$ for some i, so the same holds for its direct summand, because the inclusion $Shv(Y_i)^c \subset Shv(Y_i)$ is closed under retracts by (HTT, 5.3.4.16).

2) is similar. □

1.1.40. For 4.3.3. If $f : Y^0 \hookrightarrow Y$ is an open embedding (so schematic morphism) in PreStk_{lft} then $f_* : Shv_{\mathcal{G}}(Y^0) \to Shv_{\mathcal{G}}(Y)$ is defined and continuous, so its left adjoint $f^!$ preserves compact objects (as in [21], 1.4.8).

Lemma 1.1.39 remains valid when we twist $Shv(Y)$ by a gerbe. This implies that $Shv_{\mathcal{G}}(Y)^c \subset Shv_{\mathcal{G}}(Y)^{loc.c}$.

Let us explain the definition of the equivalence $\mathbb{D} : (Shv_{\mathcal{G}}(Y)^{loc.c})^{op} \widetilde{\to} Shv_{\mathcal{G}^{-1}}(Y)^{loc.c}$. Assume Y is an ind-scheme. For each i and each quasi compact open subscheme $Y_i^0 \subset Y_i$ we have the Verdier duality

$$\mathbb{D} : (Shv_{\mathcal{G}}(Y_i^0)^c)^{op} \widetilde{\to} Shv_{\mathcal{G}^{-1}}(Y_i^0)^c \tag{10}$$

Further,

$$Shv_{\mathcal{G}}(Y_i) \widetilde{\to} \lim_{Y_i^0 \in \mathcal{C}^{op}} Shv_{\mathcal{G}}(Y_i^0)$$

Here $\mathcal{C}$ is the category of quasi-compact open subschemes of Y_i. For QCoh such an equivalence is in ([20], I.3, 1.4.4). This comes from the fact that Shv satisfies the etale descent for morphisms in PreStk_{lft}. By definition,

$$Shv_{\mathcal{G}}(Y_i)^{loc.c} \widetilde{\to} \lim_{Y_i^0\in \mathcal{C}^{op}} Shv_{\mathcal{G}}(Y_i^0)^c$$

The corresponding restrictions preserve compact objects because for an open immersion $j: V^0\hookrightarrow V$ in PreStk_{lft}, j_* is continuous ([21], 1.4.8). So,

$$(Shv_{\mathcal{G}}(Y_i)^{loc.c})^{op} \widetilde{\to} \lim_{Y_i^0\in \mathcal{C}^{op}} (Shv_{\mathcal{G}}(Y_i^0)^c)^{op}$$

The desired equivalence is obtained by passing to the limit over $Y_i^0 \in \mathcal{C}^{op}$ in the equivalences (10), and then to the colimit over I, here $Y\widetilde{\to} \mathrm{colim}_{i\in I} Y_i$ and I is filtered. We used the fact that the projection $1-\mathcal{C}\mathrm{at}^{St}\to 1-\mathcal{C}\mathrm{at}$ preserves limits.

We also use the following: $(\mathrm{colim}_{i\in I} Shv_{\mathcal{G}}(Y_i)^{loc.c})^{op} \widetilde{\to} \mathrm{colim}_{i\in I} (Shv_{\mathcal{G}}(Y_i)^{loc.c})^{op}$, where the colimit is calculated, say in $1-\mathcal{C}\mathrm{at}^{St}$ (the latter category admits filtered colimits). Indeed, the functor $D\mapsto D^{op}$ is an autoequivalence of $1-\mathcal{C}\mathrm{at}^{St}$. Moreover, the natural map $\mathrm{DGCat}^{non-cocmpl}\to 1-\mathcal{C}\mathrm{at}^{St}$ preserves filtered colimits. Recall also that $1-\mathcal{C}\mathrm{at}^{St}\to 1-\mathcal{C}\mathrm{at}$ preserves filtered colimits by ([36], 1.1.4.6).

1.1.41. For 4.4.1. Since the action of $\Lambda^\sharp$ on $Conf_{\infty x}$ commutes with the action of $Conf$, for $\lambda\in\Lambda^\sharp$, $(\mathrm{Tr}^\lambda)^*\mathcal{G}^\Lambda$ is naturally a factorization module gerbe over $(Conf, \mathcal{G}^\Lambda)$. Then it is required that (4.8) is an isomorphism of factorization module gerbes over $(Conf, \mathcal{G}^\Lambda)$.

1.1.42. For 4.5.3. One may get (4.13) from the universal property of the tensor product. The composition

$$\mathrm{Rep}(T_{\tilde H})\otimes Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x})\to \mathrm{Rep}(T_{\tilde H})\otimes_{\mathrm{Rep}(T_H)} Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x})\to Shv_{\mathcal{G}^{\tilde\Lambda}}(\widetilde{\mathrm{Conf}}_{\infty x})$$

is the action of $\mathrm{Rep}(T_{\tilde H})$ on $Shv_{\mathcal{G}^{\tilde\Lambda}}(\widetilde{\mathrm{Conf}}_{\infty x})$.

1.1.43. For Lm. 4.5.5, proof. The assumption implies that $\tilde\Lambda^\sharp/\Lambda^\sharp\widetilde{\to}\tilde\Lambda/\Lambda$ is a lattice, so torsion free. Pick a decomposition $\tilde\Lambda^\sharp=\Lambda^\sharp\oplus\tilde\Lambda_1$, where $\tilde\Lambda_1$ is a lattice. Then $\widetilde{\Lambda}_1\oplus\Lambda=\tilde\Lambda$.

Let T_1 be the torus whose weight lattice is $\widetilde{\Lambda}_1$, so $T_{\tilde H}\widetilde{\to} T_H\times T_1$. We have $\mathrm{QCoh}(B(T_H))\otimes \mathrm{QCoh}(B(T_1))\widetilde{\to}\mathrm{QCoh}(B(T_{\tilde H}))$ accordingly, so

$$\mathrm{Rep}(T_{\tilde H})\otimes_{\mathrm{Rep}(T_H)} Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x})\widetilde{\to}\mathrm{Rep}(T_1)\otimes Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x})$$

Further,

$$Shv_{\mathcal{G}^{\tilde\Lambda}}(\widetilde{\mathrm{Conf}}_{\infty x})\widetilde{\to}\prod_{\mu\in\tilde\Lambda} Shv_{\mathcal{G}^{\tilde\Lambda}}(\widetilde{\mathrm{Conf}}^\mu_{\infty x}),$$

where $\widetilde{\mathrm{Conf}}^\mu_{\infty x}\subset\widetilde{\mathrm{Conf}}_{\infty x}$ is the connected component given by fixing the degree to be μ. For $\lambda'\in\widetilde{\Lambda}_1$ and $\lambda\in\Lambda$ let $\mu=\lambda+\lambda'$. Then the map $\mathrm{Tr}^{\lambda'}: Conf^\lambda_{\infty x}\to\widetilde{\mathrm{Conf}}^\mu_{\infty x}$ is an isomorphism.

We have $\mathrm{Rep}(T_1)=\oplus_{\tilde\lambda_1\in\widetilde{\Lambda}_1}\mathrm{Vect}$, so

$$\mathrm{Rep}(T_1)\otimes Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x})\widetilde{\to}\bigoplus_{\tilde\lambda_1\in\widetilde{\Lambda}_1} Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x}),$$

the $\tilde{\lambda}_1$-summand here is the desired category of sheaves on the union of components corresponding to $\Lambda + \tilde{\lambda}_1 \subset \tilde{\Lambda}$. In other words, the action of the summand Vect corresponding to $\tilde{\lambda}_1$ gives an isomorphism

$$Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x}) \tilde{\to} Shv_{\mathcal{G}^{\tilde{\Lambda}}}(\widetilde{\mathrm{Conf}}^{\lambda_1+\Lambda}_{\infty x})$$

1.1.44. For 4.6.2 Here is a proof of a simpler claim.

Lemma 1.1.45. *Let T be a torus. Then*
i) the unit section $\mathrm{Ran} \to \mathrm{Gr}_{T,\mathrm{Ran}}$ *is a closed immersion.*
ii) if $1 \to G \to G' \to T \to 1$ is an exact sequence of reductive groups then $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G',\mathrm{Ran}}$ *is a closed immersion.*

Proof. i) We may assume $T = \mathbb{G}_m$. Then, by ([59], Lemma 31.18.9) an S-point of $\mathrm{Gr}_{T,\mathrm{Ran}}$ over a S-point $\mathcal{I} \in \mathrm{Ran}(S)$ is a relative Cartier divisor D on $S \times X$ over S such that D is contained set-theoretically in $\Gamma_{\mathcal{I}}$. Write $D = D_1 - D_2$, where D_i are relative effective Cartier divisors over S. Then the desired closed subscheme is given by the condition that $D_1 = D_2$. More precisely, D_1, D_2 give rise to a closed subscheme of $S \times X \times X$, which we intersect with $S \times X$, let $D_{12} \to S \times X$ be the resulting scheme. Then $D_{12} \to S$ is proper, and the required closed subscheme of S is the image of the proper map $D_{12} \to S$.
ii) follows from i). □

Write $S \mapsto \mathrm{Div}(X)(S)$ for the functor sending $S \in \mathrm{Sch}^{aff}$ to the set of relative Cartier divisors on $S \times X$ over S. Let $\mathrm{Div}(X)_{\mathrm{Ran}} : (\mathrm{Sch}^{aff})^{op} \to Sets$ be the functor sending S to the set of $(\mathcal{I} \in \mathrm{Ran}(S), D \in \mathrm{Div}(X)(S)$ such that D is set-theoretically contained in $\Gamma_{\mathcal{I}}$. Then $\mathrm{Gr}_{\mathbb{G}_m,\mathrm{Ran}} \tilde{\to} \mathrm{Div}(X)_{\mathrm{Ran}}$, and $\mathrm{Gr}_{T,\mathrm{Ran}} \tilde{\to} \mathrm{Hom}(\check{\Lambda}, \mathrm{Div}(X)_{\mathrm{Ran}})$. This means that an S-point of $\mathrm{Gr}_{T,\mathrm{Ran}}$ is $\mathcal{I} \in \mathrm{Map}(S, \mathrm{Ran})$, and a homomorphism from $\check{\Lambda}$ to the group of relative Cartier divisors D on $S \times X$ over S such that D is set-theoretically included into $\Gamma_{\mathcal{I}}$.

By definition, a relative Cartier divisor on $S \times X$ over S is written as $D_1 - D_2$, where D_i are relative effective Cartier divisors on $S \times X$ over S. We identify $D_1 - D_2 = D_1' - D_2'$ iff $D_1 + D_2' = D_1' + D_2$ as relatiive effective Cartier divisors on $S \times X$ over S.

Lemma 1.1.46. *Let $T' \to T$ be a surjective homomorphism of tori with a finite kernel. Then* $\mathrm{Gr}_{T',\mathrm{Ran}} \to \mathrm{Gr}_{T,\mathrm{Ran}}$ *is a closed immersion.*

Proof. Write $T' = \Lambda' \otimes \mathbb{G}_m, T = \Lambda \otimes \mathbb{G}_m$, where $\Lambda' \subset \Lambda$ is a sublattice of finite index. There is a base $e_1, \dots, e_n \in \Lambda$ and positive integers $m_1, \dots, m_n$ such that $\{m_i e_i\}$ is a base of Λ'. So, we are reduced to the case of the map $\mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto z^n$.

We show that the multiplication $\mathrm{Div}(X)_{\mathrm{Ran}} \to \mathrm{Div}(X)_{\mathrm{Ran}}$ by n is a closed immersion. For this it suffices to show that $\mathrm{Div}(X) \xrightarrow{n} \mathrm{Div}(X)$ is a closed immersion. The latter follows from the fact that $\mathrm{Div}^{eff}(X) \tilde{\to} \sqcup_m X^{(m)}$. □

1.1.47. For 4.6.2. Let us underline the definition of $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{non-pos}$. In this definition we assume $G = G_{sc}$. For $S \in \mathrm{Sch}_{ft}$, its S-point is a datum of a T-torsor $\mathcal{F}_T$ on $S \times X$, $\mathcal{J} \in \mathrm{Hom}(S, \mathrm{Ran})$, a trivialization $\mathcal{F}_T \tilde{\to} \omega^\rho \mid_{S \times X - \Gamma_{\mathcal{J}}}$ such that for any $\check{\lambda} \in \check{\Lambda}^+$,

$\check{\lambda}(\mathcal{F}_T) \to \check{\lambda}(\omega^\rho)$ is regular over X. We do not have to require that the quotient is flat over S, this is automatic due to the following result from ([59], Lemma 31.18.9).

Claim 1.1.48. *Let $\phi : X \to S$ be a flat morphism of schemes which is locally of finite presentation. Let $Z \subset X$ be a closed subscheme. Let $x \in Z$ with image $s \in S$.*

i) If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, then there exists an open $U \subset X$ and a relative effective Cartier divisor $D \subset U$ such that $Z \cap U \subset D$ and $Z_s \cap U = D_s$.

ii) If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, the morphism $Z \to X$ is of finite presentation, and $Z \to S$ is flat at x, then we can choose U and D such that $Z \cap U = D$.

iii) If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x and Z is a locally principal closed subscheme of X in a neighbourhood of x, then we can choose U and D such that $Z \cap U = D$.

At the level of k-points, a point $\omega^\rho(\lambda y)$ with $\lambda \in \Lambda$ and natural trivialization outside y is in $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{non-pos}$ iff $\lambda \in \Lambda^{neg}$.

1.1.49. For 4.6.4. This is analogous to [GLys, 4.1.2], which claimed that the map from the combinatorial grassmanian to the usual grassmanian over Ran induces an isomorphism after sheafification in the topology of finite surejctive maps.

For any map $S \to X$ its graph $\Gamma \subset S \times X$ is an effective Cartier divisor on $S \times X$. Indeed, the diagonal $X \subset X \times X$ is a Cartier divisor, so its preimage under $S \times X \to X \times X$ is also a Cartier divisor. Assume now given $J \in \mathrm{Ran}(S)$ given by a collection $S \xrightarrow{j} X$, $j \in J$. Since each $\Gamma_i \subset S \times X$ is a Cartier divisor, their sum is also a Cartier divisor. Assume in addition given a subsheaf $\beta : L \subset \mathcal{O}_{S\times X}$, where L is a line bundle, and β is an isomorphism over $U_I = S \times X - \Gamma_I$. Then $\mathcal{O}/L$ is flat over S by Claim 1.1.48, so $(L \subset \mathcal{O})$ defines a relative Cartier divisor on $S \times X$ over S.

For this reason we get a morphism $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{non-pos} \to \mathrm{Conf}$ sending $(\mathcal{F}_T, \mathcal{J} \in \mathrm{Ran}, \beta : \mathcal{F}_T \widetilde{\to} \omega^\rho \mid_{S\times X - \Gamma_{\mathcal{J}}})$ to D such that β induces an isomorphism $\mathcal{F}(T) \widetilde{\to} \mathcal{O}(D)$.

By the way, for a closed subscheme $Y \subset S \times X$, the extension of vector bundles from $S \times X - Y$ to $S \times X$ is discussed here:
`https://mathoverflow.net/questions/22111/extending-vector-bundles-on-a-given-open-subscheme`
It is related to Serre's condition S_2.

Lemma 1.1.50. *The map $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg} \to \mathrm{Conf}$ is surjective in the topology of finite surjective maps.*

Proof. For each $\lambda \in \Lambda^{neg}$ written as $\lambda = -\sum n_i\alpha_i$, where α_i are simple coroots, we have a symmetrization map $\prod_i X^{n_i} \to \mathrm{Conf}^\lambda$. It decomposes as $\prod_i X^{n_i} \to (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{non-pos} \to \mathrm{Conf}$ with the image Conf^λ. □

We have $(\mathrm{Conf} \times \mathrm{Ran})^{\subset} \sqcup \mathrm{Ran} \widetilde{\to} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{non-pos}$. Here Ran corresponds to the locus of those S-points of $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{non-pos}$, where the trivialization $\alpha : \mathcal{F}_T \widetilde{\to} \omega^\rho$ extends to $S \times X$.

In fact, $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg}$ classifies $D \in \mathrm{Conf}$, $\mathcal{J} \in \mathrm{Ran}$ such that $(S\times X) - \Gamma_{\mathcal{J}} = (S \times X) - \operatorname{supp} D$, in particular $(D, \mathcal{J}) \in (\mathrm{Conf} \times \mathrm{Ran})^{\subset}$. So, $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg} \hookrightarrow (\mathrm{Conf} \times \mathrm{Ran})^{\subset}$.

I don't see if $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg} \to \mathrm{Conf}$ is an isomorphism after sheafification in the topology of finite surjective maps, though it is surjective. The projection $(\mathrm{Conf}\times\mathrm{Ran})^{\subset} \to \mathrm{Conf}$ defines an isomorphism on spaces of gerbes, and the !-pull-back

$$Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}) \to Shv_{\mathcal{G}^G}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg})$$

is fully faithful. So, (4.17) is fully faithful (but maybe an equivalence indeed).

1.1.51. For 4.6.5 and (4.17). Dennis claims that if $Y_1 \to Y_2$ is a map in PreStk_{lft} which becomes an isomorphism after sheafification in the topology of finite surjectuve maps that $Shv(Y_2) \to Shv(Y_1)$ is an equivalence. This was used also without proof in our first joint paper. What is the reference?

1.1.52. For 5.3.8. Explanation of the fact that the collection $\mathcal{M}^{\mu,!}_{\mathrm{Conf}} \in Shv_{\mathcal{G}^\Lambda}(Conf_{\infty x})$ is a set of compact generators. First, for $\mathrm{Conf}_{=\mu x} \stackrel{j_\mu}{\to} \mathrm{Conf}_{\le \mu x} \stackrel{i_\mu}{\hookrightarrow} \mathrm{Conf}_{\infty x}$ the functors $(i_\mu)_!, (j_\mu)_!$ preserve compact objects, as their right adjoints are continuous.

Now, given $\lambda, \mu \in \Lambda$ with $\lambda - \mu \le 0$, the stratification of $\mathrm{Conf}^\lambda_{\le \mu x}$ by the subschemes $\mathrm{Conf}^\lambda_{=\nu x}$ is finite. It is indexed by $\lambda \le \nu \le \mu$. So, if for $K \in Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}^\lambda_{\le\mu x})$ its !-restriction to each stratum vanishes then K vanishes itself. The claim follows now from the isomorphism $Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x}) \widetilde{\to} \lim_{\mu\in\Lambda} Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\le \mu x})$.

It is not clear why $\mathcal{M}^{\mu,*}_{\mathrm{Conf}}$, $\mu \in \Lambda$ co-generate $\mathcal{A} - \mathrm{FactMod}(Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x}))$. Let $K \in \mathcal{A} - \mathrm{FactMod}(Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x}))$ with $\mathcal{H}om(K, \mathcal{M}^{\mu,*}_{\mathrm{Conf}}) = 0$ for all $\mu \in \Lambda$. This shows that $\iota^*_\mu K = 0$ for all μ.It is not clear if this really implies that $K = 0$, because the map $Shv(\mathrm{Conf}_{\infty x}) \to \lim_{\mu\in\Lambda} Shv(\mathrm{Conf}_{\le\mu})$ given by the system of functors ι^*_μ could maybe have a "kernel". Maybe Verdier duality could help here to finish the argument.

1.1.53. For 5.4. For $Y \in \mathrm{Sch}_{ft}$ there is a t-structure on $Shv(Y)$ that we think of as being perverse. It is important that this t-structure is accessible.

The t-structure on $Shv(Y)$ for Y an ind-scheme is defined as follows. If $Y = \mathrm{colim}_{i\in I} Y_i$ with I filtered and $Y_i \in \mathrm{Sch}_{ft}$ then $Shv(Y)^{\le 0} \subset Shv(Y)$ should be the smallest full subcategory containing $Shv(Y_i)^{\le 0}$ for any i, closed under extensions and closed under small colimits. By (HA, 1.4.4.11), $Shv(Y)^{\le 0}$ is then presentable and defines an accessible t-structure on Y. We use here the fact that $Shv(Y_i)$ is generated by a small set of objects.

So, for $F \in Shv(Y)$ we have $F \in Shv(Y)^{\ge 0}$ iff for any i and the closed immersion $i_i : Y_i \to Y$, $i_i^! F \in Shv(Y_i)^{\ge 0}$. This implies that the t-structure on $Shv(Y)$ is compatible with filtered colimits. Recall that for a closed immersion $f : Y_1 \to Y_2$ with $Y_i \in \mathrm{Sch}_{ft}$, $f^!$ is left exact for the perverse t-structure.

1.1.54. For 5.4.1. The property for $F \in Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x})$ the property

$$\mathrm{Map}_{\mathcal{A}-\mathrm{FactMod}}(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}, F) = *, \text{ for any } \mu \in \Lambda$$

means that $i^!_\mu F \in Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\le\mu x})^{\ge 0}$ for any μ, that is, $F \in Shv_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty x})^{\ge 0}$.

1.1.55. For 5.4.2. The following observation is used. If $f : C \to C'$ is a t-exact functor, a map in DGCat_{cont} and f is conservative, assume $c \in C$ and $f(c) \in (C')^{\heartsuit}$. Then $c \in C^{\heartsuit}$.

1.1.56. Dennis proposed essentially the following.

Definition 1.1.57. *Let C be a* e*-linear abelian category, Λ be a partially ordered set. Assume given for $\lambda \in \Lambda$ a full subcategory $(i_\lambda)_! : C_{\leq\lambda} \subset C$. Assume that this functor admits both left i^*_λ and right $i^!_\lambda$ adjoint. For $\lambda \in \Lambda$ set $C_{<\lambda} = \operatorname{colim}_{\mu<\lambda} C_{\leq\mu}$, where the colimit is calculated in a* suitable category *(to be precised). We also assume that* $\mathrm{Vect}^\heartsuit$ *is isomorphic to the cofibre of $C_{<\lambda} \to C_{\leq\lambda}$ in the same category. Besides, the functor $(j_\lambda)^! : C_{\leq\lambda} \to C_{\leq\lambda}/C_{<\lambda}$ admits both left $(j_\lambda)_!$ and right $(j_\lambda)_*$ adjoints in the same category. Finally, we assume that $\operatorname{colim}_{\lambda\in\Lambda} C_{\leq\lambda} \to C$ is an equivalence, again for colimit calculated in* the same category*.*

The above notion is adopted for a given $(\infty, 2)$-category $\mathcal{E}$ (let's assume $\mathcal{E}^{1-\mathcal{C}\mathrm{at}}$ pointed for simplicity):

Definition 1.1.58. *Let $C \in \mathcal{E}$. Then a structure of a h.w. category on C with respect to $\mathcal{E}$ is a datum as in the previous definition, where now $(i_\lambda)_!$ and its adjoints are understood in C, the colimits are calculated in C (that is, in $C^{1-\mathcal{C}\mathrm{at}}$).*

Since we want to apply this to a Grothendieck abelian category C, one option to make the above precise is as follows: consider the 2-category $\mathcal{E}$, whose objects are presentable abelian categories, and morphisms are continuous functors. Then apply definition in this particular case to get a notion of an abelian h.w. category.

We may also apply the above to DGCat viewed as a 2-category, and get a notion of h.w. DG-category.

Remark 1.1.59. *Let now $C \in$* DGCat *with an accessible t-structure, which is compatible with filtered colimits. Assume $C_{\leq\lambda} \subset C$ for $\lambda \in \Lambda$ defines a structure of a h.w. category with respect to* DGCat *(the latter was denoted* DGCat_{cont} *in your book). We assume $C_{\leq\lambda}$ equipped with the (unique possible) t-structure such that the inclusion $C_{\leq\lambda} \to C$ is t-exact. Set $D = C^\heartsuit$ and $D_{\leq\lambda} = (C_{\leq\lambda})^\heartsuit$. The inclusion $D_{\leq\lambda} \subset D$ is continuous, both $D_{\leq\lambda}$, D are presentable by (HA, Remark 1.3.5.23). Is it true that this defines a h. w. category structure on D?*

It was not explained in the proof of 5.4.4 why in a h.w. abelian category C one has $\mathrm{Ext}^i(c^{\mu',!}, c^{\mu,*}) = 0$ for $i \geq 1$. Why this is so?

1.2. **For Part II.**

1.2.1. About invariants/coinvariants. If G is a group ind-scheme of ind-finite type then $(Shv(G), m_*)$ is monoidal (convolution monoidal structure).

The functor $Shv(G) \otimes Shv(G) \to Shv(G \times G)$ sends a compact object $F_1 \otimes F_2$ to a compact object $F_1 \boxtimes F_2$. So, this functor admits a continuous right adjoint. In the contstructible context the functor $m_* : Shv(G \times G) \to Shv(G)$ admits a continuous right adjoint. Besides, the dual to m_* is the functor $m^!$. Thus, passing to the dual in $(Shv(G), m_*)$, in the constructible context we get a coalgebra $(Shv(G), m^!)$ in DGCat_{cont}. Recall that $(Shv(G), m_*) - mod \widetilde{\to} (Shv(G), m^!) - comod$ (cf. [46]).

For any ind-scheme of ind-finite type Y, Y is a cocommutative coalgebra in PreStk_{lft} via the maps $Y \to Y \times Y$ and $Y \to \operatorname{Spec} k$, hence a commutative algebra in $(\mathrm{PreStk}_{lft})^{op}$. Applying the right-lax monoidal functor Shv, we get on $Shv(Y)$ a commutative algebra structure in $CAlg(\mathrm{DGCat}_{cont})$. The product is $Shv(Y)\otimes Shv(Y)\to Shv(Y\times Y)\stackrel{\vartriangle^!}{\to} Shv(Y)$. We denote this algebra $(Shv(Y),\vartriangle^!)$. It makes sense for any sheaf theory. Applying the duality, we get a coalgebra structure on $Shv(Y)$, which we denote $(Shv(Y),\vartriangle_*)$ following [8]. Recall that this duality exchanges the functors $\vartriangle_*$ and $\vartriangle^!$.

Then $(Shv(G),\vartriangle^!,m^!)$ is a Hopf algebra in the D-module case, but this is maybe wrong in the constructible context. The same for $(Shv(G),m_*,\vartriangle_*)$ (as in [8]).

In the case of $\mathcal{D}$-modules, $(Shv(G),m_*)-mod$ becomes a monoidal category.

1.2.2. If $Y\in\mathrm{PreStk}_{lft}$ is equipped with a G-action then the action map $a: G\times Y\to Y$ is ind-schematic (isomorphic to the projection $Y\times G\to Y$). So, $(Shv(G),\star)$ acts on $Shv(Y)$ on the left via $F\in Shv(G), K\in Shv(Y)\mapsto a_*(F\boxtimes K)$. If $f: Y_1\to Y_2$ is an ind-schematic morphism in PreStk_{lft} commuting with G-actions then $f_*: Shv(Y_1)\to Shv(Y_2)$ is a map of $(Shv(G),\star)$-modules. Besides, $f^!$ is a map of $(Shv(G),\star)$-modules. Indeed, consider the category of correspondences $Corr(\mathrm{PreStk}_{lft})_{all,ind-sch}$, where for $Y_1,Y_2\in\mathrm{PreStk}_{lft}$ a map in this category from Y_1 to Y_2 is given by a diagram $Y_1\stackrel{a}{\leftarrow} Y_{12}\stackrel{b}{\to} Y_2$, where b is ind-schematic (of ind-finite type). Then

$$Shv: Corr(\mathrm{PreStk}_{lft})_{all,ind-sch}\to\mathrm{DGCat}_{cont},\ \ Y\mapsto Shv(Y)$$

sending the above morphism to $b_*a^!$ is right-lax symmetric monoidal ([20], Chapter 3, Section 6.1). Now if $f: Y_1\to Y_2$ is a morphism of G-modules in PreStk_{lft} then not only the horizotal map f but also the vertical map $Y_2\to Y_1$ given by the diagram $Y_2\leftarrow Y_1\stackrel{\mathrm{id}}{\to} Y_1$ in $Corr(\mathrm{PreStk}_{lft})_{all,ind-sch}$ is a morphism of G-modules in $Corr(\mathrm{PreStk}_{lft})_{all,ind-sch}$. This reduces to the fact that the corresponding diagrams are cartesian.

Consider the prestack quotient $Y/G\in\mathrm{PreStk}_{lft}$. The map $f: Y\to Y/G$ commutes with G-actions, where G acts trivially on Y/G. So, $f^!: Shv(Y/G)\to Shv(Y)$ is a map of $(Shv(G),\star)$-modules. Thus, it induces a functor

$$Shv(Y/G)\to\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect},Shv(Y)) \tag{11}$$

Is it an equivalence?

Assuming G smooth of finite type, as in ([45], 3.0.22) one shows that $Shv(Y/G)\,\widetilde{\to}\, e-comod(Shv(Y))$. Namely, e here is the constant sheaf on G, it is a coalgebra in $(Shv(G),\star)$, and we consider the corresponding category of comodules with the convolution action of $Shv(G)$ on $Shv(Y)$. The forgetful functor $e-comod(Shv(Y))\to Shv(Y)$ is $p^!$ for $p: Y\to Y/G$.

By the universal property of $\mathrm{Fun}_{(Shv(G),\star)}$, we have a canonical forgetful functor $\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect},Shv(Y))\to Shv(Y)$ (whose composition with (11) is $f^!$). Is its right adjoint continuous?

The answer to the question is yes for all sheaf theories. By definition, $Shv(Y/G)$ identifies with the limit of

$$Shv(Y)\rightrightarrows Shv(G\times Y)\stackrel{\to}{\stackrel{\to}{\to}} Shv(G^2\times Y)\ldots,$$

while $\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, Shv(Y))$ is the limit of

$$Shv(Y) \rightrightarrows Shv(G)\otimes Shv(Y) \overset{\to}{\underset{\to}{\to}} Shv(G)^{\otimes 2}\otimes Shv(Y)\dots$$

The map $Shv(G)^{\otimes n}\otimes Shv(Y)\to Shv(G^n\times Y)$ is an equivalence for $\mathcal{D}$-modules, so in this case this is easy. For all the 4 sheaf theories by ([46], 0.0.20), $\mathrm{Fun}_{Shv(G)}(\mathrm{Vect}, Shv(Y))\widetilde{\to} e_G-$ *comod*$(Shv(Y))$ also, and the functor (11) is an equivalence. So, $\mathrm{oblv}_G : Shv(Y)^G\to Shv(Y)$ identifies with $f^! : Shv(Y/G)\to Shv(Y)$.

We see that in the constructible context the functor $f^!$ admits a left adjoint $f_!$, that is, we get a dual pair $\mathrm{Av}_!^G : Shv(Y)\rightleftarrows Shv(Y)^G : \mathrm{oblv}_G$.

1.2.3. With the notations of the previous section, assume G smooth group scheme of finite type. Consider now the adjunction

$$(\mathrm{pr}_G)^L : \mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, Shv(Y)) \leftrightarrows Shv(Y) : \mathrm{pr}_G$$

given by (12) below for $C = Shv(Y)$. The functor $(\mathrm{pr}_G)^L$ comonadic by ([47], 1.3.6).

1.2.4. Let $C\in (Shv(G),\star)-mod$. Assume G smooth of finite type. Then we have the adjoint pair $p^* : \mathrm{Vect}\leftrightarrows Shv(G) : p_*$ for $p : G\to \mathrm{Spec}\, k$. Note that $Shv(G)$ is equipped with a left and right $(Shv(G),\star)$-actions. Besides, $p_* : Shv(G)\to \mathrm{Vect}$ is a monoidal functor. So, we may view the above adjunction as an adjunction in $(Shv(G),\star)-mod$ and also in $(Shv(G),\star)-mod^r$.

Applying the functor $\mathrm{Fun}_{Shv(G)}(\bullet, C)$ for the above adjunction in $(Shv(G),\star)-mod$, we get an adjoint pair

$$(\mathrm{pr}_G)^L : C^G = \mathrm{Fun}_{Shv(G)}(\mathrm{Vect}, C)\rightleftarrows C : \mathrm{pr}_G \tag{12}$$

Viewing $p^* : \mathrm{Vect}\leftrightarrows Shv(G) : p_*$ as an adjunction in $(Shv(G),\star)-mod^r$ and applying $\bullet\otimes_{Shv(G)} C$, we get an adjunction

$$\mathrm{oblv}^G : C_G = \mathrm{Vect}\otimes_{Shv(G)} C\leftrightarrows C : Av_*^G$$

Since p^* is a map of $(Shv(G),\star)$-bimodules, the functor oblv^G inherits a structure of a map of left $(Shv(G),\star)$-modules, where on C_G the action is trivial (that is, sending $F\in Shv(G), K$ to $(p_*F)\otimes K$). By the definition of $\mathrm{Fun}_{Shv(G)}$, this yields a functor $\theta_G : C_G\to C^G$. For $\mathcal{D}$-modules this is an equivalence by ([8], 2.3.12). It is an equivalence for all the 4 sheaf theories by ([24], Th. B.1.2, where H the group is assumed smooth of finite type).

1.2.5. If G is an ind-scheme of ind-finite type, assume $m : G\times G\to G$ ind-proper. Then $(Shv(G),\star)$ is rigid for any sheaf theory. My understanding is that there is no hope for it to be rigid without the ind-properness assumption.

Claim 1: let $f : G\to H$ be a surjective homomorphism of smooth group schemes of finite type. Then $f^* : Shv(H)\leftrightarrows Shv(G) : f_*$ is an adjoint pair in $Shv(G)-mod$. Namely, f_* is monoidal, and $(Shv(G),*)$ acts on itself by convolutions on the left.

Proof. We have to check that f^* is a morphism of $Shv(G)$-module categories. The square is cartesian

$$\begin{array}{ccc} G\times G & \stackrel{m}{\to} & G \\ \downarrow \mathrm{id}\times f & & \downarrow f \\ G\times H & \stackrel{f\times\mathrm{id}}{\to} H\times H \stackrel{m}{\to} & H \end{array}$$

So, for $F\in Shv(H), K\in Shv(G)$ one gets $m_*(K\boxtimes f^*F)\widetilde{\to} f^*m_*(f_*K\boxtimes F)$. □

For $C\in Shv(G)-mod$ it gives an adjoint pair $\mathrm{Fun}_{Shv(G)}(Shv(H),C)\leftrightarrows C$. If $K=\mathrm{Ker}(G\to H)$ and f is surjective then we may view $Shv(H)\widetilde{\to} Shv(G)^K\widetilde{\to} Shv(G)_K$, so

$$\mathrm{Fun}_{Shv(G)}(Shv(H),C)\widetilde{\to}\mathrm{Fun}_{Shv(K)}(\mathrm{Vect},C)\widetilde{\to} C^K$$

Claim 2 Let H,G be placid group schemes, $G\hookrightarrow H$ be a subgroup (not necessarily a placid closed immersion). Assume $G\widetilde{\to}\lim_{i\in I^{op}} G_i$, where G_i is a group scheme of finite type, I is filtered, for $i\to j$ in I the map $G_j\to G_i$ is smooth affine surjective. Write $K_i=\mathrm{Ker}(G\to G_i)$. Assume $H\widetilde{\to}\lim_{i\in I^{op}} H/K_i$ in PreStk. Assume H/G is a pro-smooth placid scheme. Consider the projection $p: H/G\to\mathrm{Spec}\, k$ as H-equivariant map. Then

i) the adjoint pair $p^*:\mathrm{Vect}\leftrightarrows Shv(H/G): p_*$ takes place in $Shv(H)-mod$;
ii) assume $C\in Shv(H)-mod(\mathrm{DGCat}_{cont})$ and G is pro-smooth. Then the above adjoint pair gives an adjoint pair in DGCat_{cont}

$$\mathrm{oblv}: C^H\leftrightarrows C^G:\mathrm{Av}_*^{H/G}$$

Proof. i) Since p is H-equivariant map, $p_*: Shv(H/G)\to\mathrm{Vect}$ is a morphism of $Shv(H)$-modules. Now the diagram is cartesian

$$\begin{array}{ccc} H\times H/G & \stackrel{act}{\to} & H/G \\ \downarrow \mathrm{pr} & & \downarrow \\ H & \to & \mathrm{Spec}\, k \end{array}$$

So, for $K\in Shv(H)$, $\mathrm{act}_*(K\boxtimes p^*e)\widetilde{\to} p^*\mathrm{R}\Gamma(H,K)$ canonically by ([46], Lemma 0.0.20).
ii) Applying $\mathrm{Fun}_{Shv(H)}(\cdot,C)$, we get the adjoint pair $\mathrm{oblv}:\mathrm{Fun}_{Shv(H)}(\mathrm{Vect},C)\leftrightarrows\mathrm{Fun}_{Shv(H)}(Shv(H/G),C):\mathrm{Av}_*^{H/G}$. Now using the assumption $H\widetilde{\to}\lim_{i\in I^{op}} H/K_i$ in PreStk from ([46], 0.0.36) we get $Shv(H/G)\widetilde{\to} Shv(H)^G$ with respect to the G-action on H by right translations.

Recall that $Shv(H)^G\widetilde{\to} Shv(H)_G$, because G is placid group scheme. Finally,

$$\mathrm{Fun}_{Shv(H)}(Shv(H/G),C)\widetilde{\to}\mathrm{Fun}_{Shv(H)}(Shv(H)\otimes_{Shv(G)}\mathrm{Vect},C)\widetilde{\to} C^G$$

□

An example of this situation: $H=G\ltimes\bar{H}$, where $\bar{H}\subset H$ is a normal subgroup, $\bar{H}$ is a placid group scheme, and G acts on $\bar{H}$ by conjugation. For example, assume moreover $\bar{H}$ prounipotent. Then the functor $\mathrm{oblv}: C^H\to C^G$ is fully faithful.

1.2.6. If H is a group scheme of finite type, L is a local system on H equipped with associative isomorphism $m^*L\widetilde{\to} L\boxtimes L$ and a compatible trivialization $i^*L\widetilde{\to} e$ for $i:\operatorname{Spec} k\to H$ then $f:(Shv(H),\star)\to(Shv(H),\star)$, $F\mapsto F\otimes L$ is a monoidal functor. Indeed,

$$f(F_1*F_2)\widetilde{\to}(F_1*F_2)\otimes L\widetilde{\to}(F_1\otimes L)*(F_2\otimes L)=f(F_1)*f(F_2)$$

Now given $C\in(Shv(H),*)-mod$, we twist the action by L as follows. The object $C_L\in H-mod$ is defined as $C\in \mathrm{DGCat}_{cont}$ with the new action given by $Shv(H)\otimes C\stackrel{f\otimes\mathrm{id}}{\to} Shv(H)\otimes C\stackrel{act}{\to} C$.

Note that for $K\in Shv(H)$ we have $L*K\widetilde{\to} L\otimes \mathrm{R}\Gamma(H,L^{-1}\otimes K)\widetilde{\to} K*L$ by Lemma 1.3.16 below.

Remark 1.2.7. *If we are not in the constructible context, it is better not to use the $\otimes$, but only $\otimes^!$. For this we should start with an object $\mathcal{L}\in Shv(H)$ invertible for the $\otimes^!$-monoidal structure and satisfying $m^!\mathcal{L}\widetilde{\to}\mathcal{L}\boxtimes\mathcal{L}$ associatively and $i^!\mathcal{L}\widetilde{\to} e$ for $i: Speck\stackrel{1}{\to} H$. Then the functor $(Shv(H),*)\to(Shv(H),*)$, $F\mapsto F\otimes^!\mathcal{L}$ is a monoidal equivalence. If moreover we are in the constructible context then for a multiplicative local system L in the usual sense, $\mathcal{L}:=L\otimes\omega_H$ satisfies the above properties.*

Proposition 1.2.8. *1) If Y is an ind-scheme of ind-finite type then for $p:Y\to\operatorname{Spec} k$ the functor $p_*:Shv(Y)\to\mathrm{Vect}$ does not admit a left adjoint unless Y is a scheme of finite type.*

2) Assume $Y=\operatorname{colim}_{i\in I}Y_i$, where Y_i is an algebraic stack locally of finite type, I is filtered and for $i\to j$, $Y_i\to Y_j$ is a closed immersion. So, Y is an ind-algebraic stack. Then $p_*:Shv(Y)\to\mathrm{Vect}$ does not admit a left adjoint unless Y is an algebraic stack locally of finite type.

Proof. 1) Write $Y=\operatorname{colim}_{i\in I}Y_i$, where Y_i is a scheme of finite type, I is filtered, and for $i\to j$, $Y_i\to Y_j$ is a closed immersion. Assume a left adjoint $p^*:\mathrm{Vect}\to Shv(Y)$ of p_* exists, let $K=p^*e$. Assume Y is not a scheme of finite type. Then for $F\in Shv(Y)$, $\mathrm{Map}_{Shv(Y)}(K,F)\widetilde{\to}\mathrm{Map}_{\mathrm{Vect}}(e,p_*F)$, so $K\in Shv(Y)^c$. By Lemma 1.1.39, K is of the form $(i_i)_*K'$ for some $i\in I$, $K'\in Shv(Y_i)^c$, here $i_i:Y_i\to Y$ is the natural map. We see that $p_*F\widetilde{\to}\mathcal{H}om_{Shv(Y_i)}(K',i_i^!F)$, the inner hom with respect to Vect-action. Pick $i\to j$ in I and a point $\eta: Speck\to Y_j$ such that η does not factor through Y_i. Consider $\eta_*e\in Shv(Y)$. We get $p_*(\eta_*e)\widetilde{\to} e$ on one hand. On the other hand, $i_i^!(\eta_*e)=0$, a contradiction.

2) the same proof. □

Remark. let $G\widetilde{\to}\operatorname{colim}_{i\in I}G_i$ be a placid ind-group scheme, here $I\in 1-\mathcal{C}at$ is small filtered, G_i is a placid group scheme, if $i\to j$ in I then $G_i\to G_j$ is a placid closed immersion and a homomorphism of group schemes. Then by ([43], 9.2.56), the natural functor

$$Shv(G)-mod\to\lim_{i\in I^{op}} Shv(G_i)-mod$$

is an equivalence, because $Shv(G)\widetilde{\to}\operatorname{colim}_{i\in I} Shv(G_i)$ with respect to the $*$-push-outs. Note that for $M, N\in Shv(G)-mod$ one has

$$\operatorname{Fun}_{Shv(G)}(M,N)\widetilde{\to}\lim_{i\in I^{op}}\operatorname{Fun}_{Shv(G_i)-mod}(M,N)$$

naturally.

1.2.9. For 6.1.4. It is used that $\mathrm{DGCat}_{cont}\to 1-\mathcal{C}at$ preserves limits.

For 6.1.9. The (eventually disconnected) right adjoint to (6.9) exists, because (6.9) is continuous. A trial to explain the formula: let $G\in Grp(\mathrm{PreStk})$, G be a placid ind-scheme, assume $G=\operatorname{colim}_{i\in I} G_i$, where G_i is a placid group scheme, I is filtered. Assume for $i\to j$ in I the map $G_i\to G_j$ is a placid closed immersion, and a homomorphism of group schemes. Let $C\in G-mod$. Recall that $C^G\widetilde{\to}\lim_{i\in I^{op}} C^{G_i}$ in DGCat_{cont} by ([47], Sect. 1.2.3). The functor $\mathrm{oblv}_G: C^G\to C$ is obtained by passing to the limit over I^{op} in the family $\mathrm{oblv}_{G_i}: C^{G_i}\to C$. The functor $\mathrm{oblv}_{G_i}: C^{G_i}\to C$ has a continuous right adjoint $\mathrm{Av}_*^{G_i}: C\to C^{G_i}$ given by the comonad $C\to C, c\mapsto e_{G_i}*c$.

For $i\to j$ let $\alpha: G_i\to G_j$ be the closed immersion. The natural map $e_{G_j}\to\alpha_* e_{G_i}$ in $Shv(G)$ is a morphism of comonads in C. It induces the morphism $e_{G_j}-comod(C)\to e_{G_i}-comod(C)$. Is there any formula for Av_*^G, which is maybe discontinuous? The system of functors $\mathrm{Av}_*^{G_i}: C\to C^{G_i}$ for $i\in I$ is not compatible with the transition functors in the inverse system $\lim_{i\in I^{op}} C^{G_i}$.

Note that we may consider $E:=\lim_{i\in I^{op}} e_{G_i}$ taken in $CoAlg(\operatorname{Fun}_{e,cont}(C,C))$. We get a natural projection $E-comod(C)\to\lim_{i\in I^{op}} C^{G_i}\widetilde{\to} C^G$. Is it an equivalence?

We may add somewhere the following.

Remark 1.2.10. *Let $f: H\to G$ be a map in $Grp(\mathrm{PreStk})$. Assume H, G are placid ind-schemes. Note that $f_*: (Shv(H),*)\to(Shv(G),*)$ is monoidal. We get a morphism of functors $(G-mod)^{op}\times(G-mod)\to\mathrm{DGCat}_{cont}$,*

$$(D,C)\mapsto\operatorname{Fun}_{(Shv(G),*)}(D,C)\to\operatorname{Fun}_{(Shv(H),*)}(D,C)$$

In particular, a map $C^G\to C^H$ functorial in $C\in G-mod$ (and whose composition with $\mathrm{oblv}: C^H\to C$ is $\mathrm{oblv}: C^G\to C$).

1.2.11. For 6.1.9. Explanation of the formula for the functor $\mathrm{Av}_*^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N}$.

Lemma 1.2.12. *Let $C\in\mathrm{DGCat}_{cont}$. Assume given a diagram $I^{op}\to\mathrm{DGCat}_{cont}$, $i\mapsto C_i$, where I is filtered, and a full embedding $\mathrm{oblv}_i: C_i\subset C$ in DGCat_{cont} functorial in i. So, if $i\to j$ in I then $C_j\subset C_i$ is a full subcategory. Assume for each i we have an adjoint pair $\mathrm{oblv}_i: C_i\leftrightarrows C:\mathrm{Av}_*^i$. Let $D=\lim_{i\in I^{op}} C_i=\cap_i C_i$. If $i\to j$ in I then we have a natural map $\mathrm{Av}_*^j\to\mathrm{Av}_*^i$ of functors $C\to C$. Consider the functor $\mathrm{Av}_*: C\to C$ equal to $\lim_{i\in I^{op}}\mathrm{Av}_*^i$ in $\operatorname{Fun}(C,C)$. We claim that Av_* takes values in D and is the right adjoint to $\mathrm{oblv}: D\to C$.*

Proof. Since $\mathrm{Av}_*^i\circ\mathrm{Av}_*^i\widetilde{\to}\mathrm{Av}_*^i$, the natural map $\mathrm{Av}_*^i\mathrm{Av}_*\to\mathrm{Av}_*$ is an isomorphism (as I is filtered). For $i\in I$ the inclusion $I_{i/}\hookrightarrow I$ is cofinal, and $\lim_{j\in I_{i/}}\mathrm{Av}_*^j\widetilde{\to}\lim_{j\in I}\mathrm{Av}_*^j$. However, Av_*^j for $j\in I_{i/}$ takes values in C_i, and the limit of the diagram $\mathrm{Av}_*^j: C\to C_i$ for $j\in I_{i/}$ gives a functor $\mathrm{Av}_*: C\to C_i$ by ([43], 2.7.9). So, Av_* takes values in D. We may also use the fact that $\mathrm{Av}^i_* : C \to C_i$ preserves limits, so $\mathrm{Av}^i_*(\lim_{j\in I^{op}} \mathrm{Av}^j_*(x)) \widetilde{\to} \lim_{j\in I^{op}} \mathrm{Av}^i_* \mathrm{Av}^j_*(x) \widetilde{\to} \mathrm{Av}_*(x)$.

Now for $d \in D, c \in C$ we have

$$\mathrm{Map}(d, \lim_i \mathrm{Av}^i(c)) \widetilde{\to} \lim_i \mathrm{Map}(d, \mathrm{Av}^i(c)) \widetilde{\to} \lim_i \mathrm{Map}(d, c) \widetilde{\to} \mathrm{Map}(d, c),$$

because I is contractible. □

It is maybe not true that Av_* is given by the action of $\lim_{k\in(\mathbb{Z}_+)^{op}} \chi_k \in Shv(\mathfrak{L}(N)^{\omega^\rho}_x)$, where χ_k is the $*$-restriction of χ_N to N_k.

How to define the object χ_N in $Shv(\mathfrak{L}(N)^{\omega^\rho}_x)$? I think it should not be defined as $\lim_{k\in(\mathbb{Z}_+)^{op}} \chi_k$. Consider the functor

$$Shv(\mathfrak{L}(N)^{\omega^\rho}_x) \to \mathrm{Vect},\ K \mapsto \mathcal{H}om_{Shv(\mathbb{A}^1)}(\mathcal{L}_\psi, \chi_* K),$$

where $\chi : \mathfrak{L}(N)^{\omega^\rho}_x \to \mathbb{A}^1$ is our nondegenerate homomorphism, and $\mathcal{L}_\psi$ is the Artin-Shreier sheaf. Is this functor representable by an object that should be called $\chi_N \in Shv(\mathfrak{L}(N)^{\omega^\rho}_x)$? This object does not exist, because otherwise χ_N would be compact.

In other words, we should think of χ_N as an object of

$$Shv(\mathfrak{L}(N)^{\omega^\rho}_x)^\vee = \mathrm{Fun}(Shv(\mathfrak{L}(N)^{\omega^\rho}_x), \mathrm{Vect})$$

given by $K \mapsto \mathcal{H}om_{Shv(\mathbb{A}^1)}(\mathcal{L}_\psi, f_* K)$. (We know that this category is dualizable, as for any placid ind-scheme). For any placid ind-scheme Z, there is a self-duality $Shv(Z) \widetilde{\to} Shv(Z)^\vee$ ([46], Section 0.0.23), however one should not try at this point to think of χ_N as a sheaf on $\mathfrak{L}(N)^{\omega^\rho}_x$.

1.2.13. For 6.2.1.

Lemma 1.2.14. *Let U be a pro-unipotent group scheme, $U = \lim_{i\in I^{op}} U_i$, where U_i is a unipotent group scheme of finite type, I is filtered, for $i \to j$ in I, the map $U_j \to U_i$ is smooth surjective homomorphism. Let $p : U \to \mathrm{Spec}\, k$ be the natural map. Then the functor $p^* : \mathrm{Vect} \to Shv(U)$ in the constructible context admits a left adjoint $(p^*)^L : Shv(U) \to \mathrm{Vect}$. Moreover, $((p^*)^L)^\vee$ identifies with the right adjoint to $p_* : Shv(U) \to \mathrm{Vect}$. We used here the self-duality on $Shv(U)$ from ([47], 1.1.10). In addition, $(p^*)^L \circ p^* \to \mathrm{id}$ is the identity, so $p_* \circ (p_*)^R \to \mathrm{id}$ is the identity.*

Proof. Write $Shv(U) = \lim_{i\in I^{op}} Shv(U_i)$ where for $i \to j$ in I and the corresponding map $f_{ij} : U_j \to U_i$ we use $(f_{ij})_* : Shv(U_j) \to Shv(U_i)$ as transition maps. For each i let $p_i : U_i \to \mathrm{Spec}\, k$ be the map. Then $p_i^* : \mathrm{Vect} \to Shv(U_i)$ form a compatible system of maps giving the functor $p^* : \mathrm{Vect} \to \lim_{i\in I^{op}} Shv(U_i)$. Since each $p_i^* = p_i^![-2\dim U_i]$ admits a left adjoint $(p_i)_![2\dim U_i]$, passing to left adjoints the compatible system of functors $(p_i)_![2\dim U_i] : Shv(U_i) \to \mathrm{Vect}$ yields a functor $\mathrm{colim}_{i\in I} Shv(U_i) \to \mathrm{Vect}$, where we use the functors f_{ij}^* in this colimit system. The latter functor is the desired left adjoint by (by [43], 9.2.6). Now by ([47], 1.2.11), the dual of $p^* : \mathrm{Vect} \to Shv(U)$ identifies with $p_* : Shv(U) \to \mathrm{Vect}$. □

Let U be a pro-unipotent group scheme, $C \in U - mod$. Then we have the adjoint pair $p_* : Shv(U) \leftrightarrows \mathrm{Vect} : (p_*)^R$ in the constructible context. However, $(p_*)^R$ is not a strict morphism of $Shv(U)$-module categories, only a lax one, see Lemma 1.3.7. In general in the constructible context the functor $\mathrm{oblv}_U : C^U \to C$ does not admit a left adjoint. An example given by Sam: take $U = \mathbb{G}_a$ and the sheaf theory to be the holonomic $\mathcal{D}$-modules. Take $C = \mathcal{D} - mod(\mathbb{G}_a)$, all $\mathcal{D}$-modules. Let $\mathbb{G}_a$ act on itself by translations. Then he claims the functor $\mathrm{oblv}_U : C^U \to C$ does not admit a left adjoint.

Assume now $N \in Grp(\mathrm{PreStk})$, and $N \widetilde{\to} \mathrm{colim}_{i\in I} N_i$, where N_i is a pro-unipotent group scheme, if $i \to j$ in I then $N_i \to N_j$ is a placid closed immersion, and a map of group schemes, and I is filtered. Let $C \in N - mod$. Recall that $C^N \widetilde{\to} \lim_{i\in I^{op}} C^{N_i} = \cap_i C^{N_i}$. If we have an adjoint pair $\mathrm{Av}_!^{N_i} : C \rightleftarrows C^{N_i} : \mathrm{oblv}_{N_i}$ for each i, here C^{N_i} is a localization of C, then by (HTT, 5.5.4.18), $\cap_i C^{N_i} \subset C$ is a strongly reflective subcategory, so the functor $\mathrm{Av}_!^N$ exists.

For $c \in C$ and $i \to j$ in I we get in this case the localization map $\mathrm{Av}_!^{N_i}(c) \to \mathrm{Av}_!^{N_j}(c)$ with respect to $C^{N_j} \subset C^{N_i}$. We claim that in this case the functor $C \to C^N$, $c \mapsto \mathrm{colim}_{i\in I} Av_!^{N_i}(c)$ is the left adjoint to the inclusion $C^N \hookrightarrow C$. This is a special case of the following.

Lemma 1.2.15. *Let $C \in \mathrm{DGCat}_{cont}$, $C_i \subset C$ be a full subcategory, this is a map in DGCat_{cont} for $i \in I$. Here $I \in 1 - \mathcal{C}\mathrm{at}$ is filtered. Assume for $i \to j$ in I, $C_j \subset C_i$. Set $D = \cap_i C_i = \lim_{i\in I^{op}} C_i$, where the limit is calculated in DGCat_{cont}. Assume $L_i : C \to C_i$ is a left adjoint to the inclusion. Then D is a localization of C, and the localization functor $L : C \to D$ is given by $L(c) = \mathrm{colim}_{i\in I} L_i(c)$, where the transition maps are the localization morphisms for $C_j \subset C_i$, and the colimit is calculated in C.*

Proof. For $x \in \cap_i C_i$, $c \in C$ we get

$$\mathrm{Map}_C(\mathrm{colim}_i L_i(c), x) \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Map}(L_i(c), x) \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Map}(c, x)$$
$$\widetilde{\to} \mathrm{Map}(c, x) \widetilde{\to} \mathrm{Fun}(I^{op}, \mathrm{Map}(c, x))$$

For $J \in 1 - \mathcal{C}\mathrm{at}$, $Z \in \mathrm{Spc}$ we have $\mathrm{Fun}(J, Z) \widetilde{\to} \mathrm{Fun}(\mid J \mid, Z)$, where $\mid J \mid \in \mathrm{Spc}$ is obtained by inverting all arrows. Since a filtered category is contractible, we are done.

To explain that L takes values in $\cap C_i$, note that we may equally understand $\mathrm{colim}_i L_i(c)$ as taken in C_j over $i \in I_{j/}$, because the inclusion $C_j \subset C$ is continuous, so the colimit lies in C_j for any j. □

Claim: Let now $Y = \mathrm{colim}_{j\in J} Y_j$ be an ind-scheme of ind-finite type, here J is filtered, Y_j is a scheme of finite type. For $i \to j$ in J, $Y_i \to Y_j$ is a closed immersion. Let U be a prounipotent group scheme acting on Y preserving each Y_j. Assume that for any $j \in J$, the U-action on Y_j factors through a finite-dimensional quotient unipotent group $U \to U_0$. Then $\mathrm{oblv} : Shv(Y)^U \to Shv(Y)$ in the constructible context admits a left adjoint $\mathrm{Av}_!^U$.

Proof. For any $j \in J$ pick a finite-dimensional quotient $U \to U_j$ such that the U-action on Y_j factors through U_j, so we have the quotient map $h : Y_j \to Y_j/U_j$. The functor $\mathrm{oblv} : Shv(Y_j)^U \to Shv(Y_j)$ identifies with $h^! : Shv(Y_j/U_j) \to Shv(Y_j)$, it has the left adjoint $h_!$ in the constructible context. For $i \to j$ in J let $f_{ij} : Y_i \hookrightarrow Y_j$ be the closed immersion. We get the diagram $\tau : J^{op} \times [1] \to \mathrm{DGCat}_{cont}$ sending j to oblv : $Shv(Y_j)^U \hookrightarrow Shv(Y_j)$, with the transition functors $f^!_{ij}$. Passing to the limit over J^{op}, this gives the functor oblv : $Shv(Y)^U \to Shv(Y)$.

If $i \to j$ is a map in J, pick a finite-dimensional quotient $U \to U_0$ such that on both Y_i, Y_j the U-action factors through U_0. For the projections $h_i : Y_i \to Y_i/U_0$ we get commutative diagram

$$\begin{array}{ccc} Shv(Y_i) & \stackrel{(h_i)_!}{\to} & Shv(Y_i/U_0) \\ \downarrow (f_{ij})_! & & \downarrow (f_{ij})_! \\ Shv(Y_j) & \stackrel{(h_j)_!}{\to} & Shv(Y_j/U_0) \end{array}$$

So, we may pass to left adjoints in the diagram τ. Passing to the colimit over I in DGCat_{cont}, this gives a functor $\mathrm{Av}^U_! : Shv(Y) \to \mathrm{colim}_{i\in I} Shv(Y_i)^U \widetilde{\to} Shv(Y)^U$. We are in the situation of ([43], 9.2.39). According to *loc.cit*, $\mathrm{Av}^U_!$ is left adjoint to oblv : $Shv(Y)^U \to Shv(Y)$. □

Claim 2: Let Y, Y' be ind-schemes of ind-finite type, $f : Y \to Y'$ be a schematic morphism of finite type. Let U be a prounipotent group scheme acting on Y, Y' so that f is U-equivariant. Then in the constructible context there are functors $f^* : Shv(Y') \to Shv(Y)$ and $f^* : Shv(Y')^U \to Shv(Y)^U$ commuting with oblv : $Shv(Y)^U \to Shv(Y)$, oblv : $Shv(Y)^U \to Shv(Y)$, that is, $f^*\,\mathrm{oblv} \widetilde{\to} \mathrm{oblv}\circ f^*$. Besides,

$$\mathrm{Av}^U_! \circ f^* \widetilde{\to} f^* \circ \mathrm{Av}^U_!$$

naturally as functors $Shv(Y') \to Shv(Y)^U$.

Proof. Write $Y' \widetilde{\to} \mathrm{colim}_{j\in J} Y'_j$, where J is small filtered, if $j \in J$ then $Y'_j \in \mathrm{Sch}_{ft}$, and for $j \to j'$ in J the map $Y'_j \to Y'_{j'}$ is a closed immersion. Set $Y_j = Y'_j \times_{Y'} Y$. Then $Y \widetilde{\to} \mathrm{colim}_{j\in J} Y_j$. If $j \in J$ then $Y_j \in \mathrm{Sch}_{ft}$, and for $j \to j'$ in J the map $Y_j \to Y_{j'}$ is a closed immersion. For $j \in J$ let $f_j : Y_j \to Y'_j$ be the restriction of f. Then $f_j^* : Shv(Y'_j) \to Shv(Y_j)$ are compatible with the !-extensions giving the transition maps in $Shv(Y) \widetilde{\to} \mathrm{colim}_{j\in J} Shv(Y_j)$ and similarly for Y'. In the colimit over $j \in J$ then give the functor $f^* : Shv(Y') \to Shv(Y)$.

Now for each j let $U \to U_j$ be the finite type quotient group scheme such that U acts on Y_j and on Y'_j via U_j. We get the cartesian square

$$\begin{array}{ccc} Y_j & \stackrel{f_j}{\to} & Y'_j \\ \downarrow h & & \downarrow h \\ Y_j/U_j & \stackrel{f_j}{\to} & Y'_j/U_j, \end{array}$$

where h is the quotient map in the sense of stacks. Then $f_j^* h_! \widetilde{\to} h_! f_j^*$. Passing to the colimit over j in this isomorphism, we get $\mathrm{Av}^U_! \circ f^* \widetilde{\to} f^* \circ \mathrm{Av}^U_!$. The second claim comes from $f_j^* h^! \widetilde{\to} h^! f_j^*$ by passing to the colimit. □

1.2.16. For 6.2.2. The functor $\mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x, \chi_N}$ admits a continuous right adjoint, so sends a compact object (on which it is defined) to a compact object. Besides, $\delta_{t^\lambda,\mathrm{Gr}}$ is compact in $Shv_{\mathfrak{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$.

Why the shift $[-\langle\lambda, 2\check{\rho}\rangle]$ in the definition of $W^{\lambda,!}$? The relation with the global definition should explain this.

1.2.17. For 6.2.5. If Y_i are ind-schemes of ind-finite type, $f : Y_1 \to Y_2$ is a closed immersion if for any $S \to Y_2$ with $S \in \mathrm{Sch}^{aff}_{ft}$, $S \times_{Y_2} Y_1 \to Y_2$ is a closed immersion.

This is the case for $\bar{S}^\mu \hookrightarrow \bar{S}^\lambda$ for $\mu \le \lambda$.

I propose to formulate here the following.

Remark 1.2.18. *Let G be a placid ind-scheme, $G \in Grp(\mathrm{PreStk})$. Assume $G = \mathrm{colim}_{i\in I} G_i$ in $Grp(\mathrm{PreStk})$, where G_i is a placid scheme, I filtered, and for $i \to j$ in I, $G_i \to G_j$ is a placid closed embedding. Let $C \in G-mod$. Recall that $G^G \widetilde{\to} \lim_{i\in I^{op}} C^{G_i}$. Assume for each $i \to j$ in I the functor $C^{G_j} \to C^{G_i}$ admits a left adjoint. Consider the functor $I \to \mathrm{DGCat}_{cont}$, $i \mapsto C^{G_i}$ obtained from the above one by passing to left adjoints. Then we have $\mathrm{colim}_{i\in I} C^{G_i} \widetilde{\to} \lim_{i\in I^{op}} C^{G_i}$ in DGCat_{cont}.*

Assume G is a placid ind-scheme, $G \in Grp(\mathrm{PreStk})$. Assume given an adjoint pair $f : C \leftrightarrows C' : g$ in $Shv(G)-mod$. Then applying $\mathrm{Fun}_{Shv(G)}(\mathrm{Vect}, \cdot)$, one gets an adjoint pair $C^G \leftrightarrows C'^G$ in DGCat_{cont}. This gives in our case the desired equivalence

$$Whit_{q,x}(G) \widetilde{\to} \mathrm{colim}_{\lambda\in\Lambda} Shv_{\mathcal{G}^G}(\bar{S}^\lambda)^{\mathfrak{L}(N)^{\omega^\rho}_x, \chi_N}$$

The explanation of the fact that $(\bar{i}_\lambda)_! : Shv_{\mathcal{G}^G}(\bar{S}^\lambda)^{\mathfrak{L}(N)^{\omega^\rho}_x, \chi_N} \to Whit_{q,x}(G)$ is fully faithful is not a good one. Here is one: we have an adjoint pair $((\bar{i}_\lambda)_!, (\bar{i}_\lambda)^!)$ between the categories of sheaves on $\bar{S}^\lambda$ and $\mathrm{Gr}^{\omega^\rho}_{G,x}$, and the left adjoint is fully faithful. Applying the functor of invariants, we get an adjoint pair $((\bar{i}_\lambda)_!, (\bar{i}_\lambda)^!)$ between the categories of invariants, and the left adjoint is still fully faithful. Indeed, invariants send an identity functor to the identity.

An alternative would be to apply ([23], Lm. 1.3.6).

1.2.19. For 6.2.5. To see that $W^{\lambda,!} \in Shv_{\mathcal{G}^G}(\bar{S}^\lambda)^{\mathfrak{L}(N)^{\omega^\rho}_x, \chi_N}$, note that we have an adjoint pair

$$\mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x, \chi_N} : Shv_{\mathcal{G}^G}(\bar{S}^\lambda) \leftrightarrows \mathrm{Whit}_{q,x}(G)_{\le\lambda} : \mathrm{oblv},$$

where the left adjoint is partially defined for $\mathcal{D}$-modules (always defined in the constructible context). It is defined here, because $\delta_{t^\lambda,\mathrm{Gr}}$ is holonomic.

1.2.20. If $G \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, which is a placid ind-scheme, assume $C \in G-mod$. For $p : G \to \mathrm{Spec}\, k$ the functor $p_* : Shv(G) \to \mathrm{Vect}$ does not have a left adjoint (unless G is a scheme). The functor p_* is monoidal, hence a map in $G-mod$, it induces applying $\mathrm{Fun}_{(Shv(G),*)}(\bullet, C)$ the functor $\mathrm{oblv}_G : C^G \to C$. However, it is not clear if oblv_G has a right adjoint, this may depend on C maybe.

For any map $C_1 \to C_2$ in $G-mod$ the diagram commutes

$$\begin{array}{ccc} C_1^G & \overset{\mathrm{oblv}_G}{\to} & C_1 \\ \downarrow & & \downarrow \\ C_2^G & \overset{\mathrm{oblv}_G}{\to} & C_2 \end{array}$$

1.2.21. For 6.2.6. The commutativity of square both both circuit follows from the previous section.

1.2.22. For 6.2.6. We explain the fact that the essential images of $\mathrm{Whit}_{q,x}(G)_{\leq\mu}$ for $\mu<\lambda$ generate the full subcategory of $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$ of objects extended by zero under the closed immersion $\bar{S}^\lambda - S^\lambda \hookrightarrow \bar{S}^\lambda$.

Let $i_i : Y_i \hookrightarrow Y$ for $i=1,\dots,n$ be diagrams of ind-schemes of ind-finite type, where i_i is a closed immersion, and $\sqcup Y_i \to Y$ is surjective (say, pointwise for k-points). Then $(i_i)_! : Shv(Y_i) \to Shv(Y)$ is fully faithful, and the essential images of $(i_i)_!$ for $i=1,\dots,n$ generate $Shv(Y)$. Indeed, by induction we may assume $n=2$. In this case for $K\in Shv(Y)$ we have a fibre sequence $(i_{12})_!(i_{12})^!K \to (i_1)_!(i_1)^!K \oplus (i_2)_!(i_2)^!K \to K$ in $Shv(Y)$. Indeed, Shv satisfies the proper descent by ([46], 0.0.32), and $Y_1\sqcup Y_2\to Y$ is a surjective on field-valued points (it is also a covering in the topology of finite surjective maps). So, to check that this is a fibre sequence, it suffices to do this after !-restriction to Y_i, Y_{ij} for all i,j, which is clear. If $\mathrm{Map}(F,K)=*$ for any $F\in Shv(Y_j)$ for $j=1,2$ then $i_i^!K=0$, hence $K\widetilde{\to}0$.

This implies that the essential images of $Shv(\bar{S}^{\lambda-\alpha_i})$ for all simple coroots α_i generate $Shv(\bar{S}^\lambda - S^\lambda)$.

For $K\in \mathrm{Whit}_{q,x}(G)_{\leq\lambda}$ extended by zero under $\bar{S}^\lambda - S^\lambda \hookrightarrow \bar{S}^\lambda$ we use a similar fibre sequence defined by the closed subschemes $i_{\lambda-\alpha_i} : \bar{S}^{\lambda-\alpha_i}$ for all simple coroots. The point is that the functors $i^!_{\lambda-\alpha_i}$ are between the corresponding Whittaker categories, so our fibre sequence will take place in $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$.

1.2.23. For 6.2.7. In the definition of $\overset{\circ}{W}{}^\lambda$ the functor used is $\mathrm{Av}_!^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N} : Shv_{\mathcal{G}^G}(S^\lambda) \to \mathrm{Whit}_{q,x}(G)_{=\lambda}$.

In (6.13) replace *Maps* by $\mathcal{H}om$. In addition, the formula (6.13) should say the answer is e for $\lambda=\lambda'$ dominant, and zero otherwise. I propose to say it follows from Prop. 6.2.9.

It would be useful for a reader if in this section it would be mentioned that $\overset{\circ}{W}{}^\lambda$ has a simplier definition: for the corresponding map say $\bar{\chi}^\lambda : S^\lambda \to \mathbb{A}^1$ one has

$$\overset{\circ}{W}{}^\lambda \widetilde{\to} (\bar{\chi}^\lambda)^! \mathcal{L}_\psi[2-\langle 2\check{\rho},\lambda\rangle],$$

where $\mathcal{L}_\psi$ is the Artin-Schreier sheaf (refer then to Thm. 7.4.2 to explain this formula).

1.2.24. For 6.2.9. It is better to say in (b) that the continuous functor $\mathrm{Vect}\to \mathrm{Whit}_{q,x}(G)_{=\lambda}$ sending e to $\overset{\circ}{W}{}^\lambda$ is an equivalence.

1.2.25. For proof of 6.2.9. We check that (6.14) admits a left adjoint given by $V\mapsto V\otimes \mathrm{colim}_k \mathrm{Av}_!^{N_k,\chi_k}(\delta_{t^\lambda,\mathrm{Gr}})$, where the colimit is calculated in $Shv_{\mathcal{G}^G}(S^\lambda)$. For $F\in \mathrm{Whit}_{q,x}(G)_{=\lambda}, V\in \mathrm{Vect}$ one has

$$\mathrm{Map}(V\otimes \mathrm{colim}_k \mathrm{Av}_!^{N_k,\chi_k}(\delta_{t^\lambda,\mathrm{Gr}}),F) \widetilde{\to} \lim_k \mathrm{Map}_{Shv_{\mathcal{G}^G}(S^\lambda)^{N_k,\chi_k}}(V\otimes \mathrm{Av}_!^{N_k,\chi_k}(\delta_{t^\lambda,\mathrm{Gr}}),F)$$

$$\lim_k \mathrm{Map}_{Shv_{\mathcal{G}^G}(S^\lambda)}(V\otimes \delta_{t^\lambda,\mathrm{Gr}},F) \widetilde{\to} \lim_k \mathrm{Map}_{\mathrm{Vect}}(V, i^!_{t^\lambda}F) \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}}(V, i^!_{t^\lambda}F),$$

because we calculate a limit over a contractible category.

To understand the proof, consider the following situation. Let U be a pro-unipotent group scheme, $U = \lim_{i\in I^{op}} U_i$, where I is filtered, for $i \to j$ the map $U_j \to U_i$ in a smooth, affine surjective homomorphism of group schemes, and U_i is a smooth group scheme of finite type. We assume $i_0 \in I$ is the initial object.

Let S be a scheme of finite type, $x \in S$, the action of U on S is transitive. Let L be a character local system on U coming from a local system L_0 on U_{i_0}. Let St be the stabilizor of x in U. Then St is a placid group scheme, and $St \to U$ is a placid closed immersion. Moreover, we may assume that there is a closed subscheme $St_{i_0} \subset U_{i_0}$ such that $St = St_{i_0} \times_{U_{i_0}} U$.

For $h : U_{i_0} \to S$, $u \mapsto ux$ we have the functor $h^* : Shv(S) \to Shv(U_{i_0})$, because h is smooth. Further, $Shv(S)^{U,L} \widetilde{\to} Shv(S)^{U_0,L_0}$ by ([47], Lemma 1.3.11). Besides, $Shv(U_0)^{U_0,L_0} \widetilde{\to}$ Vect with the generator L_0 by ([47], Sect. 1.3.15). The functor h^* gives the full embedding

$$Shv(S)^{U_{i_0},L_0} \subset Shv(U_{i_0})^{U_{i_0},L_0} \widetilde{\to} \mathrm{Vect}$$

We see that if $L_0 \widetilde{\to} h^*\bar{L}$ is in the essential image of $h^* : Shv(S) \to Shv(U_{i_0})$ then $Shv(S)^{U,L} \widetilde{\to}$ Vect with the generator $\bar{L}$, and zero otherwise.

Consider now the functor α given as the composition $Shv(S)^{U,L} \subset Shv(S) \stackrel{\delta_x^!}{\to} \mathrm{Vect}$. It has a left adjoint sending e to $\mathrm{Av}_!^{U,L}(\delta_x)$. Here $\delta_x = (i_x)_! e$ for $i_x : \operatorname{Spec} k \stackrel{x}{\to} S$. Then $\mathrm{Av}_!^{U,L}(\delta_x)[-2\dim S] \widetilde{\to} \bar{L}$ if $h^*\bar{L} \widetilde{\to} L$ for some $\bar{L}$, and zero otherwise. Indeed, $i_x^! \bar{L} \widetilde{\to} e[-2\dim S]$.

1.2.26. For 6.2.10. To see (c), note that $\bar{S}^\lambda$ is an ind-scheme of ind-finite type. For any $\mu \le \lambda$, $\bar{S}^\mu \hookrightarrow \bar{S}^\lambda$ is a closed immersion. Now if $Y \subset \bar{S}^\lambda$ is a closed subscheme, Y is of finite type that Y meets only a finite number of $\mathfrak{L}(N)_x^{\omega^\rho}$-orbits $Y \cap S^\mu$. Let $F \in Shv(\bar{S}^\lambda)^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N}$, let F_Y be its !-restriction to Y. To see that $F = 0$, it suffices to show that $F_Y = 0$ for any closed subscheme of finite type $Y \subset \bar{S}^\lambda$. Since we know this for the !-restriction to $Y \cap S^\mu$, F_Y vanishes indeed. So, $Shv(\bar{S}^\lambda)^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N} = 0$ unless λ is dominant.

Note that for λ dominant, $\mathrm{Whit}_{q,x}(G)_{\le\lambda} = Shv(\bar{S}^\lambda)^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N}$ admits a finite filtration with the graded pieces Vect. Indeed, $\{\mu \in \Lambda^+ \mid \mu \le \lambda\}$ is finite. By filtration we mean here that there are full subcategories $\mathrm{Whit}_{q,x}(G)_{\le\mu}$ for $\mu \le \lambda$, the functor $\mathrm{Whit}_{q,x}(G)_{\le\mu} \hookrightarrow \mathrm{Whit}_{q,x}(G)_{\le\lambda}$ is a map in DGCat_{cont}, we also have functors

$$j_\mu^* : \mathrm{Whit}_{q,x}(G)_{\le\mu} \to \mathrm{Whit}_{q,x}(G)_{=\mu} \widetilde{\to} \mathrm{Vect}$$

for μ dominant, and $\mathrm{Whit}_{q,x}(G)_{=\mu}$ is a localization of $\mathrm{Whit}_{q,x}(G)_{\le\mu}$. Besides, $\mathrm{Whit}_{q,x}(G)_{=\mu}$ is the the right orthogonal to the full subcategory of $\mathrm{Whit}_{q,x}(G)_{\le\mu}$ of those objects, which are extensions by zero under the closed immersion

$$S^{<\mu} \hookrightarrow \bar{S}^\mu$$

The latter category is generated by $\mathrm{Whit}_{q,x}(G)_{\le\nu}$ with $\nu < \mu$ in the sense of ([20], ch. I.1, 5.4.1).

To see that $W^{\lambda,!}$ generate $\mathrm{Whit}_{q,x}(G)$ for λ dominant, let $K \in \mathrm{Whit}_{q,x}(G)$ with $\mathrm{Map}(W^{\lambda,!}[n], K) = *$ for any $n \in \mathbb{Z}, \lambda \in \Lambda^+$. Then $(\bar{i}^\lambda)^! K = 0$ for any $\lambda \in \Lambda$. Since $\mathrm{Gr}^{\omega^\rho}_{G,x} = \mathrm{colim}_{\nu \in \Lambda} \bar{S}^\nu$, $K = 0$.

(b) follows from the fact that for λ minimal and $\mu < \lambda$, $(i^\mu)^! W^{\lambda,*} = 0$. This means that $W^{\lambda,*}$ is the extension by zero from S^λ.

(a) For any $\lambda \in \Lambda^+$ the object $W^{\lambda,!}$ admits a finite filtration by the objects $W^{\mu,*}$ with $\mu \in \Lambda^+$, $\mu \leq \lambda$. This comes from the geometry, from the !-restrictions of $W^{\lambda,!}$ to the strata S^μ. So, the smallest stable subcategory containing $W^{\mu,*}$ for $\mu \in \Lambda^+$, also contains $W^{\lambda,!}$. This is why the objects $W^{\mu,*}$ generate $\mathrm{Whit}_{q,x}(G)$.

1.2.27. For 6.3. The fact that this indeed defines a t-structure: we may say that $\mathrm{Whit}_{q,x}(G)^{\leq 0}$ is the smallest full subcategory containing $W^{\lambda,!}$ for $\lambda \in \Lambda^+$, closed under colimits and extensions. Then $\mathrm{Whit}_{q,x}(G)^{\leq 0}$ is presentable, and indeed defines an accessible t-structure by (HA, 1.4.4.11).

The t-structure on $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$ can be defined in two ways: by the fact that the inclusion into $\mathrm{Whit}_{q,x}(G)$ is compatble with the t-structure. The second way is to say that $\mathrm{Whit}_{q,x}(G)^{\leq 0}_{\leq\lambda}$ is the smallest full subcategory containing $W^{\mu,!}$ for $\mu \in \Lambda$, closed under the colimits and extensions. This definition also shows that $\bar{i}^!_\lambda : Whit_{q,x}(G) \to \mathrm{Whit}_{q,x}(G)_{\leq\lambda}$ is left t-exact. Indeed, for $F \in Whit_{q,x}(G)$ and $\mu \leq \lambda$,

$$\mathrm{Hom}_{\mathrm{Whit}_{q,x}(G)}(W^{\mu,!}, F) \widetilde{\to} \mathrm{Hom}_{\mathrm{Whit}_{q,x}(G)}(W^{\mu,!}, \bar{i}^!_\lambda F)$$

1.2.28. For 6.3.2. In (a) we use the following definition. Let $C \subset D$ be a full embedding in DGCat_{cont}, D equipped with a t-structure. We say that C is compatible with the t-structure on D if the truncation functors preserve C. This implies that $C^{\geq 0} := C \cap D^{\geq 0}$ and $C^{\leq 0} := C \cap D^{\leq 0}$ define a t-structure on C. Moreover, $C^\heartsuit \subset D^\heartsuit$ is a full subcategory closed under extensions, and $C \hookrightarrow D$ is t-exact.

Conversely, assume $C \subset D$ is a t-exact full embedding in DGCat_{cont}, C,D equipped with t-structures. Then the truncation functors of D preserve C. Indeed, if $c \in C$, let $\tau^{\leq n}_D c \to c \to \tau^{> n}_D c$ be the corresponding fibre sequence in D. Consider the fibre sequence $\tau^{\leq n}_C c \to c \to \tau^{> n}_C c$ in C. This is a fibre sequence in D also, and $\tau^{\leq n}_C c \in D^{\leq n}, \tau^{> n}_C c \in D^{> n}$. Thus, the two fibre sequences are isomorphic. So, C is compatible with the t-structure on D.

If the t-structure on D is compatible with filtered colimits then the same holds for the induced t-structure on C.

Proof of 6.3.2(a). For $F \in \mathrm{Whit}_{q,x}(G)_{\leq\lambda}$ the condition $F \in \mathrm{Whit}_{q,x}(G)^{\geq 0}_{\leq\lambda}$ is equivalent to

$$\mathrm{Hom}_{\mathrm{Whit}_{q,x}(G)}(W^{\mu,!}, F) = 0$$

for any $\mu \leq \lambda$. So, $(\mathrm{Whit}_{q,x}(G)_{\leq\lambda})^{\leq 0}$ is the smallest full subcategory containing $W^{\mu,!}$ for $\mu \leq \lambda \in \Lambda^+$, closed under colimits and extensions. It is presentable by (HA, 1.4.4.11). This gives $(\mathrm{Whit}_{q,x}(G)_{\leq\lambda})^{\leq 0} \subset (\mathrm{Whit}_{q,x}(G)^{\leq 0}$.

If $K \in (\mathrm{Whit}_{q,x}(G)_{\leq\lambda})^{\geq 0}$ and $\mu \in \Lambda$, and μ is not less then λ then $i^!_\mu K = 0$. So, $\mathrm{Hom}_{\mathrm{Whit}_{q,x}(G)}(W^{\mu,!}, F) = 0$ and $K \in (\mathrm{Whit}_{q,x}(G)^{\geq 0}$. We see that the inclusion $\mathrm{Whit}_{q,x}(G)_{\leq\lambda} \hookrightarrow \mathrm{Whit}_{q,x}(G)$ is t-exact, we are done.

1.2.29. For Remark 6.3.3. Misprint, you meant $\mathrm{D}^+(\mathrm{Whit}_{q,x}(G)^\heartsuit)$, not $\mathrm{D}^+(\mathrm{Whit}_{q,x}(G)$.

You can not talk about $\mathrm{D}^+(\mathrm{Whit}_{q,x}(G)^\heartsuit)$ before you justify the fact that $\mathrm{Whit}_{q,x}(G)^\heartsuit$ has enough injective objects. Since you want to use the universal property of D^+, you have first to justify that $\mathrm{Whit}_{q,x}(G)$ is right complete for this t-structure.

1.2.30. For 6.3.5. In general, let $C \subset D$ be a full embedding in DGCat_{cont}, D equipped with a t-structure and C compatible with this t-structure. If $c \in C^\heartsuit$ is irreducible, then its image in $D^\heartsuit$ is not necessarily irreducible.

Example: Let $Y \widetilde{\to} \mathbb{A}^n$, $p : Y \to \operatorname{Spec} k$. Then $p^* : \mathrm{Vect} \to Shv(Y)$ is fully faithful. Consider $Shv(Y)$ with the usual, not the perverse t-structure. This t-structure is compatible with Vect. However, $e \in \mathrm{Vect}$ is irreducible in $\mathrm{Vect}^\heartsuit$, and p^*e is not irreducible in $Shv(Y)^\heartsuit$.

The proof of 6.3.5 is absent, as it is not justified in the proof that $L \in \mathrm{Whit}_{q,x}(G)_{\le\lambda}$ is not justified (see Lemma below).

Let L be an irreducible object of $\mathrm{Whit}_{q,x}(G)^\heartsuit$. For some $\lambda \in \Lambda^+$ there is a nonzero map $W^{\lambda,!} \to L$, by definition of the t-structure. It gives a morphism $W^{\lambda,!} \to (j_\lambda)^*(\bar{i}^\lambda)^! L$. Since $\mathrm{Whit}_{q,x}(G)_{=\lambda} \widetilde{\to} \mathrm{Vect}$, so is semisimple, we get a nonzero morphism

$$(j_\lambda)^*(\bar{i}^\lambda)^! L \to \overset{\circ}{W}{}^\lambda$$

over S^λ, which gives a nonzero map $(\bar{i}^\lambda)^! L \to (j_\lambda)_* \overset{\circ}{W}{}^{\lambda,!}$, hence a nonzero map

$$\mathrm{H}^0((\bar{i}^\lambda)^! L) \to (j_\lambda)_* \overset{\circ}{W}{}^{\lambda,!},$$

but this does not give the map $L \to W^{\lambda,*}$ that you wanted, we only have $(i^\lambda)_! \mathrm{H}^0((\bar{i}^\lambda)^! L) \to L$, which is surjective.

Here is how it should have been written.

Lemma 1.2.31. *The objects $W^{\lambda,!*}$, $\lambda \in \Lambda^+$ are irreducible, and each irreducible object of $(\mathrm{Whit}_{q,x}(G)^\heartsuit$ is of this form. Moreover, the unique irreducible quotient of $W^{\lambda,!}$ is $W^{\lambda,!*}$.*

Proof. **Step 1**. Let us show that $W^{\lambda,!*}$ is irreducible. We have a fibre sequence $W^{\lambda,!*} \to W^{\lambda,*} \to K$ in $\mathrm{Whit}_{q,x}(G)_{\le\lambda}$ with $K \in (\mathrm{Whit}_{q,x}(G)_{\le\lambda})^\heartsuit$. Now for $\mu \in \Lambda$, $\mu < \lambda$ we get

$$\mathcal{H}om(W^{\mu,!}, W^{\lambda,!*}) \widetilde{\to} \mathcal{H}om(W^{\mu,!}, K[-1])$$

In particular, $\mathrm{Hom}(W^{\mu,!}, W^{\lambda,!*}) = 0$. Besides, $\mathrm{Hom}(W^{\lambda,!}, W^{\lambda,!*}) \widetilde{\to} e$. Let now $L \subset W^{\lambda,!*}$ be a subobject in $\mathrm{Whit}_{q,x}(G)^\heartsuit$ with $W^{\lambda,!*}/L \ne 0$. There is $\mu \le \lambda$ and a nonzero map $W^{\mu,!} \to L$. Since the composition $W^{\mu,!} \to L \hookrightarrow W^{\lambda,!*}$ is not surjective, $\mu \ne \lambda$. But we have seen that for $\mu < \lambda$ this Hom vanishes. So, $W^{\lambda,!*}$ is irreducible.

Step 2. Let $W^{\lambda,!} \to L$ be a nonzero map in $(\mathrm{Whit}_{q,x}(G)_{\le\lambda})^\heartsuit$ with L irreducible. We claim that this map coincides up to a multiple with the canonical map $W^{\lambda,!} \to W^{\lambda,!*}$. Indeed, we have a nonzero surjection $\overset{\circ}{W} \to j_\lambda^* L$, which shows that we may pick a nonzero map $L \to W^{\lambda,*}$ in $(\mathrm{Whit}_{q,x}(G)_{\le\lambda})^\heartsuit$, which is injective. Since $\mathcal{H}om(W^{\lambda,!}, W^{\lambda,*}) \widetilde{\to} e$, this implies the claim.

Step 3. Let us show there are no other irreducibles. Let L be an irreducible object. Pick $\lambda \in \Lambda^+$ and a nonzero map $\tau : W^{\lambda,!} \to L$, this gives a nonzero map $W^{\lambda,!} \to \bar{i}_\lambda^! L$

in $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$. The functor $\bar{i}^!_\lambda$ is left t-exact, so $\mathrm{H}^0(\bar{i}^!_\lambda L) \neq 0$, and we get a nonzero map $W^{\lambda,!} \to \mathrm{H}^0(\bar{i}^!_\lambda L)$ in $(\mathrm{Whit}_{q,x}(G)_{\leq\lambda})^\heartsuit$, whose restriction to S^λ is nonzero. Since the composition

$$W^{\lambda,!} \to (\bar{i}_\lambda)_! \mathrm{H}^0(\bar{i}^!_\lambda L) \xrightarrow{\eta} L$$

is τ, the map η is nonzero. Let K be an irreducible quotient of $\mathrm{H}^0(\bar{i}^!_\lambda L)$ in $(\mathrm{Whit}_{q,x}(G)_{\leq\lambda})^\heartsuit$ such that η factors through a (nonzero) map $(\bar{i}_\lambda)_! K \to L$. So, $W^{\lambda,!} \xrightarrow{\nu} K$ is nonzero over $\bar{S}^\lambda$. By Step 2, ν identifies with $W^{\lambda,!} \to W^{\lambda,!*}$. We obtained a surjection $(\bar{i}_\lambda)_! W^{\lambda,!*} \to L$. By Step 1, $W^{\lambda,!*}$ is irreducible in $\mathrm{Whit}_{q,x}(G)^\heartsuit$. □

Lemma 1.2.32. *Let $\lambda \in \Lambda^+$. Then $W^{\lambda,!}$ admits a unique irreducible quotient isomorphic to $W^{\lambda,!*}$. Any other irreducible subquotient is of the form $W^{\mu,!*}$ for $\mu < \lambda$.*

Proof. The first claim was proved in the previous lemma.

