1. Comments to: The stack of local systems with restricted variation and geometric Langlands theory with nilpotent singular support

1.1. **For version of Jan 24, 2022.**

1.1.1. For 1.1.9. Let S be a scheme of finite type, X be a smooth scheme of finite type. To see that $\mathrm{QLisse}(X)$ is left-complete, argue as follows. We know that $Shv(X)$ is left complete, so for $x \in \mathrm{QLisse}(X)$ the natiral map $x \to \lim_n \tau^{\geq -n}x$ with limit taken in $Shv(X)$ is an isomorphism. So, this also holds in $\mathrm{QLisse}(X)$, hence $\mathrm{QLisse}(X)$ has convergent Postnikov towers. Let now $\{c^n\} \in \lim_n \mathrm{QLisse}^{[-n,0]}$ and $c = \lim_n c^n$ calculated in $Shv(X)$. Then we know already that for any n, $\tau^{\geq -n}c \to c^n$ is an isomorphism. This shows that $c \in \mathrm{QLisse}(X)$, so the limit of the diagram $\{c^n\}$ taken in $\mathrm{QLisse}(X)$ and in $Shv(X)$ are the same. Moreover, the truncation functor for $\mathrm{QLisse}(X)$ is the restriction of the truncation functor for $Shv(X)$. So, the natural map $\tau^{\geq -n}c \to c^n$ in $\mathrm{QLisse}(X)$ is an isomorpism. That is, their functor $\hat{C} \to C$ given by (B.2) is fully faithful. So, $\mathrm{QLisse}(X)$ is left complete. We did not need the fact that $\mathrm{QLisse}(X)^{\leq 0} \subset \mathrm{QLisse}(X)$ is closed under countable product, nor to apply (HA, 1.2.1.19).

The t-structure on $Shv(S)$ is accessible and compatible with filtered colimits (see [11], ch. II.1, Lm. 1.2.4) proved in ([16], 10.3.3). Moreover, the t-structure on $Shv(S)$ is compactly generated in the sense of ([21], 6.3.8). Recall that each object of $Shv(S)^{constr}$ is bounded. So, by ([18], 1.2.36), $Shv(S)$ is right complete. We also know it is left complete by ([1], Th. 1.1.4).

$\mathrm{QLisse}(X)$ is equipped with a t-structure. By ([13], 5.3.5.11), $\mathrm{Ind}(Lisse(X)) \subset \mathrm{QLisse}(X)$ is a full subcategory. We equip $\mathrm{Ind}(Lisse(X))$ with the t-structure defined by ([11], ch. II.1, Lm. 1.2.4) by ind-extention of the t-structure on $Lisse(X)$. Then $\mathrm{Ind}(Lisse^{\leq 0}(X)) \subset \mathrm{Ind}(Shv(X)^{\leq 0,constr})$ and $\mathrm{Ind}(Lisse^{\geq 0}(X)) \subset \mathrm{Ind}(Shv(X)^{\geq 0,constr})$, so the inclusion $\mathrm{Ind}(Lisse(X)) \to \mathrm{QLisse}(X)$ is t-exact. In particular, for $n \in \mathbb{Z}$ the functor

$$\mathrm{IndLisse}(X)^{\geq -n} \subset \mathrm{QLisse}(X)^{\geq -n} \qquad (1)$$

is fully faithful. Let us show it is essentially surjective. Its image is closed under filtered colimits and clearly containes $\mathrm{QLisse}(X)^{\heartsuit}[m]$ for $m \leq n$. Finally, for any $z \in \mathrm{QLisse}(X)^{\geq -n}$ we have $z \widetilde{\to} \operatorname{colim}_{m\in\mathbb{Z}} \tau^{\leq m}z$ in $Shv(X)$, because $Shv(X)$ is right complete, see ([16], 4.0.10). But $\tau^{\leq m}z \in \mathrm{IndLisse}(X)^{\geq -n}$, and (1) is closed under filtered colimits, so (1) is an isomorphism.