Recall that $W^{\lambda,!*}$ is the extension by zero from $\bar{S}^\lambda$, because we defined the image of $W^{\lambda,!} \to W^{\lambda,*}$ in $(\mathrm{Whit}_{q,x}(G)_{\leq\lambda})^\heartsuit$ first and used the fact that $(\bar{i}_\lambda)_*$ is t-exact.

Assume by induction our claim true for $\mu < \lambda$. We check the same for λ. The base of the induction follows from 6.2.10.

Choose a filtration on $W^{\lambda,!}$ with simple quotients. Assume i is the first index such that for the i-th subquotient $W^{\nu,!*}$, the inequality $\nu \leq \lambda$ doesn't hold. We have a short exact sequence $0 \to K' \to K \to W^{\nu,!*} \to 0$, where K is a subobject of $W^{\lambda,!}$. We have a non-zero map $W^{\nu,!} \to W^{\nu,!*}$.

We claim this map can be lifted to a map $W^{\nu,!} \to K$. Indeed, the obstruction to the lift is in $\mathrm{Ext}^1(W^{\nu,!}, K')$. By assumption, the simple subquotients of K' are of form $W^{\mu,!*}$ for $\mu < \lambda$. So it's enough to show that for $\mu < \lambda$, we have $\mathrm{Ext}^1(W^{\nu,!}, W^{\mu,!*}) = 0$. By (2), $W^{\mu,!*}$ is supported on $\bar{S}^\lambda$. By the assumption on ν,

$$\mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(W^{\nu,!}, \mathcal{M}) = 0 \tag{13}$$

for any $\mathcal{M}$ supported on $\bar{S}^\lambda$.

Thus, we got a non-zero map $W^{\nu,!} \to W^{\lambda,!}$. Now by (13), $\nu = \lambda$. In the latter case, the map $W^{\nu,!} \to W^{\lambda,!}$ is the scalar multiple of the identity map, so the compostion $W^{\nu,!} \to K \to W^{\lambda,!}$ is surjective, and we were dealing with the last quotient. □

1.2.33. For 6.3.5. Since $W^{\lambda,!}$ are of finite length and their irreducible subquotients are compact, $W^{\lambda,!}$ lies in the subcategory of $\mathrm{Whit}_{q,x}(G)$ generated by all $W^{\mu,!*}$. Since $W^{\lambda,!}$ generate $\mathrm{Whit}_{q,x}(G)$, we see that the collection $W^{\mu,!*}$, $\mu \in \Lambda^+$ generate $\mathrm{Whit}_{q,x}(G)$.

1.2.34. For 6.3.7. It is better to say that $\mathrm{Av}_!^{N_k,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}})$ is placed in usual degree $-2\dim(N_k t^\lambda)$. This follows from Section 1.2.25 of this file. Since $Shv(S^\lambda)^{\leq m} \subset Shv(S^\lambda)$ is closed under colimits, $\mathrm{Av}_!^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N}(\delta_{t^\lambda,\mathrm{Gr}})$ is placed in degrees $\leq m$ for any $m \in \mathbb{Z}$.

I think for an ind-scheme Y of ind-finite type, $Shv(Y)^{\leq 0}$ is stable under countable products, right? Then this shows that $Shv(S^\lambda)$ is not left complete by (HA, 1.2.1.19).

1.2.35. For 6.3.8. There it is assumed $C \in \mathrm{DGCat}_{cont}$.

In the definition of a *Artinian* t-structure the finite length is understood in $C^{\heartsuit}$ (not in the abelian subcategory $C^c \cap C^{\heartsuit}$).

Note that if C^c is preserved by truncation functors and the t-structure is compatible with filtered colimits then the t-structure is compactly generated. Indeed, $\tau^{\leq 0} : C \to C^{\leq 0}$ preserves filtered colimits. So, if $c \in C^{\leq 0}$, pick a functor $I \to C$, $i \mapsto c_i$ with I small filtered such that $\mathrm{colim}_i\, c_i \widetilde{\to} c$ and $c_i \in C^c$. Applying $\tau^{\leq 0}$, one gets $c \widetilde{\to} \mathrm{colim}_i\, \tau^{\leq 0} c_i$.

Remark: in the definition of *noetherian* t-structure you write in parenthesis "in particular is abelian". For any coherent t-structure, C^c inherits a t-structure, hence $C^c \cap C^{\heartsuit}$ is abelian by (HA, 1.2.1.12). So, it is better to make this remark in the definition of a noetherian t-structure.

1.2.36. Let us prove Cor. 6.3.10, we check that the t-structure on $\mathrm{Whit}_{q,x}(G)$ is Artinian.

Recall that $\mathrm{Whit}_{q,x}(G) \widetilde{\to} \mathrm{colim}_{\lambda \in \Lambda} \mathrm{Whit}_{q,x}(G)_{\leq \lambda}$. Note also that

$$\mathrm{Whit}_{q,x}(G) = \prod_{\mu \in \pi_1(G)} \mathrm{Whit}_{q,x}(G)_{\mu},$$

where $\mathrm{Whit}_{q,x}(G)_{\mu}$ is the Whittaker category on the connected component $\mathrm{Gr}^{\omega^{\rho},\mu}_{G,x}$ of $\mathrm{Gr}^{\omega^{\rho}}_{G,x}$. Over each connected component $\mathrm{Gr}^{\omega^{\rho},\mu}_{G,x}$ this colimit is filtered, so as in Lemma 1.1.39 of this file, each compact object of $\mathrm{Whit}_{q,x}(G)_{\mu}$ is the extension by zero from some $\bar{S}^{\nu}$ for ν over μ. Moreover, any compact object of $\prod_{\mu \in \pi_1(G)} \mathrm{Whit}_{q,x}(G)_{\mu}$ is of the form $(c_{\mu})_{\mu \in \pi_1(G)}$, where $c_{\mu} \in \mathrm{Whit}_{q,x}(G)^c_{\mu}$ and $c_{\mu} = 0$ for all but finite number of μ by ([43], 9.2.28).

Lemma 1.2.37 ([20], ch. II.1, Lm. 1.2.4)**.** *Let C_0 be a (non-cocomplete)* DG-*category, endowed with a t-structure. Then $C := \mathrm{Ind}(C_0)$ carries a unique accessible t-structure, which is compatible with filtered colimits, and for which the tautological inclusion $C_0 \to C$ is t-exact. Moreover, the subcategory $C^{\leq 0}$ (resp., $C^{\geq 0}$) is compactly generated under filtered colimits by $C_0^{\leq 0}$ (resp., $C_0^{\geq 0}$). In addition, if C_0 is bounded from above then C is right-complete.*

Proof. The proof of all but the last claim are given in ([43], 10.3.3). To see that C is right complete, note first that C is presentable, as C_0 admits finite colimits. Besides, the t-structure on C is accessible. So, by ([43], 4.0.10), it suffices to show that for any $z \in C$ the natural map $z \to \mathrm{colim}_{n \in \mathbb{Z}}\, \tau^{\leq n} z$ is an isomorphism in C. Pick a presentation $z \widetilde{\to} \mathrm{colim}_{i \in I}\, z_i$ with $z_i \in C_0$. Then

$$\underset{n \in \mathbb{Z}}{\mathrm{colim}}\, \tau^{\leq n} z \widetilde{\to} \underset{n \in \mathbb{Z}}{\mathrm{colim}}\, \underset{i \in I}{\mathrm{colim}}\, \tau^{\leq n} z_i \widetilde{\to} \underset{i \in I}{\mathrm{colim}}\, \underset{n \in \mathbb{Z}}{\mathrm{colim}}\, \tau^{\leq n} z_i \widetilde{\to} \underset{i \in I}{\mathrm{colim}}\, z_i \widetilde{\to} z,$$

because $\tau^{\leq n}$ preserves filtered colimits. □

Proposition 1.2.38. *Let $C \in \mathrm{DGCat}_{cont}$ with a t-structure compatible with filtered colimits. The condition that each irreducible object of $C^{\heartsuit}$ is compact and they generate C is equivalent to the t-structure on C be Artinian.*

Proof. i) Assume each irreducible object of $C^\heartsuit$ is compact and they generate C. Let I be the set of irreducible objects in $C^\heartsuit$, we write c_i for the corresponding object. Let D be the smallest stable subcategory of C containing c_i for all i. So, each object of D is a finite extension of objects of the form $c_i[n_i]$.

Claim: 1) $D \subset C$ is the full subcategory of those $d \in C$, which are cohomologically bounded, and whose all cohomologies are of finite length in $C^\heartsuit$.
2) The inclusion $D \subset C$ is closed under direct summands.

Proof. 1) Let $d \in D$. We claim that each $\mathrm{H}^i(d)$ is of finite length, and its subquotients are of the form c_j for some $j \in I$. This is proved by induction on the length of a filtration on d. Assume $d_1 \to d \to c_i[n]$ is a fibre sequence, where we know this claim already for d_1 by induction hypothesis. Then $H^i(d_1) \to H^i(d) \to \mathrm{H}^i(c_i[n])$ is exact, and we are done.

The converse inclusion is obvious.
2) Let $d \in C$, $d = z \oplus z'$ with $z, z' \in C$ then z, z' are cohomologically bounded, because H^0 preserves finite products, which are also finite coproducts. Moreover, $\mathrm{H}^i(z) \oplus \mathrm{H}^i(z') \widetilde{\to} \mathrm{H}^i(d)$ is of finite length, hence the same holds for $\mathrm{H}^i(z)$ and $\mathrm{H}^i(z')$. Thus, $z, z' \in D$. $\square$

By ([20], ch. I.1, 7.2.4(3)), $\mathrm{Ind}(D) \widetilde{\to} C$. So, $C^c \widetilde{\to} D$ by (HTT, 5.4.2.4). By Lemma 1.2.37, the t-structure on $\mathrm{Ind}(D)$ $C^{\le 0}$ is compactly generated under filtered colimits by $D^{\le 0}$. So, the t-structure is compactly generated. By the above, the t-structure is coherent.

If $d \in C^c \cap C^\heartsuit$ then d is of finite length by the above *claim*, hence its subquotients also lie in $C^c \cap C^\heartsuit$. That is, the t-structure is noetherian and artinian. We are done.

ii) Conversely, assume the t-structure is Artinian. The category $C^c \cap C^\heartsuit$ is abelian, let I denote the set of its irreducible objects. For $i \in I$ we denote by $d_i \in C^c \cap C^\heartsuit$ the corresponding object. Then C^c is the smallest stable subcategory of C containing d_i for all i.

Since the t-structure is coherent, C^c is equipped with the induced t-structure. Since the t-structure on C is compatible with filtered colimits, the t-structure on C is the one defined on $\mathrm{Ind}(C^c) \widetilde{\to} C$ in Lemma 1.2.37. In particular, $C^{\le 0} = \mathrm{Ind}(C^c \cap C^{\le 0})$.

Let $c \in C^\heartsuit$. Pick a diagram $J \to C^{\le 0}$, $j \mapsto c_j$ such that J is small filtered, $c_j \in C^c \cap C^{\le 0}$ and $c \widetilde{\to} \mathrm{colim}_j \, c_j$. Then $c \widetilde{\to} \tau^{\ge 0} c \widetilde{\to} \mathrm{colim}_j \, \tau^{\ge 0} c_j$ in $C^\heartsuit$, because $\tau^{\ge 0} : C^{\le 0} \to C^\heartsuit$ preserves colimits. This shows that $\mathrm{Ind}(C^c \cap C^\heartsuit) \widetilde{\to} C^\heartsuit$. (The notation Ind is that of [36]).

Let now $c \in C^\heartsuit$ be irreducible. Pick a presentation $c \widetilde{\to} \mathrm{colim}_{j \in J} \, c_j$ with $c_j \in C^\heartsuit \cap C^c$. By the above, there is $i \in I$ such that $\mathrm{Hom}_{C^\heartsuit}(d_i, c) \neq 0$. Then a nonzero map $d_i \to c$ is surjective. Since $C^c \cap C^\heartsuit \subset C^\heartsuit$ is stable under subquotients, $c \in C^c$. $\square$

1.2.39. Example, take $C = Shv(\mathbb{A}^1)$, the ℓ-adic sheaves. Assume k algebraically closed for simplicity in this example. Equip C with the usual t-structure. The collection $e_x = (i_x)_! e$ (for $x \in A^1$ closed points) does not generate C. Indeed, write e_Y for the constant sheaf on $Y = \mathbb{A}^1$. For $x \in Y$ we have $i_x^! e_Y \widetilde{\to} e[-2]$. Consider the map

$\oplus_x (i_x)_! (i_x)^! e \to e$ in C. Applying $i_x^!$ for any x, it becomes an isomorphism. However, it is not an isomorphism. So, the collection e_x, ($x \in \mathbb{A}^1$ closed) does not generate C.

Dennis claims in the case of ℓ-adic sheaves to get a system of generators, it suffices to add $j_* L$ for any irreducible representation L of the Galois group of $\eta \in \mathbb{A}^1$, the generic point of $\mathbb{A}^1$ (and in the case of $\mathcal{D}$-modules, to add $j_* \mathcal{D}$). Here $j : \eta \to \mathbb{A}^1$ the inclusion.

1.2.40. Let $G = \operatorname{colim}_{i \in I} G_i$ in PreStk, where I is a filtered small category, each G_i is a placid scheme, a group scheme, and for $i \to j$ in I the map $i_{ij} : G_i \to G_j$ is a homomorphism of group schemes and a placid closed embedding. So, G is a placid ind-scheme. Recall that $Shv(G) \widetilde{\to} \operatorname{colim}_{i \in I} Shv(G_i)$. Let $M \in G - mod^r, C \in G - mod$. Then one has

$$\operatorname{colim}_{i \in I} M \otimes_{Shv(G_i)} C \widetilde{\to} M \otimes_{Shv(G)} C$$

Indeed, I is sifted, so $\operatorname{colim}_{i \in I} Shv(G_i)^{\otimes n} \widetilde{\to} Shv(G)^{\otimes n}$. So,

$$M \otimes_{Shv(G)} C \widetilde{\to} \operatorname*{colim}_{[n] \in \boldsymbol{\Delta}^{op}} M \otimes Shv(G)^{\otimes n} C \widetilde{\to}$$
$$\operatorname*{colim}_{i \in I} \operatorname*{colim}_{[n] \in \boldsymbol{\Delta}^{op}} M \otimes Shv(G_i)^{\otimes n} C \widetilde{\to} \operatorname*{colim}_{i \in I} M \otimes_{Shv(G_i)} C$$

In particular, $C_G \widetilde{\to} \operatorname{colim}_{i \in I} C_{G_i}$ in DGCat_{cont}, the transition maps $C_{G_i} \to C_{G_j}$ for $i \to j$ in I come from $(i_{ij})_* : Shv(G_i) \to Shv(G_j)$.

1.2.41. Let $f : H \to G$ be a map in $\mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, where H, G are placid ind-schemes. Recall that $f_* : (Shv(H), *) \to (Shv(G), *)$ is monoidal. Let $D \in G - mod^r, C \in G - mod$ then we have a natural functor $D \otimes_{Shv(H)} C \to D \otimes_{Shv(G)} C$. Indeed, this holds for any morphism $A \to B$ in $Alg(\mathrm{DGCat}_{cont})$ and $D \in B - mod^r, C \in B - mod$.

In particular, we have a natural functor $C_H \to C_G$. For $H = \operatorname{Spec} k$ and f the unit map we denote the corresponding functor by $\mathrm{Av}_{G,*} : C \to C_G$. So, the composition $C \stackrel{\mathrm{Av}_{H,*}}{\to} C_H \to C_G$ is $\mathrm{Av}_{G,*}$.

1.2.42. Let $p : G \to \operatorname{Spec} k$ be a placid scheme, and a group scheme. Let $C \in G - mod$. Viewing $p^* : \mathrm{Vect} \leftrightarrows Shv(G) : p_*$ as an adjoint pair in $G - mod^r$ and applying $\bullet \otimes_{Shv(G)} C$, we get an adjoint pair $\mathrm{oblv}_G : C_G \leftrightarrows C : \mathrm{Av}_{G,*}$ in DGCat_{cont}.

If in addition G is pro-unipotent then $\mathrm{id} \to p_* p^*$ is an isomorphism, so oblv_G is fully faithful and $\mathrm{Av}_{G,*} \mathrm{oblv}_G \widetilde{\to} \mathrm{id}$.

Let now G be an placid ind-scheme, an object of $\mathcal{G}\mathrm{rp}(\mathrm{PreStk})$ written as $\operatorname{colim}_{i \in I} G_i$, where I is small filtered, G_i is a pro-unipotent group scheme, for $i \to j$ in I the transition map $f_{ij} : G_i \to G_j$ is a placid closed immersion and a homomorphism of group schemes. For any $C \in G - mod$ and $i \to j$ in I the composition $C \stackrel{\mathrm{Av}_{G_i,*}}{\to} C_{G_i} \to C_{G_j}$ is $\mathrm{Av}_{G_j,*}$. Moreover, $\mathrm{Av}_{G_i,*}, \mathrm{Av}_{G_j,*}$ have fully faithful left adjoints. By ([43], 9.2.35) the functor $\mathrm{oblv}_{G_j} : C_{G_j} \to C$ factors through $\mathrm{oblv}_{G_i} : C_{G_i} \hookrightarrow C$. Denote the functor so obtained by $\mathrm{oblv}_{G_i, G_j} : C_{G_j} \to C_{G_i}$.

We obtained an adjoint pair $\mathrm{oblv}_{G_i, G_j} : C_{G_j} \leftrightarrows C_{G_i} : \mathrm{Av}_{G_i, G_j, *}$, where oblv_{G_i, G_j} is fully faithful!

1.2.43. Let G be a unipotent group scheme $C \in \mathrm{DGCat}_{cont}$ a nonunital G-module category. Let $D = \mathrm{Fib}(C \stackrel{\mathrm{Av}^G_*}{\to} C^G)$, this is a full subcategory in C. Consider the functor $\xi : C \to D$ sending c to $\mathrm{cofib}(\mathrm{Av}^G_*(c) \to c)$. We use here the adjoint pair $\mathrm{oblv}_G : C^G \rightleftarrows C : \mathrm{Av}^G_*$, where $\mathrm{Av}^G_*(c) = e_G * c$. It gives the above morphism $\mathrm{Av}^G_*(c) \to c$. We want to check that the essential image of ξ generates D.

Clearly, ξ is left adjoint to the inclusion $j : D \to C$. Since j is conservative, the essential image of ξ generates D by ([20], ch. I.1, 5.4.3).

Now consider the natural functor $\mathrm{Av}_{G,*} : C \to C_G$. Clearly, D is contained in its kernel.

For any smooth group scheme G of finite type, $\mathrm{Av}^G_* : C \to C^G$ factors through $C_G \to C^G$, and the latter is an equivalence ([24], B.1.2). This implies that the kernel of $\mathrm{Av}_{G,*} : C \to C_G$ is precisely D.

1.2.44. Let $G \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$ be a placid scheme, $C \in G-mod$. Assume G prounipotent, write $G = \lim_{i \in I^{op}} G_i$, where G_i is a unipotent group scheme, I is small filtered, and for $i \to j$ in I the map $\alpha_{ij} : G_j \to G_i$ is a smooth surjective affine homomorphism, whose kernel K_{ij} is a unipotent group scheme. Recall that $Shv(G) \widetilde{\to} \mathrm{colim}_{i \in I} Shv(G_i)$, where for $i \to j$ the transition map $Shv(G_i) \to Shv(G_j)$ is α^*_{ij}. The natural functor $Shv(G_i) \to Shv(G)$ coming from this inductive system is nonunital monoidal. Indeed, the square is cartesian

$$\begin{array}{ccc} G_j \times_{K_{ij}} G_j & \to & G_j \\ \downarrow & & \downarrow \\ G_i \times G_i & \stackrel{m}{\to} & G_i, \end{array}$$

where we denoted by $G_j \times_{K_{ij}} G_j$ the quotient of $G_j \times G_j$ by the action of K_{ij}, where $z \in K_{ij}$ acts on (g_1, g_2) as $(g_1 z, z^{-1} g_2)$. This implies that for $F_i \in Shv(G_i)$ one has $\alpha^*_{ij} F_1 * \alpha^*_{ij} F_2 \widetilde{\to} \alpha^*_{ij}(F_1 * F_2)$. Let $\alpha_i : G \to G_i$ be the projection. Now given $M \in G-mod^r$, the morphism

$$\mathrm{id} \otimes \alpha^*_i \otimes \mathrm{id} : M \otimes Shv(G_i)^{\otimes n} \otimes C \to M \otimes Shv(G)^{\otimes n} \otimes C$$

becomes a morphism of functors in $\mathrm{Fun}(\boldsymbol{\Delta}^{op}_s, \mathrm{DGCat}_{cont})$. Here $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ is the subcategory with the same objects and only injective maps. By ([35], 6.5.3.7), $\boldsymbol{\Delta}^{op}_s \to \boldsymbol{\Delta}^{op}$ is cofinal, so $M \otimes_{Shv(G)} C \widetilde{\to} \mathrm{colim}_{[n] \in \boldsymbol{\Delta}^{op}_s} M \otimes Shv(G)^{\otimes n} \otimes C$. Restricting the action, we may view M as a nonunital right $Shv(G_i)$-module, and C as a nonunital let $Shv(G_i)$-module, and we get a morphism

$$M \otimes_{Shv(G_i)} C \to M \otimes_{Shv(G)} C$$

for each $i \in I$. Moreover,

$$M \otimes_{Shv(G)} C \widetilde{\to} \mathrm{colim}_{i \in I} M \otimes_{Shv(G_i)} C,$$

because I is sifted.

In particular, we get $C_G \widetilde{\to} \mathrm{colim}_{i \in I} C_{G_i}$. This implies formula (7.2) in the paper.

Related question: is the category $G-mod$ equivalent to the category of nonunital modules over $Shv(G)$?

1.2.45. For 7.1.1. The description of $\mathrm{Whit}_{q,x}(G)_{co}$ comes from Section 1.10.8.

The claim for any continuous idempotent comonad acting on some $D \in \mathrm{DGCat}_{cont}$ that he has in mind in 7.1.3 seems to be precisely Lemma 1.8.17 from this file.

For 7.1.5. For $k \le k'$, $N_{k'}/N_k$ is a smooth scheme of finite type, its dualizing sheaf is $e[2\dim(N_{k'}/N_k)]$.

Definition of (7.3) of the paper: For $k \le k'$ we have by definitions for the closed immersion $i : N_k \to N_{k'}$ the map $i_*i^! e_{N_{k'}} \to e_{N_{k'}}$. Since $i^! e_{N_{k'}} \widetilde{\to} e_{N_k}[-2\dim(N_{k'}/N_k)]$ by ([46], 0.0.21), this gives a map $i_* e_{N_k} \to e_{N_{k'}}[2\dim(N_{k'}/N_k)]$, hence for $F \in Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ a map

$$i_* e_{N_k} * F \to e_{N_{k'}}[2\dim(N_{k'}/N_k)] * F$$

We have $\mathrm{Av}^{N_k,\chi_k}_*(F) \widetilde{\to} \chi_k * F$ for $F \in Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$. We have denoted by χ_k the $*$-restriction of χ_N to N_k. One similarly has $i^!\chi_{k'} \widetilde{\to} \chi_k[-2\dim(N_{k'}/N_k)]$. This gives a map

$$\mathrm{Av}^{N_k,\chi_k}_*(F) \widetilde{\to} (i_*\chi_k) * F \to \chi_{k'}[2\dim(N_{k'}/N_k)] * F \widetilde{\to} \mathrm{Av}^{N_{k'},\chi_{k'}}_*(F)[2\dim(N_{k'}/N_k)]$$

Let us assume $N_0 = \mathfrak{L}^+(N)^{\omega^\rho}_x$. The above map yields a morphism

$$h' : \chi_k[2\dim(N_k/N_0)] \to \chi_{k'}[2\dim(N_{k'}/N_0)]$$

in $Shv(\mathfrak{L}(N)^{\omega^\rho}_x)$, so (7.4) of the paper is the functor of action by

$$E' := \operatorname*{colim}_{k\ge 0} \chi_k[2\dim(N_k/N_0)] \in Shv(\mathfrak{L}(N)^{\omega^\rho}_x) \tag{14}$$

As in Section 1.2.9 of this file, we may also consider $E := \lim\limits_{k\in(\mathbb{Z}_+)^{op}} \chi_k$ in $Shv(\mathfrak{L}(N)^{\omega^\rho}_x)$. By ([43], 4.0.12) E fits into a fibre sequence

$$E \to \prod_{k\ge 0} \chi_k \stackrel{f}{\to} \prod_{k\ge 0} \chi_k,$$

where f is given by the collection of maps $f_m : \prod_{k\ge 0} \chi_k \to \chi_m$. Here f_m is the composition

$$\prod_{k\ge 0} \chi_k \to \chi_{m+1} \oplus \chi_m \stackrel{h-\mathrm{id}}{\to} \chi_m$$

and h is the natural map $\chi_{m+1} \to i_*i^*\chi_{m+1} \widetilde{\to} \chi_m$ for $i : N_m \hookrightarrow N_{m+1}$.

Question Is E isomorphic to E'? I think no!!

For $C = Shv_{\mathcal{G}^G}(\mathfrak{L}(N)^{\omega^\rho}_x)$, C^{op} is stable, and we may calculate sequential limits in C^{op} by the above recipe. So, E' fits into a fibre sequence

$$\bigoplus_{k\ge 0} \chi_k[2\dim(N_k/N_0)] \stackrel{g}{\to} \bigoplus_{k\ge 0} \chi_k[2\dim(N_k/N_0)] \to E',$$

where g is given by a collection of maps

$$g_m : \chi_m[2\dim(N_m/N_0)] \to \bigoplus_{k\ge 0} \chi_k[2\dim(N_k/N_0)]$$

Here g_m is the composition

$$\chi_m[2\dim(N_m/N_0)] \stackrel{-\mathrm{id}+h'}{\to} \chi_m[2\dim(N_m/N_0)] \oplus \chi_{m+1}[2\dim(N_{m+1}/N_0)] \to \bigoplus_{k\ge 0} \chi_k[2\dim(N_k/N_0)]$$

By ([43], 9.2.6),

$$E \widetilde{\to} \operatorname*{colim}_{k\in\mathbb{Z}_+} (i_k)_* i_k^! E$$

for natural maps $i_k : N_k \to \mathfrak{L}(N)_x^{\omega^\rho}$. Since $i_k^!$ admits a left adjoint,

$$i_m^! E \widetilde{\to} \lim_{k\geq m} i_m^! \chi_k$$

For $k \geq m$ one has $i_m^! \chi_k \widetilde{\to} \chi_m[-2\dim(N_k/N_m)]$.

A better idea: one has

$$i_m^! E' \widetilde{\to} \operatorname*{colim}_{k\geq m} i_m^! \chi_k[2\dim(N_k/N_0)] \widetilde{\to} \operatorname*{colim}_{k\geq m} \chi_m[2\dim(N_m/N_0)] \widetilde{\to} \chi_m[2\dim(N_m/N_0)],$$

because the corresponding inductive system is constant. So, E' is given by the inverse system with terms $\chi_m[2\dim(N_m/N_0)]$ in $\lim_{i\in(\mathbb{Z}_+)^{op}} Shv(Z_i)$, where the transition maps are given for $k \geq m$ by the isomorphisms

$$i_m^! \chi_k[2\dim(N_k/N_0)] \widetilde{\to} \chi_m[2\dim(N_m/N_0)]$$

Proposition 1.2.46.

$$\operatorname*{colim}_{k\geq 0} \chi_k^{-1}[2\dim(N_k/N_0)] \in Shv(\mathfrak{L}(N)_x^{\omega^\rho})$$

corresponds to $\chi_N \in Shv(\mathfrak{L}(N)_x^{\omega^\rho})^\vee$ under the self-duality on $Shv(\mathfrak{L}(N)_x^{\omega^\rho})$. The latter self-duality uses a particular element $0 \in \mathbb{Z}_+$ to apply the general framework of ([46]*, 0.0.23).*

Proof. Note that $\dim(N_k/N_0) = \operatorname{codim}_{N_k}(N_0)$ in the notation of ([46], 0.0.23). So, it suffices to show that for each $k > 0$, the image of χ_k^{-1} under the self-duality $Shv(N_k)\widetilde{\to}Shv(N_k)^\vee$ identifies with the composition $Shv(N_k) \overset{i_*}{\hookrightarrow} Shv(\mathfrak{L}(N)_x^{\omega^\rho}) \overset{\chi_N}{\to}$ Vect given by $K \mapsto \mathcal{H}om(\chi_k, K)$. This follows from ([46], 0.0.19). Namely, for $k > 0$ our nondegenerate character $f : N_k \to \mathbb{A}^1$ can be seen as a projection on a zero term of a placid presentation $N_k \widetilde{\to} \lim N_{k,r}$ with $N_{k,0} = \mathbb{A}^1$. Then for $K \in Shv(N_k)$,

$$\mathrm{R}\Gamma(\mathbb{A}^1, (f_*K) \otimes^! \mathcal{L}_\psi^{-1}) \widetilde{\to} \mathcal{H}om_{Shv(\mathbb{A}^1)}(\mathcal{L}_\psi, f_*K)$$

□

This proposition explains why E' is good for the definition of the pseudo-identity functor (7.5).

By ([47], 1.2.5), for $F \in Shv(N_k)$ one has $\chi_k * F \widetilde{\to} \chi_k \otimes \mathrm{R}\Gamma(N_k, F \otimes \chi_k^{-1})$. For any $m \geq 0$ one has $\chi_m * E' \widetilde{\to} E'$ in $Shv(\mathfrak{L}(N)_x^{\omega^\rho})$, because for $k \geq m$, $\chi_m * \chi_k \widetilde{\to} \chi_k$, and the convolution preserves colimits. For this reason the functor Ps-Id takes values in $\mathrm{Whit}_{q,x}(\mathrm{Gr}_{G,x}^{\omega^\rho})$.

For any $m \geq 0$, $F \in Shv(N_m)$ applying $E' * \cdot$ to $\chi_m * F \to F$, one gets an isomorphism. Indeed, applying $E' * \cdot$ to $\chi_m \to \delta_1$ one gets an isomorphism. This follows from the fact that applying $\mathrm{R}\Gamma(N_m, \chi_m^{-1} \otimes \cdot)$ to $\chi_m \to \delta_1$, one gets an isomorphism. For this reason Ps-Id factors through the coinvariants.

1.2.47. For 7.1.6. Proof of the formula (7.6): let $\mu\in\Lambda^+$, $\mu\neq\lambda$. Since $W^{\mu,!}$ is compact, we have

$$\mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(W^{\mu,!},\text{Ps-Id}(\delta_{t^\lambda,\mathrm{Gr}}))\widetilde{\to}$$
$$\mathrm{colim}_k\,\mathcal{H}om_{Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{N_k,\chi_k}}((i_\mu)_!\mathring{W}^\mu,\mathrm{Av}_*^{N_k,\chi_k}(\delta_{t^\lambda,\mathrm{Gr}})[2\dim(N_k/N_0)])\widetilde{\to}$$
$$\mathrm{colim}_k\,\mathcal{H}om_{Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})}((i_\mu)_!\mathring{W}^\mu,\delta_{t^\lambda,\mathrm{Gr}}[2\dim(N_k/N_0)])=0\in\mathrm{Vect}$$

because $i_\mu^!\delta_{t^\lambda,\mathrm{Gr}}=0$. Recall that here $i_\mu:S^\mu\hookrightarrow\mathrm{Gr}^{\omega^\rho}_{G,x}$.

Note that the $*$-direct image of χ_k under $N_k\to\mathrm{Gr}^{\omega^\rho}_{G,x}$, $x\mapsto xt^\lambda$ is the $*$-extension of a local system $\chi_{k,\lambda}$ under $i:N_kt^\lambda\hookrightarrow\mathrm{Gr}^{\omega^\rho}_{G,x}$. For the latter embedding i we get

$$i^!\,\text{Ps-Id}(\delta_{t^\lambda,\mathrm{Gr}})\widetilde{\to}\underset{m\geq k}{\mathrm{colim}}\,i^!\chi_{m,\lambda}[2\dim(N_m/N_0)]\widetilde{\to}\chi_{k,\lambda}[2\dim(N_k/N_0)]$$

We used that $Stab_{\mathfrak{L}(N)^{\omega^\rho}_x}(t^\lambda)\subset\mathfrak{L}^+(N)^{\omega^\rho}_x$, so if $m\geq k$ then $\mathrm{codim}_{N_mt^\lambda}(N_kt^\lambda)=\mathrm{codim}_{N_m}(N_k)$. This gives $i^!\chi_{m,\lambda}\widetilde{\to}\chi_{k,\lambda}[-2\dim(N_m/N_k)]$ in the above displayed formula. Now

$$\mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(W^{\lambda,!},\text{Ps-Id}(\delta_{t^\lambda,\mathrm{Gr}}))\widetilde{\to}\lim_k\mathcal{H}om(\mathrm{Av}_!^{N_k,\chi_k}(\delta_{t^\lambda,\mathrm{Gr}})[-\langle\lambda,2\check\rho\rangle],\text{Ps-Id}(\delta_{t^\lambda,\mathrm{Gr}}))$$
$$\lim_k\mathcal{H}om_{Shv(N_kt^\lambda)}(\mathrm{Av}_!^{N_k,\chi_k}(\delta_{t^\lambda,\mathrm{Gr}})[-\langle\lambda,2\check\rho\rangle],\chi_{k,\lambda}[2\dim(N_k/N_0)])\widetilde{\to}$$
$$\lim_k\mathcal{H}om_{Shv(N_kt^\lambda)}(\chi_{k,\lambda}[\langle\lambda,2\check\rho\rangle],\chi_{k,\lambda})\widetilde{\to}e[-\langle\lambda,2\check\rho\rangle]$$

We have used the fact that $\mathrm{Av}_!^{N_k,\chi_k}(\delta_{t^\lambda,\mathrm{Gr}})\widetilde{\to}\chi_{k,\lambda}[2\dim(N_kt^\lambda)]$ over N_kt^λ obtained as in Section 1.2.25 of this file. Further, $\dim(N_0t^\lambda)=\langle\lambda,2\check\rho\rangle$ and $\dim(N_kt^\lambda)=\dim(N_k/N_0)+\langle\lambda,2\check\rho\rangle$. This finishes the proof of the formula (7.6) of the paper.

1.2.48. For 7.2.2. For $K\in Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x})$, $F\in Shv_{(\mathcal{G}^G)}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ and $k\geq 0$ one has

$$\mathrm{R\Gamma}(\mathrm{Gr}^{\omega^\rho}_{G,x},K\otimes^!(\chi_k*F))\widetilde{\to}\mathrm{R\Gamma}(\mathrm{Gr}^{\omega^\rho}_{G,x},(\chi_k^{-1}*K)\otimes^!F)$$

canonically. Indeed, consider the map $f:\mathrm{Gr}\times N_k\to\mathrm{Gr}\times N_k\times\mathrm{Gr}$, $(g,n)\mapsto(g,n,n^{-1}g)$. The LHS identifies with $\mathrm{R\Gamma}\,f^!(K\boxtimes\chi_k\boxtimes F)$. Consider now the isomorphism $\tau:\mathrm{Gr}\times N\to\mathrm{Gr}\times N$ sending (x,n) to (nx,n). Then $\mathrm{R\Gamma}\,\tau^!f^!(K\boxtimes\chi_k\boxtimes F)$ identifies with the RHS. This shows that the dual of the functor

$$\mathrm{Av}_*^{N_k,\chi_k}:Shv_{(\mathcal{G}^G)}(\mathrm{Gr}^{\omega^\rho}_{G,x})\to Shv_{(\mathcal{G}^G)}(\mathrm{Gr}^{\omega^\rho}_{G,x})$$

is the functor $\mathrm{Av}_*^{N_k,\chi_k^{-1}}:Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x})\to Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x})$.

Consider the equivalence $Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x})\to Shv_{(\mathcal{G}^G)}(\mathrm{Gr}^{\omega^\rho}_{G,x})^\vee$ sending K to the functor $F\mapsto f_K(F)=\mathrm{R\Gamma}(\mathrm{Gr}^{\omega^\rho}_{G,x},K\otimes^!F)$. If $K\in Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{N_k,\chi_k^{-1}}$ then f_K sends each map $\mathrm{Av}_*^{N_k,\chi_k}(F)\to F$ to an equivalence. So, if $K\in\mathrm{Whit}_{q^{-1},x}(G)$ then $f_K\in(\mathrm{Whit}_{q,x}(G)_{co})^\vee$. This defines a functor $\mathrm{Whit}_{q^{-1},x}(G)\to(\mathrm{Whit}_{q,x}(G)_{co})^\vee$, which is an equivalence. Indeed, it is fully faithful by construction. It is also essentially surjective.

Indeed, if $C_0 \subset C$ is a map in DGCat_{cont}, which is a full embedding, let $D = \mathrm{cofib}(C_0 \to C)$. Then $\mathrm{Fun}(D, \mathrm{Vect}) \widetilde{\to} \mathrm{Fun}(C, \mathrm{Vect}) \times_{\mathrm{Fun}(C_0,\mathrm{Vect})} 0$, so $\mathrm{Fun}(D, \mathrm{Vect}) \to \mathrm{Fun}(C, \mathrm{Vect})$ is fully faithful.

Let now $K \in Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ such that f_K lies in the full subcategory $(\mathrm{Whit}_{q,x}(G)_{co})^\vee \subset Shv_{(\mathcal{G}^G)}(\mathrm{Gr}^{\omega^\rho}_{G,x})^\vee$. Then $K \in \mathrm{Whit}_{q^{-1},x}(G)$. Indeed, it suffices to show that for any k the map $\mathrm{Av}_*^{N_k,\chi_k^{-1}}(K) \to K$ is an isomorphism. This map is transformed by f to the morphism $f_{\mathrm{Av}_*^{N_k,\chi_k^{-1}}(K)}(F) \to f_K(F)$, that is, to the morphism $f_K(\mathrm{Av}_*^{N_k,\chi_k}(F)) \stackrel{f_K(\xi)}{\to} f_K(F)$ for the natural map $\xi : \mathrm{Av}_*^{N_k,\chi_k}(F) \to F$. However, ξ is an isomorphism in $\mathrm{Whit}_{q,x}(G)_{co}$, hence $f_K(\xi)$ is also an isomorphism. Since F was arbitrary, we are done.

1.2.49. For 7.2.3. Denote also, by abuse of notation, by $\text{Ps-Id} : Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ the functor given by (7.4) in the paper. What is missing is the claim that the dual of this functor is the corresponding pseudo-identity functor $Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ for q replaced by q^{-1}. It is given as $\mathrm{colim}_k \mathrm{Av}_*^{N_n,\chi_k^{-1}}[2\dim(N_k/N_0)]$. This is also related with Proposition 1.2.46 above. This implies that (7.9) is an involution. This also removes a potential abmiguity in the definition of (7.9), as one could compose from one side or the pother side!

For the sake of completeness, the diagram commutes

$$\begin{array}{ccc} \mathrm{Whit}_{q^{-1},x}(G)^\vee & \widetilde{\to} & \mathrm{Whit}_{q,x}(G)_{co} \\ \uparrow (\text{Ps-Id})^\vee & & \downarrow \text{Ps-Id} \\ (\mathrm{Whit}_{q^{-1},x}(G)_{co})^\vee & \widetilde{\to} & \mathrm{Whit}_{q,x}(G) \end{array}$$

here the horizontal arrows are (7.7).

Definition of Verdier duality $\mathbb{D}^{Verdier}$: the image of $K \in (\mathrm{Whit}_{q,x}(G)^c)^{op}$ is $\mathbb{D}(K) \in \mathrm{Whit}_{q^{-1},c}(G)^c$ iff for any $L \in \mathrm{Whit}_{q,x}(G)$,

$$\mathcal{H}om(K, L) \widetilde{\to} \mathrm{R}\Gamma(\mathrm{Gr}, L \otimes^! \text{Ps-Id}^{-1}(\mathbb{D}(K)))$$

It is equivalently characterized by the property. that for any $S \in \mathrm{Whit}_{q,x}(G)_{co}$,

$$\mathcal{H}om(K, \text{Ps-Id}(S)) \widetilde{\to} \mathrm{R}\Gamma(\mathrm{Gr}, (\mathbb{D}K) \otimes^! S)$$

1.2.50. For the proof of 7.2.5. It is understood that the map $(\mathrm{Whit}_{q,x}(G)^c)^{op} \to \mathrm{Whit}_{q,x}(G)^\vee$ sends K to the functor $F \mapsto \mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(K, F)$. We must check that the image of $W^{\lambda,*}_{co}$ under the canonical equivalence $\mathrm{Whit}_{q,x}(G)_{co} \to \mathrm{Whit}_{q^{-1},x}(G)^\vee$ is the functor $F \mapsto \mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(W^{\lambda,!}, F)$. One has

$$\mathcal{H}om_{Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})}(\delta_{t^\lambda,\mathrm{Gr}}, F) \widetilde{\to} i^!_{t^\lambda} F,$$

where $i_{t^\lambda} : \mathrm{Spec}\, k \to \mathrm{Gr}^{\omega^\rho}_{G,x}$ is the point t^λ.

1.2.51. For 7.3.5. The following seems relevant here. Let $Y \in \mathrm{PreStk}_{lft}$, $i : Y' \hookrightarrow Y$ a closed immersion (so, schematic), and $j : U \hookrightarrow Y$ an open immersion, the complement to Y' (so, j is schematic). Let N be a unipotent group scheme acting on Y and preserving U, Y'. Let χ be a character local system on N. Then we have the full embeddings $Shv(U)^{N,\chi} \subset Shv(U), Shv(Y)^{N,\chi} \subset Shv(Y), Shv(Y')^{N,\chi} \subset Shv(Y')$ in DGCat_{cont}. The functors $j_*, j^!, i^!, i_!$ restrict to functors

$$j^! : Shv(Y)^{N,\chi} \leftrightarrows Shv(U)^{N,\chi} : j_*, \;\; i_! : Shv(Y')^{N,\chi} \leftrightarrows Shv(Y)^{N,\chi} : i^!$$

This is because the functors $i^!, i_!, j^!, j_*$ commute with the actions of $Shv(N)$. Moreover, for $K \in Shv(Y)$ we have $K \in Shv(Y)^{N,\chi}$ iff $j^!K \in Shv(U)^{N,\chi}$ and $i^!K \in Shv(Y')^{N,\chi}$.

Indeed, K fits into a fibre sequence $i_!i^!K \to K \to j_*j^!K$. Suppose $i^!K \in Shv(Y')^{N,\chi}$, $j^!K \in Shv(U)^{N,\chi}$. Since $Shv(Y)^{N,\chi} \subset Shv(Y)$ is closed under colimits, $K \in Shv(Y)^{N,\chi}$. This explains ([24], Lm. 4.6.2).

1.2.52. Note that $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x} = \mathrm{colim}_{\lambda\in\Lambda}(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le\lambda x}$, so that

$$Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}) \widetilde{\to} \lim_{\lambda\in\Lambda^{op}} Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le\lambda x}),$$

the transition functors are !-pullbacks. Since limits commute with invariants by definition,

$$\mathrm{Whit}_{q,glob}(G) \widetilde{\to} \lim_{\lambda\in\Lambda^{op}} \mathrm{Whit}_{q,glob}(G)_{\le\lambda} \tag{15}$$

In the latter limit we may pass to left adjoints and get

$$\mathrm{Whit}_{q,glob}(G) \widetilde{\to} \underset{\lambda\in\Lambda}{\mathrm{colim}}\, \mathrm{Whit}_{q,glob}(G)_{\le\lambda},$$

where the transition functors are !-pushforwards (the colimit taken in DGCat_{cont}, this is not the colimit it $1-\mathcal{C}\mathrm{at}$).

For a finite collection of points $\underline{y} = \{y_1, \ldots, y_m\}$ on $X - x$, let $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\, \mathrm{good\ at}\, \underline{y}} \subset (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}$ be the open substack given by requiring that the maps

$$\kappa^{\check{\lambda}} : \omega^{\langle\rho,\check{\lambda}\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}(\infty x)$$

have no zeros at $\underline{y}$. One first defines $\mathrm{Whit}_q((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\, \mathrm{good\ at}\, \underline{y}})$ as in ([24], 4.5.1). The full embedding $\mathrm{Whit}_q((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\, \mathrm{good\ at}\, \underline{y}}) \hookrightarrow Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\, \mathrm{good\ at}\, \underline{y}})$ admits a continuous right adjoint (this is proved as in Section 1.2.11 of this file), and similarly for

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le\lambda x,\, \mathrm{good\ at}\, \underline{y}}) \hookrightarrow Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le\lambda x,\, \mathrm{good\ at}\, \underline{y}})$$

This right adjoint commutes with the !-restriction under

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le\lambda x,\, \mathrm{good\ at}\, \underline{y}} \hookrightarrow (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le\lambda' x,\, \mathrm{good\ at}\, \underline{y}}$$

for $\lambda \le \lambda'$.

In ([24], 4.6.5) Dennis uses the following general remark: Let I be the index small category, we are given $I \times [1] \to \mathrm{DGCat}_{cont}$ sending i to $\mathrm{oblv}_i : C_i \subset E_i$, a full subcategory. Assume oblv_i is included into an adjoint pair $\mathrm{oblv}_i : C_i \leftrightarrows E_i : \mathrm{Av}^i_*$ in DGCat_{cont}. Let $\mathrm{oblv} : C = \lim_i C_i \to E = \lim_i E_i$ be obtained by passing to the limit in DGCat_{cont}. Then C is a full subcategory of E by ([43], 2.2.17), because $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}\mathrm{at}$ preserves limits. Assume for $i \to j$ in I the diagram commutes

$$\begin{array}{ccc} C_j & \stackrel{\mathrm{Av}^j_*}{\leftarrow} & E_j \\ \uparrow & & \uparrow \\ C_i & \stackrel{\mathrm{Av}^i_*}{\leftarrow} & E_i \end{array}$$

Then the right adjoint to oblv is continuous by ([20], ch. I.1, Lm. 2.6.4) and ([43], Lm. 2.2.68), and for any i the diagram commutes

$$\begin{array}{ccc} C & \stackrel{\mathrm{Av}_*}{\leftarrow} & E \\ \downarrow & & \downarrow \\ C_i & \stackrel{\mathrm{Av}_*}{\leftarrow} & E_i \end{array}$$

If $\underline{y} = \underline{y}' \cup \underline{y}''$ then we have the open immersion $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}} \subset (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}'}$. As in ([24], 4.6.7), the restriction functor along the latter map sends $\mathrm{Whit}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}'})$ to $\mathrm{Whit}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}})$, and the diagram commutes

$$\begin{array}{ccc} Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}'}) & \to & Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}}) \\ \downarrow & & \downarrow \\ \mathrm{Whit}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}'}) & \to & \mathrm{Whit}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}}), \end{array}$$

where the vertical arrows are the corresponding right adjoints to the inclusions.

By definition, $\mathrm{Whit}_{q,glob}(G) \subset Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ is the full subcategory of those objects whose restriction to $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}}$ lies in

$$\mathrm{Whit}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}}) \subset Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x,\,\text{good at}\,\underline{y}})$$

for any finite non-empty collection of points $\underline{y} \subset X - x$.

From (15) we see that given $K \in Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$, it lies in $\mathrm{Whit}_{q,glob}(G)$ iff for any $\mu \in \Lambda$, the !-restriction to $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le \mu x}$ lies in $\mathrm{Whit}_{q,glob}(G)_{\le \mu}$.

For $K \in Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le \mu x})$ the condition to lies in $\mathrm{Whit}_{q,glob}(G)_{\le\mu}$ is equivalent to the property that its !-restriction to any locally closed substack given by fixing $\nu \le \mu$ such that in a neighbourhood of x the map

$$\kappa^{\check{\lambda}} : \omega^{\langle \rho, \check{\lambda}\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}(\langle \nu, \check{\lambda}\rangle x)$$

has no zero, lies in the corresponding Whittaker category. (The latter stratum is bigger than $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\nu x}$).

1.2.53. For 7.3.5 more: the fact that the inclusion $\mathrm{Whit}_{q,glob}(G) \hookrightarrow Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ is compatible with the perverse t-structure on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}$ comes as follows. For a finite collection of points $\underline{y}$ and closed group subschemes $\mathfrak{L}^+(N)_{\underline{y}} = N_0 \subset N_m \subset \mathcal{L}(N)_{\underline{y}}$, the stack

$$N_0 \backslash N_m \times^{N_0} (\overline{\mathrm{Bun}}_N^{\omega^\rho})^{N-level\ \infty \underline{y}}_{\infty x,\,\text{good at}\,\underline{y}}$$

has a structure of groupoid acting on $(\overline{\operatorname{Bun}}_N^{\omega^\rho})^{N-level\ \infty\underline{y}}_{\infty x,\text{ good at }\underline{y}}$. The corresponding action map is smooth, and for this reason the truncation functors preserve the equivariance condition.

In Section 7.3.5 the perverse t-structure on $Shv_{\mathcal{G}^G}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x})$ is mentioned without any definition. In the convention section a definition of the perverse t-structure for an ind-algebraic stack should be given. My understanding is as follows: if $Y = \operatorname{colim}_{i\in I} Y_i$ with Y_i an algebraic stack locally of finite type, I filtered then $Shv(Y)^{\le 0}$ should be the smallest full subcategory of $Shv(Y)$ containing $Shv(Y_i)^{\le 0}$ for any i, closed under extensions and small colimits. Then by (HA, 1.4.4.11), $Shv(Y)^{\le 0}$ is then presentable and defines an accessible t-structure on $Shv(Y)$. For $K \in Shv(Y)$ we have $K \in Shv(Y)^{\ge 0}$ iff for any i, the !-restriction of K to Y_i lies in $Shv(Y_i)^{\ge 0}$. As in the case of ind-schemes of ind-finite type, this t-structure is compatible with filtered colimits.

1.2.54. For 7.3.5. We may apply ([43], 2.7.6) to describe $Shv_{\mathcal{G}^G}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x})$. Namely, let I be the set of finite subsets in $X - x$ ordered by reversed inclusion. We have a functor $I \to \operatorname{PreStk}$ sending $\underline{y}$ to $(\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y}}$. As in ([43], 2.7.6), we get a functor $F : \boldsymbol{\Delta}^{op} \to \operatorname{PreStk}$ sending $[n]$ to

$$\sqcup_{\underline{y}} (\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y}},$$

the coproduct in PreStk taken over all maps of sets $\underline{y} : [n] \to X - x$, that is, $\underline{y} = \{y_0, \dots, y_n\} \subset X - x$. It is understood that if $\alpha : [m] \to [n]$ is a map in $\boldsymbol{\Delta}$ then for $\underline{y} = \{y_0, \dots, y_n\} \subset X - x$ one get $\underline{y'} = \{y_{\alpha(0)}, \dots, y_{\alpha(m)}\} \subset \underline{y}$, and

$$(\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y}} \subset (\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y'}}$$

Then $\operatorname{colim}_{\underline{y}\in I} (\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y}}$ identifies with $\operatorname{colim}_{[n]\in\boldsymbol{\Delta}^{op}} F$ by ([43], 2.7.6). Its sheafification in etale topology is $(\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x}$. This is similar to ([43], 10.2.2). So, applying $Shv_{\mathcal{G}^G}$, we get

$$Shv_{\mathcal{G}^G}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x}) \widetilde{\to} \lim_{\underline{y}\in I^{op}} Shv_{\mathcal{G}^G}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y}})$$

and passing to the full subcategories Whit, we get

$$\operatorname{Whit}_{q,glob}(G) \widetilde{\to} \lim_{\underline{y}\in I^{op}} \operatorname{Whit}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y}})$$

Now for each $\underline{y}$, we have the continuous right adjoint

$$\operatorname{Av}_{*,\underline{y}} : Shv_{\mathcal{G}^G}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y}}) \to \operatorname{Whit}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x,\text{ good at }\underline{y}})$$

to the inclusion. Passing to the limit over $\underline{y}$, we get the continuous right adjoint $\operatorname{Av}_*^{N_{glob},\chi_N} : Shv_{\mathcal{G}^G}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x}) \to \operatorname{Whit}_{q,glob}(G)$ to the inclusion.

1.2.55. For 7.3.10. It is not clear if $Shv_{\mathcal{G}^G}(((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le\mu x})$ is compactly generated for D-modules. However, it is compactly generated in the constructible context. Namely, for any $Y\in\mathrm{PreStk}_{lft}$, $Shv(Y)$ is compactly generated in the constructible context by ([2], C.1.1).

Note that $\mathrm{Whit}_{q,glob}(G)$ is compactly generated by objects of the form $W^{\lambda,!}_{glob}$ for $\lambda\in\Lambda^+$. We check that

$$\mathrm{Whit}_{q,glob}(G)^c\subset Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})^{loc.c} \tag{16}$$

Consider the smallest stable subcategory $C\subset\mathrm{Whit}_{q,glob}(G)$ containing $W^{\lambda,!}_{glob}$ for all $\lambda\in\Lambda^+$. Then $\mathrm{Ind}(C)\to\mathrm{Whit}_{q,glob}(G)$ is an equivalence, so any object of $\mathrm{Whit}_{q,glob}(G)^c$ is a direct summand in $\mathrm{Whit}_{q,glob}(G)$ of some $K\in C$. So, over the connected component of $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}$ given by $\bar\mu\in\pi_1(G)$, $K\in\mathrm{Whit}_{q,glob}(G)_{\le\mu}$ for some $\mu\in\Lambda^+$ over $\bar\mu$, and its !-restriction to each $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda x}$ lies in $\mathrm{Whit}_{q,glob}(G)^c_{=\lambda}$. We see that $C\subset\mathrm{Whit}_{q,glob}(G)$ is stable under direct summands, so $C=\mathrm{Whit}_{q,glob}(G)^c$.

Let us check that $\lambda\in\Lambda^+$, $W^{\lambda,!}_{glob}\in Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})^{loc.c}$. It is reduced to showing that its restriction to $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda}$ lies in $(Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda})^c$. However, $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda}$ is isomorphic to $\mathbb{A}^m/\mathbb{A}^r$ for some $m,r\ge 0$, where we view $\mathbb{A}^r$ as a group scheme. Since it is smooth, for the projection $p:\mathbb{A}^m/\mathbb{A}^r\to\mathrm{Spec}\,k$ the functor p_* is continuous?

We claim that for $Y\in\mathrm{PreStk}_{lft}$ equipped with a trivial action of a unipotent group scheme U for the projection $q:Y\to Y/U$ the functor $q^!:Shv(Y/U)\to Shv(Y)$ is an equivalence. Indeed, it is fully faithful as invariants under $Shv(U)$ with U unipotent group scheme. The composition $Y\xrightarrow{q}Y/U\xrightarrow{\mathrm{pr}}Y$ is id, where pr is the projection, so $q^!\,\mathrm{pr}^!\,\widetilde{\to}\,\mathrm{id}$. Thus, $q^!$ is essentially surjective.

Since for $\bar p:\mathbb{A}^m\to\mathrm{Spec}\,k$ the functor $p_*:Shv(\mathbb{A}^m)\to\mathrm{Vect}$ is continuous, p_* is continuous. Thus, we proved the inclusion (16).

Now the equivalence (7.15) should be the claim that under the Verdier duality equivalence

$$(Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})^{loc.c})^{op}\,\widetilde{\to}\,Shv_{(\mathcal{G}^G)^{-1}}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})^{loc.c}$$

the category $(\mathrm{Whit}_{q,glob}(G)^c)^{op}$ is identified with $\mathrm{Whit}_{q^{-1},glob}(G)^c$.

1.2.56. For 7.4.1. The proof of ([24], 5.4.1(a)) uses a remark: for a finite subset $\underline{y}\subset X$, $X-\underline{y}$ is affine. (If X is smooth proper then $X-x$ is affine. Indeed, the line bundle $\mathcal{O}_X(nx)$ for n large enough defines an inclusion $X\hookrightarrow\mathbb{P}^N$ for some N, and the section $1\in\mathcal{O}_X(nx)$ vanishes only at x, its complement is an affine open embedding).

As in ([24], 5.2.4), we have the following. For $\mu\in\Lambda$ the map $\pi_\mu:S^\mu\to(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x}$ is a torsor under N_{X-x}. The action maps

$$\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)\times S^\mu\to(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x} \tag{17}$$

and $\mathfrak{L}_y(N)\times S^\mu\to(\overline{\mathrm{Bun}}_N^{\omega^\rho})^{N-level_{\infty y}}_{=\mu x}$ are torsors under $N_{X-\{x,y\}}$. Here the condition $=\mu x$ includes the property "good elsewhere" by the definitions from [32]. It is understood here that $N_{X-\{x,y\}}$ acts diagonally on $\mathfrak{L}_y(N)/\mathfrak{L}^+_y(N)\times S^\mu$.

For example, $\mathrm{Bun}_N \widetilde{\to} N_{X-\{x,y\}}\backslash(\mathrm{Gr}_{N,y}\times \mathrm{Gr}_{N,y})$ with respect to the diagonal action. This corresponds to a trivialization of a given N-torsor over $X-\{x,y\}$.

The group $N_{X-\{x,y\}}$ acts here on $\mathfrak{L}_y(N)\times S^\mu$ diagonally, where on the factor $\mathfrak{L}_y(N)$ it acts by left multiplication. Recall that $(\overline{\mathrm{Bun}}_N^{\omega^\rho})^{N-level_{\infty y}}_{=\mu x}$ is equipped with a left action of $\mathfrak{L}_y(N)$ by regluing. We let $v\in\mathfrak{L}_y(N)$ act on $(z, gG(\mathcal{O}_x))\in \mathfrak{L}_y(N)\times S^\mu$ as $(zv^{-1}, gG(\mathcal{O}_x))$. Then the map

$$\pi^{level}_{y,\mu}: \mathfrak{L}_y(N)\times S^\mu\to (\overline{\mathrm{Bun}}_N^{\omega^\rho})^{N-level_{\infty y}}_{=\mu x}$$

is $\mathfrak{L}_y(N)$-equivariant. Taking the quotient under $\mathfrak{L}^+_y(N)$ (acting by right translations on the $\mathfrak{L}_y(N)$-factor and trivially on S^μ) in the map $\pi^{level}_{y,\mu}$ one gets the map (17).

This is why in ([24], (a') and (b)) the character $-\chi_y$ appears!! (Because when we talk about action by right translations, we still mean a left action!)

Consider the perverse irreducible sheaf W^λ_{glob} on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x}$, which is a generator of $\mathrm{Whit}_{q,glob}(G)_{=\mu}$. That is, we have a map $ev_\mu: (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x}\to\mathbb{A}^1$ and $W^\lambda_{glob} = ev_\mu^*\mathcal{L}_\psi[\dim]$, where

$$\dim = \dim(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x} = (g-1)(d-\langle 2\check{\rho}, 2\rho\rangle)+\langle 2\check{\rho},\mu\rangle$$

with $d=\dim\mathfrak{n}$. For the map $\pi_\mu: S^\mu\to(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x}$ we verify that $\pi_\mu^! W^\lambda_{glob}[d_g]\widetilde{\to}\overset{\circ}{W}{}^\mu$, where $d_g=\dim \mathrm{Bun}_N^{\omega^\rho}$ is that of thm. 7.4.2.

In Thm. 7.4.2 the shift is correct. Here is a detailed explanation. Let $\mu\in\Lambda^+$. Consider the composition $\mathrm{Spec}\, k\overset{i_0}{\to} S^\mu\overset{\pi_\mu}{\to}(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x}\overset{ev_\mu}{\to}\mathbb{A}^1$. Recall that

$$\dim(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x} = d_g+\langle 2\check{\rho},\mu\rangle$$

with $d=\dim\mathfrak{n}$ and $d_g=(g-1)(d-\langle 2\check{\rho},2\rho\rangle)$. Let

$$W^\mu_{glob} = ev_\mu^*\mathcal{L}_\psi[d_g+\langle 2\check{\rho},\mu\rangle] = ev_\mu^!\mathcal{L}_{\psi^{-1}}[2-d_g-\langle 2\check{\rho},\mu\rangle]$$

so W^μ_{glob} is perverse.

We know from the proof of 6.2.9 in the paper that the composition $\mathrm{Whit}_{q,x}(G)_{=\mu}\to Shv_{\mathcal{G}^G}(S^\mu)\overset{i_0^!}{\to}\mathrm{Vect}$ is an equivalence and has a left adjoint sending e to

$$\mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}})$$

By definition,

$$\overset{\circ}{W}{}^\mu = \mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\delta_{t^\mu,\mathrm{Gr}})[-\langle\mu,2\check{\rho}\rangle]$$

So, $i_0^!(\overset{\circ}{W}{}^\mu)\widetilde{\to} e[-\langle\mu,2\check{\rho}\rangle]$. Now to verify the isomorphism

$$\pi_\mu^! W^\mu_{glob}[d_g]\widetilde{\to}\overset{\circ}{W}{}^\mu, \tag{18}$$

it suffices to apply $i_0^!$ to both sides. The result follows now from the fact that for $i:\mathrm{Spec}\, k\overset{0}{\to}\mathbb{A}^1$ one has $i^!\mathcal{L}_\psi\widetilde{\to} e[-2]$.

1.2.57. For 7.4.1. The following observation from ([24], 2.3.5) is used essentially in the proof. For $\mu \in \Lambda$, let N^μ be the stabilizer of $t^\mu \in \mathrm{Gr}_G$ in $\mathfrak{L}(N)$. There is an ind-group scheme N' with a closed immersion $N' \subset \mathfrak{L}(N)$ such that $N'N^\mu = \mathfrak{L}(N)$ and $N' \cap N^\mu = \{1\}$.

1.2.58. For 7.4.3. The fact that $\pi_x^! W^{\mu,*}_{glob}[d_g] \widetilde{\to} W^{\mu,*}_{glob}$ follows by base change from (18).

The map (7.16) is defined as follows. If $f : X \to X'$ is obtained from $g : Y \to Y'$ by the base change via $f' : X' \to Y'$ let $f : X \to Y$ be the corresponding map. We have a canonical map $g'_! f^! \to (f')^! g_!$ and apply it to $W^\mu_{glob}[d_g]$.

Assume $\pi_x^!$ is fully faithful. Then (7.16) indeed induces an isomorphism on all $\mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(\cdot, W^{\lambda,*})$. Indeed, the functor $\pi^!_\mu : \mathrm{Whit}_{q,glob}(G)_{=\mu} \to \mathrm{Whit}_{q,x}(G)_{=\mu}$ is then also fully faithful, namely obtained by restricting $\pi_x^!$ to a full subcategory via $(i_\mu)_! : \mathrm{Whit}_{q,glob}(G)_{=\mu} \to \mathrm{Whit}_{q,glob}(G)$. We have

$$\mathcal{H}om_{\mathrm{Whit}_{q,glob}(G)}(W^{\mu,!}_{glob}[d_g], W^{\lambda,*}_{glob}[d_g]) \widetilde{\to} \mathcal{H}om_{\mathrm{Whit}_{q,glob}(G)_{=\mu}}(W^\mu_{glob}[d_g], (i^\mu)^! W^{\lambda,*}_{glob}[d_g]) \overset{\pi_x^!}{\to}$$
$$\mathcal{H}om_{\mathrm{Whit}_{q,x}(G)_{=\mu}}(\overset{\circ}{W}{}^\mu, (i_\mu)^! W^{\lambda,*}) \widetilde{\to} \mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(W^{\mu,!}, W^{\lambda,*})$$

which is an isomorphism in Vect, because the arrow with $\pi_x^!$ over it is an isomorphism.

We see also directly that

$$\pi_x^![d_g] : \mathrm{Whit}_{q,glob}(G)^{\le 0} \widetilde{\to} \mathrm{Whit}_{q,x}(G)^{\le 0}$$

Moreover $K \in \mathrm{Whit}_{q,glob}(G)$ lies in $\mathrm{Whit}_{q,glob}(G)^{\ge 0}$ iff $\pi_x^! K[d_g] K$ lies in $\mathrm{Whit}\, q, x(G)^{\ge 0}$. So, $\pi_x^![d_g]$ is t-exact.

1.2.59. The !-pullback functors for maps $Z' \to Z$ are missing, where $Z \in \mathrm{PreStk}_{lft}$, and Z' is a placid ind-scheme.

Important phenomenon for $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}$. If $\mu \in \Lambda$ and $K \in Shv((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le \mu x}$ is the extension by zero from a quasi-compact open substack of $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le \mu x}$ then

$$Shv((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}) \to \mathrm{Vect},\ L \mapsto \mathrm{R}\Gamma((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}, (\bar{i}_\mu)_* K \otimes^! L)$$

is continuous! This is essentially because "there are $\mathbb{G}_m$ factors in the stabilizors of points of this stack".

For this reason, an object of $\mathrm{Whit}_{q,glob}(G)$ is compact iff it is an object of the smallest full stable subcategory containing $W^{\lambda,!}_{glob}$ for all $\lambda \in \Lambda^+$.

Let $K \in \mathrm{Whit}_{q,glob}(G)$. Then the functor

$$Shv_{\mathcal{G}^{-1}}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}) \to \mathrm{Vect},\ L \mapsto \mathrm{R}\Gamma((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}, K \otimes^! L)$$

is continuous. Indeed, $K \widetilde{\to} \underset{\mu \in \Lambda}{\mathrm{colim}} (\bar{i}_\mu)_* (\bar{i}_\mu)^! L$, and for any $\mathcal{F} \in Shv((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$,

$$\mathrm{R}\Gamma((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}, \mathcal{F}) \widetilde{\to} \mathrm{colim}_\mu \mathrm{R}\Gamma((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\le \mu}, (\bar{i}_\mu)^! \mathcal{F})$$

Comment to the proof of ([24], 4.8.3). The formula (4.7) there is proved for $\mathcal{F} \in \mathrm{Whit}((\overline{\mathrm{Bun}}_N)^{G-level_{nx}}_{\infty x})^c$ only. It holds for non-compact objects also. Indeed, let us show that for any $\mathcal{F} \in Shv_{\mathcal{G}^{-1}}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ the functor

$$\mathrm{Whit}_{q,glob}(G) \to \mathrm{Vect},\ K \mapsto \mathrm{R}\Gamma(((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}, K \otimes^! \mathcal{F})$$

is continuous. Write $\mathcal{F} \widetilde{\to} \operatorname{colim}_{i\in I} \mathcal{F}_i$ with I is small filtered, $\mathcal{F}_i \in Shv_{\mathcal{G}^{-1}}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})^c$. Then

$$\mathrm{R}\Gamma(((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}, K \otimes^! \mathcal{F}) \widetilde{\to} \operatorname*{colim}_{i\in I} \mathrm{R}\Gamma(((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}, K \otimes^! \mathcal{F}_i)$$

by the above. Our stacks appearing are duality adapted, to $\mathbb{D}(\mathcal{F}_i) \in Shv_{\mathcal{G}^{-1}}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})^c$ for each i. So, for each i the functor

$$K \mapsto \mathrm{R}\Gamma(((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}, K \otimes^! \mathcal{F}_i)$$

is continuous, hence their colimit is also continuous.

Conclusion: for any $K \in \mathrm{Whit}_{q,glob}(G), \mathcal{F} \in Shv_{\mathcal{G}^{-1}}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ one has

$$\mathrm{R}\Gamma((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}), K \otimes^! \mathcal{F}) \widetilde{\to} \mathrm{R}\Gamma((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}), K \otimes^! \mathrm{Av}_*^{N_{glob},\chi_N} \mathcal{F})$$

1.2.60. For 7.4.4. For completeness, N_{X-x} is a group ind-scheme, for $S \in \mathrm{Sch}_{ft}^{aff}$, its S-points is the set of maps $S \times (X - x) \to N$. This shows that π_x is ind-schematic. The group N_{X-x} can be written as a colimit of unipotent group schemes. Namely, for a faithful representation $N \subset \mathrm{GL}(V)$ we may take those sections of N_{X-x} which gives regular maps $V \to V(mx)$ over the whole of X and vary m. So, $Shv(S^\mu)^{N_{X-x}} \subset Shv(S^\mu)$ is a full subcategory.

To calculate the functor dual to $\mathrm{Whit}_{q^{-1},glob}(G) \overset{\pi_x^!}{\to} \mathrm{Whit}_{q^{-1},x}(G) \hookrightarrow Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_{G,x}^{\omega^\rho})$, note that for $F_1 \in \mathrm{Whit}_{q^{-1},glob}(G)$, $F_2 \in Shv_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho})$ one has

$$\mathrm{R}\Gamma(\pi_x^! F_1 \otimes^! F_2) \widetilde{\to} \mathrm{R}(F_1 \otimes^! (\pi_x)_* F_2) \widetilde{\to} \mathcal{H}om_{Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})}(\mathbb{D}(F_1), (\pi_x)_* F_2) \widetilde{\to}$$
$$\mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(\mathbb{D}(F_1), \mathrm{Av}_*^{N_{glob},\chi_N}) \widetilde{\to} \mathrm{R}\Gamma(F_1 \otimes^! \mathrm{Av}_*^{N_{glob},\chi_N})$$

Thus, the above dual is the composition

$$Shv_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho}) \overset{(\pi_x)_*}{\to} Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}) \overset{\mathrm{Av}_*^{N_{glob},\chi_N}}{\to} \mathrm{Whit}_{q,x}(G)$$

Now the dual to the inclusion $\mathrm{Whit}_{q^{-1},x}(G) \hookrightarrow Shv_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_{G,x}^{\omega^\rho})$ is the projection $Shv_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho}) \to \mathrm{Whit}_{q,x}(G)_{co}$.

([24], 5.4.2(a)) in our setting reduces to the following claim: let $N' \subset \mathfrak{L}(N)_x^{\omega^\rho}$ be a group subscheme large enough such that $N' N_{X-x} = \mathfrak{L}(N)_x^{\omega^\rho}$. Then for any $\mu \in \Lambda$ the natural map $\mathrm{Av}_*^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N} \to \mathrm{Av}_*^{N',\chi_N}$ of functors $Shv(S^\mu)^{N_{X-x}} \to \mathrm{Whit}_{q,x}(G)_{=\mu}$ is an isomorphism.

This is a claim like this: there are unipotent group schemes of finite type $N'' \subset N'$, a map $\pi : S \to Y$, where Y is an affine space, S is an ind-scheme of ind-finite type, and

π is N'-equivariant. So, we have a cartesian square

$$\begin{array}{ccc} N''\backslash S & \stackrel{b}{\to} & N'\backslash S \\ \downarrow\pi & & \downarrow\pi \\ N''\backslash Y & \stackrel{b}{\to} & N'\backslash Y \end{array}$$

Then first $b_*\pi^! \widetilde{\to} n\pi^! b_*$. Combine this with the fact that $Shv(S)^{N''} \stackrel{\text{oblv}}{\to} Shv(S) \stackrel{\text{Av}_*^{N'}}{\to} Shv(S)^{N'}$ is b_*. This gives ([24], 5.4.2(a)) on each stratum.

Note that for the map $\pi_\mu : S^\mu \to (\overline{\text{Bun}}_N^{\omega^\rho})_{=\mu x}$ the functor $(\pi_\mu)^!(\pi_\mu)_* : Shv(S^\mu) \to Shv(S^\mu)$ is the functor of action by $\omega_{N_{X-x}}$. It takes values in the full subcategory $Shv(S^\mu)^{N_{X-x}} \subset Shv(S^\mu)$.

Before commenting on ([24], 5.4.2(b)), which is badly explained, we claim that the composition

$$\text{Whit}_{q,x}(G) \hookrightarrow Shv_{\mathcal{G}^G}(\text{Gr}_{G,x}^{\omega^\rho}) \to Whit_{q,x}(G)_{co}$$

vanishes. We will see this aposteriori from this section and ([24], 5.4.5). And the same holds for each orbit: the composition

$$\text{Whit}_{q,x}(G)_{=\lambda} \hookrightarrow Shv_{\mathcal{G}^G}(S^\lambda) \to Shv_{\mathcal{G}^G}(S^\lambda)_{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N}$$

vanishes (this is probably also explained somewhere in Sam).

It is not true that $\text{Av}_{*,glob}^{\text{Whit}} \circ \pi_*$ identifies with $\pi_* \text{Av}_*^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N}$. In fact, if we apply π_* to $W^{\lambda,!}$, we will get zero! Indeed, by ([24], 5.2.4(a)) it suffices to show that $\pi^!\pi_* W^{\lambda,!} = 0$. The latter identifies with

$$\omega_{N_{X-x}} * W^{\lambda,!},$$

where we mean the action of $Shv(\mathfrak{L}(N)_x^{\omega^\rho})$ on $Shv(S^\lambda)$ here. The result vanishes, because the isomorphism

$$N_{X-x} \times S^\lambda \widetilde{\to} N_{X-x} \times S^\lambda,\ (z,y) \mapsto (z, zy)$$

identifies $\omega_{N_{X-x}} * W^{\lambda,!}$ with $\text{R}\Gamma(N_{X-x},\omega) \otimes W^{\lambda,!}$, and $\text{R}\Gamma(N_{X-x},\omega) = 0$, because N_{X-x} is a colimit of unipotent group schemes.

The composition

$$Shv_{\mathcal{G}^G}(\text{Gr}_{G,x}^{\omega^\rho}) \stackrel{\text{Av}_*^{N',\chi_N}}{\to} Shv_{\mathcal{G}^G}(\text{Gr}_{G,x}^{\omega^\rho}) \stackrel{(\pi_x)_*}{\to} Shv_{\mathcal{G}^G}((\overline{\text{Bun}}_N^{\omega^\rho})_{\infty x})$$

takes values in $\text{Whit}_{q,glob}(G)$. It suffices to check that after applying $i_\mu^!$ for any $\mu \in \Lambda$. To check the latter, it suffices to apply $\pi_\mu^!$ and show that the composition

$$Shv(S^\mu) \stackrel{\text{Av}_*^{N',\chi_N}}{\to} Shv(S^\mu) \stackrel{(\pi_\mu)_*}{\to} Shv_{\mathcal{G}^G}((\overline{\text{Bun}}_N^{\omega^\rho})_{=\mu}) \stackrel{\pi_\mu^!}{\to} Shv(S^\mu)^{N_{X-x}}$$

takes values in $Shv(S^\mu)^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N}$. We know that $\pi_\mu^! \circ (\pi_\mu)_*$ is the action of $\omega_{N_{X-x}}$. So, the composition is the action by $\chi_{N'} * \omega_{N_{X-x}} \in Shv(\mathfrak{L}(N)_x^{\omega^\rho})$. In ([24], 5.4.5) he claims in particular the following.

Lemma 1.2.61. *One has*

$$\omega_{N_{X-x}} * \chi_{N'} \widetilde{\to} \chi_{N'} * \omega_{N_{X-x}} \widetilde{\to} E'[2d] \in Shv(\mathfrak{L}(N)_x^{\omega^\rho})$$

with $d = \dim \mathrm{Bun}_N^{\omega^\rho}$. *Here* E' *is given by the formula (14) of this file.*

Proof. We establish the second isomorphism, the first being similar. Recall that $N_0 = \mathfrak{L}^+(N)_x^{\omega^\rho}$, and $\mathfrak{L}(N)_x^{\omega^\rho} \widetilde{\to} \mathrm{colim}_k N_k$. Write temporarily $i_k : N_k \hookrightarrow \mathfrak{L}(N)_x^{\omega^\rho}$ for the inclusion. Assume k is large enough so that $N' \subset N_k$. Then $i_k^!(\chi_{N'} * \omega_{N_{X-x}}) \widetilde{\to} \chi_{N'} * \omega_{N_{X-x} \cap N_k}$ by base change. Here $N_{X-x} \cap N_k$ is a group scheme of finite type. Since $N' N_{X-x} = \mathfrak{L}(N)_x^{\omega^\rho}$, we get $N'(N_{X-x} \cap N_k) = N_k$. Besides,

$$\omega_{N_{X-x} \cap N_k} \widetilde{\to} e_{\omega_{N_{X-x} \cap N_k}}[2 \dim(N_{X-x} \cap N_k)]$$

So, $\chi_{N'} * \omega_{N_{X-x} \cap N_k} \widetilde{\to} \chi_k[2 \dim(\omega_{N_{X-x} \cap N_k})]$. So, the claim is reduces to the equality

$$\dim(N_{X-x} \cap N_k) = d + \dim(N_k/N_0)$$

for such k. The natural map $N_k/N_k \cap N_{X-x} \to \mathfrak{L}(N)_x^{\omega^\rho}/N_{X-x}$ is an isomorphism by assumption, so $N_0 \backslash N_k / N_k \cap N_{X-x} \widetilde{\to} \mathrm{Bun}_N^{\omega^\rho}$ as stack quotients. This gives an equality $\dim(N_0 \backslash N_k) - \dim(N_k \cap N_{X-x}) = \dim \mathrm{Bun}_N^{\omega^\rho}$. □

This lemma shows finally that indeed the functor $\mathrm{Av}_*^{N', \chi_N} \circ \pi_*$ from ([24], 5.4.2(b)) takes values in $\mathrm{Whit}_{q,glob}(G)$.

It should be true that for an object of $Shv(S^\mu)^{N_{X-x}}$ the (N', χ_N)-equivariance implies already $(\mathfrak{L}(N)_x^{\omega^\rho}, \chi_N)$-equivariance.

The idea of the proof of ([24], Pp. 5.4.2(b)) is to check that $\circ \pi_* \circ \mathrm{Av}_*^{N', \chi_N}$ factors through $Shv_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho}) \to \mathrm{Whit}_{q,x}(G)_{co}$. To see this let $K \in Shv_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho})$ and $k \geq 0$. We check that the map $\mathrm{Av}_*^{N_k, \chi_N}(K) \to K$ becomes an isomorphism after applying $\pi_* \circ \mathrm{Av}_*^{N', \chi_N}$. Since we will know later that $\pi^!$ is fully faithful, we may replace the latter functor by $\pi^! \circ \pi_* \circ \mathrm{Av}_*^{N', \chi_N} = \omega_{N_{X-x}} * \mathrm{Av}_*^{N', \chi_N}$. By Lemma 1.2.61 above, $\omega_{N_{X-x}} * \chi_{N'} \widetilde{\to} E'[2d]$, and we know that the action of E' on $Shv_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho})$ factors through $Shv_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho}) \to \mathrm{Whit}_{q,x}(G)_{co}$.

1.2.62. A general observation about categories of invariants. Let $Y = \mathrm{colim}_{j \in J} Y_j$ be an ind-scheme of ind-finite type, here J is a filtered category, Y_j is a scheme of finite type, and for $j \to j'$ in J the map $Y_j \to Y_{j'}$ is a closed immersion.

Let $\alpha : H \to G$ be a homomorphism of group schemes, which are placid schemes. Assume I is a filtered category and $H \widetilde{\to} \lim_{i \in I^{op}} H_i$, $G \widetilde{\to} \lim_{i \in I^{op}} G_i$, where H_i, G_i is a smooth group scheme of finite type, for $i \to j$ in I the transition maps $H_j \to H_i$, $G_j \to G_i$ are smooth, affine, surjective homomorphisms. Besides, we are given a diagram $I^{op} \times [1] \to Grp(\mathrm{Sch})$, sending i to $\alpha_i : H_i \to G_i$, where α_i is a closed subgroup. We assume $\alpha = \lim_{i \in I^{op}} \alpha_i$.

Assume G acts on Y. Moreover, for any $j \in J$, Y_j is G-stable, and G acts on Y_j through the quotient $G \to G_i$ for some $i \in I$. We claim that (for any of our 4 sheaf theories) $oblv : Shv(Y)^G \to Shv(Y)^H$ admits a continuous right adjoint Av_*.

Proof. We have $Shv(Y)^G \widetilde{\to} \lim_{j \in J^{op}} Shv(Y_j)^G$ with respect to the !-pullbacks, similarly $Shv(Y)^H \widetilde{\to} \lim_{j \in J^{op}} Shv(Y_j)^H$, and $oblv : Shv(Y)^G \to Shv(Y)^H$ is the limit over

$j \in J^{op}$ of $\mathrm{oblv}_j : Shv(Y_j)^G \to Shv(Y_j)^H$. For given $j \in J$ the functor oblv_j admits a continuous right adjoint $\mathrm{Av}_{j,*}$. Indeed, pick $i \in I$ such that G-action on Y_j factors through G_i. Then oblv_j identifies with the functor $f^! : Shv(Y_j/G_i) \to Shv(Y_j/H_i)$ for the projection $f : Y_j/H_i \to Y_j/G_i$. Since G_i/H_i is smooth, f is smooth. So, $f^!$ admits a continuous right adjoint (as f is schematic of finite type).

Let now $j \to j'$ be a map in J. Pick i such that the G-action on $Y_j, Y_{j'}$ factors through G_i. Then we get a cartesian square

$$\begin{array}{ccc} Y_j/H_i & \stackrel{h}{\to} & Y_{j'}/H_i \\ \downarrow f_j & & \downarrow f_{j'} \\ Y_j/G_i & \stackrel{h'}{\to} & Y_{j'}/G_i, \end{array}$$

where h, h' are closed immersions. We have $(h')^! f_{j',*} \widetilde{\to} f_{j,*} h^!$. Since $f_j, f_{j'}$ are of the same relative dimension, we see that the diagram commutes

$$\begin{array}{ccc} Shv(Y_j)^H & \stackrel{h^!}{\leftarrow} & Shv(Y_{j'})^H \\ \downarrow \mathrm{Av}_{j,*} & & \downarrow \mathrm{Av}_{j',*} \\ Shv(Y_j)^G & \stackrel{(h')^!}{\leftarrow} & Shv(Y_{j'})^G \end{array}$$

By ([20], ch. I.1, 2.6.4), oblv admits a right adjoint Av_*, and for the evaliation maps $ev_j : Shv(Y)^G \to Shv(Y_j)^G$, $ev_j : Shv(Y)^H \to Shv(Y_j)^H$ one gets $ev_j \mathrm{Av}_* \widetilde{\to} \mathrm{Av}_{j,*} ev_j$. So, Av_* is continuous. □

An example of an application: if $H_1 \subset G_1$ is a closed subgroup, G_1 is a smooth affine group scheme of finite type then take $\alpha : H_1(\mathcal{O}) \to G_1(\mathcal{O})$ for $\mathcal{O} = k[[t]]$.

Lemma 1.2.63. *Let G be a smooth group scheme of finite type over $\mathrm{Spec}\, k$, $U \subset G$ be a normal unipotent group subscheme. Then for $F \in Shv(G)$ one has $F * e_U \widetilde{\to} e_U * F * e_U$. So, if $C \in Shv(G) - mod$ then $Shv(G)$-action on C preserves the full subcategory $C^U \subset C$, so we get a $Shv(G)$-action on C^U.*

Proof. We have a cartesian square

$$\begin{array}{ccc} G & \stackrel{\mathrm{pr}}{\leftarrow} & G \times U \\ \downarrow q & & \downarrow m \\ G/U & \stackrel{q}{\leftarrow} & G, \end{array}$$

where q is the quotient map, and m is the product. Thus, $F * e_U \widetilde{\to} q^* q_* F$. Similarly, $e_U * F \widetilde{\to} q^* q_* F$. The claim follows from $q_* q^* \mathcal{F} \widetilde{\to} \mathcal{F}$ for $\mathcal{F} \in Shv(G/U)$. The category $C^U \subset C$ is the essential image of the functor $C \to C$, $c \mapsto e_U * c$. The second claim follows. □

Lemma 1.2.64. *Let H be a pro-smooth placid group scheme, U be a priounipotent group scheme with an action of H by automorphisms of U such that the semi-direct product $G = U \rtimes H$ is a pro-smooth placid group scheme. Then for $F \in Shv(G)$ we have $F * e_U \widetilde{\to} e_U * F * e_U$. Let $C \in Shv(G) - mod(\mathrm{DGCat}_{cont})$. Then C^U is stable under the $Shv(G)$-action of C, so inherits such an action (where U acts trivially).*

Similarly, $e_U * F \widetilde{\to} e_U * F * e_U$.

Proof. It suffices to show the desired isomorphism for $F \in Shv(G)^c$. We have the cartesian square

$$\begin{array}{ccc} G & \stackrel{\mathrm{pr}}{\leftarrow} & G\times U \\ \downarrow q & & \downarrow m \\ H & \stackrel{q}{\leftarrow} & G, \end{array}$$

where q is the quotient map, and m is the product. By ([46], Lm. 0.0.19), we have the base change $q^*q_*(\mathcal{F}) \widetilde{\to} m_* \operatorname{pr}^*(\mathcal{F})$ for any $\mathcal{F} \in Shv(G)^c$. Indeed, we may assume given an exact sequence $1 \to H' \to H \to \bar{H} \to 1$, where H$'$ is prounipotent, and $\bar{H}$ is a smooth group scheme of finite type with $\mathcal{F} \in Shv(G/H')$. Then we actually deal with the diagram

$$\begin{array}{ccc} G & \stackrel{\mathrm{pr}}{\leftarrow} & G\times U \\ \downarrow q & & \downarrow m \\ H & \stackrel{q}{\leftarrow} & G \\ \downarrow & & \downarrow \\ \bar{H} & \leftarrow & G/H' \end{array}$$

(Here H' is not necessarily normal in G). So, we may repeat the argument of the previous lemma. □

In the situation of the above lemma for $C \in Shv(G)-mod(\mathrm{DGCat}_{cont})$ the category C^U inherits an action of $Shv(H)$.

Lemma 1.2.65. *In the situation of Lemma 1.2.64 given* $C \in Shv(G)-mod(\mathrm{DGCat}_{cont})$*, the functor* $(C^U)^H \to C^H$ *(obtained from* $\mathrm{oblv}: C^U \to C$ *by functoriality of invariants) is fully faithful.*

Proof. We have a morphism of cosimplicial diagrams

$$\mathrm{Fun}(Shv(H)^{\otimes n}, C^U) \to \mathrm{Fun}(Shv(H)^{\otimes n}, C)$$

for $[n] \in \boldsymbol{\Delta}$ whose limit is the desired functor. Each functor in the diagram is fully faithful, because $Shv(H)$ is dualizable, so that we may apply ([23], 1.5.1). So, passing to the limit we get a fully faithful embedding by ([43], Lemmas 2.2.16, 2.2.17), because $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}\mathrm{at}$ preserves limits. □

In the situation of the last lemma $(C^U)^H \widetilde{\to} C^G$?

Remark 1.2.66. *Let G be a placid group scheme, $1 \to U \to G \stackrel{q}{\to} H \to 1$ be a surjective group homomorphism, where H is a smooth group scheme of finite type, U is prounipotent. Then for $K_i \in Shv(H)$ one has $q^*K_1 * q^*K_2 \widetilde{\to} q^*(K_1 * K_2)$ naturally. However, q^* is not monoidal.*

1.2.67. Let $P \subset G$ be a parabolic in a connected reductive group with Levi M and unipotent radical U. Let $F = k((t)), \mathcal{O} = k[[t]]$. Let $H = M(\mathcal{O})U(F)$. This is a placid ind-scheme, closed in $P(F)$. We have also $P(F)/H \widetilde{\to} M(F)/M(\mathcal{O})$. Since the object $\delta_1 \in Shv(\mathrm{Gr}_M)$ is H-invariant, the functor $\mathrm{Vect} \to Shv(\mathrm{Gr}_M)$, $e \mapsto \delta_1$ is $Shv(H)$-linear. Now the $Shv(H)$-action on $Shv(\mathrm{Gr}_M)$ comes as the restriction of a $Shv(P(F))$-action, hence we get by adjointness a canonical functor

$$Shv(P(F)) \otimes_{Shv(H)} \mathrm{Vect} \to Shv(\mathrm{Gr}_M)$$

Let us show this is an equivalence.

Proof. Pick a presentation $U(F) = \operatorname{colim}_{n\in\mathbb{N}} U_n$, where U_n is a placid group scheme, for $n \le m$, $U_n \to U_m$ is a placid closed immersion. Assume $M(\mathcal{O})$ normalizes each U_n, so $M(\mathcal{O})U_n =: H_n$ is a placid group scheme, and $M(F) = \operatorname{colim}_{n\in\mathbb{N}} H_n$. We have $P(F) \widetilde{\to} \operatorname{colim}_{n\in\mathbb{N}} M(F)U_n$ in PreStk, as colimits in PreStk are universal. It should ne true that now $Shv(P(F)) \widetilde{\to} \operatorname{colim}_{n\in\mathbb{N}} Shv(M(F)U_n)$ with respect to the $*$-push-forwards. This gives

$$Shv(P(F)) \otimes_{Shv(H)} \mathrm{Vect} \to Shv(\mathrm{Gr}_M) \widetilde{\to} \operatorname*{colim}_{(n\le m)\in \mathrm{Fun}([1],\mathbb{N})} Shv(M(F)U_m) \otimes_{Shv(H_n)} \mathrm{Vect}$$

The diagonal map $\mathbb{N} \to \mathrm{Fun}([1], \mathbb{N})$ is cofinal, so the above identifies with

$$\operatorname{colim}_{n\in\mathbb{N}} Shv(M(F)U_n) \otimes_{Shv(H_n)} \mathrm{Vect}$$

Now each term of the latter diagram identifies with $Shv(M(F)/M(\mathcal{O}))$ using ([46], 0.0.36), and we are done. Indeed, for any $I \in 1-\mathcal{C}\mathrm{at}$ the natural map $I \to\mid I \mid$ is cofinal, and for I filtered we get $\mid I \mid \widetilde{\to} *$. □

Further, let $C \in Shv(P(F)) - mod(\mathrm{DGCat}_{cont})$. We get

$$C^H = \mathrm{Fun}_{Shv(H)}(\mathrm{Vect}, C) \widetilde{\to} \mathrm{Fun}_{Shv(P(F))}(Shv(P(F)) \otimes_{Shv(H)} \mathrm{Vect}, C)$$

Thus, $\mathrm{Fun}_{Shv(P(F))}(Shv(\mathrm{Gr}_M), Shv(\mathrm{Gr}_M))$ acts on C^H. Now

$$\mathrm{Fun}_{Shv(P(F))}(Shv(\mathrm{Gr}_M), Shv(\mathrm{Gr}_M)) \widetilde{\to} Shv(\mathrm{Gr}_M)^H \widetilde{\to} Shv(\mathrm{Gr}_M)^M$$

1.3. **For Section 8.**

1.3.1. For 8.1.4. If $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a map in PreStk_{lft} which is universally homologically contractible, $\mathcal{G}$ is a gerbe on $\mathcal{X}_2$ then $f^! : Shv_{\mathcal{G}}(\mathcal{X}_2) \to Shv_{\mathcal{G}}(\mathcal{X}_1)$ is fully faithful.

Indeed, pick an etale schematic cover $Y_2 \to \mathcal{X}_2$ such that $\mathcal{G}$ trivializes over Y_2, let $\bar{f} : Y_1 \to Y_2$ be obtained by base change. Then $Shv_{\mathcal{G}}(\mathcal{X}_2) \widetilde{\to} \operatorname{Tot} Shv_{\mathcal{G}}((Y_2)^{\bullet}_{\mathcal{X}_2})$ and similarly for $\mathcal{X}_1$. Since for each $n \ge 0$, $Shv_{\mathcal{G}}((Y_2)^n_{\mathcal{X}_2}) \to Shv_{\mathcal{G}}((Y_1)^n_{\mathcal{X}_1})$ is fully faithful, passing to the limit we get the desired claim.

In fact, the map (8.2) is universally homologically contractible.

1.3.2. For 8.2.2. More details on the definition of $\mathrm{Whit}_{q,\mathrm{Ran}_x}(G)$. First, recall that $\mathrm{Ran}_x = \operatorname{colim}_I (X^I \times_X \{x\})$, where the colimit is taken over $(fSets_*)^{op}$, here $fSets_*$ is the category of pointed finite sets, and surjections preserving the distinguished point ([28], 2.5.2). It is understood that the projection $X^I \to X$ is given by the distinguished point. Now for $I \in fSets_*$ we let

$$\mathfrak{L}(N)^{\omega^\rho}_I = \mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}_x} \times_{\mathrm{Ran}_x} (X^I \times_X \{x\})$$

We have a canonical character $ev : \mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}_x} \to \mathbb{A}^1$. It gives the functor $Shv(\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}_x}) \to \mathrm{Vect}$, $K \mapsto \mathcal{H}om_{Shv(\mathbb{A}^1)}(ev_* K, \mathcal{L}_\psi)$. By definition, χ_N is this functor, which is an object of the dual category.

Now $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x} = \operatorname*{colim}_{I\in(fSets_*)} \mathrm{Gr}^{\omega^\rho}_{G,I}$ with

$$\mathrm{Gr}^{\omega^\rho}_{G,I} = \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x} \times_{\mathrm{Ran}_x} (X^I \times_X \{x\})$$

and

$$\mathrm{Whit}_{q,\mathrm{Ran}_x}(G)\widetilde{\to}\lim_{I\in(fSets_*)^{op}} Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})^{\mathfrak{L}(N)^{\omega^\rho}_I,\chi_N}$$

Here

$$Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})^{\mathfrak{L}(N)^{\omega^\rho}_I,\chi_N}\subset Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})$$

is a full subcategory (compare with ([47], 1.2.1) for incorporating the character). We used here the formalism from Sections 1.3.3-1.3.13.

Namely, $\mathfrak{L}(N)^{\omega^\rho}_I\in Grp(\mathrm{PreStk}_{/X^I})$ is a placid ind-scheme over X^I written as

$$\mathrm{colim}_{\alpha\in A}\, N^\alpha_I,$$

where N^α_I is a placid group scheme over X^I, for $\alpha\to\alpha'$ in A the map $N^\alpha_I\to N^{\alpha'}_I$ is a placid closed immersion and a homomorphism of group schemes over X^I, and A is filtered (we may take $A=\mathbb{N}$). Then for each α we have the full subcategory

$$Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})^{N^\alpha_I,\chi_N}\subset Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})$$

and

$$Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})^{\mathfrak{L}(N)^{\omega^\rho}_I,\chi_N}=\bigcap_{\alpha\in A} Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})^{N^\alpha_I,\chi_N}$$

1.3.3. In the rest of Section 1.3 we develop the theory of group schemes over a base S acting on categories (for any of our 4 sheaf theories) extending some results of ([33], Appendix B) established for $\mathcal{D}$-modules.

Let S be a scheme of finite type, $f: G\to S$ be a group scheme of finite type and smooth over S. Then $Shv(G)$ is a monoidal category with the convolution given by

$$Shv(G)\otimes Shv(G)\stackrel{\boxtimes}{\to} Shv(G\times G)\stackrel{q^!}{\to} Shv(G\times_S G)\stackrel{m_*}{\to} Shv(G),$$

for the diagram $G\times G\stackrel{q}{\leftarrow} G\times_S G\stackrel{m}{\to} G$. Here m is the product. Let $i: S\to G$ be the unit section. Recall that $(Shv(S),\otimes^!)$ is monoidal. Then $i_*: (Shv(S),\otimes^!)\to(Shv(G),*)$ and $f_*: (Shv(G),*)\to(Shv(S),\otimes^!)$ are monoidal functors.

Let $\mathrm{PreStk}_{ind-sch,/S}\subset(\mathrm{PreStk}_{lft})_{/S}$ be the 1-full subcategory where we restrict the morphism to be ind-schematic of ind-finite type. Consider the functor $\mathrm{PreStk}_{ind-sch,/S}\to \mathrm{DGCat}_{cont}$ sending $Y\to S$ to $Shv(Y)$, and a map $f: Y\to Y'$ over S to $f_*: Shv(Y)\to Shv(Y')$. Then this functor is right-lax symmetric monoidal. For $Y_i\in\mathrm{PreStk}_{ind-sch,/S}$ the corresponding morphism $Shv(Y_1)\otimes Shv(Y_2)\to Shv(Y_1\times_S Y_2)$ is the composition $q^!\circ\boxtimes$, where $q: Y_1\times_S Y_2\to Y_1\times Y_2$ is the natural map. So, the above functor sends algebras to algebras. We could also instead consider the above functor with values in $Shv(S)-mod(\mathrm{DGCat}_{cont})$, then it is still right-lax symmetric monoidal. So, $Shv(G)\in Alg(Shv(S)-mod)$.

For $K\in Shv(G), F\in Shv(S)$ one has canonically $K*i_*F\widetilde{\to} K\otimes^! f^!F\widetilde{\to} i_*F*K$. So, the two possible structures of a $Shv(S)$-module on $Shv(G)$ coincide, and $Shv(S)$ is central in $(Shv(G),*)$.

My understanding is that the following actually is true. Let $\mathrm{Corr}(\mathrm{PreStk}_{lft}/S)_{ind-sch,all}$ be the category of correspondences, whose objects are $(\mathcal{Y}\to S)\in\mathrm{PreStk}_{lft}/S$, and a morphism from $\mathcal{Y}_1$ to $\mathcal{Y}_2$ is a diagram $\mathcal{Y}_1 \stackrel{g}{\leftarrow} \mathcal{Y}_{12} \stackrel{f}{\to} \mathcal{Y}_2$ in PreStk_{lft}/S with g any and f ind-schematic of ind-finite type. Then in the constructible context we get the functor

$$Shv_{\mathrm{Corr}} : \mathrm{Corr}(\mathrm{PreStk}_{lft}/S)_{ind-sch,all} \to \mathrm{DGCat}_{cont}$$

sending $\mathcal{Y}$ to $Shv(Y)$, and sending the above map to $f_*g^! : Shv(\mathcal{Y}_1) \to Shv(\mathcal{Y}_2)$. Then this functor has a natural right-lax symmetric monoidal structure. In the case $S = pt$ this is ([19], A.1.7).

If now $H \to S$ is another group scheme smooth of finite type over S and $\alpha : G \to H$ is a morphism of group schemes over S then $\alpha_* : (Shv(G), *) \to (Shv(H), *)$ is monoidal and moreover a morphism in $Alg(Shv(S) - mod)$. So, α_* is $Shv(S)$-linear.

Let now $Y \in \mathrm{PreStk}_{lft}$ with a map $Y \to S$. Assume G acts on Y over S. Then $Shv(G)$ acts on $Shv(Y)$ on the left as follows. Consider the diagram

$$G \times Y \stackrel{\bar{q}}{\leftarrow} G \times_S Y \stackrel{\mathrm{act}}{\to} Y,$$

where act is the action map. For $F \in Shv(G), K \in Shv(Y)$ let $F * K = \mathrm{act}_*(\bar{q}^!(F \boxtimes K))$. In fact, $(Shv(G), Shv(Y)) \in Alg + mod(Shv(S) - mod)$.

Restricting this action under $i_* : Shv(S) \to Shv(G)$, we get the action of $Shv(S)$ on $Shv(Y)$ such that $F \in Shv(S)$ sends $K \in Shv(Y)$ to $(\mathrm{pr}^! F) \otimes^! K$ for $\mathrm{pr} : Y \to S$. Now given $C \in (Shv(G), *) - mod$, we may consider

$$\mathrm{Fun}_{(Shv(G),*)}(Shv(S), C) \in Shv(S) - mod$$

The theory of group ind-schemes (over a base) acting on categories is developed in ([33], Appendix B), where it is explained that the latter is a good definition of invariants of $Shv(G)$ on C. By ([43], 9.2.36), the category of invariants is defined for (and depends only on) a non-unital $Shv(G)$-module category C.

If $h : Y \to Y'$ is a morphism in $(\mathrm{PreStk}_{lft})_{/S}$ assume G acts on Y and Y' over S, and h intertwines the G-actions. Then $h^! : Shv(Y') \to Shv(Y)$ commutes with $(Shv(G), *)$-actions. A way to see it should be to say that the map $Y' \to Y$ in $\mathrm{Corr}(\mathrm{PreStk}_{lft}/S)_{ind-sch,all}$ given by $h : Y \to Y'$ is a morphism of G-modules from Y' to Y in $\mathrm{Corr}(\mathrm{PreStk}_{lft}/S)_{ind-sch,all}$. If h is ind-schematic (of ind-finite type) then h_* commutes with the $(Shv(G), *)$-actions.

Write Y/G for the quotient of Y by G in the sense of prestacks over S, that is,

$$Y/G = \operatorname*{colim}_{[n] \in \boldsymbol{\Delta}^{op}} G \times_S G \times_S \ldots \times_S Y$$

Let $\tau : Y \to Y/G$ be the natural map, $\bar{p} : Y/G \to S$ be the projection. The functor $Shv(S) \otimes Shv(Y/G) \to Shv(Y)$, $(F, K) \mapsto \tau^!(K \otimes^! \bar{p}^! F)$ by the universal property gives a functor

$$Shv(Y/G) \to \mathrm{Fun}_{(Shv(G),*)}(Shv(S), Shv(Y)) \tag{19}$$

(by [43], 9.2.56).

Proposition 1.3.4. *The functor (19) is an equivalence.*

Proof. Let $G^n_S = G \times_S G \times_S \ldots \times_S G$ product over S taken n times. Recall that $Y/G = \operatorname{colim}_{[n]\in\boldsymbol{\Delta}^{op}} G^n_S \times_S Y$. Consider

$$Shv(Y/G) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} Shv(G^n_S \times_S Y),$$

limit in DGCat_{cont}. As in the case when S is $\operatorname{Spec} k$, this cosimplicial diagram satisfies the comonadic Beck-Chevalley conditions, so that the functor $ev^0 : Shv(Y/G) \to Shv(Y)$ is comonadic. Indeed, for any $n \geq 0$ assume the map attached to the last face map $\partial_n : [n] \to [n+1]$ avoiding $n+1$ is $\mathrm{id} \times \mathrm{act} : G^n_S \times_S G \times_S Y \to G^n_S \times_S Y$. The functor $(\mathrm{id} \times \mathrm{act})^!$ admits a right adjoint, because G is smooth over S of some relative dimension d. For any $\alpha : [m] \to [n]$ in $\boldsymbol{\Delta}$ then the diagram

$$\begin{array}{ccc} G^n_S \times_S Y & \stackrel{f_\alpha}{\to} & G^m_S \times_S Y \\ \uparrow \delta_n & & \uparrow \delta_m \\ G^{n+1}_S \times_S Y & \stackrel{f_{\alpha+1}}{\to} & G^{m+1}_S \times_S Y \end{array}$$

is cartesian, where we denoted by δ_n the map attached to ∂_n, and by f_α the map attached to α. So, $f^!_\alpha(\delta_m)_* \widetilde{\to} (\delta_n)_* f^!_{\alpha+1}$. The corresponding comonad is $(\mathrm{act}^!)^R \circ \mathrm{pr}^!$ for the maps $\mathrm{pr}, \mathrm{act} : G \times_S Y \to Y$. Here $(\mathrm{act}^!)^R \widetilde{\to} \mathrm{act}_*[-2d]$ is the right adjoint to $\mathrm{act}^!$, and $f : G \to S$ is smooth of dimension d. The comonad is $\mathrm{act}_* \mathrm{pr}^*$, it is given by the action of $f^*\omega_S \in Shv(G)$ for $f : G \to S$.

Recall that $Shv(G)$ has just one natural structure of a $Shv(S)$-module.

Write $Shv(G)^{\otimes n}_{Shv(S)}$ for the n-th tensor power of $Shv(G)$ over $Shv(S)$. Let us check that the cosimplicial category $[n] \mapsto \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes n}_{Shv(S)}, Shv(Y))$ satisfies the comonadic Beck-Chevalley conditions. By ([43], 9.2.36) its totalization identifies with the RHS of (19). For $\partial_n : [n] \to [n+1]$ the corresponding functor

$$T^{\partial_n} : \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes n}_{Shv(S)}, Shv(Y)) \to \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes n+1}_{Shv(S)}, Shv(Y))$$

sends h to the functor

$$F_1 \otimes \ldots \otimes F_{n+1} \mapsto h(F_1 \otimes \ldots \otimes F_{n-1} \otimes (F_n * f_* F_{n+1})),$$

where $F_n * f_* F_{n+1}$ is the convolution in the monoidal category $Shv(G)$. Note that $f_* : Shv(G) \to Shv(S)$ is a map of $Shv(G)$-modules, because it is monoidal. It has a left adjoint $f^* : Shv(S) \to Shv(G)$, which is a strict morphism of $Shv(G)$-modules (this is just base change).

Since $Shv(S) - mod(\mathrm{DGCat}_{cont})$ is a 2-category, we get an adjoint pair

$$\mathrm{id} \otimes f^* : Shv(G)^{\otimes n}_{Shv(S)} \rightleftarrows Shv(G)^{\otimes n+1}_{Shv(S)} : \mathrm{id} \otimes f_*$$

in $Shv(S) - mod$. Let $(T^{\partial_n})^R$ be the functor obtained from $\mathrm{id} \otimes f^*$ by applying

$$\mathrm{Fun}_{Shv(S)}(\cdot, Shv(Y))$$

We get the adjoint pair in DGCat_{cont}

$$T^{\partial_n} : \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes n}_{Shv(S)}, Shv(Y)) \rightleftarrows \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes n+1}_{Shv(S)}, Shv(Y)) : (T^{\partial_n})^R$$

Let now $\alpha : [m] \to [n]$ be a map in $\boldsymbol{\Delta}$. Consider the corresponding diagram

$$\begin{array}{ccc} \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes n}_{Shv(S)}, Shv(Y)) & \stackrel{(T^{\partial_n})^R}{\leftarrow} & \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes n+1}_{Shv(S)}, Shv(Y)) \\ \uparrow T^{\alpha} & & \uparrow T^{\alpha+1} \\ \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes m}_{Shv(S)}, Shv(Y)) & \stackrel{(T^{\partial_m})^R}{\leftarrow} & \mathrm{Fun}_{Shv(S)}(Shv(G)^{\otimes m+1}_{Shv(S)}, Shv(Y)) \end{array}$$

We check that it commutes. It suffices to prove this for α injective, because of the following. Let $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ be the full subcategory with the same class of object, where we keep only injective maps. Then $\boldsymbol{\Delta}_s^{op} \to \boldsymbol{\Delta}^{op}$ is cofinal by ([35], 6.5.3.7).

If $\alpha : [m] \to [n]$ is injective, and $0, n$ are in the image then the desired commutativity follows from the commutativity of

$$\begin{array}{ccc} Shv(G)^{\otimes n}_{Shv(S)} & \stackrel{\mathrm{id} \otimes f^*}{\to} & Shv(G)^{\otimes n+1}_{\otimes S} \\ \downarrow (m_\alpha)_* & & \downarrow (m_{\alpha+1})_* \\ Shv(G)^{\otimes m}_{Shv(S)} & \stackrel{\mathrm{id} \otimes f^*}{\to} & Shv(G)^{\otimes m+1}_{Shv(S)}, \end{array}$$

where $(m_\alpha)_*$ is the product along α in the monoidal category $Shv(G)$. We used the observation that the convolution in $Shv(G)$ factors through a map

$$Shv(G) \otimes_{Shv(S)} Shv(G) \to Shv(G),$$

which is a morphism of $Shv(S)$-modules.

If $\alpha : [n-1] \to [n]$ is the last face map then $\alpha + 1 : [n] \to [n+1]$ avoids n then $T^{\alpha+1}$ is the composition with

$$Shv(G)^{\otimes n+1}_{Shv(S)} \to Shv(G)^{\otimes n}_{Shv(S)},\ K_1 \otimes \ldots \otimes K_{n+1} \mapsto K_1 \otimes \ldots \otimes K_{n-1} \otimes K_n * K_{n+1}$$

In this case the desired commutativity follows from $K * (f^*\omega_S) \widetilde{\to} (i_* f_* K) * (f^*\omega_S)$. Indeed, one has $K * (f^*\omega_S) \widetilde{\to} m_* \mathrm{pr}_1^* K \widetilde{\to} f^* f_* K$ for the cartesian square

$$\begin{array}{ccc} G & \stackrel{\mathrm{pr}_1}{\leftarrow} & G \times_S G \\ \downarrow f & & \downarrow m \\ S & \stackrel{f}{\leftarrow} & G \end{array}$$

Now we get $f^* f_* K \widetilde{\to} i_*(f_* K) * (f^*\omega_S)$.

If $\alpha : [n-1] \to [n]$ is injective and avoids 0 then $T\alpha$ sends f to the functor

$$K_1 \otimes \ldots \otimes K_n \mapsto K_1 * f(K_2 \otimes \ldots \otimes K_n)$$

and the commutativity is tautological. So, it always hold. Thus,

$$ev_0 : \mathrm{Fun}_{(Shv(G),*)}(Shv(S), Shv(Y)) \to Shv(Y)$$

is comonadic, and the corresponding comonad is $(T^{\partial_0})^R T^{\alpha}$ for $\alpha : [0] \to [1], \alpha(0) = 1$. This is the functor $Shv(Y) \to Shv(Y), F \mapsto f^*\omega_S * F$. $\square$

1.3.5. Let now $G \to S$ be as in the previous section and χ be a character local system on G, so $m^*\chi \widetilde{\to} q^*(\chi \boxtimes \chi)$, and $i^*\chi \widetilde{\to} e_S$ (at least in the constructible context). For $\mathcal{D}$-modules see the remark below.

We get the auto-equivalence $Shv(G) \to Shv(G)$ sending F to $F \otimes \chi$. This is a monoidal functor. Indeed, $(i_*\omega_S) \otimes \chi \widetilde{\to} i_*\omega_S$, and for $F_i \in Shv(Y)$ we get

$$\chi \otimes (F_1 * F_2) \widetilde{\to} (\chi \otimes F_1) * (\chi \otimes F_2)$$

Now for $C \in Shv(G) - mod$ we define $C_{\chi^{-1}}$ as the object of $Shv(G) - mod$ equal to $C \in \mathrm{DGCat}_{cont}$ with the new action such that $F \in Shv(G)$ acts on $c \in C$ as $(F \otimes \chi) * c$. Here $*$ denotes the original action of $Shv(G)$. This definition agrees with ([47], 1.3.1). Finally we set

$$C^{G,\chi} = \mathrm{Fun}_{(Shv(G),*)}(Shv(S), C_{\chi^{-1}}) \in Shv(S) - mod$$

By Section 1.3.17 below, we may equivalently define it as

$$((f^*\omega_S) \otimes \chi) - comod(C)$$

Now for any sheaf theory, maybe e_S does not make sense. In this case by a character local system on G we mean an object $\mathcal{L} \in Shv(G)$ invertible for the !-monoidal structure on $Shv(G)$ and satisfying: $m^!\mathcal{L} \widetilde{\to} q^!(\mathcal{L} \boxtimes \mathcal{L})$ associatively, and $i^!\mathcal{L} \widetilde{\to} \omega_S$. Then the functor $Shv(G) \to Shv(G), K \mapsto K \otimes^! \mathcal{L}$ is a monoidal equivalence, which preserves the full subcategory $Shv(S)$ and induces the identity on $Shv(S)$ (cf. also Remark 1.2.7). If we actually in the constructible context and L is a character local system on G in the initial sense then $\mathcal{L} := L \otimes \omega_G$ is a character local system in this new sense.

1.3.6. Let $S \in \mathrm{Sch}_{ft}$, $p : G \to S$ be a group scheme smooth of finite type over S. In the constructible context the functor $p_* : Shv(G) \to Shv(S)$ admits a continuous right adjoint $p_*^R : Shv(S) \to Shv(G)$ equal to $(p_!)^\vee$. Since $p_* : (Shv(G), *) \to (Shv(S), \otimes^!)$ is monoidal, p_*^R is right-lax monoidal. In particular, it is a right-lax morphism of $Shv(G)$-module categories. That is, for $\mathcal{V} \in Shv(S), M \in Shv(G)$ we have a canonical map $M * p_*^R(\mathcal{V}) \to p_*^R(\mathcal{V} \otimes^! p_*M)$.

Lemma 1.3.7. *In general, this map is not an isomorphism, and p_*^R is not a strict morphism in $Shv(G) - mod$.*

Proof. The argument is due to Sam. Assume $S = \mathrm{Spec}\, k$.

0) For $F, M \in Shv(G)$ write $\langle F, M\rangle = \mathrm{R}\Gamma(G, F \otimes^! M)$. Let $inv(F)$ denote the preimage of F under the inversion $G \to G$. Then $i_0^!(inv(F) * M) \widetilde{\to} \langle F, M\rangle$ for the unit section $i_0 : \mathrm{Spec}\, k \to G$. For $V_1, V_2 \in \mathrm{Vect}$ write also $\langle V_1, V_2\rangle = V_1 \otimes V_2 \in \mathrm{Vect}$.

1) For $F \in Shv(G), V \in \mathrm{Vect}$ we have $\langle F, p_*^R(V)\rangle \widetilde{\to} (p_!F) \otimes V \widetilde{\to} \langle p_!F, V\rangle$. Indeed, it suffices to prove this for $F \in Shv(G)^c$ and pass to filtered colimits, as $Shv(G)$ is compactly generated. For F compact, we get

$$\mathrm{R}\Gamma(F \otimes^! p_*^R(V)) \widetilde{\to} \mathrm{Hom}(\mathbb{D}(F), p_*^R(V)) \widetilde{\to} \mathrm{Hom}(p_*\mathbb{D}(F), V) \widetilde{\to} \mathrm{Hom}(\mathbb{D}(p_!F), V) \widetilde{\to} (p_!F) \otimes^! V$$

2) For $F, M \in Shv(G), V \in$ Vect we have $\langle F, M * p_*^R(V)\rangle \widetilde{\to} \langle inv(M) * F, p_*^R(V)\rangle$. Indeed, $inv(F * M) \widetilde{\to} inv(M) * inv(F)$, so

$$\langle F, M * p_*^R(V)\rangle \widetilde{\to} i_0^!(inv(F) * M * p_*^R(V)\rangle \widetilde{\to} i_0^!(inv(inv(M) * F) * p_*^R(V)\rangle$$
$$\widetilde{\to} \langle inv(M) * F, p_*^R(V)\rangle$$

3) Assume our map $M * p_*^R(e) \to p_*^R(p_* M)$ is an isomorphism for $\mathcal{V} = e$. Then for $F \in Shv(G)$ we get

$$\langle F, M * p_*^R(e)\rangle \widetilde{\to} \langle inv(M) * F, p_*^R(e)\rangle \widetilde{\to} p_!(inv(M) * F)$$

On the other hand, $\langle F, p_*^R(p_* M)\rangle \widetilde{\to} (p_! F) \otimes p_* M$. Taking $F = (i_0)_! e$, our map becomes $p_!(inv(M)) \to p_* M$. This is not an isomorphism in general, for example, for $U = \mathbb{A}^n$ abelian. □

However, if $p : G \to \operatorname{Spec} k$ is proper, the same argument shows that the map p_*^R is a strict morphism in $Shv(G) - mod$.

1.3.8. Assume now $f : G \to S$ is a group scheme over S written as $\lim_{i \in I^{op}} G_i$, where I is small filtered category, G_i is a group scheme of finite type over S. For $i \to j$ in I the map $f_{ij} : G_j \to G_i$ is smooth affine surjective homomorphism of group schemes over S. By definition, $Shv(G) \widetilde{\to} \lim_{i \in I^{op}} Shv(G_i)$ with the transition functors $(f_{ij})_*$. (If each $G_i \to S$ is smooth, we say that G is prosmooth over S).

Then for $i \to j$ the functor $(f_{ij})_* : (Shv(G_j), *) \to (Shv(G_i), *)$ is monoidal, so $Shv(G) \widetilde{\to} \lim_{i \in I^{op}} Shv(G_i)$ can be understood as a limit in $Alg(\mathrm{DGCat}_{cont})$, that is, a monoidal category denoted $(Shv(G), *)$.

The category $Shv(G_i)$ is naturally a $Shv(S)$-module (both structures of $Shv(S)$-module coincides as we have seen above). Then for $i \in I$, $Shv(G_i) \in Alg(Shv(S) - mod)$ naturally. Namely, the product on $Shv(G_i)$ is $Shv(S)$-bilinear, and yields a functor $Shv(G_i) \otimes_{Shv(S)} Shv(G_i) \to Shv(G_i)$ in $Shv(S) - mod$. So we may think of $\lim_{i \in I^{op}} Shv(G_i)$ as a limit in $Alg(Shv(S) - mod)$, so $Shv(G) \in Alg(Shv(S) - mod)$. This structure comes of course from the monoidal functor $i_* : Shv(S) \to Shv(G)$, the push-out via the unit section i.

For a map $i \to j$ in I, the adjoint pair $f_{ij}^* : Shv(G_i) \leftrightarrows Shv(G_j) : (f_{ij})_*$ takes place in $Shv(S) - mod$.

The functor $f_* : (Shv(G), *) \to (Shv(S), \otimes^!)$ is monoidal. For the projection $ev_i : G \to G_i$ the functor $(ev_i)_* : (Shv(G), *) \to (Shv(G_i), *)$ is monoidal by construction.

We have $G \times_S G \widetilde{\to} \lim_{i \in I^{op}} G_i \times_S G_i$, because I is sifted, so $G \times_S G$ is also a placid scheme. The map $q : G \times_S G \to G \times G$ is a placid closed embedding for S separated, but even if it is not, the functor $q^!$ is defined.

Let $q_i : G_i \times_S G_i \to G_i \times G_i$ be the natural map. In fact, the system of functors $q_i^!$ is compatible with the transition functors f_{ij}^*, hence in the colimit over $i \in I$ yields the functor $q^! : Shv(G \times G) \to Shv(G \times_S G)$.

(More generally, for a placid scheme Y over $S \in \mathrm{Sch}_{ft}$ and a map $S' \to S$ in Sch_{ft}, let $h : Y' \to Y$ be obtained by base change $S' \to S$. Then $h^!$ is defined, cf. [46], 0.0.43).

The convolution on $Shv(G)$ is finally given by the diagram $G\times G\stackrel{q}{\leftarrow} G\times_S G\stackrel{m}{\to} G$ and the product is the composition

$$Shv(G)\otimes Shv(G)\to Shv(G\times G)\stackrel{q^!}{\to} Shv(G\times_S G)\stackrel{m_*}{\to} Shv(G)$$

where the first functor is that of ([24], C.2.8), the exteriour product. Since $m: G\times_S G\to G$ is a morphism of placid schemes over S, m_* is well-defined by ([24], Appendix C).

Assume G prosmooth. Then $m^*: Shv(G)\to Shv(G\times_S G)$ is well-defined. Indeed, for each $i\in I$, the product $m_i: G_i\times_S G_i\to G_i$ is smooth, so we have $m_i^*: Shv(G_i)\to Shv(G_i\times_S G_i)$. These functors are compatible with the transition functors f_{ij}^* in the corresponding colimit systems, and yield m^* in the colimit.

Consider the map $\nu: G\to G\times S$. For $F\in Shv(G), K\in Shv(S)$, one has

$$\nu^!(F\boxtimes K)\widetilde{\to}(i_*K)*F\widetilde{\to}F*(i_*K),$$

and $Shv(S)$ is central in $(Shv(G),*)$.

1.3.9. Let us for simplicity understand by a character local system on G a character local system on G_i for some i, that is, its $*$-pull-back to G. Assume G placid prosmooth over S.

Let χ be a character local system on G and $C\in Shv(G)-mod$. Consider $C_{\chi^{-1}}\in Shv(G)-mod$ as in the previous subsection. To be more precise, for a map $i\to j$ in I recall the morphism of group schemes $f_{ij}: G_j\to G_i$. For $K\in Shv(G_j)$ one has canonically $(f_{ij})_*(K\otimes f_{ij}^*\chi)\widetilde{\to}((f_{ij})_*K)\otimes\chi$. So, the collection of monoidal functors $Shv(G_j)\to Shv(G_j), K\mapsto K\otimes f_{ij}^*\chi$ is compatible with the transition maps given by $*$-direct images, and we may pass to the limit over $(I_{i/})^{op}$. Now the limit of the above functors becomes a monoidal functor $Shv(G)\to Shv(G)$ still denoted $K\mapsto K\otimes ev_i^*K$. Here $ev_i: G\to G_i$ is the evaluation map. Then $C_{\chi^{-1}}\in Shv(G)-mod$ is defined as C, where the new action of $Shv(G)$ is given as above: $F\in Shv(G)$ acts on $c\in C$ as $(F\otimes ev_i^*\chi)*c$.

We have the adjoint pair

$$f^*: Shv(S)\rightleftarrows Shv(G): f_* \tag{20}$$

If G is prosmooth, this is an adjoint pair in $Shv(G)-mod$ and also in $Shv(G)-mod^r$ by Lemma 1.3.16 below. Applying the functor $\mathrm{Fun}_{Shv(G)}(\bullet, C_{\chi^{-1}})$, we get an adjoint pair

$$\mathrm{oblv}_{G,\chi}: C^{G,\chi}\rightleftarrows C:\mathrm{Av}_*^{G,\chi}$$

in $Shv(S)-mod$. The composition $\mathrm{oblv}_{G,\chi}\mathrm{Av}_*^{G,\chi}$ is the functor $c\mapsto((f^*\omega_S)\otimes\chi)*c$, where we use the original action of $Shv(G)$ on C.

Applying the functor $\bullet\otimes_{Shv(G)} C$, we get an adjoint pair in DGCat_{cont}

$$\mathrm{oblv}^G: C_G := Shv(S)\otimes_{Shv(G)} C\leftrightarrows C:\mathrm{Av}_*^G$$

Since f^* is a map of $Shv(G)$-bimodules, oblv^G inherits a structure of a morphism in $Shv(G)-mod$, hence factors as $C_G\to C^G\stackrel{\mathrm{oblv}}{\to} C$. Dennis claims the so obtained map $C_G\to C^G$ is always an equivalence (for all the 4 sheaf theories) for any placid group scheme G over S.

Lemma 1.3.10. *In the situation of Section 1.3.8 assume each G_i is a unipotent group scheme over S. Then the functor* $\operatorname{oblv}_{G,\chi} : C^{G,\chi} \to C$ *is fully faithful.*

Proof. Recall the adjoint pair (20). The unit of this adjunction is an isomorphism $\mathrm{id} \to f_*f^*$. Thus, f^* is fully faithful. This gives $\mathrm{Av}_*^{G,\chi} \circ \operatorname{oblv}_{G,\chi} \widetilde{\to} \mathrm{id}$. □

1.3.11. Let S be a scheme of finite type, $f : G \to H$ be a morphism of group schemes over S, both being as in the previous subsection. The functor $f_* : Shv(G) \to Shv(H)$ is monoidal.

Indeed, write $H = \lim_{j\in J^{op}} H_j$ and $G = \lim_{i\in I^{op}} G_i$ as in the previous section. It suffices to show that for any i the composition $Shv(G) \stackrel{f_*}{\to} Shv(H) \stackrel{(ev_j)*}{\to} Shv(H_j)$ is monoidal. Pick $i \in I$ such that this composition factors through $ev_i : G \to G_i$. Such factorization exists by ([47], 1.1.3). The induced map $\bar{f} : G_i \to H_j$ is automatically a morphism of group schemes over S, hence $\bar{f}_* : (Shv(G_i), *) \to (Shv(H_j), *)$ is monoidal by Section 1.3.3. We are done.

If H is a placid group scheme over S, $f : G \to H$ is a placid closed immersion over S, and a group subscheme over S then in the constructible context $f^* : Shv(H) \to Shv(G)$ is defined. Moreover, f^* is left-lax monoidal.

1.3.12. Let S be a scheme of finite type, $G \to S$ be an object of $Grp(\mathrm{PreStk}_{/S})$ written as $G = \mathrm{colim}_{i\in I} G_i$ with G_i a placid group scheme over S, I small filtered, and for $i \to j$ in I the map $h_{ij} : G_i \to G_j$ is a placid closed immersion and a homomorphism of group schemes over S. Then for $i \to j$ in I the functor $(h_{ij})_* : (Shv(G_i), *) \to (Shv(G_j), *)$ is monoidal. Indeed, the square is cartesian

$$\begin{array}{ccc} G_j \times G_j & \leftarrow & G_j \times_S G_j \\ \uparrow h_{ij}\times h_{ij} & & \uparrow \\ G_i \times G_i & \leftarrow & G_i \times_S G_i \end{array}$$

So, $Shv(G) = \mathrm{colim}_{i\in I} Shv(G_i)$ taken in $Alg(\mathrm{DGCat}_{cont})$ in view of (HA, 3.2.3.1) equips $Shv(G)$ with a monoidal structure (the convolution). Namely, the projection $Alg(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves filtered colimits.

Moreover, if $H \to S$ is another object of $Grp(\mathrm{PreStk}_{/S})$ with the same properties (thus, a placid ind-scheme) and $\alpha : G \to H$ is any morphism in $\mathcal{G}\mathrm{rp}(\mathrm{PreStk}_{/S})$ then $\alpha_* : Shv(G) \to Shv(H)$ is monoidal. Indeed, write $G \widetilde{\to} \mathrm{colim}_{i\in I} G_i$, $H \widetilde{\to} \mathrm{colim}_{j\in J} H_j$ as above. Then for any $i \in I$ the map $G_i \to G \to H$ factors through $H_j \hookrightarrow H$ for some j by ([47], 1.2.6). Besides, the functors $Shv(G_i) \to Shv(H_j) \to Shv(H)$ are monoidal, and form a compatible family giving a monoidal functor α_*.

For $C \in Shv(G) - mod$ and a character local system χ on G (in the sense below) we have

$$C^{G,\chi} = \mathrm{Fun}_{Shv(G)}(Shv(S), C_{\chi^{-1}}) \widetilde{\to} \lim_{i\in I^{op}} C^{G_i,\chi}$$

as in ([47], 1.3.8). It is more convenient to twist the action of $Shv(G)$ on $Shv(S)$, that is, get a new monoidal functor $Shv(G) \to Shv(S)$ using a character local system on G. Namely, if $G \stackrel{v}{\to} H \stackrel{u}{\to} S$ is a homomoprhism of group prestacks over S, where H is a group scheme of finite type, and χ is a character local system on H, we get the monoidal functor $Shv(G) \to Shv(S), K \mapsto u_*(v_*(K) \otimes \chi)$.

Assume in addition that each G_i is a prounipotent group scheme over S. Then each $\mathrm{oblv}_{G_i,\chi}: C^{G_i,\chi}\to C$ is fully faithful, and for $i\to j$ in I the corresponding functors

$$C^{G_j,\chi}\to C^{G_i,\chi}\stackrel{\mathrm{oblv}}{\to} C$$

are fully faithful, and $\lim_{i\in I^{op}} C^{G_i,\chi}$ amounts to the intersection $\cap_{i\in I}C^{G_i,\chi}$ in C by ([43], 2.7.7), because the forgetful functor $\mathrm{DGCat}_{cont}\to 1-\mathcal{C}at$ preserves limits. The natural functor $\mathrm{oblv}_{G,\chi}: C^{G,\chi}\to C$ is fully faithful, it admits a (maybe discontinuous) right adjoint given by $\lim\limits_{i\in I^{op}}\mathrm{Av}_*^{G_i,\chi}$ by Lemma 1.2.12 of this file.

1.3.13. Let $\alpha: S'\to S$ be a morphism in Sch_{ft}, $Y\to S$ be a prestack locally of finite type over S, G be a group scheme of finite type and smooth over S. Assume G acts on Y over S. Let G', Y' be obtained from G, Y by the base change via α. Let χ be a character local system on G, χ' its $*$-restriction to G'. Let $\bar\alpha: Y'\to Y$, $\beta: G'\to G$ be natural maps.

The functor $\beta^!: Shv(G)\to Shv(G')$ is monoidal, it actually induces a functor $Shv(S')\otimes_{Shv(S)} Shv(G)\to Shv(G')$. For $\mathcal{D}$-modules this kind of sheaves was discussed in ([31], Sect. 1.6).

The functor $\bar\alpha^!: Shv(Y)\to Shv(Y')$ commutes with the actions of $Shv(G)$, where it acts on $Shv(Y')$ via $\beta^!: Shv(G)\to Shv(G')$.

It is not true in the constructible context that $Shv(G)\otimes_{Shv(S)} Shv(S')\widetilde{\to} Shv(G')$, already for $S=\operatorname{Spec} k$.

The functor $Shv(Y)^{G,\chi}\to Shv(Y')^{G',\chi'}$ can be defined as the functor

$$((f^*\omega_S)\otimes\chi)-comod(Shv(Y))\to((f'^*\omega_{S'})\otimes\chi')-comod(Shv(Y')) \tag{21}$$

Here $f: G\to S, f': G'\to S$ are the projections. The latter functor is induced by $\bar\alpha^!$. Indeed,

$$\beta^!(f^*\omega_S)\otimes\chi)\widetilde{\to}(f'^*\omega_{S'})\otimes\chi'$$

naturally. In details, $\bar\alpha^!: Shv(Y)\to Shv(Y')$ is a map of $Shv(G)$-modules. So, for any coalgebra $\mathcal{A}$ in $Shv(G)$, the functor $\bar\alpha^!$ upgrades to a functor $\mathcal{A}-comod(Shv(Y))\to \mathcal{A}-comod(Shv(Y'))$ by ([43], 3.0.49).

Since the colimits in a topos are universal, $Y'/G'\widetilde{\to}(Y/G)\times_S S'$ in PreStk_{lft} canonically. In particular, we have the projection $\tilde\alpha: Y'/G'\to Y/G$. It gives the functor $\tilde\alpha^!: Shv(Y/G)\to Shv(Y'/G')$. For χ trivial the functor $\tilde\alpha^!$ identifies with (21).

We may also consider the $Shv(S')$-linear functor

$$u: Shv(S')\otimes_{Shv(S)}\mathrm{Fun}_{Shv(G)}(Shv(S), Shv(Y))\to Shv(Y')$$

coming from the $Shv(S)$-linear functor

$$\mathrm{Fun}_{Shv(G)}(Shv(S), Shv(Y))\stackrel{\circ f_*}{\to}\mathrm{Fun}_{Shv(G)}(Shv(G), Shv(Y))\widetilde{\to} Shv(Y)\stackrel{\bar\alpha^!}{\to} Shv(Y')$$

Here $f: G\to S$. Then u is $Shv(G')$-linear? I think so, but don't see a formal proof!!! Then by adjointness ([43], 9.2.56), it gives a $Shv(S)$-linear functor

$$\mathrm{Fun}_{Shv(G)}(Shv(S), Shv(Y))\to\mathrm{Fun}_{Shv(G')}(Shv(S'), Shv(Y'))$$

1.3.14. Let us convent that by a unipotent group scheme over $S \in \mathrm{Sch}_{ft}$ we mean in particular, that this group scheme is smooth over S. The analog of Lemma 1.2.14 holds also over a base:

Lemma 1.3.15. *Let $S \in \mathrm{Sch}_{ft}$, U be a pro-unipotent group scheme over S, $U = \lim_{i\in I^{op}} U_i$, where U_i is a unipotent smooth group scheme of finite type over S, I is filtered, for $i \to j$ in I, the map $f_{ij} : U_j \to U_i$ is smooth affine surjective homomorphism of group schemes over S. Let $p : U \to S$ be the projection. Then $p^* : Shv(S) \to Shv(U)$ in the constructible context admits a left adjoint $(p^*)^L : Shv(U) \to Shv(S)$. Moreover, $((p^*)^L)^\vee : Shv(S) \to Shv(U)$ identifies with the right adjoint to $p_* : Shv(U) \to Shv(S)$. We used here the self-duality on $Shv(U)$ from ([47], 1.1.10). In addition, $(p^*)^L \circ p^* \to \mathrm{id}$ is the identity, so $p_* \circ (p_*)^R \to \mathrm{id}$ is the identity.*

Proof. Same proof, we have to replace $\dim U_i$ by d_i, where $U_i \to S$ is smooth of relative dimension d_i. □

Lemma 1.3.16. *Let $S \in Sch_{ft}$, $p : G \to S$ be a placid pro-smooth group scheme over S. We have for $K \in Shv(G), F \in Shv(S)$ naturally $(p^*F) * K \widetilde{\to} p^*(F \otimes^! p_*K)$. So, $p^* : Shv(S) \to Shv(G)$ is a morphism in $Shv(G) - mod$ naturally (that is, the left-lax $Shv(G)$-monoidal structure on p^* is strict).*

Similarly for the left action of $Shv(G)$, we have $K * (p^*F) \widetilde{\to} p^*(F \otimes^! p_*K)$, so we may view p^* as a map in $Shv(G) - mod - Shv(G)$, the category of bimodules.

Proof. 1) First, assume $p : G \to S$ is a smooth group scheme of finite type over S, of relative dimension d over S. Then $m \times \mathrm{pr}_2 : G \times_S G \to G \times_S G$ is an isomorphism. Here $m : G \times_S G \to G$ is the product. So, for $\mathcal{K} \in Shv(G)$, $m_* \mathrm{pr}_2^* \mathcal{K} \widetilde{\to} p^* p_* \mathcal{K}$. For $K \in Shv(G), F \in Shv(S)$ we get for $q : G \times_S G \to G \times G$ the isomorphism $q^!((p^*F) \boxtimes K) \widetilde{\to} \mathrm{pr}_2^*((p^!F) \otimes^! K)$. It gives

$$(p^*F) * K \widetilde{\to} m_* q^!((p^*F) \boxtimes K) \widetilde{\to} m_* \mathrm{pr}_2^*((p^!F) \otimes^! K) \widetilde{\to} p^* p_*((p^!F) \otimes^! K)$$

By the projection formula, $p_*((p^!F) \otimes^! K) \widetilde{\to} F \otimes^! (p_*K)$, and we are done.

2) The general case. The map p^* is left-lax monoidal. This gives a canonical map $p^*(F \otimes^! p_*K) \to (p^*F) * K$. We check it is an isomorphism. We have a diagram, where both squares are cartesian

$$\begin{array}{ccccc} G \times G & \stackrel{q}{\leftarrow} & G \times_S G & \stackrel{m}{\to} & G \\ \downarrow p\times \mathrm{id} & & \downarrow \mathrm{pr}_2 & & \downarrow p \\ S \times G & \leftarrow & G & \stackrel{p}{\to} & S \end{array}$$

Now we apply ([46], Lemma 0.0.19, 0.0.20) to get the desired base change. To see that the assumptions of Lemma 0.0.20 holds, we may write $G = \lim_{i\in I^{op}} G_i$, G_i is a smooth group scheme of finite type over S, I is filtered, and for $i \to j$ the transition map $G_j \to G_i$ is a smooth affine surjective morphism of group schemes over S. Then the left square is a limit over I^{op} is the cartesian squares

$$\begin{array}{ccc} G_i \times G_i & \stackrel{q}{\leftarrow} & G_i \times_S G_i \\ \downarrow p_i\times \mathrm{id} & & \downarrow \mathrm{pr}_2 \\ S \times G_i & \stackrel{b}{\leftarrow} & G_i \end{array}$$

So, $q^!(p^*F\boxtimes K)\widetilde{\to}\operatorname{pr}_2^* b^!(F\boxtimes K)$. We have $b^!(F\boxtimes K)\widetilde{\to}(i_*F)*K$, where $i:S\to G$ is the unit section (this is the usual $Shv(S)$-module structure on $Shv(G)$). Further, $m_*\operatorname{pr}_2^*\widetilde{\to}p^*p_*$. Finally, $p_*((i_*F)*K)\widetilde{\to}F\otimes^! p_*K$, as p_* is monoidal. We are done. □

Let now $U\to S$ be as in Lemma 1.3.15 . Consider the adjoint pair $p_*: Shv(U)\rightleftarrows Shv(S):(p_*)^R$ in the constructible context. This is not an adjoint pair in $Shv(U)-mod$ in general, as we have seen above. Let $C\in Shv(U)-mod$.

The functor $\operatorname{oblv}_U: C^U\to C$ does not admit a left adjoint in the constructible context in general. This happens already for $S=\operatorname{Spec} k$ and $U=\mathbb{G}_a$.

1.3.17. Let $S\in\operatorname{Sch}_{ft}$, $f:G\to S$ be a placid prosmooth group scheme over S, $C\in Shv(G)-mod$. Arguing as in Proposition 1.3.4 and using Lemma 1.3.16 in addition, one shows that cosimplicial category $[n]\mapsto\operatorname{Fun}_{Shv(S)}(Shv(G)^{\otimes n}_{Shv(S)}, C)$ satisfies the comonadic Beck-Chevalley conditions. So, the functor $\operatorname{oblv}_G: C^G\to C$ is comonadic, and the corresponding comonad is $c\mapsto (f^*\omega_S)*c$.

More generally, if $f:G\to S$ is a placid group ind-scheme over S, $C\in Shv(G)-mod$ then $C^G\to C$ is comonadic by ([36], 4.7.5.1), but we can say less about the corresponding comonad.

1.3.18. Let $S\in\operatorname{Sch}_{ft}$, I filtered, $U\in Grp(\operatorname{PreStk}_{/S})$ a placid ind-scheme over S written as $U\widetilde{\to}\operatorname{colim}_{i\in I} U_i$, where $U_i\to S$ is a pro-unipotent placid group scheme over S, for $i\to j$ in I the map $U_i\to U_j$ is a placid closed immersion, and a homomorphism of group schemes. Let $C\in Shv(U)-mod$.

The forgetful functor $\operatorname{oblv}_{U_i}: C^{U_i}\to C$ is fully faithful for any i. If it admits a left adjoint $\operatorname{Av}_!^{U_i}$ then the fully faithful embedding $C^U\widetilde{\to}\lim_{i\in I^{op}} C^{U_i}\to C$ also admits a left adjoint $\operatorname{Av}_!^U$ by (HTT, 5.5.4.18) as in Section 1.2.14-1.2.15 of this file. In this case by Lemma 1.2.15, $\operatorname{Av}_!^U\widetilde{\to}\operatorname{colim}_{i\in I}\operatorname{Av}_!^{U_i}$.

1.3.19. Let S be a scheme of finite type, $f:Y\to S$ a map in $\operatorname{PreStk}_{lft}$ which is ind-schematic of ind-finite type say. Let $U\in Grp(\operatorname{PreStk}_{/S})$ be written as $U=\operatorname{colim}_{i\in I} U_i$, where U_i is a prounipotent placid group scheme over S, for $i\to j$ in I, $f_{ij}:U_i\to U_j$ is a placid closed immerion and a homomorphism of group schemes over S. Assume U acts on Y over S, and the action is transitive on each fibre of f. Besides, there is a section $s:S\to Y$, whose stabilizer is a prounipotent placid closed subscheme of U.

Then $f^!: Shv(S)\to Shv(Y)^U$ is an equivalence? This kind of claim was used in ([26], 1.4.2). What are the precise assumptions to require???

We apply Proposition 1.3.4 of this file. Namely, consider first the following case: let $f:Y\to S$ be a morphism in Sch_{ft}, $p:U\to S$ a unipotent group scheme of finite type over S, U acts on Y over S, and the action is transitive over each fibre of f. Let $s:S\to Y$ be a section of f, whose stabilizer in U is a closed subgroup scheme $U'\subset U$ over S. Here U' is defined as $U\times_{Y\times Y} Y$, namely by the "equation" $us(\bar u)=s(\bar u)$ for $u\in U$, here $\bar u\in S$ is the projection of u. By Proposition 1.3.4, $Shv(Y)^U\widetilde{\to}Shv(Y/U)$. The diagram $U\to Y\to S$ yields by passing to the stack quotients the diagram $S\stackrel{\bar s}{\to} Y/U\stackrel{\bar f}{\to} S$ with $\bar f\bar s=\mathrm{id}$. By the assumption, $Y/U\widetilde{\to}S/U'$, where the action of U' on S trivial. Since U' is unipotent, the map $\bar s$ yields an equivalence $\bar s^!: Shv(S/U')\widetilde{\to}Shv(S)$.

We want a version of this result for $U \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk}_{/S})$ a placid ind-scheme over S as in the beginning of this section.

1.3.20. Let $S \in Sch_{ft}$ separated, $G \in Grp(\mathrm{PreStk}_{/S})$ be a relative placid ind-scheme over S written as $G = \mathrm{colim}_{i\in I} G_i$ in $\mathrm{PreStk}_{/S}$, where I is filtered, G_i is a placid prounipotent group scheme over S, for $i \to j$ in I the map $G_i \to G_j$ is a placid closed immersion, and a homomorphism of group schemes over S. Assume $0 \in I$ is an initial element, and $G_0 \to S$ is a prounipotent group scheme over S. Recall that $Shv(G/G_0)^G \subset Shv(G/G_0)$ is fully faithful. Let $\bar{i} : S \to G/G_0$ be the canonical section.

Lemma 1.3.21. *Under the assumptions of Section 1.3.20, the composition*

$$Shv(G/G_0)^{Shv(G)} \hookrightarrow Shv(G/G_0) \stackrel{\bar{i}^!}{\to} Shv(S)$$

is an equivalence.

Proof. (compare with [33], Lemma B.4.1). We have $Shv(G/G_0) \widetilde{\to} \lim_{i\in I^{op}} Shv(G_i/G_0)$ with respect to the !-pullbacks, and $Shv(G/G_0) \widetilde{\to} \mathrm{colim}_{i\in I} Shv(G_i/G_0)$ via $*$-pushforwards. So,

$$Shv(G/G_0)^{Shv(G)} \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \lim_{j\in I^{op}} \mathrm{Fun}(Shv(G_j)^{\otimes n} \otimes Shv(S), Shv(G/G_0))$$

Consider the category $\mathrm{Fun}([1], I)$. Note that the map $I \to \mathrm{Fun}([1], I)$ sending i to $(i \stackrel{\mathrm{id}}{\to} i)$ is cofinal. Indeed, for any i, $I_{i/}$ is contractible. For each j we may write $Shv(G/G_0) \widetilde{\to} \lim_{i\in(I_{j/})^{op}} Shv(G_i/G_0)$, and the above limit identifies with

$$\lim_{[n]\in\boldsymbol{\Delta}} \lim_{j\in I^{op}, i\in(I_{j/})^{op}} \mathrm{Fun}(Shv(G_j)^{\otimes n} \otimes Shv(S), Shv(G_i/G_0)) \widetilde{\to}$$
$$\lim_{[n]\in\boldsymbol{\Delta}} \lim_{j\in I^{op}} \mathrm{Fun}(Shv(G_j)^{\otimes n} \otimes Shv(S), Shv(G_j/G_0))$$

Now fix j and calculate

$$\lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}(Shv(G_j)^{\otimes n} \otimes Shv(S), Shv(G_j/G_0)) \widetilde{\to} Shv(G_j/G_0)^{G_j}$$

By assumption, G_j/G_0 is a scheme of finite type over S. Pick a placid closed subgroup $H \subset G_0$ such that $H \subset G_j$ is normal. So, G_j/H is a group scheme of finite type over S, and the G_j-action on G_j/G_0 factors through an action of G_j/H. Then H is also prounipotent placid group scheme over S. For the projection $p : G_j \to G_j/H$ the functor $p_* : Shv(G_j) \to Shv(G_j/H)$ is monoidal. The $Shv(G_j)$-action on $Shv(G_j/G_0)$ factors through a $Shv(G_j/H)$-action. The prestack quotient of G_j/G_0 by G_j/H identifies with $B(G_0/H)$, and $Shv(B(G_0/H)) \widetilde{\to} Shv(S)$. Our claim follows from the next lemma. □

Lemma 1.3.22. *Let $S \in \mathrm{Sch}_{ft}$, $1 \to U \to G \to G_1 \to 1$ an exact sequence of placid prosmooth group schemes over S, where $U \hookrightarrow G$ is a placid closed immersion, and U is a prounipotent group scheme over S. Let $E \in Shv(G_1)-mod$, which we view by restriction as $Shv(G)$-module. Then $C^G \widetilde{\to} C^{G_1}$ canonically.*

Proof. Let $p : G \to S$ and $p_1 : G_1 \to S$ be the projections. By Section 1.3.17, $C^{G_1} \widetilde{\to} p_1^*\omega_S - comod(E)$ and $C^G \widetilde{\to} p^*\omega_S - comod(E)$. We have a canonical isomorphism of the corresponding comonads on E, because for the projection $h : G \to G_1$ we have $h_*h^*p_1^*\omega_S \widetilde{\to} p_1^*\omega_S$. □

1.3.23. We generalize the situation of Section 1.3.13 as follows. Let $\alpha : S' \to S$ be a map in Sch_{ft}, $Y \to S$ be a prestack locally of finite type over S. Let $G \in Grp(\mathrm{PreStk}_{/S})$ be a placid ind-scheme written as $G = \mathrm{colim}_{i\in I} G_i$, where I is filtered, $G_i \to S$ is a placid prosmooth group scheme over S, for $i \to j$ in I the map $G_i \to G_j$ is a placid closed immersion, and a homomorphism of group schemes. Let $\bar{\alpha} : Y' \to Y$ and $\beta : G' \to G$ be obtained by the base change via α. Set $G'_i = G_i \times_S S'$, so $G' \widetilde{\to} \mathrm{colim}_{i\in I} G'_i$ in PreStk. Assume G acts on Y over S.

The functor $\bar{\alpha}^!$ yields a functor $Shv(Y)^G \to Shv(Y')^{G'}$ defined as follows. First, for each $i \in I$, we have a functor $Shv(Y)^{G_i} \to Shv(Y')^{G'_i}$ defined as in Section 1.3.13 by (21). Namely, let $p_i : G_i \to S$, $p'_i : G'_i \to S'$ be the projections, $\beta_i : G'_i \to G_i$ be obtained from G_i by the base change $S' \to S$. Since G_i is prosmooth over S, $\beta_i^! p_i^*\omega_S \widetilde{\to} (p'_i)^*\omega_{S'}$ by ([46], 0.0.21). The functor $\bar{\alpha}^!$ is $Shv(G_i)$-linear, where $Shv(G_i)$ acts on $Shv(Y')$ via $\beta_i^! : Shv(G_i) \to Shv(G'_i)$. This gives the functor

$$Shv(Y)^{G_i} \widetilde{\to} p_i^*\omega_S - comod(Shv(Y)) \to (p'_i)^*\omega_{S'} - comod(Shv(Y')) \widetilde{\to} Shv(Y')^{G'_i}$$

Set $v_i = p_i^*\omega_S$, this is a coalgebra in $Shv(G)$. The coalgebra structure comes from the fact that p_i^* is left-lax monoidal. For $i \to j$ in I write $f_{ij} : G_i \hookrightarrow G_j$ for the closed immersion. If $i \to j$ is a map in I then we have a morphism of coalgebras $v_j \to (f_{ij})_*v_i$ in $Shv(G)$ for any of the 4 sheaf theories. In the constructible context it is given by $\mathrm{id} \to (f_{ij})_*(f_{ij})^*$. In other contexts it comes from the natural map

$$\omega_S \to (p_j)_*(f_{ij})_*p_i^*\omega_S \widetilde{\to} (p_i)_*p_i^*\omega_S$$

The fact that this is indeed a morphism of coalgebras comes from the fact that the morphism $p_j^* \to (f_{ij})_*p_i^*$ is a morphism of left-lax functors, so automatically gives a morphism of coalgebras when evaluated on a coalgebra by ([43], Example in 3.0.12).

It yields a functor $(v_j) - comod(Shv(Y)) \to (v_i) - comod(Shv(Y))$. The diagram commutes

$$\begin{array}{ccc} (v_i) - comod(Shv(Y)) & \to & (\beta^! v_i) - comod(Shv(Y')) \\ \uparrow & & \uparrow \\ (v_j) - comod(Shv(Y)) & \to & (\beta^! v_j) - comod(Shv(Y')) \end{array}$$

So, we get a morphism of inverse systems and passing to the limit, we get a functor

$$Shv(Y)^G \widetilde{\to} \lim_{i\in I^{op}} Shv(Y)^{G_i} \to \lim_{i\in I^{op}} Shv(Y')^{G'_i} \widetilde{\to} Shv(Y')^{G'}$$

1.3.24. In practice, we deal especially with the following case. Let $\alpha : S' \to S$ be a map in Sch_{ft}, $G \in Grp(\mathrm{PreStk}_{/S})$ be a placid ind-scheme written as $G = \mathrm{colim}_{i\in I} G_i$, where I is filtered, $G_i \to S$ is a placid prounipotent group scheme over S, for $i \to j$ the map $G_i \to G_j$ is a placid closed embedding and a homomorphism of group schemes over S. Let $Y\to S$ be an ind-scheme of ind-finite type with an action $G\times_S Y\to Y$ of G over S. Let $G'=G\times_S S'$, $Y'=Y\times_S S'$. Recall that

$$Shv(Y)^G \widetilde{\to} \lim_{i\in I^{op}} Shv(Y)^{G_i}.$$

Here $Shv(Y)^{G_i}\subset Shv(Y)$ is a full subcategory, and the above limit amounts to the intersection (by [43], 2.7.7).

For each i we may write $Y\widetilde{\to} \operatorname{colim}_{j\in J} Y_j$, where $Y_j\subset Y$ is a closed subscheme of finite type, for $j\to j'$ in J the map $h_{jj'}: Y_j\to Y_{j'}$ is a closed immersion, and Y_j is stable under the G_i-action on Y. Then $Shv(Y)^{G_i}\widetilde{\to}\lim_{j\in J^{op}} Shv(Y_j)^{G_i}$. For $j\to j'$ in J the functor $h^!_{jj'}: Shv(Y_{j'})\to Shv(Y_j)$ sends $Shv(Y_{j'})^{G_i}$ to $Shv(Y_j)^{G_i}$ as for any $(C'\to C)\in Shv(G_i)-mod$. Let $G'_i=G_i\times_S S'$, $Y'_j=Y_j\times_S S'$ for $j\in J$.

The additional phenimenon is that on each Y_j the G_i-action factors through an action of some finite dimensional quotient group scheme $G_i\to G_{i,m}$, so that by Lemma 1.3.22

$$Shv(Y_j)^{G_i}\widetilde{\to} Shv(Y_j)^{G_{i,m}}$$

Assume that on both $Y_j, Y_{j'}$ the G_i-action factors through $G_i\to G_{i,m}$. Then we have the cartesian square

$$\begin{array}{ccc} Y_j & \stackrel{h_{jj'}}{\to} & Y_{j'}\\ \downarrow & & \downarrow\\ Y_j/G_{i,m} & \stackrel{\bar h_{jj'}}{\to} & Y_{j'}/G_{i,m}, \end{array}$$

and the functor $Shv(Y_{j'})^{G_i}\to Shv(Y_j)^{G_i}$ idenifies with

$$\bar h^!_{jj'}: Shv(Y_{j'}/G_{i,m})\to Shv(Y_j/G_{i,m})$$

In this case the functor $Shv(Y)^{G_i}\to Shv(Y')^{G'_i}$ is also geometric essentially. Namely, it suffices to understand each functor $Shv(Y_j)^{G_i}\to Shv(Y'_j)^{G'_i}$. Let $G'_{i,m}=G_{i,m}\times_S S'$. Then we have a canonical isomorphism of prestacks $(Y_j/G_{i,m})\times_S S'\widetilde{\to} Y'_j/G'_{i,m}$. So, for the projection $\bar h: Y'_j/G'_{i,m}\to Y_j/G_{i,m}$ we get the desired functor

$$\bar h^!: Shv(Y_j)^{G_i}\widetilde{\to} Shv(Y_j/G_{i,m})\to Shv(Y'_j/G'_{i,m})\widetilde{\to} Shv(Y'_j)^{G'_i}$$

Important addition: assume that $\alpha: S'\to S$ is a closed immersion in Sch_{ft}. Let $Y\to S$ be an ind-scheme of ind-finite type over S, $Y\widetilde{\to}\operatorname{colim}_{i\in I} Y_i$ with I small filtered, $Y_i\hookrightarrow Y_j$ a closed immersion in Sch_{ft}. Let G be a placid prosmooth group scheme over S acting on Y over S. Set $G'=G\times_S S'$, $Y'=Y\times_S S'$. Then the so obtained functor

$$Shv(Y)^G\otimes_{Shv(S)} Shv(S')\to Shv(Y')^{G'}$$

is an equivalence (for both $\mathcal{D}$-modules and the constructible context).

Proof: we may assume each Y_i is G-stable. The G-action on each Y_i factors through some finite type smooth quotient group scheme G_i over S such that $\mathrm{Ker}(G\to G_i)$ is prounipotent over S. Then $(Y_i/G_i)\times_S S'\widetilde{\to} Y'_i/G'_i$, so

$$Shv(Y_i/G_i)\otimes_{Shv(S)} Shv(S')\to Shv(Y'_i/G'_i)$$

is an equivalence by ([46], 0.3.1). Here $G'_i=G_i\times_S S'$. Thus,

$$Shv(Y_i)^G\otimes_{Shv(S)} Shv(S')\to Shv(Y'_i)^{G'}$$

is an equivalence. Since $Shv(S')$ is dualizable in $Shv(S)-mod$, we get

$$Shv(S')\otimes_{Shv(S)}\lim_{i\in I^{op}} Shv(Y_i)^G \widetilde{\to} \lim_{i\in I^{op}} Shv(S')\otimes_{Shv(S)} Shv(Y_i)^G \widetilde{\to}$$
$$\lim_{i\in I^{op}} Shv(Y'_i)^{G'} \widetilde{\to} Shv(Y')^{G'}$$

We are done. □

Claim 1 Let $S\in \mathrm{Sch}_{ft}$, $f: Y\to S$ be a map in PreStk_{lft}. Let G be a placid prosmooth group scheme over S acting on Y over S. Let $\alpha: S'\to S$ be a closed immersion, $G'=G\times_S S', Y'=Y\times_S S'$. Then one has canonically in $Shv(S')-mod$

$$Shv(Y)^G\otimes_{Shv(S)} Shv(S')\widetilde{\to} Shv(Y')^{G'}.$$

Proof. Let $f': Y'\to S'$ be obtained from f by the base change $\alpha: S'\to S$. Let $\bar\alpha: Y'\to Y$ and $\beta: G'\to G$ be the natural maps. By Section 1.3.17, one has canonically $Shv(Y)^G\widetilde{\to} f^*\omega_S-comod(Shv(Y))$. By ([46], 0.3.1), one has canonically

$$Shv(Y)\otimes_{Shv(S)} Shv(S')\widetilde{\to} Shv(Y').$$

The comonad $f^*\omega_S$ on $Shv(Y)$ is $Shv(S)$-linear. Now apply ([46], Lemma 0.3.6). It gives

$$(f^*\omega_S-comod(Shv(Y)))\otimes_{Shv(S)} Shv(S')\widetilde{\to}\beta^! f^*\omega_S-comod(Shv(Y'))$$

Finally, $\beta^! f^*\omega_S\widetilde{\to}(f')^*\omega_{S'}$ by ([46], 0.0.21), and

$$(f')^*\omega_{S'}-comod(Shv(Y'))\widetilde{\to} Shv(Y')^{G'}$$

canonically by Section 1.3.17. □

Corollary 1 Assume $S\widetilde{\to}\mathrm{colim}_{j\in J} S_j$ in PreStk with $S_j\in \mathrm{Sch}_{ft}$ such that for $j\to j'$ in J the transtion map $S_j\to S_{j'}$ is a closed immersion. Let $Y\to S$ be a map in PreStk_{lft}. Assume G is a placid prosmooth group scheme over S, which means by definition that for each j, $G\times_S S_j\to S_j$ is a placid prosmooth group scheme over S_j. Assume G acts on Y over S. Then we get a sheaf of categories on S given by the compatble collection of $Shv(Y\times_S S_j)^{G\times_S S_j}\in Shv(S_j)-mod$. This works for both $\mathcal{D}$-modules and the constructible context.

Claim 2. Let $S\in \mathrm{Sch}_{ft}$, $f: Y\to S$ be a map in PreStk_{lft}. Let $G\to S$ be a placid group ind-scheme written as $G\widetilde{\to}\underset{i\in I}{\mathrm{colim}} G_i$, where I is small filtered, G_i is a placid prosmooth group scheme over S, and for $i\to j$ in I the map $G_i\to G_j$ is a placid closed immersion and a homomorphism of group schemes over S. Assume G acts on Y over S. Let $\alpha: S'\to S$ be a closed immersion. Set $Y'=Y\times_S S'$, $G'=G\times_S S'$. Then one has canonically

$$Shv(Y)^G\otimes_{Shv(S)} Shv(S')\widetilde{\to} Shv(Y')^{G'}$$

in $Shv(S')-mod$.

Proof. One has $Shv(Y)^G\widetilde{\to}\lim_{i\in I^{op}} Shv(Y)^{G_i}$ by ([47], 1.3.8). Recall that $Shv(S')$ is dualizable in $Shv(S)-mod$. So,

$$Shv(Y)^G\otimes_{Shv(S)} Shv(S')\widetilde{\to}\lim_{i\in I^{op}} Shv(Y)^{G_i}\otimes_{Shv(S)} Shv(S')$$

For each i by Claim 1 just above we get

$$Shv(Y)^{G_i}\otimes_{Shv(S)} Shv(S')\widetilde{\to} Shv(Y')^{G'_i}$$

with $G'_i=G_i\times_S S'$. Finally,

$$\lim_{i\in I^{op}} Shv(Y')^{G'_i}\widetilde{\to} Shv(Y')^{G'}$$

in $Shv(S')-mod$. □

Claim 3 Let $\alpha: S'\to S$ be a closed immersion in Sch_{ft}, $Y\to S$ be a ind-scheme of ind-finite type over S, $Y'=Y\times_S S'$. Let G be a placid prosmooth group scheme over S acting on Y over S. Assume $Y\widetilde{\to}\underset{i\in I}{\mathrm{colim}} Y_i$, where I is small filtered, and for $i\to j$ in I the map $Y_i\to Y_j$ is G-equivariant closed immersion in Sch_{ft}. Let $\bar\alpha: Y'\to Y$ be obtained by base change from α. Let $G'=G\times_S S'$. Then the diagram commutes

$$\begin{array}{ccc} Shv(Y)^G & \stackrel{\bar\alpha^!}{\to} & Shv(Y')^{G'}\\ \downarrow \mathrm{oblv}[dimrel] & & \downarrow \mathrm{oblv}[dimrel]\\ Shv(Y) & \stackrel{\bar\alpha^!}{\to} & Shv(Y')\end{array}$$

Proof. The functor in the top row is obtained by passing to the limit over $i\in I^{op}$ with respect to the !-restrictions in the functors $Shv(Y_i)^G\stackrel{\bar\alpha_i^!}{\to} Shv(Y'_i)^G$, where $\bar\alpha_i: Y'_i\to Y_i$ is obtained from α by the base change $Y_i\to S$. So, the claim follows from the fact that forr each i the diagram commutes

$$\begin{array}{ccc} Shv(Y_i)^G & \stackrel{\bar\alpha_i^!}{\to} & Shv(Y'_i)^{G'}\\ \downarrow \mathrm{oblv}[dimrel] & & \downarrow \mathrm{oblv}[dimrel]\\ Shv(Y_i) & \stackrel{\bar\alpha_i^!}{\to} & Shv(Y'_i)\end{array}$$

Indeed, give i pick a quotient smooth group scheme of finite type $G\to G_0$ such that $\mathrm{Ker}(G\to G_0)$ is prounipotent group scheme over S, and the G-action on Y_i factors through G_0. Then for $G'_0=G_0\times_S S'$ the square is cartesian

$$\begin{array}{ccc} Y'_i & \stackrel{\bar\alpha_i}{\to} & Y_i\\ \downarrow & & \downarrow\\ Y'_i/G'_0 & \stackrel{\bar\alpha_i}{\to} & Y_i/G_0\end{array}$$

and $\dim.\mathrm{rel}(G_0/S)=\dim.\mathrm{rel}(G'_0/S')$. □

1.3.25. *More general character local systems.* Let $S\in\mathrm{Sch}_{ft}$, let $G\to S$ be a placid group ind-scheme over S written as $G\widetilde{\to}\mathrm{colim}_{i\in I} G_i$, where G_i is a placid group scheme over S, and for $i\to j$ in I the map $\beta_{ij}: G_i\to G_j$ is a placid closed immersion (over S), and a homomorphism of group schemes over S.

Assume we are in the constructible context. By ([46], 0.0.53), each $(Shv(G_i),\otimes)\in CAlg(\mathrm{DGCat}_{cont})$. Pick a map $i\to j$ in I, let $\beta_{ij}: G_i\to G_j$ be the transition map. Let E be a character local system on G_j in the usual sense, so for $m: G_j\times_S G_j\to G_j$ one has $m^*E\widetilde{\to}\mathrm{pr}_1^* E\otimes\mathrm{pr}_2^* E$, and for the unit section $u: S\to G_j$ the local system

u^*E is trivialized. Then β_{ij}^*E is a character local system on G_i. Moreover, the functor $Shv(G_j)\to Shv(G_j), K\mapsto K\otimes E$ is monoidal??

Assume for each i we are given a character local system E_i on G_i with an isomorphism $\beta_{ij}^*E_j\widetilde{\to}E_i$ of character local systems. Consider the self-functors $Shv(G_i)\to Shv(G_i), K\mapsto K\otimes E_i$. These functors are compatible with the direct images transition functors $(\beta_{ij})_*$, because for $K\in Shv(G_i)$ we have $((\beta_{ij})_*K)\otimes E_j\widetilde{\to}(\beta_{ij})_*(K\otimes E_i)$ canonically by ([46], 0.0.55). So, passing to the colimit, they yield a functor of the tensor product on $Shv(G)$ by the projective system $\{E_i\}$. My hope is that the latter functor is monoidal.

For example, if each G_i is a scheme of finite type then each functor $Shv(G_i)\to Shv(G_i), K\mapsto K\otimes E_i$ is monoidal. In this case passing to the colimit over $i\in I$ in $\mathrm{Alg}(\mathrm{DGCat}_{cont})$ the above functors yield the monoidal functor of the tensor product on $Shv(G)$ by the projective system $\{E_i\}$.

1.3.26. An idea coming from ([24], 2.5.7). Let Y be an ind-scheme of ind-finite type, G be a placid group scheme acting on Y. Write $G\widetilde{\to}\lim_{i\in I^{op}} G_i$, where I is small filtered, G_i is a group scheme of finite type, and for $i\to j$ in I, $G_j\to G_i$ is smooth affine surjective. Let $K_i=\mathrm{Ker}(G\to G_i)$. We assume K_i prounipotent for $i\neq 0$, where $0\in I$ is initial (it seems this is automatic). Then the essential images of the functors $\mathrm{oblv}: Shv(Y)^{K_i}\to Shv(Y)$ for $i\in I$ generate $Shv(Y)$.

Indeed, let $Y'\subset Y$ be a closed G-invariant subscheme of finite type. Then G acts on Y' through a quotient G_i for some i. So, K_i acts trivially on Y'. We see that $Shv(Y')\widetilde{\to}Shv(Y')^{K_i}\subset Shv(Y)^{K_i}$ a full subcategory. The claim follows.

For each i consider the right adjont $\mathrm{Av}_*^{K_i}: Shv(Y)\to Shv(Y)^{K_i}$. We conclude that the functor $Shv(Y)\to\prod_i Shv(Y)^{K_i}$ whose i-th component is $\mathrm{Av}_*^{K_i}$ is conservative, so the intersection of kernels of $\mathrm{Av}_*^{K_i}$ is zero.

1.3.27. For 8.2.7. The following points from [24] need an explanations. In the notations of [24] it is claimed in (*loc.cit.*, 2.5.3) that if $j\geq 1$ then $I^j=(I^j\cap\mathfrak{L}^+(B^-))(I^j\cap\mathfrak{L}(N))$. This follows from the Iwahori decomposition of ([34], Section 3). Namely,

$$\mathring{I}^j=(\mathfrak{L}^+(N^-)\cap K_j)(\mathfrak{L}^+(T)\cap K_j)\mathfrak{L}^+(N)$$

(see also [57], 2.2.6). Recall that here $K_j=\mathrm{Ker}(\mathfrak{L}^+(G)\to\mathfrak{L}^+(G)_j)$, and $\mathfrak{L}^+(G)_j$ has k-points $\mathrm{Hom}(k[t]/t^j, G)$. By definition, $\mathring{I}^j$ is the preimage of $\mathfrak{L}^+(N)_j$ under $\mathfrak{L}^+(G)\to\mathfrak{L}^+(G)_j$ and $I^j=Ad_{t^{-j\rho}}(\mathring{I}^j)$. Note also that $\mathring{I}^j=K_j\mathfrak{L}^+(N)$. Now

$$Ad_{t^{-j\rho}}(\mathfrak{L}^+(N^-)\cap K_j)\subset\mathfrak{L}^+(N^-)$$

because for any negative root $\check{\alpha}$ and the corresponding root subgroup $x_{\check{\alpha}}:\mathbb{A}^1\to N^-$ for $y\in t^j\mathcal{O}$ we have $t^{-j\rho}x_{\check{\alpha}}(y)t^{j\rho}=x_{\check{\alpha}}(t^{-\langle\check{\alpha},j\rho\rangle}y)\in x_{\check{\alpha}}(\mathcal{O})$. So,

$$I^j\subset\mathfrak{L}^+(N^-)(\mathfrak{L}^+(T)\cap K_j)Ad_{t^{-j\rho}}(\mathfrak{L}^+(N))$$

1.3.28. in ([24], 2.5.7) the following is used. If $D, C_j\in\mathrm{DGCat}_{cont}$ for $j\in J$, where J is a small set, let $f_j: C_j\to D$ be continuous functors with right adjoints f_j^R. Then $\prod_j f_j^R: D\to\prod_j C_j$ is the right adjoint to the functor $f:\oplus_j C_j\to D$ whose j-th component is f_j.

1.3.29. Assume given a diagram $X \stackrel{i}{\to} Y \stackrel{p}{\to} X \stackrel{\pi}{\to} Z$ in PreStk_{lft} with $pi \widetilde{\to} \mathrm{id}$. Let $f = \pi p$. Here X is a retract of Y. If $f^! : Shv(Z) \to Shv(Y)$ is fully faithful then $\pi^! : Shv(Z) \to Shv(X)$ is also fully faithful as a retract of $f^!$. A similar idea may work when instead of usual sheaves, we consider sheaves that change under the action by some group scheme by a given character local system.

1.4. For 8.3.3. The argument about the "retract of a fully faithful functor" is wrong. We should say instead: Ran_x is universally homologically contractible, hence the !-pullback along the projection $\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Gr}^{\omega^\rho}_{G,x}$ is fully faithful. The claim follows.

1.4.1. For 8.4.1. We may introduce $S^0_{\mathrm{Ran}} \subset \bar{S}^0_{\mathrm{Ran}}$, the open subfunctor given by the property that each map $\omega^{\langle \rho, \check{\lambda}\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ is regular and has no zeros over X. So, we have the projections $\bar{S}^0_{\mathrm{Ran}} \to \overline{\mathrm{Bun}}^{\omega^\rho}_N$ and S^0_{Ran} is the preimage of $\mathrm{Bun}^{\omega^\rho}_N$ under this map. Any object of $\mathrm{Whit}_{q,\mathrm{Ran}}(G)^{\leq 0}$ is the extension by zero under $S^0_{\mathrm{Ran}} \subset \bar{S}^0_{\mathrm{Ran}}$.

Verification of the shift in the formula: The claim that !-restriction of $\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}$ to $\mathrm{Gr}^{\omega^\rho}_{G,x}$ is $W^{0,!}$ is equivalent to $\delta^!_{t^0} W^{0,!} \widetilde{\to} e$. For the corresponding map $\bar{\chi}^\lambda : S^\lambda \to \mathbb{A}^1$ one has

$$\mathring{W}^\lambda \widetilde{\to} (\bar{\chi}^\lambda)^! \mathcal{L}_\psi[2 - \langle 2\check{\rho}, \lambda\rangle],$$

where $\mathcal{L}_\psi$ is the Artin-Schreier sheaf (by Thm. 7.4.2). So, $\delta^!_{t^0} W^{0,!} \widetilde{\to} e$ indeed.

1.4.2. Recall that for a prestack $\mathcal{X} \in \mathrm{PreStk}_{lft}$ the property of $\mathcal{X} \to pt$ being universally homologically contractible is equivalent to homologically contractible, that is, to the fact that the !-pullback $\mathrm{Vect} \to Shv(\mathcal{X})$ is fully faithful (equivalently, $\mathrm{R}\Gamma_c(\mathcal{X}, \omega) \to e$ is an isomorphism).

Let us prove ([26], Lemma A.2.5). Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map in PreStk_{lft}, which is pseudo-proper. We claim that f is universally homologically contractible iff all its fibres (over field valued points including extensions of fields) are homologically contractible.

In one direction this is clear: after base change for $x \in \mathcal{X}_2$ the map $(\mathcal{X}_2)_x \to x$ is universally homologically contractible. Conversly, assume each fibre is homologically contractible. Let $Y \to \mathcal{X}_2$ be a map with $Y \in \mathrm{Sch}_{ft}$. Let $f_Y : Y \to Y \times_{\mathcal{X}_2} \mathcal{X}_1$ be obtained by base change. By the projection formula, it suffices to show that the map $(f_Y)_! f_Y^! \omega \to \omega$ is an isomorphism on $Y \times_{\mathcal{X}_2} \mathcal{X}_1$. For this it suffices to show that it becomes an isomorphism after any base change by a field valued point $\mathrm{Spec}\, k' \to Y \times_{\mathcal{X}_2} \mathcal{X}_1$. Our claim follows from the fact that the pseudo-proper maps f satisfy the base change $(f_!, g^!)$ for any map g.

1.5. **For Section 9.**

1.5.1. For 9.1.1. The quotients $\mathfrak{L}^+(G)_x \backslash \mathrm{Gr}_{G,x}$ and $\mathfrak{L}^+(G)^{\omega^\rho}_x \backslash \mathrm{Gr}^{\omega^\rho}_{G,x}$ are naturally isomorphic.

About the normalization of the action, is it canonical? Example: let $G \subset G'$ be a closed subgroup of an algebraic group G', G' acts on $Z \in \mathrm{PreStk}_{lft}$ on the right. Here say $G, G' \in Sch_{ft}$. Then $Shv(G'/G)^G$ acts on the right on $Shv(Z/G)$ as follows. Write

$G\backslash G'/G$ for the quotient of G'/G by G in the sense of prestacks. We have the diagram

$$\begin{array}{ccc} Z/G \stackrel{p_1}{\leftarrow} & Z\times^G (G'/G) & \stackrel{\mathrm{act}}{\to} Z/G \\ & \downarrow p_2 & \\ & G\backslash G'/G & \end{array}$$

Given $F\in Shv(Z/G), K\in Shv(G\backslash G'/G)$ one may let $F*K=\mathrm{act}_*(p_1\times p_2)^!(F\boxtimes K)$. A similar convolution gives a monoidal structure on $Shv(G\backslash G'/G)$, so $Shv(G\backslash G'/G)\in Alg(\mathrm{DGCat}_{cont})$. This defines a right action of $Shv(G\backslash G'/G)$ on $Shv(Z/G)$.

But we actually mean a different normalization, which is well-adapted to the perverse t-structure on G'/G. Namely, assume now G' is a placid ind-scheme, G a placid group scheme closed in G'. For $F\in Shv(Z/G)$ and $K\in Shv(G'/G)^{\heartsuit}$ for the perverse t-structure, which is G-equivariant, we first define $F\widetilde{\boxtimes}K\in Shv(Z\times^G(G'/G))$ by the property that for the diagram of projections

$$(Z/G)\times(G'/G)\stackrel{\alpha}{\leftarrow} Z\times G'/G\stackrel{\beta}{\to} Z\times^G(G'/G)$$

and has $\alpha^*(F\boxtimes K)\widetilde{\to}\beta^*(F\widetilde{\boxtimes}K)$. Then we let

$$F*K\widetilde{\to}\mathrm{act}_*(F\widetilde{\boxtimes}K)$$

This definition is well-adapted to the case when G' is a placid ind-scheme, and G a placid group scheme, because the functors α^*,β^* are well-defined.

Let $p: G'/G\to G\backslash G'/G$ be the natural map. The functor p^* is well-defined. Our normalization is for $\mathcal{K}\in Shv(G\backslash G'/G), F\in Shv(Z/G)$ to let $F*\mathcal{K}=\mathrm{act}_*\,\bar p^*(F\boxtimes K)$, where $\bar p: Z\times^G(G'/G)\to(Z/G)\times(G\backslash G'/G)$ is the natural map.

1.5.2. For 9.2.2. The action of E' given by (14) in this file satisfies $(E'*F)*K\widetilde{\to}E'*(F*K)$. Thus, Ps-Id intertwines the desired actions.

1.5.3. For 9.2.3 and 9.2.4 the tensor product in the left columns makes no sense, write $\times$ instead.

Maybe instead of "proper push-forward" say ind-proper?

The explanation of the commutativity of the square: given $\mathcal{S}\in Shv_{q,x}(G)^c, K\in \mathrm{Whit}_{q,x}(G)^c, L\in\mathrm{Whit}_{q,x}(G)$, one has $\mathcal{H}om(K*\mathcal{S},L)\widetilde{\to}\mathcal{H}om(K,L*\mathbb{D}^{Verdier}\,\mathrm{inv}^G(\mathcal{S}))$. So,

$$\mathrm{R}\Gamma(\mathrm{Gr}_G, L\otimes^!\text{Ps-Id}^{-1}(\mathbb{D}(K*\mathcal{S})))\widetilde{\to}\mathcal{H}om(K*\mathcal{S},L)\widetilde{\to}\mathcal{H}om(K,L*(\mathbb{D}\,\mathrm{inv}^G(\mathcal{S})))$$

$$\widetilde{\to}\mathrm{R}\Gamma(\mathrm{Gr}_G,(L*(\mathbb{D}\,\mathrm{inv}^G(\mathcal{S})))\otimes^!\text{Ps-Id}^{-1}(\mathbb{D}K))\widetilde{\to}\mathrm{R}\Gamma(\mathrm{Gr}_G,L\otimes^!\text{Ps-Id}^{-1}((\mathbb{D}K)*\mathbb{D}(\mathcal{S})))$$

We used that $\mathbb{D}\,\mathrm{inv}^G(\mathcal{S})\widetilde{\to}\mathrm{inv}^G\,\mathbb{D}(\mathcal{S})$. This shows that $(\mathbb{D}K)*(\mathbb{D}\mathcal{S})\widetilde{\to}\mathbb{D}(K*\mathcal{S})$.

1.5.4. For 9.3.3. Recall that χ_N is an object in $Shv(\mathfrak{L}(N))^{\vee}$, so is χ_N^{λ}. This is not a problem, of course.

Line 8: we say "χ_N^{λ} descends...", this is not precise, because the map under which it descends is not indicated. Say that λ is arbitrary, and we consider the map $\mathfrak{L}(N)\to S^{\mu-\lambda}$, $z\mapsto zt^{\mu-\lambda}$. Our nondegenerate character $ev:\mathfrak{L}(N)\to\mathbb{A}^1$ descends under this map to a morphism $\bar{ev}: S^{\mu-\lambda}\to\mathbb{A}^1$, and we get the object $\chi_N^{\lambda}\in Shv(S^{\mu-\lambda})^{\vee}$ given by $\bar{ev}^*(\mathcal{L}_{\psi})$. Over each closed subscheme of $S^{\mu-\lambda}$ it is a true object (the corresponding functor is representable). We may also refer to ([17], 7.1.5) to explain this. So, χ^λ_N is the analog of the function denoted by $\chi^{\mu-\lambda}_\lambda$ in ([17], 7.1.5).

In (9.3) we again do not precise which restriction in $Sat_{q,G}(V)\mid_{S^{\mu-\lambda}}$ is meant!

By $\mathcal{F}\mid_{t^\mu}$ we denote the !-fibre in (9.3).

To verify the formula (9.3) it is easier to establish it for $Shv((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ first using the definition of the Hecke action from ([17], 5.3.6-5.3.8). We will see below that for $\pi_x : \mathrm{Gr}^{\omega^\rho}_{G,x} \to (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}$ the functor $\pi_x^!$ commutes with the right actions of $\mathrm{Rep}(H)$.