1.1.2. Let S be a scheme of finite type, work in the constructible context.

If $K \in Shv(S)^{constr}$ let $K^\vee = \mathcal{H}om(K, e)$, where e is the constant sheaf on S. We have a natural map $K \to \mathcal{H}om(K^\vee, e)$ corresponding to the natural map $K \otimes \mathcal{H}om(K, e) \to e$. The full subcategory of $K \in Shv(S)^{constr}$ for which this map is an isomorphism is closed under extensions and shifts and contains local systems. So, it contains the full subcategory $Lisse(S) \subset Shv(S)^{constr}$. So, for $E \in Lisse(S)$, the map $E \otimes \omega \to \mathbb{D}(E^\vee)$ is an isomorphism. Recall that $Lisse(S) \subset Shv(S)$ is the full subcategory of objects dualizable with respect to the $\otimes$-monoidal structure, and $E^\vee$ is the dual of $E \in Lisse(S)$ with respect to this monoidal structure. Let $E \in Lisse(S)$, $K \in Shv(S)$. Then the above easily gives $K \otimes^! (E \otimes \omega) \widetilde{\to} K \otimes E$.

Let us show that $E\otimes\omega$ is dualizable with respect to the $\otimes^!$-monoidal structure on $Shv(S)$. For $F_i\in Shv(S)$ we get

$$\mathcal{H}om(F_1, F_2\otimes^! (E\otimes\omega))\widetilde{\to}\mathcal{H}om(F_1,\mathcal{H}om(E^*,F_2))\widetilde{\to}\mathcal{H}om(F_1\otimes E^*,F_2)\widetilde{\to}$$
$$\mathcal{H}om(F_1\otimes^! (E^*\otimes\omega),F_2),$$

here $\mathcal{H}om(F_1,F_2)\in \mathrm{Vect}$ means relative inner hom for Vect-action on $Shv(S)$. I don't see why the dualizability of $K\in Shv(S)$ for the $\otimes^!$-monoidal structure implies that $K\in Lisse(S)$.

Note that $\mathrm{IndLisse}(S)^c = Lisse(S)$, and each object of $Lisse(S)$ is dualizable in $(\mathrm{IndLisse}(S),\otimes^!)$. Besides, $\otimes^!$ sends $Lisse(S)\times Lisse(S)$ to $Lisse(S)$, and $\omega\in \mathrm{IndLisse}(S)^c$ because $\omega\in Lisse(S)$. So, by ([11], ch. I.1, 9.1.5), $\mathrm{IndLisse}(S)$ is rigid.

1.1.3. For the proof of B.1.9. The presentation changed in the new version.

If C is a stable category with a t-structure, $a\in C^{\heartsuit}, c\in C$ then $\mathrm{Ext}^k_C(a,c) = \mathrm{Hom}_{hC}(a[-k],c)\widetilde{\to}\mathrm{Hom}_{hC}(a[-k],\tau^{\geq k}c)$.

Let now $C=\mathrm{Ind}(D^b(\mathcal{A}))$, where $\mathcal{A}$ is an abelian category, say with enough projective objects. Let $a\in\mathcal{A}$, $c\in \mathrm{Ind}(D^b(\mathcal{A}))$ and $k>0$. Then a map $a\to \mathrm{H}^k(c)$ is the same as a map $a[-k]\to\tau^{\geq k}c$ in C, and $\mathrm{H}^0(\mathcal{H}om(a[-k],\tau^{\geq k}c)$ is a nonzero vector space. Pick a surjection $a'\to a$, where a' is projective. Then $\mathrm{Ext}^{k+1}_C(a',\tau^{<k}(c))=0$ by ([14], 1.3.3.7), so the map $\mathrm{Ext}^k(a',c)\to \mathrm{Ext}^k(a',\tau^{\geq k}(c))$ is surjective, in particular, $\mathrm{Ext}^k(a',c)\neq 0$. This constradicts the fact that $\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a',c)\in \mathrm{Vect}^{\leq 0}$.

1.1.4. It follows from Appendix C about categorical $K(\pi,1)$ that the t-structure on $\mathrm{QLisse}(X)$ is compactly generated for any smooth complete curve X.