Let $\mathcal{H}_x$ be the Hecke stack classifying $(\mathcal{F}_G, \mathcal{F}'_G, \beta)$, where $\mathcal{F}_G, \mathcal{F}'_G$ are G-torsors on X, $\beta : \mathcal{F}_G \widetilde{\to} \mathcal{F}'_G$ is an isomorphism over $X - x$. We have the diagram of projections

$$\mathrm{Bun}_G \stackrel{h^{\leftarrow}}{\leftarrow} \mathcal{H}_x \stackrel{h^{\rightarrow}}{\to} \mathrm{Bun}_G$$

where $h^{\leftarrow}$ (resp., $h^{\rightarrow}$) sends the above point to $\mathcal{F}_G$ (resp., $\mathcal{F}'_G$). Set

$$Z = \mathcal{H}_x \times_{\mathrm{Bun}_G} (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x},$$

where we used the map $h^{\rightarrow}$ to define the fibre product. We have the projections

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x} \stackrel{'h^{\leftarrow}}{\leftarrow} Z \stackrel{'h^{\rightarrow}}{\to} (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}$$

extending the above projections $h^{\leftarrow}, h^{\rightarrow}$. Using these projections for $\mathcal{S} \in Sph_q(G)$ which is perverse on Gr_G and $\mathcal{T} \in Shv((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ one defines $(\mathcal{T}\widetilde{\boxtimes}\mathcal{S})^l, (\mathcal{T}\widetilde{\boxtimes}\mathcal{S})^r$ on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}$ as in ([17], 5.3.6).

For $\mathcal{S} \in Sph_q(G)$ which is perverse on Gr_G and $\mathcal{T} \in Shv((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ by definition

$$\mathcal{T} * \mathcal{S} = {'h}^{\leftarrow}_*((\mathcal{T}\widetilde{\boxtimes}\mathcal{S})^r)$$

We calculate the !-restriction of $W^{\lambda,*}_{glob} * Sat_{q,G}(V)$ to $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x}$. To do this apply ([17], Lemma 7.2.4). As in [17], we have the substacks $Z^{\mu,?}, Z^{?,\mu'}, Z^{\mu,\mu'}, Z^{\mu,\mu',\lambda}$ and so on given in ([17], 7.2.2), here $\mu, \mu' \in \Lambda$ and λ is dominant.

Let W^λ_{glob} be the perverse sheaf on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda x}$ for which $W^{\lambda,*}_{glob}$ is its $*$-extension. Then $W^{\lambda,*}_{glob} * Sat_{q,G}(V)$ is the $*$-extension of $W^\lambda_{glob} * Sat_{q,G}(V)$, so we are calculating the !-restriction to $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x}$ of $W^\lambda_{glob} * Sat_{q,G}(V)$. The map

$$'h^{\leftarrow} : Z^{\mu,\lambda} \to (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x} \tag{22}$$

is a fibration with fibre $S^{\lambda-\mu}$ by ([17], Lemma 7.2.4). That is, if we trivialize $\mathcal{F}_G$ over D_x then the resulting $(\mathcal{F}'_G, \beta)$ lies in $S^{\lambda-\mu}$. On the other hand, the fibre of $'h^{\rightarrow} : Z^{\mu,\lambda} \to (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda x}$ identifies with $S^{\mu-\lambda}$. Further, write $Z^{\mu,\lambda}_{\mathcal{F}_G}$ for the fibre of (22) over $(\mathcal{F}_G, \kappa)$.

We apply ([17], Lemma 7.2.7) to understand the composition

$$Z^{\mu,\lambda}_{\mathcal{F}_G} \hookrightarrow Z^{\mu,\lambda} \stackrel{'h^{\rightarrow}}{\to} (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda x} \stackrel{ev}{\to} \mathbb{A}^1$$

When we identify $Z^{\mu,\lambda}_{\mathcal{F}_G} \widetilde{\to} S^{\lambda-\mu}$ then the above function becomes $\bar{\chi}^{\lambda-\mu}_\mu$ in the notations of ([17], 7.2.7(2)).

So, the !-restriction to $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu x}$ of $W^\lambda_{glob} * Sat_{q,G}(V)$ is

$$W^\mu_{glob} \otimes \mathrm{R}\Gamma(S^{\lambda-\mu}, (*Sat_{q,G}(V)) \mid_{S^{\lambda-\mu}} \otimes (\bar{\chi}^{\lambda-\mu}_\mu)^* \mathcal{L}_\psi)[\langle \mu - \lambda, 2\check{\rho}\rangle]$$

for μ dominant and vanishes otherwise. Here $\bar{\chi}^{\lambda-\mu}_\mu : S^{\lambda-\mu} \to \mathbb{A}^1$ is the function sending $zt^{\lambda-\mu}G(\mathcal{O})$ to $\chi(\mu(t)z\mu(t)^{-1})$ for $z \in \mathfrak{L}(N)$, and by

$$(*Sat_{q,G}(V)) \mid_{S^{\lambda-\mu}}$$

we mean the !-restriction. Here $\mathcal{L}_\psi$ is the Artin-Schreier sheaf.

Thus, the !-fibre of $W^{\lambda,*} * Sat_{q,G}(V) \in \mathrm{Whit}_{q,x}(G)$ at t^μ is

$$\mathrm{R}\Gamma(S^{\lambda-\mu}, (*Sat_{q,G}(V)) \mid_{S^{\lambda-\mu}} \otimes (\bar{\chi}^{\lambda-\mu}_\mu)^* \mathcal{L}_\psi)[-\langle \lambda, 2\check{\rho}\rangle], \tag{23}$$

because the !-fibre of $\overset{\circ}{W}{}^\mu$ at t^μ is $e[-\langle\mu, 2\check{\rho}\rangle]$. As in Section 9.3.4 of the "small FLE" paper, it is easy to see that (23) isn placed in degrees $\geq \langle \mu, 2\check{\rho}\rangle$. This means that the functor $\_ * \mathrm{Sat}_q(V)$ is left t-exact.

We may also compare with the 2nd displayed formula on p. 747 of the published version of [17].

Let us calculate the $*$-restriction of $W^{\lambda,!}_{glob} * Sat_{q,G}(V)$ to $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu}$. The calculation is done in ([44], after Lemma 2.1.4). The answer is

$$W^\mu_{glob} \otimes \mathrm{R}\Gamma_c(S^{\lambda-\mu}, (*Sat_{q,G}(V)) \mid_{S^{\lambda-\mu}} \otimes (\chi^{\lambda-\mu}_\mu)^* \mathcal{L}_\psi)[\langle \lambda - \mu, 2\check{\rho}\rangle],$$

where $(*Sat_{q,G}(V)) \mid_{S^{\lambda-\mu}}$ denotes the $*$-restriction. Here

$$\chi^{\lambda-\mu}_\mu(zt^{\lambda-\mu}) = \chi(\mu(t)z\mu(t)^{-1})\chi^{\lambda-\mu}_\mu(t^{\lambda-\mu})$$

The complex

$$\mathrm{R}\Gamma_c(S^{\lambda-\mu}, (*Sat_{q,G}(V)) \mid_{S^{\lambda-\mu}} \otimes (\chi^{\lambda-\mu}_\mu)^* \mathcal{L}_\psi)[\langle \lambda - \mu, 2\check{\rho}\rangle]$$

is placed in cohomological degrees ≤ 0, this shows that $W^{\lambda,!}_{glob} * Sat_{q,G}(V)$ is placed in perverse degrees ≤ 0.

The same calculated is also done in ([48], Theorem 7.1.1).

Our $\mathrm{Gr}^{\omega^\rho}_{G,x}$ classifies $(\mathcal{F}_G, \eta)$, where $\mathcal{F}_G$ is a G-torsor on X, $\eta : \omega^\rho \to \mathcal{F}_G$ is an isomorphism over $X - x$. The map $\pi_x : \mathrm{Gr}^{\omega^\rho}_{G,x} \to (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}$ sends it to $(\mathcal{F}_G, \kappa)$, where κ is the collection of maps

$$\kappa^{\check{\lambda}} : \omega^{\langle \rho, \check{\lambda}\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}(\infty x)$$

for $\check{\lambda} \in \check{\Lambda}^+$ given as

$$\omega^{\langle \rho, \check{\lambda}\rangle} \to \mathcal{V}^{\check{\lambda}}_{\omega^\rho} \stackrel{\eta}{\to} \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}(\infty x)$$

We sometimes write $\mathrm{Gr}_G, G(\mathcal{O}), G(F)$ meaning actually their twists by ω^ρ. We identify $\mathrm{Gr}_G \widetilde{\to} G(F)/G(\mathcal{O})$ by the map sending $\mathcal{F}_G$ with a trivialization $\eta : \omega^\rho \to \mathcal{F}_G$ over $X - x$ to $\eta^{-1}\gamma \in G(F)/G(\mathcal{O})$ for a given $\gamma : \omega^\rho \widetilde{\to} \mathcal{F}_G \mid_{D_x}$.

The convolution diagram is Conv_G, it is the prestack classifying G-torsors $\mathcal{F}, \mathcal{F}'$ on X and isomorphisms $\eta_1 : \omega^\rho \widetilde{\to} \mathcal{F}' \mid_{X-x}$, $\eta_2 : \mathcal{F}' \widetilde{\to} \mathcal{F} \mid_{X-x}$. The map $m : \mathrm{Conv}_G \to \mathrm{Gr}_G$ sends this point to $(\mathcal{F}, \eta_2 \circ \eta_1)$.

Let $\widetilde{\mathrm{Conv}}_G$ be the prestack classifying a point of Conv_G as above together with a trivialization $\mu_1 : \omega^\rho \to \mathcal{F}' \mid_{D_x}$. Let $q : \widetilde{\mathrm{Conv}}_G \to \mathrm{Conv}_G$ be the map forgetting μ_1.

We identify $\widetilde{\mathrm{Conv}}_G \widetilde{\to} G(F) \times \mathrm{Gr}_G$ via the map sending the above collection to $(\eta_1^{-1}\mu_1 \in G(F), (\eta_2\mu_1, \mathcal{F}) \in \mathrm{Gr}_G)$. We identify

$$\mathrm{Conv}_G \widetilde{\to} G(F) \times^{G(\mathcal{O})} \mathrm{Gr}_G$$

as follows. Let $G(\mathcal{O})$ act on $G(F) \times \mathrm{Gr}_G$ so that $h \in G(\mathcal{O})$ sends $(g_1, gG(\mathcal{O}))$ to $(g_1 h^{-1}, hgG(\mathcal{O}))$. This gives via the above isomorphism an action of $G(\mathcal{O})$ on $\widetilde{\mathrm{Conv}}_G$. Namely, $h \in G(\mathcal{O})$ sends the above collection to the same collection with μ_1 replaced by $\mu_1 h^{-1}$.

Write $m_1 : \mathrm{Conv}_G \to \mathrm{Gr}_G$ for the map sending the above collection to $(\mathcal{F}'_G, \eta_1)$. We get the commutative diagram

$$\begin{array}{ccccc} \mathrm{Gr}_G & \stackrel{m_1}{\leftarrow} & \mathrm{Conv}_G & \stackrel{m}{\to} & \mathrm{Gr}_G \\ \downarrow \pi_x & & \downarrow & & \downarrow \pi_x \\ (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x} & \stackrel{'h^{\to}}{\leftarrow} & Z & \stackrel{'h^{\leftarrow}}{\to} & (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}, \end{array}$$

where both squares are cartesian. This diagram shows that for $\mathcal{T} \in Shv((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x}), \mathcal{S} \in Shv(\mathrm{Gr}_G)^{\mathfrak{L}^+(G)}$ one gets $\pi_x^!(\mathcal{T} * \mathcal{S}) \widetilde{\to} (\pi_x^! \mathcal{T}) * \mathcal{S}$.

If I pick a trivialization $\tau : \omega^\rho \widetilde{\to} \mathcal{F}_G \mid_{D_x}$ then our fibre of $'h^{\leftarrow}$ over $(\mathcal{F}_G, \kappa)$ identifies with Gr via the map sending a point of Z to $(\mathcal{F}'_G, \eta_2^{-1}\tau)$. This way we get the subscheme $S^{\lambda-\mu} \subset \mathrm{Gr}_G$ over which we integrate.

How can we get $S^{\mu-\lambda}$ as the fibre? Consider ${}^\lambda \mathrm{Conv}_G \subset \mathrm{Conv}_G$ given by imposing the condition that $(\mathcal{F}'_G, \eta_1) \in S^\lambda$. For a point of ${}^\lambda \mathrm{Conv}_G$, $\mathcal{F}'_G$ gets a B-structure $\mathcal{F}'_B$ on X together with an isomorphism $\mathcal{F}'_B \times_B T \widetilde{\to} \omega^\rho(-\lambda x)$ on X.

The scheme $m^{-1}(t^\mu)$ classifies $\mathcal{F}' \in \mathrm{Bun}_G$ and isomorphisms $\eta_1 : \omega^\rho \widetilde{\to} \mathcal{F}' \mid_{X-x}$, $\eta_2 : \mathcal{F}' \widetilde{\to} \omega^\rho(-\mu x) \mid_{X-x}$ such that $\eta_2\eta_1 : \omega^\rho \widetilde{\to} \omega^\rho(-\mu x) \mid_{X-x}$ is the identity.

For a point of ${}^\lambda \mathrm{Conv}_G \cap m^{-1}(t^\mu)$ pick any trivialization

$$\bar{\mu}_1 : \omega^\rho(-\lambda x) \widetilde{\to} \mathcal{F}'_B \mid_{D_x}$$

of B-torsors inducing the identity on the corresponding T-torsors and define μ_1 as the composition $\omega^\rho \stackrel{t^\lambda}{\to} \omega^\rho(-\lambda x) \stackrel{\bar{\mu}_1}{\to} \mathcal{F}'_B \mid_{D_x}$. Then

$$(\mathcal{F} = \omega^\rho(-\mu x), \eta_2\mu_1) \in S^{\mu-\lambda}$$

Here we view $(\eta_2\mu_1 : \omega^\rho \widetilde{\to} \mathcal{F} \mid_{X-x}) \in \mathrm{Gr}_G$ according to our above convention. This is how you Dennis wants to identify the fibre with $S^{\mu-\lambda}$, and gets the !-restriction $Sat_{q,G}(V) \mid_{S^{\mu-\lambda}}$ in (9.3).

Another idea to get (9.3). The inclusion $\mathrm{Whit}_{q,x}(G) \hookrightarrow Shv_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho})$ is a map of right $\mathrm{Rep}(H)$-modules, and in the constructible context has a left adjoint $\mathrm{Av}_!^{\mathfrak{L}(N)_x^{\omega^\rho}, \chi_N}$. Since $\mathrm{Rep}(H)$ is rigid, in the constructible context $\mathrm{Av}_!^{\mathfrak{L}(N)_x^{\omega^\rho}, \chi_N}$ is a strict morphism of $\mathrm{Rep}(H)$-modules, not just a left-lax morphism. So,

$$W^{\lambda,!} * Sat_{q,G}(V) \widetilde{\to} \mathrm{Av}_!^{\mathfrak{L}(N)_x^{\omega^\rho}, \chi_N}(\delta_{t^\lambda, \mathrm{Gr}} * Sat_{q,G}(V))[-\langle \lambda, 2\check{\rho} \rangle]$$

1.5.5. One more way to calculate the same expression. In the formula below we write $Sat(V)$ for brevity instead of $Sat_{q,G}(V)$. Recall that $i_\lambda : S^\lambda \hookrightarrow \mathrm{Gr}_G$ is the inclusion. Assume $V \in \mathrm{Rep}(H)^c$. We have

$$\mathcal{H}om(e, i^!_{t^\mu}(W^{\lambda,*} * Sat(V))) \widetilde{\to} \mathcal{H}om(\delta_{t^\mu}, W^{\lambda,*} * Sat(V)) \widetilde{\to}$$
$$\mathcal{H}om(\delta_{t^\mu} * Sat(V^*), (i_\lambda)_* \overset{\circ}{W}{}^\lambda) \widetilde{\to} \mathcal{H}om_{Shv(S^\lambda)}((i_\lambda)^*(\delta_{t^\mu} * Sat(V^*)), \overset{\circ}{W}{}^\lambda)$$

Recall that for our $\bar{\chi}^\lambda : S^\lambda \to \mathbb{A}^1$ one has

$$\overset{\circ}{W}{}^\lambda \widetilde{\to} (\bar{\chi}^\lambda)^! \mathcal{L}_\psi[2 - \langle 2\check{\rho}, \lambda\rangle]$$

Here $\bar{\chi}^\lambda$ is what is χ^λ_0 in "Whittaler patterns". The composition $S^{\lambda-\mu} \overset{t^\mu}{\to} S^\lambda \overset{\bar{\chi}^\lambda}{\to} \mathbb{A}^1$ equals $\chi^{\lambda-\mu}_\mu$. The isomorphism $S^{\lambda-\mu} \widetilde{\to} S^\lambda$ given by multiplication by t^μ identifies the above object of Vect with

$$\mathcal{H}om_{Shv(S^{\lambda-\mu})}(i^*_{\lambda-\mu}(Sat(V^*)), (\chi^{\lambda-\mu}_\mu)^! \mathcal{L}_\psi)[2 - \langle 2\check{\rho}, \lambda\rangle]$$

The latter identifies with the Verdier dual of

$$\mathrm{R}\Gamma_c(S^{\lambda-\mu}, i^*_{\lambda-\mu}(Sat(V^*)) \otimes (\chi^{\lambda-\mu}_\mu)^* \mathcal{L}_\psi^{-1})[\langle\lambda, 2\check{\rho}\rangle]$$

Recall that $\mathbb{D}(*Sat(V)) \widetilde{\to} Sat(V^*)$. Thus, the result becomes

$$\mathrm{R}\Gamma(S^{\lambda-\mu}, i^!_{\lambda-\mu}(*Sat(V)) \otimes (\chi^{\lambda-\mu}_\mu)^* \mathcal{L}_\psi)[-\langle\lambda, 2\check{\rho}\rangle]$$

We recovered my formula (23) from the previous section once again.

1.5.6. For 9.3.3. Here is the proof of (9.3) of Dennis up to shift. Recall that $i_\lambda : S^\lambda \hookrightarrow \mathrm{Gr}_G$ is the inclusion. Write N_{out} for the group of maps $X - x \to N^{\omega^\rho}$. Recall that N_{out} is an ind-scheme of ind-finite type, and it acts transitively on S^λ. We ignore the twist by ω^ρ in the notation below for this subsection.

Write χ_{out} for the composition $N_{out} \hookrightarrow N(F) \overset{\chi}{\to} \mathbb{A}^1$. Note that $N_{out} \cap t^\lambda N(\mathcal{O}) t^{-\lambda}$ is the global sections on X of the group scheme of automorphisms of the B-torsor $\omega^\rho(-\lambda x)$ acting trivially on the induced T-torsor. This is also the stabilizor of $t^\lambda \in \mathrm{Gr}_G$ in N_{out}. Let $n_\lambda = \dim N_{out} \cap t^\lambda N(\mathcal{O}) t^{-\lambda}$. If $\lambda \in \Lambda^+$ then $N_{out} \cap t^\lambda N(\mathcal{O}) t^{-\lambda} = N(M^\lambda)$ is the unipotent radical of the Borel of M^λ, where $M^\lambda \subset G$ is the standard Levi whose set of simple roots is the set of those $\check{\alpha}_i$ satisfying $\langle\lambda, \check{\alpha}_i\rangle = 0$.

First, we claim that

$$\chi^!_{out} \mathcal{L}_\psi * \delta_{t^\lambda} \widetilde{\to} W^{\lambda,*}[\langle 2\check{\rho}, \lambda\rangle - 2 + 2n_\lambda]$$

Indeed, it suffices to prove this over S^λ. Let $\bar{\alpha} : N_{out} \to \mathrm{Gr}_G$, $z \mapsto zt^\lambda$. By definition, $\chi^!_{out} \mathcal{L}_\psi * \delta_{t^\lambda} \widetilde{\to} \bar{\alpha}_* \chi^!_{out} \mathcal{L}_\psi$. Consider the commutative diagram

$$\begin{array}{ccc} N_{out} & \overset{\chi_{out}}{\to} & \mathbb{A}^1 \\ \downarrow \alpha & \nearrow \bar{\chi}^\lambda & \\ S^\lambda, & & \end{array}$$

where $\alpha(z) = zt^\lambda \in S^\lambda$, and $\bar{\chi}^\lambda(nt^\lambda) = \chi(n)$ for $n \in N(F)$. We have $\alpha_* \omega \widetilde{\to} \omega[2n_\lambda]$. So, over S^λ we get

$$\chi^!_{out} \mathcal{L}_\psi * \delta_{t^\lambda} \widetilde{\to} \alpha_* \chi^!_{out} \mathcal{L}_\psi \widetilde{\to} \alpha_* \alpha^! (\bar{\chi}^\lambda)^! \mathcal{L}_\psi \widetilde{\to} (\bar{\chi}^\lambda)^! \mathcal{L}_\psi[2n_\lambda],$$

and our formula follows.

For $V \in \mathrm{Rep}(H)$ and $F = Sat(V)$ consider $\chi^!_{out}\mathcal{L}_\psi * \delta_{t^\lambda} * F$. We have the associativity and can put the parenthesis as we like here. Write $N^\lambda_{out} = t^{-\lambda} N_{out} t^\lambda$. Let $\chi_{out,\lambda} : N^\lambda_{out} \to \mathbb{A}^1$ be the map sending $t^{-\lambda} z t^\lambda$ to $\chi(z)$ for $z \in N_{out}$. Write $a : \mathrm{Gr}_G \to \mathrm{Gr}_G$ for the multiplication by t^λ.

Next step is the isomorphism

$$\chi^!_{out}\mathcal{L}_\psi * \delta_{t^\lambda} * F \widetilde{\to} a_*(\chi^!_{out,\lambda}\mathcal{L}_\psi * F)$$

Now the !-fibre of this complex at t^μ is

$$i^!_{t^{\mu-\lambda}}(\chi^!_{out,\lambda}\mathcal{L}_\psi * F) \widetilde{\to} i^!_{t^{\mu-\lambda}} i^!_{\mu-\lambda}(\chi^!_{out,\lambda}\mathcal{L}_\psi * F)$$

By base change, we have

$$i^!_{\mu-\lambda}(\chi^!_{out,\lambda}\mathcal{L}_\psi * F) \widetilde{\to} i^!_{\mu-\lambda}(\chi^!_{out,\lambda}\mathcal{L}_\psi * F'),$$

where F' is the !-restriction of F to $S^{\mu-\lambda}$.

Note that the expression $\chi^!_{out,\lambda}\mathcal{L}_\psi * F'$ makes sense for any sheaf on $S^{\mu-\lambda}$, no equivariance condition on F' is needed. The object $\chi^*_{out,\lambda}\mathcal{L}_\psi$ makes sense naturally as an object of $Shv(N^\lambda_{out})^\vee$, or a projective system of local systems on subschemes of finite type, cf. Sect. 1.3.25.

We have $\chi^!_{out,\lambda}\mathcal{L}_\psi[2] \widetilde{\to} \omega_{N^\lambda_{out}} \otimes \chi^*_{out,\lambda}\mathcal{L}_\psi$, where the tensor product in the RHS is in the sense of Section 1.3.25.

Let $\alpha : N^\lambda_{out} \to S^{\mu-\lambda}$ be the map $z \mapsto z t^{\mu-\lambda}$.

Lemma 1.5.7. *For $F' \in Shv(S^{\mu-\lambda})$ one has $i^!_{\mu-\lambda}(\omega_{N^\lambda_{out}} * F') \widetilde{\to} (\alpha_*\omega) \otimes \mathrm{R}\Gamma(S^{\mu-\lambda}, F')$ canonically.*

Proof. The group N^λ_{out} acts transitively on $S^{\mu-\lambda}$. Write $N^\lambda_{out} = \mathrm{colim}_k N_k$, where N_k is a unipotent group of finite type. Then $\omega_{N^\lambda_{out}} = \mathrm{colim}_k \omega_{N_k}$ in $Shv(N^\lambda_{out})$. It suffices to prove our claim after !-restriction to $N_k t^{\mu-\lambda}$ for each k. Fix such k. We may assume that F' is the extension by zero under $N_{k'} t^{\mu-\lambda} \hookrightarrow S^{\mu-\lambda}$ for some k'. Moreover, we assume the stabilizor of $t^{\mu-\lambda}$ in N^λ_{out}, which is finite-dimensional, is contained in $N_{k'}$. We may and do assume $k > k'$. The inductive system $m \mapsto i^!_{N_k t^{\mu-\lambda}}(\omega_{N_m} * F')$ stabilizes starting from $m = k$, and it suffices to calculate the !-restriction of $(\omega_{N_k} * F')$ to $N_k t^{\mu-\lambda}$. We have $\omega_{N_k} = e_{N_k}[2 \dim N_k]$, where e_{N_k} is the constant sheaf on N_k, and the !-restriction of $e_{N_k} * F'$ to $N_k t^{\mu-\lambda}$ is $e_{N_k t^{\mu-\lambda}} \otimes \mathrm{R}\Gamma(F')$. So, for the map $\alpha_k : N_k \to N_k t^{\mu-\lambda}$, $z \mapsto z t^{\mu-\lambda}$ we get $((\alpha_k)_*\omega) \otimes \mathrm{R}\Gamma(F') \widetilde{\to} i^!_{N_k t^{\mu-\lambda}}(\omega_{N_m} * F')$ for all $m \geq k$, so $((\alpha_k)_*\omega) \otimes \mathrm{R}\Gamma(F') \widetilde{\to} i^!_{N_k t^{\mu-\lambda}}(\omega_{N^\lambda_{out}} * F')$. The claim follows, as $N^\lambda_{out} \times_{S^{\mu-\lambda}} N_k t^{\mu-\lambda} \to N_k t^{\mu-\lambda}$ identifies with α_k. □

A version of the above lemma with a character is as follows. Recall that for $\mu \in \Lambda^+$, $\chi^{\mu-\lambda}_\lambda$ is the composition $S^{\mu-\lambda} \xrightarrow{t^\lambda} S^\mu \xrightarrow{\bar{\chi}^\mu} \mathbb{A}^1$. This is the map sending $z t^{\mu-\lambda}$ to $\chi(t^\lambda z t^{-\lambda})$ for $z \in N(F)$. Then $\chi^{\mu-\lambda}_\lambda \alpha = \chi_{out,\lambda}$.

Lemma 1.5.8. *For $F \in Shv(S^{\mu-\lambda})$ and $\mu \in \Lambda^+$ one has canonically*

$$i^!_{\mu-\lambda}(\chi^!_{out,\lambda}\mathcal{L}_\psi) * F) \widetilde{\to} (\alpha_*\alpha^!(\chi^{\mu-\lambda}_\lambda)^!\mathcal{L}_\psi) \otimes \mathrm{R}\Gamma(S^{\mu-\lambda}, F \otimes^! ((\chi^{\mu-\lambda}_\lambda)^! \mathbb{D}\mathcal{L}_\psi))$$

Proof. This follows from the general claim below. □

Lemma 1.5.9. *Let* $N \widetilde{\to} \operatorname{colim}_{i\in I} N_i$*, where* N_i *is a smooth group scheme of finite type,* I *is small filtered, for* $i \to j$ *in* I *the map* $N_i \to N_j$ *is a closed subgroup scheme. Let* S *be a* N*-homogeneous space with a* k*-point* $s \in S$*, whose stabilizor in* N *is a closed unipotent subgroup of finite type. Let* $\chi : S \to H$ *be a map, where* H *is a group scheme of finite type. Let* $\mathcal{E}$ *be a character local system on* H*. Let* $q : N \to S, z \mapsto zs$ *and* $F \in Shv(S)$*. Assume* χq *is a homomorphism of group ind-schemes. Then*

$$q^!\chi^!\mathcal{E} * F \widetilde{\to} (q_*(q^!\chi^!\mathcal{E})) \otimes \mathrm{R}\Gamma(S, (\chi^!\mathbb{D}\mathcal{E}) \otimes^! F)$$

Proof. We may and do assume that F is the extension by zero under $N_k s \hookrightarrow S$ for some fixed k, as both sides are continuous functors of F. We may aslo assume the stabilizor of s in N is contained in N_k.

Let $i_i : N_i \hookrightarrow N$ be the embedding. Let $N_i \stackrel{q_i}{\to} N_i s \stackrel{\chi_i}{\to} H$ be obtained by restriction. Then $q_i^*\chi_i^*\mathcal{E}$ is a character local system on N_i, and $q_i^!\chi_i^!\mathcal{E} \widetilde{\to} q_i^*\chi_i^*\mathcal{E}[2\dim N_i - 2\dim H]$. From the cartesian square

$$\begin{array}{ccc} N_i \times N_i & \stackrel{m}{\to} & N_i \\ \downarrow \mathrm{id} \times q_i & & \downarrow q_i \\ N_i \times N_i s & \stackrel{\mathrm{act}}{\to} & N_i s \end{array}$$

For $i \geq k$ we see that

$$(q_i^*\chi_i^*\mathcal{E}) * F \widetilde{\to} \chi_i^*\mathcal{E} \otimes \mathrm{R}\Gamma(N_i s, F \otimes \chi_i^*\mathcal{E}^{-1}) \widetilde{\to} \chi_i^*\mathcal{E} \otimes \mathrm{R}\Gamma(N_i s, (\chi_i^!\mathbb{D}\mathcal{E}) \otimes^! F)$$

We get $\mathrm{R}\Gamma(S, (\chi^!\mathbb{D}\mathcal{E}) \otimes^! F) \widetilde{\to} \mathrm{R}\Gamma(N_i s, (\chi_i^!\mathbb{D}\mathcal{E}) \otimes^! F)$, and the above isomorphism becomes

$$(q_i^!\chi_i^!\mathcal{E}) * F \widetilde{\to} ((q_i)_*\omega_{N_i} \otimes^! \chi_i^!\mathcal{E}) \otimes \mathrm{R}\Gamma(S, (\chi^!\mathbb{D}\mathcal{E}) \otimes^! F) \widetilde{\to} ((q_i)_*(q_i^!\chi_i^!\mathcal{E})) \otimes \mathrm{R}\Gamma(S, (\chi^!\mathbb{D}\mathcal{E}) \otimes^! F)$$

Since $q^!\chi^!\mathcal{E} \widetilde{\to} \operatorname{colim}_{i\in I} (i_i)_! i_i^! q^!\chi^!\mathcal{E} \widetilde{\to} \operatorname{colim}_{i\in I} q_i^!\chi_i^!\mathcal{E}$ in $Shv(N)$, we get passing to the colimit over i

$$q^!\chi^!\mathcal{E} * F \widetilde{\to} (q_*(q^!\chi^!\mathcal{E})) \otimes \mathrm{R}\Gamma(S, (\chi^!\mathbb{D}\mathcal{E}) \otimes^! F)$$

□

Combining with the above we obtain the following.

Proposition 1.5.10. *For* $V \in \mathrm{Rep}(H)$*,* $\lambda \in \Lambda^+$*,* $\mu \in \Lambda$ *the complex* $i_{t^\mu}^!(W^{\lambda,*} * Sat_q(V))$ *vanishes unless* $\mu \in \Lambda^+$*. In the latter case it identifies with*

$$\mathrm{R}\Gamma(S^{\mu-\lambda}, (i_{\mu-\lambda}^! Sat_q(V)) \otimes^! (\chi_\lambda^{\mu-\lambda})^! \mathbb{D}\mathcal{L}_\psi)[-\langle \lambda, 2\check{\rho}\rangle - 2n_\lambda + 2n_\mu].$$

□

For λ, μ dominant in the above Proposition we get $n_\lambda = \dim N(M^\lambda)$, $n_\mu = \dim N(M^\mu)$.

1.5.11. Note the following. Let $\lambda\in\Lambda^+,\mu\in\Lambda$, $V\in\operatorname{Rep}(H)^\heartsuit$ finite-dimensional. Using the notations of (23), since $W^{\lambda,*}*(*Sat(V))$ is not compact in $Shv(\mathrm{Gr}_G)$,

$$i^!_{t^\mu}(W^{\lambda,*}*(*Sat(V)) \tag{24}$$

does not identify with

$$\mathcal{H}om_{\mathrm{Vect}}(i^*_{t^\mu}(W^{\lambda,!}*Sat(V^*)),e)\widetilde{\to}\mathcal{H}om_{Shv(\mathrm{Gr}_G)}(W^{\lambda,!}*Sat(V^*),\delta_{t^\mu})$$
$$\widetilde{\to}\mathcal{H}om_{\mathrm{Whit}_{q,x}(G)}(W^{\lambda,!}*Sat(V^*),\mathrm{Av}_*^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\delta_{t^\mu})).$$

In fact, the latter complex vanishes by lemma below.

Lemma 1.5.12. *For any* $\mu\in\Lambda$, $\mathrm{Av}_*^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\delta_{t^\mu})=0$.

Proof. It suffices to show that $\mathcal{H}om(W^{\nu,!},\mathrm{Av}_*^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\delta_{t^\mu}))=0$ for all $\nu\in\Lambda^+$. This is clear for $\nu\neq\mu$. For $\nu=\mu$ we show that $\mathrm{Hom}_{Shv(S^\mu)}(\omega_{S^\mu},\delta_{t^\mu})=0$.

Indeed, let $N(F)=\mathrm{colim}_k N_k$, where N_k is a pro-unipotent group scheme, and each $N_k\to N_{k'}$ is a placid closed immersion. For each k consider $N_kt^\mu\subset S^\mu$. Then $\omega_{S^\mu}=\mathrm{colim}_k\,\omega_{N_kt^\mu}$, and $\omega_{N_kt^\mu}=e[2\dim(N_kt^\mu)]$, so $\mathrm{Hom}(\omega_{N_kt^\mu},\delta_{t^\mu})\widetilde{\to}e[-2\dim(N_kt^\mu)]$. Thus, $\mathrm{Hom}_{Shv(S^\mu)}(\omega_{S^\mu},\delta_{t^\mu})=\lim_k\mathrm{Hom}(\omega_{N_kt^\mu},\delta_{t^\mu})=0$.

Since $\overset{\circ}{W}{}^\mu\widetilde{\to}(\bar\chi^\mu)^*\mathcal{L}_\psi\otimes\omega_{S^\mu}$ up to a shift, we get $\mathrm{Hom}_{Shv(S^\mu)}(\overset{\circ}{W}{}^\mu,\delta_{t^\mu})=0$. □

1.5.13. For 9.4.2. We may introduce $\Lambda^\sharp_0=\{\lambda\in\Lambda^\sharp\mid\langle\lambda,\check\alpha_i\rangle=0\text{ for all }i\}$. Then we may add in Lemma 9.4.2 that for $\lambda\in\Lambda^+$ the elements μ,γ are defined uniquely up to an action of $\Lambda^\sharp_0$.

1.5.14. For 9.4.6. When you write in line 5 "expression"

$$W^{\lambda,*}*Sat_{q,G}(V^\gamma)\mid_{t^\mu},$$

precise that this is the !-fibre.

1.5.15. For 9.5.1. For the convenience of the reader, recall the following. Let A be a torsion abelian group whose elements are of orders coprime to *chark*. To describe the multiplicative A-torsors on T (also known as Kummer local systems), we have to analyse

$$\mathrm{Map}_{\mathcal{G}rp(\mathrm{PreStk})}(T,B_{et}(A))\widetilde{\to}\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(T),B^2_{et}(A))$$

This is the relative cohomology $\mathrm{Map}_{\mathrm{PreStk}}(B(T),B^2_{et}(A))\times_{\mathrm{Map}_{\mathrm{PreStk}}(*,B^2_{et}(A))}*$. Let $q:*\to B(T)$ be the natural map in PreStk. Define K by the fibre sequence $K\to A\to q_*A$ in the corresponding stable category of sheaves on $B(T)$. The corresponding long exact sequence in cohomology gives $0\to\mathrm{H}^2_{et}(B(T),K)\to\mathrm{H}^2_{et}(B(T),A)\to0$. The map in the middle is an isomorphism, so

$$\mathrm{H}^2_{et}(B(T),K)\widetilde{\to}\mathrm{Hom}(\Lambda,A(-1))$$

by ([31], Th. 3.2.6). So,

$$\pi_0\,\mathrm{Map}_{\mathcal{G}rp(\mathrm{PreStk})}(T,B_{et}(A))\widetilde{\to}\mathrm{Hom}(\Lambda,A(-1))$$

If $\mathcal{G}$ is an A-gerbe over $*$, to provide its descent datum under the map $*\to B(T)$ means essentially to provide a point of $\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(T),B^2_{et}(A))$. Indeed, we may assume our gerbe on $*$ trivial. The corresponding multiplicative A-torsor on T is obtained as follows: we have $\Omega B(T)\widetilde{\to}T$. So, for $h : T\to *$ we get an automorphism of $h^*\mathcal{G}$, which is given by a A-torsor on T.

If $\mathcal{E} : T\to B_{et}(A)$ is a Kummer local system corresponding to $f:\Lambda\to A(-1)$ then for $\lambda:\mathbb{G}_m\to T$ with $\lambda\in\Lambda$ its restriction to $\mathbb{G}_m$ is a local system corresponding to $f(\lambda)\in A(-1)$. Namely, for $n\geq 1$ coprime with $chark$, $\mathbb{G}_m\to\mathbb{G}_m$, $z\mapsto z^n$ defines a map $\mathbb{G}_m\to B_{et}(\mu_n)$, and an element $f\in\operatorname{Hom}(\mu_n, A)$ allows to compose it with $B_{et}(\mu_n)\to B_{et}(A)$.

1.5.16. For 9.5.2, line 2: $\chi^{\mu}_0 : S^{\mu}\to\mathbb{G}_a$ is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ by multiplication by scalars and on S^{μ} via $\rho:\mathbb{G}_m\to T$ and the T-action. Here $\chi^{\mu}_0(zt^{\mu})=\chi(z)\chi^{\mu}_0(t^{\mu})$.

In line 3 we consider the push-forward of Ψ_q under $S^{\mu}\cap S^{-,\nu}\to\mathbb{G}_a$, but it is better to say here along $b\to\mathbb{G}_a$, where $b\subset S^{\mu}\cap S^{-,\nu}$ is a given irreducible component. Recall here the following. Let $ev : S^{\mu}\to\mathbb{A}^1$ be the map sending $zt^{\mu}G(\mathcal{O})$ to $\chi(z)$, where $\chi:\mathfrak{L}(N)^{\omega^\rho}_x\to\mathbb{A}^1$ is our nondegenerate character. Then for any irreducible component b of $S^{\mu}\cap S^{-,\nu}$, the map $ev : b\to\mathbb{A}^1$ is dominant?? For this ([25], Section 5.6) is not sufficient.

If Ψ_q is trivial on a given irreducible component $b\subset S^{\mu}\cap S^{-,\nu}$ then $\mu-\nu\in\Lambda^{\sharp}$, because Ψ_q is twisted T-equivariant under the Kummer local system corresponding to $b(\mu-\nu,\cdot):\Lambda\to e^*(-1)$.

Let $\chi^{\mu}_0 : S^{\mu}\to\mathbb{A}^1$ be the map sending $zt^{\mu}G(\mathcal{O})$ to $\chi(z)$, where $\chi:\mathfrak{L}(N)^{\omega^\rho}_x\to\mathbb{A}^1$ is our nondegenerate character. Let $\mu\neq\nu$ and b be an irreducible component of $S^{\mu}\cap S^{-,\nu}$. If the map $\chi^{\mu}_0 : b\to\mathbb{A}^1$ is dominant then $\chi^0_N\otimes\Psi_q$ is nontrivial on b. Indeed, for $\chi^{\mu}_0 : b\to\mathbb{A}^1$ the complex $((\chi^{\mu}_0)_!\Psi_q)\otimes\mathcal{L}_{\psi}$ in usual degree $\langle\mu-\nu,2\check{\rho}\rangle-2$ is twisted $\mathbb{G}_m$-equivariant under the Kummer local system corresponding to $b(\mu-\nu,\rho)\in e^*(-1)$. This Kummer local system can not be $\mathcal{L}_{\psi}$ at the generic point. So, $\mathrm{H}^{\langle\mu-\nu,2\check{\rho}\rangle-2}(\mathcal{L}_{\psi}\otimes(\chi^{\mu}_0)_!\Psi_q)$ is nontrivial over the generic point of $\mathbb{A}^1$. So, $\mathrm{H}^{\langle\mu-\nu,2\check{\rho}\rangle}_c(b,\chi^0_N\otimes\Psi_q)=0$.

This shows that if $\chi^0_N\otimes\Psi_q$ is trivial on b then both χ^0_N, Ψ_q are trivial on b.

By ([48], Lemma 4.12.4), for an irreducible component $b\subset S^{\mu}\cap S^{-,\nu}$ the map $\chi^{\mu}_0 : \bar{b}\to\mathbb{A}^1$ is dominant iff there is a vertex i of the Dynkin diagram such that $\phi_i(\bar{b})>\langle\mu,\check{\alpha}_i\rangle$.

1.6. For Section 10.

1.6.1. For 10.1, last line. I propose to add that H is of finite type, this is the only needed case, right? Recall that $\operatorname{Rep}(H)$ is rigid.

Namely, by [7], $B(H)$ is perfect in the sense of [GR1, ch. I.3, 3.6], hence also passable by [GR1, ch. I.3, 3.5], hence rigid by [GR1, ch. I.3, 3.4].

1.6.2. For 10.1.2. Refer to [GR1, Chapter 1, 9.3.3] for the existence of the continuous right adjoint Ψ_{univ} to $C\otimes D\to C\otimes_{\operatorname{Rep}(H)}D$. The fact that Ψ_{univ} is conservative follows from the fact that the essential image of $C\otimes D\to C\otimes_{\operatorname{Rep}(H)}D$ generates $C\otimes_{\operatorname{Rep}(H)}D$ under colimits by [GR1, ch. 1, Lemma 5.4.3]. The generation claim is [GR1, ch. 1, Lemma 8.2.6].

1.6.3. For 10.1.2. By [GR1, ch. I.1, 3.7.7], the functor $\Psi_{univ}: C\otimes_{\mathrm{Rep}(H)} D\to C\otimes D$ is monadic. The standard totalization complex representing

$$\mathrm{Fun}_{\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)}(\mathrm{Rep}(H), C\otimes D)\widetilde{\to} C\otimes_{\mathrm{Rep}(H)} D$$

gives the description of $C\otimes_{\mathrm{Rep}(H)} D$ from 10.1.2.

For the diagonal map $\vartriangle: B(H)\to B(H)\times B(H)$ one may define $\mathrm{Reg}(H)\in\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)\widetilde{\to}\mathrm{QCoh}(B(H)\times B(H))$ as $\vartriangle_*\mathcal{O}$. The functor $\vartriangle_*$ is right-lax symmetric monoidal, so sends algebras to algebras. So, $\vartriangle_*\mathcal{O}$ is an algebra.

Consider the adjoint pair $m:\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)\rightleftarrows\mathrm{Rep}(H): m^R$. Applying $\mathrm{Fun}_{\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)}(\bullet, C\otimes D)$, we get the adjoint pair

$$\Phi_{univ}: C\otimes D\leftrightarrows C\otimes_{\mathrm{Rep}(H)} D:\Psi_{univ}$$

Now $m^R\circ m$ is the monad on $A\otimes A$ for $A=\mathrm{Rep}(H)$ given by the action of the algebra $m^R(1)$. These results hold for any symmetric monoidal rigid $A\in CAlg(\mathrm{DGCat}_{cont})$.

1.6.4. For 10.1.4. If C, D are compactly generated then $C\otimes D$ is compactly geberated by objects of the form $c\boxtimes d$ with $c\in C^c, d\in D^c$ by ([20], ch. I.1, 7.4.2). By ([20], ch. I.1, 7.1.5), the functor $C\otimes D\to C\otimes_{\mathrm{Rep}(H)} D$ sends compact objects to compact ones, so $C\otimes_{\mathrm{Rep}(H)} D$ is compactly generated by the above.

1.6.5. For 10.1.5. Let $C\in\mathrm{Rep}(H)-mod^r, D\in\mathrm{Rep}(H)-mod$. Assume C, D dualizable in DGCat_{cont}. Then we have a canonical isomorphism

$$D^\vee\otimes_{\mathrm{Rep}(H)} C^\vee\widetilde{\to}(C\otimes_{\mathrm{Rep}(H)} D)^\vee$$

by ([20], ch. I.1, 9.5.4). Namely, since $\mathrm{Rep}(H)$ is symmetric monoidal, the corresponding automorphism denoted ϕ in *loc.cit* is the identity.

The fact that the adjoint pair $\Phi_{univ}: D^\vee\otimes C^\vee\leftrightarrows D^\vee\otimes_{\mathrm{Rep}(H)} C^\vee:\Psi_{univ}$ is obtained by passing to the duals in the adjoint pair $C\otimes D\leftrightarrows C\otimes_{\mathrm{Rep}(H)} D$ follows from ([43], 9.2.37) for example.

1.6.6. For 10.1.7. Let $C, D\in\mathrm{DGCat}_{cont}$ be compactly generated equipped with t-structures. Assume the t-structures compactly generated for C, D (see Sect. 6.3.8 of the paper). That is, $\mathrm{Ind}(C^c\cap C^{\le 0})\widetilde{\to} C^{\le 0}$ naturally. By ([36], 1.4.4.11) this implies that the t-structure is accessible, that is, $C^{\le 0}$ is presentable.

We equip $C\otimes D$ with the t-structure declaring $(C\otimes D)^{\le 0}$ to be the smallest full subcategory containing $c\boxtimes d$ for $c\in C^c\cap C^{\le 0}, d\in D^c\cap D^{\le 0}$, closed under extensions and small colimits. This is indeed an accessible t-structure by ([36], 1.4.4.11). Moreover, $C\otimes D$ is compactly generated by objects of the form $c\boxtimes d$ with $c\in C^c, d\in D^c$ by ([20], ch. I.1, 7.4.2).

By ([43], Section 9.3) the t-structure on $C\otimes D$ is compactly generated. Moreover, the t-structure on C and on D are compatible with filtered colimits.

By ([20], ch. I.3, 3.6.4), $V\in\mathrm{Rep}(H)$ is perfect iff its $*$-restriction under $q:\mathrm{Spec}\, k\to B(H)$ is compact in Vect, that is, bounded with finite-dimensional cohomologies. Besides, $V\in\mathrm{Rep}(H)$ is pefect iff V is compact by ([20], ch. I.1, Sect. 9). The functor q^* is t-exact. Clearly, the truncation functors on $\mathrm{Rep}(H)$ preserve $\mathrm{Rep}(H)^c$. Besides, the t-structure on $\mathrm{Rep}(H)$ is compatible with filtered colimits by ([20], ch. I.3, 1.5.7).

So, the t-structure on $\mathrm{Rep}(H)$ is compactly generated and coherent. Moreover, the product functor $m : \mathrm{Rep}(H) \otimes \mathrm{Rep}(H) \to \mathrm{Rep}(H)$ is t-exact.

Assume H affine. Then the right adjoint $m^R : \mathrm{Rep}(H) \to \mathrm{Rep}(H) \otimes \mathrm{Rep}(H)$ is also t-exact. Indeed, this follows by base change as H is affine: for any S a classical affine scheme, $\mathrm{R}\Gamma : \mathrm{QCoh}(S) \to \mathrm{Vect}$ is t-exact.

If H is reductive then the t-structure on $\mathrm{Rep}(H)$ is Artinian (as $char(e) = 0$).

The shortest way to get the desired claim about t-structure on $C \otimes_{\mathrm{Rep}(H)} D$ is to refer to ([43], Lemma 9.3.11) using the presentation $C \otimes_{\mathrm{Rep} H} D \widetilde{\to} \mathrm{Reg}(H) - mod(C \otimes D)$.

The t-structure on $C \otimes_{\mathrm{Rep}(H)} D$ can also be defined by ([36], 1.4.4.11). Namely,

$$(C \otimes_{\mathrm{Rep}(H)} D)^{\leq 0} \subset C \otimes_{\mathrm{Rep}(H)} D$$

is the smallest full subcategory containing $\Phi_{univ}(c \boxtimes d)$ for $c \in C^c \cap C^{\leq 0}, d \in D^{\leq 0} \cap D^c$ stable under colimits and extensions. We see that the t-structure on $C \otimes_{\mathrm{Rep}(H)} D$ is accessible. One gets immediately that $(C \otimes_{\mathrm{Rep}(H)} D)^{>0} = \Psi_{univ}^{-1}((C \otimes D)^{>0})$. Besides, the t-structure on $C \otimes_{\mathrm{Rep}(H)} D$ is compactly generated by construction: for $c \in C^c \cap C^{\leq 0}, d \in D^{\leq 0} \cap D^c$ the object $\Phi_{univ}(c \boxtimes d)$ is compact in $C \otimes_{\mathrm{Rep}(H)} D$.

Assume the action functors $a : C \otimes \mathrm{Rep}(H) \to C$, $b : \mathrm{Rep}(H) \otimes D \to D$ are t-exact. Why both Φ_{univ} and Ψ_{univ} are t-exact?

We may try to apply ([20], ch. I.3, 1.5.8). Namely, via the usual bar construction write $V \otimes_{\mathrm{Rep}(H)} D \widetilde{\to} \underset{[n] \in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, C \otimes \mathrm{Rep}(H)^{\otimes n} D$. Passing to right adjoint, this rewrites as

$$\lim_{[n] \in \boldsymbol{\Delta}} C \otimes \mathrm{Rep}(H)^{\otimes n} D$$

Let $a^R : C \to C \otimes \mathrm{Rep}(H)$, $b^R : D \to \mathrm{Rep}(H) \otimes D$ be the right adjoints to a, b. Recall that a^R, b^R are continuous by ([20], ch. I.1, 9.3.2). By ([43], Remark 10.1.6), they are left t-exact. Recall also that there is an explicit formula for a^R given in ([20], ch. I.1, 9.3.2). Let $A = \mathrm{Rep}(H)$. Then a^R is the composition

$$\mathrm{Vect} \otimes C \overset{\mu \otimes \mathrm{id}}{\to} A \otimes A \otimes C \overset{\mathrm{id} \otimes \mathrm{act}}{\to} A \otimes C$$

Here μ is the unit for the self-duality on the rigid symmetric monoidal category A as in ([20], ch. I.1, 9.2.1). So, to check that a^R is right t-exact, it is sufficient to check that $\mu : \mathrm{Vect} \to A \otimes A$ is right t-exact in view of ([43], 9.3.10). This is true, because μ is the composition

$$\mathrm{Vect} \overset{1_A}{\to} A \overset{m^R}{\to} A \otimes A,$$

and both functors here are t-exact. The first is the pull-back along smooth map $B(H) \to \mathrm{Spec}\, k$.

A better idea. Recall that for $A = \mathrm{Rep}(H)$ the functor $m^R : A \to A \otimes A$ is t-exact, so $\mathrm{Reg} = m^R(1) \in (A \otimes A)^{\heartsuit}$. View $C \otimes D$ as a $A \otimes A$-module, and consider the projection $\Psi_{univ} : \mathrm{Reg} - mod(C \otimes D) \to C \otimes D$. Its left adjoint $\Phi_{univ} : C \otimes D \to \mathrm{Reg} - mod(C \otimes D)$ sends z to $\mathrm{Reg} * z$ with its natural Reg-module structure.

The tensor product of actions $A \otimes A \otimes C \otimes D \to C \otimes D$,

$$(a_1 \boxtimes a_2 \boxtimes c \boxtimes d) \mapsto (a_1 * c) \boxtimes (a_2 * d)$$

is t-exact by ([43], 9.3.10). Since the functor $\Psi_{univ}\Phi_{univ}$ is t-exact, we apply ([43], Lemma 9.3.11) and get the desired t-structure on $C\otimes_{\mathrm{Rep}(H)} D\widetilde{\to}\mathrm{Reg}-mod(C\otimes D)$.

1.6.7. For 10.2.2. Say a more standard thing here: for a morphism $A\to B$ in $Alg(\mathrm{DGCat}_{cont})$, $M\in A-mod$, $N\in B-mod$ we get $\mathrm{Fun}_A(M,N)\widetilde{\to}\mathrm{Fun}_B(B\otimes_A M,N)$ by adjointness.

1.6.8. For 10.2.5. We must assume H affine of finite type here, as we need $B(H)$ to be 1-affine in the sense of ([22], Th. 2.2.2).

Note that $\mathrm{QCoh}(H)$ is naturally a coalgebra in DGCat_{cont}, and we define the category of categories "acted on by H" as $\mathrm{QCoh}(H)-comod(\mathrm{DGCat}_{cont})$, as in ([22], Section 10.2.1). Moreover, by ([22], Section 10.2.1) one has the equivalence

$$\mathrm{Rep}(H)-mod\widetilde{\to}\mathrm{QCoh}(H)-comod \tag{25}$$

The map of coalgebras $\mathrm{Vect}\to\mathrm{QCoh}(H)$, $e\mapsto\mathcal{O}_H$ defines an augmentation of the coalgebra $\mathrm{QCoh}(H)$, and the corresponding cobar complex co-$\mathrm{Bar}^\bullet(\mathrm{QCoh}(H))$ is obtained from $[\dots H^2 \substack{\to\\ \to\\ \to} H \rightrightarrows pt]$ by applying the functor $\mathrm{QCoh}(\cdot)$. We get

$$\mathrm{Tot}(\text{co-Bar}^\bullet(\mathrm{QCoh}(H)))\widetilde{\to}\mathrm{QCoh}(B(H)),$$

as $\mathrm{QCoh}:\mathrm{PreStk}^{op}\to\mathrm{DGCat}_{cont}$ preserves limits. According to ([43], Section 3.3.1), this totalization gets a structure of an augmented algebra in DGCat_{cont}. Is this the pointwise tensor product on $B(H)$? I think so, because in ([22], 10.2.2) the obtained category $\mathrm{QCoh}(B(H))$ is denoted $\mathrm{Rep}(H)$.

The equivalence (25) sends $\mathrm{Vect}\in\mathrm{QCoh}(H)-comod$ to $\mathrm{QCoh}(B(H))\in\mathrm{Rep}(H)-mod$. More generally, it sends $M\in\mathrm{QCoh}(H)-comod$ to

$$\mathrm{Tot}(\text{co-Bar}^\bullet(\mathrm{QCoh}(H),M))\in\mathrm{QCoh}(B(H))-mod$$

In particular, it sends $\mathrm{QCoh}(H)\in\mathrm{QCoh}(H)-comod$ to $\mathrm{Vect}\in\mathrm{Rep}(H)-mod$.

Note also that $\mathrm{QCoh}(H)-comod(\mathrm{DGCat}_{cont})$ is naturally an $(\infty,2)$-category. Recall that $\mathrm{QCoh}(H)$ is naturally a commutative Hopf algebra in the sense of ([22], Appendix E), the algebra structure is given by the pointwise tensor product $\mathrm{QCoh}(H)\otimes\mathrm{QCoh}(H)\to\mathrm{QCoh}(H\times H)\overset{\triangle^*}{\to}\mathrm{QCoh}(H)$ for $\triangle\colon H\to H\times H$. The coalgebra structure is given by

$$m^*:\mathrm{QCoh}(H)\to\mathrm{QCoh}(H\times H)\widetilde{\to}\mathrm{QCoh}(H)\otimes\mathrm{QCoh}(H)$$

for the product $m:H\times H\to H$. For this reason, $\mathrm{QCoh}(H)-comod(\mathrm{DGCat}_{cont})$ gets a monoidal structure: for $C,D\in\mathrm{QCoh}(H)-comod(\mathrm{DGCat}_{cont})$, $C\otimes D$ is a $\mathrm{QCoh}(H)\otimes\mathrm{QCoh}(H)$-comodule, and the $\mathrm{QCoh}(H)$-comodule structure is given by the extension of scalars via the map of coalgebras $\mathrm{QCoh}(H)\otimes\mathrm{QCoh}(H)\overset{\triangle^*}{\to}\mathrm{QCoh}(H)$. That is, $C\otimes D$ is equipped with the composition map

$$C\otimes D\to C\otimes D\otimes\mathrm{QCoh}(H)\otimes\mathrm{QCoh}(H)\overset{\mathrm{id}\otimes\triangle^*}{\to}C\otimes D\otimes\mathrm{QCoh}(H)$$

We also have the antipode $\mathrm{inv}^*:\mathrm{QCoh}(H)\widetilde{\to}\mathrm{QCoh}(H)$ for the inversion map $\mathrm{inv}:H\to H$. The unit object of $\mathrm{QCoh}(H)-comod$ is Vect.

On the other hand, $\mathrm{Rep}(H) - mod$ also has a symmetric monoidal structure, as $\mathrm{QCoh}(B(H))$ is symmetric monoidal. This is the symmetric monoidal structure of $ShvCat(B(H))$. My understanding is that it corresponds to the above symmetric monoidal structure on $\mathrm{QCoh}(H) - comod$ via the equivalence (25).

For $C, D \in \mathrm{QCoh}(H){-}comod(\mathrm{DGCat}_{cont})$ we may consider the inner hom $\underline{\mathcal{H}om}(C, D)$ in this monoidal category. According to ([22], 10.2.2), one has $\mathrm{Rep}(H) \widetilde{\to} \underline{\mathcal{H}om}(\mathrm{Vect}, \mathrm{Vect})$, where Vect is considered as an object of $\mathrm{QCoh}(H) - comod(\mathrm{DGCat}_{cont})$. (This comes from the fact that in the symmetric monidal category $\mathrm{Rep}(H){-}mod$ the inner hom from $\mathrm{Rep}(H)$ to iself is $\mathrm{Rep}(H)$). Besides, the functor $\mathrm{QCoh}(H) - comod \widetilde{\to} \mathrm{Rep}(H) - mod$ can be understood as the functor $C \mapsto \underline{\mathcal{H}om}(\mathrm{Vect}, C)$, which is naturally a right $\underline{\mathcal{H}om}(\mathrm{Vect}, \mathrm{Vect})$-module.

Now Vect has commuting structures of $\mathrm{Rep}(H)$-module and $\mathrm{QCoh}(H)$-comodule, hence the functor $C \mapsto C \otimes_{\mathrm{Rep}(H)} \mathrm{Vect}$ can be seen as the functor $\mathrm{Rep}(H) - mod \to \mathrm{QCoh}(H) - comod$.

Note also that $\mathrm{Vect} \otimes_{\mathrm{Rep}(H)} \mathrm{Vect} \widetilde{\to} \mathrm{QCoh}(H)$ by ([43], ch. I.3, 3.3.5).

1.6.9. For 10.2.8. By Λ_H-graded algebra A we mean $A \in Alg(\mathrm{Rep}(H))$. The only case needed in that $A \in \mathrm{Rep}(H)^\heartsuit$ I think.

Let H be a torus with weight lattice Λ_H. Given A a Λ_H-graded algebra in $\mathrm{Vect}^\heartsuit$, the isomorphism

$$\overset{\bullet}{A} - mod \otimes_{\mathrm{Rep}(H)} \mathrm{Vect} \widetilde{\to} A - mod$$

follows from ([20], ch. I.1, 8.5.7). Namely, $A \in Alg(\mathrm{Rep}(H))$ and by definition $\overset{\bullet}{A} - mod = A - mod(\mathrm{Rep}(H))$. Now Vect is a $\mathrm{Rep}(H)$-module, so

$$A - mod(\mathrm{Rep}(H)) \otimes_{\mathrm{Rep}(H)} \mathrm{Vect} \widetilde{\to} A - mod(\mathrm{Vect})$$

The functor $oblv_{\mathrm{Hecke}} : A - mod \to \overset{\bullet}{A} - mod$ sends V to $V \otimes \mathcal{O}_H = \oplus_{\lambda \in \Lambda_H} V_\lambda$, where $\mathcal{O}_H$ is the ring of functions on H and $V_\lambda = V$. Write $A = \oplus_\lambda A_\lambda$. The A-action on V_λ is given by the old one $A_\mu \otimes V_\lambda \to V_{\lambda+\mu}$ with the difference that it changes the graded component.

1.6.10. Recall that for H reductive, $\mathrm{Rep}(H) \widetilde{\to} \prod_{\lambda \in \Lambda_H^+} \mathrm{Vect}$ according to ([22], 7.2.4).

For 10.4.1. It is understood that $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H) \to \mathrm{Rep}(T_H)$ is the map of algebras sending $V \boxtimes W$ to $\mathrm{Res}^{T_H}(V) \otimes W$.

1.6.11. For 10.5.1, line 1: add $C \in \mathrm{DGCat}_{cont}$.

Recall here the equivalence $(C^c)^{op} \to (C^\vee)^c$, $c \mapsto c^\vee$ from Section 10.1.5. So, for $c \in C^c$, $c^\vee : C \to \mathrm{Vect}$ is the functor $\mathcal{H}om_C(c, \cdot)$. Mention that the $\mathrm{Rep}(H)$-action on $C^\vee$ is the natural one in the sense of ([20], ch. I.1, 4.1.7). The first displayed formula in this Section 10.5.1 is wrong. Indeed, we want to consider the natural action of $\mathrm{Rep}(H)$ on $C^\vee$, as the formula $\mathrm{Hecke}(C)^\vee \widetilde{\to} \mathrm{Hecke}(C^\vee)$ from 10.1.5 is established for the natural action.

Given $c \in C^\vee$, the natural action by $V \in \mathrm{Rep}(H)$ sends $c^\vee$ to $c^\vee * V : C \to \mathrm{Vect}$, $x \mapsto \mathcal{H}om_C(c, V * x)$. By ([36], 4.6.2.1), for $V \in \mathrm{Rep}(H)^c$ the functor $C \to C, c \mapsto c * V$ is both left and right adjoint to $C \to C, c \mapsto V^* * c$. So, for the natural actions $c^\vee * V \widetilde{\to} (c * V^*)^\vee$.

In general, I think there is no reason for an isomorphism

$$\mathcal{H}om_C(\tau^H(V^*) * c, x) \widetilde{\to} \mathcal{H}om_C(c, V * x)$$

The square given by the displayed formula commutes for the following reason: write $\mathrm{Reg}_H = (\mathrm{id} \otimes \mathrm{Res}^H)(\mathrm{Reg}(H)) \in Alg(\mathrm{Rep}(H))$. Then for $c \in C^c$, $M \in \mathrm{Reg}_H - mod$,

$$\mathcal{H}om_{\mathrm{Reg}_H - mod}(\mathrm{Reg}_H * c, M) \widetilde{\to} \mathcal{H}om_C(c, oblv(M))$$

1.6.12. For 10.5.2, the formulation is very bad! We just apply the 2nd displayed formula from 10.1.5 identifying $\mathrm{Rep}(T_H)^\vee \widetilde{\to} \mathrm{Rep}(T_H)$ via the natural pairing that we have since $\mathrm{Rep}(T_H)$ is rigid.

Lemma 1.6.13. *Let* $\mathrm{Rep}(H)$ *act naturally on* $\mathrm{Rep}(T_H)$ *via the restriction, consider then the induced* $\mathrm{Rep}(H)$*-action on* $\mathrm{Rep}(T_H)^\vee$*. Let us transfer the latter* $\mathrm{Rep}(H)$*-action to a* $\mathrm{Rep}(H)$*-action on* $\mathrm{Rep}(T_H)$ *by the canonical equivalence* $\mathrm{Rep}(T_H) \widetilde{\to} \mathrm{Rep}(T_H)^\vee$ *coming from the fact that* $\mathrm{Rep}(T_H)$ *is rigid. We claim that the so obtained* $\mathrm{Rep}(H)$*-action on* $\mathrm{Rep}(T_H)$ *is the natural one. Namely,* $V_1 \in \mathrm{Rep}(H)$ *sends* $V \in \mathrm{Rep}(T_H)$ *to* $V \otimes \mathrm{Res}^{T_H}(V_1)$*.*

Proof. We identify $\mathrm{Rep}(T_H) \widetilde{\to} \mathrm{Rep}(T_H)^\vee$ sending V to $f_V : \mathrm{Rep}(T_H) \to \mathrm{Vect}$, where $f_V(V') = \mathcal{H}om_{\mathrm{Rep}(T_H)}(k, V \otimes V')$. The natural action of $\mathrm{Rep}(H)$ on $\mathrm{Rep}(T_H)$ induces an action of $\mathrm{Rep}(H)$ on $\mathrm{Rep}(T_H)^\vee$, namely $V_1 \in \mathrm{Rep}(H)$ sends f_V to the functor

$$V' \mapsto f_V(\mathrm{Res}^{T_H}(V_1) \otimes V') = \mathcal{H}om_{\mathrm{Rep}(T_H)}(k, V \otimes \mathrm{Res}^{T_H}(V_1) \otimes V')$$

So, $(f_V) * V_1 \widetilde{\to} f_{V \otimes \mathrm{Res}^{T_H}(V_1)}$. □

On the other hand, we have the map $(\mathrm{Rep}(H)^c)^{op} \to (\mathrm{Rep}(H)^\vee)^c$, $V \mapsto V^\vee$ as for any compactly generated category. We have $(V^*)^\vee \widetilde{\to} f_V$ for $V \in \mathrm{Rep}(H)^c$. Here $f_V : \mathrm{Rep}(H) \to \mathrm{Vect}$ denotes the functor $f_V(W) = \mathcal{H}om_{\mathrm{Rep}(H)}(e, V \otimes W)$.

Apply 10.1.5 for C any and $D = \mathrm{Rep}(T_H)$, we get the equivalence

$$\overset{\bullet}{\mathrm{Hecke}}(C)^\vee \widetilde{\to} C^\vee \otimes_{\mathrm{Rep}(H)} \mathrm{Rep}(T_H)^\vee$$

and the commutative square

$$\begin{array}{ccc} ((C \otimes \mathrm{Rep}(T_H))^c)^{op} & \overset{\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}}{\to} & (\overset{\bullet}{\mathrm{Hecke}}(C)^c)^{op} \\ \downarrow & & \downarrow \\ (C^\vee \otimes \mathrm{Rep}(T_H)^\vee)^c & \overset{\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}}{\to} & (C^\vee \otimes_{\mathrm{Rep}(H)} \mathrm{Rep}(T_H)^\vee)^c, \end{array}$$

where the left vertical arrow sends $c \boxtimes V$ to $c^\vee \boxtimes V^\vee$, and the right vertical arrow sends z to $z^\vee$.

By lemma, the pairing coming from rigidity is an equivalence of $\mathrm{Rep}(H)$-modules $\mathrm{Rep}(T_H) \widetilde{\to} \mathrm{Rep}(T_H)^\vee$, so gives the equivalence $(\overset{\bullet}{\mathrm{Hecke}}(C))^\vee \widetilde{\to} \overset{\bullet}{\mathrm{Hecke}}(C^\vee)$.

The square that you wrote in 10.5.2 commutes if the left vertical arrow sends $c \boxtimes V$ to $c^\vee \boxtimes V^*$, and we use evrywhere the natural actions! That is, for $c \in C^c, V \in \mathrm{Rep}(T_H)^c$ we get

$$(\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(c \boxtimes V))^\vee \widetilde{\to} \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(c^\vee \boxtimes V^*)$$

I propose to add this formula and remove the twisted actions. For example, in the spacial case $G = T$ it is particularly clear that we get the formula I have just written above for the natural actions.

1.6.14. For 10.6, line 5: I think you meant $\mathrm{Hecke}(C)^\heartsuit$ instead of $\mathrm{Hecke}(C)$.
We must assume H reductive in Section 10.6.

1.6.15. For 10.6.3. If $c' \in \mathrm{Hecke}(C)^\heartsuit$ then the natural map $\mathrm{ind}_{\mathrm{Hecke}} \mathrm{oblv}_{\mathrm{Hecke}}(c') \to c'$ is surjective. Indeed, viewing $\mathrm{Hecke}(C)$ as $\mathrm{Reg}_H -mod(C)$, where

$$\mathrm{Reg}_H = (\mathrm{id} \otimes \mathrm{Res})(\mathrm{Reg}(H)) \in Alg(\mathrm{Rep}(H)),$$

the corresponding map $\mathrm{Reg}_H * c' \to c'$ is the action map for this Reg_H-module. It is surjective, as the composition $c' \widetilde{\to} k * c' \to \mathrm{Reg}_H * c' \to c'$ is the identity.

Recall that

$$\mathrm{Reg}_H \widetilde{\to} \oplus_{V \in Irrep(H)} V \otimes \underline{V}^*$$

Indeed, for $V \in Irrep(H)$ and $q : \mathrm{Spec}\, k \to B(H)$ we have $\mathcal{H}om(V, q_*\mathcal{O}) \widetilde{\to} \mathcal{H}om(\underline{V}, k)$. This gives a map $V \otimes \underline{V}^* \to q_*\mathcal{O}$, hence taking the direct sum over $V \in Irrep(H)$, we get a morphism

$$\epsilon : \bigoplus_{V \in Irrep(H)} V \otimes \underline{V}^* \to \mathrm{Reg}_H \widetilde{\to} q_*\mathcal{O}$$

For any $V' \in Irrep(H)$ the map ϵ induces an isomorphism

$$\mathcal{H}om(V', \oplus_{V \in Irrep(H)} V \otimes \underline{V}^*) \to \mathcal{H}om(V', \mathrm{Reg}_H)$$

Assume now H reductive. Then for any $V' \in \mathrm{Rep}(H)^c$ the latter map is an isomorphism. Thus, ϵ is an isomorphism.

In (10.7) there is $V \in Irrep(H)$ such that the component $c_1 \to (c * V) \otimes \underline{V}^*$ is nonzero, because otherwise the map $ind(c_1) \to ind(c)$ would vanish.

For each $V \in Irrep(H)$ we have a canonical isomorphism $\alpha_V : V * Reg_H \widetilde{\to} Reg_H \otimes \underline{V}$. Indeed, consider the $Rep(H)$-action on itself coming from the symmetric monoidal structure. We get the adjoint pair

$$\mathrm{ind}_{\mathrm{Hecke}} : \mathrm{Rep}(H) \leftrightarrows Reg_H - mod(\mathrm{Rep}(H)) : \mathrm{oblv}_{\mathrm{Hecke}}$$

Then for $k \in \mathrm{Rep}(H)$ we get $\mathrm{ind}_{\mathrm{Hecke}}(k) = \mathrm{Reg}_H$, so according to Section 10.2.4 of the paper, we get the desired isomorphisms. As in ([5], Sect. 2.2) it is constructed explicitly as follows. We have the inclusion $V \otimes \underline{V}^* \hookrightarrow \mathrm{Reg}_H$ via the matrix coefficient, now the composition

$$\mathrm{Reg}_H * (V \otimes \underline{V}^*) \hookrightarrow \mathrm{Reg}_H * \mathrm{Reg}_H \stackrel{m}{\to} \mathrm{Reg}_H$$

yields by adjointness the map $\alpha_V : \mathrm{Reg}_H * V \to \mathrm{Reg}_H \otimes \underline{V}$, which is an isomorphism. Here m is the product in the algebra Reg_H.

The desired map $\mathrm{ind}_{\mathrm{Hecke}}(c * V) \widetilde{\to} \underline{V} \otimes \mathrm{ind}_{\mathrm{Hecke}}(c) \to \mathrm{ind}_{\mathrm{Hecke}}(c)$ identifies indeed with $\xi \times \mathrm{id}$. This follows from the commutativity of the diagram

$$\begin{array}{ccccc} V \otimes \mathrm{Reg}_H & \stackrel{\mathrm{id} \otimes \xi \otimes \mathrm{id}}{\to} & V \otimes \underline{V}^* \otimes \mathrm{Reg}_H \subset & \mathrm{Reg}_H \otimes \mathrm{Reg}_H \\ \downarrow \alpha_V & & & \downarrow m \\ \mathrm{Reg}_H \otimes \underline{V} & \stackrel{\xi}{\to} & & \mathrm{Reg}_H \end{array}$$

This is general: let E_i be vector spaces and $\alpha : E_1 \to E_2 \otimes \underline{V}$ correspond by adjointness to $\bar{\alpha} : E_1 \otimes \underline{V}^* \to E_2$. Then for $\xi \in \underline{V}^*$ the diagram commutes

$$\begin{array}{ccc} E_1 & \stackrel{\xi}{\to} & E_1 \otimes \underline{V}^* \\ \downarrow \alpha & & \downarrow \bar{\alpha} \\ E_2 \otimes V & \stackrel{\xi}{\to} & E_2 \end{array}$$

1.6.16. For 10.6.6. Let $c' \in \mathrm{Hecke}(C)^{\heartsuit}$ be irreducible. Then $\mathrm{ind}_{\mathrm{Hecke}}(\mathrm{oblv}_{\mathrm{Hecke}}(c') \to c'$ is nonzero. Since the t-structure on $C^{\heartsuit}$ is Artinian, we may pick a presentation $\mathrm{oblv}_{\mathrm{Hecke}}(c') \widetilde{\to} \mathrm{colim}_{i\in I}\, c_i$, where $c_i \in C^{\heartsuit} \cap C^c$, and I is small filtered. So, there is $i \in I$ and a nonzero map $c_i \to \mathrm{oblv}_{\mathrm{Hecke}}(c')$, hence the corresponding map $\mathrm{ind}_{\mathrm{Hecke}}(c_i) \to c'$ in $\mathrm{Hecke}(C)^{\heartsuit}$ is nonzero.

For the end of the proof of (b): assume $V \in \mathrm{Rep}(H)^{\heartsuit}$, $c_1 \in C^{\heartsuit}$ restricted irreducible with an isomorphism $c_1 * V \widetilde{\to} c_1$. So, $\underline{V} \otimes \mathrm{ind}_{\mathrm{Hecke}}(c_1) \widetilde{\to} \mathrm{ind}_{\mathrm{Hecke}}(c_1 * V) \widetilde{\to} \mathrm{ind}_{\mathrm{Hecke}}(c_1)$ is an irreducible object of $\mathrm{Hecke}(C)^{\heartsuit}$ by Pp. 10.6.3. So, $\underline{V}$ is 1-dimensional.

1.6.17. For 10.6.7. We may apply Proposition 1.2.38 of this file. Indeed, we know by ([43], Lemma 9.3.7) that $\mathrm{Hecke}(C)$ is compactly generated and its t-structure is compactly generated, so the t-structure on $\mathrm{Hecke}(C)$ is in particular compatible with filtered colimits by ([43], 9.3.5).

By 10.6.6 of the paper we know that each irreducible object of $\mathrm{Hecke}(C)^{\heartsuit}$ is compact. So, it suffices to show they generate $\mathrm{Hecke}(C)$. Since $\mathrm{oblv}_{\mathrm{Hecke}} : \mathrm{Hecke}(C) \to C$ is conservative, the essential image of $\mathrm{ind}_{\mathrm{Hecke}} : C \to \mathrm{Hecke}(C)$ generates $\mathrm{Hecke}(C)$ under colimits. Thus, $\mathrm{Hecke}(C)$ is compactly generated by objects of the form $\mathrm{ind}_{\mathrm{Hecke}}(c)$ for $c \in C^c$. By 6.3.8 of the paper, such c is cohomologically bounded, its cohomologies lie in $C^c \cap C^{\heartsuit}$. Moreover, each object of $C^c \cap C^{\heartsuit}$ is of finite length. Thus, $\mathrm{Hecke}(C)$ is generated by objects of the form $\mathrm{ind}_{\mathrm{Hecke}}(c)$ with $c \in C^c \cap C^{\heartsuit}$ such that c is irreducible in $C^{\heartsuit}$. So, $\mathrm{Hecke}(C)$ is generated by objects of the form $\mathrm{ind}_{\mathrm{Hecke}}(c * V) \widetilde{\to} \mathrm{ind}_{\mathrm{Hecke}}(c) \otimes \underline{V}$ with c restricted and $V \in \mathrm{Rep}(H)^{\heartsuit}$ irreducible. So, irreducible objects of $\mathrm{Hecke}(C)^{\heartsuit}$ generate $\mathrm{Hecke}(C)$.

1.6.18. For 10.7.3. Our assumptions are: C is compactly generated with compactly generated t-structure, the action $C \otimes \mathrm{Rep}(H) \to C$ is t-exact, H is reductive. Then both $\mathrm{Hecke}(C), \dot{\mathrm{Hecke}}(C)$ are compactly generated with compactly generate t-structures by ([43], Lemma 9.3.13). Now any irreducible object of $\dot{\mathrm{Hecke}}(C)^{\heartsuit}$ is restricted. So, by Prop. 10.6.3, $\mathrm{Res}^{T_H}(c) \in \mathrm{Hecke}(C)^{\heartsuit}$ is irreducible for $c \in \dot{\mathrm{Hecke}}(C)^{\heartsuit}$ irreducible. The functor $\mathrm{Res}^{T_H} : \dot{\mathrm{Hecke}}(C) \to \mathrm{Hecke}(C)$ is the induction: $\dot{\mathrm{Hecke}}(C) \to \dot{\mathrm{Hecke}}(C) \otimes_{\mathrm{Rep}(T_H)} \mathrm{Vect}$, and $\mathrm{coInd}^{T_H} : \mathrm{Hecke}(C) \to \dot{\mathrm{Hecke}}(C)$ is its right adjoint (that is, $\mathrm{oblv}_{\mathrm{Hecke}}$ for the $\mathrm{Rep}(T_H)$-action on $\dot{\mathrm{Hecke}}(C)$). Both $\mathrm{Res}^{T_H}, \mathrm{coInd}^{T_H}$ are t-exact by 10.1.7 of the paper. So, 10.7.3(a) is proved.

For $\gamma \in \Lambda_H$ write $e^{\gamma} \in \mathrm{Rep}(T_H)$ for the corresponding 1-dim representation of T_H.

For $V \in \mathrm{Rep}(H)$ write $\overline{V} := \mathrm{Res}^{T_H}(V) \in \mathrm{Rep}(T_H)$ for brevity. Let

$$\mathrm{Reg}_{H,T_H} = (\mathrm{id} \boxtimes \mathrm{Res}^{T_H}) Reg(H) \in \mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$$

So, $\mathrm{Reg}_{H,T_H} \widetilde{\to} \oplus_{V\in Irrep(H)} V\otimes \bar{V}^*$. For the adjoint pair

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}} : \mathrm{Rep}(H)\otimes \mathrm{Rep}(T_H) \leftrightarrows \mathrm{Reg}_{H,T_H} -mod(\mathrm{Rep}(H)\otimes \mathrm{Rep}(T_H)) : \mathrm{oblv}_{\overset{\bullet}{\mathrm{Hecke}}}$$

we get $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(k\boxtimes k) \widetilde{\to} \mathrm{Reg}_{H,T_H}$. By 10.3.4 of the paper for $V\in \mathrm{Rep}(H)$ it is equipped with an isomorphism

$$V * \mathrm{Reg}_{H,T_H} \widetilde{\to} \mathrm{Reg}_{H,T_H} * \mathrm{Res}^{T_H}(V),$$

where we write the $\mathrm{Rep}(T_H)$ action on the right, and $\mathrm{Rep}(H)$-action on the left.

Let $c\in C^\heartsuit$ be restricted, let us first show that for $\gamma\in\Lambda_H$, $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(c\boxtimes e^\gamma)$ is irreducible in $\overset{\bullet}{\mathrm{Hecke}}(C)^\heartsuit$. This is similar to 10.6.3. Let $c'\in \overset{\bullet}{\mathrm{Hecke}}(C)^\heartsuit$ with a given nonzero map $c'\to \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(c\boxtimes e^\gamma)$. We must show it is surjective. We may assume it comes from some nonzero map

$$c''\to \mathrm{oblv}_{\overset{\bullet}{\mathrm{Hecke}}}\,\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(c\boxtimes e^\gamma) = \underset{V\in Irrep(H)}{\oplus} (V*c)\boxtimes (e^\gamma\otimes \bar{V}^*)$$

for some $c''\in (C\otimes \mathrm{Rep}(T_H))^\heartsuit$ by adjointness. Since the t-structure on C is compactly generated, we may pick $c_1\in C^{\le 0}\cap C^c$ and $\mu\in\Lambda_H$ and a nonzero map

$$c_1\boxtimes e^\mu \to \underset{V\in Irrep(H)}{\oplus} (V*c)\boxtimes (e^\gamma\otimes \bar{V}^*)$$

So, there is $V\in Irrep(H)$ such that its component

$$c_1\boxtimes e^\mu \to (V*c)\boxtimes (e^\gamma\otimes \bar{V}^*)$$

is nonzero. Replace c_1 by $\tau^{\ge 0}c_1$, the latter map is still nonzero. We may assume $c_1 = V*c$ and the first component is the identity, and the weight $\gamma-\mu$ appears in $\bar{V}$. Recall that $V*c\in C^\heartsuit$ is irreducible. The corresponding map

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}((V*c)\boxtimes e^\mu) \widetilde{\to} (\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(c\boxtimes e^\mu)) * \bar{V} \to \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(c\boxtimes e^\gamma)$$

is surjective, because when we apply $\mathrm{id}\boxtimes \mathrm{Res}^{T_H}$ it becomes surjective.

Note that for $z\in \overset{\bullet}{\mathrm{Hecke}}(C)^\heartsuit$ if $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(z)=0$ then $z=0$. Indeed, $\mathrm{Reg}_{T_H} = \oplus_\mu e^\mu$, and $z*\mathrm{Reg}_{T_H} \widetilde{\to} \oplus_\mu z*e^\mu$. Since each e^μ is one-dimensional, this means that $z*e^\mu=0$ for all $\mu\in\Lambda_H$, so $z=0$.

Let us verify 10.7.3(b) now. Let $c'\in \mathrm{Hecke}(C)^\heartsuit$ be irreducible. Then there is $c_1\in \overset{\bullet}{\mathrm{Hecke}}(C)^\heartsuit$ with a nonzero map $c_1 * \mathrm{Reg}_{T_H}\to c'$. Here we view

$$\mathrm{Hecke}(C) \widetilde{\to} \mathrm{Reg}_{T_H} -mod(\overset{\bullet}{\mathrm{Hecke}}(C))$$

The above map comes from a nonzero map $c_1\to \mathrm{coInd}^{T_H}(c')$ in $\overset{\bullet}{\mathrm{Hecke}}(C)^\heartsuit$. Since the t-structure on $\overset{\bullet}{\mathrm{Hecke}}(C)$ is compactly generated, we may assume $c_1\in \overset{\bullet}{\mathrm{Hecke}}(C)^c$.

We could finish if we new that we may assume c_1 irreducible in addition. This is not clear in general, and maybe wrong.

So, **let us make for 10.7.3(b) and (c) and for 10.7.4(b) the assitional assumption**: the t-structure on C is Artinian, and the $\mathrm{Rep}(H)$-action on C is accessible.

Then by 10.6.6 we may assume $c' \widetilde{\to} \mathrm{ind}_{\mathrm{Hecke}}(c)$ for some $c \in C^{\heartsuit}$ restricted. The diagram commutes

$$\begin{array}{ccc} C \otimes \mathrm{Rep}(T_H) & \overset{\mathrm{ind}_{\dot{\mathrm{Hecke}}}}{\to} & \dot{\mathrm{Hecke}}(C) \\ \downarrow \mathrm{id} \otimes \mathrm{Res} & & \downarrow \mathrm{Res}^{T_H} \\ C & \overset{\mathrm{ind}_{\mathrm{Hecke}}}{\to} & \mathrm{Hecke}(C) \end{array}$$

This is just the functoriality of the relative tensor product. We have already shown that $\mathrm{ind}_{\dot{\mathrm{Hecke}}}(c \boxtimes e)$ is irreducible. We get $\mathrm{Res}^{T_H}(\mathrm{ind}_{\dot{\mathrm{Hecke}}}(c \boxtimes e)) \widetilde{\to} \mathrm{ind}_{\mathrm{Hecke}}(c)$. So, 10.7.3(b) is proved.

Proof of 10.7.3(c). Let $c_1 \in \dot{\mathrm{Hecke}}(C)^{\heartsuit}$ be irreducible with $\mathrm{Res}^{T_H}(c_1) \widetilde{\to} \mathrm{ind}_{\mathrm{Hecke}}(c_0)$ for some $c_0 \in C^{\heartsuit}$ restricted. Recall that $\mathrm{Res}^{T_H}(\mathrm{ind}_{\dot{\mathrm{Hecke}}}(c \boxtimes e)) \widetilde{\to} \mathrm{ind}_{\mathrm{Hecke}}(c)$. So, we may take $c_2 = \mathrm{ind}_{\dot{\mathrm{Hecke}}}(c_0 \boxtimes e)$, and we get an isomorphism $c_1 * \mathrm{Reg}_{T_H} \widetilde{\to} c_2 * \mathrm{Reg}_{T_H}$ in $\mathrm{Reg}_{T_H} -mod(\dot{\mathrm{Hecke}}(C))$. It comes from a nonzero morphism $c_2 \to c_1 * \mathrm{Reg}_{T_H} = \oplus_\mu c_1 * e^\mu$ in $\dot{\mathrm{Hecke}}(C)^{\heartsuit}$. Since c_2 is compact in $\dot{\mathrm{Hecke}}(C)$, there is a nonzero map $c_2 \to c_1 * e^\mu$ for some $\mu \in \Lambda_H$. It is an isomorphism, as both objects are irreducible. The claim 10.7.3(c) is proved. This also proves 10.7.4(b)(ii), because $\mathrm{ind}_{\dot{\mathrm{Hecke}}}$ is a map of $\mathrm{Rep}(T_H)$-modules.

We verify 10.7.4(b)(iii). Let $c_i \in C^{\heartsuit}$ be restricted irreducible, $\gamma_i \in \Lambda_H$ and

$$\mathrm{ind}_{\dot{\mathrm{Hecke}}}(c_1 \boxtimes e^{\gamma_1}) \widetilde{\to} \mathrm{ind}_{\dot{\mathrm{Hecke}}}(c_2 \boxtimes e^{\gamma_2})$$

in $\dot{\mathrm{Hecke}}(C)$. Such an isomorphism comes from a nonzero map

$$c_1 \boxtimes e^{\gamma_1} \to \mathrm{oblv}_{\dot{\mathrm{Hecke}}} \mathrm{ind}_{\dot{\mathrm{Hecke}}}(c_2 \boxtimes e^{\gamma_2})$$

in $C \otimes \mathrm{Rep}(T_H)$. As the t-structure on C is Artinian, $c_1 \boxtimes e^{\gamma_1} \in (C \otimes \mathrm{Rep}(T_H))^c$, so there is $V \in Irrep(H)$ and a nonzero map $c_1 \boxtimes e^{\gamma_1} \to (V * c_2) \boxtimes (e^{\gamma_2} \otimes \bar{V}^*)$. The latter is the tensor product of an isomorphism $c_1 \widetilde{\to} V * c_2$ in $C^{\heartsuit}$ with a nonzero map $e^{\gamma_1} \to e^{\gamma_2} \otimes \bar{V}^*$ in $\mathrm{Rep}(T_H)$. Since $V * c_2$ is restricted, for any $V' \in \mathrm{Rep}(H)$, $(V' \otimes V) * c_2$ is irreducible in $C^{\heartsuit}$, hence V is 1-dimensional. Indeed, $V \otimes V^*$ must be irreducible and contains e. So, $\gamma_1 - \gamma_2$ is a character of H, and 10.7.4(b)(iii) is proved.

We verify 10.7.4(b)(i). For $c \in C^{\heartsuit}$ restricted irreducible, $\gamma \in \Lambda_H$, $\mathrm{ind}_{\dot{\mathrm{Hecke}}}(c \boxtimes e^\gamma) \in \dot{\mathrm{Hecke}}(C)^c$, because $c \in C^c$. So, each irreducible object of $\dot{\mathrm{Hecke}}(C)^{\heartsuit}$ is compact in $\dot{\mathrm{Hecke}}(C)$. We know already by ([43], 9.3.13) that the t-structure on $\dot{\mathrm{Hecke}}(C)$ is compactly generated. We check that the objects $\mathrm{ind}_{\dot{\mathrm{Hecke}}}(c \boxtimes e^\gamma)$ for c restricted irreducible in $C^{\heartsuit}$ and $\gamma \in \Lambda_H$ generate $\dot{\mathrm{Hecke}}(C)$. Let $z \in \dot{\mathrm{Hecke}}(C)$ with

$$\mathrm{Map}_{C \otimes \mathrm{Rep}(H)}(c \boxtimes e^\gamma[n], \mathrm{oblv}_{\dot{\mathrm{Hecke}}}(z)) \widetilde{\to} \mathrm{Map}_{\dot{\mathrm{Hecke}}(C)}(\mathrm{ind}_{\dot{\mathrm{Hecke}}}(c \boxtimes e^\gamma)[n], z) \widetilde{\to} *$$

for any c, γ as above and $n \in \mathbb{Z}$. Since the t-structure on C is Artinian, the objects $c \boxtimes e^\gamma$ for c, γ as above generate $C \otimes \mathrm{Rep}(H)$, so $\mathrm{oblv}_{\dot{\mathrm{Hecke}}}(z) \widetilde{\to} 0$. Since $\mathrm{oblv}_{\dot{\mathrm{Hecke}}}$ is conservative, $z \widetilde{\to} 0$. We are done by Proposition 1.2.38 of this file.