1.1.5. in 8.1.1, $\mathrm{QLisse}(X)$ is considered with respect to the $*$-monoidal structure. In the definition of a smooth action in 8.1.1, $fSet$ is the category of finite sets (possibly empty) and any maps between them. The functor $fSet\to \mathrm{DGCat}^{Mon}_{cont}$, $I\mapsto \mathrm{Rep}(G)^{\otimes I}$ sends $\alpha: I\to J$ to the product map along α, similarly for $I\mapsto \mathrm{QLisse}(X)^{\otimes I}$.

1.1.6. In 11.1.1, $fSet$ denotes the category of finite sets (maybe empty) and all morphisms between them. We have an adjoint pair $l:(fSet)^{op}\leftrightarrows Tw(fSet): r$, where $l(J)=(\emptyset\to J)$ and $r(I\to J)=J$.

For 11.1.5. Let $C\in CAlg(\mathrm{DGCat}_{cont})$, let us show that $C_{\mathrm{Ran}}\widetilde{\to}\mathrm{Vect}$. We have the functor $q:(fSet)^{op}\to \mathrm{DGCat}_{cont}$, $J\mapsto Shv(X^J)$. For a map $\alpha: J_2\to J_1$ in $fSet$, that is, to $\alpha: J_1\to J_2$ in $(fSet)^{op}$ it attached the morphism $\vartriangle_*: Shv(X^{J_1})\to Shv(X^{J_2})$ for $\vartriangle: X^{J_1}\to X^{J_2}$. By ([16], 2.2.39), qr is the LKE of q along $l:(fSet)^{op}\to Tw(fSet)$. So,

$$\underset{(I\to J)\in Tw(fSet)}{\mathrm{colim}}\; qr(I\to J)\widetilde{\to}\underset{J\in (fSet)^{op}}{\mathrm{colim}}\; q(J)\widetilde{\to} q(\emptyset)\widetilde{\to}\mathrm{Vect},$$

because $\emptyset\in fSet$ is initial.

For 11.1.6. The symmetric monoidal structure on $Tw(fSet)$ is given by sending $(I_1\to J_1),(I_2\to J_2)$ to $(I_1\sqcup I_2\to J_1\sqcup J_2)$, see ([19], 1.2.3). Their functor (11.1) is indeed right-lax symetric, and sends the unit object $(\emptyset\to\emptyset)$ to $\mathrm{Vect}\in \mathrm{DGCat}_{cont}$. So, by their B.1.8, its colimit lies in $CAlg(\mathrm{DGCat}_{cont})$. In case, this could use the fact that X is proper, as we need that for two maps of finite sets $J_2\to J_1, J_2'\to J_1'$ and the corresponding $\triangle$: $X^{J_1}\to X^{J_2}$, $\triangle'$: $X^{J'_1}\to X^{J'_2}$ and $s: X^{J_1\sqcup J'_1}\to X^{J_2\sqcup J'_2}$ the diagram commutes

$$\begin{array}{ccc} Shv(X^{J_1})\otimes Shv(X^{J'_1}) & \stackrel{\boxtimes}{\to} & Shv(X^{J_1\sqcup J'_1}) \\ \downarrow{\scriptstyle \triangle_*\otimes\triangle'_*} & & \downarrow{\scriptstyle s_*} \\ Shv(X^{J_2})\otimes Shv(X^{J'_2}) & \stackrel{\boxtimes}{\to} & Shv(X^{J_2\sqcup J'_2}) \end{array}$$

1.1.7. For B.1.1: my understanding is that the notation $\mathrm{Fun}^{rlax}(O,A)$ means non-unital right-lax monoidal functors, that is, the map $1\to f(1)$ is not necessarily an isomorphism.

Their Pp. B.2.9 is precisely ([19], 1.3.4).

1.1.8. For 8.2.3. Let O be a symmetric monoidal presentable category, in which the tensor product preserves colimits separately in each variable. Let $A\in CAlg(O)$, B be a unital cocommutative coalgebra in O, which is dualizable in O. Then the commutative algebra structure on $coEnd(A,B)=\mathrm{colim}_{(I\to J)\in Tw(fSet)} A^{\otimes I}\otimes B^{\otimes J}$ in O is as follows. Let $ins_{(I\to J)}: A^{\otimes I}\otimes B^{\otimes J}\to coEnd(A,B)$ be the natural map. For $(I_1\to J_1),(I_2\to J_2)\in Tw(fSet)$ the composition