1.7. **For Section 11.**

1.7.1. For 11.1.3, in the displayed square remove τ^{T_H}.

This section should be rewritten as follows. I assume that no twisted actions have appeared in 10.5.1-10.5.2 according to my suggestions above. First, $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^\vee$ identifies with $\mathrm{Whit}_{q,x}(G)^\vee \otimes_{\mathrm{Rep}(H)} \mathrm{Rep}(T_H)^\vee$, where we use everywhere the actions through $Sat_{q,x}$. By Lemma 1.6.13 of this file, we identify $\mathrm{Rep}(T_H)^\vee \widetilde{\to} \mathrm{Rep}(T_H)$ via the canonical self-diality coming from the rigidity of $\mathrm{Rep}(T_H)$, and the $\mathrm{Rep}(H)$-action on $\mathrm{Rep}(T_H)$ becomes the natural one. Now we identify $\mathrm{Whit}_{q,x}(G)^\vee \widetilde{\to} \mathrm{Whit}_{q^{-1},x}(G)$ via (7.9) in the paper. Under this equivalence the above $\mathrm{Rep}(H)$-action on $\mathrm{Whit}_{q,x}(G)^\vee$ identifies not with the $\mathrm{Rep}(H)$-action via $Sat_{q^{-1},G}$ but by a twist of the latter. Namely, for $\mathcal{F} \in \mathrm{Whit}_{q^{-1},x}(G)$, $V \in \mathrm{Rep}(H)$ we have (according to Section 1.7.15 of this file)

$$\mathcal{F} * Sat_{q^{-1},G}(V) \widetilde{\to} \mathcal{F} * Sat_{q,G}(\tau_H(V))$$

1.7.2. For 11.2.1. In general, the composition $\Lambda^\sharp \hookrightarrow \Lambda \to \tilde{\Lambda}$ does not factor through $\tilde{\Lambda}^\sharp$.

Example: take $\tilde{G} = \tilde{T} = \mathbb{G}_m^2$, $T = \mathbb{G}_m$ given by the first factor. Let $\alpha \in e^{*,tors}(-1)$. We get $\tilde{\Lambda} = \mathbb{Z}^2$. Let $\tilde{q} : \tilde{\Lambda} \to e^{*,tors}(-1)$ be given by $\tilde{q}(a_1, a_2) = \alpha a_1 a_2$ for $(a_1, a_2) \in \mathbb{Z}^2$. Then $\Lambda^\sharp = \Lambda$ with base e_1. However, $e_1 \notin \tilde{\Lambda}^\sharp$ if α is nontrivial, because $\tilde{b}(e_1, e_2) = \alpha$.

As we discussed by email, I suppose we include the property $\Lambda^\sharp \subset \tilde{\Lambda}^\sharp$ in the definition of *strictly compatible with the geometric metaplectic data* from 11.2.2.

In addition, $\tilde{H} \to H$ is a surjection, and its kernel is a torus equal to $\mathrm{Ker}(\tilde{T}_H \to T_H)$, so $\mathrm{Rep}(H) \subset \mathrm{Rep}(\tilde{H})$ is fully faithful.

The reference for the existence of the map $\tilde{H} \to H$ attached to the the corresponding morphism of root data is (SGA3, XXV, 1.1). This reference uses the notion of *données radicielles réduites épinglées* defined in (SGA3, XXIII, 1.5). The usual references like Springer, Linear Alegbraic groups, 2nd edition (2009) only treat the case of isogenies with a finite quotient of lattices!

Note also that the cocartesian square in 11.2.2 is needed to garantee that the natural inclusion $\Lambda/\Lambda_H \to \tilde{\Lambda}/\Lambda_{\tilde{H}}$ is bijective. Later we will identify Λ/Λ_H with the set of irreducible objects of $\mathrm{Hecke}(\mathrm{Whit}_{q,x}(G))$. So, this condition assures that the irreducible do not augment when we make our "generalized isogeny".

1.7.3. For 11.2.3. The image of $\mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Gr}^{\omega^\rho}_{\tilde{G},x}$ is the union of some connected components (up to nilpotents). Just after (11.4) you claim that $\mathrm{Sph}_{q,x}(G) \to \mathrm{Sph}_{q,x}(\tilde{G})$ is fully faithful. In fact, there is no such natural functor at all.

Indeed, we may consider the local version of the Hecke stack $\mathrm{Hecke}^{loc}_{G,x}$ classifying $\mathcal{F}_G, \mathcal{F}'_G$ over D_x together with an isomorphism $\mathcal{F}_G \widetilde{\to} \mathcal{F}'_G$ over $\overset{\circ}{D}_x$. Then we have the natural map $f : \mathrm{Hecke}^{loc}_{G,x} \to \mathrm{Hecke}^{loc}_{\tilde{G},x}$ compatible with the corresponding gerbes $\mathcal{G}^{G,G,ratio}$, $\mathcal{G}^{\tilde{G},\tilde{G},ratio}$. One could try to define the desired functor as f_*, this is a bad approach as already the case $G = 1, \tilde{G} = T$ shows. Namely, this would produce $\mathrm{R}\Gamma(\mathfrak{L}^+(T)_x, e)$ instead of the constant sheaf.

There is no natural map $\mathfrak{L}^+(G)^{\omega^\rho}_x \backslash \mathrm{Gr}^{\omega^\rho}_{G,x} \overset{a}{\leftarrow} \mathfrak{L}^+(\tilde{G})^{\omega^\rho}_x \backslash \mathrm{Gr}^{\omega^\rho}_{G,x}$ in general, so no hope to define it as a pull-back.

The only thing we need is the following. Let $h : \mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Gr}^{\omega^\rho}_{\tilde{G},x}$ be the natural map. Then h_* commutes with $\mathrm{Rep}(H)$-actions on both sides. Here on the target it acts through the morphism $\mathrm{Rep}(H) \to \mathrm{Rep}(\tilde{H})$, which we have because we do have a morphism $\tilde{H} \to H$.

This is general: let $Y = Y_1 \sqcup Y_2$ be a disjoint union of two prestacks. Assume $\mathrm{Rep}(\tilde{H})$ acts on $Shv(Y)$, and the induced $\mathrm{Rep}(H)$-action preserves the full subcategory $Shv(Y_1)$. Then the inclusion $Shv(Y_1) \subset Shv(Y)$ commutes with $\mathrm{Rep}(H)$-actions.

We have however the fully faithful functor $\mathrm{Sph}_{q,x}(G)^\heartsuit \to \mathrm{Sph}_{q,x}(\tilde{G})^\heartsuit$.

1.7.4. For 11.2.4. Given a morphism of algebraic groups $\tilde{H} \to H$, we get a monoidal functor $\mathrm{Rep}(H) \to \mathrm{Rep}(\tilde{H})$. Let now $C \in \mathrm{Rep}(H) - mod(\mathrm{DGCat}_{cont})$, $\tilde{C} \in \mathrm{Rep}(\tilde{H}) - mod(\mathrm{DGCat}_{cont})$ and $C \to \tilde{C}$ be a map in $\mathrm{Rep}(H) - mod(\mathrm{DGCat}_{cont})$. It yields a morphism $\mathrm{Hecke}(C) \to \tilde{C} \otimes_{\mathrm{Rep}(H)} \mathrm{Vect} \to \tilde{C} \otimes_{\mathrm{Rep}(\tilde{H})} \mathrm{Vect}$.

If H is reductive then $\mathrm{Rep}(H) \widetilde{\to} \prod_{V \in Irrep(H)} \mathrm{Vect}$. This is obtained from ([20], ch. I.3, 2.4.2) by taking left completions on both sides. Indeed, $\mathrm{QCoh}(H)$ is left-complete by ([20], ch. I.3, 1.5.7).

If we think of an object of $\mathrm{Hecke}(Shv_{\mathcal{G}^{\tilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\tilde{G},x}))$ as $F \in Shv_{\mathcal{G}^{\tilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\tilde{G},x})$ with a Hecke property then the isomorphism (11.5) says in words then when restricting it to $\mathrm{Gr}^{\omega^\rho}_{G,x}$, it inherits the Hecke property with respect to H.

1.7.5. For 11.2.5. Assume that $\Lambda_H \subset \tilde{\Lambda}_{\tilde{H}}$, so we have the functor (11.5). Indeed, to show it is fully faithful, it suffices to show that it induces an isomorphism on the map spaces for any pair of objects $\mathrm{ind}_{\mathrm{Hecke}}(\mathcal{F}_0), \mathrm{ind}_{\mathrm{Hecke}}(\mathcal{F}_1)$ with $\mathcal{F}_0, \mathcal{F}_1 \in Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^c$. The reason is that $\mathrm{Hecke}(Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}))$ is compactly generated by the image of $\mathrm{ind}_{\mathrm{Hecke}} : Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^c \to \mathrm{Hecke}(Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}))$. So, any object of $\mathrm{Hecke}(Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}))$ writes as filtered colimit of objects of the form $\mathrm{ind}_{\mathrm{Hecke}}(F)$ for $F \in Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^c$.

Assume now the isogeny strictly compatible with the geometric metaplectic data in the sense that $\Lambda_H \subset \tilde{\Lambda}_{\tilde{H}}$ and the square in 11.2.2 is cocartesian. Then indeed we get an exact sequence $1 \to \check{T}_0 \to \tilde{H} \to H \to 1$.

For (b): by ([20], ch. I.1, 5.4.5), it suffices to show that for $0 \neq F' \in \mathrm{Hecke}(Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{\tilde{G},x})$ there is $F \in Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ and a nonzero map $\mathrm{ind}_{\mathrm{Hecke}}(F) \to F'$ in $\mathrm{Hecke}(Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{\tilde{G},x})$.

"For point (b) we note that the condition in Sect. 11.2.2 imply that for every $0 \neq \mathcal{F}_1 \in Shv_{\mathcal{F}^{\tilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\tilde{G},x})$ there is $V \in Irrep(\tilde{H})$ so that $\mathcal{F}_1 * Sat_{q,G}(V)$ is non-zero when restricted to $\mathrm{Gr}^{\omega^\rho}_{G,x}$". The explanation of this: let $\gamma \in \pi_1(\tilde{G})$ whose image γ_0 in Λ_0 is such that $\mathcal{F}_1$ is nonzero over the component $\mathrm{Gr}^{\omega^\rho,\gamma}_{\tilde{G},x}$. Pick $V \in Irrep(\tilde{H})$ such that $\check{T}_0$ acts on V by $-\gamma_0$ then $\mathcal{F}_1 * Sat_{q,G}(V)$ is non-zero hen restricted to $\mathrm{Gr}^{\omega^\rho}_{G,x}$. Indeed, e appears in $V^* \otimes V$, so $\mathcal{F}_1$ appears in $(\mathcal{F}_1 * Sat_{q,G}(V)) * Sat_{q,G}(V^*)$ as a direct summand.

Remark: let $\tilde{H}$ be a split reductive group with a central torus $\check{T}_0 \subset \tilde{H}$. Then for any character $\lambda : \check{T}_0 \to \mathbb{G}_m$ there is a dominant weight of $\tilde{H}$ whose restriction to $\check{T}_0$ is λ. Indeed, let $\tilde{\lambda} : \tilde{T}_H \to \mathbb{G}_m$ be any extension of λ. If necessary, correct $\tilde{\lambda}$ by adding a dominant character of $H = \tilde{H}/\check{T}_0$.

1.7.6. For 11.2.6. First, let $C \in \mathrm{DGCat}_{cont}$ be compactly generated equipped with a $\mathrm{Rep}(T)$-action, where T is a torus. Then $\mathrm{ind}_{\mathrm{Hecke}} : C \to \mathrm{Hecke}(C) = C \otimes_{\mathrm{Rep}(T)} \mathrm{Vect}$ is 1-fully faithful.

Indeed, for $c \in C^c, c' \in C$ the map

$$\mathrm{Map}_C(c, c') \to \mathrm{Map}_{\mathrm{Hecke}(C)}(\mathrm{ind}_{\mathrm{Hecke}}(c), \mathrm{ind}_{\mathrm{Hecke}}(c')) \widetilde{\to} \mathrm{Map}_C(c, \bigoplus_\mu c' * e^\mu) \widetilde{\to} \sqcup_\mu \mathrm{Map}(c, c' * e^\mu)$$

is a full subspace, where μ runs through the weights of T. Now for any $c = \mathrm{colim}_{i \in I}\, c_i$, where I is small filtered and $c_i \in C^c$ we get

$$\mathrm{Map}_C(c, c') \widetilde{\to} \lim_i \mathrm{Map}_C(c_i, c) \subset \lim_i \mathrm{Map}_{\mathrm{Hecke}(C)}(\mathrm{ind}_{\mathrm{Hecke}}(c_i), \mathrm{ind}_{\mathrm{Hecke}}(c'))$$

is also a full subspace.

Lemma 1.7.7. *Let* $f : C_1 \to C_2$ *be a map in* $\mathrm{Rep}(T) - mod(\mathrm{DGCat}_{cont})$. *If the induced functor* $\mathrm{ind}(f) : \mathrm{Hecke}(C_1) \to \mathrm{Hecke}(C_2)$ *is an equivalence then* f *is an equivalence.*

Proof. The functor $\mathrm{Rep}(T) - mod(\mathrm{DGCat}_{cont}) \to T - mod$, $C \mapsto \mathrm{Hecke}(C)$ is an equivalence, see Section 10.2.6 of the paper. □

We apply the lemma to get Cor. 11.2.6. Namely, applying $\mathrm{Vect} \otimes_{\mathrm{Rep}(T_{\tilde{H}})} \cdot$ to both sides of (a), one gets an equivalence by 11.2.5, so (a) itself is an equivalence.

11.2.6(b) follows from 11.2.6(a). Namely, if we pick any splitting of (11.3), this gives a splitting of the exact sequence $1 \to \check{T}_0 \to T_{\tilde{H}} \to T_H \to 1$ and an equivalence $\mathrm{Rep}(T_{\tilde{H}}) \widetilde{\to} \mathrm{Rep}(\check{T}_0) \otimes \mathrm{Rep}(T_H)$, hence the desired equivalence by base change.

1.7.8. For 11.2.8(a). The same proof as in 11.2.5 applies and gives 11.2.8(a). As a consequence, 11.2.8(b,c) are obtained in the same way as Cor. 11.2.6 (using my above Lemma 1.7.7).

1.7.9. For 11.3.4. Here we assume $[H, H]$ simply-connected. The bijection between the irreducible objects of $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ and Λ is as follows. To a pair (λ, γ), where $\lambda \in \Lambda^+$ is restricted and $\gamma \in \Lambda_H$ we associate $\lambda + \gamma \in \Lambda$. The corresponding irreducible is $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,!*} \boxtimes e^\gamma)$. We have an action of $\Lambda_0^\sharp$ on pairs (λ, γ) such that $\tau \in \Lambda_0^\sharp$ sends (λ, γ) to $(\lambda + \tau, \gamma - \tau)$. The orbits identify with Λ. Here $\Lambda_0^\sharp = \{\lambda \in \Lambda^\sharp \mid \langle \lambda, \check{\alpha}_i \rangle = 0 \text{ for all } i\}$.

Now the irreducible of $\mathrm{Hecke}(\mathrm{Whit}_{q,x}(G))$ are in bijection with $\mathcal{M}/\Lambda_0^\sharp$, here $\mathcal{M} = \{\lambda \in \Lambda^+ \mid \lambda \text{ is restricted}\}$. Namely, to $\lambda \in \mathcal{M}$ is assocated the object $\mathrm{ind}_{\mathrm{Hecke}}(W^{\lambda,!*})$. If $\tau \in \Lambda_0^\sharp$ then $\mathrm{ind}_{\mathrm{Hecke}}(W^{\lambda+\tau,!*}) \widetilde{\to} \mathrm{ind}_{\mathrm{Hecke}}(W^{\lambda,!*})$. The functor

$$\mathrm{Res}^{T_H} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \mathrm{Hecke}(\mathrm{Whit}_{q,x}(G))$$

sends $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,!*} \boxtimes e^\gamma)$ to $\mathrm{ind}_{\mathrm{Hecke}}(W^{\lambda,!*})$.

1.7.10. For 11.3.5. Suppose we know that the t-structure on $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))$ is Artinian. We check that the t-structure on $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ is also Artinian. Pick a splitting of (11.3), so that we get an equivalence of 11.2.8(c)

$$\mathrm{Rep}(\check{T}_0)\otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))\widetilde{\to}\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))$$

Using the forgetful functor $\mathrm{Rep}(\check{T}_0)\to \mathrm{Vect}$, we get

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))\widetilde{\to}\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))\otimes_{\mathrm{Rep}(\check{T}_0)}\mathrm{Vect}$$

The $\mathrm{Rep}(\check{T}_0)$-action on $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))$ is automatically accessible (as in 10.7.1 of the paper). So, the t-structure on $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ is Artinian by Cor. 10.7.4(b).

Assume $[\tilde{H},\tilde{H}]$ simply-connected. Then we know already by (11.3.3, first case) that the irreducibles of $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))^{\heartsuit}$ are in bijection with $\tilde{\Lambda}$, that is, with pairs (σ,γ), where $\sigma\in\tilde{\Lambda}^+$ is restricted and $\gamma\in\Lambda_{\tilde{H}}$ up to the action of the lattice $\Lambda_{\tilde{H},0}$ of characters of $\tilde{H}_{ab}$. Namely, for such pair (σ,γ) the object $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*}\boxtimes e^{\gamma})$ is irreducible in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))^{\heartsuit}$, here $e^{\gamma}\in\mathrm{Rep}(T_{\tilde{H}})$ is 1-dimensional. We pick a splitting of (11.3) given by $s:\Lambda_0\hookrightarrow\tilde{\Lambda}^{\sharp}=\Lambda_{\tilde{H}}$. Then the forgetful functor $\mathrm{Rep}(\check{T}_0)\to\mathrm{Vect}$ yields a functor

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))\to\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)),$$

we may view the target as the Hecke category of the source with respect to the $\mathrm{Rep}(\check{T}_0)$-action. Now we apply 10.7.3 to describe the irreducibles of $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\heartsuit}$. We see that every irreducible object of the target is the image of some irreducible object $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*}\boxtimes e^{\gamma})$ of the source. More over, the pairs (σ_1,γ_1) and (σ_2,γ_2) give isomorphic irreducible objects in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\heartsuit}$ iff there is $\nu\in\Lambda_0=\mathrm{Hom}(\check{T}_0,\mathbb{G}_m)$ such that $\sigma_1=\sigma_2$ and $\gamma_2=\gamma_1+s(\nu)$.

We underline that the splitting of (11.3) in general is not compatible with root systems, it is just a splitting of an exact sequence of abelian groups.

Write $\mathcal{M}=\{\lambda\in\Lambda^+\mid\lambda \text{ is restricted}\}$. In general, the map $\mathcal{M}\to\Lambda/\Lambda^{\sharp}$ is not surjective, this is why we need isogenies. Write $\tilde{\mathcal{M}}=\{\lambda\in\tilde{\Lambda}^+\mid\lambda\text{ is restricted}\}$. We have a bijection between equivalence classes of pairs $(\sigma,\gamma)\in\tilde{\mathcal{M}}\times\Lambda_{\tilde{H}}$ and $\tilde{\Lambda}$. Here the pairs (σ_1,γ_1) and (σ_2,γ_2) here are equivalent if there is $\tau\in\Lambda_{\tilde{H},0}$ such that $\sigma_2=\sigma_1+\tau$, $\gamma_2=\gamma_1-\tau$. Consider inside the set of equivalence classes of pairs $(\sigma,\gamma)\in\tilde{\mathcal{M}}\times\Lambda_{\tilde{H}}$ such that $\sigma+\gamma\in\Lambda$. Under the above bijection it identifies with Λ. This is the desired bijection between Λ and irreducibles of $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$. In the notations of 11.3.7 of the paper the image of $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*}\boxtimes e^{\gamma})$ in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ is $\mathcal{M}_{\mathrm{Whit}}^{\sigma+\gamma,!*}$ provided that $\sigma+\gamma\in\Lambda$.

To have a notation independent of s, let's adopt the following. Given $\sigma\in\tilde{\mathcal{M}},\gamma\in\Lambda_{\tilde{H}}$ with $\sigma+\gamma\in\Lambda$ write

$$\mathcal{M}_{\mathrm{Whit}}^{\sigma+\gamma,!*}\in\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\heartsuit}$$

for the unique irreducible object such that the image of

$$e \boxtimes \mathcal{M}_{\mathrm{Whit}}^{\sigma+\gamma,!*} \in \mathrm{Rep}(T_{\tilde{H}}) \otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$$

under

$$\mathrm{Rep}(T_{\tilde{H}}) \otimes_{\mathrm{Rep}(T_H)} \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))$$

is $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*} \boxtimes e^{\gamma})$. Then for $\lambda \in \Lambda, \gamma \in \Lambda_H$ we have indeed

$$\mathcal{M}_{\mathrm{Whit}}^{\lambda,!*} * e^{\gamma} \widetilde{\to} \mathcal{M}_{\mathrm{Whit}}^{\lambda+\gamma,!*},$$

because the previous functor is $\mathrm{Rep}(T_{\tilde{H}})$-linear.

Similarly, there is a unique $\mathcal{M}_{\mathrm{Whit}}^{\sigma+\gamma,*} \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\heartsuit}$ such that the image of

$$e \boxtimes \mathcal{M}_{\mathrm{Whit}}^{\sigma+\gamma,*} \in \mathrm{Rep}(T_{\tilde{H}}) \otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$$

under

$$\mathrm{Rep}(T_{\tilde{H}}) \otimes_{\mathrm{Rep}(T_H)} \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))$$

is $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,*} \boxtimes e^{\gamma})$.

Another way: we have canonically $\mathrm{Rep}(T_H) \subset \mathrm{Rep}(T_{\tilde{H}})$. It yields a canonical fully faithful functor

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \mathrm{Rep}(T_{\tilde{H}}) \otimes_{\mathrm{Rep}(T_H)} \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \widetilde{\to} \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{\tilde{q},x}(\tilde{G}))$$

The composition is t-exact, so

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\heartsuit} \to \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{\tilde{q},x}(\tilde{G}))^{\heartsuit}$$

is a full abelian subcategory stable under extensions. My understanding is that given $(\sigma, \gamma) \in \tilde{\mathcal{M}} \times \Lambda_{\tilde{H}}$, we have

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*} \boxtimes e^{\gamma}) \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\heartsuit}$$

iff $\sigma + \lambda \in \Lambda$, and this way we get all the irreducibles of the latter abelain category.

Proof of the existence of $\mathcal{M}_{\mathrm{Whit}}^{\sigma+\gamma,*}$. Let. $\sigma \in \tilde{\mathcal{M}}, \gamma \in \Lambda_{\tilde{H}}$ with $\sigma + \gamma \in \Lambda$. Then the cokernel of $W^{\sigma,!*} \to W^{\sigma,*}$ admits a finite filtration by objects of the form $W^{\sigma',!*}$ with $\sigma' \leq \sigma$. For any such σ' we have $\sigma' + \gamma \in \Lambda$. So, the cokernel of

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*} \boxtimes e^{\gamma}) \to \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,*} \boxtimes e^{\gamma})$$

has a finite filtration with subquotients of the form $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma',!*} \boxtimes e^{\gamma})$ for $\sigma' \leq \sigma, \sigma' \in \Lambda^+_{\tilde{H}}$. For such σ' pick a presentation $\sigma' = \sigma_1 + \gamma_1$ with $\sigma' \in \tilde{\mathcal{M}}, \gamma_1 \in \Lambda^+_{\tilde{H}}$. Then $W^{\sigma',!*} \widetilde{\to} W^{\sigma_1,!*} * V^{\gamma_1}$. This gives

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma',!*} \boxtimes e^{\gamma}) \widetilde{\to} \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma_1} \boxtimes e^{\gamma} \otimes \mathrm{Res}^{T_{\tilde{H}}}(V^{\gamma_1}))$$

We see that all the irerducible subquotient of the latter lie in the subcategory

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\heartsuit} \subset \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{\tilde{q},x}(\tilde{G}))^{\heartsuit}$$

Thus, $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,*}\boxtimes e^\gamma)$ also lies in this subcategory, as it is closed under extensions.

$\square$

Maybe $\mathcal{M}^{\sigma+\gamma,*}_{\mathrm{Whit}}$ is a bad notation...

1.7.11. For 11.3.7. The displayed formula should be: if $\sigma\in\Lambda^+$ is restricted, $\gamma\in\Lambda_H$ then $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*}\boxtimes e^\gamma)\widetilde{\to}\mathcal{M}^{\sigma+\gamma,!*}_{\mathrm{Whit}}$ in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$.

More generally, the diagram commutes

$$\begin{array}{ccc}
\mathrm{Rep}(T_H)\otimes\mathrm{Whit}_{q,x}(G) & \to & \mathrm{Rep}(T_{\tilde H})\otimes\mathrm{Whit}_{\tilde q,x}(\tilde G)\\
\downarrow \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}} & & \downarrow \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}\\
\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) & \to & \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{\tilde q,x}(\tilde G)),
\end{array}$$

where the vertical arrow denotes respectivly the induction for G and $\tilde G$.

1.7.12. For 11.3.8, line 1: $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,*})$ does not make sense, we meant $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,*}\boxtimes e^?)$ I think. The same for line 1 of the proof: we meant $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,!*}\boxtimes e^?)$.

The functor

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}:\mathrm{Whit}_{q,x}(G)\otimes\mathrm{Rep}(T_H)\to\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$$

is t-exact, so $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*}\boxtimes e^\gamma)\to\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,*}\boxtimes e^\gamma)$ is injective for any $\sigma\in\Lambda^+,\gamma\in\Lambda_H$, and the quotient admits a finite filtration with the subquotients

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma',!*}\boxtimes e^\gamma)$$

for $\sigma'<\sigma$.

The displayed formula in the proof of 11.3.8 is wrong, it should be

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda_1,!*}\boxtimes e^\gamma)\widetilde{\to}\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}$$

What is the correct formulation? First case is as follows.

Lemma 1.7.13. *Assuming $[H,H]$ simply-connected. Let $\lambda\in\Lambda$ be written as $\lambda=\lambda_1+\gamma$ with $\lambda_1\in\Lambda^+$ restricted and $\gamma\in\Lambda_H$. Then in the notations of Section 1.7.10 of this file,*

$$\mathcal{M}^{\lambda,*}_{\mathrm{Whit}}=\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda_1,*}\boxtimes e^\gamma)\in\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^\heartsuit$$

receives a non-zero map from $\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}$, and the Jordan-Holder constituents of the quotient are of the form $\mathcal{M}^{\lambda',!*}_{\mathrm{Whit}}$ for $\lambda'<\lambda$.

Proof. The object $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda_1,!*}\boxtimes e^\gamma)$ is irreducible in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^\heartsuit$, now $W^{\lambda_1,!*}\to W^{\lambda_1,*}$ gives the desired injection, and the quoitient is equipped with a finite filtration whose subquotients are

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda'_1,!*}\boxtimes e^\gamma)$$

for some $\lambda'_1<\lambda_1$ with $\lambda'_1\in\Lambda^+$. Let now $\lambda'_1<\lambda_1$ with $\lambda'_1\in\Lambda^+$. It suffices to show that $\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda'_1,!*}\boxtimes e^\gamma)$ has a finite filtration with the successive quotients $\mathcal{M}^{\lambda',!*}_{\mathrm{Whit}}$

for some $\lambda' < \lambda$. Pick a decomposition $\lambda'_1 = \lambda'_2 + \gamma_2$, where $\lambda'_2 \in \Lambda^+$ is restricted and $\gamma_2 \in \Lambda^+_H$. Then

$$W^{\lambda'_1,!*} \widetilde{\to} W^{\lambda'_2,!*} * V^{\gamma_2}$$

in $\mathrm{Whit}_{q,x}(G)^{\heartsuit}$. Here $V^{\gamma_2} \in Irrep(H)$ with h.w. γ_2. We get

$$\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda'_1,!*} \boxtimes e^{\gamma}) \widetilde{\to} \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda'_2,!*} \boxtimes (e^{\gamma} \otimes \mathrm{Res}^{T_H}(V^{\gamma_2})))$$

Clearly, this object has the desired finite filtration. □

Let now $[H, H]$ by any. We apply 11.3.6 and chose an isogeny strictly compatible with the geometric metaplectic data. Then as in 11.2.8(c), we have a fully faithful embedding $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\heartsuit} \hookrightarrow (\mathrm{Rep}(\check{T}_0) \otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)))^{\heartsuit} \widetilde{\to} \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(\tilde{G}))^{\heartsuit}$. Then the above lemma for $\tilde{G}$ gives the following for G.

Lemma 1.7.14. *By Prop. 11.3.6 pick an isogeny strictly compatible with the geometric metaplectic data such that $[\tilde{H}, \tilde{H}]$ is simply-connected. Let $\lambda \in \Lambda$ be written as $\lambda = \lambda_1 + \gamma$ with $\lambda_1 \in \tilde{\Lambda}^+$ restricted for $\tilde{G}$, and $\gamma \in \Lambda_{\tilde{H}}$. Then in the notations of Section 1.7.10 of this file, $\mathcal{M}^{\lambda,*}_{\mathrm{Whit}}$ receives an injective map from $\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}$, and the quotient has a finite filtration with the sucessive quotients of the form*

$$\mathcal{M}^{\lambda',!*}_{\mathrm{Whit}}$$

with $\lambda' < \lambda$.

Proof. We may pick a splitting $s : \Lambda_0 \to \Lambda_{\tilde{H}}$ of (11.3) if necessary. The proof of the previous lemma goes though, since given $\lambda'_1 \in \tilde{\Lambda}^+$ with $\lambda'_1 < \lambda_1$ in $\tilde{\Lambda}^+$ we still have $\lambda'_1 + \gamma \in \Lambda$. □

1.7.15. For 11.3.9. In the very beginning of this section the following should be explained first. Consider G equipped with the factorizable gerbe $(\mathcal{G}^G)^{-1}$. Then the corresponding metaplectic Langlands dual group is again H canonically. We considered before the equivalence $\mathrm{Whit}_{q,x}(G)^{\vee} \widetilde{\to} \mathrm{Whit}_{q^{-1},x}(G)$ given by (7.9) in the paper.

Under this equivalence the action of $\mathrm{Rep}(H)$ via $Sat_{q^{-1},G}$ on $\mathrm{Whit}_{q^{-1},x}(G)$ correspond not to the natural action of $\mathrm{Rep}(H)$ on $\mathrm{Whit}_{q,x}(G)^{\vee}$ via $Sat_{q,G}$ but to a twist of this natural action. This is the true reason to introduce twists, and this should be well-explained! Namely, we have for $\mathcal{F} \in \mathrm{Whit}_{q,x}(G)^{\vee}, V \in \mathrm{Rep}(H)$

$$\mathcal{F} * Sat_{q^{-1},G}(V) \widetilde{\to} \mathcal{F} * Sat_{q,G}(\tau_H(V))$$

where in the RHS we mean the action of $\mathrm{Rep}(H)$ coming from its action on $\mathrm{Whit}_{q,x}(G)$ via $Sat_{q,G}$ by passing to the dual category.

Using 11.1.3 of the paper we get in the notations of Section 1.7.10 of this file the following. Given $\sigma \in \tilde{\Lambda}^+$ restricted, $\gamma \in \Lambda_{\tilde{H}}$ with $\sigma + \gamma \in \Lambda$ we get

$$\mathbb{D}(\mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*} \boxtimes e^{\gamma})) \widetilde{\to} \mathrm{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\sigma,!*} \boxtimes e^{\gamma})$$

in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q^{-1},x}(\tilde{G}))$ first. So, $\mathbb{D}(\mathcal{M}^{\sigma+\gamma,!*}_{\mathrm{Whit}}) \widetilde{\to} \mathcal{M}^{\sigma-\gamma,!*}_{\mathrm{Whit}}$ in the case when $[H, H]$ is simply-connected.

Now let $[H,H]$ be any. Then we apply the recipe of Sections 1.7.10 and Sect. 11.1.3 of the paper (and Section 1.7.1 of this file) to calculate the dual. For this we first need to answer

Question: how the equivalence 11.2.8(b) interacts with passing to dual categories? I think this should be explained in the paper, this is not clear!

Your formula for $\mathbb{D}(\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}})$ is not clear in the case when $[H,H]$ is not simply-connected. Indeed, for the definition of an irreducible we used the full subcategory $Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \subset Shv_{\mathcal{G}^{\tilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\tilde{G},x})$, and also a canonical functor

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \mathrm{Rep}(T_{\tilde{H}}) \otimes_{\mathrm{Rep}(T_H)} \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$$

sending z to the image of $e \boxtimes z$. How this interacts with the duality?

Namely, consider for the dual metaplectic data the natural inclusion $\mathrm{Whit}_{q^{-1},x}(G) \hookrightarrow \mathrm{Whit}_{q^{-1},x}(\tilde{G})$ commuting with $\mathrm{Rep}(H)$-actions via $Sat_{q^{-1},G}$ and pass to the dual categories, we get a functor $\mathrm{Whit}_{q,x}(\tilde{G}) \to \mathrm{Whit}_{q,x}(G)$ commuting with the induced $\mathrm{Rep}(H)$-actions via $Sat_{q^{-1},G}$. Since it commutes with $\mathrm{Rep}(H)$-actions via $Sat_{q^{-1},G}$, it also commutes with $\mathrm{Rep}(H)$-actions via $Sat_{q,G}$. So, we get a diagram

$$\mathrm{Whit}_{q,x}(G) \to \mathrm{Whit}_{q,x}(\tilde{G}) \to \mathrm{Whit}_{q,x}(G)$$

of functors commuting with $\mathrm{Rep}(H)$-actions via $Sat_{q,G}$. Is the composition the identity?

The formula

$$\mathbb{D}^{Verdier}(\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}) \widetilde{\to} \mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}$$

should be better explained in the case when $[H,H]$ is not simply-connected. One can simply say the following I hope. Assume we have chosen the isogeny strictly compatible with the metaplectic data for G. Then the diagram commutes

$$\begin{array}{ccc} (\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^c)^{op} & \hookrightarrow & (\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{\tilde{q},x}(\tilde{G}))^c)^{op} \\ \downarrow \mathbb{D} & & \downarrow \mathbb{D} \\ \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q^{-1},x}(G))^c & \hookrightarrow & \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{\tilde{q}^{-1},x}(\tilde{G}))^c, \end{array}$$

the horizontal arrows being natural fully faithful embeddings.

For 11.3.10: the category $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^{\le 0}$ is the smallest full subcategory of $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ containing $\mathcal{M}^{\mu,!*}_{\mathrm{Whit}}$ for $\mu \in \Lambda$ and closed under extensions and colimits. This implies 11.3.10.

1.7.16. For 11.4.2. Let us show that for a coroot α of G the element $\tilde{\check{\alpha}} := \ell_\alpha \tilde{\check{\alpha}}_H$, which is apriori a map $\tilde{\Lambda} \to \mathbb{Q}$ takes values in $\mathbb{Z}$. Since $\tilde{\check{\alpha}}_H$ is a coroot of $\tilde{H}$, $\ell_\alpha \tilde{\check{\alpha}}_H : \Lambda_{\tilde{H}} \to \mathbb{Z}$. So, it remains to show that for $\lambda \in \Lambda$, $\langle \ell_\alpha \tilde{\check{\alpha}}_H, \lambda \rangle \in \mathbb{Z}$. However, the composition $\Lambda_H \subset \Lambda_{\tilde{H}} \overset{\tilde{\check{\alpha}}_H}{\to} \mathbb{Z}$ equals $\check{\alpha}_H$, the corresponding coroot of H and $\check{\alpha}_H = \frac{\check{\alpha}}{\ell_\alpha}$ by construction of H. We get

$$\langle \ell_\alpha \tilde{\check{\alpha}}_H, \lambda \rangle = \langle \check{\alpha}, \lambda \rangle \in \mathbb{Z},$$

we are done.

Why $\{\tilde{\alpha} \in \tilde{\Lambda}, \tilde{\check{\alpha}} \in \tilde{\check{\Lambda}}\}$, as α runs through the coroots of G, forms a root datum? The equality $\langle \tilde{\alpha}, \tilde{\check{\alpha}} \rangle = 2$ is clear. Now, to see that $s_{\tilde{\check{\alpha}}} : \tilde{\Lambda} \to \tilde{\Lambda}$ preserves the set $\{\tilde{\alpha}\}$

(as α runs through the coroots of G), we use the following. First, this is clear for the root datum of $(\Lambda_{\tilde{H}}, \tilde{\alpha}_H, \check{\Lambda}_{\tilde{H}}, \tilde{\check{\alpha}}_H)$ of $\tilde{H}$. Now, to check an equality in some lattice $\mathcal{N}$, it suffices to check it in $\mathcal{N} \otimes \mathbb{Q}$, so this is automatic.

1.7.17. For 11.4.4. It is not true that for an affine curve the geometric metaplectic data are classified up to an isomorphism by the associated quadratic form. Namely, by [GLys, Cor. 3.3.6],

$$\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\pi_{1,alg}(G), e^{\times,tors}))) \widetilde{\to} \mathrm{FactGe}^0(\mathrm{Gr}_G)$$

For an affine curve $\mathrm{H}^2_{et}(X, \mathrm{Hom}(\pi_{1,alg}(G), e^{\times,tors}))$ is nonzero in general. So, in addition to the construction of $\tilde{q}$, we should extend some gerbe from the structure group $\mathrm{Hom}(\pi_{1,alg}(G), e^{\times,tors})$ to a gerbe with the structure group $\mathrm{Hom}(\pi_{1,alg}(\tilde{G}), e^{\times,tors})$. However, since $\mathrm{Ext}^1(\Lambda_0, e^{\times,tors}) = 0$, the map

$$\mathrm{Hom}(\pi_{1,alg}(\tilde{G}), e^{\times,tors}) \to \mathrm{Hom}(\pi_{1,alg}(G), e^{\times,tors})$$

is an isomorphism, and we are done.

1.7.18. For 11.4.5 line 4: replace "a map $\tilde{q}^\sharp : \tilde{\Lambda}^\sharp \to \pm 1$..." by "a linear map...".

Correct the last displayed formula, it should be

$$Quad(\tilde{\Lambda}, e^{\times,tors}(-1))^W_{restr} \subset Quad(\tilde{\Lambda}, e^{\times,tors}(-1))$$

If you want to write $q(\lambda) + \tilde{q}^\sharp(\tilde{\lambda}^\sharp)$ then you should say that we denote the operation on $e^{\times,tors}(-1)$ additively, apriori it is the product.

If we take for $\tilde{q}^\sharp : \tilde{\Lambda}^\sharp \to \pm 1$ any linear map extending $q^\sharp$ then it is indeed W-invariant. It suffices to show that $\tilde{q}^\sharp(s_\alpha(\tilde{\lambda}^\sharp)) = \tilde{q}^\sharp(\tilde{\lambda}^\sharp)$ for any simple coroot α of G. We denote by $\alpha_H = \ell_\alpha \alpha$ the corresponding root of H, this is also the corresponding root of $\tilde{H}$ via $\Lambda^\sharp \subset \tilde{\Lambda}^\sharp$. Write $\tilde{\check{\alpha}}_H$ for the corresonding coroot of $\tilde{H}$. Then $s_\alpha(\tilde{\lambda}^\sharp) = \tilde{\lambda}^\sharp - \langle \tilde{\lambda}^\sharp, \tilde{\check{\alpha}}_H \rangle \alpha_H$. Since $\tilde{q}^\sharp$ is linear, it suffices to show that $\tilde{q}^\sharp(\alpha_H) = 1$ in the multiplicative notation. But we have $\tilde{q}^\sharp(\ell_\alpha \alpha) = q(\alpha_H) = q(\alpha)^{\ell_\alpha^2} = 1$, as $\ell_\alpha = ord(q(\alpha))$.

By the way, for any central extension $1 \to \check{T}_0 \to \tilde{H} \to H \to 1$ the map

$$\mathrm{Hom}(\tilde{\Lambda}^\sharp, \mathbb{Z}/2\mathbb{Z})^W \to \mathrm{Hom}(\Lambda^\sharp, \mathbb{Z}/2\mathbb{Z})^W$$

is surjective. Indeed, we may view $\mathrm{Hom}(\Lambda^\sharp, \mathbb{Z}/2\mathbb{Z})^W$ as the subgroup Z_2 of elements of order 2 in the center $Z \subset H$. Similarly, $\mathrm{Hom}(\tilde{\Lambda}^\sharp, \mathbb{Z}/2\mathbb{Z})^W$ is the subgroup $\tilde{Z}_2$ of elements of order 2 in the center $\tilde{Z}$ of $\tilde{H}$.

We view $q^\sharp : \Lambda^\sharp \to \pm 1 \subset e^*$ as an element $\epsilon \in Z_2$. We have an exact sequence $1 \to \check{T}_0 \to \tilde{Z} \to Z \to 1$ of algebraic groups over e. We must show that $\tilde{Z}_2 \to Z_2$ is surjective. This follows from ([12], Theorem 1.1). Namely, any extension $1 \to \check{T}_0 \to ? \to \mu_2 \to 1$ splits.

So, we assume now $\tilde{q}^\sharp : \tilde{\Lambda}^\sharp \to \pm 1$ is W-invariant and extending $q^\sharp$. Then $\tilde{q}$ is correctly defined, extends q and $\tilde{q} \in Quad(\tilde{\Lambda}, e^{\times,tors}(-1))^W$. Let $\tilde{b} : \tilde{\Lambda} \times \tilde{\Lambda} \to e^{\times,tors}(-1)$ be the attached bilinear form. For $\lambda_i \in \Lambda, \tilde{\lambda}_i^\sharp \in \tilde{\Lambda}^\sharp$ we get

$$\tilde{b}(\lambda_1 + \tilde{\lambda}_1^\sharp, \lambda_2 + \tilde{\lambda}_2^\sharp) = b(\lambda_1, \lambda_2)$$

So, the kernel of $\tilde{b}$ is indeed $\tilde{\Lambda}^\sharp$.

It remains to verify that $\tilde{q}$ is restricted. We must show that for every coroot α of G, $\lambda\in\Lambda, \tilde{\lambda}^\sharp\in\tilde{\Lambda}^\sharp$ we have

$$\tilde{b}(\alpha,\lambda+\tilde{\lambda}^\sharp)=\langle \ell_\alpha\tilde{\check{\alpha}}_H, \lambda+\tilde{\lambda}^\sharp\rangle\tilde{q}(\alpha)$$

Here $\tilde{q}(\alpha)=q(\alpha)$ and $\tilde{b}(\alpha,\lambda+\tilde{\lambda}^\sharp)=b(\alpha,\lambda)=\langle\check{\alpha},\lambda\rangle q(\alpha)$. Since $\tilde{\check{\alpha}}_H:\tilde{\Lambda}^\sharp\to\mathbb{Z}$, $\langle\tilde{\check{\alpha}}_H,\tilde{\lambda}^\sharp\rangle\in\mathbb{Z}$ and $\ell_\alpha q(\alpha)=1$. We have $\langle\check{\alpha},\lambda\rangle=\langle\ell_\alpha\tilde{\check{\alpha}}_H,\lambda\rangle$, because the restriction of $\tilde{\check{\alpha}}$ to $\Lambda\subset\tilde{\Lambda}$ is $\check{\alpha}$ by definition. Prop. 11.3.6 is proved.

1.8. **For Part IV.**

1.8.1. For 12.1. Write $\mathfrak{L}(N)^{\omega^\rho}_x$ as a union of closed subschemes N_k, $k\ge 1$. We assume N_k is a placid group scheme, and for $i<j$, $N_i\subset N_j$ is a placid closed immersion. As in ([47], 1.2.8), we get the full embedding $Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{N_k}\subset Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ admitting a continuous right adjoint $\mathrm{Av}^{N_k}_*$, and

$$\bigcap_k Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{N_k}\widetilde{\to}\mathrm{SI}_{q,x}(G)$$

Note that $Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{N_k}\widetilde{\to} e_{N_k}-comod(Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}))$ by ([47], 1.3.12).

The full subcategory $Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{N_k}$ consists of $F\in Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ such that $\mathrm{Av}^{N_k}_*(F)\to F$ is an isomorphism, as for any colocalization. By Lemma 1.2.12 of this file, the inclusion $\mathrm{SI}_{q,x}(G)\subset Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ admits a maybe discontinuous right adjoint $\mathrm{Av}^{\mathfrak{L}(N)^{\omega^\rho}_x}_*$ given by $\mathrm{Av}^{\mathfrak{L}(N)^{\omega^\rho}_x}_*=\lim_{k\in\mathbb{N}^{op}}\mathrm{Av}^{N_k}_*$.

By Lemma 1.2.14 and the section just after it of this file, in the constructible context we have the left adjoint

$$\mathrm{Av}^{N_k}_!: Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\to Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{N_k}$$

to the inclusion. Moreover, the left adjoint

$$\mathrm{Av}^{\mathfrak{L}(N)^{\omega^\rho}_x}_!: Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\to Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{\mathfrak{L}(N)^{\omega^\rho}_x}$$

to the inclusion also exists and is given by $\mathrm{Av}^{\mathfrak{L}(N)^{\omega^\rho}_x}_!\widetilde{\to}\mathrm{colim}_{k\in\mathbb{N}}\mathrm{Av}^{N_k}_!$ by Lemma 1.2.15 of this file.

For $\lambda\le\lambda'\in\Lambda$ we have the commutative diagram

$$\begin{array}{ccc} Shv_{\mathcal{G}^G}(\bar{S}^{\lambda'}) & \leftarrow & Shv_{\mathcal{G}^G}(\bar{S}^{\lambda'})^{\mathfrak{L}(N)^{\omega^\rho}_x}\\ \downarrow && \downarrow\\ Shv_{\mathcal{G}^G}(\bar{S}^{\lambda}) & \leftarrow & Shv_{\mathcal{G}^G}(\bar{S}^{\lambda})^{\mathfrak{L}(N)^{\omega^\rho}_x}, \end{array}$$

where the vertical arrows are !-pull-backs, and the diagram

$$\begin{array}{ccc} Shv_{\mathcal{G}^G}(\bar{S}^{\lambda'}) & \leftarrow & Shv_{\mathcal{G}^G}(\bar{S}^{\lambda'})^{\mathfrak{L}(N)^{\omega^\rho}_x}\\ \uparrow && \uparrow\\ Shv_{\mathcal{G}^G}(\bar{S}^{\lambda}) & \leftarrow & Shv_{\mathcal{G}^G}(\bar{S}^{\lambda})^{\mathfrak{L}(N)^{\omega^\rho}_x}, \end{array}$$

where the vertival arrows are $*$-pushouts. Now $\mathrm{SI}_{q,x}(G)\widetilde{\to}\lim_\lambda Shv_{\mathcal{G}^G}(\bar{S}^\lambda)^{\mathfrak{L}(N)^{\omega^\rho}_x}$ with respect to the !-pullbacks. Passing to left adjoints, this rewrites as

$$\mathrm{SI}_{q,x}(G)\widetilde{\to}\mathrm{colim}_\lambda Shv_{\mathcal{G}^G}(\bar{S}^\lambda)^{\mathfrak{L}(N)^{\omega^\rho}_x}$$

with respect to the $*$-pushouts. The functor $(\bar{i}_\lambda)_! : \mathrm{SI}_{q,x}(G)_{\le\lambda} := Shv_{\mathcal{G}^G}(\bar{S}^\lambda)^{\mathfrak{L}(N)^{\omega^\rho}_x} \to \mathrm{SI}_{q,x}(G)$ is fully faithful, as this is so before taking the invariants.

For $j_\lambda : S^\lambda \to \bar{S}^\lambda$ we get the adjoint pair

$$j_\lambda^* : \mathrm{SI}_{q,x}(G)_{\le\lambda} \leftrightarrows Shv_{\mathcal{G}^G}(S^\lambda)^{\mathfrak{L}(N)^{\omega^\rho}_x} =: \mathrm{SI}_{q,x}(G)_{=\lambda} : (j_\lambda)_*$$

with $(j_\lambda)_*$ fully faithful.

As in Section 1.2.22 of this file, the subcategories $\mathrm{SI}_{q,x}(G)_{\le\lambda-\alpha_i} \subset \mathrm{SI}_{q,x}(G)_{\le\lambda}$ for all simple coroots α_i, generate the full subcategory $\mathrm{SI}_{q,x}(G)_{<\lambda} \subset \mathrm{SI}_{q,x}(G)_{\le\lambda}$ consisting of F such that F is the extension by zero under $\bar{S}^\lambda - S^\lambda \hookrightarrow \bar{S}^\lambda$. The essential image of $(j_\lambda)_* : \mathrm{SI}_{q,x}(G)_{=\lambda} \to \mathrm{SI}_{q,x}(G)_{\le\lambda}$ is the right orthogonal to $\mathrm{SI}_{q,x}(G)_{<\lambda}$.

As in the proof of 6.2.9, for any $\lambda \in \Lambda$ and $k \ge 1$ we have

$$\mathrm{Av}_!^{N_k}(\delta_{t^\lambda,\mathrm{Gr}}) \widetilde{\to} e_{N_k t^\lambda}[2\dim N_k t^\lambda] \widetilde{\to} \omega_{N_k t^\lambda},$$

and the corresponding monad on Vect is the identity. So, the functor $\mathrm{Vect} \to \mathrm{SI}_{q,x}(G)_{=\lambda}$ sending e to $\overset{\circ}{\mathcal{W}}{}^{\lambda,!} := \mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x} \delta_{t^\lambda,\mathrm{Gr}} \widetilde{\to} \mathrm{colim}_k(\mathrm{Av}_!^{N_k} \delta_{t^\lambda,\mathrm{Gr}}) \widetilde{\to} \omega_{S^\lambda}$ is an equivalence. Since $e \in Vect^c$, $\overset{\circ}{\mathcal{W}}{}^{\lambda,!} \in \mathrm{SI}_{q,x}(G)_{=\lambda}$ is compact.

Let $\mathcal{W}^{\lambda,!} = (j_\lambda)_! \overset{\circ}{\mathcal{W}}{}^{\lambda,!} \in \mathrm{SI}_{q,x}(G)_{\le\lambda}$. Then $\mathcal{W}^{\lambda,!} \in \mathrm{SI}_{q,x}(G)^c$, because $(j_\lambda)_! : \mathrm{SI}_{q,x}(G)_{=\lambda} \to \mathrm{SI}_{q,x}(G)_{\le\lambda}$ preserve compact objects. Similarly, $(i_\lambda)_! : \mathrm{SI}_{q,x}(G)_{=\lambda} \to \mathrm{SI}_{q,x}(G)$ preserves compact objects, it has a continuous right adjoint $(i_\lambda)^!$. (This is both for the constructible context and for $\mathcal{D}$-modules, as ω is holonomic on a scheme of finite type). The existence of $(j_\lambda)_! : \mathrm{SI}_{q,x}(G)_{=\lambda} \to \mathrm{SI}_{q,x}(G)_{\le\lambda}$ is explained in. ([27], 1.4.2).

1.8.2. For 12.1.3. In Sect. 6.2.2 of the paper we have chosen trivializations of the fibres of the gerbe $\mathcal{G}^G$ at t^λ for $\lambda \in \Lambda$. For $\lambda = 0$ this gerbe is already trivialized, for this reason the equivalence $\mathrm{SI}_{q,x}(G)_{=0} \widetilde{\to} \mathrm{Vect}$ is canonical.

1.8.3. For 12.1.4. For $\lambda \in \Lambda$ consider $\mathcal{W}^{\lambda,*} := (j_\lambda)_* \overset{\circ}{\mathcal{W}}{}^{\lambda,!} \in \mathrm{SI}_{q,x}(G)_{\le\lambda}$. Why they are not compact? (This is affirmed in [27]).

Let $A = \{\mu \in \Lambda \mid \mu \le \lambda\}$ with the usual order. If $\mu' \le \mu \le \lambda$ then $\bar{S}^\lambda - \bar{S}^\mu \subset \bar{S}^\lambda - \bar{S}^{\mu'}$ is open, and $\bar{S}^\lambda = \cup_{\mu\in A} \bar{S}^\lambda - \bar{S}^\mu$. Let $\tau_\mu : \bar{S}^\lambda - \bar{S}^\mu \subset \bar{S}^\lambda$ be the open immersion. The natural map

$$\mathcal{W}^{\lambda,*} \to \underset{\mu\in A^{op}}{\mathrm{colim}}\, (\tau_\mu)_! \tau_\mu^* \mathcal{W}^{\lambda,*}$$

is an isomorphism in $\mathrm{SI}_{q,x}(G)$, because the property of being an isomorphism of sheaves is local in Zariski topology. Recall that $\mathcal{S} \mapsto Shv(\mathcal{S})$ satisfies the etale descent for any sheaf theory.

Here A^{op} is filtered. Is the natural map

$$\mathrm{Map}(\mathcal{W}^{\lambda,*}, \underset{\mu\in A^{op}}{\mathrm{colim}}\, (\tau_\mu)_! \tau_\mu^* \mathcal{W}^{\lambda,*}) \leftarrow \underset{\mu\in A^{op}}{\mathrm{colim}}\, \mathrm{Map}(\mathcal{W}^{\lambda,*}, (\tau_\mu)_! \tau_\mu^* \mathcal{W}^{\lambda,*})$$

in Spc an isomorphism? Any object in the RHS comes from some map $\mathcal{W}^{\lambda,*} \to (\tau_\mu)_! \tau_\mu^* \mathcal{W}^{\lambda,*}$ for some $\mu \le \lambda$ by ([43], 13.1.4). Assume the canonical map $\mathcal{W}^{\lambda,*} \to \underset{\mu\in A^{op}}{\mathrm{colim}}\, (\tau_\mu)_! \tau_\mu^* \mathcal{W}^{\lambda,*}$ factors through $(\tau_\mu)_! \tau_\mu^* \mathcal{W}^{\lambda,*}$. This would mean that $(\bar{i}_\mu)^* \mathcal{W}^{\lambda,*} = 0$, and probably this is wrong.

It is affirmed in [27] that $\mathcal{W}^{\lambda,*}$ are not compact. Maybe this can be done using the relation with the global geometry and using the resulutions from [11].

1.8.4. For 12.1.5. The definition of $\mathrm{SI}_{q,\mathrm{Ran}}(G)$ is similar to the case of $\mathrm{Whit}_{q,\mathrm{Ran}}(G)$ discussed in Sections 1.3.3-1.3.13 of this file. Recall that $\mathrm{Ran} \widetilde{\to} \mathrm{colim}_I X^I$ taken over the category $(\mathcal{F}\mathrm{in}^{surj})^{op}$, here $\mathcal{F}\mathrm{in}^{surj}$ is the category of finite nonempty sets and surjective maps. For $I \in \mathcal{F}\mathrm{in}^{surj}$ let $\mathrm{Gr}^{\omega^\rho}_{G,I} = X^I \times_{\mathrm{Ran}} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$, so $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}} \widetilde{\to} \underset{I\in(\mathcal{F}\mathrm{in}^{surj})^{op}}{\mathrm{colim}} \mathrm{Gr}^{\omega^\rho}_{G,I}$.

For each I we have a full subcategory

$$\mathrm{SI}_I = Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})^{\mathfrak{L}(N)^{\omega^\rho}_I} \subset Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})$$

and by definition

$$\mathrm{SI}_{q,\mathrm{Ran}}(G) \widetilde{\to} \underset{I\in(\mathcal{F}\mathrm{in}^{surj})}{lim} \mathrm{SI}_I \subset \underset{I\in(\mathcal{F}\mathrm{in}^{surj})}{\lim} Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,I})) \widetilde{\to} Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$$

Here we used ([46], 0.0.42).

1.8.5. For 12.1.6. The definition of $\bar{S}^0_{\mathrm{Ran}}$ should be corrected as follows. This correction is essential in the case when $[G,G]$ is not simply-connected in view of Schieder's paper ([58], Section 7.2). Fix an exact sequence $1 \to Z \to \tilde{G} \to G \to 1$ with $[\tilde{G},\tilde{G}]$ simply-connected and Z a connected central torus in $\tilde{G}$. The coroots lattices for $\tilde{G}$ and G are naturally isomorphic, so ω^ρ can be seen as a $\tilde{G}$-torsor on X. We have a natural map $\mathrm{Gr}^{\omega^\rho}_{\tilde{G},\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$, and the prestack $\bar{S}^0_{\mathrm{Ran}}$ defined for $\tilde{G}$ as in the paper. By definition, for G the prestack $\bar{S}^0_{\mathrm{Ran}}$ is defined as the same prestack for $\tilde{G}$. It is independent of a choice of $\tilde{G}$.

1.8.6. For 12.2.1. When you say in • $\mathcal{P}_G$ is a G-bundle. Say G-torsor on what...

We should explain somewhere the following. Given $S \in \mathrm{Sch}_{ft}$ and $S \to \mathrm{Conf}$, we may talk about $(S \times X) - \mathrm{supp}\, D$. Namely, it is understood that we pick a homomorphism $\tau : \Lambda^{neg} \to \mathbb{Z}$ sending each negative simple coroot to a strictly positive integer. Applying this gives an S-point of $X^{(n)}$ (or maybe of finite union of such for several n). Now for $S \to X^{(n)}$ we get the corresponding relative effective Cartier divisor $D' \hookrightarrow S \times X$, here D is flat over S, then $(S \times X) - D'$ is the desired scheme. It does not depend on a choince of τ.

1.8.7. For 12.2.2. Given $D = \sum_k \mu_k x_k \in \mathrm{Conf}$, the fibre of $\bar{S}^{\mathrm{Conf}}_{\mathrm{Conf}} \to \mathrm{Conf}$ over D is $\prod_k \bar{S}^{\mu_k}_{x_k}$. The fibre of $S^{\mathrm{Conf}}_{\mathrm{Conf}} \to \mathrm{Conf}$ over D is $\prod_k S^{\mu_k}_{x_k}$.

1.8.8. For 12.2.3. The section of $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Conf}} \to \mathrm{Conf}$ used in 12.2.3 sends D to $\omega^\rho(-D)$.

1.8.9. For 12.2.4. Consider the closed subscheme in $X^I \times X^{(n)}$ given by the property that for the collection $((x_i), D)$ we have $D \le n(\sum_i x_i)$. On the other hand, for $S \in \mathrm{Sch}^{aff}$ and a pair $S \overset{\mathcal{J}}{\to} X^I, S \overset{D}{\to} X^{(n)}$ we may consider the closed subscheme $\tilde{D} \subset S \times X$ defined by D and require that $\tilde{D}$ factors through $\mathcal{D}_{\mathcal{J}} \subset S \times X$. Does this define the same subfunctor of $X^I \times X^{(n)}$? Recall that $\mathcal{D}_{\mathcal{J}}$ is the affine scheme corresponding to the formal scheme $\hat{\mathcal{D}}_{\mathcal{J}}$.

The final definition of $(Conf \times \mathrm{Ran})^{\subset}$ is as follows. Pick a homomorphism $\Lambda \to \mathbb{Z}$ sending each simple coroot to a positive integer. Let $S \in \mathrm{Sch}^{aff}$ and we are given $\mathcal{J} \subset \mathrm{Hom}(S, X), D \in \mathrm{Hom}(S, Conf)$. Let $\mathrm{Conf}_{\mathbb{Z}}$ be similarly defined scheme for Λ replaced by $\mathbb{Z}$, $D_{\mathbb{Z}} : S \to \mathrm{Conf}_{\mathbb{Z}}$ the corresponding point. Then $\mathrm{D}_{\mathbb{Z}}$ yields a closed subscheme $\tilde{D} \subset S \times X$, and we require that $(S \times X) - \mathcal{D}_{\mathcal{J}} \subset (S \times X) - \tilde{D}$.

The formula (12.2) is wrong, in the LHS there are additional factors $\mathrm{Gr}_{G,x}$ for some points x which are in the collection $\mathcal{J}$ but not in the support of the divisor D. But I think we don't need this isomorphism. The "consequence" of it in 12.2.5 is correct, I think.

1.8.10. For 12.2.5. Note that $\bar{S}^{\mathrm{Conf}}_{\mathrm{Ran}} \hookrightarrow \bar{S}^{0}_{\mathrm{Ran}} \times_{\mathrm{Ran}} (\mathrm{Conf} \times \mathrm{Ran})^{\subset}$ is a closed subfunctor.

First, a point of $\bar{S}^{0}_{\mathrm{Ran}} \times_{\mathrm{Ran}} (\mathrm{Conf} \times \mathrm{Ran})^{\subset}$ is $(D, (x_i)) \in (\mathrm{Conf} \times \mathrm{Ran})^{\subset}$, $\mathcal{F}_G$ on X with $\beta : \mathcal{F}_G \widetilde{\to} \omega^{\rho} \mid_{X-(x_i)}$ such that $\omega^{\rho} \subset \mathcal{F}_G$ defines a generalized B-structure on X. This points lies in $\bar{S}^{\mathrm{Conf}}_{\mathrm{Ran}}$ if the trivialization β extends to $\beta : \mathcal{F}_G \widetilde{\to} \omega^{\rho} \mid_{X-\mathrm{supp}(D)}$ first, and moreover β defines a generalized B-structure $\omega^{\rho}(-D) \subset \mathcal{F}_G$. Both conditions are closed.

The map $\bar{i}^{\mathrm{Conf}}_{\mathrm{Ran}}$ is ind-proper (or pseudo-proper).

1.8.11. For 12.2.7. Now an **IMPORTANT change of notations**: as we discussed by skype, I will assume from now on that the objects denoted by $S^{\lambda}_{\mathrm{Ran}}, \bar{S}^{\lambda}_{\mathrm{Ran}}$ in Section 12.2.7 for $\lambda < 0$ are denoted, say by $\mathcal{S}^{\lambda}_{\mathrm{Ran}}, \bar{\mathcal{S}}^{\lambda}_{\mathrm{Ran}}$ respectively.

Now we denote for $\lambda \leq 0$ by $\bar{S}^{\lambda}_{\mathrm{Ran}}$ the following prestack. Pick $\tilde{G}$ as above, so that $[\tilde{G}, \tilde{G}]$ is simply-connected. Then $\bar{S}^{\lambda}_{\mathrm{Ran}}$ classifies $\mathcal{J} \in \mathrm{Ran}$, $(D, \mathcal{J}) \in (\mathrm{Conf}^{\lambda} \times \mathrm{Ran})^{\subset}$, a $\tilde{G}$-torsor $\mathcal{F}$ on X with an isomorphism $\mathcal{F} \widetilde{\to} \omega^{\rho} \mid_{X-\Gamma_{\mathcal{J}}}$ such that for each $\check{\lambda} \in \check{\Lambda}^{+}_{\tilde{T}}$ the map

$$\check{\lambda}(\omega^{\rho}(-D)) \to \tilde{\mathcal{V}}^{\check{\lambda}}_{\mathcal{F}}$$

is regular over X. Here $\tilde{\mathcal{V}}^{\check{\lambda}}$ is the corresponding Weyl module for $\tilde{G}$, and the above map is over $(\mathrm{Conf}^{\lambda} \times \mathrm{Ran})^{\subset} \times X$.

The difference with $\bar{\mathcal{S}}^{\lambda}_{\mathrm{Ran}}$ is that for this new $\bar{S}^{\lambda}_{\mathrm{Ran}}$ we do not require the trivialization $\mathcal{F} \widetilde{\to} \omega^{\rho} \mid_{X-\Gamma_{\mathcal{J}}}$ to extend to $X - \mathrm{supp}(D)$.

Now $S^{\lambda}_{\mathrm{Ran}} \subset \bar{S}^{\lambda}_{\mathrm{Ran}}$ is defined by requiring that the above map has no zeros, so that it defines a $\tilde{B}$-structure on $\mathcal{F}$ with the corresponding T-torsor $\omega^{\rho}(-D)$.

It would help also to add the following. If $D = \sum_x \lambda_x x \in Conf^{\lambda}$ for some $\lambda_x \in \Lambda^{neg}$ and $(D, \mathcal{J}) \in (\mathrm{Conf} \times \mathrm{Ran})^{\subset}$ is a k-point then the fibre of the projection $\bar{\mathcal{S}}^{\lambda}_{\mathrm{Ran}} \to (\mathrm{Conf} \times \mathrm{Ran})^{\subset}$ over this point identifies with $\prod_x \bar{S}^{\lambda_x}$, and a similar claim for $\mathcal{S}^{\lambda}_{\mathrm{Ran}}$.

I propose not to define $\bar{\mathcal{S}}^{\lambda}_{\mathrm{Ran}}$ for $\lambda = 0$. However, for $\lambda = 0$, we may set $\mathcal{S}^{0}_{\mathrm{Ran}} = \mathrm{Ran}$.

For convenience, for $\lambda \in \Lambda^{neg} - 0$ the prestack $\bar{\mathcal{S}}^{\lambda}_{\mathrm{Ran}}$ classifies: $(D, (x_i)) \in (\mathrm{Conf} \times \mathrm{Ran})^{\subset}$, $\mathcal{F}_G$ on X with $\beta : \mathcal{F}_G \widetilde{\to} \omega^{\rho} \mid_{X-\mathrm{supp}(D)}$ such that β defines a generalized B-structure $\omega^{\rho}(-D) \subset \mathcal{F}_G$ lying in the true $\overline{\mathrm{Bun}}_B$. The open immersion $j^{\lambda}_{\mathrm{Ran}} : \mathcal{S}^{\lambda}_{\mathrm{Ran}} \to \bar{\mathcal{S}}^{\lambda}_{\mathrm{Ran}}$ is given by the condition that the generalized B-structure $\omega^{\rho}(-D) \subset \mathcal{F}_G$ is in fact a true B-structure.

1.8.12. For 12.2.10. After the correction of the definitions of $\bar{S}^0_{\mathrm{Ran}}$, Lemma 12.2.10 is true in general. Recall that if $\mu \in \Lambda$ and $\langle \mu, \check{\lambda} \rangle \leq 0$ for all $\check{\lambda}$ dominant, this does not imply that $\mu \in \Lambda^{neg}$. I assume the definition of $\bar{S}^0_{\mathrm{Ran}}$ is corrected in in the style of Schieder's paper ([58], Section 7).

1.8.13. For 12.3.3. The reference [Ga7, Cor. 1.4.5] should be [Ga7, Cor. 1.5.3] for the last version of [Ga7]. It is important for using [26] that for a finite nonempty set I the projection $p^\lambda_I : S^\lambda_I \to (\mathrm{Conf}^\lambda \times X^I)^\subset$ is ind-schematic of ind-finite type, so the functor $(p^\lambda_I)^!$ is defined, we remain in the category PreStk_{lft}. For $F \in Shv((\mathrm{Conf}^\lambda \times X^I)^\subset)$, $K \in Shv(S^\lambda_I)$ we have the projection formula $(p^\lambda_I)_*((p^\lambda_I)^! F \otimes^! K) \widetilde{\to} F \otimes^! (p^\lambda_I)_* K$. This was used in ([26], 1.5.3).

In the constructible context the existence of $(i^\lambda_{\mathrm{Ran}})^*$ follows from the fact that i^λ_{Ran} is schematic of finite type.

In ([26], proof of 1.7.3) Dennis claimed that any object of $Shv(\bar{S}^0_I)$ is $\mathbb{G}_m$-monodromic. He meant instead that any object of $\mathrm{SI}^{\leq 0}_I$ is $\mathbb{G}_m$-monodromic (here $\mathbb{G}_m$-action comes from T-action by restricting via a regular character).

Definition: if G is an algebraic group of finite type acting on $Z \in \mathrm{PreStk}_{lft}$ then $Shv(Z)^{G-mon}$ is the full subcategory generated by the essential image of $Shv(Z/G) \to Shv(Z)$ in the case of $\mathcal{D}$-module. In the constructible context according to ([16], Sect. 0.4), the definition changes as follows: $Shv(Z)^{G-mon} \subset Shv(Z)$ is obtained from the essential image of $Shv(Z/G) \to Shv(Z)$ by adding objects obtained by finite iteration of the procedure of taking cone of a morphism.

In ([26], Pp. 1.5.3) the following is proved. For a finite set consider the corresponding versions $S^\lambda_I, S^{-,\lambda}_I, (X^\lambda \times X^I)^\subset$ as in *loc.cit.* and the diagram

$$(X^\lambda \times X^I)^\subset \overset{p^{-,\lambda}_I}{\leftarrow} S^{-,\lambda}_I \overset{i^{-,\lambda}}{\to} \mathrm{Gr}_{G,I} \overset{i^\lambda}{\leftarrow} S^\lambda_I \overset{p^\lambda_I}{\to} (X^\lambda \times X^I)^\subset$$

Then $i^\lambda_* : \mathrm{SI}^{=\lambda}_I \to \mathrm{SI}^{\leq 0}_I$ has a left adjoint given by $(p^\lambda_I)^! (p^{-,\lambda}_I)_* (i^{-,\lambda})^!$. This immediately gives the base change property in ([26], Pp. 1.5.3, (c)).

1.8.14. For 12.3.4. We meant here the reference to ([26], 1.5.6). The explanation in ([26], 1.5.6) is insufficient, Dennis should explain what he means by "a formal Cousin argument" in ([26], 1.5.3). In the case of stratification with two strata this is Lemma 1.8.16 below.

One has the following.

Lemma 1.8.15. *Let $i_* = i_! : C' \to C$ be a fully faithful functor in* DGCat_{cont}.
1) The following conditions are equivalent:
A) $i_!$ is a colocalization, that is, admits a right adjoint $i^!$ such that $\mathrm{id} \to i^! i_!$ *is an isomorphism;*
B) for any $c \in C$ there is an exact triangle $c' \to c \to d$ in C with $c' \in C'$ such that for each $c_1 \in C'$, $\mathcal{H}om(c_1, d) = 0$ in Vect.

If these conditions hold, we may identify C/C' with the right orthogonal

$$C'^r = \{z \in C \mid \text{for any } c_1 \in C', \mathcal{H}om(c_1, z) = 0 \in \mathrm{Vect}\}$$

and obtain a pair of adjoint functors $j^*: C \rightleftarrows C'^r : j_*$ *such that* $j^*j_* \to \mathrm{id}$ *is an isomorphism, here* j_* *is the inclusion. The exact triangle from B) then becomes* $i_!i^!c \to c \to j_*j^*c$.
2) Dually, the following conditions are equivalent.
A') i_* is a localization, that is, admits a left adjoint i^* such that $i^*i_* \to \mathrm{id}$ is an isomorphism;
B') for any $c \in C$ there is an exact triangle $d \to c \to c'$ with $c' \in C'$ such that for each $c_1 \in C'$, $\mathcal{H}om(d, c_1) = 0$ in Vect.

If these conditions hold, we identify the left orthogonal $C'^l = \{z \in C \mid \text{for any } c_1 \in C', \mathcal{H}om(z, c_1) = 0 \in \mathrm{Vect}\}$ *with C/C' and obtain an adjoint pair* $j_! : C'^l \leftrightarrows C : j^!$ *such that* $\mathrm{id} \to j^!j_!$ *is an isomorphism, here $j_!$ is the inclusion. The exact triangle from B') then becomes* $j_!j^!c \to c \to i_*i^*c$. □

Proof. In 2) it is clear that A') and B') are equivalent, and we get the functors $j_! : C'^l \leftrightarrows C : j^!$, where $j^!$ is maybe discontinuous. Besides, for each $c \in C$ the fibre sequence from B') becomes $j_!j^!c \to c \to i_*i^*c$. In partcular, $j^!i_* = 0$. The functor $j^!$ is continuous. To see this, it suffices to show that $j_!j^! : C \to C$ is continuous. However, $j_!j^!$ is $c \mapsto Cofib(c[-1] \to i_*i^*c[-1])$. Since i_*, i^* are continuous, we conclude that $j^!$ is continuous.

Since $j^!i_* = 0$, the functor $j^!$ factors as $C \xrightarrow{h} C/C' \xrightarrow{\tau} C'^l$, where τ is continuous, and h is the projection. Let $\eta = hj_!$. Now $\mathrm{id} \widetilde{\to} j^!j_!$ gives an isomorphism $\tau\eta \widetilde{\to} \mathrm{id}$. Since h^R is fully faithful, h is a localization functor, so is essentially surjective. For $c \in C$ the above fibre sequence gives $\eta j^!(c) \widetilde{\to} hj_!j^!(c) \widetilde{\to} h(c)$, so η is essentially surjective.

We have $\eta\tau h \widetilde{\to} hj_!j^! \widetilde{\to} h$ naturally as functors $C \to C/C'$. Indeed, the above fibre sequence gives $hj_!j^!(c) \widetilde{\to} h(c)$ fucntorially for $c \in C$. Since h^R is fully faithful, multiplying the above isomorphism by h^R on the right, we get an isomorphism $\eta\tau hh^R \widetilde{\to} hh^R$. Now, $hh^R \widetilde{\to} \mathrm{id}$, so this gives an isomorphism $\eta\tau \widetilde{\to} \mathrm{id}$. Thus, τ and η are mutually inverse equivalences. □

In Lemma 1.8.15 1), the functor j_* is a map in DGCat, it is not necessarily continuous (if $i^!$ is continuous then j_* is also continuous).

Lemma 1.8.16. *Let $C^0, C, C' \in \mathrm{DGCat}_{cont}$ and we are given adjoint pairs* $j^! = j^* : C \leftrightarrows C^0 : j_*$, $i_! = i_* : C' \leftrightarrows C : i^!$, *and* $i^* : C \leftrightarrows C' : i_*$. *Assume that for any $F \in C$ the triangle is exact* $i_*i^!F \to F \to j_*j^*F$. *Assume in addition that* $i^*i_* \to \mathrm{id}$, *and* $\mathrm{id} \to i^!i_!$, *and* $j^*j_* \to \mathrm{id}$ *are isomorphisms, so i_* and j_* are fully faithful. Consider the functor* $h : C_0 \to C$ *sending F to*

$$\mathrm{Fib}(j_*F \to i_*i^*j_*F)$$

Then h is left adjoint to $j^!$.

Proof. Note that $Cofib(C' \xrightarrow{i_!} C)$ in DGCat_{cont} identifies with C_0, namely we may pass to left adjoint in the diagram $C_0 \widetilde{\to} \mathrm{Fib}(C \xrightarrow{i^!} C')$ in DGCat_{cont}. So, we may think the input datum is just a fully faithful embedding $i_! : C' \to C$ admitting a continuous right adjoint $i^!$ and a left adjoint i^* such that both A, A' of Lemma 1.8.15 hold.

Note that $j_* : C_0 \widetilde{\to} \{F \in C \mid i^!F = 0\} = C'^r$ is an equivalence. So, we identify C_0 with C'^r via j_*. From now on, $j^* : C \rightleftarrows C'^r : j_*$, and $h : C'^r \to C$.

Write temporarily $j_{l!} : C'^l \to C$ for the inclusion and $j^{l!} : C \to C'^l$ for its left adjoint given by Lemma 1.8.15. Then from Lemma 1.8.15 we see that the composition $C'^r \stackrel{j_*}{\hookrightarrow} C \stackrel{j^{l!}}{\to} C'^l$ factors as $C'^r \stackrel{j_*}{\hookrightarrow} C \to C/C' \widetilde{\to} C'^l$, where the composition of the first two arrows is an equivalence. Thus, $j^{l!} j_* : C'^r \to C'^l$ is an equivalence. Again from Lemma 1.8.15 we see that $j_{l!} j^{l!} = \mathrm{Fib}(\mathrm{id} \to i_* i^*)$, and our claim follows.

Once again, the functor $C'^r \to C'^l$ sending $c \in C'^r$ to $\mathrm{Fib}(c \to i_* i^* c)$ is an equivalence. □

If in addition $A \in CAlg(\mathrm{DGCat}_{cont})$ and all the categories and functors in the inputs of Lemma 1.8.16 are maps in $A-mod(\mathrm{DGCat}_{cont})$ then h is also a map in $A-mod(\mathrm{DGCat}_{cont})$. This is why in ([26], Cor. 1.5.6(b)) the functor $(i^\lambda)_!$ commutes with the actions of $Shv(X^I)$.

We used here the fact that $\mathrm{oblv} : A-mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves limits and colimits.

In the constructible context the existence of $(i^\lambda_{\mathrm{Ran}})_! : \mathrm{SI}^{=\lambda}_{q,\mathrm{Ran}} \to \mathrm{SI}^{\le 0}_{q,\mathrm{Ran}}$ is automatic, this is the usual !-extension for sheaves, and it preserves the equivariance condition.

Lemma 1.8.17. *Let $C \in \mathrm{DGCat}$, assume given an adjoint pair $i : B \leftrightarrows C : i^!$ in DGCat_{cont} with i fully faithful. Set $\mathcal{A} = ii^!$, this is a continuous e-linear comonad on C. Let $D = \mathrm{Ker}(i^!)$. . Then $i^!$ factors naturally as $C \stackrel{h}{\to} C/D \stackrel{a}{\to} B$, where a is an equivalence.*

Proof. By assumptions, $i^!$ is continuous. We have an equvalence $C/D \widetilde{\to} \mathrm{Ker}(j^R)$, where $j : D \to C$ is the inclusion, and $j^R : C \to D$ is the right adjoint of j.

We claim that a and hi are quasi-inverses of each other. Indeed, on one hand, the natural map $\mathrm{id} \to i^! i \widetilde{\to} ahi$ is an isomorphism, because i is fully faithful. To show that $(hi)a$ is isomorphic to id, it suffices to show that $hiah \widetilde{\to} h$, because h is a localization functor. For $x \in C$ we have a functorial in $x \in C$ fibre sequence $ii^! x \to x \to y$ with $y \in D$, hence $h(ii^! x) \to h(x) \to h(y)$ is a fibre sequence in C/D. Here $h(y) = 0$, so $hii^! \widetilde{\to} h$. □

Any idempotent continuous e-linear comonad on C is of the form $ii^!$ as in the previous lemma, where $B = \mathcal{A} - comod(C)$.

Lemma 1.8.17 can be strenthened as follows.

Lemma 1.8.17': Let $L : C \leftrightarrows D : R$ be an adjoint pair in DGCat with R fully faithful, so L is a map in DGCat_{cont}. Let $C_0 = \mathrm{Ker}(L)$. Then L factors naturally as $C \stackrel{h}{\to} C/C_0 \stackrel{a}{\to} D$, where a is an equivalence.

Proof. First, h has a fully faithful right adjoint h^R, because $C/C_0 = \mathrm{Ker}(j^R)$, where $j : C_0 \to C$ is the inclusion. Now a also has a right adjoint a^R and $h^R a^R \widetilde{\to} R$ is fully faithful, so a^R is fully faithful. We claim that hR and a are quasi-inverse to each other. Indeed, on one hand, $ahR \widetilde{\to} LR \widetilde{\to} \mathrm{id}$. Now we have to show that $(hR)a \widetilde{\to} \mathrm{id}$. For this it suffices to show that $hRah \widetilde{\to} h$, because h is a localization functor. For $c \in C$ we have a functorial fibre sequence $x \to c \to RL(c)$ in C with $x \in C_0$. Applying h we get a fibre sequence $h(x) \to h(c) \to hRL(c)$ in C/C_0, here $h(x) = 0$. Thus, $h \widetilde{\to} hRL$ as desired. □

1.8.18. For 12.4.2. In Section 1.1.15 of this file I explained that one has to correct the definition of the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,x}$. Proposition 12.4.2 can be true only after my correction of the definition of $\mathcal{G}^G$. In (12.4), p_{Ran} should be $p^{\mathrm{Conf}}_{\mathrm{Ran}}$.

Prop. 12.4.2 is however wrong as stated, the corresponding gerbes are opposite, not the same.

Indeed, recall the isomorphism $(\mathrm{Conf}\times\mathrm{Ran})^{\subset}\widetilde{\to}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg}$, it sends $(D,(x_i))$ with $\mathrm{supp}(D)\subset(x_i)$ to $\mathcal{F}_T=\omega^\rho(D)$. So, by the actual definition the fibre of $\mathcal{G}^\Lambda$ at D is the fibre of $\mathcal{G}^G$ at

$$(\mathcal{F}_T=\omega^\rho(D),(x_i),\alpha:\mathcal{F}_T\widetilde{\to}\omega^\rho\mid_{X-(x_i)})\in\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}$$

Consider a point of $S^{\mathrm{Conf}}_{\mathrm{Ran}}$ given by $(D,(x_i),\mathrm{supp}\,D\subset(x_i)),\mathcal{F}_B$ a B-torsor on X with the corresponding T-torsor identified with $\omega^\rho(-D)$. The image of this point under $S^{\mathrm{Conf}}_{\mathrm{Ran}}\to\mathrm{Gr}^{\omega^\rho}_{B,\mathrm{Ran}}\to\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}$ is

$$\mathcal{F}_T=\omega^\rho(-D),\alpha:\mathcal{F}_T\widetilde{\to}\omega^\rho\mid_{X-(x_i)},(x_i)\in\mathrm{Ran}$$

What seems natural is to ask that whatever definitions are, the pullback of $\mathcal{G}^\Lambda$ under $S^{\mathrm{Conf}}_{\mathrm{Ran}}\to\mathrm{Conf}$ identifies with $\mathcal{G}^G$. This is assumed in 12.4.3 for example.

The proof of 12.4.2 should be simplified. The proof is simply the fact that the diagram should commute (if you change the definition of $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg}$ replacing $\mathcal{F}_T$ by its opposite torsor):

$$\begin{array}{ccc} S^{\mathrm{Conf}}_{\mathrm{Ran}} & \to (\mathrm{Conf}\times\mathrm{Ran})^{\subset}\widetilde{\to} & (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg} \\ \downarrow & & \downarrow \\ \mathrm{Gr}^{\omega^\rho}_{B,\mathrm{Ran}} & \to & \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \end{array}$$

To be able to continue reading, **from now on I assume** that the definition of $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg}$ is corrected as follows: we assume for this definition $G=G_{sc}$. Then its S-point is a collection: $\mathcal{J}\in\mathrm{Hom}(S,\mathrm{Ran}),\mathcal{F}_T$ on $S\times X$ with a trivialization $\omega^\rho\mid_{S\times X-\Gamma_{\mathcal{J}}}\widetilde{\to}\mathcal{F}_T$ such that for any $\check\lambda\in\check\Lambda^+$ the map $\omega^{\langle\rho,\check\lambda\rangle}\to\check\lambda(\mathcal{F}_T)$ is regular over $S\times X$.

I also assume that $\mathcal{G}^\Lambda$ is defined as the descent of $\mathcal{G}^G$ under $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg}\to\mathrm{Conf}$.

Now Prop. 12.4.2 is correct, but is not sufficient to get the functor $(p^\lambda_{\mathrm{Ran}})^!$: $Shv_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda\times\mathrm{Ran})^{\subset})\to Shv_{\mathcal{G}^G}(S^\lambda_{\mathrm{Ran}})$ in Section 12.4.3 of the paper. For such applications, it is better to change the formulation of Prop. 12.4.2 to adopt it to S-versions as opposed to $\mathcal{S}$-versions. Namely, we have a commutative diagram

$$\begin{array}{ccc} S^{\lambda}_{\mathrm{Ran}} & \to (\mathrm{Conf}\times\mathrm{Ran})^{\subset}\widetilde{\to} & (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{neg} \\ \downarrow & & \downarrow \\ \mathrm{Gr}^{\omega^\rho}_{B,\mathrm{Ran}} & \to & \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \end{array}$$

1.8.19. For 12.4.3. The map p_{Ran} in the 1st displayed fromula was instead denoted $p^{\mathrm{Conf}}_{\mathrm{Ran}}$ in 12.2.6. Choose one of the two notations to use throughout.

1.8.20. For 12.4.4. The proof of [Ga7, 1.4.8] is not given in [Ga7]. I have written a proof in ([44], Lemma 1.3.19) for any sheaf theory. For $\mathcal{D}$-modules a closed claim is ([33], Lemma B.4.1). Some reference for the proof is needed here. Maybe the upcoming paper by Lin Chen generalizing [33] will be already available at the moment of revising of our paper, and we could refer to his upcoming paper.

1.8.21. For 12.5.1. Replace ϕ_{small} by $\phi^!_{small}$ in the 1st displayed formula.

The prestack $\mathrm{Gr}^{\omega^\rho}_{G,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}}$ has never been defined. It has appeared in the proof of 12.4.2 but without a definition. You should write what you mean: this is the prestack classifying $(\mathcal{J}\subset\mathcal{J}')\in(\mathrm{Ran}\times\mathrm{Ran})^{\subset}$, $\mathcal{P}_G$ a G-torsor on X and an isomorphism $\mathcal{P}_G\widetilde{\to}\omega^\rho_{S\times X-\Gamma_{\mathcal{J}}}$. Then the map $\phi_{big}:\mathrm{Gr}^{\omega^\rho}_{G,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}}\to\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ is the one denoted by ϕ_{big} in Sect. 1.6 of the paper.

The definition of $Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})_{untl}$ should be corrected I think. This is just the limit in DGCat_{cont} of the diagram

$$Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})\stackrel{\phi^!_{small}}{\to}\mathrm{Gr}^{\omega^\rho}_{G,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}}\stackrel{\phi^!_{big}}{\leftarrow}Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$$

I mean one should not impose in addition the property that $\vartriangle^!$ applied to the isomoprhism $\phi^!_{small}(\mathcal{F})\widetilde{\to}\phi^!_{big}(\mathcal{F})$ gives the identity. Indeed, if $F;F'\in Shv_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$ and $\phi^!_{small}(F')\widetilde{\to}\phi^!_{big}(F)$ then applying $\vartriangle^!$ this yields an isomorphism $F'\widetilde{\to}F$, and we identify F' with F.

In 1.6.5 we defined a unital structure for any prestack $Z\to\mathrm{Ran}$ over Ran. It would be easier to understand the definition if you give it in this generality. I propose to define the following more general notion making things clearer. The next section justifies the fact that the unital category should indeed be a full subcategory of $Shv(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$.

1.8.22. *Generality about invariants under category objects.* Let $\mathcal{X}:\boldsymbol{\Delta}^{op}\to\mathrm{PreStk}_{lft}$ be a category object with $S=\mathcal{X}[0],H=\mathcal{X}[1]$, so H acts on S. Then one may define the category of H-equivariant objects $Shv(S)^H$ of $Shv(S)$ as $\mathrm{Tot}(Shv(\mathcal{X}([\bullet])))$. Here we applied the functor $Shv:(\mathrm{PreStk}_{lft})^{op}\to\mathrm{DGCat}_{cont}$ to $\mathcal{X}$. Namely, denote $\mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}}\mathcal{X}[n]$ by S/H, we think of it as the quotient of S by H. Then by definition $Shv(S)^H\widetilde{\to}Shv(S/H)$.

In this generality, $Shv(S)^H\to Shv(S)$ is comonadic by ([36], 4.7.5.1).

Let's call the unit category object acting on S the constant functor $\boldsymbol{\Delta}^{op}\to\mathrm{PreStk}_{lft}$ with value S. The unit section yields a morphism from the unit category object acting on S to H. Note that $Shv(S)^S\widetilde{\to}Shv(S)$. Applying the invariants, we get a functor $Shv(S)^H\to Shv(S)^S\widetilde{\to}Shv(S)$.

Is your definition of the unital category equivalent to the above definition of the category of invariants under the action of $(\mathrm{Ran}\times\mathrm{Ran})^{\subset}$ on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$?