$$A^{\otimes I_1}\otimes B^{\otimes J_1}\otimes A^{\otimes I_2}\otimes B^{\otimes J_2}\stackrel{ins_{(I_1\to J_1)}\otimes ins_{(I_2\to J_2)}}{\to} coEnd(A,B)\otimes coEnd(A,B)\stackrel{mult}{\to} coEnd(A,B)$$

coincides with

$$ins_{(I_1\sqcup I_2\to J_1\sqcup J_2)}: A^{\otimes I_1\sqcup I_2}\otimes B^{\otimes J_1\sqcup J_2}\to coEnd(A,B)$$

The unit of $coEnd(A,B)$ is $ins_{(\emptyset\to\emptyset)}: 1\to coEnd(A,B)$.

1.1.9. In 8.2.4 they claim the following. Let $O\in CAlg(1-\mathcal{C}\mathrm{at})$, which is cocomplete and such that the monoidal product preserves colimits separately in each variable. Let B be a unital cocommutative coalgebra in O, which is dualizable, so $B^\vee$ is a unital commutative algebra. The functor $R: O\to B^\vee-mod(O)$, $M\mapsto B^\vee\otimes M$ admits a left adjoint L sending N to $B\otimes_{B^\vee}N$ by ([16], 3.2.5). Now, the functor R gives $\mathcal{R}: CAlg(O)\to CAlg(B^\vee-mod(O))$. The left adjoint $\mathcal{L}$ to $\mathcal{R}$ is given by their Pp. B.2.9. So,

$$\mathcal{L}(N)\widetilde{\to}\mathop{\mathrm{colim}}_{(I\to J)\in Tw(fSet)}\ \bigotimes_{j\in J} L(N^{\otimes I_j}),$$

where the tensor power of N is taken in $B^\vee-mod(O)$. Here $fSet$ is the category of finite sets. Now for $A\in CAlg(O)$ we get

$$\mathcal{L}(B^\vee\otimes A)\widetilde{\to}\mathop{\mathrm{colim}}_{(I\to J)\in Tw(fSet)} B^{\otimes J}\otimes A^{\otimes I}$$

Indeed, $B\otimes_{B^\vee}(B^\vee\otimes A)\widetilde{\to}B\otimes A$. They define $coEnd(A,B):=\mathcal{L}(B^\vee\otimes A)$ and use the fact that

$$\mathrm{Map}_{CAlg(O)}(A,A'\otimes B^\vee)\widetilde{\to}\mathrm{Map}_{CAlg(B^\vee-mod(O))}(B^\vee\otimes A,A'\otimes B^\vee)\widetilde{\to}\mathrm{Map}_{CAlg(O)}(\mathcal{L}(B^\vee\otimes A),A')$$

They also write $\underline{\mathrm{coHom}}(A,B^\vee)=\mathcal{L}(B^\vee\otimes A)$.

Let us describe the counit map of the adjunction $\mathcal{L}(B^\vee\otimes A)\to A$ in $CAlg(O)$. We do this using ([19], 1.3.6). It is given by a compatible system of maps $B^{\otimes J}\otimes A^{\otimes I}\to A$ for $(I\stackrel{\psi}{\to} J)\in Tw(fSet)$, which are the compositions

$$B^{\otimes J}\otimes A^{\otimes I}\stackrel{counit^{\otimes J}\otimes \mathrm{id}}{\to} A^{\otimes I}\stackrel{m}{\to} A, \tag{2}$$

where the first map comes from the J-th tensor power of the counit $B\to 1_O$ of B. The second map in (2) is the product $m: A^{\otimes I}\to A$.

1.1.10. In Lemma 8.2.7, the coalgebra B and algebra C are assumed unital. By a compatible collection of maps $A^{\otimes I}\to C\otimes (B^\vee)^{\otimes I}$ in (a) they mean the space of natural transformations of functors $fSet\to Alg(\mathrm{DGCat})$ (or $fSet\to CAlg(\mathrm{DGCat})$ respectively) from $I\mapsto A^{\otimes I}$ to the functor $I\mapsto (B^\vee)^{\otimes I}$, I think.