As in Section 1.1.12 of this file, we have a natural right action of H on itself, so that the map $t:H\to S$ attached to $[0]\stackrel{1}{\to}[1]$ is H-equivariant.

Recall from ([43], 3.0.73) the category $\boldsymbol{\Delta}_{-\infty}$ and the map $\phi:\boldsymbol{\Delta}^{op}_{-\infty}\to\boldsymbol{\Delta}^{op}$. Restricting $\mathcal{X}$ along this map, we get a split augmented simplicial object. The corresponding augmented simplicial object is a colimit diagram by ([36], 4.7.2.3), namely, $\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, H\times_{t,S,s}H^n_S\widetilde{\to}S$ in PreStk_{lft}. This says that the quotient of H by the natural right action of H on itself identifies with S. Here $H^n_S=H\times_{t,S,s}H\times_{t,S,s}\ldots\times_{t,S,s}H$, where H appears n times.

Consider the inclusion $[n]\hookrightarrow\{-\infty\}\star[n]$ functorial in $[n]\in\boldsymbol{\Delta}$, it gives a morphism of simplicial diagrams $\alpha_n:H\times_{t,S,s}H^n_S\to H^n_S$ (functorial in $[n]\in\boldsymbol{\Delta}$). Passing to the colimit, this gives the map

$$S \widetilde{\to} \underset{[n]\in \boldsymbol{\Delta}^{op}}{\operatorname{colim}} H \times_{t,S,s} H_S^n \to \underset{[n]\in \boldsymbol{\Delta}^{op}}{\operatorname{colim}} H_S^n \widetilde{\to} S/H$$

which is the natural map $f : S \to S/H$. Now for $s, t : [0] \to [1]$ write $\alpha_s, \alpha_t : H \to S$ for the corresponding maps. Assume $\alpha_t : H \to S$ universally homologically contractible. So, for any $n \geq 0$ the functor $\alpha_n^! : Shv(H_S^n) \to Shv(H \times_{t,S,s} H_S^n)$ is fully faithful. Passing to the limit we conclude that $f^! : Shv(S/H) \to Shv(S)$ is fully faithful.

My understanding is that an object $K \in Shv(S)$ lies in the full subcategory $Shv(S/H)$ iff $\alpha_s^!(K)$ lies in the essential image of the full embedding $\alpha_t^! : Shv(S) \to Shv(H)$. Is this correct?

Remark: assume now $\mathfrak{X}$ is such that the source map $\alpha_s : H \to S$ attached to $[0] \stackrel{0}{\to} [1]$ is universally homologically contractible. Then consider $\mathfrak{X}^{rm}$, which is $\mathfrak{X}$ with reversed multiplication. Applying the above, we also see that $f^! : Shv(S/H) \to Shv(S)$ is fully faithful.

1.8.23. For 12.5.3. The right action of the category object $(\mathrm{Ran} \times \mathrm{Ran})^{\subset}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ preserves $\bar{S}^0_{\mathrm{Ran}}$. It also acts naturally on $\bar{S}^\lambda_{\mathrm{Ran}}$ for $\lambda \leq 0$ preserving the open part S^λ_{Ran}.

The inclusions $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{untl} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}$, $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda}_{untl} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda}$,

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{untl} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}$$

are full subcategories.

We may define the full subcategories $Shv_q(\bar{S}^\lambda_{\mathrm{Ran}})_{untl} \subset Shv_q(\bar{S}^\lambda_{\mathrm{Ran}}))$, $Shv_q(S^\lambda_{\mathrm{Ran}})_{untl} \subset Shv_q(S^\lambda_{\mathrm{Ran}})$, and the corresponding semi-infinite categories are by definition the intersections

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda}_{untl} = \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda} \cap Shv_q(\bar{S}^\lambda_{\mathrm{Ran}})_{untl},$$

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{untl} = \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda} \cap Shv_q(S^\lambda_{\mathrm{Ran}})_{untl}.$$

For example, $Shv(\bar{S}^\lambda_{\mathrm{Ran}})_{untl}$ is the full subcategory of those $K \in Shv(\bar{S}^\lambda_{\mathrm{Ran}})$ such that for the diagram

$$\bar{S}^\lambda_{\mathrm{Ran}} \stackrel{\phi_s}{\leftarrow} \bar{S}^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \stackrel{\phi_b}{\to} \bar{S}^\lambda_{\mathrm{Ran}}$$

the object $\phi_b^! K$ lies in the essential image of the fully faithful functor $\phi_s^! : Shv(\bar{S}^\lambda_{\mathrm{Ran}}) \to Shv(\bar{S}^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}})$

The fact that for $\lambda < 0$ the functor $(\bar{i}^\lambda)_*$ preserves unital subcategories follows from the fact that both diagrams are cartesian

$$\begin{array}{ccc} \bar{S}^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} & \stackrel{\bar{i}^\lambda_{\mathrm{Ran}}}{\to} & \bar{S}^0_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \\ \downarrow \varphi_{small} & & \downarrow \varphi_{small} \\ \bar{S}^\lambda_{\mathrm{Ran}} & \stackrel{\bar{i}^\lambda_{\mathrm{Ran}}}{\to} & \bar{S}^0_{\mathrm{Ran}}, \end{array} \qquad \begin{array}{ccc} \bar{S}^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} & \stackrel{\bar{i}^\lambda_{\mathrm{Ran}}}{\to} & \bar{S}^0_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \\ \downarrow \varphi_{big} & & \downarrow \varphi_{big} \\ \bar{S}^\lambda_{\mathrm{Ran}} & \stackrel{\bar{i}^\lambda_{\mathrm{Ran}}}{\to} & \bar{S}^0_{\mathrm{Ran}} \end{array}$$

Similar thing happens for $(j^\lambda_{\mathrm{Ran}})_*$.

The fact that $(\bar{i}^\lambda)^!$ preserves unital subcategories follows from the fact that the diagram commutes

$$\begin{array}{ccc} \bar{S}^\lambda_{\mathrm{Ran}} & \overset{\bar{i}^\lambda_{\mathrm{Ran}}}{\to} & \bar{S}^0_{\mathrm{Ran}} \\ \uparrow \phi_{big} & & \uparrow \phi_{big} \\ \bar{S}^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^\subset} & \overset{\bar{i}^\lambda_{\mathrm{Ran}}}{\to} & \bar{S}^0_{(\mathrm{Ran}\times\mathrm{Ran})^\subset} \\ \downarrow \phi_{small} & & \downarrow \phi_{small} \\ \bar{S}^\lambda_{\mathrm{Ran}} & \overset{\bar{i}^\lambda_{\mathrm{Ran}}}{\to} & \bar{S}^0_{\mathrm{Ran}} \end{array}$$

Similar thing happens for $(j^\lambda)^!$.

We may similarly define the prestack $(\mathrm{Conf}^\lambda \times \mathrm{Ran}\times\mathrm{Ran})^\subset$, it classifies $(D,(x_i)) \in (\mathrm{Conf}^\lambda\times\mathrm{Ran})^\subset$, $((x_i)\subset(x_j)) \in (\mathrm{Ran}\times\mathrm{Ran})^\subset$. Then define $Shv_{\mathcal{G}^\lambda}((\mathrm{Conf}^\lambda\times\mathrm{Ran})^\subset)_{untl}$ in a similar way.

For $\lambda \le 0$ we have the diagram, where both squares are cartesian

$$\begin{array}{ccccc} \mathcal{S}^\lambda_{\mathrm{Ran}} & \overset{\varphi_{small}}{\leftarrow} & \mathcal{S}^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^\subset} & \overset{\varphi_{big}}{\to} & \mathcal{S}^\lambda_{\mathrm{Ran}} \\ \downarrow p^\lambda_{Ran} & & \downarrow p^\lambda_{Ran} & & \downarrow p^\lambda_{Ran} \\ (\mathrm{Conf}^\lambda\times\mathrm{Ran})^\subset & \overset{\varphi_{small}}{\leftarrow} & (\mathrm{Conf}^\lambda\times\mathrm{Ran}\times\mathrm{Ran})^\subset & \overset{\varphi_{big}}{\to} & (\mathrm{Conf}^\lambda\times\mathrm{Ran})^\subset \end{array}$$

However, the corresponding diagram for S-versions is not cartesian!

For this reason for $\lambda < 0$ the functor

$$(p^\lambda_{Ran})_* : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda} \to Shv_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda\times\mathrm{Ran})^\subset)$$

preserves the corresponding unital categories. We prove this using the $\mathcal{S}$-versions! Besides, $(p^\lambda_{\mathrm{Ran}})^!$ and $(s^\lambda_{\mathrm{Ran}})^!$ preserve unital subcategories for $\lambda < 0$, here $s^\lambda_{\mathrm{Ran}} : (\mathrm{Conf}^\lambda\times\mathrm{Ran})^\subset \to S^\lambda_{\mathrm{Ran}}$ is the canonical section sending $(D,(x_i))$ to $\mathcal{F} = \omega^\rho(-D)$ with $\mathcal{F}\widetilde{\to}\omega^\rho\mid_{X-\mathrm{supp}(D)}$. This first implies that (12.6) is indeed an equivalence for $\lambda<0$. It is easy to see that the functor $(s^0_{\mathrm{Ran}})^! : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=0} \to Shv_{\mathcal{G}^G}(\mathrm{Ran})$ preserves unital subcategories, hence gives an equivalence $(s^0_{\mathrm{Ran}})^! : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=0}\widetilde{\to}\mathrm{Vect}$, where $\omega_{S^0_{\mathrm{Ran}}}$ goes to ω_{Ran}. Here $Shv_{\mathcal{G}^G}(\mathrm{Ran})_{untl}\widetilde{\to}\mathrm{Vect}$ with the canonical generator ω_{Ran}.

1.8.24. For ([26], 4.2.2): it is correct. Namely, $\bar{S}^0_{\mathrm{Ran}} \in \mathrm{PreStk}_{lft}$. Let us be given $K \in Shv_{\mathcal{G}^G}(\bar{S}^0_{\mathrm{Ran}})$. For any finite collection $\lambda_i \le 0$, the union of $\sqcup_i \bar{S}^{\lambda_i}_{\mathrm{Ran}}$ maps to $\bar{S}^0_{\mathrm{Ran}}$, write $\bar{S}^{\le\{\lambda_i\}}_{\mathrm{Ran}}$ for the image. Then this image and its complement in $\bar{S}^0_{\mathrm{Ran}}$ are stable under the action of $(\mathrm{Ran}\times\mathrm{Ran})^\subset$. So, we have the category $Shv_{\mathcal{G}^G}(\bar{S}^0_{\mathrm{Ran}} - \bar{S}^{\le\{\lambda_i\}}_{\mathrm{Ran}})_{untl}$.

Claim: let $K \in Shv_{\mathcal{G}^G}(\bar{S}^0_{\mathrm{Ran}})$. Let $\{\lambda_i\}$ be a finite collection of elements of Λ^{neg} such that

$$\Lambda^{neg} - \cup_i\{\mu\in\Lambda \mid \mu\le\lambda_i\}$$

is finite. Assume that for each such collection the restriction of K to $\bar{S}^0_{\mathrm{Ran}} - \bar{S}^{\le\{\lambda_i\}}_{\mathrm{Ran}}$ is unital. Then $K \in Shv_{\mathcal{G}^G}(\bar{S}^0_{\mathrm{Ran}})_{untl}$.

Proof. Consider the diagram $\bar{S}^0_{\mathrm{Ran}} \overset{\phi_s}{\leftarrow} \bar{S}^0_{(\mathrm{Ran}\times\mathrm{Ran})^\subset} \overset{\phi_b}{\to} \bar{S}^0_{\mathrm{Ran}}$. By our assumption, for each finite collection of elements of Λ^{neg} such that $\Lambda^{neg} - \cup_i\{\mu\in\Lambda\mid\mu\le\lambda_i\}$ is finite, the restriction of $\phi_b^! K$ to

$$(\bar{S}^0_{\mathrm{Ran}} - \bar{S}^{\le \{\lambda_i\}}_{\mathrm{Ran}}) \times_{\mathrm{Ran},\phi_s} (\mathrm{Ran}\times\mathrm{Ran})^{\subset}$$

descends under ϕ_s to an object of $Shv_{\mathcal{G}^G}((\bar{S}^0_{\mathrm{Ran}} - \bar{S}^{\le \{\lambda_i\}}_{\mathrm{Ran}})$.

We have $cup_{\{\lambda_i\}} S^0_{\mathrm{Ran}} - \bar{S}^{\le \{\lambda_i\}}_{\mathrm{Ran}} = \bar{S}^0_{\mathrm{Ran}}$, as this is an open covering. Since Shv satisfies Zariski descent,

$$Shv_{\mathcal{G}^G}(\bar{S}^0_{\mathrm{Ran}}) \widetilde{\to} \lim Shv(\bar{S}^0_{\mathrm{Ran}} - \bar{S}^{\le \{\lambda_i\}}_{\mathrm{Ran}})$$

The objects we get on $\bar{S}^0_{\mathrm{Ran}} - \bar{S}^{\le \{\lambda_i\}}_{\mathrm{Ran}}$ clearly organize into an object of the above limit. □

The above claim implies ([26], 4.2.2).

To obtain ([26], 4.2.5), argue as follows. Consider the diagram

$$(X^\lambda \times \mathrm{Ran})^{\subset} \stackrel{p^\lambda_{\mathrm{Ran}}}{\leftarrow} S^\lambda_{\mathrm{Ran}} \stackrel{s^\lambda_{\mathrm{Ran}}}{\leftarrow} (X^\lambda \times \mathrm{Ran})^{\subset}$$

Since $(p^\lambda_{\mathrm{Ran}})^!$ and $(s^\lambda_{\mathrm{Ran}})^!$ preserve unital subcategories, we get the diagram

$$Shv((X^\lambda \times \mathrm{Ran})^{\subset})_{untl} \stackrel{(p^\lambda_{\mathrm{Ran}})^!}{\to} Shv(S^\lambda_{\mathrm{Ran}})_{untl} \stackrel{(s^\lambda_{\mathrm{Ran}})^!}{\leftarrow} Shv((X^\lambda \times \mathrm{Ran})^{\subset})_{untl}, \tag{26}$$

where the first functor is fully faithful, because $(p^\lambda_{\mathrm{Ran}})^! : Shv(X^\lambda \times \mathrm{Ran})^{\subset}) \widetilde{\to} \mathrm{SI}(G)^{=\lambda}$ is an equivalence. This implies that

$$(p^\lambda_{\mathrm{Ran}})^! : Shv((X^\lambda \times \mathrm{Ran})^{\subset})_{untl} \to SI(G)^{=\lambda}_{untl}$$

is fully faithful. Let us show it is an equivalence. Since $(s^\lambda_{\mathrm{Ran}})^! : SI(G)^{=\lambda} \to Shv((X^\lambda \times \mathrm{Ran})^{\subset})$ is an equivalence, the functor

$$(s^\lambda_{\mathrm{Ran}})^! : SI(G)^{=\lambda}_{untl} \to Shv((X^\lambda \times \mathrm{Ran})^{\subset})_{untl}$$

is fully faithful. The latter is an equivalence, because $(s^\lambda_{\mathrm{Ran}})^!(p^\lambda_{\mathrm{Ran}})^! = \mathrm{id}$. Now since the composition in (26) is an equivalence, the first functor in (26) is also an equivalence.

1.8.25. For 12.5.4 and for ([26], Cor. 4.2.3). We check that $(i^\lambda)^* : SI(G)^{\le 0}_{\mathrm{Ran}} \to SI(G)^{=\lambda}_{\mathrm{Ran}}$ preserves the unital subcategories, then for $i^\lambda_!$ this is a formal consequence.

Recall the definition of $\bar{S}^{-,\lambda}_{\mathrm{Ran}}$. It is the subfunctor of $(X^\lambda \times \mathrm{Ran})^{\subset} \times_{\mathrm{Ran}} \mathrm{Gr}_{G,\mathrm{Ran}}$. Its S-point is a collection $(D, \mathcal{I}, \mathcal{F}_{\tilde{G}})$ with an ismorphism $\mathcal{F}_{\tilde{G}} \widetilde{\to} \mathcal{F}^0_{\tilde{G}} \mid_{(S\times X)-D_{\mathcal{I}}}$ of $\tilde{G}$-torsors such that for any $\check{\lambda}$ dominant for $\tilde{G}$, the map $\mathcal{V}^{\check{\lambda}}_{\mathcal{F}_{\tilde{G}}} \to \mathcal{O}(-\langle\check{\lambda}, D\rangle)$ is regular over $S\times X$. Here $\mathcal{V}^{\check{\lambda}}$ is the Weyl module for $\tilde{G}$ (cf. [49], 0.5).

Using ([26], proof of 1.5.3) let us show that the functor $(i^\lambda_{\mathrm{Ran}})^*$ preserves unital subcategories for $\lambda < 0$. One has

$$(i^\lambda)^* \widetilde{\to} (p^\lambda_{\mathrm{Ran}})^! (p^{-,\lambda}_{\mathrm{Ran}})_* (i^{-,\lambda})^!$$

for the diagram

$$S^\lambda_{\mathrm{Ran}} \stackrel{p^\lambda_{\mathrm{Ran}}}{\leftarrow} (X^\lambda \times \mathrm{Ran})^{\subset} \stackrel{p^{-,\lambda}_{\mathrm{Ran}}}{\leftarrow} S^{-,\lambda}_{\mathrm{Ran}} \cap \bar{S}^0_{\mathrm{Ran}} \stackrel{i^{-,\lambda}}{\to} \bar{S}^0_{\mathrm{Ran}},$$

here of course we mean the intersection over Ran.

Since $(i^{-,\lambda})^!, (p^\lambda_{\rm Ran})^!$ preserve the unital categories, it suffices to show that the functor $(p^{-,\lambda}_{\rm Ran})_*: Shv_{\mathcal{G}^G}(S^{-,\lambda}_{\rm Ran}\cap \bar S^0_{\rm Ran})\to Shv_{\mathcal{G}^G}({\rm Conf}^\lambda\times{\rm Ran})^{\subset}$ preserves the unital categories.

In the diagram

$$\begin{array}{ccccc} S^{-,\lambda}_{\rm Ran}\cap\bar S^0_{\rm Ran} & \stackrel{\phi_s}{\leftarrow} & (S^{-,\lambda}_{\rm Ran}\cap\bar S^0_{\rm Ran})\times_{{\rm Ran},\phi_s}({\rm Ran}\times{\rm Ran})^{\subset} & \stackrel{\phi_b}{\to} & S^{-,\lambda}_{\rm Ran}\cap\bar S^0_{\rm Ran}\\ \downarrow p^{-,\lambda}_{\rm Ran} & & \downarrow & & \downarrow p^{-,\lambda}_{\rm Ran}\\ (X^\lambda\times{\rm Ran})^{\subset} & \stackrel{\phi_s}{\leftarrow} & (X^\lambda\times{\rm Ran}\times{\rm Ran})^{\subset} & \stackrel{\phi_b}{\to} & (X^\lambda\times{\rm Ran})^{\subset}\end{array}$$

both squares are cartesian.[1] Indeed, for an S-point $(D,\mathcal{J}\subset\mathcal{J}'),\mathcal{F}_{\tilde G},\mathcal{F}_{\tilde G}\widetilde{\to}\mathcal{F}^0_{\tilde G}\mid_{(S\times X)-D_{\mathcal{J}'}}$ lying in the fibred product the trivialization automatically extends to $(S\times X)-D_{\mathcal{J}}$, because at a point $x\in X$ one has $S^{-,0}\cap\bar S^{\le 0}=t^0\in{\rm Gr}_{G,x}$. We apply this to points of $\mathcal{J}'$ not lying in $\mathcal{J}$.

Since $(i^\lambda_{\rm Ran})^*:{\rm SI}_{q,{\rm Ran}}(G)^{\le\lambda}\to{\rm SI}_{q,{\rm Ran}}(G)^{=\lambda}$ preserves unital subcategories, this should imply that $(i^\lambda_{\rm Ran})_!$ preserves unital categories for $\lambda<0$ formally via Lemma 1.8.16 of this file, because there is an explicit formula for this functor in terms of other functors, which are already known to preserve unital categories.

One has separately to verify all the claims from 12.5.3-12.5.7 for $\lambda=0$. It is clear that $(p^0_{\rm Ran})^!,(\bar p^0_{\rm Ran})^!,(j^0_{\rm Ran})^!,(j^0_{\rm Ran})_*$ preserve unital subcategories.

1.8.26. In ([26], Cor. 4.2.2) it is claimed that $F\in{\rm SI}^{\le 0}_{\rm Ran}$ lies in ${\rm SI}^{\le 0}_{{\rm Ran},untl}$ iff for any $\lambda\le 0$, $(i^\lambda_{\rm Ran})^!F\in{\rm SI}^{=\lambda}_{{\rm Ran},untl}$.

To prove this, we present $\bar S^0_{\rm Ran}\widetilde{\to}{\rm colim}_{k\ge 0}Y_k$ in PreStk, where $Y_i\subset\bar S^0_{\rm Ran}$ is the union those $S^\lambda_{\rm Ran}$ for which $ht(\lambda)\le k$. Here for $\lambda=\sum_\alpha n_\alpha\alpha$, the sum being over simple coroots, we let $ht(\lambda)=\sum_\alpha -n_\alpha\in\mathbb{Z}_+$. This holds, because $\tau_{\le 0}{\rm Spc}\subset{\rm Spc}$ is closed under filtered colimits, so for $S\in{\rm Sch}^{aff}$, ${\rm Map}(S,{\rm colim}_k Y_k)$ is the union of the sets ${\rm Map}(S,Y_k)$. Now any element in ${\rm Map}(S,\bar S^0_{\rm Ran})$ lies in some ${\rm Map}(S,Y_k)$, because S is quasi-compact.

The complement of Y_k in $\bar S^0_{\rm Ran}$ is closed, and $Y_k\subset Y_{k+1}$. So,

$$Shv_{\mathcal{G}^G}(\bar S^0_{\rm Ran})\widetilde{\to}\lim_{k\in\mathbb{Z}_+^{op}}Shv_{\mathcal{G}^G}(Y_k)$$

For each Y_k we have the unital subcategory $Shv_{\mathcal{G}^G}(Y_k)_{untl}$ defined similarly. For $F\in Shv_{\mathcal{G}^G}(Y_k)$ the already establised results show that F lies in ${\rm SI}_q(Y_k)_{untl}$ iff for any $\lambda\le 0$ with $ht(\lambda)\le k$, $(i^\lambda_{\rm Ran})^!F$ lies in the unital category.

Let now $F\in{\rm SI}^{\le 0}_{\rm Ran}$ with $(i^\lambda_{\rm Ran})^!F\in{\rm SI}_{q,{\rm Ran}}(G)^{=\lambda}_{untl}$ for any $\lambda\le 0$. Its restriction to the open part Y_k is unital for any k. So, over $Y_k\times_{{\rm Ran},\varphi_{small}}({\rm Ran}\times{\rm Ran})^{\subset}\subset\bar S^0_{({\rm Ran}\times{\rm Ran})^{\subset}}$ we get the desired isomorphism $\varphi^!_{small}F\widetilde{\to}\varphi^!_{big}F$, they are automatically compatible and yield in the limit over k the desired isomorphism.

Another thing, the proof of ([26], 4.2.7) is correct, but we need to know that Shv satisfies the descent for a morphism $f:Y\to Z$ in ${\rm Sch}_{ft}$, which is finite and surjective. This follows from ([46], 0.0.30).

[1] The right square would not be cartesian if we considered $S^{-,\lambda}_{\rm Ran}$ instead of the intersection $S^{-,\lambda}_{\rm Ran}\cap\bar S^0_{\rm Ran}$.

1.8.27. *t-structures on gluings of categories.* Let us be in the situation of Lemma 1.8.16. Assume both C^0, C' are equipped with accessible t-structures. Then we define $C^{\geq 0} \subset C$ as the full subcategory of those $F \in C$ for which $i^! F \in C'^{\geq 0}$ and $j^!(F) \in (C^0)^{\geq 0}$. Then $F \in C^{<0}$ iff both $i^* F \in (C')^{<0}$ and $j^* F \in (C^0)^{<0}$. Indeed $\{F \in C \mid i^* F \in (C')^{<0}$ and $j^* F \in (C^0)^{<0}\}$ contains $i_*(C'^{<0}), j_!((C^0)^{<0})$ and is closed under colimits and extensions. Since both $C'^{<0}$ and $(C^0)^{<0}$ are presentable, each of them is generated by a small set of objects. Now $C^{<0} \subset C$ is the smallest full subcategory closed under colimits, closed under extensions and containing $j_! F_1$ for $F_1 \in (C^0)^{<0}$ and $i_* F_2$ for $F_2 \in C'^{<0}$. Thus, by ([36], 1.4.4.11), $C^{<0}$ is presentable and defines an accessible t-structure on C. Besides, $i_! : C' \to C$ is t-exact, $j_!$ is right t-exact and j_* is left t-exact.

Let $K \in C^{\heartsuit}$. Then $i^* K \in C'^{\leq 0}$, and $j^* K \in (C^0)^{\heartsuit}$. So, $i_* i^* K \in C^{\leq 0}$. The exact triangle $j_! j^* K \to K \to i_* i^* K$ gives an exact sequence $\mathrm{H}^0(j_! j^* K) \to K \to \mathrm{H}^0(i_* i^* K) \to 0$ in $C^{\heartsuit}$. If $i^* K \in C^{<0}$ then the map $\mathrm{H}^0(j_! j^* K) \to K$ in $C^{\heartsuit}$ is surjective. We have $j_* j^* K \in C^{\geq 0}$ and $i^! K \in C'^{\geq 0}$. The exact triangle $i_! i^! K \to K \to j_* j^* K$ gives an exact sequence $0 \to \mathrm{H}^0(i_! i^! K) \to K \to \mathrm{H}^0(j_* j^* K)$. So, if $i_! i^! K \in C^{>0}$ then $K \to \mathrm{H}^0(j_* j^* K)$ is injective.

In particular, if $K \in C^{\heartsuit}$ is such that $i^! K \in C^{>0}$ and $i^* K \in C^{<0}$ then K is the intermediate extension of $j^* K$, which is defined as the image of the map $\mathrm{H}^0(j_! j^* K) \to \mathrm{H}^0(j_* j^* K)$.

If the t-structures on C', C_0 are compatible with fltered colimits, the same holds for the t-structure on C.

1.8.28. *t-structures on gluings of categories: more.* Assume we are given $C = C_{\leq 0} \in \mathrm{DGCat}_{cont}$ and for any $\lambda \in \Lambda^{neg}$ a full embedding $(\bar{i}^\lambda)_* = (\bar{i}^\lambda)_! : C_{\leq \lambda} \to C$, which admits a continuous right adjoint $(\bar{i}^\lambda)^!$. Assume that for $\lambda \leq \mu$ we have $C_{\leq \lambda} \subset C_{\leq \mu}$, that is, $(\bar{i}^\lambda)_!$ factors through $(\bar{i}^\mu)_!$. Assume also given a full embedding $(j^\lambda)_* : C_{=\lambda} \to C_{\leq \lambda}$ admitting a left adjoint $(j^\lambda)^* = (j^\lambda)^! : C_{\leq \lambda} \to C_{=\lambda}$ in DGCat_{cont}. Assume also $(j^\lambda)^!$ has a left adjoint $(j^\lambda)_!$, then this left adjoint is automatically fully faithful. Set $(i^\lambda)^! = (j^\lambda)^! (\bar{i}^\lambda)^!$. Assume the composition $(\bar{i}^\lambda)_* (j^\lambda)_*$ admits a left adjoint $(i^\lambda)^*$.

For $\lambda \in \Lambda^{neg}$ let $C_{<\lambda}$ be the full DG-subcategory generated by $C_{\leq \lambda - \alpha_i}$ for all simple coroots α_i. We assume in addition that $(j^\lambda)_* C_{=\lambda}$ is the right orthogonal to $C_{<\lambda}$ in $C_{\leq \lambda}$, and $C_{<\lambda}$ is the left orthogonal to $(j^\lambda)_* C_{=\lambda}$. In particular,

$$C_{<\lambda} = \mathrm{Ker}(C_{\leq \lambda} \stackrel{(j^\lambda)^*}{\to} C_{=\lambda})$$

by Lemma 1.8.15. To be safe, assume that if $\lambda \neq \mu, F \in C_{=\mu}$ then $(i^\lambda)^! j^\mu_* F = 0$ (is it automatic??). Let us also assume that if $\lambda \neq \mu, F \in C_{=\mu}$ then $(i^\lambda)^* (j^\mu)_! F = 0$.

Assume now each $C_{=\lambda}$ is equipped with an accessible t-structure. Let $C^{\leq 0}$ be the smallest full subcategory closed under colimits, closed under extensions and containing for each $\lambda \in \Lambda^{neg}$ and $F \in C^{\leq 0}_{=\lambda}$ the object $(\bar{i}^\lambda)_! (j^\lambda)_! F$. Since each $C^{\leq 0}_{=\lambda}$ is presentable, by ([36], 1.4.4.11), $C^{\leq 0}$ is presentable and defines an accessible t-structure on C.

Lemma 1.8.29. *Under the assumptions of the previous subsection, we have the following.*
1) For $F \in C$ we have $F \in C^{\geq 0}$ iff for any $\lambda \in \Lambda^{neg}$ one has $(i^\lambda)^! F \in C^{\geq 0}_{=\lambda}$.

2) If for any λ the t-structure on $C_{=\lambda}$ is compatible with filtered colimits then the t-structure on C is also compatible with filtered colimits.

Proof. 1) is immediate from definitions.
2) follows from 1). □

The functors $i_!^\lambda : C_{=\lambda} \to C$ are right t-exact, the functors $(i^\lambda)_* : C_{=\lambda} \to C$ are left t-exact, hence $(i^\lambda)^* : C \to C_{=\lambda}$ are right t-exact.

In our special case the following holds in addition. For $k \geq 0$ let ${}_{\geq k}C \subset C$ be the full subcategory generated by $C_{\leq\lambda}$ for $\lambda \in \Lambda^{neg}$ with $ht(\lambda) \geq k$. Note that ${}_{\geq k+1}C \subset {}_{\geq k}C$, so we get the functor $(\mathbb{Z}_+)^{op} \to \mathrm{DGCat}_{cont}$, $k \mapsto C/{}_{\geq k}C$. Then

$$C \to \lim_{k\in(\mathbb{Z}_+)^{op}} C/{}_{\geq k}C$$

is an equivalence. This additional assumption allows to conclude the following: if $F \in C$ and $(i^\lambda)^! F = 0$ for all $\lambda \leq 0$ then $F = 0$. Indeed, the image of F in $C/{}_{\geq k}C$ vanishes for each k, hence $F = 0$. We used that the right orthogonal $({}_{\geq k}C)^r$ to ${}_{\geq k}C$ in C is the full subcategory generated by the objects of the form $(i^\lambda)_*F$ for $ht(\lambda) < k$, $F \in C_{=\lambda}$. Then $C/{}_{\geq k}C \widetilde{\to} ({}_{\geq k}C)^r$. Note that the set $\{\mu \in \Lambda^{neg} \mid ht(\mu) = n\}$ is finite for any n.

Each category $({}_{\geq k}C)^r$ has a similar filtration indexed by λ with $ht(\lambda) < k$. We also define a t-structure on $({}_{\geq k}C)^r$ similarly. Then the evaluation $C \to C/{}_{\geq k}C$ is t-exact, and the t-structure on C can be seen as the t-structure obtained from the t-structures on each $C/{}_{\geq k}C$ via ([20], ch. I.3, Lm. 1.5.8).

Then, I think, as in Section 1.8.27 of this file one gets the following. Let $K \in C^\heartsuit$ such that for any $\lambda \neq 0$ one has $(i^\lambda)^! K \in C_{=\lambda}^{>0}$ and $(i^\lambda)^* K \in C_{=\lambda}^{<0}$ then K is the intermediate extension of $(j^0)^! K \in C_{=\lambda}^\heartsuit$.

Under our additional assumption, we also get the following. If $(i^\lambda)^* F = 0$ for all λ then $F = 0$. This gives in turn: an object $F \in C$ is connective iff $(i^\lambda)^* F \in C_{=\lambda}$ is connective for any $\lambda \leq 0$. Indeed, one direction is evident. Assume that $(i^\lambda)^* F \in C_{=\lambda}$ is connective for any $\lambda \leq 0$. Assume $F \in C^{>0}$, we have to show that $F = 0$. Assume $F \neq 0$. By the above, there is $\lambda \leq 0$ such that $(i^\lambda)^* F \neq 0$. Take λ maximal with this property. Then $(i^\lambda)^* F \widetilde{\to} (i^\lambda)^! F$, a contradiction.

1.8.30. Let Λ be any partially ordered set, assume given a diagram $\Lambda \to \mathrm{DGCat}_{cont}$, $\lambda \mapsto C_{\leq\lambda}$ such that for $\lambda < \mu$, $(\bar{i}_{\lambda,\mu})_! : C_{\leq\lambda} \subset C_{\leq\mu}$ is fully faithful and has a continuous right adjoint $(\bar{i}_{\lambda,\mu})^!$. Assume also given a full embedding $j_*^\lambda : C_{=\lambda} \hookrightarrow C_{\leq\lambda}$ admitting a left adjoint $(j^\lambda)^* = (j^\lambda)^!$ in DGCat_{cont}. Assume also $(j^\lambda)^!$ has a left adjoint $j_!^\lambda : C_{=\lambda} \hookrightarrow C_{\leq\lambda}$, this $j_!^\lambda$ is automatically fully faithful.

For $\lambda \in \Lambda$ let $C_{<\lambda}$ be the full subcategory generated by $C_{\leq\mu}$ for $\mu < \lambda$. Assume the inclusion $(i_{<\lambda})_! : C_{<\lambda} \to C_{\leq\lambda}$ admits a continuous right adjoint $(i_{<\lambda})^!$, and the essential image of j_*^λ is the right orthogonal to $C_{<\lambda}$ in $C_{\leq\lambda}$. So, for each $K \in C_{\leq\lambda}$ we get a fibre sequence $(i_{<\lambda})_!(i_{<\lambda})^! K \to K \to j_*^\lambda (j^\lambda)^* K$ in $C_{\leq\lambda}$ with $K_{<\lambda} \in C_{<\lambda}$.

Let $C = \mathrm{colim}_{\lambda\in\Lambda} C_{\leq\lambda}$ with respect to the transition functor $(\bar{i}_{\lambda,\mu})_!$, equivalently $C \widetilde{\to} \lim_{\lambda\in\Lambda^{op}} C_{\leq\lambda}$ with respect to the functors $(\bar{i}_{\lambda,\mu})^!$. Let $\bar{i}_!^\lambda : C_{\leq\lambda} \to C$ be the natural functor, it is fully faithful (by [23], Lemma 1.3.6), write $(\bar{i}^\lambda)^! : C \to C_{\leq\lambda}$ for its right adjoint, this is the projection in the above projective system.

Assume each $C_{=\mu}$ is equipped with an accessible t-structure. Let $C^{\leq 0} \subset C$ be the smallest full subcategory closed under extensions and small colimits and containing for $\lambda \in \Lambda$ the objects $\bar{i}^{\lambda}_! j^{\lambda}_!(K)$ for $K \in C^{\leq 0}_{=\lambda}$. By ([36], 1.4.4.11), $C^{\leq 0}$ is presentable and defines an accessible t-structure. Let $(i^{\lambda})^* = (j^{\lambda})^*(\bar{i}^{\lambda})^!$ for $\lambda \in \Lambda$.

Is it true that for $K \in C$ one has $K \in C^{\leq 0}$ iff for any λ, $(i^{\lambda})^* K \in C^{\leq 0}_{=\lambda}$?

1.8.31. *Example.* (Sam). Consider $Y = \operatorname{colim}_{n \in \mathbb{N}} \mathbb{A}^n$ with respect to the closed immersions $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$. Let $i_n : \mathbb{A}^n \hookrightarrow Y$ be the natural embedding. Then for any n, $i_n^* \omega = 0$. Indeed, $\omega \widetilde{\to} \operatorname{colim}_{n \in \mathbb{N}} (i_n)_! i_n^! \omega$. So, for any $K \in Shv(\mathbb{A}^n)$,

$$\mathcal{H}om(\omega, (i_n)_* K) \widetilde{\to} \lim_{m \in \mathbb{N}^{op}} \mathcal{H}om((i_m)_! \omega_{\mathbb{A}^n}, (i_n)_* K)$$

is placed in degrees $> N$ for any $N \in \mathbb{Z}$. Note that $\omega[r] \in Shv(Y)^{\leq 0}$ for any r.

If $K \in Shv(Y)$ and $i_n^* K \in Shv(\mathbb{A}^n)^{\leq 0}$ for all n, does it imply that $K \in Shv(Y)^{\leq 0}$?

1.8.32. The full subcategory of connective objects in $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{untl}$ is the smallest full subcategory containing for each $\lambda \in \Lambda^{neg}$ and a connective $F \in \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{untl}$ the object $(i^{\lambda}_{\mathrm{Ran}})_! F$, closed under colimits and extensions. The previous section shows that the t-structure on $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{untl}$ is accessible and compatible with filtered colimits.

1.8.33. For 13.2.2 line 1: replace $\mathrm{Rep}(H)$ by $\mathrm{Rep}(H)^{\heartsuit}$.

The objects V^{γ} for $\gamma \in \Lambda^{\sharp}$ dominant were already defined in 2.4.6 of the paper.

It is claimed that we get (13.2) by adjunction. This is not correct, it it obtained by appying the functor $F \mapsto \delta_{t^{-\gamma},\mathrm{Gr}} * F$ to (13.1).

1.8.34. For 13.2.3. By $(\Lambda^{\sharp})^+$ we mean dominant coweights of H. It is not true that $(\Lambda^{\sharp})^+$ becomes a poset with the definition $\gamma_1 \prec \gamma_2$ iff $\gamma_2 - \gamma_1 \in (\Lambda^{\sharp})^+$.

It is better to say that we get just a category $(\Lambda^{\sharp,+} \prec)$, not a poset, and this category is indeed filtered, this is all we need. The same correction for [27].

1.8.35. For ([26], 1.6.3). An example: let $Y \to S$ be a map with $S \in \mathrm{Sch}_{ft}$, Y an ind-scheme of ind-finite type which can be written as $Y \widetilde{\to} \operatorname{colim}_{i \in I} Y_i$, where I is small filtered, $Y_i \subset Y$ is a closed subscheme of finite type, if $i \to j$ in I then $Y_i \hookrightarrow Y_j$. Assume each Y_i smooth over S. Then ω_Y is ULA with respect to the $Shv(S)$-action on $Shv(Y)$.

1.9. **Comments to the paper the semi-infinite IC-sheaf** [27].

1.9.1. The description of compact objects in $\mathrm{SI}(\mathrm{Gr}_G)$ in ([27], 1.4.10): by ([27], 1.4.7), $\mathrm{SI}(\mathrm{Gr}_G) \widetilde{\to} \mathrm{Ind}(C)$, where $C \subset \mathrm{SI}(\mathrm{Gr}_G)$ is the smallest stable subcategory containing $\boldsymbol{\Delta}^{\lambda}$ for all $\lambda \in \Lambda$. Here $C \subset \mathrm{SI}(\mathrm{Gr}_G)$ is idempotent complete, as any direct summand K of an object of C satisfies: $(i^{\mu})^* K = 0$ for all but finite number of μ, hence K is in C. The description of $\mathrm{Ind}(C)^c$ is given in [35].

any object of C is a finite extension objects of the form $\triangle^{\lambda}[m]$ for some λ, m. Such compact object F satisfies the property that $i_{\lambda}^* F$ vanishes for all but finite number of λ, and $i_{\lambda}^* F$ is compact in $SI(\mathrm{Gr}_G)_{=\lambda}$. Conversely, let $F \in SI(\mathrm{Gr}_G)$ be such that $i_{\lambda}^* F$ vanishes for all but finite number of λ, and $i_{\lambda}^* F$ is compact in $SI(\mathrm{Gr}_G)_{=\lambda}$. Then there is a locally closed ind-subscheme $i : U \subset \mathrm{Gr}_G$, which is a union of finite number of the orbits S^{λ} such that $F = i_! F_U$ for some $F_U \in SI(U)$. Moreover, F_U admits a finite filtration in $SI(U)$ with the successive quotients $(i_\lambda)_! i_\lambda^* F$. Since each $(i_\lambda)_! i_\lambda^* F \in SI(\mathrm{Gr}_G)^c$, we get $F \in SI(\mathrm{Gr}_G)^c$.

For ([27], 2.1.3). In my file [49] I explained that for λ dominant coweight for (G, B) one has $\mathrm{coind}_{\check{B}}^{\check{G}}(e^{-\lambda}) \widetilde{\to} (V^\lambda)^*$. This implies formally that for λ dominant coweight for (G, B) one has $\mathrm{coind}_{\check{B}^-}^{\check{G}}(e^\lambda) \widetilde{\to} V^\lambda$.

The map (2.1) from ([27], 2.1.4) is equally determined by requiring that $e \stackrel{v^{\lambda_1} \otimes v^{\lambda_2}}{\to} V^{\lambda_1} \otimes V^{\lambda_2} \to V^{\lambda_1 + \Lambda_1}$ equals $v^{\lambda_1+\lambda_2}$.

For ([27], 2.3.1). His $(\Lambda^+, \le)$ is not a poset, but a filtered category, this is sufficient.

For ([27], 2.3.7). In point (c) we use the following fact. Given $\lambda_i \in \Lambda^+$, we have a canonical inclusion $\mathcal{A}^{\lambda_1+\lambda_2} \hookrightarrow \mathcal{A}^{\lambda_1} * \mathcal{A}^{\lambda_2}$, where $\mathcal{A}^\lambda \in \mathrm{Sph}(G)$ is the IC-sheaf of $\bar{\mathrm{Gr}}_G^\lambda$. It simply comes from the fact that the $*$-restriction of $(\mathcal{A}^{\lambda_1} * \mathcal{A}^{\lambda_2}) \mid_{\mathrm{Gr}_G^\lambda}$ is canonically $\mathrm{IC}(\mathrm{Gr}_G^\lambda)$.

Note that for $g \in G(F)$, $K \in \mathrm{Sph}(G)$, $\delta_g * K \widetilde{\to} g \cdot K$, where $g : \mathrm{Gr}_G \to \mathrm{Gr}_G$ is the multiplication by g, and by $g \cdot K$ we mean the direct image under this map. We apply the functor $\bullet \mapsto \bullet * \mathcal{A}^{\lambda_2}$ for the canonical map $\mathcal{A}^{\lambda_1} \to (\delta_{t^{\lambda_1}})_* e[\langle \lambda_1, 2\check{\rho}\rangle]$, compose with the map $\mathcal{A}^{\lambda_2} \to (\delta_{t^{\lambda_2}})_* e[\langle \lambda_2, 2\check{\rho}\rangle]$, and precompose with $\mathcal{A}^{\lambda_1+\lambda_2} \hookrightarrow \mathcal{A}^{\lambda_1} * \mathcal{A}^{\lambda_2}$. The result is the same map for $\lambda_1 + \lambda_2$. This is why in ([27], 2.3.4) the two compositions coincide in the homotopy category.

For ([27], 2.4.4). Their Section 2.4.4 he actually shows that

$$i_\mu^* \mathrm{IC}^{\frac{\infty}{2}} \widetilde{\to} i_\mu^*(\vartriangle^\mu) \otimes \mathrm{colim}_{\lambda \in \Lambda^+} V^\lambda(\lambda + \mu)$$

Each term in this inductive system for λ deep enough in Λ^+ is $U(\check{\mathfrak{n}}^-)_\mu$. However, it is not clear if the transition maps are the identities. Here $\check{\mathfrak{n}}^-$ is the Lie algebra of $\check{N}^-$, and $U(\check{\mathfrak{n}}^-)$ is its envelopping algebra. Indeed, for λ deep enough in Λ^+, $V^\lambda(\lambda+\mu) \widetilde{\to} U(\check{\mathfrak{n}}^-)_\mu$ via the action of $U(\check{\mathfrak{n}}^-)$ on v^λ. Compare with ([27], 2.5.4).

For ([27], 2.5.4), the answer is correct. Somehow, the map $\delta_{t^\lambda} \to \mathcal{A}^\lambda[\langle \lambda, 2\check{\rho}\rangle]$ "corresponds" to the map $e \to V^\lambda$ given by v^λ. In the sense that the map $\delta_{t^\lambda} * \mathcal{A}^{\lambda_1} \to \mathcal{A}^\lambda * \mathcal{A}^{\lambda_1}[\langle \lambda, 2\check{\rho}\rangle]$ obtained by applying $\bullet * \mathcal{A}^{\lambda_1}$ induces by applying $\mathrm{R}\Gamma_c(S^{\lambda_2+\mu}, i_{\mu+\lambda_2}^*(\bullet))$ the morphism

$$V^{\lambda_1}(\lambda_1 + \mu) \widetilde{\to} V^{\lambda_1}(\lambda_1 + \mu) \otimes V^\lambda(\lambda) \to (V^{\lambda_1} \otimes V^\lambda)(\lambda_2 + \mu)$$

For $v \in V^{\lambda_1}(\lambda_1 + \mu)$ with λ_1 dominant for G, the function $n \mapsto \langle (v^{\lambda_1})^*, nv\rangle$ lies in $\mathcal{O}(\check{N})(\mu)$, where $t \in \check{T}$ acts on $f \in \mathcal{O}(\check{N})$ as $(tf)(n) = f(t^{-1}nt)$.

The composition $V^{\lambda_1} \widetilde{\to} V^\lambda(\lambda) \otimes V^{\lambda_1} \to V^{\lambda_1} \otimes V^\lambda \to V^{\lambda_2}$ in the proof of ([27], 2.5.4) is a map of $\check{N}$-modules. So, $\mathrm{colim}_{\lambda \in \Lambda^+} V^\lambda$ is naturally a $\check{N}$-module. Such colimit is described more generally in ([44], 7.6.16). It is better to write this colimit as $\mathrm{colim}_{\lambda \in \Lambda^+} V^\lambda \otimes (V^\lambda(\lambda))^*$, where the transition map for $\lambda_2 = \lambda_1 + \lambda$ with $\lambda_i, \lambda \in \Lambda^+$ is the composition

$$V^{\lambda_1} \otimes (V^{\lambda_1}(\lambda_1))^* \widetilde{\to} V^{\lambda_1} \otimes (V^\lambda(\lambda)) \otimes (V^{\lambda_2}(\lambda_2))^* \to (V^{\lambda_1} \otimes V^\lambda) \otimes (V^{\lambda_2}(\lambda_2))^*$$
$$\to V^{\lambda_2} \otimes (V^{\lambda_2}(\lambda_2))^*$$

Then $\operatorname*{colim}_{\lambda\in\Lambda^+} V^\lambda\otimes(V^\lambda(\lambda))^*\widetilde{\to}\mathcal{O}(\check{N})$ as $\check{N}$-module, this is a version of ([4], Proposition-Construction 3.1.2).

For ([27], 2.8.2). We see moreover that for $\mathcal{F}\in \mathrm{Sph}(G), \lambda\in\Lambda$ and any $\mu\in\Lambda$, $i^*_\mu(\vartriangle^\lambda *\mathcal{F})$ lies in $\mathrm{SI}(\mathrm{Gr}_G)^{\heartsuit}_{=\mu}$ actually.

The fact that for $\mathcal{F}\in\mathrm{Sph}(G)$ the functor $\mathrm{SI}(\mathrm{Gr}_G)\to\mathrm{SI}(\mathrm{Gr}_G), K\mapsto K\star\mathcal{F}$ is left adjoint to $K\mapsto K\star\mathbb{D}(*\mathcal{F})$ follows from ([17], 5.3.9). Here it is important that by $\bullet\star\mathcal{F}$ we mean the right action of $\mathcal{F}\in\mathrm{Sph}(G)$ on $\mathrm{SI}(\mathrm{Gr}_G)$. Here $*$ denotes as in [17] the functor $\mathrm{Sph}(G)\widetilde{\to}\mathrm{Sph}(G)$ induced by the map $G(F)\to G(F), g\mapsto g^{-1}$. Note that $*\mathbb{D}(\mathcal{F})\widetilde{\to}\mathbb{D}(*\mathcal{F})$, because $*\mathbb{D}(\bullet)$ is an involution.

1.9.2. For 13.2.5 of [32]. First, if $\mathcal{A}$ is a monoidal ∞-category, $C\in 1-\mathcal{C}at$ then a lax action of $\mathcal{A}$ on the left on C is a right lax monoidal functor $\mathcal{A}\to\mathrm{End}(C)$, here $\mathrm{End}(C)$ is the monoidal ∞-category $\mathrm{Fun}(C,C)$. So, for $c\in C$, $a_i\in\mathcal{A}$ this gives functorial morphisms $c\to 1*c$ and $a_1*(a_2*c)\to(a_1*a_2)*c$. Similarly for bimodules. This is used in ([27], 2.7.1).

For bimodules this means that we are given a right-lax monoidal functor $\mathcal{A}\times\mathcal{A}^{rm}\to\mathrm{End}(C)$, here $\mathcal{A}^{rm}$ is $\mathcal{A}$ with reversed multiplication.

In the situation of 2.7.1 either he means non-unital $\mathcal{A}$ or, if it is unital then it satisfies in addition the property that the canonical map $c\to 1*c*1$ is an isomorphism for $c\in C$.

For ([27], 2.7.3), $\tilde{A}$ in general is not monoidal, I think. Namely, if a map $a_1\to a_2$ in $\tilde{A}$ is given by $a_2\widetilde{\to} a*a_1$, $b\in\mathcal{A}$ then it does not induce a map $b*a_1\to b*a_2$. He rather assumes each a admits a right dual, as $1\to a^\vee * a$ is given. But anyway this is applied to a symmetric monoidal $\mathcal{A}$.

His setting does not apply as is, because any $0\neq\lambda\in\Lambda^+$ is not dualizable in Λ^+.

Actually a simplier thing is used. Let $\mathcal{A}$ be an abelian group in *Sets*, $\mathcal{A}^+\subset\mathcal{A}$ be a submonoid. Assume given a lax $\mathcal{A}\times(\mathcal{A}^+)^{rm}$-action on $\mathcal{C}\in 1-\mathcal{C}at$. Assume the left action of $\mathcal{A}$ on $\mathcal{C}$ is a strict action, not a lax one. Note that $(\mathcal{A}^+)^{rm}=\mathcal{A}^+$ as $\mathcal{A}$ is abelian (rm stands for the reversed multiplication).

Let $c\in C$ be a lax central element in the sence of ([27], 2.7.1). We assume for any $a\in\mathcal{A}^+$ the space $\mathrm{Map}_C(ac, ca)$ is discrete. View now $\mathcal{A}^+$ as a category, where for $a_i\in\mathcal{A}^+$ we let

$$\mathrm{Map}_{\mathcal{A}^+}(a_1,a_2)=\begin{cases} a_2a_1^{-1}, & \text{if } a_2a_1^{-1}\in\mathcal{A}^+\\ \emptyset, & \text{otherwise}\end{cases}$$

We want to check that the map $\mathcal{A}^+\to C$, $a\mapsto a^{-1}ca$ is well-defined as a functor of ∞-categories. For $a_i\in\mathcal{A}^+$ let $a=a_2-a_1\in\mathcal{A}^+$ also. Then the map $a_1^{-1}ca_1\to a_2^{-1}ca_2$ is defined as the composition

$$a_1^{-1}ca_1\widetilde{\to} a_1^{-1}a^{-1}aca_1\widetilde{\to}(aa_1)^{-1}(ac)a_1\stackrel{\phi(a,c)}{\to} a_2^{-1}(ca)a_1\to a_2^{-1}ca_2$$

Then this is well-defined as a functor of ∞-categories. There is "no room for higher homotopies". In our situation for $b\in\mathcal{A}^+$, $\mathrm{Map}_C(acb,(ca)b)$ is no more discrete possibly. However, what really counts is the discreteness of spaces of the form $\mathrm{Map}_C(b(ac), c(ba))$, and more generally, of the spaces $\mathrm{Map}_C(a_1\star(a_2\star\ldots(a_n\star c)\ldots), c\star(a_1\ldots a_n))$ which we assume. They n-morphisms relating the compositions (for $n \geq 2$) lie in the latter spaces, so need not be provided. Thus, we get a functor.

For example, consider the composition of the morphisms $a_1^{-1}ca_1 \overset{\alpha_{12}}{\to} a_2^{-1}ca_2 \overset{\alpha_{23}}{\to} a_3^{-1}ca_3$ and the map $\alpha_{13}: a_1^{-1}ca_1 \to a_3^{-1}ca_3$. Let $a = a_2a_1^{-1}$, $b = a_3a_2^{-1}$. They both lie in $\mathrm{Map}_C(b(a(ca_1)), c(baa_1))$, because the multiplication by a_3^{-1} is an equivalence. Since the latter space is discrete, we automatically get a 2-isomorphism $\alpha_{23}\alpha_{12} \widetilde{\to} \alpha_{13}$.

1.9.3. For ([27], 1.2). The inclusion $SI(\mathrm{Gr}_G) \hookrightarrow Shv(\mathrm{Gr}_G)$ admits a partially defined left adjoint $\mathrm{Av}_!^{\mathfrak{L}(N)}: Shv(Gr_G) \to SI(\mathrm{Gr}_G)$, which is always defined in the constructible context (by my claim after Lemma 1.2.15), and in the $\mathcal{D}$-module context it is defined on the holonomic objects. (This is similar to the situation with the Whittaker category).

For ([27], 1.5.2). He claims $F \in SI(\mathrm{Gr}_G)$ lies in $SI(\mathrm{Gr}_G)^{\leq 0}$ iff for any λ, $i_\lambda^* F \in SI(\mathrm{Gr}_G)^{\leq 0}_{=\lambda}$. This is probably wrong! The following is true: consider a closed ind-subscheme $Y \subset \mathrm{Gr}_G$ stable under $\mathfrak{L}(N)$ and such that if $S^\lambda \subset Y$ then there is at most a finite number of μ such that $\lambda < \mu$ and $S^\mu \subset Y$. Then indeed $F \in SI(Y)$ lies in $SI(Y)^{\leq 0}$ iff for any $S^\lambda \subset Y$, $i_\lambda^* F \in SI(\mathrm{Gr}_G)^{\leq 0}_{=\lambda}$. This is proved as in [26], Lemma 2.1.9).

1.9.4. For ([27], 2.6.2). In fact, $G(F) \widetilde{\to} \lim_{n \in \mathbb{N}^{op}} G(F)/K_n$ as prestacks, here $F = k((t))$ and $K_n = \mathrm{Ker}(G(\mathcal{O}) \to G(\mathcal{O}/t^n))$. This follows from ([42], 4.4.2). The category $Shv(G(F))$ is defined in ([24], C.3.1), and $G(F)$ is a placid ind-scheme. For the map $f: G(F) \to \mathrm{Gr}_G$ the functor f^* is well-defined.

1.9.5. For ([27], 2.8.2). The argument is wrong as stated, because 1.5.2(iii) is wrong as stated probably. To correct, one has to assume first that $\mathcal{F} \in Shv(G)^\heartsuit$ is compact in $Shv(G)$, that is, of finite length. Then the given argument garantees indeed that $\boldsymbol{\Delta}^\lambda * \mathcal{F} \in SI(\mathrm{Gr}_G)^\heartsuit$, because $i_\mu^*(\boldsymbol{\Delta}^\lambda * \mathcal{F}) = 0$ for all but finite number of μ. Dennis has corrected this in the revised version of Oct 31, 2021.

It is useful to note that for $V \in \mathrm{Rep}(\check{G})^\heartsuit$ of finite length, $\boldsymbol{\Delta}^\lambda * Sat(V)$ has a finite filtration in $SI(\mathrm{Gr}_G)^\heartsuit$ with successive quotients $\boldsymbol{\Delta}^\mu \otimes V(\mu - \lambda)$, $\mu \in \Lambda$.

To get ([27], 2.8.2) note that any $V \in \mathrm{Rep}(\check{G})^\heartsuit$ is a filtered colimit of objects V_i, where $V_i \in \mathrm{Rep}(\check{G})^\heartsuit$ is of finite length.

1.9.6. For ([27], 2.8.3). There we have to assume not Th. 1.5.5 but 1.5.7, that is, the fact that each object $\vartriangle^{-\lambda}$ lies in the heart $\mathrm{SI}(\mathrm{Gr}_G)^\heartsuit$. Moreover, the claim is that for any $\lambda \in \Lambda^+$, the functor $\mathrm{Av}_!^{\mathfrak{L}(N)}$ is defined on $\delta_{t^{-\lambda}} * \mathrm{Sat}(V^\lambda)$, and one has canonically in $\mathrm{SI}(\mathrm{Gr}_G)$

$$\mathrm{Av}_!^{\mathfrak{L}(N)}(t^{-\lambda} \star \mathrm{Sat}(V^\lambda)) \widetilde{\to} \mathrm{Av}_!^{\mathfrak{L}(N)}(t^{-\lambda}) \star \mathrm{Sat}(V^\lambda)$$

Indeed, for any $K \in \mathrm{SI}(\mathrm{Gr}_G)$, one has

$$\mathrm{Map}_{\mathrm{SI}(\mathrm{Gr}_G)}(\mathrm{Av}_!^{\mathfrak{L}(N)}(t^{-\lambda}) \star \mathrm{Sat}(V^\lambda), K) \widetilde{\to} \mathrm{Map}_{\mathrm{SI}(\mathrm{Gr}_G)}(\mathrm{Av}_!^{\mathfrak{L}(N)}(t^{-\lambda}), K \star (\mathbb{D}(*\mathrm{Sat}(V^\lambda))))$$
$$\widetilde{\to} \mathrm{Map}_{Shv(\mathrm{Gr}_G)}(\delta_{t^{-\lambda}}, K \star (\mathbb{D}(*\mathrm{Sat}(V^\lambda)))) \widetilde{\to} \mathrm{Map}_{Shv(\mathrm{Gr}_G)}(\delta_{t^{-\lambda}} \star \mathrm{Sat}(V^\lambda), K)$$

(We may also note that $\delta_{t^{-\lambda}} * \mathrm{Sat}(V^\lambda)$ is holonomic, hence $\mathrm{Av}_!^{\mathfrak{L}(N)}$ is defined on it).

So,

$$\vartriangle^{-\lambda} \star \mathrm{Sat}(V^\lambda) \widetilde{\to} \mathrm{Av}_!^{\mathfrak{L}(N)}(t^{-\lambda})[\langle \lambda, 2\check{\rho} \rangle] \star \mathrm{Sat}(V^\lambda) \widetilde{\to} \mathrm{Av}_!^{\mathfrak{L}(N)}(t^{-\lambda} \star \mathrm{Sat}(V^\lambda))[\langle \lambda, 2\check{\rho} \rangle]$$

For $g\in T(F)$ consider the automorphism $g:\mathrm{Gr}_G\widetilde{\to}\mathrm{Gr}_G$. The functor $g_*: Shv(\mathrm{Gr}_G)\to Shv(\mathrm{Gr}_G)$ preserves $\mathrm{SI}(\mathrm{Gr}_G)$, as $T(F)$ normalizes $N(F)$. Taking $g=t^\mu$ for $\lambda\in\Lambda$ we get $t^\mu\omega_{S^\lambda}\widetilde{\to}\omega_{S^{\lambda+\mu}}$. Consider the autoequivalence $\mathrm{SI}(\mathrm{Gr}_G)$, $K\mapsto t^\mu K[-\langle\mu,2\check{\rho}\rangle]$. It sends $\boldsymbol{\Delta}^\lambda$ to $\boldsymbol{\Delta}^{\lambda+\mu}$, so is right t-exact. It is in fact t-exact: let $F\in\mathrm{SI}(\mathrm{Gr}_G)^{>0}$ and $\lambda\in\Lambda$. It suffices to show that for any $n\geq 0$, $\mathrm{Hom}(\boldsymbol{\Delta}^\lambda, t^\mu F[-\langle\mu,2\check{\rho}\rangle])=0$. The latter Hom identifies with $\mathrm{Hom}(\boldsymbol{\Delta}^{\lambda-\mu}[n],\mathcal{F})=0$. So, if we know that $\boldsymbol{\Delta}^0\in\mathrm{SI}(\mathrm{Gr}_G)^\heartsuit$ then the same holds for all $\boldsymbol{\Delta}^\nu$, $\nu\in\Lambda$.

1.9.7. For ([27], 3.1). Schieder's correction is needed in the definition of the stacks $\overline{\mathrm{Bun}}_N, (\overline{\mathrm{Bun}}_N)_{\infty x}, (\overline{\mathrm{Bun}}_N)_{\leq\lambda x}$.

Let us mean by $\overline{\mathrm{Bun}}_N$ the corrected definition now. The map $\mathrm{Gr}_G\to(\overline{\mathrm{Bun}}_N)_{\infty x}$ from ([27], 3.1.6) is defined as follows. Pick an exact sequence $1\to Z\to\tilde{G}\to G\to 1$, where Z is a central torus in $\tilde{G}$, $[\tilde{G},\tilde{G}]$ is simply-connected. Let $\tilde{N}\subset\tilde{G}$ be the preimage of N. The correct definition of $\overline{\mathrm{Bun}}_N$ is just $\overline{\mathrm{Bun}}_{\tilde{N}}$, where the latter is attached to the pair $(\tilde{N}\subset\tilde{G})$. In other words, this is $\mathrm{Map}_{gen}(X, B(\tilde{N})\subset\tilde{G}\backslash\overline{(\tilde{G}/\tilde{N})})$.

Then $\overline{\mathrm{Bun}}_{\tilde{B}}\times_{\mathrm{Bun}_{\tilde{T}}}\mathrm{Bun}_Z\widetilde{\to}\overline{\mathrm{Bun}}_{\tilde{N}}\times\mathrm{Bun}_Z$ naturally. Consider the prestack $\mathcal{X}$ classifying $\mathcal{F}_{\tilde{G}}$ on X, $\mathcal{F}_Z$ on X and an isomorphism $\mathcal{F}_G\widetilde{\to}\mathcal{F}_Z\mid_{X-x}$. We have a natural map $v:\mathcal{X}\to(\overline{\mathrm{Bun}}_N)_{\infty x}\times\mathrm{Bun}_Z$ commuting with Bun_Z-actions. Let $\mathcal{X}'$ be the prestack classifying $\mathcal{F}_{\tilde{G}}$ on X, $\mathcal{F}_Z$ on $X-x$ and an isomorphism $\mathcal{F}_G\widetilde{\to}\mathcal{F}_Z\mid_{X-x}$. The projection $\mathcal{X}'\to\mathrm{Gr}_G$ is a Bun_Z-torsor in etale topology. We also have a projection $q:\mathcal{X}\to\mathcal{X}'$ commuting with Bun_Z-actions. The map q is a torsor under Gr_Z. Since Z acts trivially on $\tilde{G}\backslash\overline{(\tilde{G}/\tilde{N})}$, the map v is Gr_Z-invariant, hence yields a morphism $\bar{v}:\mathcal{X}'\to(\overline{\mathrm{Bun}}_N)_{\infty x}\times\mathrm{Bun}_Z$, which is still Bun_Z-equivariant. Taking the quotient by Bun_Z, one gets the desired morphism $\mathrm{Gr}_G\to\overline{\mathrm{Bun}}_N$.

Note also that $\boldsymbol{\Delta}^\lambda_{glob}, \nabla^\lambda_{glob}$ from ([27], 3.1.4) are perverse, as the inclusion $(\overline{\mathrm{Bun}}_N)_{=\lambda x}\hookrightarrow(\overline{\mathrm{Bun}}_N)_{\leq\lambda x}$ is affine by ([17], 3.3.1).

1.9.8. The map (3.1) in ([27], 3.2.3) comes from $\pi_!\pi^!\omega\to\omega$ for $\pi: S^\lambda\to(\overline{\mathrm{Bun}}_N)_{=\lambda x}$, here $\pi_!$ is defined on ω_{S^λ}, because the latter is holonomic.

Writing temporary $\bar{\pi}:\bar{S}^\lambda\to(\overline{\mathrm{Bun}}_N)_{\leq\lambda x}$ we see that $i_{\lambda !}\pi_!\omega[-\langle\lambda,2\check{\rho}\rangle]\widetilde{\to}\bar{\pi}_!\boldsymbol{\Delta}^\lambda$. So, $\bar{\pi}_!$ is defined on $\boldsymbol{\Delta}^\lambda\in SI(\bar{S}^\lambda)$.

In ([27], 3.3.1) a misprint in the 1st displayed formula: it should be

$$\mathrm{colim}_{\lambda\in\Lambda^+}\mathrm{H}_{t^{\lambda+\mu}}(\mathrm{Gr}_G,\mathrm{IC}_{\overline{\mathrm{Gr}}_G^\lambda})[\langle\lambda,2\check{\rho}\rangle]$$

([27], 3.3.4) is proved only under the assumption that $[G,G]$ is simply-connected, while ([27], 2.5.2) is claimed without this assumption.

1.9.9. For ([27], 3.3.8). He wants to use ([29], 3.5.2). More precisely, here some Koszul duality is need, so that the cited result should imply that $j_!\,\mathrm{IC}_{\mathrm{Bun}_B}\widetilde{\to} coBar(U^\vee(\check{\mathfrak{n}}^-_X),\mathrm{IC}_{\overline{\mathrm{Bun}}_B})$. Then we want to use the description of the complex $i^!_\mu\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ given by ([27], 3.3.4) essentially, though the latter is for B replaced by N.

1.9.10. For ([27], 3.4.1). My understanding is that he claims that $\pi_!$ is defined on $t^{-\lambda}Sat(V^\lambda)$, as the latter is holonomic. Note that for $\lambda\in\Lambda$ the image of $t^{-\lambda}$ under $\pi:\mathrm{Gr}_T\to(\overline{\mathrm{Bun}}_N)_{\infty x}$ comes from the T-torsor $\mathcal{F}^0_T(\lambda x)$ by extending the structure group to B.

For ([27], 3.4.3), there λ is dominant. His nonstardard notation $\mathrm{Gr}_G^{-\lambda}$ means $\mathrm{Gr}_G^{-w_0(\lambda)}$, same for $\overline{\mathrm{Gr}}_G^{-\lambda}$. To help a reader, the stack $(\overline{\mathrm{Bun}}_N)_{\infty x}\tilde{\times}\overline{\mathrm{Gr}}_G^{-w_0(\lambda)}$ here classified $(\mathcal{F}_G,\mathcal{F}'_G)$, where $\mathcal{F}_G\in(\overline{\mathrm{Bun}}_N)_{\infty x}$ and $\mathcal{F}'_G$ is in the position $\le -w_0(\lambda)$ w.r.t. $\mathcal{F}_G$ at x. The map act sends this point to $\mathcal{F}'_G$ and pr sends it to $\mathcal{F}_G$. Then $\mathrm{IC}_{glob}*\mathrm{IC}_{\overline{\mathrm{Gr}}_G^{-w_0(\lambda)}}\widetilde{\to}\mathrm{act}_*(\mathrm{IC}\boxtimes\mathcal{A}_G^{-w_0(\lambda)})$ by definition.

The preimage of $t^{-\lambda}$ under $\bar{S}^{\le 0}\tilde{\times}\overline{\mathrm{Gr}}_G^{-w_0(\lambda)}\stackrel{\mathrm{act}}{\to}\mathrm{Gr}_G$ over $t^{-\lambda}$ does not lies in $\bar{S}^{\le 0}\tilde{\times}(\overline{\mathrm{Gr}}_G^{-w_0(\lambda)}\cap S^{-\lambda})$. In fact, it lies in many $N(F)$-orbits. Recall that $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)}\cap S^{-\lambda}$ is the point $t^{-\lambda}$. It is not true that

$$\mathrm{act}^{-1}(\pi(t^{-\lambda}))\cap(\overline{\mathrm{Bun}}_N\tilde{\times}\overline{\mathrm{Gr}}_G^{-w_0(\lambda)})=\mathrm{act}^{-1}(\pi(t^{-\lambda}))\cap(\mathrm{Bun}_N\tilde{\times}\mathrm{Gr}_G^{-w_0(\lambda)})$$

However, the 0-th cohomology of the desired !-fibre is indeed e, and this gives the desired map

$$\delta_{\pi(t^{-\lambda})}\to\mathrm{IC}_{glob}*\mathrm{IC}_{\overline{\mathrm{Gr}}_G^{-w_0(\lambda)}}[(g-1)\dim N-\langle\lambda,2\check{\rho}\rangle]$$

1.9.11. For ([27], 3.4.6). The inclusion $Sets\hookrightarrow Spc$ preserves limits. So if $C\in 1-\mathcal{C}\mathrm{at}$, $c\widetilde{\to}\mathrm{colim}_{i\in I}c_i$ in C, $c'\in C$, assume $\mathrm{Map}_C(c_i,c')\in Sets$ for any i. Then $\lim_{i\in I^{op}}\mathrm{Map}_C(c_i,c')$ can be calculated in Sets. Its element is a collection of maps $c_i\to c'$ such that for any $i\to j$ in I, the composition $c_i\to c_j\to c'$ is homotopic to $c_i\to c'$. This gives a map $c\to c'$ in C.

For ([27], 3.4.8). We consider here the closed immersion $i_s:S^0\cap t^{-\lambda}\mathrm{Gr}_G^\lambda\hookrightarrow S^0$ say. Then the natural map $i_{s!}i_s^!\omega\to\omega$ gives the desired morphism $j_0^!(t^{-\lambda}Sat(V^\lambda))[\langle\lambda,2\check{\rho}\rangle]\to\omega_{S_0}$, and we apply $(j_0)_*$ to the latter map.

1.9.12. For ([27], 3.5.3). If $K\in Shv(\bar{S}^{\le 0})$ is the extension by zero from $S^{\le\mu}$ for any $\mu\le 0$ then $K=0$. Indeed, the open subschemes $S^{\le 0}-S^{\le\mu}$ for $\mu<0$ cover $S^{\le 0}$, and Shv satisfies the Zarizki descent. In particular the functor $Shv(\bar{S}^{\le 0})\to\prod_{\mu<0}Shv(\bar{S}^{\le 0}-\bar{S}^{\le\mu})$ given by the product of restrictions, is conservative.

For ([27], 3.5.3 and 3.2.4). We have an action of Λ on Gr_G by automorphisms, namely λ acts by $t^\lambda:\mathrm{Gr}_G\to\mathrm{Gr}_G$. Consider the action of Λ on $Shv(\mathrm{Gr}_G)$ such that λ sends F to $t^\lambda F[-\langle\lambda,2\check{\rho}\rangle]$, we mean here the direct image under $t^\lambda:\mathrm{Gr}_G\to\mathrm{Gr}_G$. Since this action preserves the set of standard objects $\boldsymbol{\Delta}^\mu$ with $\mu\in\Lambda$, we get an action of Λ on $\mathrm{SI}(\mathrm{Gr}_G)$, because these are the compact generators. For $\lambda\in\Lambda$ the corresponding autoequivalence of $\mathrm{SI}(\mathrm{Gr}_G)$ is t-exact, as we have seen above.

Similarly, consider the smallest full stable cocomplete DG-subcategory $\mathcal{C}$ generated in $Shv((\overline{\mathrm{Bun}}_N)_{\infty x})$ by $\boldsymbol{\Delta}^\lambda_{glob}$ for $\lambda\in\Lambda$. Then $\mathcal{C}$ contains $\mathrm{IC}_{\overline{\mathrm{Bun}}_N}$, also because Shv satisfies the Zariski descent. We can consider the action of Λ on $\mathcal{C}$ coming from its action on the set of objects $\boldsymbol{\Delta}^\lambda_{glob}$ for $\lambda\in\Lambda$. Namely, $\lambda\in\Lambda$ sends $\boldsymbol{\Delta}^\mu_{glob}$ to $\boldsymbol{\Delta}^{\mu+\lambda}_{glob}$. Recall that for $\lambda\in\Lambda$ we should have $\boldsymbol{\Delta}^\mu\widetilde{\to}\pi^!\boldsymbol{\Delta}^\mu_{glob}[(g-1)\dim N]$ according to ([27], Thm. 3.2.4). So, $\pi^!$ should induce a functor $\mathcal{C} \to SI(\mathrm{Gr}_G)$ commuting with Λ-actions on both sides.

He claims that to prove Thm. 3.2.4, it suffices to show that $(i_\mu)^*\pi^!(\boldsymbol{\Delta}^0_{glob}) = 0$ for $\mu \neq 0$. One should similarly prove actually that $(i_\mu)^*\pi^!(\boldsymbol{\Delta}^\lambda_{glob}) = 0$ for $\mu \neq \lambda$.

The group Λ acts on $(\overline{\mathrm{Bun}}_N)_{\infty x}$. Namely, $\mu \in \Lambda$ sends $(\mathcal{F}_G, \kappa) \in (\overline{\mathrm{Bun}}_N)_{\infty x}$ to $(\mathcal{F}_G, \kappa')$, where for each $\check{\lambda}$ dominant, $(\kappa')^{\check{\lambda}} : \mathcal{O}_X \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ equals $t_x^{-\langle \mu, \check{\lambda}\rangle}\kappa^{\check{\lambda}}$. This automorphism $t^\mu : (\overline{\mathrm{Bun}}_N)_{\infty x} \to (\overline{\mathrm{Bun}}_N)_{\infty x}$ sends $(\overline{\mathrm{Bun}}_N)_{\le \lambda}$ to $(\overline{\mathrm{Bun}}_N)_{\le \lambda+\mu}$ and identifies $(\overline{\mathrm{Bun}}_N)_{=\lambda}$ to $(\overline{\mathrm{Bun}}_N)_{=\lambda+\mu}$. For $\mu \in \Lambda$ the diagram commutes

$$\begin{array}{ccc} \mathrm{Gr}_G & \stackrel{t^\mu}{\to} & \mathrm{Gr}_G \\ \downarrow \pi & & \downarrow \pi \\ (\overline{\mathrm{Bun}}_N)_{\infty x} & \stackrel{t^\mu}{\to} & (\overline{\mathrm{Bun}}_N)_{\infty x} \end{array}$$

So, $t^\mu \boldsymbol{\Delta}^\lambda_{glob} \widetilde{\to} \boldsymbol{\Delta}^{\lambda+\mu}_{glob}$ and $t^\mu \nabla^\lambda_{glob} \widetilde{\to} \nabla^{\lambda+\mu}_{glob}$. We get $\pi^! \boldsymbol{\Delta}^\lambda_{glob} \widetilde{\to} t^\lambda \pi^! \boldsymbol{\Delta}^0_{glob}$. So,

$$(i_\mu)^*\pi^!(\boldsymbol{\Delta}^\lambda_{glob}) \widetilde{\to} i^*_{\mu-\lambda}\pi^! \boldsymbol{\Delta}^0_{glob}$$

1.9.13. In ([27], 3.6.2) Dennis uses the description of Zastava spaces via Weyl modules and dual Weyl modules, see ([49], 0.5) where I explain what they are.

The version of Zastava space from ([27], 3.6) is obtained from the usual one from [9] by interchanging B and B^-.

In his Prop. 3.6.6 he refers to ([9], Remark just after Pp. 5.7). By $H_{\mathfrak{F}^\mu}(\mathcal{Z}^\mu, \mathrm{IC}_{\mathcal{Z}^\mu})$ he means $\mathrm{R}\Gamma(\mathfrak{F}^\mu, i^! \,\mathrm{IC}_{\mathcal{Z}^\mu})$ for the closed immersion $i : \mathfrak{F}^\mu \hookrightarrow \mathcal{Z}^\mu$. Then his Prop. 3.6.6 comes from ([9], Pp. 5.8).

For ([27], 3.6.7). It is used that the $\mathbb{G}_m$-action on $\mathcal{Z}^\mu$ contracting it to X^μ can be chosen so that it preserves the open subscheme $\mathring{\mathcal{Z}}^\mu$.

1.9.14. For ([27], 3.7.2). First, for λ dominant and $\mu \le 0$, $S^{-,\mu+\lambda} \cap \overline{\mathrm{Gr}}^\lambda_G$ is of pure dimension $-\langle \check{\rho}, \mu\rangle$ by ([53], Th. 3.2). Further, $Irr(S^{-,\mu+\lambda} \cap \overline{\mathrm{Gr}}^\lambda_G) = Irr(S^{-,\mu+\lambda} \cap \mathrm{Gr}^\lambda_G)$ again by ([53], Th. 3.2), because the complement is of smaller dimension. It is known that $S^\lambda \cap S^{-,\mu+\lambda}$ is of pure dimension $-\langle \mu, \check{\rho}\rangle$. By Anderson's theorem ([1], Pp. 3), one has a bijection

$$\{a \in Irr(S^\lambda \cap S^{-,\mu+\lambda}) \mid a \subset \overline{\mathrm{Gr}}^\lambda_G\} \widetilde{\to} Irr(\mathrm{Gr}^\lambda_G \cap S^{-,\mu+\lambda})$$

sending a to the closure of $a \cap \mathrm{Gr}^\lambda_G$. So, indeed each irreducible component $\mathrm{Gr}^\lambda_G \cap S^{-,\mu+\lambda}$ is the closure of a unique irreducible component of $S^\lambda \cap S^{-,\mu+\lambda} \cap \mathrm{Gr}^\lambda_G$.

Applying ([9], Pp. 6.4) with B and B^- exchanged, we get that for μ fixed and λ deep enough in the dominant chamber Λ^+ one has $S^\lambda \cap S^{-,\mu+\lambda} \subset \mathrm{Gr}^\lambda_G$.

To obtain the last map in ([27], 3.7.2), he uses for any K on $S^{-,\mu} \cap (t^{-\lambda}\overline{\mathrm{Gr}}^\lambda_G)$ the natural map $K \to j_*j^*K$ for the open immersion

$$j : S^0 \cap S^{-,\mu} \cap (t^{-\lambda}\,\mathrm{Gr}^\lambda_G) \to S^{-,\mu} \cap (t^{-\lambda}\overline{\mathrm{Gr}}^\lambda_G)$$

It induces his map on the level of cohomologies.

Recall that in his notations $\dim \mathcal{Z}^\mu = -\langle \mu, 2\check{\rho}\rangle$ for $\mu \le 0$, and $\dim S^0 \cap S^{-,\mu} = -\langle \mu, \check{\rho}\rangle$. He uses several times the commutative diagram

$$\begin{array}{ccc} \bar S^0 & \leftarrow & \bar S^0 \cap S^{-,\mu} = \mathfrak{F}^\mu \\ \downarrow \pi & & \downarrow i \\ \overline{\mathrm{Bun}}_N & \stackrel{q}{\leftarrow} & \mathcal{Z}^\mu, \end{array}$$

where i is the inclusion of the central fibre $\mathfrak{F}^\mu$.

1.9.15. For ([27], 3.8.3). The action of $N_{X-(x,y)}$ on S^λ_x is transitive, as already the action of N_{X-x} is transitive.

Consider the category I whose objects are open subschemes $U \subset \bar S^0$ consisting of finite number of $N(F)$-orbits, and maps are open immersions. The natural map $Shv(\bar S^0) \to \lim_{U\in I^{op}} Shv(U)$ is an equivalence. Indeed, let $i : S \subset \bar S^0$ be a closed subscheme of finite type. It suffces to show that the natural map $Shv(S) \to \lim\limits_{U\in I^{op}} Shv(I \cap S)$ is an isomorphism. However, S is covered by a finite number of $N(F)$-orbits.

1.9.16. For ([27], 3.9). Two things have to be added here: first, the fact that $j_! \mathrm{IC}_{\mathrm{Bun}_N}$ is perverse on $\overline{\mathrm{Bun}}_N$, and similarly for $j_! \mathrm{IC}_{\overset{\circ}{\mathcal{Z}}{}^\mu}$ on $\mathcal{Z}^\mu$. The second one is the fact that the fibres in the maps in the diagram (3.9) used for the descent of perverse sheaves are connected.

1.9.17. For ([27], 4.1.1). Note that I is a placid group scheme over $\operatorname{Spec} k$, so we have an adjoint pair $\mathrm{oblv} : Shv(\mathrm{Gr}_G)^I \leftrightarrows Shv(\mathrm{Gr}_G) : \mathrm{Av}_*$ in DGCat_{cont} for the usual category of invariants $Shv(\mathrm{Gr}_G)^I$. Justin proposes to define the renormalized version as $\mathrm{Ind}(C)$, where $C = \{K \in Shv(\mathrm{Gr}_G)^I \mid oblv(K) \in Shv(\mathrm{Gr}_G)^c\}$. Note that $C \in \mathrm{DGCat}^{non-cocmpl}$, so that $\mathrm{Ind}(C) \in \mathrm{DGCat}_{cont}$ by ([43], 9.2.14). Indeed, $\mathrm{DGCat}^{non-cocmpl}$ admits limits, and the oblivion functor $\mathrm{DGCat}^{non-cocmpl} \to 1-\mathcal{C}\mathrm{at}$ preserves limits.

Let $Shv(\mathrm{Gr}_G)^{I,constr} \subset Shv(\mathrm{Gr}_G)^I$ be the full subcategory of objects that pull-back to a compact object of $Shv(\mathrm{Gr}_G)$. Then $Shv(\mathcal{F}l_G)^I$ acts on $\mathrm{Ind}(Shv(\mathrm{Gr}_G)^{I,constr})$ by left convolutions. Indeed, $Shv(\mathcal{F}l_G)^I$ is compactly generated. Any compact object K of $Shv(\mathcal{F}l_G)^I$ is the extension by zero from some I-invariant closed subscheme of finite type $Y \subset \mathcal{F}l_G$, and K restricts to a compact object in $Shv(Y)$. This is why $(Shv(\mathcal{F}l_G)^I)^c$ acts on $Shv(\mathrm{Gr}_G)^{I,constr}$. Passing to the ind-completion, we get an action of $Shv(\mathcal{F}l_G)^I$ on $\mathrm{Ind}(Shv(\mathrm{Gr}_G)^{I,constr})$.

Here is a model situation. Let $Y \in \mathrm{Sch}_{ft}$, G be a group scheme of finite type, and $H \subset G$ a closed group subscheme, assume G acts on Y. Consider $f : Y \to Y/H$. We then have an inclusion $Shv(Y/H)^c \subset Shv(Y/H)^{constr}$, which is not an equality in general. Assume that we are in the constructible context or that G/H is proper. Then $Shv(H\backslash G/H)$ acts on $\mathrm{Ind}(Shv(Y/H)^{constr})$. Namely, the monoidal structure on $Shv(H\backslash G/H)$ is as in my Section 1.10.1. The action map $m : G \times^H Y \to Y$ identifies with the compositioin $G \times^H Y \widetilde{\to} (G/H) \times Y \stackrel{\mathrm{pr}}{\to} Y$, where the first map comes from the$(g, y) \mapsto (g, gy)$. In the constructible context the map pr_* has a continuous right adjoint, so preserves compactness. So, under our assumptions m_* preserves compactness. If $K \in Shv(H\backslash G/H)^c$, $L \in Shv(Y/H)^{constr}$ then consider

$K\widetilde{\boxtimes}L \in Shv(G\times^H Y)$. Since its restriction to $G\times Y$ is compact, $K\widetilde{\boxtimes}L \in Shv(G\times^H Y)^c$. So, $m_*(K\widetilde{\boxtimes}L) \in Shv(Y)^c$. This defines an action of $Shv(H\backslash G/H)^c$ on $Shv(Y/H)^{constr}$, and the desired action is then obtained by Ind-extension.

The t-structure on $Shv(\mathcal{F}l_G)^I$ is defined in ([46], 0.0.40). However, Dennis wants to use the renormalized version of $Shv(\mathcal{F}l_G)^I$ instead.

1.9.18. The reference for ([27], 4.1.2) in ([3], Lemma 8).

For ([27], 4.1.2). He used without a proof the following. Let W^{aff} denote the extended affine Weyl group. For $w \in W^{aff}$, $j_{w,!}$, $j_{w,*}$ the standard and costandard objects, $w_0 \in W$ the longuest element of the finite Weyl group. On the orbit itself $\mathcal{F}l_G^w$ we take the IC-sheaf and extend by ! or $*$. Recall that the length $\ell(w)$ of $w \in W^{aff}$ is the dimension of the I-orbit on $\mathcal{F}l_G$ through w.

Lemma 1.9.19. *Let $\lambda \in \Lambda^+$. One has $j_{w_0,!} * j_{\lambda,*} * j_{w_0,*} \widetilde{\to} j_{w_0(\lambda),*}$ for the convolution on $Shv(\mathcal{F}l_G)^I$.*

Proof. The rule of the game is that given $w, w' \in W$ with $\ell(w) + \ell(w') = \ell(ww')$ then $j_{w,*} * j_{w',*} \widetilde{\to} j_{ww',*}$. Besides, $j_{w,*} * j_{w^{-1},!} \widetilde{\to} \delta_1$. For any $\lambda \in \Lambda$, the set $E_\lambda = \{t^\lambda w \mid w \in W\}$ has a unique element of minimal length, and $\dim O^\lambda = \min_{w\in E_\lambda} \ell(w)$. Here O^λ is the I-orbit on Gr_G through t^λ. If λ is dominant then t^λ is of minimal length in E_λ.

Assume λ dominant. Then $\ell(t^\lambda) = \langle\lambda, 2\check{\rho}\rangle$ and $\ell(w_0) = \dim(G/B)$, as $Bw_0B/B \subset G/B$ is open. Besides, $\mathcal{F}l_G^{t^\lambda w_0} \subset \pi^{-1}(\mathrm{Gr}_G^\lambda)$ is open, where $\pi : \mathcal{F}l_G \to \mathrm{Gr}_G$ is the projection. So, $\ell(t^\lambda w_0) = \ell(t^\lambda) + \ell(w_0)$, hence $j_{t^\lambda w_0,*} \widetilde{\to} j_{\lambda,*} * j_{w_0,*}$. One has $t^\lambda w_0 = w_0 t^{w_0\lambda}$.

If μ is antidominant then t^μ is the unique element of minimal length in Wt^μ. For this reason, $t^{w_0\lambda}$ is minimal in $Wt^{w_0\lambda}$, hence similarly $\ell(w_0) + \ell(t^{w_0\lambda}) = \ell(w_0 t^{w_0(\lambda)})$. So, $j_{w_0,*} * j_{t^{w_0\lambda},*} \widetilde{\to} j_{t^\lambda w_0,*}$. Multiplying on the left the isomorphism

$$j_{w_0,*} * j_{t^{w_0(\lambda)},*} \widetilde{\to} j_{t^\lambda,*} * j_{w_0,*}$$

by $j_{w_0,!}$, we get the result. □

1.9.20. For ([27], 4.2.3). The reformulation of the main result of [ABG] is not clear, should be explained. For $\lambda \in \Lambda^+$ the projection $\mathcal{F}l_G^{t^\lambda} \to O^\lambda$ is an isomorphism. Here for $\nu \in \Lambda$ we denote by O^ν the I-orbit through t^ν on Gr_G, and $\mathcal{F}l_G^w$ is the I-orbit on $\mathcal{F}l_G$ through w in the extended affine Weyl group. So, the object $j_{\lambda,*} * \delta_{1,\mathrm{Gr}_G}$ is simply $j_*\,\mathrm{IC}$ for the open immersion $O^\lambda \hookrightarrow \overline{\mathrm{Gr}}_G^\lambda$, hence a natural map $\mathrm{IC}_{\overline{\mathrm{Gr}}_G^\lambda} \to j_{\lambda,*} * \delta_{1,\mathrm{Gr}_G}$ in $Shv(\mathrm{Gr}_G)^I$.

The product $\check{\mathfrak{n}}^- \times_{\check{\mathfrak{g}}} \{0\}$ is taken in the category of derived affine schemes. Then $\check{B}^-$ acts on it in the sense of prestacks. I think by $(\check{\mathfrak{n}}^- \times_{\check{\mathfrak{g}}} \{0\})/\check{B}^-$ he means the following: first take the quotient in the sense of prestacks and then take etale sheafification. As in ([20], ch. I.2, 4.3.8) the result is a 1-Artin stack.

1.9.21. For ([27], 4.2.4). Misha Finkelberg says that

$$\mathrm{colim}_{\lambda\in\Lambda^+} \mathrm{H}_{t^{\lambda+\mu}}(\mathrm{Gr}_G, Sat(V^\lambda)[\langle\lambda+\mu, 2\check{\rho}\rangle]$$

is calculated in [41] with the answer $\mathrm{Sym}((\check{\mathfrak{g}}/\check{\mathfrak{b}}^-)[-2])(-\mu)$ as in ([27], Pp. 2.5.2).

To understand Dennis' calculation, recall first that for H an algebraic group of finite type, $Shv(B(H))\tilde{\to} C(H)-comod(\mathrm{Vect})\tilde{\to} C_{\cdot}(H)-mod(\mathrm{Vect})$, where $C(H)=\mathrm{R}\Gamma(H,e)$ is a coalgebra in Vect, and $C_{\cdot}(H)=\mathrm{R}\Gamma_c(H,\omega)$ is the dual algebra in Vect.

For a split torus T we get $C_{\cdot}(T)\tilde{\to}\mathrm{Sym}(\mathfrak{t}[1])$, where $\mathfrak{t}=\mathrm{Lie}\,T$. We used here the fact that for $V_i\in\mathrm{Vect}^{\le 0}$ we have $\mathrm{Sym}(V_1\oplus V_2)\tilde{\to}\mathrm{Sym}(V_1)\otimes\mathrm{Sym}(V_2)$ naturally in Vect. Indeed, by ([43], 3.0.40), the functor $\mathrm{oblv}: CAlg(\mathrm{Vect}^{\le 0})\to\mathrm{Vect}^{\le 0}$ admits a left adjoint sending V to $\mathrm{Sym}(V)=\oplus_{n\ge 0}\mathrm{Sym}^n(V)$. This implies that $\mathrm{Sym}(V_1\oplus V_2)$ is the coproduct of $\mathrm{Sym}(V_1)$ and $\mathrm{Sym}(V_2)$ in $CAlg(\mathrm{Vect}^{\le 0})$.

We have $C_{\cdot}(\mathbb{G}_m)\tilde{\to} e[1]\oplus e\tilde{\to}\mathrm{Sym}(e[1])$, so if we pick an isomorphism $T\tilde{\to}\mathbb{G}_m^n$, we get $C_{\cdot}(T)\tilde{\to}\otimes_{i=1}^n\mathrm{Sym}(e[1])\tilde{\to}\mathrm{Sym}(e^{\oplus n}[1])\tilde{\to}\mathrm{Sym}(\mathfrak{t}[1])$ via the induced isomorphism $\mathfrak{t}\tilde{\to} e^{\oplus n}$. I think the resulting isomorphism does not depend on a choice of $T\tilde{\to}\mathbb{G}_m^n$.

By $\mathcal{H}om_{Shv(\mathrm{Gr}_G)^I}(K_1,K_2)\in\mathrm{Vect}$ in the formula (4.4) he means the relative inner hom in $Shv(\mathrm{Gr}_G)^I$ with respect to the Vect-action. Further,

$$RG(B(T),e)\tilde{\to}\mathcal{H}om_{Shv(B(T))}(e,e)\tilde{\to}\mathrm{Sym}(\check{\mathfrak{t}}[-2])$$

(the latter isomorphism holds even $char(k)>0$ for $\bar{\mathbb{Q}}_\ell$-sheaves for example).

For $\lambda,\lambda+\mu\in\Lambda^+$,

$$\mathcal{H}om_{Shv(\mathrm{Gr}_G)^I}(Sat(V^\lambda),j_{\lambda+\mu,*}*\delta_{1,\mathrm{Gr}_G})\tilde{\to}\mathcal{H}om_{\mathrm{Vect}}(i^*_{t^{\lambda+\mu}}Sat(V^\lambda),e)\otimes\mathrm{Sym}(\check{\mathfrak{t}}[-2])[\langle\lambda+\mu,2\check{\rho}\rangle]$$

Here $i_{t^{\lambda+\mu}}:\mathrm{Spec}\,k\to\mathrm{Gr}_G$ is the inclusion of the point $t^{\lambda+\mu}$. Indeed,

$$\mathcal{H}om_{Shv(O^{\lambda+\mu})^I}(e,e)\tilde{\to}\mathrm{Sym}(\check{\mathfrak{t}}[-2])$$

Now $\mathcal{H}om_{\mathrm{Vect}}(i^*_{t^{\lambda+\mu}}Sat(V^\lambda),e)\tilde{\to} i^!_{t^{\lambda+\mu}}Sat(V^\lambda)$. So,

$$\mathcal{H}om_{Shv(\mathrm{Gr}_G)^I}(Sat(V^\lambda),j_{\lambda+\mu,*}*\delta_{1,\mathrm{Gr}_G})\otimes_{\mathrm{Sym}(\check{\mathfrak{t}}[-2])}e\tilde{\to} i^!_{t^{\lambda+\mu}}Sat(V^\lambda)[\langle\lambda+\mu,2\check{\rho}\rangle]$$

Inside the proof in Sect. 4.2.4 in formula (4.6) inside the proof and ALL the remaining formulas inside the proof of Pp. 2.5.2 replace $\mathfrak{g}/\mathfrak{b}$ by $\check{\mathfrak{g}}/\check{\mathfrak{b}}$. He gets the answer

$$\mathrm{Sym}((\check{\mathfrak{g}}/\check{\mathfrak{n}}^-)[-2])(-\mu)\tilde{\to}\mathrm{Sym}((\check{\mathfrak{g}}/\check{\mathfrak{b}}^-)[-2])(-\mu)\otimes\mathrm{Sym}(\check{\mathfrak{t}}[-2]),$$

which via extension of scalars $\mathrm{Sym}(\check{\mathfrak{t}}[-2])\to e$ gives the desired result.

1.9.22. For ([27], 4.3.1). For. $q:\mathrm{Spec}\,e\to B(M)$ we have an adjoint pair $q^*:\mathrm{Rep}(M)\leftrightarrows\mathrm{Vect}:q_*$ in $\mathrm{Rep}(M)$-modules, see ([20], I.3). For this reason $\mathrm{oblv}_{\mathrm{Hecke}_M}:\mathrm{Hecke}_M(\mathcal{C})\to\mathcal{C}$ is continuous.

For ([27], 4.3.2). For $C\in Shv(B(M))-mod(\mathrm{DGCat}_{cont})$ we have

$$C\otimes_{\mathrm{Rep}(M)}\mathrm{Vect}\tilde{\to}\mathrm{Fun}_{\mathrm{Rep}(M)}(\mathrm{Vect},C)$$

by ([43], 9.2.43), as Vect is self-dual in DGCat_{cont}, and $\mathrm{Rep}(M)$ is rigid, here M is an algebraic group over e. For the natural map $q:\mathrm{Spec}\,k\to B(M)$ under the canonical self-dualities on the rigid categories Vect and $\mathrm{QCoh}(B(M))$, the dual of $q^*:\mathrm{QCoh}(B(M))\to\mathrm{Vect}$ identifies with $q_*:\mathrm{Vect}\to\mathrm{QCoh}(B(M))$. The functor $\mathrm{oblv}_{\mathrm{Hecke}_M}:\mathrm{Fun}_{\mathrm{Rep}(M)}(\mathrm{Vect},\mathcal{C})\to\mathrm{Fun}_{\mathrm{Rep}(M)}(\mathrm{Rep}(M),\mathcal{C})$ comes from the composition with $q^*:\mathrm{Rep}(M)\to\mathrm{Vect}$. The same functor $\mathrm{oblv}_{\mathrm{Hecke}_M}:\mathcal{C}\otimes_{\mathrm{Rep}(M)}\mathrm{Vect}\to\mathcal{C}\otimes_{\mathrm{Rep}(M)}\mathrm{Rep}(M)$ equals $\mathrm{id}\otimes q_*$ for $q_*:\mathrm{Vect}\to\mathrm{Rep}(M)$.

For ([27], 4.3.3). Let $q: B(\check{T})\to B(\check{G})$ be the natural map then $q^*: \operatorname{Rep}(\check{G})\leftrightarrows \operatorname{Rep}(\check{T}): q_*$ is an adjoint pair in $\operatorname{Rep}(\check{G})$-modules, because q is schematic and quasi-compact. Tensoring with $C\in \operatorname{Rep}(\check{G})-mod$, this gives an adjoint pair $\operatorname{ind}: C\leftrightarrows C\otimes_{\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T}):\operatorname{oblv}$ in DGCat_{cont}. By ([43], 9.2.43) we similarly get

$$C\otimes_{\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T})\widetilde{\to}\operatorname{Fun}_{\operatorname{Rep}(\check{G})}(\operatorname{Rep}(\check{T}),C)$$

By ([43], 6.1.10) we also get the following: consider the natural functor $l: C\otimes\operatorname{Rep}(\check{T})\to C\otimes_{\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T})$. Since $\operatorname{Rep}(\check{G})$ is rigid, its right adjoint r is continuous and monadic. So, $C\otimes_{\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T})\widetilde{\to}\mathcal{A}-mod(C\otimes\operatorname{Rep}(\check{T}))$ for $\mathcal{A}=rl$.

For ([27], 4.3.4). The fact that $\mathcal{C}\otimes_{\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T})$ is as described in 4.3.4 follows from ([44], A.2.23). Note that $\mathcal{C}\otimes\operatorname{Rep}(\check{T})\widetilde{\to}\oplus_{\lambda\in\Lambda}\mathcal{C}$. The we consider the graph $\Gamma_\pi: B(\check{T})\to B(\check{T})\times B(\check{G})$ of the natural map $\pi: B(\check{T})\to B(\check{G})$. We have $(\Gamma_\pi)_*\mathcal{O}\widetilde{\to}\mathcal{O}_{\check{G}}$, where $\check{G}$ acts by right translations, and $\check{T}$ by left translations. Here we have identified $\mathcal{O}_{\check{G}}\widetilde{\to}\mathcal{O}_{(\check{T}\times\check{G})/\check{T}}$, where $\check{T}$ acts diagonally on the product. Then

$$\mathcal{C}\otimes_{\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T})\widetilde{\to}\mathcal{O}_{(\check{T}\times\check{G})/\check{T}}-mod(C\otimes\operatorname{Rep}(\check{T}))$$

This gives the description as graded Hecke objects. In more details, we have an adjoint pair $\Gamma_\pi^*: \operatorname{Rep}(\check{T}\times\check{G})\leftrightarrows\operatorname{Rep}(\check{T}): (\Gamma_\pi)_*$ in $\operatorname{Rep}(\check{T}\times\check{G})$-modules. Tensoring with C, this gives an adjoint pair

$$l: C\otimes\operatorname{Rep}(\check{T})\leftrightarrows C\otimes_{\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T}): r$$

So, the monad $\mathcal{A}=rl$ on $\operatorname{Rep}(\check{T}\times\check{G})$ is tensoring with $@(\Gamma_\pi)_*\mathcal{O}$.

The functor $\operatorname{oblv}_{\operatorname{Hecke}_{\check{T}}}: \operatorname{Hecke}_{\check{G}}(\mathcal{C})\to \dot{\operatorname{Hecke}}_{\check{G}}(\mathcal{C})$ is just the restriction of scalars

$$\mathcal{O}_{\check{T}\times\check{G}}-mod(C\otimes\operatorname{Rep}(\check{T}))\to\mathcal{O}_{(\check{T}\times\check{G})/\check{T}}-mod(C\otimes\operatorname{Rep}(\check{T}))$$

with respect to the homomorphism of algebras $\mathcal{O}_{(\check{T}\times\check{G})/\check{T}}\to\mathcal{O}_{\check{T}\times\check{G}}$ coming from the quotient map $\check{T}\times\check{G}\to(\check{T}\times\check{G})/\check{T}$. The equivalence $\operatorname{Hecke}_{\check{G}}(\mathcal{C})\widetilde{\to}\mathcal{O}_{\check{T}\times\check{G}}-mod(C\otimes\operatorname{Rep}(\check{T}))$ sends c with a Hecke property to the graded object $\{c_\lambda\}_{\lambda\in\Lambda}$ with $c_\lambda=c$ for all λ and the same Hecke property for $\check{G}$ and the evident Hecke property along $\check{T}$. This is why $\operatorname{oblv}_{\operatorname{Hecke}_{\check{T}}}$ sends c to the graded object $\{c_\lambda\}_{\lambda\in\Lambda}$ with $c_\lambda=c$ for all λ.

For ([27], 4.3.5). Let $\mathcal{C}\in\mathrm{DGCat}_{cont}$ be equipped with an action of $\operatorname{Rep}(\check{G})\otimes\operatorname{Rep}(\check{T})$. Then $\mathcal{C}\otimes_{\operatorname{Rep}(\check{G})\otimes\operatorname{Rep}(\check{T})}\operatorname{Rep}(\check{T})\widetilde{\to}\mathcal{O}_{\check{G}}-mod(\mathcal{C})$, where $\mathcal{O}_{\check{G}}$ is viewed as a $\check{G}$-module say via right translations, and as $\check{T}$-module via left translations. This gives the description from ([27], 4.3.5).

1.9.23. For ([27], 4.3.6), he continues to assume that $\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})$ acts on $\mathcal{C}$. In 4.3.6 line 1: Φ comes from the monoidal functor $\operatorname{Rep}(\check{G})\to\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})$, $V\mapsto e\boxtimes V$, where e is the trivial $\check{T}$-module.

Consider $\mathcal{C}\otimes_{\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{T})$, where we used the monoidal functor $\operatorname{id}\otimes\operatorname{Res}^{\check{G}}_{\check{T}}: \operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{G})\to\operatorname{Rep}(\check{T})\otimes\operatorname{Rep}(\check{T})$ to form the tensor product. This tensor product identifies canonically with $\mathcal{C}\otimes_{\operatorname{Rep}(\check{G})}\operatorname{Rep}(\check{T})$. Now the dual pair (Φ,Ψ)

comes from the dual pair in $\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{T})-mod$

$$mult:\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{T})\leftrightarrows\mathrm{Rep}(\check{T}):mult^R,$$

where $mult$ is the product in the symmetric monoidal category $\mathrm{Rep}(\check{T})$. The latter dual pair identifies with

$$d^*:\mathrm{QCoh}(B(\check{T}\times\check{T}))\leftrightarrows\mathrm{QCoh}(B(\check{T})):d_*$$

for the diagonal map $d:B(\check{T})\to B(\check{T}\times\check{T})$.

Let us describe Φ and Ψ. Think of $\overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathcal{C})$ as the category of graded objects $\{c_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{C}$ equipped with isomorphisms $c_\lambda\otimes V\widetilde{\to}\oplus_\mu V(\mu)\otimes c_{\lambda-\mu}$ for any $\lambda\in\Lambda$ (Dennis used $\lambda+\mu$ instead in the latter formula, which is another normalization!). Recall that $\mathcal{C}\otimes_{\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{G})}\mathrm{Rep}(\check{T})$ is identified with the category of $c\in\mathcal{C}$ together with a collection of isomorphisms

$$c*V\widetilde{\to}\mathrm{Res}(V)*c \tag{27}$$

for $V\in\mathrm{Rep}(\check{G})$, where $\mathrm{Res}:\mathrm{Rep}(\check{G})\to\mathrm{Rep}(\check{T})$ is the restriction, and we write the $\mathrm{Rep}(\check{T})$-action (resp., $\mathrm{Rep}(\check{G})$-action) on the left (resp., on the right).

Then Φ sends the above object $\{c_\lambda\}$ to $c:=\oplus_\lambda e^\lambda*c_\lambda$ equipped with the isomorphisms (27) obtained as the composition

$$c*V\widetilde{\to}\bigoplus_\lambda e^\lambda*(c_\lambda*V)\widetilde{\to}\bigoplus_\lambda e^\lambda*(\bigoplus_\mu V(\mu)\otimes c_{\lambda-\mu})\widetilde{\to}\bigoplus_\mu V(\mu)\otimes e^\mu*(\bigoplus_\lambda e^{\lambda-\mu}*c_{\lambda-\mu})$$
$$\widetilde{\to}\bigoplus_\mu V(\mu)\otimes e^\mu*c$$

Here the second isomorphism comes from the Hecke structure on $\{c_\lambda\}$.

In this normalization the functor Ψ sends c to $\{c_\lambda\}$ with $c_\lambda=e^{-\lambda}*c$ with the Hecke property obtained from that of c. (This normalization differs from that of Dennis).

Note that we have $\overset{\bullet}{\mathrm{Hecke}}_{\check{G}}(\mathcal{C})\widetilde{\to}(\mathcal{C}\otimes\mathrm{Rep}(\check{T})\otimes_{\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{G})}\mathrm{Rep}(\check{T})$, where we do not use the $\mathrm{Rep}(\check{T})$-action on $\mathcal{C}$ at all. Now $act:\mathcal{C}\otimes\mathrm{Rep}(\check{T})\to\mathcal{C}$ is a $\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{G})$-linear functor, where on the source $\mathrm{Rep}(\check{T})$ acts via its action on the factor $\mathrm{Rep}(\check{T})$. Applying $\bullet\otimes_{\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{G})}\mathrm{Rep}(\check{T})$ to act, we get Φ. The right adjoint to act is continuous and given explicitly by ([20], ch. I.1, 9.3.2).

1.9.24. For ([27], 4.3.7). He continues to assume that $\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{G})$ acts on C. Then on $\mathrm{Hecke}_{\check{T}}(C)$ we consider the remaining action of $\mathrm{Rep}(\check{G})$ and the trivial action of $\mathrm{Rep}(\check{T})$, so we may form $\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{Hecke}_{\check{T}}(C))$, and it identifies with

$$\mathrm{Hecke}_{\check{G}}(\mathrm{Hecke}_{\check{T}}(C))\widetilde{\to}C\otimes_{\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{G})}\mathrm{Vect}$$

Indeed, in the diagram below both squares are cartesian

$$\begin{array}{ccccc} pt & \to & B(\check{G}) & \to & pt \\ \downarrow & & \downarrow \mathrm{id}\times v & & \downarrow \\ B(\check{T}) & \overset{i\times\mathrm{id}}{\to} & B(\check{G})\times B(\check{T}) & \overset{\mathrm{pr}}{\to} & B(\check{T}), \end{array}$$

here i comes from $\check{T}\hookrightarrow\check{G}$, and $v:pt\to B(\check{T})$ is the natural map.

In the diagram (4.10) the map $\mathrm{ind}_{\mathrm{Hecke}_{\check{T}}}$ sends $\{c_\lambda\}_{\lambda\in\Lambda}$ to $\oplus_\lambda c_\lambda$ with the induced Hecke property. Besides, the functor $\mathrm{Hecke}_{\check{G}}(\mathrm{oblv}_{\mathrm{Hecke}_{\check{T}}})$ sends c to c, it forgets the Hecke property with respect to $\check{T}$.

1.9.25. For ([27], 4.4.1) by ([20], ch. I.3, 3.3.5 one has

$$\mathrm{QCoh}(B(\check{B}^-))\otimes_{\mathrm{Rep}(\check{G})}\mathrm{Vect}\,\widetilde{\to}\,\mathrm{QCoh}(\check{G}/\check{B}^-)$$

The formula (4.13) was wrong, it should be $\mathrm{QCoh}(B(\check{N}^-))\,\widetilde{\to}\,\mathrm{Hecke}_{\check{T}}(\mathrm{QCoh}(B(\check{B}^-)))$, corrected in the version of Oct 31, 2021.

Taking the quotient of the cartesian square

$$\begin{array}{ccc}(\check{T}\backslash\check{G})\times(\check{G}/\check{B}^-) & \to & \check{T}\backslash\check{G}\\ \downarrow & & \downarrow\\ \check{G}/\check{B}^- & \to & pt\end{array}$$

by the action of $\check{G}$, we get $\check{T}\backslash\check{G}/\check{B}^-\,\widetilde{\to}\,B(\check{B}^-)\times_{B(\check{G})}B(\check{T})$.

We do have

$$\mathrm{Hecke}_{\check{G}}(\mathrm{Hecke}_{\check{T}}(\mathrm{QCoh}(pt/\check{B}^-)))\,\widetilde{\to}\,\mathrm{QCoh}(\check{G}/\check{N}^-)$$

Consider the diagonal embedding $\check{B}^-\hookrightarrow\check{T}\times\check{G}$. Taking the quotient under the right $\check{B}^-$-action (via the diagonal embedding), we get a cartesian square

$$\begin{array}{ccc}(\check{T}\times\check{G})/\check{B}^- & \to & pt\\ \downarrow & & \downarrow\\ B(\check{B}^-) & \to & B(\check{T}\times\check{G})\end{array}$$

Let in addition $\check{T}$ act on the left diagonally on $(\check{T}\times\check{G})/\check{B}^-$. Taking the quotient by this action, we get a cartesian square

$$\begin{array}{ccc}\check{T}\backslash(\check{T}\times\check{G})/\check{B}^- & \to & B(\check{T})\\ \downarrow & & \downarrow d\\ B(\check{B}^-) & \to & B(\check{T}\times\check{G}),\end{array}$$

where d comes from the diagonal inclusion $\check{T}\hookrightarrow\check{T}\times\check{G}$. One has naturally

$$\check{T}\backslash(\check{T}\times\check{G})/\check{B}^-\,\widetilde{\to}\,(\check{G}/\check{N}^-)/Ad_{\check{T}}$$

This justifies the formula (4.14). The last displayed diagram in 4.4.1 is correct, the sense of the functor Ψ given by (4.9) is taking direct image along the quotient map by the $\check{T}$-action, this is why along the horizontal arrows in that diagram we get direct images. The last displayed diagram in 4.4.1 comes from the commutative diagram

$$\begin{array}{ccc}\check{T}\backslash(\check{T}\times\check{G})/\check{B}^- & \stackrel{\mathrm{pr}}{\to} & \check{T}\backslash\check{G}/\check{B}^-\\ \uparrow & & \uparrow\\ (\check{T}\times\check{G})/\check{B}^- & \stackrel{\mathrm{pr}}{\to} & \check{G}/\check{B}^-,\end{array}$$

where the vertical arrows are the stack quotients.

The direct image along $\check{T}\backslash(\check{T}\times\check{G})/\check{B}^-\to B(\check{B}^-)$ is the forgetful functor

$$\mathrm{Hecke}_{\check{G},\check{T}}(\mathrm{QCoh}(pt/\check{B}^-))\to\mathrm{QCoh}(pt/\check{B}^-)$$

So, the image of $\mathcal{M}_{\check{G},\check{T}}$ under the latter functor is the direct image of $\mathcal{O}_{pt/\check{T}}$ under $B(\check{T})\to B(\check{B}^-)$, it identifies with $\mathcal{O}(\check{B}^-/\check{T})$ with the action of $\check{B}^-$ by left translations.

The diagonal embedding $\check{T}\hookrightarrow \check{T}\times\check{G}$ gives a closed immersion

$$\check{T}\backslash\check{T}/\check{T}\to\check{T}\backslash(\check{T}\times\check{G})/\check{B}^-$$

Composing with $\mathrm{pr}:\check{T}\backslash(\check{T}\times\check{G})/\check{B}^-\to\check{T}\backslash\check{G}/\check{B}^-$ we get the closed immersion $\check{T}\backslash\check{B}^-/\check{B}^-\to\check{T}\backslash\check{G}/\check{B}^-$. Taking the direct image of $\mathcal{O}$ under this closed immersion and then the pull-back under $\check{G}/\check{B}^-\to\check{T}\backslash\check{G}/\check{B}^-$ we get the sheaf $\delta_1\in\mathrm{QCoh}(\check{G}/\check{B}^-)$, which corresponds to $\mathcal{M}_{\check{G}}$ in his Section 4.4.2.

1.9.26. For ([27], 4.4.3). Let $q:B(\check{T})\to B(\check{B}^-)$ be the projection. The isomorphism between the 2nd and 3rd line in the displayed formula is just the projection formula $W\otimes(q_*e)\widetilde{\to}q_*q^*W$ for $W\in\mathrm{QCoh}(B(\check{B}^-)$. Indeed, if we denote by e^μ the 1-dimensional representation of $\check{B}^-$ then $q^*W\widetilde{\to}\oplus_\mu\,(q^*e^\mu)\otimes W(\mu)$, and $q_*q^*e^\mu\widetilde{\to}(q_*\mathcal{O})\otimes e^\mu$.

For 4.4.4. To get an isomorphism

$$\underset{\lambda\in\Lambda^+}{\mathrm{colim}}\, e^\lambda\otimes\mathrm{Res}^{\check{G}}_{\check{B}^-}(V^\lambda)^*\widetilde{\to}q_*e,$$

let $p:B(\check{B}^-)\to B(\check{G})$ be the natural map. The corresponding map $e^\lambda\otimes p^*(V^\lambda)^*\to q_*e$ is by adjunction a map $q^*e^\lambda\otimes q^*p^*(V^\lambda)^*\to e$ on $B(\check{T})$. The latter is just $v^\lambda:e^\lambda\hookrightarrow V^\lambda$.

This isomorphism is precisely ([4], Proposition-Construction 3.1.2).

1.9.27. The last displayed formula in ([27], 4.5.3) is wrong as stated in arxiv version 5. The problem here is that μ appearing in V is not necessarily dominant, so it is not garanteed that $j_{\lambda,*}*j_{\mu,*}\widetilde{\to}j_{\lambda+\mu,*}$. It is clear how to correct. The correct formula in the RHS is

$$\underset{\mu}{\oplus}(j_{-\lambda-\mu,!}*j_{\lambda,*}\otimes V(-\mu))\otimes(j_{\lambda+\mu,*}*Sat((V^{\lambda+\mu})^*)$$

(corrected in the version of Oct 31, 2021).

In arxiv version 6: let λ be dominant coweight. By the monoidal dual of the map $\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G}\to j_{\lambda,*}*\delta_{1,\mathrm{Gr}_G}$ he means the fact that if $\mathrm{Rep}(\check{G})$ acts on some C on the right, and $\mathrm{Rep}(\check{T})$ acts on it on the left, so that $C\in\mathrm{Rep}(\check{G})\otimes\mathrm{Rep}(\check{T})$-module then for $c\in C$, $V\in\mathrm{Rep}(G)^c$ and $\lambda\in\Lambda^+$ one has $\mathrm{Map}_C(c*V,e^\lambda*c)\widetilde{\to}\mathrm{Map}_C(e^{-\lambda}*c,c*V^\vee)$, where $V^\vee$ is the dual of V with respect to the monoidal structure on $\mathrm{Rep}(\check{G})$ (see HA, 4.6.1.5). For any $V\in\mathrm{Rep}(\check{G})^\heartsuit$ finite-dimensional, V is dualizable in $\mathrm{Rep}(\check{G})$ with the dual V^*. Namely, the usual unit and counit maps $e\to V^*\otimes V, V^*\otimes V\to e$ provide this structure. Now take $C=Shv(\mathrm{Gr}_G)^I$ and $c=\delta_{1,\mathrm{Gr}_G}$. We get the morphism $j_{-\lambda,!}*\delta_{1,\mathrm{Gr}_G}\to Sat((V^\lambda)^*)$.

1.9.28. For ([27], 5.1.1). First, the category $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ is defined as the category of $\mathfrak{L}^+(T)$-invariants in $\mathrm{SI}(\mathrm{Gr}_G)=Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$. The category $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$ inherits an action of $Shv(\mathfrak{L}^+(T))$ by my Lemma 1.2.64. On the other hand, $\mathfrak{L}(N)\mathfrak{L}^+(T)$ is a placid ind-scheme, and one may also define $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ as the category of invariants under this group.

But further he assumes that $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ is renormalized as follows. First, we consider $C\in\mathrm{DGCat}^{non-cocompl}$, where $C\subset Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ is the full category of F whose image in $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$ is compact. The renormalized category is defined as $\mathrm{Ind}(C)$.

For ([27], 5.1.4). By my Section 1.9.6, for $\lambda \in \Lambda$ the functor $F \mapsto t^{\lambda} F[-\langle \lambda, 2\check{\rho}\rangle]$ is t-exact.

Dennis claims first that $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ has a natural t-structure such that $\mathrm{oblv} : Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)} \to \mathrm{SI}(\mathrm{Gr}_G)$ is t-exact. For any $\lambda \in \Lambda$, S^{λ} is $\mathfrak{L}^+(T)$-invariant, so $\omega_{S^{\lambda}}$ is naturally $\mathfrak{L}^+(T)$-equivariant.

Write $\mathfrak{L}(N) \widetilde{\to} \mathrm{colim}_{\alpha \in A} N_{\alpha}$, where A is a filtered category, and N_{α} is a placid group scheme, and for $\alpha \to \alpha'$ in A the map $N_{\alpha} \to N_{\alpha'}$ is a placid closed immersion and a homomorphism of group schemes. Moreover, we may assume each N_{α} is $\mathcal{L}^+(T)$-invariant. Then $\mathfrak{L}(N)\mathfrak{L}^+(T) \widetilde{\to} \mathrm{colim}_{\alpha} N_{\alpha}\mathfrak{L}^+(T)$, here $N_{\alpha}\mathfrak{L}^+(T)$ is the semi-direct product of the two factors. Moreover, $N_{\alpha}\mathfrak{L}^+(T)$ is a placid group-scheme.

We may equivalently define $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ via geometry I think. Before any renormalization,

$$Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)} \widetilde{\to} \lim_{\alpha \in A^{op}} Shv(\mathrm{Gr}_G)^{N_{\alpha}\mathfrak{L}^+(T)}$$

Fix $\alpha \in A$. Pick a presentation $\mathrm{Gr}_G \widetilde{\to} \mathrm{colim}_{i \in I} Y_i$, where I is filtered, and $Y_i \subset \mathrm{Gr}_G$ is a closed $N_{\alpha}\mathfrak{L}^+(T)$-invariant subscheme of finite type. We assume for $i \to j$ in I the map $Y_i \to Y_j$ is a closed immersion. Then $Shv(\mathrm{Gr}_G)^{N_{\alpha}\mathfrak{L}^+(T)} \widetilde{\to} \lim_{i \in I^{op}} Shv(Y_i)^{N_{\alpha}\mathfrak{L}^+(T)}$. Now the group $N_{\alpha}\mathfrak{L}^+(T)$ acts on Y_i through a quotient of finite type $G_{\alpha,i}$ such that $\mathrm{Ker}(N_{\alpha}\mathfrak{L}^+(T) \to G_{\alpha,i})$ is prounipotent. Then we define $Shv(Y_i)^{N_{\alpha}\mathfrak{L}^+(T)}$ as $Shv(Y_i)^{G_{\alpha,i}}$. This gives the desired category. The functor $\mathrm{oblv} : Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)} \to Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$ is also geometric given by !-pullback. We may similarly define "the stratification by $\mathfrak{L}(N)$-orbits" on $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ and the objects $\Delta^{\lambda}, \nabla^{\lambda} \in Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ equipped with $\mathrm{oblv}(\Delta^{\lambda}) \widetilde{\to} \Delta^{\lambda}$, $\mathrm{oblv}(\nabla^{\lambda}) \widetilde{\to} \nabla^{\lambda}$.

Then we define the t-structure on $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ in a way similar to that for $\mathrm{SI}(\mathrm{Gr}_G)$. Namely, connective objects is the smallest full subcategory stable under extensions, colimits and containing Δ^{λ} for all $\lambda \in \Lambda$. By definition, oblv is right t-exact. It is also left t-exact. Indeed, let $F \in Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ be coconnective. Then for any $\lambda \in \Lambda$, $i_{\lambda}^! F$ is coconnective in $Shv(S^{\lambda})^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$, hence $\mathrm{oblv}(i_{\lambda}^! F)$ is coconnective in $\mathrm{SI}(\mathrm{Gr}_G)_{=\lambda}$. So, $\mathrm{oblv}(F)$ is coconnective. My understanding is that $Shv(S^{\lambda})^{\mathfrak{L}(N)\mathfrak{L}^+(T)} \widetilde{\to} Shv(B(T))$.

The renormalized version of $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)}$ is similarly equipped with a t-structure. Namely, we first equip the above C with a t-structure, which in turn gives one on $\mathrm{Ind}(C)$ by Lemma 1.2.37 of this file.

Using the geometry as above we also see that $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)} \subset Shv(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$ is a full subcategory (before any renormalization). We then get $(Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)})^{constr} \subset (Shv(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)})^{constr}$ is a full subcategory, hence a natural functor between the renormalizations (=the ind-completions of the constructible subcategories). By ([35], 5.3.5.11(1)) the functor between the renormalizations is also fully faithful.

1.9.29. For ([27], 5.2.1). The fact that oblv : $Shv(\mathrm{Gr}_G)^I \to Shv(\mathrm{Gr}_G)^{\mathcal{L}^+(T)}$ (before any renormalization) admits a continuous right adjoint follows from Section 1.2.62 of this file.

1.9.30. For ([27], 5.2.3). By definition, $\mathrm{Av}_!^{\mathfrak{L}(N)}$ is the partially defined left adjoint to the full embedding $Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)} \subset Shv(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$. In the constructible context the functor $\mathrm{Av}_!^{\mathfrak{L}(N)}$ is everywhere defined. Indeed, in the notations of my Section 1.9.28, pick α. Then for $i \in I$ we may assume $\mathfrak{L}^+(T)$ acts on Y_i through its quotient $\mathfrak{L}^+(T)_i$ of finite type, and $\mathfrak{L}^+(T)_i \subset G_{\alpha,i}$ is a subgroup. Then we have the projection of stack quotients $f : Y_i/\mathcal{L}^+(T)_i \to Y_i/G_{\alpha,i}$, and the left adjoint in question comes from $f_!$, which is everywhere defined in the constructible context.

To show that $\mathrm{Av}_!^{\mathfrak{L}(N)}$ is defined on the essential image of oblv : $Shv(\mathrm{Gr}_G)^I \to Shv(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$, use my Lemma 1.2.15. Namely, for each $\lambda \in \Lambda^+$ set $N_\lambda = Ad_{t^{-\lambda}}(\mathfrak{L}^+(N))$, so $N_\lambda \mathfrak{L}^+(T)$ is a placid group scheme, and we get

$$Shv(\mathrm{Gr}_G)^{\mathfrak{L}(N)\mathfrak{L}^+(T)} \widetilde{\to} \lim_{\lambda \in (\Lambda^+)^{op}} Shv(\mathrm{Gr}_G)^{N_\lambda \mathfrak{L}^+(T)}$$

The key thing is is the following. Let $\lambda \in \Lambda^+$. Then it is known that $N(\mathcal{O})t^\lambda G(\mathcal{O})/G(\mathcal{O}) \subset \mathrm{Gr}_G$ is an affine space of dimension $\langle \lambda, 2\check{\rho}\rangle$, it coincides with the I-orbit O^λ on Gr_G through t^λ. For $\tilde{w}$ in the affine extended Weyl group write $\mathcal{F}l_G^{\tilde{w}}$ for the corresponding I-orbit on $\mathcal{F}l_G$. It is well known that the natural projection $\mathcal{F}l_G^{t^\lambda} \to O^\lambda$ is an isomorphism. So,

$$t^{-\lambda} I t^\lambda I/I = N_\lambda G(\mathcal{O})/G(\mathcal{O})$$

is the N_λ-orbit of 1 on Gr_G. Here $N_\lambda = t^{-\lambda} N(\mathcal{O}) t^\lambda$.

Consider the functor $\mathrm{Av}_!^{N_\lambda} : Shv(\mathrm{Gr}_G)^{T(\mathcal{O})} \to Shv(\mathrm{Gr}_G)^{T(\mathcal{O})N_\lambda}$ left adjoint to the inclusion. Given $K \in Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}$ the object $\mathrm{Av}_!^{N_\lambda}(K)$ is nothing but $\mathrm{act}_!(e \widetilde{\boxtimes} K)$ for the map

$$\mathrm{act} : N_\lambda T(\mathcal{O}) \times^{T(\mathcal{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G,$$

more precisely this is the image of $\mathrm{Av}_!^{N_\lambda}(K)$ under oblv : $Shv(\mathrm{Gr}_G)^{T(\mathcal{O})N_\lambda} \to Shv(\mathrm{Gr}_G)$.

Let now $F \in Shv(\mathrm{Gr}_G)^I$. Note that $\mathrm{IC} = e[\langle \lambda, 2\check{\rho}\rangle]$ is the IC-sheaf of the affine space $t^{-\lambda} I t^\lambda I/I$. Then the object

$$t^{-\lambda} j_{\lambda,!} * F[\langle \lambda, 2\check{\rho}\rangle]$$

writes as $a_!(\mathrm{IC} \widetilde{\boxtimes} F)$ for the action map

$$a : t^{-\lambda} I t^\lambda I \times^I \mathrm{Gr}_G \to \mathrm{Gr}_G$$

So, the latter may be calculated as $\mathrm{Av}_!^{N_\lambda}(F')$, where F' is the image of F under oblv : $Shv(\mathrm{Gr}_G)^I \to Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}$.

The shift he uses does not mean too much, because a shift is an equivalence of $Shv(\mathrm{Gr}_G)^I$, and the way to identify abstract functors (such as oblv) with geometricn ones is not made precise! Normalization of shifts is really not clear in his paper!

For ([27], 5.2.4). The 1st isomorphism in his Section 5.2.4 follows from the fact that for the left adjoint $\mathrm{Av}_!^{N(F)} : Shv(\mathrm{Gr}_G)^{T(\mathcal{O})} \to Shv(\mathrm{Gr}_G)^{T(\mathcal{O})N(F)}$ to the inclusion, and any $K\in Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}$ we have

$$t^{-\lambda}\,\mathrm{Av}_!^{N(F)}(t^\lambda K)\widetilde{\to}\mathrm{Av}_!^{N(F)}(K)$$

Indeed, for $L\in Shv(\mathrm{Gr}_G)^{N(F)T(\mathcal{O})}$

$$\begin{aligned}\mathcal{H}om_{Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}}(t^{-\lambda}\,\mathrm{Av}_!^{N(F)}(t^\lambda K),L)&\widetilde{\to}\mathcal{H}om_{Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}}(\mathrm{Av}_!^{N(F)}(t^\lambda K),t^\lambda L)\\ &\widetilde{\to}\mathcal{H}om_{Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}}(t^\lambda K,t^\lambda L)\widetilde{\to}\mathcal{H}om_{Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}}(K,L)\end{aligned}$$

1.9.31. For ([27], 5.2.5). If $F\in Shv(\mathrm{Gr}_G)^I$ and $\lambda\in\Lambda^+$ then $t^{-\lambda}F$ is equivariant under $I^\lambda:=t^{-\lambda}It^\lambda$. Now the composition

$$Shv(\mathrm{Gr}_G)^{I^\lambda}\stackrel{\mathrm{oblv}}{\to}Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}\stackrel{\mathrm{Av}_*^{I/T(\mathcal{O})}}{\to}Shv(\mathrm{Gr}_G)^I$$

identifies with $Shv(\mathrm{Gr}_G)^{I^\lambda}\stackrel{\mathrm{oblv}}{\to}Shv(\mathrm{Gr}_G)^{I^\lambda\cap I}\stackrel{\mathrm{Av}_*^{I/I\cap I^\lambda}}{\to}Shv(\mathrm{Gr}_G)^I$, because for a prounipotent group the inclusion of invariants is fully faithful. The latter functor writes as $K\mapsto \mathrm{act}_*(e\widetilde{\boxtimes}K)$ for the action map act : $II^\lambda\times^{I^\lambda}\mathrm{Gr}_G\to\mathrm{Gr}_G$. Now for $F\in Shv(\mathrm{Gr}_G)^I$ we get

$$\mathrm{act}_*(e\widetilde{\boxtimes}t^{-\lambda}F)\widetilde{\to}j_{-\lambda,*}*F \tag{28}$$

up to a shift, because $II^\lambda=It^{-\lambda}It^\lambda$, and due to the following. Consider the isomorphism

$$II^\lambda\times\mathrm{Gr}_G\widetilde{\to}It^{-\lambda}I\times\mathrm{Gr}_G,(v,gG(\mathcal{O}))\mapsto(vt^{-\lambda},t^\lambda gG(\mathcal{O}))$$

Let $y\in I$ act on $It^{-\lambda}I\times\mathrm{Gr}_G$ diagonally, where on $u\in It^{-\lambda}I$ it acts as uy^{-1}, and on $gG(\mathcal{O})\in\mathrm{Gr}_G$ it acts as $ygG(\mathcal{O})$. Let also $y\in I$ act on $II^\lambda\times\mathrm{Gr}_G$ diagonally, where on $v\in II^\lambda$ it acts as $vt^{-\lambda}y^{-1}t^\lambda$, and on $gG(\mathcal{O})$ as $t^{-\lambda}yt^\lambda gG(\mathcal{O})$. Then the above isomorphism is I-equivariant, and this gives (28).

This proves that for $F\in Shv(\mathrm{Gr}_G)^I$ one has

$$\mathrm{Av}^{I/T(\mathcal{O})}(t^{-\lambda}F)\widetilde{\to}j_{-\lambda,*}*F[-\langle\lambda,2\check{\rho}\rangle] \tag{29}$$

as claimed in his paper. I don't understand the shift however!

For 5.2.6. He uses the fact from ([24], D.1.2) that for any $C\in\mathrm{DGCat}_{cont}$ with an action of $Shv(G(F))$, $C\widetilde{\to}\mathrm{colim}_n\, C^{K_n}$, where $K_n=\mathrm{Ker}(G(\mathcal{O})\to G(\mathcal{O}/t^n))$. So, for any $c\in C$, $c\widetilde{\to}\mathrm{colim}_n,\mathrm{oblv}_n\,\mathrm{Av}_*^{K_n}(c)$, where $\mathrm{oblv}_n: C^{K_n}\to C$ and $\mathrm{Av}_*^{K_n}: C\to C^{K_n}$ are adjoint functors.

The reference for the Iwahori factorization $N^-(\mathcal{O})_1T(\mathcal{O})(\mathcal{O})=I$ is ([14], Section 3). Here $N^-(\mathcal{O})_1=\mathrm{Ker}(N^(\mathcal{O})\to N^-)$ is the first congruence subgroup. For $\lambda\in\Lambda^+$ he thinks of $\mathrm{Av}_*^{t^{-\lambda}N^-(\mathcal{O})_1t^\lambda}$ here as a functor

$$Shv(\mathrm{Gr}_G)^{T(\mathcal{O})}\to Shv(\mathrm{Gr}_G)^{T(\mathcal{O})t^{-\lambda}N^-(\mathcal{O})_1t^\lambda}$$

In the end of the proof there are misprints. A correct argument: given $F\in Shv(\mathrm{Gr}_G)^{N(F)T(\mathcal{O})}$ nonzero, there is $\lambda\in\Lambda^+$ deep enough such that $\mathrm{Av}_*^{t^{-\lambda}N^-(\mathcal{O})_1t^\lambda}(F)\neq 0$, the latter is in $Shv(\mathrm{Gr}_G)^{I^\lambda}$. Now

$$\mathrm{Av}_*^{N^-(\mathcal{O})_1}\mathrm{Av}_*^{t^{-\lambda}N^-(\mathcal{O})_1t^\lambda}(F)\widetilde{\to}j_{-\lambda,*}*(t^\lambda\,\mathrm{Av}_*^{t^{-\lambda}N^-(\mathcal{O})_1t^\lambda}(F))[-\langle\lambda,2\check{\rho}\rangle]$$

by formula (29) of this file.

1.9.32. *Example.* For ([56], 6.4.1). Let $I \subset G(\mathcal{O})$ be the Iwahori. For $\lambda \in \Lambda$ let $I^\lambda = Ad_{t^{-\lambda}}(I)$. Let $C \in Shv(G(F)) - mod$. For any $\lambda, \mu \in \Lambda^+$ the composition $C^{I^\lambda} \stackrel{\text{oblv}}{\to} C^{I^\lambda \cap I^\mu} \stackrel{\text{Av}_*}{\to} C^{I^\mu}$ is an equivalence.

Proof. Since $I^\lambda / I^\lambda \cap I^\mu$ is contractible (a tower of $\mathbb{A}^1$-torsors), oblv : $C^{I^\lambda} \to C^{I^\lambda \cap I^\mu}$ is fully faithful. Up to conjugation, we may assume $\lambda = 0$ and μ any. Then this composition is the functor $F \mapsto t^{-\mu} j_{\mu,*} * F$ for $F \in C^I$. Here $j_{\mu,*} = j_{t^\mu,*}$ is the corresponding object in $Shv(I\backslash G(F)/I)$, the $*$-extension of the constant sheaf from $It^\mu I$ to $G(F)/I$. Note also that $j_{\mu,!}$ exists for any sheaf theory, because I-orbit through t^μ is an affine space and the dualizing sheaf on an affine space is holonomic. So, the inverse functor makes sense. The object $j_{t^\mu,*}$ is invertible in $Shv(I\backslash G(F)/I)$, its inverse is $j_{t^{-\mu},!}$. The forgetting to $I \cap I^\mu$ appears, because the stabilizer of $1 \in \mathcal{F}l_G$ inside I^μ is $I \cap I^\mu$. □

1.9.33. Let V be a finite-dimensional e-vector space. Then $e \otimes_{\mathrm{Sym}\, V} e \widetilde{\to} \mathrm{Sym}(V[1])$ canonically (the Koszul complex). Let now $0 \to E \to V \to W \to 0$ be an exact sequence of vector spaces, this gives a surjective map of algebras $\mathrm{Sym}\, V \to \mathrm{Sym}\, W$. Now $\mathrm{Sym}\, W \otimes_{\mathrm{Sym}\, V} e \widetilde{\to} \mathrm{Sym}(E[1])$ canonically. Indeed, if we fix a splitting $W \to V$ of the above exact sequence then it gives an isomorphism $\mathrm{Sym}\, V \widetilde{\to} \mathrm{Sym}\, W \otimes \mathrm{Sym}\, E$, and $\mathrm{Sym}\, W \otimes_{(\mathrm{Sym}\, W \otimes \mathrm{Sym}\, E)} e \widetilde{\to} e \otimes_{\mathrm{Sym}\, E} e$. I think the so obtained isomorphism is independent of a splitting. We used here ([43], 9.2.10).

1.9.34. For [27]. Remarks 6.1.5, 6.1.7 are correct in arxiv version 6. Their proof is essentially as follows. Assume $\mathrm{Rep}(\check{G})$ acts on C, and $c \in \mathcal{O}(\check{G}/\check{N}^-) - mod(C)$, where we view now $\mathcal{O}(\check{G}/\check{N}^-)$ as an algebra in $\mathrm{Rep}(\check{G})$. For each $\lambda_i \in \Lambda^+$ we have a commutative diagram

$$\begin{array}{ccc} (c * V^{\lambda_1}) * V^{\lambda_2} & \stackrel{u^{\lambda_1,\lambda_2}}{\to} & c * V^{\lambda_1+\lambda_2} \\ \downarrow a_{\lambda_1} & & \downarrow a_{\lambda_1+\lambda_2} \\ c * V^{\lambda_2} & \stackrel{a_{\lambda_2}}{\to} & c, \end{array}$$

where a_λ denotes the corresponding action map $c * V^\lambda \to c$. We denote by $u^{\lambda_1,\lambda_2} : V^{\lambda_1} \otimes V^{\lambda_2} \to V^{\lambda_1+\lambda_2}$ and $v^{\lambda_1,\lambda_2} : V^{\lambda_1+\lambda_2} \to V^{\lambda_1} \otimes V^{\lambda_2}$ the maps fixed in his Section 2.1.4 (as well as their duals). We must show that the diagram obtained by passing to adjoints

$$\begin{array}{ccc} c & \stackrel{b_{\lambda_1+\lambda_2}}{\to} & c * (V^{\lambda_1+\lambda_2})^* \\ \downarrow b_{\lambda_1} & & \uparrow v^{\lambda_1,\lambda_2} \\ c * (V^{\lambda_1})^* & \stackrel{b_{\lambda_2}}{\to} & c * (V^{\lambda_2})^* \otimes (V^{\lambda_1})^* \end{array}$$

also commutes naturally, where $b_\lambda : c \to c * (V^\lambda)^*$ is the map obtained from a_λ by adjointness.

This is easy to check. For this we use the following. First, the composition $V^{\lambda_1+\lambda_2} \stackrel{v^{\lambda_1,\lambda_2}}{\to} V^{\lambda_1} \otimes V^{\lambda_2} \stackrel{u^{\lambda_1,\lambda_2}}{\to} V^{\lambda_1+\lambda_2}$ is id. Second, the diagram commutes

$$\begin{array}{ccc} V^{\lambda_1} \otimes V^{\lambda_2} \otimes (V^{\lambda_1})^* \otimes (V^{\lambda_2})^* & \stackrel{u\otimes u}{\leftarrow} & e \\ \downarrow u^{\lambda_1,\lambda_2} & & \downarrow u \\ V^{\lambda_1+\lambda_2} \otimes (V^{\lambda_1})^* \otimes (V^{\lambda_2})^* & \stackrel{u^{\lambda_1,\lambda_2}}{\leftarrow} & V^{\lambda_1+\lambda_2} \otimes (V^{\lambda_1+\lambda_2})^*, \end{array}$$

where u every time denotes the unit of the corresponding duality.

We may see here c as a lax central element, where the left $\mathrm{Rep}(\check{T})$-action on c is trivial. For a nontrivial $\mathrm{Rep}(\check{T})$-action the situatioin is similar.

1.9.35. For 6.2.1. Let $q : B(\check{T}) \to B(\check{T} \times \check{G})$ come from the diagonal map $\check{T} \to \check{T} \times \check{G}$. We have an adjoint pair $q^* : \mathrm{QCoh}(B(\check{T} \times \check{G})) \leftrightarrows \mathrm{QCoh}(B(\check{T})) : q_*$ in $\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G}) - mod$. Tensoring by C over $\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$, one gets the desired right adjoint to $C \to \mathrm{Hecke}_{\check{G},\check{T}}(C)$. This right adjoint is monadic, because q fits into a diagram $B(\check{T}) \stackrel{q}{\to} B(\check{T} \times \check{G}) \to B(\check{T})$, whose composition is id. So, $C \to \mathrm{Hecke}_{\check{G},\check{T}}(C)$ generates $\mathrm{Hecke}_{\check{G},\check{T}}(C)$ under colimits.

We may also use ([20], ch. I.1, 8.5.7) and the fact that $q_* : \mathrm{Rep}(\check{T}) \to \mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$ is monadic. So, for the algebra $\mathcal{A} = q_*q^*\mathcal{O} \in \mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$ we have $\mathcal{A} - mod(\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})) \widetilde{\to} \mathrm{Rep}(\check{T})$ and

$$C \otimes_{\mathrm{Rep}(\check{T})\otimes\mathrm{Rep}(\check{G})} \mathcal{A} - mod(\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})) \widetilde{\to} \mathcal{A} - mod(C)$$

1.9.36. For 6.2.2. Consider the diagram

$$\begin{array}{ccc} B(\check{B}^-) & \stackrel{\eta}{\leftarrow} & B(\check{T}) \\ & \searrow q & \downarrow p \\ & & B(\check{T} \times \check{G}), \end{array}$$

where we use the diagonal maps $\check{T} \to \check{B}^- \to \check{T} \times \check{G}$. Using ([20], ch. I.1, 3.3.3), one has $\mathrm{QCoh}(B(\check{B}^-)) \widetilde{\to} q_*\mathcal{O} - mod(\mathrm{QCoh}(B(\check{T} \times \check{G})))$ and $\mathrm{QCoh}(B(\check{T})) \widetilde{\to} p_*\mathcal{O} - mod(\mathrm{QCoh}(B(\check{T} \times \check{G})))$. After the base change by $pt \to B(\check{T} \times \check{G})$ the diagram becomes

$$\begin{array}{ccc} \check{G}/\check{N}^- & \stackrel{\bar{\eta}}{\leftarrow} & \check{G} \\ & \searrow & \downarrow \\ & & pt \end{array}$$

The adjoint pair he considers comes from the adjoint pair $\eta^* : \mathrm{QCoh}(B(\check{B}^-) \leftrightarrows \mathrm{QCoh}(B(\check{T})) : \eta_*$.

The functor $\eta^* : \mathcal{O}(\check{G}/\check{N}^-) - mod \to \mathcal{O}(\check{G}) - mod$, where modules are taken in $\mathrm{Rep}(\check{T}) \otimes \mathrm{Rep}(\check{G})$ is given by

$$c \mapsto \mathcal{O}(\check{G}) \otimes_{\mathcal{O}(\check{G}/\check{N}^-)} c$$

This is why it is sufficient to present $\mathcal{O}(\check{G}) \widetilde{\to} \operatorname*{colim}_{\lambda\in\Lambda^+} e^\lambda \otimes \mathcal{O}(\check{G}/\check{N}^-) \otimes (V^\lambda)^*$.

1.9.37. For 6.2.4. The first two isomorphisms come from ([20], ch. I.1, 8.5.7). Let us check that the maps $e^\lambda\otimes \mathcal{O}(\check{G}/\check{N}^-)\otimes (V^\lambda)^*\to \mathcal{O}(\check{G})$ that he suggests are compatible with the transition maps in our inductive system. The above map sends $e^\lambda\otimes f\otimes u$ with $y\in (V^\lambda)^*, f\in \mathcal{O}(\check{G}/\check{N}^-)$ to the function $g\mapsto f(g)\langle y, gv^\lambda\rangle$ or maybe to $f(g)\langle y, g^{-1}v^\lambda\rangle$

The map $V^\lambda\otimes e^{-\lambda}\to \mathcal{O}(\check{G}/\check{N}^-)$ sends $v\otimes e^{-\lambda}$ to $\langle (v^\lambda)^*, g^{-1}v\rangle$. These maps are evidently compatible with the product in $\mathcal{O}(\check{G}/\check{N}^-)=\underset{\mu\in\Lambda^+}{\oplus} V^\mu\otimes e^{-\mu}$ given in his Section 6.1.2.

Similarly, we have the maps $(V^\mu)^*\otimes e^\mu\to \mathcal{O}(\check{G}/\check{N})$ sending $y\otimes e^\mu$ to $\langle y, g^{-1}v^\lambda\rangle$. They are similarly compatible with the product in $\mathcal{O}(\check{G}/\check{N})=\underset{\nu\in\Lambda^+}{\oplus}(V^\nu)^*\otimes e^\nu$ given by

$$(V^{\nu_1})^*\otimes e^{\nu_1}\otimes (V^{\nu_2})^*\otimes e^{\nu_2}\overset{v^{\nu_1,\nu_2}}{\to}(V^{\nu_1+\nu_2})^*\otimes e^{\nu_1+\nu_2}$$

The claim reduces to the commutativity for any $\lambda_1,\lambda_2,\mu\in\Lambda^+$ of the diagram

$$\begin{array}{ccc}
e^{\lambda_1}*[V^\mu\otimes e^{-\mu}]*(V^{\lambda_1})^* & \to & \mathcal{O}(\check{G})\\
\downarrow unit & & \\
e^{\lambda_1+\lambda_2}*e^{-\lambda_2}*[V^\mu\otimes e^{-\mu}]*V^{\lambda_2}\otimes (V^{\lambda_2})^*\otimes (V^{\lambda_1})^* & & \uparrow\\
\downarrow u^{\mu,\lambda_2} & & \\
e^{\lambda_1+\lambda_2}*[V^{\mu+\lambda_2}\otimes e^{-\mu-\lambda_2}]*(V^{\lambda_2})^*\otimes (V^{\lambda_1})^* & \overset{v^{\lambda_1,\lambda_2}}{\to} & e^{\lambda_1+\lambda_2}*[V^{\mu+\lambda_2}\otimes e^{-\mu-\lambda_2}]*(V^{\lambda_1+\lambda_2})^*
\end{array}$$

Here we view parts in [ ] parenthesis as those of $\mathcal{O}(\check{G}/\check{N}^-)$, the lowest vertical arrow is the product in $\mathcal{O}(\check{G}/\check{N}^-)$ with the term $V^{\lambda_2}\otimes e^{-\lambda_2}$, and the remaining matrices are taking the matrix coefficients. The decomposition of $\mathcal{O}(\check{G})$ and related things are discussed in *Roe Goodman, Nolan R. Wallach, Symmetry, Representations, and Invariants*, 12.1.4.

1.9.38. In ([27], 6.3.2 line 5) he means the direct image of $\mathcal{O}_{pt/\check{T}}$ under the closed immersion $B(\check{T})\hookrightarrow (\check{G}/\check{N}^-)/\check{T}$.

1.9.39. In ([27], 5.3.3 and 5.3.4) there is a mistake: given λ dominant and regular, it is not true that $\ell(t^{-w_0(\lambda)})=\ell(w_0)+\ell(t^{-\lambda}w_0)$ as stated, so the isomorphism $j_{w_0,!}*j_{-w_0(\lambda),*}\widetilde{\to} j_{t^{-\lambda}w_0,*}$ does not hold.

1.9.40. In ([27], diagram (3.9)) there is a mistake: the right vertical map $\mathfrak{q}$ does not exists, only the composition of $\mathfrak{q}$ with (3.8) exists and is smooth. The argument holds however.

1.10. **FLE again.**

1.10.1. *Action of Hecke algebras on invariants.* If G is a placid group scheme over $\operatorname{Spec} k$ and $C\in Shv(G)-mod(\mathrm{DGCat}_{cont})$ then Dennis claims that the natural functor $C_G\to C^G$ is an equivalence for any of the 4 sheaf theories (in [33] this is proved for D-modules). Assume G prosmooth. Then oblv : $C^G\to C$ admits a continuous right adjoint $\mathrm{Av}_*^G: C\to C^G$ by Section 1.3.9. The composition $C\to C_G\to C^G$ is Av_*^G.

Let $H \in Grp(\mathrm{PreStk})$ be a placid ind-scheme, and G its closed placid group subscheme. By ([46], 0.0.37), $Shv(H/G) \widetilde{\to} Shv(H)^G$, where G acts on H by right translations. Note that H/G is an ind-scheme of ind-finite type. So, we have naturally

$$Shv(H)_G \widetilde{\to} Shv(H)^G \widetilde{\to} Shv(H/G)$$

More precisely, the functor oblv : $Shv(H)^G \to Shv(H)$ may be identified with $a^* : Shv(H/G) \to Shv(H)$ for $a : H \to H/G$, so its right adjoint $\mathrm{Av}_*^G : Shv(H) \to Shv(H/G)$ is a_*. It factors as $Shv(H) \to Shv(H)_G \to Shv(H/G)$, where the second arrow is an equivalence.

Let now $C \in Shv(H)-mod(\mathrm{DGCat}_{cont})$. This gives an equivalence

$$C^G \widetilde{\to} \mathrm{Fun}_{Shv(H)}(Shv(H) \otimes_{Shv(G)} \mathrm{Vect}, C) \widetilde{\to} \mathrm{Fun}_{Shv(H)}(Shv(H/G), C)$$

The monoidal category $\mathrm{Fun}_{Shv(H)}(Shv(H/G), Shv(H/G))$ acts on $\mathrm{Fun}_{Shv(H)}(Shv(H/G), C)$ by compositions. Now

$$\mathrm{Fun}_{Shv(H)}(Shv(H/G), Shv(H/G)) \widetilde{\to} Shv(H/G)^G,$$

where G acts on H/G by left translations.

The so obtained monoidal structure on $Shv(H/G)^G$ is as follows. We accept the convention of ([46], 0.0.40). So, $Shv(H/G)^G \widetilde{\to} Shv(G\backslash H/G)$ in such a way that for $q : H/G \to G\backslash H/G$ the functor q^* identifies with oblv : $Shv(H/G)^G \to Shv(H/G)$. Then the monoidal structure on $Shv(G\backslash H/G)$ is as in Section 1.5.1 I think. Namely, consider the diagram

$$\begin{array}{ccccc} G\backslash H/G & \stackrel{p_1}{\leftarrow} & G\backslash H \times_G (H/G) & \stackrel{m}{\to} & G\backslash H/G \\ & & \downarrow p_2 & & \\ & & G\backslash H/G, & & \end{array}$$

where p_i is the projection on i-th term. Given $K_1, K_2 \in Shv(G\backslash H/G)$, we get $K_1 * K_2 = m_*(p_1 \times p_2)^*(K_1 \boxtimes K_2)$. The functor $(p_1 \times p_2)^*$ makes sense, because $p_1 \times p_2$ is a G-torsor. The needed base change result is ([46], Lemma 0.0.20).

1.10.2. Let G be a placid group scheme, $H \subset G$ be a placid closed immersion and a group subscheme. Let $C \in Shv(G)-mod(\mathrm{DGCat}_{cont})$. We claim that oblv : $C^G \to C^H$ admits a continuous right adjoint Av_{H*}^G.

Proof. For $q : H \to \mathrm{Spec}\, k$ consider the dual pair $q^* : \mathrm{Vect} \leftrightarrows Shv(H) : q_*$ in $Shv(H)-mod$, where H acts on itself by left translations. Tensoring by $Shv(G)$ over $Shv(H)$, we get a dual pair $L : Shv(G) \otimes_{Shv(H)} \mathrm{Vect} \leftrightarrows Shv(G) : R$ in $Shv(G)$, where G acts by left translations on G. Here R is the natural functor appearing in the bar construction of the tensor product, it sends K to $K \boxtimes e$.

Consider the functor $\mathcal{R} : Shv(G) \otimes_{Shv(H)} \mathrm{Vect} \to \mathrm{Vect}$ corresponding under the isomorphism $\mathrm{Fun}_{Shv(G)}(Shv(G) \otimes_{Shv(H)} \mathrm{Vect}, \mathrm{Vect}) \widetilde{\to} \mathrm{Fun}_{Shv(H)}(\mathrm{Vect}, \mathrm{Vect})$ to id. So, $\mathcal{R}$ sends $K \boxtimes e$ to $\mathrm{R}\Gamma(G, K)$. By Section 1.10.1, $\gamma : Shv(G) \otimes_{Shv(H)} \mathrm{Vect} \widetilde{\to} Shv(G/H)$ canonically. Here for $a : G \to G/H$ we get $\gamma(K \boxtimes e) \widetilde{\to} a_* K$ naturally for $K \in Shv(G)$. Now the functor $\mathcal{R} \circ \gamma^{-1}$ identifies with $\mathrm{R}\Gamma(G/H, \bullet) : Shv(G/H) \to \mathrm{Vect}$. Since G/H is a smooth scheme of finite type, $\mathcal{R}$ admits a continuous left adjoint $\mathcal{L} : \mathrm{Vect} \to$

$Shv(G/H)$ sending e to $e_{G/H}$. Moreover, the adjoint pair $\mathcal{L} : \mathrm{Vect} \leftrightarrows Shv(G/H) : \mathcal{R}$ takes place in $Shv(G) - mod$.

Applying the functor $\mathrm{Fun}_{Shv(G)}(\bullet, C)$ to the latter adjoint pair, we get an adjoint pair $\mathrm{oblv} : \mathrm{Fun}_{Shv(G)}(\mathrm{Vect}, C) \leftrightarrows \mathrm{Fun}_{Shv(H)}(\mathrm{Vect}, C) : \mathrm{Av}^G_{H*}$ in DGCat_{cont}. □

Note also that if G/H is isomorphic to an affine space then $\mathrm{oblv} : C^G \to C^H$ is fully faithful, because $\mathrm{id} \to \mathrm{Av}^G_{H*} \circ \mathrm{oblv}$ is an isomorphism.

According to Section 1.10.1, $Shv(H\backslash G/H)$ acts on C^H. The functor $\mathrm{oblv} \circ \mathrm{Av}^G_{H*}$ identifies with the action of $e_G \in Shv(H\backslash G/H)$ (maybe up to a shift).

1.10.3. Let $G \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$ be a placid indscheme, $C \in Shv(G) - mod(\mathrm{DGCat}_{cont})$. Then $\mathrm{Fun}(C, \mathrm{Vect})$ is naturally a right $Shv(G)$-module category. One has naturally $\mathrm{Fun}({}_GC, \mathrm{Vect}) \widetilde{\to} \mathrm{Fun}(C, \mathrm{Vect})^G$. Under this isomorphism the functor $C \to {}_GC$ is dual to $\mathrm{oblv} : \mathrm{Fun}(C, \mathrm{Vect})^G \to \mathrm{Fun}(C, \mathrm{Vect})$. If C is dualizable this gives $\mathrm{Fun}({}_GC, \mathrm{Vect}) \widetilde{\to} (C^\vee)^G$.

The notation ${}_GC$ is supposed to recall that G acts on C on the left.

Proof. One has ${}_GC \widetilde{\to} \mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \mathrm{Vect} \otimes Shv(G)^{\otimes n} \otimes C$, so

$$\mathrm{Fun}({}_GC, \mathrm{Vect}) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}(Shv(G)^{\otimes n} \otimes C, \mathrm{Vect}) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}(Shv(G)^{\otimes n}, \mathrm{Fun}(C, \mathrm{Vect})) \widetilde{\to} \mathrm{Fun}(C, \mathrm{Vect})^G$$

Here Fun means $\mathrm{Fun}_{\mathrm{Vect}}$ □

Note that for any $D \in \mathrm{DGCat}_{cont}$, if $C \in Shv(G){-}mod(\mathrm{DGCat}_{cont})$ then $\mathrm{Fun}_{k,cont}(C, D)$ is a right G-module.

More generally, let $f : H \to G$ be a homomorphism of placid group ind-schemes, $C \in Shv(G) - mod$. Under the above isomorphisms the functor $\mathrm{oblv}^G_H : \mathrm{Fun}(C, \mathrm{Vect})^G \to \mathrm{Fun}(C, \mathrm{Vect})^H$ is dual to the natural functor $\mathrm{Av}_{G,H*} : {}_HC \to {}_GC$. Here 'dual' means obtained by applying $\mathrm{Fun}_{k,cont}(\bullet, \mathrm{Vect})$ (it is not necessarily the dual functor in the sense of the monoidal structure on DGCat_{cont}, but a weaker notion).

Assume in addition that $\mathrm{Av}_{G,H*}$ admits a continuous left adjoint $\mathrm{oblv}_{G,H} : {}_GC \to {}_HC$. Then the dual to $\mathrm{oblv}_{G,H}$ in the above sense is the right adjoint $\mathrm{Av}^G_{H*} : \mathrm{Fun}(C, \mathrm{Vect})^H \to \mathrm{Fun}(C, \mathrm{Vect})^G$ of oblv^G_H.

1.10.4. Let G, H be placid group schemes and $f : H \to G$ a homomorphism of group schemes (we do not assume it is a placid closed embedding). Let $\mathcal{R} : Shv(G) \otimes_{Shv(H)} \mathrm{Vect} \to \mathrm{Vect}$ be the continuous e-linear functor sending $F \boxtimes e$ to $\mathrm{R}\Gamma(G, F)$. This is a morphism of $Shv(G)$-module categories. If $\mathcal{R}$ admits a continuous left adjoint $\mathcal{L}$ in $Shv(G){-}mod$ then for $C \in Shv(G){-}mod$ applying $\mathrm{Fun}_{Shv(G)}(\bullet, C)$, we get the functor $\mathrm{Av}^G_{H*} : C^H \to C^G$ right adjoint to $\mathrm{oblv} : C^G \to C^H$.

1.10.5. Let $G = U \rtimes H$ be as in Lemma 1.2.64, $C \in Shv(H) - mod$. Recall that $C^U \in Shv(G) - mod$, and we have $(C^U)^G \widetilde{\to} (C^U)^H$ by ([47], Lemma 1.3.7). Since $\mathrm{oblv} : C^U \to C$ is a map in $Shv(G) - mod$, its continuous right adjoint $\mathrm{Av}^U_* : C \to C^U$ is a right-lax morphism of $Shv(G)$-module categories. We claim it is a strict morphism of $Shv(G)$-module categories.

Indeed, let $F\in Shv(G)$, $c\in C$. We have to show that the natural map $F*\mathrm{Av}^U_*(c)\to \mathrm{Av}^U_*(F*c)$ is an isomorphism in C^U. It suffices to show this after applying $\mathrm{oblv}: C^U\to C$. We get $\mathrm{oblv}(F*\mathrm{Av}^U_*(c))\,\widetilde{\to}\, F*e_U*c$ and $\mathrm{oblv}\,\mathrm{Av}^U_*(F*c)\,\widetilde{\to}\, e_U*F*c$. Our claim follows from Lemma 1.2.64.

Applying the functor $\mathrm{Fun}_{Shv(G)}(\mathrm{Vect},\bullet)$ to the adjoint pair $\mathrm{oblv}: C^U\leftrightarrows C:\mathrm{Av}^U_*$, we get an adjoint pair $L:(C^U)^G\leftrightarrows C^G: R$. From $\mathrm{id}\,\widetilde{\to}\,\mathrm{Av}^U_*\mathrm{oblv}$ we learn that $\mathrm{id}\to RL$ is an isomorphism, so L is fully faithful. We check below that $L:(C^U)^G\to C^G$ is an equivalence.

The composition $\mathrm{Vect}\stackrel{e_H\boxtimes e_U}{\to} Shv(H)\otimes Shv(U)\stackrel{h}{\to} Shv(G)$ sends e to e_G, here $h=\boxtimes$. Each functor in this diagram has a continuous right adjoint, the diagram of right adjoints is $Shv(G)\stackrel{h^R}{\to} Shv(H)\otimes Shv(U)\stackrel{\mathrm{R}\Gamma\otimes\mathrm{R}\Gamma}{\to}\mathrm{Vect}$, their composition is $\mathrm{R}\Gamma$. Moreover, $\mathrm{R}\Gamma: Shv(G)\to\mathrm{Vect}$ factors naturally as $Shv(G)\to Shv(G)\otimes_{Shv(U)}\mathrm{Vect}\stackrel{\mathcal{R}}{\to}\mathrm{Vect}$, here $\mathcal{R}(F\boxtimes e)\,\widetilde{\to}\,\mathrm{R}\Gamma(G,F)$ for $F\in Shv(G)$. Since h is a map of $Shv(U)^{rm}$-modules, h^r is a right-lax morphism in $Shv(U)^{rm}$-modules. Is it strict? This looks plausible, but here is a simplier argument.

Consider the map $q: G=U\rtimes H\to U$ sending (u,h) to u. The functor q_* writes as the composition $Shv(G)\stackrel{h^R}{\to} Shv(H)\otimes Shv(U)\stackrel{\mathrm{R}\Gamma(H,\cdot)\otimes\mathrm{id}}{\to} Shv(U)$. The functor q_* is a morphism of right $Shv(U)$-modules, where U acts by convolutions on the right. So, we get a dual pair $q^*: Shv(U)\leftrightarrows Shv(G): q_*$ in $Shv(U)-mod^r(\mathrm{DGCat}_{cont})$. Passing to conivariants for U, we get a dual pair $\mathcal{L}:\mathrm{Vect}\leftrightarrows Shv(G)\otimes_{Shv(U)}\mathrm{Vect}:\mathcal{R}$ in DGCat_{cont}. The map $\mathcal{L}$ is a left-lax morphism of $Shv(G)$-modules by construction, and $\mathcal{L}(e)\,\widetilde{\to}\, e_G\boxtimes e$ canonically. For $F\in Shv(G)$ we have $F*e_G\,\widetilde{\to}\,\mathrm{R}\Gamma(G,F)\otimes e_G$ in $Shv(G)$, so $\mathcal{L}$ is a strict morphism of $Shv(G)$-modules. Now by Section 1.10.4 we learn that $\mathrm{oblv}: C^G\to C^U$ admits a continuous right adjoint $\mathrm{Av}^G_{U*}: C^U\to C^G$. The composition $\mathrm{oblv}\,\mathrm{Av}^G_{U*}$ sends $c\in C^U$ to e_G*c.

Now we claim that $\mathrm{oblv}: C^G\to C^U$ is comonadic. Since the composition $C^G\to C^U\to C$ is conservative, $\mathrm{oblv}: C^G\to C^U$ is conservative. Let V be a simplicial object of $(C^G)^{op}$, which becomes a split simplicial object V' in $(C^U)^{op}$. Then V' has a colimit in $(C^U)^{op}$ and $\mathrm{oblv}:(C^U)^{op}\to C^{op}$ preserves this colimit automatically. Let V'' be the obtained split simplicial object in C^{op}. Since $C^G\to C$ is comonadic, V admits a colimit in $(C^G)^{op}$, and $(C^G)^{op}\to C^{op}$ preserves this colimit. Since $(C^U)^{op}\subset C^{op}$ is a full subcategory, the colimit of V'' lies in $(C^U)^{op}$. So, $\mathrm{oblv}:(C^G)^{op}\to (C^U)^{op}$ preserves the colimit of V. By ([36], Th. 4.7.3.5), $\mathrm{oblv}: C^G\to C^U$ is comonadic, the corresponding comonad is $c\mapsto e_G*c$. This is the same comonad as for the comonadic functor $\mathrm{oblv}:(C^U)^G\to C^U$. Since we have a diagram $(C^U)^G\to C^G\to C^U$, and the comonads are the same, the above functor $L:(C^U)^G\to C^G$ is an equivalence for al our 4 sheaf theories.

1.10.6. Let U,Q be group ind-schemes, whose underlying ind-schemes are placid. Assume Q acts on U by conjugation, set $G=Q\ltimes U$, the semi-direct product, so G is a placid ind-scheme. We claim that $Shv(G)\otimes_{Shv(U)}\mathrm{Vect}\,\widetilde{\to}\, Shv(Q)$ naturally. In the setting of $\mathcal{D}$-modules this should be in ([13], B.2).

Proof. Pick a presentation $U\widetilde{\to}\operatorname{colim}_{i\in I} U_i$, where $I\in 1-\mathcal{C}at$ is small filtered, U_i is a placid group scheme, for $i\to j$ in I the morphism $U_i\to U_j$ is a placid closed immersion and a homomorphism of group schemes. Then $G\widetilde{\to}\operatorname{colim}_{i\in I} Q\times U_i$, because the colimits in PreStk are universal. So, $Shv(G)\widetilde{\to}\operatorname{colim}_{i\in I} Shv(Q\times U_i)$ with respect to the $*$-direct images. Now

$$Shv(G)_U\widetilde{\to}\operatorname*{colim}_{i\in I} Shv(G)\otimes_{Shv(U_i)}\mathrm{Vect}\widetilde{\to}$$
$$\operatorname*{colim}_{(i\to j)\in \mathrm{Fun}([1],I)} Shv(Q\times U_j)\otimes_{Shv(U_i)}\mathrm{Vect}\widetilde{\to}\operatorname*{colim}_{i\in I} Shv(Q\times U_i)\otimes_{Shv(U_i)}\mathrm{Vect},$$

because the diagonal map $\mathbb{N}\to\mathrm{Fun}([1],\mathbb{N})$ is cofinal. Finally,

$$Shv(Q\times U_i)_{U_i}\widetilde{\to} Shv(Q\times U_i)^{U_i}\widetilde{\to} Shv(Q),$$

and the corresponding maps are the identities. Since $I\to\mid I\mid$ is cofinal and I is contractible, we are done. □

This gives the fact that for any $C\in Shv(G)-mod(\mathrm{DGCat}_{cont})$, $C^U\in Shv(Q)-mod^r(\mathrm{DGCat}_{cont})$ naturally. Indeed,

$$\mathrm{Fun}_{Shv(U)}(\mathrm{Vect},C)\widetilde{\to}\mathrm{Fun}_{Shv(G)}(Shv(G)\otimes_{Shv(U)}\mathrm{Vect},C)$$
$$\widetilde{\to}\mathrm{Fun}_{Shv(G)}(Shv(Q),C)$$

The latter is a right $Shv(Q)$-module.

Similarly, $C_U\in Shv(Q)-mod(\mathrm{DGCat}_{cont})$. Indeed,

$$\mathrm{Vect}\otimes_{Shv(U)}C\widetilde{\to}\mathrm{Vect}\otimes_{Shv(U)}Shv(G)\otimes_{Shv(G)}C\widetilde{\to} Shv(Q)\otimes_{Shv(G)}C$$

Now if $C\to C'$ is a map in $Shv(G)-mod$ then it yields morphisms $C^U\to C'^U$ and $C_U\to C'_U$ in $Shv(Q)-mod$ by the above.

1.10.7. Let H,G be placid group schemes, $f:G\to H$ be a closed subgroup scheme such that $H\widetilde{\to} Y\times G$ as right G-modules for some placid scheme Y. Then for $q:H\to H/G$ the functor $q^*:Shv(H/G)\to Shv(H)$ is $Shv(H)$-linear. Here $Y\widetilde{\to} H/G$. So, the dual pair $q^*:Shv(H/G)\leftrightarrows Shv(H):q_*$ takes place in $Shv(H)-mod$.

Proof. We have a cartesian square

$$\begin{array}{ccc} H\times H & \stackrel{m}{\to} & H\\ \downarrow \mathrm{id}\times q & & \downarrow q\\ H\times H/G & \stackrel{\mathrm{act}}{\to} & H/G,\end{array}$$

where m is the mutliplication. Now we have $m_*(\mathrm{id}\times q)^*\widetilde{\to} q^*\mathrm{act}_*$ by ([46], 0.0.52). Indeed, m identifies with the product $\mathrm{act}\times\mathrm{id}_G$. We are done. □

If now $C\in Shv(H)-mod(\mathrm{DGCat}_{cont})$, in the above situation applying the functor $\mathrm{Fun}_{Shv(H)}(\cdot,C)$ gives an adjoint pair $\mathrm{Fun}_{Shv(H)}(Shv(H/G),C)\leftrightarrows C$ in DGCat_{cont}. Assume G prosmooth. Then $Shv(H/G)\widetilde{\to} Shv(H)^G\widetilde{\to} Shv(H)_G$, because G is placid. So, we get canonically $\mathrm{Fun}_{Shv(H)}(Shv(H/G),C)\widetilde{\to} C^G$. We reobtained the adjoint pair $\mathrm{oblv}:C^G\leftrightarrows C:\mathrm{Av}^G_*$. Let $p:G\to\mathrm{Spec}\, k$. In fact, q^* is obtained from $p^*:\mathrm{Vect}\to Shv(G)$ by applying $Shv(H)\otimes_{Shv(G)}\cdot$.

1.10.8. Let G, H be pro-smooth placid group schemes, $f : H \to G$ a homomorphism of group schemes and a placid closed immersion. Let $C \in Shv(G) - mod(\mathrm{DGCat}_{cont})$. Recall that the canonical functor $C_G \to C^G$ is an equivalence, and similarly for H. Consider the natural functor $\mathrm{pr} : C_H \to C_G$. Composing with the above equivalences, it gives a functor $F : C^H \to C^G$. Gurbir claims that F is the right adjoint to $\mathrm{oblv} : C^G \to C^H$.

We check this under the additional assumptions that $H\backslash G$ is smooth. Let $q : H\backslash G \to \mathrm{Spec}\, k$ be the projection. Then the functor $q^* : \mathrm{Vect} \to Shv(H\backslash G)$ is a map of right $Shv(G)$-modules. This follows from ([46], Lemma 0.0.20) by base change, because $H\backslash G$ is smooth. We get the dual pair $q^* : \mathrm{Vect} \leftrightarrows Shv(H\backslash G) : q_*$ in right $Shv(G)$-modules. Applying the functor $\cdot \otimes_{Shv(G)} C$, this gives an adjoint pair

$$\mathrm{pr}^L : C_G \leftrightarrows C_H : \mathrm{pr}$$

We used here the isomorphism $Shv(H\backslash G) \widetilde{\to} \mathrm{Vect} \otimes_{Shv(H)} Shv(G)$ of right $Shv(G)$-modules. (The functor pr exists for any morphism of placid group ind-schemes $H \to G$). Passing to the left adjoints in the diagram $C \overset{\mathrm{pr}_H}{\to} C_H \overset{\mathrm{pr}}{\to} C_G$ with $\mathrm{pr} \circ \mathrm{pr}_H \widetilde{\to} \mathrm{pr}_G$, one gets $C_G \overset{\mathrm{pr}^L}{\to} C_H \overset{\mathrm{pr}_H^L}{\to} C$ with $\mathrm{pr}_H^L \mathrm{pr}^L \widetilde{\to} \mathrm{pr}_G^L$. The functor $\mathrm{pr}_H^L : C_H \to C$ identifies with $\mathrm{oblv}_H : C^H \to C$, and similarly for pr_G^L. So, pr^L identifies with $\mathrm{oblv} : C^G \to C^H$.

1.10.9. Let U be a pro-unipotent group scheme, $p : U \to \mathrm{Spec}\, k$. The dual pair $p^* : \mathrm{Vect} \leftrightarrows Shv(U) : p_*$ and $C \in Shv(U) - mod$ give an adjoint pair $\mathrm{pr}^L : C_U \leftrightarrows C : \mathrm{pr}$, where pr is the natural functor (existing for any placid group ind-scheme). Clearly, pr^L is fully faithful. Passing to the right adjoints in the isomorphism $\mathrm{id} \widetilde{\to} \mathrm{pr} \circ \mathrm{pr}^L$, we see that the right adjoint $\mathrm{pr}^R : C_U \to C$ of pr is also fully faithful. So, pr is a localization functor.

More generally, let $U \widetilde{\to} \mathrm{colim}_{i \in I} U_i$ be an ind-pro-unipotent group scheme, here I is small filtered, U_i is pro-unipotent, and for $i \to j$ in I the map $U_i \to U_j$ is a placid closed immersion and a homomorphism of group schemes. Then for any $C \in Shv(U) - mod$, $C \to C_U$ is a localizatoin functor (has a fully faithful right adjoint). By Lemma 1.8.17' of this file, if $C_0 = \mathrm{Ker}(C \to C_U)$ then $C_U \widetilde{\to} C/C_0$ naturally. Here C_0 is the smallest full DG-subcategory of C containing $\mathrm{Ker}(C \to C_{U_i})$ for each i. If $i \to j$ is a map in I then $C_{U_i} \to C_{U_j}$ is a localization functor.

Proof. For each $i \in I$, $C \to C_{U_i}$ is a localization functor by the above, it has a fully faithful right adjoint. Recall that $C_U \widetilde{\to} \mathrm{colim}_{i \in I} C_{U_i}$ in DGCat_{cont}, hence also in $1 - \mathcal{C}\mathrm{at}_{cont}^{St, cocmpl}$ and in $\mathcal{P}\mathrm{r}^L$, here $\mathcal{P}\mathrm{r}^L$ is the notation from (HTT, 5.5.3.1). So, $C_U \widetilde{\to} \lim_{i \in I^{op}} C_{U_I}$, where we passed to right adjoint in DGCat, these right adjoint are maybe discontinuous, and the limit is taken in DGCat (or in $1 - \mathcal{C}\mathrm{at}^{St, cocmpl}$). It also coincides with $\cap_{i \in U} C_{U_i}$ taken inside C, because the corresponding limit can be calculated in $1 - \mathcal{C}\mathrm{at}$ by ([20], ch. I.1, 2.5.7), so we apply ([43], 2.7.7).

For $i \to j$ is a map in I then $C \to C_{U_i} \to C_{U_j}$ admits right adjoints, and the composition of this right adjoints is fully faithful, so $C_{U_i} \to C_{U_j}$ is a localization functor. $\square$

1.10.10. Let $f : H \to G$ be a homomorphism of placid group ind-schemes. Recall that $f_* : Shv(H) \to Shv(G)$ is monoidal, hence gives the restriction functor $Shv(G)-mod \to Shv(H)-mod$. If $C \in Shv(H)-mod$ then $\mathrm{Fun}_{Shv(H)}(Shv(G), C)$ is naturally an object of $Shv(G) - mod$, a left module. Moreover the functor $C \mapsto \mathrm{Fun}_{Shv(H)}(Shv(G), C)$ is right adjoint to the above restriction functor.

Proof. We view here $Shv(G)$ as a $Shv(H)$-module via the left action $Shv(H) \otimes Shv(G) \to Shv(G), (K, L) \mapsto f_*(K) * L$. Then the right action of $Shv(G)$ on itself by right convolutions yields the left $Shv(G)$-module structure on $\mathrm{Fun}_{Shv(H)}(Shv(G), C)$. Now apply ([43], 9.2.56). □

The left adjoint to the restriction functor $Shv(G) - mod \to Shv(H) - mod$ is given by the induction functor $Shv(H) - mod \to Shv(G) - mod$, $D \mapsto Shv(G) \otimes_{Shv(H)} D$.

1.10.11. Let G be a smooth group scheme of finite type, $Y \in \mathrm{PreStk}_{lft}$. Recall that the prestack quotient $Y/G \widetilde{\to} \mathrm{colim}_{[n] \in \boldsymbol{\Delta}^{op}} G^n \times Y$ in PreStk_{lft}, so $Shv(Y/G) \widetilde{\to} \lim_{[n] \in \boldsymbol{\Delta}^{op}} Shv(G^n \times Y)$ with respect to the corresponding !-restrictions.

Assume we are in the constructible context. The co-simplicial category $[n] \mapsto Shv(G^n \times Y)$ satisfies the comonadic Beck-Chevalley conditions, so $Shv(Y/G) \widetilde{\to} \mathcal{A} - comod(Shv(Y))$, where $\mathcal{A}(K) = \mathrm{act}_!(\omega_G \boxtimes K)$ for $\mathrm{act} : G \times Y \to Y$. We may also write $\mathcal{A} = q^! q_!$ for $q : Y \to Y/G$.

Let now $C \in \mathrm{DGCat}_{cont}$. Let $Shv(G)$ act on $Shv(Y) \otimes C$ via its action on $Shv(Y)$.

Lemma 1.10.12. *The natural functor $Shv(Y)^G \otimes C \to (Shv(Y) \otimes C)^G$ is an equivalence.*

Proof. For any n, $Shv(G)^{\otimes n}$ is dualizable, so

$$\mathrm{Fun}(Shv(G)^{\otimes n}, Shv(Y) \otimes C) \widetilde{\to} \mathrm{Fun}(Shv(G)^{\otimes n}, Shv(Y)) \otimes C$$

Moreover, this is an isomorphism of co-simplicial categories in $[n] \in \boldsymbol{\Delta}$, where the RHS is obtained from the co-simplicial category $\mathrm{Fun}(Shv(G)^n, Shv(Y))$ by tensoring with C.

The co-simplicial category $[n] \mapsto \mathrm{Fun}(Shv(G)^n, Shv(Y)) \otimes C$ satisfies the comonadic Beck-Chevalley conditions, so

$$(Shv(Y) \otimes C)^G \widetilde{\to} \mathcal{A} - comod(Shv(Y)) \otimes C)$$

Here $\mathcal{A} \in \mathrm{Alg}(\mathrm{Fun}(Shv(Y)) \otimes C, Shv(Y)) \otimes C)$ equal to $e_G \otimes \mathrm{id}$ for $\mathrm{id} : C \to C$, here e_G is the constant sheaf on G.

Now $Shv(Y)^G \widetilde{\to} \mathrm{Tot}(Shv(G^\bullet \times Y)$ with the transition functors given by !-inverse images. It also satisfies the comonadic Beck-Chevalley conditions with the comonad $e_G \in \mathrm{Fun}(Shv(Y), Shv(Y))$. We may pass to left adjoints in the latter totalization and get $Shv(Y)^G \widetilde{\to} \mathrm{colim}_{[n] \in \boldsymbol{\Delta}^{op}} Shv(G^n \times Y)$, because we are in the constructible context. So, $Shv(Y)^G \otimes C \widetilde{\to} \mathrm{colim}_{[n] \in \boldsymbol{\Delta}^{op}} Shv(G^n \times Y) \otimes C$. We may again pass to right adjoints in the latter colimit and get $Shv(Y)^G \otimes C \widetilde{\to} \mathrm{Tot}(Shv(G^\bullet \times Y) \otimes C)$.

Since the co-simplicial category $\mathrm{Tot}(Shv(G^\bullet \times Y)$ satisfies the comonadic Beck-Chevalley conditions, so does the co-smplicial category $\mathrm{Tot}(Shv(G^\bullet \times Y) \otimes C)$ with the comonad being $\mathcal{A}$. □

1.10.13. Let I be small filtered, $N\widetilde{\to}\operatorname{colim}_{i\in I} N_i$, where N_i is a prounipotent group scheme, for $i\to j$ in I the map $N_i\to N_j$ is a placid closed immersion, a homomorphism of group schemes. So, N is a placid ind-scheme. Let $0\in I$ be initial.

We claim that $Shv(N/N_0)_N\widetilde{\to}\mathrm{Vect}$. Indeed, $Shv(N/N_0)_N\widetilde{\to}\operatorname{colim}_i Shv(N/N_0)_{N_i}$ with respect to the natura maps $Shv(N/N_0)_{N_i}\to Shv(N/N_0)_{N_j}$ for $i\to j$ in N. Then

$$\operatorname{colim}_i Shv(N/N_0)_{N_i}\widetilde{\to}\operatorname{colim}_i\operatorname{colim}_{i\to j, j\in I} Shv(N_j/N_0)_{N_i}\widetilde{\to}\operatorname{colim}_i Shv(N_i/N_0)_{N_i},$$

because $I\to\mathrm{Fun}([1],I)$ is cofinal. Here for $j\to j'$ the map $Shv(N_j/N_0)_{N_i}\to Shv(N_{j'}/N_0)_{N_i}$ comes from the $*$-extension under $N_j/N_0\to N_{j'}/N_0$. The functor $\mathrm{R}\Gamma: Shv(N_i/N_0)\to\mathrm{Vect}$ factors as $Shv(N_i/N_0)\to Shv(N_i/N_0)_{N_i}\widetilde{\to}\mathrm{Vect}$, and the corresponding transition maps are identities. This gives $Shv(N/N_0)_N\widetilde{\to}\mathrm{Vect}$. In fact, $\mathrm{R}\Gamma: Shv(N/N_0)\to\mathrm{Vect}$ factors as $Shv(N/N_0)\to Shv(N/N_0)_N\widetilde{\to}\mathrm{Vect}$.

Besides, $\mathrm{Vect}\widetilde{\to} Shv(N/N_0)^N$, $e\mapsto\omega$. Compare with ([24], 3.4.6).

1.10.14. In my paper [52] the unital structures on chiral algebras and chiral categories are not discussed, but they are useful.

For example, if $A\in CAlg(Shv(X),\otimes^!)$ is a unital commutative algebra then $A\stackrel{1\otimes^!\mathrm{id}}{\to} A\otimes^! A\stackrel{m}{\to} A$ is the identity. This implies that restriction of the diagram $\omega_X\boxtimes A_X\to A_X\boxtimes A_X\stackrel{\bar{m}}{\to} A_{X^2}$ to the diagonal $X\hookrightarrow X^2$ is the identity, where $\bar{m}$ is the chiral multiplication. Since A is unital, one gets a map $\omega_{\mathrm{Ran}}\to\mathrm{Fact}(A)$ in $Shv(\mathrm{Ran})$ by functoriality of $\mathrm{Fact}(\cdot)$ construction.

1.11. More about [26], arxiv version 5.

1.11.1. The detailed construction given in ([26], 2.3.5) is given in details in ([50], 1.8) and works only in the constructible context.

The formula (2.3) there is simple: if $S\in\mathrm{Sch}_{ft}$, $K_i\in Shv(S)^c$, $I\to Shv(S), i\to K_i$ is given then if I is a finite diagram (defining a finite colimit) then $K=\operatorname{colim}_{i\in I} K_i\in Shv(S)^c$, as $Shv(S)^c$ is closed under finite colimits by ([35], 5.3.4.15). So,

$$\mathbb{D}(K)\widetilde{\to}\lim_{i\in I^{op}}\mathbb{D}(K_i),$$

where the limit can be understood in $Shv(S)^c$ or in $Shv(S)$.

1.11.2. In ([26], Section 2.5.3) given $\mathcal{A}\in CAlg(\mathrm{DGCat}_{cont})$ Dennis defines $\mathrm{Fact}^{alg}(\mathcal{A})_{\mathrm{Ran}}$ just as an object of DGCat_{cont}. In fact, for any of our 4 sheaf categories, there is a sheaf of categories $\mathrm{Fact}(\mathcal{A})$ defined as in ([31], 8.1.6) so that $\mathrm{Fact}^{alg}(\mathcal{A})_{\mathrm{Ran}}$ is the category of its global sections over Ran.

Indeed, for any $C\in ShvCat(\mathrm{Ran})$, $\Gamma(\mathrm{Ran}, C)\widetilde{\to}\lim_{I\in fSet}\Gamma(X^I, C)$ with respect to restrictions. Here $fSet$ is the category of finite nonempty sets and surjections. For each $I\to J$ in $fSet$ we have the adjont pair $\vartriangle_!: Shv(X^J)\leftrightarrows Shv(X^I): \vartriangle^!$ in $Shv(X^I)-mod$, here $\vartriangle: X^J\to X^I$. Tensoring with $\Gamma(X^I, C)$ yields an adjoint pair

$$\Gamma(X^I, C)\otimes_{Shv(X^I)} Shv(X^J)\leftrightarrows\Gamma(X^I, C)$$

So, passing to left adjoint, we may rewrite $\Gamma(\mathrm{Ran}, C)\widetilde{\to}\operatorname*{colim}_{i\in(fSet)^{op}}\Gamma(X^I, C)$ taken in DGCat_{cont} (equivalently, in $Shv(\mathrm{Ran})-mod$).

1.11.3. In 2.5.5 he means the following. Let $\mathcal{A} \in ComCoAlg^{nu}(\mathrm{DGCat}_{cont})$. We then get a functor $TwArr(\mathcal{A}) : TwArr^{op} \to \mathrm{DGCat}_{cont}$ sending $(I \to J)$ to $Shv(X^J) \otimes \mathcal{A}^{\otimes I}$. Given a map in $TwArr$ from $(I_1 \to J_1)$ to $(I_2 \to J_2)$ given by the diagram

$$\begin{array}{ccc} I_1 & \to & J_1 \\ \downarrow \alpha & & \uparrow \\ I_2 & \to & J_2, \end{array}$$

the attached map

$$Shv(X^{J_2}) \otimes \mathcal{A}^{\otimes I_2} \to Shv(X^{J_1}) \otimes \mathcal{A}^{\otimes I_1}$$

in DGCat_{cont} is the tensor product of $\vartriangle^!$: $Shv(X^{J_2}) \to Shv(X^{J_1})$ with the coproduct map $\mathcal{A}^{\otimes I_2} \to \mathcal{A}^{\otimes I_1}$ along α. Then he defines $\mathrm{Fact}^{coalg}(\mathcal{A})_{\mathrm{Ran}}$ as $\lim\limits_{TwArr^{op}} TwArr(\mathcal{A})$.

Does the result upgrades to a sheaf of categories on Ran?

In fact, each X^J is 1-affine for any sheaf theory, so given $(I \to J) \in TwArr$, $Shv(X^J) \otimes \mathcal{A}^{\otimes I} \in ShvCat(X^J)$.

1.11.4. In ([26], 2.5.4(ii)) he means the following functor. Let $\mathcal{A} = \oplus_{\Lambda^{neg}-0} \mathrm{Vect}$. For each $(J \to K) \in TwArr$ we have a grading on $\mathcal{A}^{\otimes J} \otimes Shv(X^K)$ coming from the $\Lambda^{neg} - 0$-grading on $\mathcal{A}$ via the product $\mathcal{A}^{\otimes J} \to \mathcal{A}$. This grading is respected by the transition functors in the diagram

$$\mathrm{Fact}^{alg}(\mathcal{A})_{\mathrm{Ran}} \widetilde{\to} \underset{(J\to I)\in TwArr}{\mathrm{colim}} \mathcal{A}^{\otimes J} \otimes Shv(X^K)$$

Since $\mathrm{colim}_{i\in I} \oplus_\lambda \mathcal{A}^i_\lambda \widetilde{\to} \oplus_\lambda \mathrm{colim}_{i\in I} \mathcal{A}^i_\lambda$, it remains to calculate the colimit of the λ-pieces. The latter becomes a colimit over the category $TwArr^\lambda$ of $(J \to K) \in TwArr$ together with a map $J \to \Lambda^{neg} - 0$, $j \mapsto \underline{\lambda}(j)$ such that $\sum_j \underline{\lambda}(j) = \lambda$. The map in the category of indices is a map in $TwArr$ such that for any $j \in J_2$,

$$\underline{\lambda}^2(j) = \sum_{j_1\in J_1, \phi(j_1)=j} \underline{\lambda}^1(j_1).$$

Here $\phi : J_1 \to J_2$ is the corresponding surjection.

For each $(J \overset{q}{\to} K, \underline{\lambda})$ of the category of indices, consider the map $f : X^K \to X^\lambda$,

$$(x_k) \mapsto \sum_{k\in K} x_k \sum_{j\in J, q(j)=k} \underline{\lambda}(j).$$

The functors $f_* : Shv(X^K) \to Shv(X^\lambda)$ are compatible with the transition functors in the diagram $\underset{TwArr^\lambda}{\mathrm{colim}}\, Shv(X^K)$, so define a functor

$$\underset{TwArr^\lambda}{\mathrm{colim}}\, Shv(X^K) \to Shv(X^\lambda).$$

He claims the latter functor is an equivalence.

Example: take $\Lambda^{neg} = \mathbb{Z}_+$ and $\lambda = 2$. Then the colimit we are calculating is $\mathrm{colim}_{B(S_2)}\, Shv(X^2)$, here S_2 is the group of 2 elements acting on X^2 by permuting the elements.

More generally, let $I = \{1, \dots, n\}$, S_n be the symmetric group of automoprhisms of I. We have an action of S_n on X^I, hence a functor $f : B(S_n) \to \mathrm{PreStk}_{lft}$, $* \mapsto X^I$.

Let $Y = \operatorname{colim}_{B(S_n)} X^I$, note that Y is a pseudo-scheme in the sense of ([21], 7.4), and the natural map $\pi : Y \to X^{(n)}$ is pseudo-proper.

By definition, $Shv(Y) \widetilde{\to} \lim_{B(S_n)^{op}} Shv(X^I) \widetilde{\to} \operatorname{colim}_{B(S_n)} Shv(X^I)$, because we may pass to the left adjoints in the formula for the limit. So, we are asking if $\pi^! : Shv(X^{(n)}) \to Shv(Y)$ is an equivalence. It is an equivalence. First, $\underset{B(S_n)}{\operatorname{colim}} X^I \to X^I/S_n$ is an isomorphism, where we used the prestack quotient (cf. [43], 2.7.24). One has naturally $Shv(B(S_n)) \widetilde{\to} \operatorname{Rep}(S_n)$, and each fibre of $\pi : X^I/S_n \to X^{(n)}$ is the classifying space of a finite group. We get an adjoint pair $\pi_! : Shv(X^I/S_n) \leftrightarrows Shv(X^{(n)}) : \pi^!$. We may check that the natural map $\pi_!\omega \to \omega$ is an isomorphism by calculating the !-fibres. So, $\pi^!$ is fully faithful by the projection formula.

However, $\pi^!$ is not an equivalence. Take a nontrivial representation V of S_n giving a skscraper sheaf $(i_x)_*V$ at $i_x : B(S_n) \hookrightarrow X^I/S_n$ for some $x \in X$. Then $\pi_!(i_x)_*V = 0$.

1.11.5. For an affine algebraic group of finite type Γ over our field e of characteristic zero, and $D \in \mathrm{DGCat}_{cont}$, $\mathrm{oblv} : \operatorname{Rep}(\Gamma) \otimes D \to D$ is comonadic by (see [55], Lemma 6.23.2). To see this quickly apply ([22], Lemma 5.5.2 and 5.5.4) to the cover $* \to B(\Gamma)$ and the quasi-coherent sheaf of categories on $B(\Gamma)$ given by $\operatorname{Rep}(\Gamma)$-module $\mathcal{C} := \operatorname{Rep}(\Gamma) \otimes D$. Apply then ([36], 4.7.5.1) to the co-simplicial category $\Gamma(\Gamma^\bullet, \mathcal{C})$, here $\Gamma^\bullet$ is the Cech nerve of $* \to B(\Gamma)$. This gives $\operatorname{Rep}(\Gamma) \otimes D \widetilde{\to} \mathcal{O}_\Gamma - comod(D)$.

Recall that for a map $f : C_1 \to C_2$ in DGCat_{cont} with C_i dualizable, $D \in \mathrm{DGCat}_{cont}$ the corresponding map $C_1 \otimes D \to C_2 \otimes D$ can be seen as

$$\operatorname{Fun}(C_1^\vee, D) \to \operatorname{Fun}(C_2^\vee, D)$$

given by the composition with $f^\vee : C_2^\vee \to C_1^\vee$. Now the dual of $\mathrm{oblv} : \operatorname{Rep}(\Gamma) \to \mathrm{Vect}$ is $p_* : \mathrm{Vect} \to \operatorname{Rep}(\Gamma)$, for $p : \operatorname{Spec} e \to B(\Gamma)$, so $(p_*)(e) = \mathcal{O}_\Gamma$. This explains why in ([55], 6.31.1) the equivalence

$$\operatorname{Fun}(\operatorname{Rep}(\Gamma), D) \widetilde{\to} \mathcal{O}_\Gamma - comod(D)$$

sends f to $f(\mathcal{O}_\Gamma)$. The inverse functor sends $A \in \mathcal{O}_\Gamma - comod(D)$ to the functor $V \mapsto (V \otimes A)^\Gamma$. By the functor of Γ-invariants $\mathcal{O}_\Gamma - comod(D) \to D$ he means the functor $q_* \otimes \mathrm{id} : \operatorname{Rep}(\Gamma) \otimes D \to D$ for $q : B(\Gamma) \to \operatorname{Spec} e$.

Assume Γ reductive and not discrete, so the set of irreducible representations Λ^+ is infinite. Then the left adjoint to *oblv* : $\operatorname{Rep}(\Gamma) \to \mathrm{Vect}$ does not exists, though it is defined in the compact part Vect^c by $V \mapsto V \otimes \mathcal{O}_\Gamma$.

For ([55], 6.31.2): Given $D \in Alg(\mathrm{DGCat}_{cont})$, $\mathcal{O}_\Gamma - comod(D)$ is equipped with a structure of an object of $Alg(\mathrm{DGCat}_{cont})$ via $\mathcal{O}_\Gamma - comod(D) \widetilde{\to} \operatorname{Rep}(\Gamma) \otimes D$, the RHS is naturally an object of $Alg(\mathrm{DGCat}_{cont})$. The claim is that we have canonically

$$\operatorname{Fun}^{rlax}_{e,cont}(\operatorname{Rep}(\Gamma), D) \widetilde{\to} Alg(\mathcal{O}_\Gamma - comod(D))$$

This is just the claim that

$$Alg(\operatorname{Fun}_{e,cont}(\operatorname{Rep}(\Gamma, D))) \widetilde{\to} Alg(\mathcal{O}_\Gamma - comod(D))$$

coming from the fact that the monoidal categories theirself are isomorphic, see ([43], 9.2.68).

1.11.6. *Chiral Hecke algebra.* Let us study the notion of chiral Hecke algebra from ([30], 7.5), which is needed for the definition of the factorizable Satake functor. It is used in ([55], Section 6.33).

For $I \in fSet$ let Gr_{G,X^I} be the corresponding version of the affine Grassmanian. As in ([53], Section 5), let G_{X^I} be the group scheme over X^I classifying $(x_i) \in X^I$ and a section of $\mathcal{F}^0_G$ over $\mathcal{D}_{\mathcal{J}}$, where $\mathcal{J}$ is the corresponding point of Ran. We have the category $\mathrm{Perv}(\mathrm{Gr}_{G,X^I})^{G_{X^I}}$ of equivariant perverse sheaves on Gr_{G,X^I}. Recall the diagram from ([53], Section 5, diagram (5.2))

$$\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X} \stackrel{p}{\leftarrow} \mathrm{Gr}_{G,X} \widetilde{\times \mathrm{Gr}}_{G,X} \stackrel{q}{\to} \mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X} \stackrel{m}{\to} \mathrm{Gr}_{G,X^2} \stackrel{\pi}{\to} X^2$$

Given $\mathcal{B}_i \in \mathrm{Perv}(\mathrm{Gr}_{G,X})^{G_X}$, one has $\mathcal{B}_1 *_X \mathcal{B}_2 \in Shv(\mathrm{Gr}_{G,X^2})^{G_{X^2}}$ defined by formula (5.6) from [53]. Let also $\tau^0 : \mathrm{Perv}(\mathrm{Gr}_G)^{G(\mathcal{O})} \to \mathrm{Perv}(\mathrm{Gr}_{G,X})^{G_X}$ be the functor defined as in ([53], Remark 5.1). Let $j : U \hookrightarrow X^2$ be the complement to the diagonal. The pullback of m under $j : U \hookrightarrow X^2$ canonically becomes the identity map

$$\mathrm{id} : (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}) \mid_U \to (\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}) \mid_U$$

in view of the factorization structure of Gr_{G,X^I}. By abuse of notations, we also write

$$(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}) \mid_U \stackrel{j}{\hookrightarrow} \mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X} \stackrel{i}{\leftarrow} (\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}) \times_{X^2} X$$

for the corresponding closed immersion and its complement.

For $V \in \mathrm{Rep}(\check{G})^{\heartsuit}$ write $\mathcal{A}_V \in \mathrm{Perv}(\mathrm{Gr}_G)^{G(\mathcal{O})}$ for the usual Satake functor at one point of our curve. Recall the convolution diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \stackrel{p}{\leftarrow} G(F) \times \mathrm{Gr}_G \stackrel{q}{\to} G(F) \times^{G(\mathcal{O})} \mathrm{Gr}_G \stackrel{m}{\to} \mathrm{Gr}_G$$

at one point of the curve. For $V, W \in \mathrm{Rep}(\check{G})^{\heartsuit}$ write $\mathcal{A}_V \tilde{\boxtimes} \mathcal{A}_W$ for the corresponding perverse sheaf on $G(F) \times^{G(\mathcal{O})} \mathrm{Gr}_G$ equipped with

$$q^*(\mathcal{A}_V \tilde{\boxtimes} \mathcal{A}_W) \widetilde{\to} p^*(\mathcal{A}_V \boxtimes \mathcal{A}_W),$$

so that $\mathcal{A}_V * \mathcal{A}_W = m_*(\mathcal{A}_V \tilde{\boxtimes} \mathcal{A}_W)$ is the usual convolution. For $V \in \mathrm{Rep}(\check{G})^{\heartsuit}$ set $\mathcal{T}_V = \tau^0(\mathcal{A}_V)$. Write also by abuse of notations

$$\tau^0 : \mathrm{Perv}(G(F) \times^{G(\mathcal{O})} \mathrm{Gr}_G)^{G(\mathcal{O})} \to \mathrm{Perv}((\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}) \times_{X^2} X)^{G_X}$$

for the corrresponding functor. Given $V, W \in \mathrm{Rep}(\check{G})^{\heartsuit}$ we define the perverse sheaf $\mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W$ on $\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}$ by the property

$$q^*(\mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W) \widetilde{\to} p^*(\mathcal{T}_V \boxtimes \mathcal{T}_W)$$

So, $\mathcal{T}_V *_X \mathcal{T}_W = m_*(\mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W)$ by definition.

Given $V, W \in \mathrm{Rep}(\check{G})^{\heartsuit}$ one has canonically $i^!(\mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W) \widetilde{\to} \tau^0(\mathcal{A}_V \tilde{\boxtimes} \mathcal{A}_W)[-1]$. Now the fibre sequence

$$i_* i^!(\mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W \to \mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W \to j_* j^*(\mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W)$$

becomes an exact sequence of perverse sheaves on $\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}$

$$0 \to \mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W \to j_* j^*(\mathcal{T}_V \tilde{\boxtimes} \mathcal{T}_W) \to \tau^0(\mathcal{A}_V \tilde{\boxtimes} \mathcal{A}_W) \to 0$$

Applying m_* it yields an exact sequence of perverse sheaves on Gr_{G,X^2}

$$0 \to \mathcal{T}_V *_X \mathcal{T}_W \to \bar{j}_*(\mathcal{T}_V \boxtimes \mathcal{T}_W) \to \bar{i}_*\mathcal{T}_{V\otimes W} \to 0 \tag{30}$$

Here we denoted by

$$(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X}) \mid_U \stackrel{\bar{j}}{\hookrightarrow} \mathrm{Gr}_{G,X^2} \stackrel{\bar{i}}{\leftarrow} \mathrm{Gr}_{G,X}$$

the corresponding closed immersion and its complement. Recall that $\mathcal{T}_V *_X \mathcal{T}_W$ is perverse, the intermediate extension under $\bar{j}$ by [53].

For $I \in fSet$ set $\lambda_I = (\underset{i\in I}{\boxtimes} e[1])[- \mid I \mid]$, so $\lambda_I \widetilde{\to} e$, but the group Aut_I acts on it by the sign character.

Let $\mathcal{O}_{\check{G}}$ be the algebra of functions on $\check{G}$ viewed as an object of $\mathrm{Rep}(\check{G})$ via the action of $\check{G}$ on itself by left translations. Let $\mathcal{R}_X = \mathcal{T}_{\mathcal{O}_{\check{G}}}$. Since $\mathcal{O}_{\check{G}} \in CAlg(\mathrm{Rep}(\check{G}))$, the product on $\mathcal{O}_{\check{G}}$ gives a map $\mathcal{T}_{\mathcal{O}_{\check{G}}\otimes\mathcal{O}_{\check{G}}} \to \mathcal{T}_{\mathcal{O}_{\check{G}}}$. Composing with the above morphism

$$\bar{j}_*(\mathcal{R}_X \boxtimes \mathcal{R}_X) \to \bar{i}_*\mathcal{T}_{\mathcal{O}_{\check{G}}\otimes\mathcal{O}_{\check{G}}}$$

we get the map

$$\bar{j}_*(\mathcal{R}_X \boxtimes \mathcal{R}_X) \to \bar{i}_*\mathcal{R}_X \tag{31}$$

on Gr_{G,X^2} denoted by (14) in ([30], Section 7.5). In fact, the construction of (31) depended on the order on the set of two elements, because the prestack $\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,X}$ is not symmetric. What we get canonically is rather the map

$$\bar{j}_*(\mathcal{R}_X^{\boxtimes I}) \otimes \lambda_I \to \bar{i}_*\mathcal{R}_X$$

for a set I of two elements.

Remark 1.11.7. *If more generally $V \in CAlg(\mathrm{Rep}(\check{G}))$, $\mathcal{T}_V$ is equipped with a similar chiral multiplication.*

Gaitsgory claims essentially that $\mathcal{R}_X$ gets a structure of a "chiral algebra" on $\mathrm{Gr}_{G,X}$, which is a synonym of a factorization algebra in $Shv(\mathrm{Gr}_{G,\mathrm{Ran}})$, and (31) is its chiral multiplication. The chiral pairing (31) satisfies the Jacobi identity, we explain this in Section 1.11.8 below.

Set $\mathrm{Sph}_{G,I} = Shv(\mathrm{Gr}_{G,X^I})^{G_{X^I}}$ for $I \in fSet$. According to the construction from ([6], 3.4.11), one forms the Chevalley-Cousin complex $C(\mathcal{R}_X)$ of $\mathcal{R}_X$, it is a collections $C(\mathcal{R}_X)_{X^I} \in \mathrm{Sph}_{G,I}$ for each $I \in fSet$ together with isomorphisms

$$\vartriangle^{(\pi)!} C(\mathcal{R}_X)_{X^J} \widetilde{\to} C(\mathcal{R}_X)_{X^I}$$

for each $\pi : J \to I$ in $fSet$. Here $\vartriangle^{(\pi)}$: $X^I \to X^J$ is the corresponding diagonal. Moreover, $C(\mathcal{R}_X)_{X^I}$ is placed in one perverse degree $- \mid I \mid$ only, and the corresponding perverse sheaf lies in $\mathrm{Perv}(\mathrm{Gr}_{G,X^I})^{G_{X^I}}$. Let $U^{(I)} \hookrightarrow X^I$ be the complement to all the diagonals. Recall the factorization isomorphism

$$\mathrm{Gr}_{G,X^I} \times_{X^I} U^{(I)} \widetilde{\to} (\mathrm{Gr}_{G,X})^I \times_{X^I} U^{(I)}$$

There is a canonical injective G_{X^I}-equivariant map of perverse sheaves

$$\mathrm{H}^{-|I|}(C(\mathcal{R}_X)_{X^I}) \hookrightarrow \bar{j}_*^{(I)}(\underset{i\in I}{\boxtimes} \mathcal{R}_X) \otimes \lambda_I,$$

where

$$\bar{j}^{(I)} : (\mathrm{Gr}_{G,X})^I \times_{X^I} U^{(I)} \hookrightarrow \mathrm{Gr}_{G,X^I}$$

is the open immersion. We used that $\bar{j}^{(I)}$ is an affine open embedding.

Besides, $C(\mathcal{R}_X)$ is equipped with the factorization isomorphisms: for any $\pi : J \to I$ in $fSet$ we have

$$(\underset{i\in I}{\boxtimes} C(\mathcal{R}_X)_{X^{J_i}}) \mid_{U^{[J/I]}} \widetilde{\to} j^{[J/I]*} C(\mathcal{R}_X)_{X^J} \tag{32}$$

as in ([6], 3.4.11). Here $j^{[J/I]} : U^{[J/I]} \hookrightarrow X^J$ is the open subset $\{(x_j) \in X^J \mid \text{if } \pi(j) \neq \pi(j') \text{ then } x_j \neq x_{j'}\}$. We denoted by the same symbol the open immersion

$$j^{[J/I]} : \prod_{i\in I} \mathrm{Gr}_{G,X^{J_i}} \times_{X^J} U^{[J/I]} \hookrightarrow \mathrm{Gr}_{G,X^J}$$

1.11.8. *Jacobi identity.* For $I = \{1,\dots,n\}$ write $(\mathrm{Gr}_{G,X})^{\tilde{\times} I}$ for the corresponding version of the convolution diagram. The linear order on I is used for the definition of the latter prestack. We still denote by $m : (\mathrm{Gr}_{G,X})^{\tilde{\times} I} \to \mathrm{Gr}_{G,X^I}$ the convolution map. For $V_i \in \mathrm{Rep}(\check{G})^{\heartsuit}$ we get the perverse sheaf $\mathcal{T}_{V_1}\tilde{\boxtimes}\dots\tilde{\boxtimes}\mathcal{T}_{V_n}$ on $(\mathrm{Gr}_{G,X})^{\tilde{\times} I}$. For the diagram

$$(\mathrm{Gr}_{G,X})^{\tilde{\times} I} \times_{X^I} X \overset{i}{\hookrightarrow} (\mathrm{Gr}_{G,X})^{\tilde{\times} I} \overset{j}{\hookleftarrow} (\mathrm{Gr}^I_{G,X} \mid_{(X^I - X)}$$

we get

$$i^!(\mathcal{T}_{V_1}\tilde{\boxtimes}\dots\tilde{\boxtimes}\mathcal{T}_{V_n}) \widetilde{\to} \tau^0(\mathcal{A}_{V_1}\tilde{\boxtimes}\dots\tilde{\boxtimes}\mathcal{A}_{V_n})[1-n]$$

Applying m_* to the fibre sequence

$$i_*i^!(\mathcal{T}_{V_1}\tilde{\boxtimes}\dots\tilde{\boxtimes}\mathcal{T}_{V_n}) \to (\mathcal{T}_{V_1}\tilde{\boxtimes}\dots\tilde{\boxtimes}\mathcal{T}_{V_n}) \to j_*(\underset{i\in I}{\boxtimes}\mathcal{T}_{V_i})$$

we get the fibre sequence

$$\bar{i}_*\mathcal{T}_{V_1\otimes\dots\otimes V_n}[1-n] \to \mathcal{T}_{V_1} *_X \dots *_X \mathcal{T}_{V_n} \to \bar{j}_*(\underset{i\in I}{\boxtimes}\mathcal{T}_{V_i})$$

for the diagram

$$(\mathrm{Gr}^I_{G,X}) \mid_{X^I - X} \overset{\bar{j}}{\hookrightarrow} \mathrm{Gr}_{G,X^I} \overset{\bar{i}}{\hookleftarrow} \mathrm{Gr}_{G,X}$$

This is precisely the property sufficient to get the exactness of the Cousin complex on Gr_{G,X^I} for the stratification coming from the diagonal stratification of X^I.

The diagonal stratification of X^I is as follows. Let for $d \geq 0$,

$$\bar{Y}_d = \underset{I\overset{\pi}{\to}T,\, |T|=n-d}{\cup} \triangle^{(\pi)}(X^T)$$

Here $\triangle^{(\pi)}\colon X^T \to X^I$. Let $Y_d = \bar{Y}_d - \bar{Y}_{d+1}$. So, Y_d is smooth of dimension $n-d$. Recall that the inclusion $Y_d \hookrightarrow X^I$ is affine.

Let $Z_d = Y_d \times_{X^I} \mathrm{Gr}_{G,X^I}$. Let $j_d : Z_d \hookrightarrow \mathrm{Gr}_{G,X^I}$ be the inclusion. We apply ([51], 1.3.3) to the perverse sheaf $\mathcal{T}_{V_1} *_X \dots *_X \mathcal{T}_{V_n}$ on Gr_{G,X^I} and the stratification $\{Z_d\}$ of Gr_{G,X^I}. The assumption of ([51], 1.3.3) says that $j_d^!(\mathcal{T}_{V_1} *_X \dots *_X \mathcal{T}_{V_n})$ is placed in perverse degree $-d$ for all $d \geq 0$. It is satisfied, so we get the exact sequence of perverse sheaves on Gr_{G,X^I}

$$\mathcal{T}_{V_1} *_X \dots *_X \mathcal{T}_{V_n} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \dots \tag{33}$$

with

$$\mathcal{F}_d = (j_d)_* j_d^!(\mathcal{T}_{V_1} *_X \dots *_X \mathcal{T}_{V_n})[d]$$

Remark 1.11.9. *When $G = \operatorname{Spec} k$, this is sufficient, as the Jacobi identity comes from the fact that the square of the differential of the latter complex on Gr_{G,X^3} vanished for $V_1 = V_2 = V_2 = e$ the trivial representation of $\check{G}$.*

Let us write down the exact sequence (33) explicitly for $n = 3$. Write $\triangle^{(12,3)}, \triangle^{(23,1)}$, $\triangle^{(13,2)}\colon X^2 \to X^3$ for the closed embedding for the corresponding equivalence relations. For example, $(12, 3)$ is the equivalence relation on $\{1, 2, 3\}$ identifying 1 and 2, and so on. Let $j^2 : X^2 - X \hookrightarrow X^2$ be the embedding.

Now (33) for $n = 3$ becomes

$$\begin{aligned} &\mathcal{T}_{V_1} *_X \mathcal{T}_{V_2} *_X \mathcal{T}_{V_3} \to (j_0)_* j_0^*(\mathcal{T}_{V_1} *_X \mathcal{T}_{V_2} *_X \mathcal{T}_{V_3}) \to \\ &\triangle_*^{(12,3)} j_* j^*(\mathcal{T}_{V_1\otimes V_2} *_X \mathcal{T}_{V_3}) + \triangle_*^{(13,2)} j_* j^*(\mathcal{T}_{V_1\otimes V_3} *_X \mathcal{T}_{V_2}) + \triangle_*^{(23,1)} j_* j^*(\mathcal{T}_{V_2\otimes V_3} *_X \mathcal{T}_{V_1}) \\ &\qquad \to \triangle_* (\mathcal{T}_{V_1\otimes V_2\otimes V_3}), \end{aligned}$$

where we denoted temporarily $j : X^2 - X \hookrightarrow X^2$, as well as its base changes, and similarly for the main diagonal $\triangle\colon X \hookrightarrow X^3$. The second term of this complex is actually

$$(j_0)_*(\mathcal{T}_{V_1} \boxtimes \mathcal{T}_{V_2} \boxtimes \mathcal{T}_{V_3}) \mid_{U^{(3)}}$$

using the factorization structure of Gr_{G,X^3}.

Assume in addition $V \in CAlg(\mathrm{Rep}(\check{G}))$. Equip $\mathcal{T}_V$ with the chiral multiplication of Remark 1.11.7.

Then $\mathcal{T}_V$ becomes a Lie algebra with this multiplication, that is, satisfies the Jacobi identity. Namely, the product in the algebra V commutes with chiral pairings. More precisely, the diagram commutes

$$\begin{array}{ccc} \triangle_*^{(12,3)} j_* j^*(\mathcal{T}_{V\otimes V} *_X \mathcal{T}_V) & \to & \triangle_* (\mathcal{T}_{V^{\otimes 3}}) \\ \downarrow & & \downarrow \\ \triangle_*^{(12,3)} j_* j^*(\mathcal{T}_V *_X \mathcal{T}_V) & \to & \triangle_* (\mathcal{T}_V), \end{array} \tag{34}$$

where the vertical arrows come from the multiplication in V, and similarly for other equivalence relations on $\{1, 2, 3\}$. Then we could get the Jacobi identity from the above exact sequence by pushing out and getting the square of the differential equal to zero. So, $\mathcal{T}_V$ becomes a chiral algebra on $\mathrm{Gr}_{G,X}$.

The commutativity of (34) follows from the more general claim: given $V, W, V' \in \mathrm{Rep}(\check{G})$ and a map $V \to V'$, the diagram on Gr_{G,X^2} commutes

$$\begin{array}{ccc} \bar{j}_* \bar{j}^*(\mathcal{T}_V *_X \mathcal{T}_W) & \to & \bar{i}_* \mathcal{T}_{V\otimes W} \\ \downarrow & & \downarrow \\ \bar{j}_* \bar{j}^*(\mathcal{T}_{V'} *_X \mathcal{T}_W) & \to & \bar{i}_* \mathcal{T}_{V'\otimes W}, \end{array}$$

where the horizontal maps are as in (30) and the vertical arrows come from the functoriality.

1.11.10. In fact, $C(\mathcal{R}_X)_{X^I}$ lifts to an object of $\mathrm{Rep}(\check{G})_{X^I}\otimes_{Shv(X^I)}\mathrm{Sph}_{G,I}$. To see this, we use the definition of $\mathrm{Rep}(\check{G})_{X^I}$ as

$$\lim_{(I\overset{p}{\to}J\to K)\in Tw(I)} Shv(X^I_{p,d})\otimes \mathrm{Rep}(\check{G})^{\otimes K}$$

So, Dennis' definition as colimit seems insufficient for this, as anyway one needs to rewrite it as the above limit.

The reason is that $\mathcal{O}_{\check{G}}\in \mathrm{Rep}(\check{G}\times\check{G})$, and moreover $\mathcal{O}_{\check{G}}\in CAlg(\mathrm{Rep}(\check{G}\times\check{G})$. Namely, for the diagonal map $q: B(\check{G})\to B(\check{G}\times\check{G})$ the functor $q_*:\mathrm{QCoh}(B(\check{G}))\to \mathrm{QCoh}(B(\check{G}\times\check{G}))$ is right-lax symmetric monoidal, so sends the commutative algebra e to the commutative algebra $\mathcal{O}_{\check{G}}$.

For this reason for $I\in fSet$ the shifted perverse sheaf $C(\mathcal{R}_X)_{X^I}$ is equipped with an action of $\check{G}$, so it is an object of $\mathrm{Rep}(\check{G})\otimes\mathrm{Sph}_{G,I}$. Now for $\pi: J\to U$ the action of $\check{G}^I$ on (32) comes as the product of actions of $\check{G}$ on each factor $C(\mathcal{R}_X)_{X^{J_i}}$ for $i\in I$. Thus, (32) lies in $\mathrm{Rep}(\check{G})^{\otimes I}\otimes\mathrm{Sph}_{G,J}$.

To get the above claim we need a version of Sam's ([55], Lemma 6.18.1):

Lemma 1.11.11. *Let $C\in CAlg^{nu}(\mathrm{DGCat}_{cont})$ be dualizable such that $m: C^{\otimes 2}\to C$ admits a continuous right adjoint. Then for any $D\in Shv(X^I)-mod$ the natural map*

$$C_{X^I}\otimes_{Shv(X^I)} D\to \lim_{(I\overset{p}{\to}J\to K)\in Tw(I)} (C^{\otimes K}\otimes Shv(X^I_{p,d})\otimes D)$$

is an equivalence. This is proved in ([52]*, 4.1.12).* □

1.11.12. Let Γ be an affine algebraic group of finite type, $C=\mathrm{Rep}(\Gamma)$. We claim that $\mathrm{oblv}_{X^I}: C_{X^I}\to Shv(X^I)$ has a continuous $Shv(X^I)$-linear right adjoint $\mathrm{oblv}^R_{X^I}$.

By ([52], 2.5.10), C_{X^I} is ULA over $Shv(X)$. Recall that by ([52], Lemma 2.2.2),

$$\mathrm{Loc}: C^{\otimes I}\otimes Shv(X^I)\to C_{X^I}$$

generates C_{X^I} under colimits, it also has a continuous $Shv(X^I)$-linear right adjoint by ([52], 2.5.2). If $c\in C^{\otimes I}$ is compact then $\mathrm{Loc}(c\otimes\omega_{X^I})$ is ULA over $Shv(X^I)$. So, it suffices to show that $\mathrm{oblv}_{X^I}\mathrm{Loc}(c\otimes\omega_{X^I})$ is ULA over $Shv(X^I)$. The latter object identifies with $\mathrm{Loc}(\mathrm{oblv}(c)\otimes\omega_{X^I})$ for the functor $\mathrm{id}=\mathrm{Loc}: Shv(X^I)\to Shv(X^I)$ for the category $\mathrm{Rep}(\Gamma)$ replaced by Vect, here $\mathrm{oblv}: C^{\otimes I}\to \mathrm{Vect}$ is the oblivion functor. We are done.

Now ([55], 6.23.2) holds also in the constructible context. Namely, for any $D\in Shv(S^I)-mod$

$$\mathrm{oblv}_{X^I}:\mathrm{Rep}(\Gamma)_{X^I}\otimes_{Shv(X^I)} D\to D$$

is comonadic.

Indeed, use the definition of $\bar{C}_{X^I}$ from [52], as $C_{X^I}\widetilde{\to}\bar{C}_{X^I}$. By ([52], 4.1.12), we have

$$\mathrm{Rep}(\Gamma)_{X^I}\otimes_{Shv(X^I)} D\widetilde{\to}\lim_{(I\overset{p}{\to}J\to K)\in \mathrm{Tw}(I)^{op}} C^{\otimes K}\otimes Shv(X^I_{p,d})\otimes_{Shv(X^I)} D$$

By ([22], Lemma 5.5.4), each functor

$$C^{\otimes K}\otimes Shv(X^I_{p,d})\otimes_{Shv(X^I)} D\to Shv(X^I_{p,d})\otimes_{Shv(X^I)} D$$

is comonadic, so is conservative and commutes with oblv-split totalizations. We used that $C^{\otimes K} \tilde{\to} \mathrm{Rep}(\Gamma^K)$. By ([43], 2.5.3) we see that oblv_{X^I} is conservative. It suffices now to show that if K is a simplicial object of $(\mathrm{Rep}(\Gamma)_{X^I} \otimes_{Shv(X^I)} D)^{op}$ which becomes split in D^{op} then K admits a colimit, which is preserved by $(\mathrm{oblv}_{X^I})^{op}$. This follows now from ([43], Lemma 2.2.68 1)).

In particular, $\mathrm{oblv}_{X^I} : \bar{C}_{X^I} \to Shv(X^I)$ is comonadic.

Lemma 1.11.13. *For $C = \mathrm{Rep}(\Gamma)$ and each I there is a natural t-structure on C_{X^I} such that $\mathrm{oblv}_{X^I} : C_{X^I} \to Shv(X^I)$ is t-exact. It is accessible, compatible with filtered colimits, left and right complete.*

Proof. For each $\Sigma = (I \xrightarrow{p} J \to K)$ the forgetful functor

$$C^{\otimes K} \otimes Shv(X^I_{p,d}) \to Shv(X^I_{p,d})$$

is comonadic with the comonad given by the coalgebra $\mathcal{O}_{\Gamma^K} \in coAlg(\mathrm{Vect})$. Since $\mathcal{O}_{\Gamma^K}$ is placed in degree zero, the functor $\cdot \otimes \mathcal{O}_{\Gamma^K} : \mathrm{Vect} \to \mathrm{Vect}$ is t-exact. We equip $C^{\otimes K} \otimes Shv(X^I_{p,d})$ with a t-structure defined in ([43], 9.3.23). So, both functors in the adjoint pair

$$\mathrm{oblv} : C^{\otimes K} \otimes Shv(X^I_{p,d}) \leftrightarrows Shv(X^I_{p,d}) : \mathrm{coind}$$

are t-exact. By definition, $(C^{\otimes K} \otimes Shv(X^I_{p,d}))^{\leq 0}$ is the preimage of $Shv(X^I_{p,d})^{\leq 0}$ under oblv, so the t-structure on $C^{\otimes K} \otimes Shv(X^I_{p,d})$ is accessible. Besides, $(C^{\otimes K} \otimes Shv(X^I_{p,d}))^{\geq 0}$ is the preimage of $Shv(X^I_{p,d})^{\geq 0}$ under oblv. So, the t-structure on $C^{\otimes K} \otimes Shv(X^I_{p,d})$ is compatible with filtered colimits.

By ([20], I.3, 1.5.8), there is a unique t-structure on

$$\lim_{(I \xrightarrow{p} J \to K) \in \mathrm{Tw}(I)^{op}} C^{\otimes K} \otimes Shv(X^I_{p,d})$$

such that each evaluation functor to $C^{\otimes K} \otimes Shv(X^I_{p,d})$ is t-exact, because the transition functors in our limit diagram are t-exact. Moreover, the t-structure on C_{X^I} is compatible with filtered colimits and accessible, as

$$\lim_{(I \xrightarrow{p} J \to K) \in \mathrm{Tw}(I)^{op}} C^{\otimes K} \otimes Shv(X^I_{p,d})^{\leq 0}$$

is presentable.

It remains to show that the t-structure on C_{X^I} is left and right complete. For $\mathcal{D}$-modules this is ([55], 6.24.1). Assume we are in the constructible context. By ([20], I.3, 1.5.8), it suffices to show that for each $(I \xrightarrow{p} J \to K) \in \mathrm{Tw}(I)$ the t-structure on $C^{\otimes K} \otimes Shv(X^I_{p,d})$ is both left and right complete.

The t-structure on $Shv(X^I_{p,d})$ is right complete by ([46], 0.0.10). So, the t-structure on $C^{\otimes K} \otimes Shv(X^I_{p,d})$ is right complete by ([43], 9.3.23). The t-structure on $Shv(X^I_{p,d})$ is left complete by ([2], Theorem 1.1.6).

To see that the t-structure on $C^{\otimes K} \otimes Shv(X^I_{p,d})$ is left complete, apply ([2], E.9.6). Namely, $Shv(X^I_{p,d}) \tilde{\to} \mathrm{Ind}(\mathrm{D}^b(\mathrm{Perv}(X^I_{p,d})))$ by ([2], E.1.2). Now the t-structure on $\mathrm{QCoh}(B(\Gamma^K))$ is left complete by ([20], I.3, 1.5.7), as $B(\Gamma^K)$ is an Artin stack. Now by ([2], E.9.6), the t-structure on $C^{\otimes K} \otimes Shv(X^I_{p,d})$ is left complete.

In turn, by ([20], I.3, 1.5.8) applied to the diagram

$$\lim_{(I\overset{p}{\to}J\to K)\in\mathrm{Tw}(I)^{op}} C^{\otimes K}\otimes Shv(X^I_{p,d})\otimes_{Shv(X^I)} D$$

we see that the t-structure on $\bar{C}_{X^I}$ is left complete. □

1.11.14. For 2.6.2. The sense of $V^{\underline{\lambda}}=\underset{(I\to J\to K)\in TwArr_{I/}}{\operatorname{colim}} V^{\underline{\lambda}}_{I\to J\to K}$ in C_{X^I} with $C=\operatorname{Rep}(\check{G})$? I would say a "spread representation" of the highest weight $\underline{\lambda}$?

The objects $V^{\underline{\lambda}}_{I\to J\to K}$ factorize naturally: if $\phi: I\to I'$ is a surjection of finite nonempty sets, $X^K\times_{X^I} X^I_{\phi,d}$ is empty unless $I'\in Q(K)$, and in the latter case we get

$$(V^{\underline{\lambda}}_{I\to J\to K})\mid_{X^I_{\phi,d}}\;\widetilde{\to}\;\boxtimes_{i\in I'} V^{\underline{\lambda}^i}_{I_i\to J_i\to K_i}\mid_{X^I_{\phi,d}}, \tag{35}$$

where $\underline{\lambda}^i: I_i\to\Lambda^+$ is the restriction of $\underline{\lambda}$. Here K_i, J_i are fibres over $i\in I'$.

Simliarly, $V^{\underline{\lambda}}$ factorize naturally: if $\phi: I\to I'$ is a surjection of finite nonempty sets, $X^K\times_{X^I} X^I_{\phi,d}$ is empty unless $I'\in Q(K)$, and in the latter case we get

$$V^{\underline{\lambda}}\mid_{X^I_{\phi,d}}\;\widetilde{\to}\;(\boxtimes_{i\in I'} V^{\underline{\lambda}^i})\mid_{X^I_{\phi,d}}$$

Proof: Write $Tw(I)=TwArr_{I/}$. Recall the full subcategory $\mathrm{Tw}(I)_\phi\subset\mathrm{Tw}(I)$ from ([52], 2.1.15). The latter inclusion is zero-cofinal, so the desired equivalence follows from (35).

1.11.15. For ([26], 2.6.4). The section $s^{-,\underline{\lambda}}_I: X^I\to\mathrm{Gr}_{T,I}$ sends (x_i) to $\mathcal{F}^0_T(\sum_{i\in I}\underline{\lambda}(i)x_i)$ with the natural trivialization outside $\cup_i x_i$.

1.11.16. For ([26], 5.2.3). If $A, A'\in CAlg(\mathrm{DGCat}_{cont})$ and $A\to A'$ is a right-lax symmetric monoidal functor then there is a right-lax symmetric monoidal functor $A_{X^I}\to A'_{X^I}$, Sam says so. Here is the construction under the assumption that A is dualizable, and both the product map $m: A\otimes A\to A$ and $u:\mathrm{Vect}\to A$ admit continuous right adjoints. Apply ([52], C.0.3). Our right-lax symmetric monoidal functor is a unital commutative algebra $\mathcal{A}$ in $\mathrm{Fun}_{e,cont}(A,A')\widetilde{\to} A^\vee\otimes A'$.

Recall that we first view A as a cocommutative coalgebra via $m^R: A\to A\otimes A$, $u^R: A\to\mathrm{Vect}$, so that its dual becomes an object of $CAlg(\mathrm{DGCat}_{cont})$.

Further, if $I\in fSets$ then $(A^\vee\otimes A')_{X^I}\widetilde{\to}(A^\vee)_{X^I}\otimes_{Shv(X^I)} A'_{X^I}$ by ([52], 2.2.14). In turn, by ([52], formula (35) after Lemma 2.6.7), we have

$$\mathrm{Fun}_{Shv(X^I)}(A_{X^I}, A'_{X^I})\widetilde{\to}(A^\vee)_{X^I}\otimes_{Shv(X^I)} A'_{X^I}\widetilde{\to}(\mathrm{Fun}_{e,cont}(A,A'))_{X^I}$$

Then $\mathcal{A}_{X^I}$ is the desired functor $A_{X^I}\to A'_{X^I}$. Since $\mathcal{A}_{X^I}\in CAlg((\mathrm{Fun}_{e,cont}(A,A'))_{X^I})$, the functor $(A_{X^I},\otimes^!)\to(A'_{X^I},\otimes^!)$ is right-lax symmetric monoidal.

1.11.17. For ([26], 5.3.2). It is essential there that $\mathcal{A}$ is symmetric.

1.11.18. For ([26], 5.3.5). Let $A, A' \in CAlg(\mathrm{DGCat}_{cont})$, let $F : A \to A'$ be a right-lax symmetric monoidal functor. Let C be a (A', A)-bimodule category, that is, $C \in A' \otimes A - mod$. Write the A' acton on the left, the A-action on the right. In ([26], 5.3.1) he defines the ∞-category $Z_F(C)$ of right-lax central objects of C with respect to F.

Let $m : A \otimes A \to A$ be the product, assume A rigid. Let $m^R : A \to A \otimes A$ be the right adjoint to m. So, $\mathcal{R} := (F \boxtimes \mathrm{id})m^R(1) \in CAlg(A' \otimes A)$.

Let us equip $\mathcal{R}$ with a structure of a right-lax central object of C with respect to F. Here we view $A' \otimes A \in A' \otimes A - mod$ naturally. For $a \in A$ recall that

$$(a \boxtimes 1) \otimes m^R(1) \widetilde{\to} m^R(a) \widetilde{\to} (1 \boxtimes a) \otimes m^R(1)$$

canonically. Define the map

$$\phi(a) : (F(a) \boxtimes 1) \otimes \mathcal{R} \to \mathcal{R} \otimes (1 \boxtimes a) \tag{36}$$

in $A' \otimes A$ as follows. Note that $(F \boxtimes \mathrm{id})m^R(a) \widetilde{\to} ((F \boxtimes \mathrm{id})m^R(1)) \otimes (1 \boxtimes a)$. Now $\phi(a)$ is the compostion

$$(F(a) \boxtimes 1) \otimes \mathcal{R} = (F(a) \boxtimes 1) \otimes (F \boxtimes \mathrm{id})m^R(1) \to$$
$$(F \boxtimes \mathrm{id})((a \boxtimes 1) \otimes m^R(1)) = (F \boxtimes \mathrm{id})m^R(a) = \mathcal{R} \otimes (1 \boxtimes a),$$

where the arrow is given by the right-lax structure on F. The lax structure with respecct to F comes now from the right-lax structure on the functor F itself.

For $a, b \in A$ let $\underline{\mathrm{Hom}}(a, b) \in A$ denote the inner hom, it exists because A is presentable. For $a \in A$ the canonical map $a \otimes \underline{\mathrm{Hom}}(a, 1) \to 1$ yields by adjointness $a \boxtimes \underline{\mathrm{Hom}}(a, 1) \to m^R(1)$ in $A \otimes A$, hence further a map $F(a) \boxtimes \underline{\mathrm{Hom}}(a, 1) \to R_A^F$ in $A' \otimes A$. So, any $c \in R_A^F - mod(C)$ we get a morphism $(F(a) \boxtimes \underline{\mathrm{Hom}}(a, 1)) \otimes c \to c$ coming from the action map.

In particular, if $a \in A^{dualizable}$ then $\underline{\mathrm{Hom}}(a, 1) \widetilde{\to} a^\vee$, and the latter map yields a morphism $F(a) \otimes c \to c \otimes a$ by duality properties. Then I think for such dualizable objects one may check hopefully the commutativity of the square defining the right-lax central structure on c with respect to F.

Indeed, for $a_i \in A^{dualizable}$ the diagram commutes

$$\begin{array}{ccc} (F(a_1) \boxtimes a_1^\vee) \otimes (F(a_2) \boxtimes a_2^\vee) & \to & F(a_1 \otimes a_2) \boxtimes (a_1 \otimes a_2)^\vee \\ \downarrow & & \downarrow \\ R_A^F \otimes R_A^F & \stackrel{m}{\to} & R_A^F, \end{array}$$

where the top horizontal arrow comes from the right-lax structure on F.

Now for an object $c \in C$ we may define a "version" of the notion of the right-lax central object with respect to F requiring:

1) that for $a \in A^{dualizable}$ we are given the map $\psi(a_1) : (F(a_1) \boxtimes a_1^\vee) \otimes c \to c$ such that the corresponding map $F(a_1) \otimes c \to c \otimes a_1$ is functorial in $a \in A^{dualizable}$;

2) For $a_i \in A^{dualizable}$ we are given the commutativity datum for the diagram

$$\begin{array}{ccc} (F(a_1) \boxtimes a_1^\vee) \otimes (F(a_2) \boxtimes a_2^\vee) \otimes c & \stackrel{\psi(a_2)}{\to} & (F(a_1) \boxtimes a_1^\vee) \otimes c \\ \downarrow & & \downarrow \psi(a_1) \\ (F(a_1 \otimes a_2) \boxtimes (a_1 \otimes a_2)^\vee) \otimes c & \stackrel{\psi(a_1 \otimes a_2)}{\to} & c, \end{array}$$

where the left vertical arrow comes from the right-lax structure on F;

3) a coherent system of higher compatibilities.

If A is compactly generated then $A^{dializable}$ generate A under filtered colimits. In this case given $a\in A$, assume c is a right-lax central in the "version" sense. Write $a\widetilde{\to}\operatorname{colim}_{i\in I} a_i$ with $a_i\in A^{dualizable}=A^c$. Then the above maps $F(a_i)\otimes c\to c\otimes a_i$ by passing to the colimit over $i\in I$ yield a map $F(a)\otimes c\to c\otimes a$, and the commutativity of the corresponding square for arbitrary $a_i\in A$ is also obtained by passing to the limit.

So, in the case of A rigid compactly generated we see that any $c\in R_A^F-mod(C)$ gets a right-lax central structure with respect to F.

Let's now drop the assumption that A is compactly generated. Clearly, for $c\in C$ the free $\mathcal{R}$-module $\mathcal{R}\otimes c$ is equipped with the right-lax central structure with respect to F, it comes from the maps (36).

Recall that any $m\in\mathcal{R}-mod(C)$ writes as $\mathcal{R}\otimes_{\mathcal{R}} m\widetilde{\to}\underset{[n]\in\boldsymbol{\Delta}^{op}}{\operatorname{colim}}\mathcal{R}^{\otimes n+1}\otimes m$, the bar resolution. For each $[n]\in\boldsymbol{\Delta}^{op}$, viewing $\mathcal{R}^{\otimes n+1}\otimes m$ as a free $\mathcal{R}$-module via the multiplication on the most left factor, the maps (36) yield morphisms

$$\phi_n:(F(a)\boxtimes 1)\otimes\mathcal{R}^{\otimes n+1}\otimes m\to(\mathcal{R}^{\otimes n+1}\otimes m)\otimes(1\boxtimes a)$$

The claim is then then for any map $\alpha:[r]\to[n]$ in $\boldsymbol{\Delta}$ the diagram commutes

$$\begin{array}{ccc}(F(a)\boxtimes 1)\otimes\mathcal{R}^{\otimes n+1}\otimes m & \stackrel{\phi_n}{\to} & (\mathcal{R}^{\otimes n+1}\otimes m)\otimes(1\boxtimes a)\\ \downarrow\mathrm{id}\otimes\bar{\alpha} & & \downarrow\bar{\alpha}\otimes\mathrm{id}\\ (F(a)\boxtimes 1)\otimes\mathcal{R}^{\otimes r+1}\otimes m & \stackrel{\phi_r}{\to} & (\mathcal{R}^{\otimes r+1}\otimes m)\otimes(1\boxtimes a)\end{array}$$

Here $\bar{\alpha}$ is the corresponding morphism in the bar resolution. (It suffices to have this for α injective for our purposes).

So, passing to the colimit over $[n]\in\boldsymbol{\Delta}^{op}$ the maps ϕ_n yield the desired map

$$\phi(a,m):(F(a)\boxtimes 1)\otimes m\to m\otimes(1\boxtimes a)$$

The only nontrivial verification here seems for the map $\alpha:[0]\to[1]$ corresponding to $\mathcal{R}\otimes\mathcal{R}\otimes m\to\mathcal{R}\otimes m$ given by the product on the algebra m tensored by the identity on m.

Question: conversely, how the right-lax central structure on c gives the $\mathcal{R}$-module structure?

Remark 1.11.19. *If in the above* $A=\mathrm{Vect}$ *then* $F:\mathrm{Vect}\to A'$ *defines an algebra object* $F(1)$. *In this case* $\mathcal{R}=F(1)\in A'$, *and it is clear that* $Z_F(C)\widetilde{\to}\mathcal{R}-mod(C)$.

1.11.20. Let $C\in CAlg(S(X)-mod)$. The for any surjection $I\to J$ of finite nonempty sets, for $\vartriangle: X^J\to X^I$ the functor $\vartriangle^!:(C_{X^I},\otimes^!)\to(C_{X^J},\otimes^!)$ is symmetric monoidal. Besides, the dual pair

$$\vartriangle_!: C_{X^J}\leftrightarrows C_{X^I}:\vartriangle^!$$

is a dual pair in $C_{X^I}-mod$. In other words, for $K\in C_{X^J}, L\in C_{X^I}$ one has the projection formula: $(\vartriangle_! K)\otimes^! L\widetilde{\to}\vartriangle_!(K\otimes^!\vartriangle^! L)$ canonically.

Indeed, first given a map $A\stackrel{v}{\to}B$ in $CAlg(Shv(X)-mod)$, the diagram commutes

$$\begin{array}{ccc}A_{X^I} & \stackrel{v}{\to} & B_{X^I}\\ \uparrow & & \uparrow\\ A_{X^J} & \stackrel{v}{\to} & B_{X^J}\end{array}$$

So, the diagram commutes

$$\begin{array}{ccccc} C_{X^I}\otimes_{Shv(X^I)} C_{X^I} & = & (C\otimes C)_{X^I} & \stackrel{\otimes^!}{\to} & C_{X^I}\\ \uparrow & & \uparrow & & \uparrow\\ C_{X^J}\otimes_{Shv(X^I)} C_{X^I} & = & (C\otimes C)_{X^J} & \stackrel{\otimes^!}{\to} & C_{X^J}\end{array}$$

Now given $M_I\in C_{X^I}-mod$ applying $\otimes_{A_{X^I}} M_I$, one get an adjoint pair

$$\vartriangle_!: C_{X^J}\otimes_{C_{X^I}} M_I \leftrightarrows M_I :\vartriangle^!$$

in $C_{X^I}-mod$. Suppose we have a compatible system $M_I\in C_{X^I}-mod$ for $I\in fSets$. That is, for $I\to J$ as above, we are given equivalences $M_J\widetilde{\to} C_{X^J}\otimes_{C_{X^I}} M_I$ together with higher compatibilities.

We have $(\mathrm{Fact}(C),\otimes^!)\widetilde{\to}\lim\limits_{I\in fSets}(C_{X^I},\otimes^!)$ in $CAlg(Shv(\mathrm{Ran})-mod)$. Then $\lim\limits_{I\in fSets} M_I$ calculated in $(\mathrm{Fact}(C),\otimes^!)-mod$ is the same as in $Shv(\mathrm{Ran})-mod$ or in DGCat_{cont}. Moreover, we see that we may pass to the left adjoints and get $\lim\limits_{I\in fSets} M_I\widetilde{\to}\operatorname*{colim}\limits_{I\in fSets^{op}} M_I$, where the colimit is calculated in DGCat_{cont} or equivalently, in $Shv(\mathrm{Ran})-mod$ or equivalently in $(\mathrm{Fact}(C),\otimes^!)-mod$.

Let

$$\mathrm{Loc}_C : (\mathrm{Fact}(C),\otimes^!)-\ \mathrm{mod}\ \to \lim_{I\in fSets} C_{X^I}-mod$$

be the functor sending M to the compatible family $\{M_I\}$ with $M_I = M\otimes_{(\mathrm{Fact}(C),\otimes^!)} C_{X^I}$ for $I\in fSets$. The functor Loc_C has a right adjoint

$$\{M_{X^I}\}\mapsto\Gamma(\mathrm{Ran},M)=\mathop{lim}_{I\in fSets} M_{X^I}$$

calculated in $(\mathrm{Fact}(C),\otimes^!)-\ \mathrm{mod}$ or equiivalently in $Shv(\mathrm{Ran})-mod$.

Lemma 1.11.21. *The functor* Loc_C *(hence also its right adjoint) is an equivalence.*

Proof. This is done as in ([52], Lemma 2.2.8). Namely, let $E\in\mathrm{Fact}(C)-mod$ and $E_I = E\otimes_{\mathrm{Fact}(C)} C_{X^I}$ for $I\in fSets$. Then $E\widetilde{\to}\lim\limits_{I\in fSets} E_I\widetilde{\to}\operatorname*{colim}\limits_{I\in fSets^{op}} E_I$ calculated in $(\mathrm{Fact}(C),\otimes^!)-mod$, because $\operatorname*{colim}\limits_{I\in fSets^{op}} C_I\widetilde{\to} C$. So, Loc_C is fully faithful.

We now check that its right adjoint is fully faithful. For this it suffices to prove the following. Let $\{E_I\}$ in the RHS then the canonical map

$$\Gamma(\mathrm{Ran},E)\otimes_{(\mathrm{Fact}(C),\otimes^!)} C_{X^I}\to E_{X^I}$$

in $C_{X^I}-mod$ is an equivalence. This follows from ([52], Lemma 2.2.8), because $C_{X^I}-mod\to Shv(X^I)-mod$ is conservative. □

1.11.22. For ([26], 5.3.9) a precise definition is needed. I think he defines $Z_F(\mathcal{C}_{\mathrm{Ran}})$ as $\mathrm{Fact}(R_A^F)-mod(\mathcal{C}_{\mathrm{Ran}})$. Here we use that $\mathcal{C}_{\mathrm{Ran}}$ is a $\mathrm{Fact}(A'\otimes A)-mod$. Then it is better to say so. We use here ([52], Lemma 2.2.22).

We have the functor $CAlg(A'\otimes A)\to CAlg((A'\otimes A)_I,\otimes^!), \mathcal{B}\mapsto\mathcal{B}_{X^I}$. The natural right-lax symmetric monoidal transformation of functors $F\to F'$ from A to A' yields a map $R_A^F\to R_A^{F'}$ of commutative algebras in $A'\otimes A$, hence a map $(R_A^F)_I\to(R_A^{F'})_I$ of commutative algebras in $((A'\otimes A)_I,\otimes^!)$, and similarly a map $\mathrm{Fact}(R_A^F)\to\mathrm{Fact}(R_A^{F'})$ of commutative algebras in $(\mathrm{Fact}(A'\otimes A),\otimes^!)$.

So, we have the adjoint pair

$$\mathrm{Fact}(R_A^F)-mod(\mathcal{C}_{\mathrm{Ran}})\leftrightarrows\mathrm{Fact}(R_A^{F'})-mod(\mathcal{C}_{\mathrm{Ran}}),$$

where the right adjoint is the restriction of scalars under $\mathrm{Fact}(R_A^F)\to\mathrm{Fact}(R_A^{F'})$, and the left adjoint is the base change $L\mapsto\mathrm{Fact}(R_A^{F'})\otimes_{\mathrm{Fact}(R_A^F)}L$.

1.11.23. In 2.2.1-2.2.2 there is a misprint, it should be as follows: $(i^\lambda)_!\omega_{S^\lambda_{\mathrm{Ran}}}[-\langle\lambda,\check{\rho}\rangle]$ lies in $(\mathrm{SI}^{\le 0}_{\mathrm{Ran}})^{\le 0}$, and $(i^\lambda)_*\omega_{S^\lambda_{\mathrm{Ran}}}[-\langle\lambda,\check{\rho}\rangle]$ lies in $(\mathrm{SI}^{\le 0}_{\mathrm{Ran}})^{\ge 0}$.

([26], Prop. 2.2.2) allows to define the semi-infinite IC-sheaf of each $T(\mathcal{O})N(F)$-orbit attached to $\lambda\in\Lambda^{neg}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$. Do we get a formula as a colimit????

1.12. **Spreading right lax symmetric monoidal functors.**

1.12.1. The following construction is maybe useful in general. Let's use the notations of [52]. Let $\Lambda\in CAlg(Sets)$. We think of it as the "category of highest weights". For $\lambda_i\in\Lambda$ the operation in Λ is denoted $\lambda_1+\lambda_2$, the neutral object is denoted $0\in\Lambda$.

Let $(C(X),\otimes^!)\in CAlg(Shv(X)-mod)$. Assume given a right-lax symmetric monoidal functor

$$Irr:\Lambda\to(C(X),\otimes^!),\ \lambda\mapsto V^\lambda$$

In particular, $V^0\tilde{\to}1_C$. We write Irr_C, V^λ_C if we need to express the dependence on $C(X)$. We think of V^λ as an analog of the irreducible representation of h.w. λ. For $\lambda_i\in\Lambda$ we have by definition a canonical map in $C(X)$

$$u:V^{\lambda_1}\otimes^! V^{\lambda_2}\to V^{\lambda_1+\lambda_2}.$$

Let now $I\in fSets$. So, we have $C_{X^I}\in Shv(X^I)-mod$. Let $\underline{\lambda}:I\to\Lambda$ be a map. We define the object $V^{\underline{\lambda}}\in C_{X^I}$ as follows. We think of it as a "spread representation of h.w. $\underline{\lambda}$".

We write $C^{\otimes J}(X)$ for the tensor power of $C(X)$ in $Shv(X)-mod$. Given objects $V_j\in C(X)$, our notation $\underset{j\in J}{\otimes}V_j$ means the corresponding object of $C^{\otimes J}(X)$ not to be confused with

$$\underset{j\in J}{\otimes^!}V_j\in C(X)$$

First, define a functor $\mathcal{F}_{\underline{\lambda},Irr}:Tw(I)\to C_{X^I}$, it sends $(I\to J\xrightarrow{\phi}K)$ to the image of

$$V^{\otimes\phi}:=\underset{k\in K}{\boxtimes}(\underset{j\in J_k}{\otimes}V^{\lambda_j})$$

under $\underset{k\in K}{\boxtimes}(C^{\otimes J_k}(X))\to C_{X^I}$, where $\lambda_j=\sum\limits_{i\in I_j}\underline{\lambda}(i)$ for $j\in J$. Given a map from $(I\to J_1\to K_1)$ to $(I\to J_2\to K_2)$ in $Tw(I)$, we get the corresponding transition morphism in $\underset{k\in K_2}{\boxtimes}C^{\otimes(J_2)_k}(X)$ and hence in C_{X^I}

$$\mathcal{F}_{\underline{\lambda},Irr}(I\to J_1\to K_1)\to\mathcal{F}_{\underline{\lambda},Irr}(I\to J_2\to K_2)$$

as follows. First, for the diagram (defining the transition functor for $\mathcal{F}_{I,C}$)

$$\underset{k\in K_1}{\boxtimes} C^{\otimes (J_1)_k}(X) \xrightarrow{m} \underset{k\in K_1}{\boxtimes} C^{\otimes (J_2)_k}(X) \to \underset{k\in K_2}{\boxtimes} C^{\otimes (J_2)_k}(X)$$

we get a natural map

$$m(\underset{k\in K_1}{\boxtimes}(\underset{j\in (J_1)_k}{\otimes} V^{\lambda_j}))\widetilde{\to} \underset{k\in K_1}{\boxtimes}(\underset{j\in (J_2)_k}{\otimes}(\underset{j'\in (J_1)_j}{\otimes^!} V^{\lambda_{j'}})) \to \underset{k\in K_1}{\boxtimes}(\underset{j\in (J_2)_k}{\otimes} V^{\lambda_j}) \tag{37}$$

in $\underset{k\in K_1}{\boxtimes} C^{\otimes (J_2)_k}(X)$. Here the second arrow in (37) comes from the right-lax structure on Irr giving the maps

$$\underset{j'\in (J_1)_j}{\otimes^!} V^{\lambda_{j'}} \to V_j^{\lambda}$$

for $j\in J_2$.

Further, for $\vartriangle\colon X^{K_1}\to X^{K_2}$ we have

$$\vartriangle^!(\underset{k\in K_2}{\boxtimes}(\underset{j\in (J_2)_k}{\otimes} V^{\lambda_j}))\widetilde{\to} \underset{k\in K_1}{\boxtimes}(\underset{j\in (J_2)_k}{\otimes} V^{\lambda_j})$$

So, we compose the previous map with

$$\vartriangle_! \underset{k\in K_1}{\boxtimes}(\underset{j\in (J_2)_k}{\otimes} V^{\lambda_j})\widetilde{\to} \vartriangle_!\vartriangle^!(\underset{k\in K_2}{\boxtimes}(\underset{j\in (J_2)_k}{\otimes} V^{\lambda_j})) \to \underset{k\in K_2}{\boxtimes}(\underset{j\in (J_2)_k}{\otimes} V^{\lambda_j})$$

Everywhere for $j\in J_2$,

$$\lambda_j = \sum_{j'\in (J_1)_{j'}} \lambda_{j'}$$

This concludes the definition of $\mathcal{F}_{\underline{\lambda},Irr}$. We write $\mathcal{F}_{\underline{\lambda},Irr_C}$ if we need to express the dependence on $C(X)$.

Finally, we set

$$V^{\underline{\lambda}} = \underset{Tw(I)}{\operatorname{colim}} \mathcal{F}_{\underline{\lambda},Irr}$$

in C_{X^I}. That is,

$$V^{\underline{\lambda}}\widetilde{\to} \underset{(I\to J\to K)\in Tw(I)}{\operatorname{colim}} \underset{k\in K}{\boxtimes}(\underset{j\in J_k}{\otimes} V^{\lambda_j}).$$

The construction explains the meaning of the objects $V^{\underline{\lambda}}$ used by Dennis in [26].

Lemma 1.12.2. *Let $I\in fSets$. The object $V^{\underline{\lambda}}\in C_{X^I}$ factorizes naturally. Namely, if $\phi: I\to I'$ is a map in $fSets$ we get canonically*

$$V^{\underline{\lambda}}\mid_{X^I_{\phi,d}} \widetilde{\to}(\underset{i\in I'}{\boxtimes} V^{\underline{\lambda}^i})\mid_{X^I_{\phi,d}}.$$

Here for $i\in I'$, $\underline{\lambda}^i: I_i\to\Lambda$ is the restriction of $\underline{\lambda}$.

Proof. Recall the full subcategory $Tw(I)_\phi\subset Tw(I)$ from ([52], Sect. 2.1.15). The latter inclusion is zero-cofinal. Now for $(I\to J\to K)\in Tw(I)$, $X^K\times_{X^I} X^I_{\phi,d}$ is empty unless $I'\in Q(K)$, so $V^{\underline{\lambda}}\mid_{X^I_{\phi,d}}$ rewrites as

$$\underset{(I\to J\to K)\in Tw(I)_\phi}{\operatorname{colim}} \underset{i'\in I'}{\boxtimes}(\underset{k\in K_{i'}}{\boxtimes} \underset{j\in (J_{i'})_k}{\otimes} V^{\lambda_j})\mid_{X^I_{\phi,d}},$$

which identifies with the RHS. ∎

1.12.3. For $I\in fSets$ view $\mathrm{Map}(I,\Lambda)$ as an object of $CAlg(Sets)$ with the pointwise operation. Consider the functor

$$fSets\to CAlg(Sets),\ I\mapsto \mathrm{Map}(I,\Lambda).$$

It sends $f:I\to I'$ in $fSets$ to the direct image map $f_*:\mathrm{Map}(I,\Lambda)\to\mathrm{Map}(I',\Lambda)$. That is, $(f_*\underline{\lambda})(i')=\sum\limits_{i\in I_{i'}}\underline{\lambda}(i)$. Let $\overline{\Lambda}=\lim\limits_{I\in fSets}\mathrm{Map}(I,\Lambda)$ be the limit of this functor in $CAlg(Sets)$ or, equivalently, in $Sets$ or in Spc.

1.12.4. If $\phi:I\to I'$ is a map in $fSets$ then for $\vartriangle: X^{I'}\to X^I$ under the equivalence $C_{X^I}\otimes_{Shv(X^I)}Shv(X^{I'})\widetilde{\to}C_{X^{I'}}$ one has canonically

$$\vartriangle^! V^{\underline{\lambda}}\widetilde{\to} V^{\underline{\lambda}'}$$

in $C_{X^{I'}}$, where $\underline{\lambda}':I'\to\Lambda$ is given by $\underline{\lambda}'=\phi_*\underline{\lambda}$. This is proved as in ([52], 2.1.7).

Moreover, for $\overline{\lambda}=\{\underline{\lambda}_I\}_{I\in fSets}\in\overline{\Lambda}$ with $\underline{\lambda}_I\in\mathrm{Map}(I,\Lambda)$, the collection $\{V^{\underline{\lambda}_I}\}_{I\in fSets}$ is equipped with a coherent system of higher compatibilites, so defines as object of $\lim\limits_{I\in fSets} C_{X^I}\widetilde{\to}\mathrm{Fact}(C)$. We denote this section by

$$V^{\overline{\lambda}}\in\mathrm{Fact}(C)$$

1.12.5. There is also a direct definition of $V^{\overline{\lambda}}$ along the lines of ([52], 2.1.8). Namely, one defines a functor

$$\mathcal{F}_{\overline{\lambda},Irr}:Tw(fSets)\to\mathrm{Fact}(C)$$

sending $(J\to K)\in Tw(fSets)$ to

$$\underset{k\in K}{\boxtimes}\big(\underset{j\in J_k}{\otimes}V^{\underline{\lambda}_J(j)}\big).$$

The transition morphisms are defined in the same way as for $\mathcal{F}_{\underline{\lambda},Irr}$. One has canonically

$$V^{\overline{\lambda}}\widetilde{\to}\lim_{Tw(fSets)}\mathcal{F}_{\overline{\lambda},Irr}$$

in $\mathrm{Fact}(C)$.

1.12.6. *Example.* For $C(X)=Shv(X)$ let $I\in fSets$ and let $\underline{\lambda}:I\to\Lambda$ be the constant zero map, which we will denote $\underline{0}$. Then $V^{\underline{0}}\widetilde{\to}\omega_{X^I}$.

1.12.7. Let $C(X)\to D(X)$ be a map in $CAlg(Shv(X)-mod)$ and $I\in fSets,\underline{\lambda}\in\mathrm{Map}(I,\Lambda)$. Let Irr_D be the composition $\Lambda\overset{\mathrm{Irr}_C}{\to}C(X)\to D(X)$. We get the functor $\mathcal{F}_{\underline{\lambda},Irr_D}:Tw(I)\to D_{X^I}$ as above and the corresponding objects

$$V_D^{\underline{\lambda}}=\underset{Tw(I)}{\mathrm{colim}}\,\mathcal{F}_{\underline{\lambda},Irr_D}\in D_{X^I}.$$

Then the image of $V_C^{\underline{\lambda}}$ under $C_{X^I}\to D_{X^I}$ identifies canonically with $V_D^{\underline{\lambda}}$.

In particular, $V^{\underline{0}}\in C_{X^I}$ is the unit of $(C_{X^I},\otimes^!)$.

1.12.8. Let us be in the situation of Section1.12.1. Since $\mathrm{Map}(I,\Lambda)$ is a set, we defined a functor

$$\mathrm{Map}(I,\Lambda)\to(C_{X^I},\otimes^!),\ \underline{\lambda}\mapsto V^{\underline{\lambda}}$$

Let us equip this functor with a right-lax symmetric monoidal structure. This is done in [52].

1.12.9. Let G be connected reductive over k, use notations of [52]. Let $I\in fSets$, $\underline{\lambda}: I\to\Lambda^+$. We have the object $V^{\underline{\lambda}}\in\mathrm{Rep}(\check{G})_{X^I}$ defined in ([26], 2.6.2). The more general pattern is proposed in ([52], Section 6). Consider now the functor $\Lambda^+\to\mathrm{Rep}(\check{G})$, $\lambda\mapsto(V^\lambda)^*$. We equip is with the right-lax symmetric monoidal structure via the maps

$$v^{\lambda,\mu}:(V^\lambda)^*\otimes(V^\mu)^*\to(V^{\lambda+\mu})^*.$$

Applying the construction of ([52], Section 6) to this functor, we get an object of $\mathrm{Rep}(\check{G})_{X^I}$ that we denote by $(V^{\underline{\lambda}})^*$.

Let us show that $(V^{\underline{\lambda}})^*$ and $V^{\underline{\lambda}}$ are canonically dual to each other in the symmetric monoidal category $(\mathrm{Rep}(\check{G})_{X^I},\otimes^!)$.

View the map $\Lambda^+\to\mathrm{Rep}(\check{G})$, $\lambda\mapsto V^\lambda\otimes(V^\lambda)^*$ as a right-lax symmetric monoidal functor (tensor product of two such). Similarly, the constant map $\lambda\mapsto e$ is symmetric monoidal. Now the unit of the duality $e\to V^\lambda\otimes(V^\lambda)^*$ becomes a morphism in $\mathrm{Fun}^{rlax}(\Lambda^+,\mathrm{Rep}(\check{G}))$. Here we used the commutativity of the diagram in ([15], proof of Lemma 2.2.11).

By ([52], Section 6) for any $\underline{\lambda}: I\to\Lambda^+$ the functor $\mathrm{Fun}^{rlax}(\Lambda^+,\mathrm{Rep}(\check{G}))\to C_{X^I}$, $Irr^V\mapsto V^{\underline{\lambda}}$ is symmetric monoidal. It is easy to see also that the functor $\Lambda\to\mathrm{Rep}(\check{G})$, $\lambda\mapsto V^\lambda$ is dualizable in the symmetric monoidal category $\mathrm{Fun}^{rlax}(\Lambda^+,\mathrm{Rep}(\check{G}))$, its dual is the functor $\lambda\mapsto(V^\lambda)^*$. It follows that $V^{\underline{\lambda}}$ and $(V^{\underline{\lambda}})^*$ are canonically dual to each other in $(C_{X^I},\otimes^!)$ for $C=\mathrm{Rep}(\check{G})$.

1.12.10. For the proof of ([26], 5.4.7). Here $I\in fSets$, $C\in\mathrm{Rep}(\check{T}\times\check{G})_{X^I}-mod$. We want to show that the composition

$$\mathcal{O}(\check{N}\backslash\check{G})_{X^I}-mod(C)\to\mathcal{O}(\check{G})_{X^I}-mod(C)\stackrel{oblv}{\to}C$$

which is defined as $c\mapsto\mathcal{O}(\check{G})_{X^I}\otimes_{\mathcal{O}(\check{N}\backslash\check{G})_{X^I}}c$ can be rewritten as

$$\underset{\underline{\lambda}\in\mathrm{Map}(I,\Lambda^+)}{\mathrm{colim}}\ e^{-\underline{\lambda}}*c*V^{\underline{\lambda}}$$

It is understood that $\mathrm{Map}(I,\Lambda^+)$ is equipped with the relation: $\underline{\lambda}_1\le\underline{\lambda}_2$ iff $\underline{\lambda}_2-\underline{\lambda}_1\in\mathrm{Map}(I,\Lambda^+)$, here $\mathrm{Map}(I,\Lambda^+)$ is equipped with the pointwise sum.

Proof. Recall that

$$\mathcal{O}(\check{G})_{X^I}-mod(C)\widetilde{\to}\mathcal{O}(\check{G})_{X^I}-mod(\mathrm{Rep}(\check{G}\times\check{T})\otimes_{\mathrm{Rep}(\check{G}\times\check{T})}C.$$

It siffuces to do the universal case and show that

$$\underset{\underline{\lambda}\in\mathrm{Map}(I,\Lambda^+)}{\mathrm{colim}}e^{-\underline{\lambda}}*\mathcal{O}(\check{N}\backslash\check{G})_{X^I}*V^{\underline{\lambda}}\widetilde{\to}\mathcal{O}(\check{G})_{X^I}$$

in $\mathrm{Rep}(\check{G}\times\check{T})$. Indeed, then

$$\mathcal{O}(\check{G})_{X^I}\otimes_{\mathcal{O}(\check{N}\backslash\check{G})_{X^I}} c\widetilde{\to} \underset{\underline{\lambda}\in \mathrm{Map}(I,\Lambda^+)}{\mathrm{colim}} (e^{-\underline{\lambda}} * \mathcal{O}(\check{N}\backslash\check{G})_{X^I} * V^{\underline{\lambda}})\otimes_{\mathcal{O}(\check{N}\backslash\check{G})_{X^I}} c\widetilde{\to} \underset{\underline{\lambda}\in \mathrm{Map}(I,\Lambda^+)}{\mathrm{colim}}\; e^{-\underline{\lambda}} * c * V^{\underline{\lambda}}$$

We used here that $\mathcal{O}(\check{N}\backslash\check{G})_{X^I}\otimes_{\mathcal{O}(\check{N}\backslash\check{G})_{X^I}} c\widetilde{\to} c$.

The proof is in the next section, where a more general fact is established. □

1.12.11. The $\overline{\mathrm{Bun}}_P$-case from [15]. Pick $I\in fSets$. Let $C\in \mathrm{Rep}(\check{G}\times\check{M}_{ab})_{X^I}-mod$. We want to show that the composition

$$\mathcal{O}(\check{G}/[\check{P}^-,\check{P}^-])_{X^I}-mod(C)\to \mathcal{O}(\check{G}/[\check{M},\check{M}])_{X^I}-mod(C)\overset{\mathrm{oblv}}{\to} C$$

defined as

$$c\mapsto \mathcal{O}(\check{G}/[\check{M},\check{M}])_{X^I}\otimes_{\mathcal{O}(\check{G}/[\check{P}^-,\check{P}^-])_{X^I}} c$$

identifies with the functor

$$c\mapsto \underset{\underline{\lambda}\in \mathrm{Map}(I,\Lambda^+_{M,ab})}{\mathrm{colim}} e^{\underline{\lambda}} * c * (V^{\underline{\lambda}})^*$$

Proof. As in the previous subsection, it suffices to do the universal case and show that

$$\underset{\underline{\lambda}\in \mathrm{Map}(I,\Lambda^+_{M,ab})}{\mathrm{colim}} e^{\underline{\lambda}} * \mathcal{O}(\check{G}/[\check{P}^-,\check{P}^-])_{X^I} * (V^{\underline{\lambda}})^*\widetilde{\to} \mathcal{O}(\check{G}/[\check{M},\check{M}])_{X^I}$$

in $\mathrm{Rep}(\check{G}\times\check{M}_{ab})$. Here we used the notations of ([52], Section 6). Set for brevity

$$A=\mathcal{O}(\check{G}/[\check{P}^-,\check{P}^-]),\; B=\mathcal{O}(\check{G}/[\check{M},\check{M}])\in CAlg(\mathrm{Rep}(\check{G}\times\check{M}_{ab})).$$

By ([52], 6.1.17) applied to $C(X)=\mathrm{Rep}(\check{G}\times\check{M}_{ab})\otimes Shv(X)$, for any $\underline{\lambda}\in\mathrm{Map}(I,\Lambda^+_{M,ab})$ we have canonically

$$e^{\underline{\lambda}}*\mathcal{O}(\check{G}/[\check{P}^-,\check{P}^-])_{X^I}*(V^{\underline{\lambda}})^*\widetilde{\to} \underset{(I\to J\to K)\in \mathrm{Tw}(I)}{\mathrm{colim}} \Big((\underset{j\in J_k}{\otimes} e^{\lambda_j})\otimes A^{\otimes J_k}\otimes (\underset{j\in J_k}{\otimes} (V^{\lambda_j})^*)\Big)\otimes\omega_{X^K},$$

here the tensor product in big brackets is taken in $\mathrm{Rep}(\check{G}\times\check{M}_{ab})$.

For each $j\in J_k$ in the above formula take the natural map

$$e^{\lambda_j}\otimes A\otimes (V_{\lambda_j})^*\to B$$

given by ([15], formula (22) in the proof of 2.2.13) and tensor them over $j\in J_k$. The resulting map

$$\Big((\underset{j\in J_k}{\otimes} e^{\lambda_j})\otimes A^{\otimes J_k}\otimes (\underset{j\in J_k}{\otimes} (V^{\lambda_j})^*)\Big)\otimes\omega_{X^K}\to B^{\otimes J_k}\otimes\omega_{X^K}$$

is a morphism in $\mathrm{Fun}(\mathrm{Tw}(I), C_{X^I})$. Taking the colimit over $\mathrm{Tw}(I)$ this gives a morphism

$$e^{\underline{\lambda}} * \mathcal{O}(\check{G}/[\check{P}^-,\check{P}^-])_{X^I} * (V^{\underline{\lambda}})^*\to B_{X^I}$$

in C_{X^I}. It remains to show this is an isomorphism.

Since all the involved objects here factorize, our claim follows from the same claim as a point given by ([15], 2.2.13). □

1.13. **For** [26] **again.**

1.13.1. The claim in his 3.5.3 that $\mathrm{Map}_Y(X - \mathcal{I}, \mathbb{A}^1)$ is represented by $\mathbb{A}^\infty \times Y$ is wrong.

Here is the correct version.

Proposition 1.13.2. *Let $Y \in \mathrm{Sch}_{ft}, \mathcal{I} \subset \mathrm{Map}(Y, X)$ be a finite subset defining a Y-point of* Ran*, so we have $\Gamma_{\mathcal{I}} \subset Y \times X$. Let $U_I = Y \times X - \Gamma_{\mathcal{I}}$. Consider the prestack $\mathcal{Y} : (\mathrm{Sch}^{aff}_{/Y})^{op} \to \mathrm{Spc}$, sending $S \to Y$ to $\mathrm{Map}_Y(S \times_Y U_I, \mathbb{A}^1)$. Then $\mathcal{Y} \to Y$ is universally homologically contractible.*

Proof. There is I filtered and a diagram $i \mapsto \mathcal{E}_i$ for $i \in I$, where $\mathcal{E}_i$ is a vector bundle on Y, and for $i \to j$ in I the map $\mathcal{E}_i \to \mathcal{E}_j$ is a map of vector bundles on Y such that $\pi_*\mathcal{O} \widetilde{\to} \mathrm{colim}_{i \in I}\, \mathcal{E}_i$. Here $\pi : U_I \to Y$.

Indeed, take $I = \{n \in \mathbb{N} \mid n \geq n_0\}$ for n_0 large enough, and $\mathcal{E}_n = \bar{\pi}_*\mathcal{O}(n\Gamma_{\mathcal{I}})$, where $\Gamma_{\mathcal{I}}$ is viewed as an effective relative Cartier divisor on $Y \times X$, and $\bar{\pi} : Y \times X \to Y$. Our functor $\mathcal{Y}$ is $\underset{i \in I}{\mathrm{colim}}\, \mathrm{Spec}_Y(\mathrm{Sym}\, \mathcal{E}_i^*)$. The claim follows now from ([23], Lemma 1.3.6). □

1.13.3. In the proof of 3.2.5 in the 3rd displayed formula the correct answer is

$$(j^\lambda_{glob})_!(p^\lambda_{glob})^!\mathcal{F} \widetilde{\to} (\bar{p}^\lambda_{glob})^!(\mathcal{F}) \otimes^! (j^\lambda_{glob})_!\omega$$

Indeed, this is a general phenomenon. Let $j : Y_0 \hookrightarrow Y$ be an open immersion, $f : Y \to S$ be a map with $S \in \mathrm{Sch}_{ft}$ smooth, assume $f_0 = f \circ j$ smooth. Assume that $j_!\omega$ is ULA over S. Then for $L \in Shv(S)^c$ one has

$$j_! f_0^! L \widetilde{\to} (j_!\omega) \otimes^! f^! L$$

This follows from the the consequences of ULA properties in [10].

Appendix A. About Schieder's correction

A.0.1. Example: $G = \mathrm{PGL}_2$, B standard Borel, $N = [B, B]$. Then G/N is the variety of nilpotent 2×2 non-zero matrices $\mathcal{N} - 0$, so $\overline{G/N}$ is the variety of nilpotent 2×2 matrices.

Let $\tilde{G} = \mathrm{GL}_2$, $\tilde{B}$ the standard Borel in $\tilde{G}$. Then $\tilde{G}/N = (E - \{0\}) \times (\det E - \{0\})$, where E is the srandard representation of G. So, the affine closure is $\overline{\tilde{G}/N} = E \times (\det E - \{0\}) \subset E \times \det E \widetilde{\to} \mathbb{A}^3$. The complement of $\tilde{G}/N$ in $E \times (\det E - \{0\})$ is $\{0\} \times (\det E - \{0\})$, and the center $\mathbb{G}_m \widetilde{\to} Z$ of GL_2 acts on $\det E - \{0\}$ by the character $z \mapsto z^2$. Now $(\det E - \{0\})/Z$ is not a point.

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