1.1.11. For 11.1. Let $\mathcal{C}\in CAlg(\mathrm{DGCat}_{cont})$. To see why in the definition of $\mathcal{C}_{\mathrm{Ran}}$ they use the $*$-direct image note the following. The formlula is analogous to the definition of $coEnd(A,B)$ in their Section 8.2.3. For a scheme of finite type S, it is understood that $Shv(S)$ is always considered with the $\otimes^!$-symmetric monoidal structure. Dualizing, $Shv(S)$ is a cocommlutative unital coalgebra with the coproduct given by $\vartriangle_*$ for $\vartriangle\colon S\to S\times S$. So, in the formula for $\mathcal{C}_{\mathrm{Ran}}$, we consider the functor $TwArr(fSet)\to \mathrm{DGCat}_{cont}$, $(I\to J)\mapsto \mathcal{C}^{\otimes I}\otimes Shv(X^J)$, and $Shv(X^J)$ secretely plays the role of the tensor power $Shv(X)^{\otimes J}$ (this is litterally true for $\mathcal{D}$-modules, but not in the constructible context). So, the functor $(fSet)^{op}\to \mathrm{DGCat}_{cont}$, $J\mapsto Shv(X^J)$ for the $*$-direct images is analogous to a unital cocommutative coalgebra in DGCat_{cont}.

1.1.12. For 11.1.7, let first $A\in \mathrm{DGCat}_{cont}$. They consider the functors $F_1,F_2: fSet\to \mathrm{DGCat}_{cont}$, $I\mapsto \mathcal{C}^{\otimes I}$ and $I\mapsto A\otimes Shv(X^I)$. The second functor here sends a map $I\to J$ to $\mathrm{id}\otimes\vartriangle^!\colon A\otimes Shv(X^I)\to A\otimes Shv(X^J)$ for the map $\vartriangle\colon X^J\to X^I$. The claim is that

$$\mathrm{Map}_{\mathrm{Fun}(fSet,\mathrm{DGCat}_{cont})}(F_1,F_2)\widetilde{\to}\mathrm{Map}_{\mathrm{DGCat}_{cont}}(\mathcal{C}_{\mathrm{Ran}},A)$$

If now $A\in CAlg(\mathrm{DGCat}_{cont})$ they get a map

$$\mathrm{Map}_{\mathrm{Fun}(fSet,CAlg(\mathrm{DGCat}_{cont}))}(F_1,F_2)\to \mathrm{Map}_{CAlg(\mathrm{DGCat}_{cont})}(\mathcal{C}_{\mathrm{Ran}},A)$$

For $A\in Alg(\mathrm{DGCat}_{cont})$ they get a map

$$\mathrm{Map}_{\mathrm{Fun}(fSet,Alg(\mathrm{DGCat}_{cont}))}(F_1,F_2)\to \mathrm{Map}_{Alg(\mathrm{DGCat}_{cont})}(\mathcal{C}_{\mathrm{Ran}},A)$$

1.1.13. For 11.1.9. Let X be a scheme of finite type. We have the functors $F_1,F_2: fSet\to CAlg(\mathrm{DGCat}_{cont})$, $F_1(I)=Shv(X)^{\otimes I}$ and $F_2(I)=Shv(X^I)$. Here for $\alpha: I\to J$ the map $F_1(\alpha): Shv(X)^{\otimes I}\to Shv(X)^{\otimes J}$ is given by the algebra structure $(Shv(X),\otimes^!)$, this is the product along α, and the map $F_2(\alpha): Shv(X^I)\to Shv(X^J)$ is $\vartriangle^!$ for $\vartriangle\colon X^J\to X^I$. Then the map $\boxtimes = h: Shv(X)^{\otimes I}\to Shv(X^I)$ is the natural transformation lying in $\mathrm{Map}_{\mathrm{Fun}(fSet,CAlg(\mathrm{DGCat}_{cont}))}(F_1,F_2)$.

Consider the unital cocommutative coalgebra $(Shv(X),\vartriangle_*)$ in DGCat_{cont} obtained by the canonical self-duality from $(Shv(X),\otimes^!)$. Cosider the functors $G_1,G_2: fSet^{op}\to \mathrm{DGCat}_{cont}$, where $G_1(I)=Shv(X)^{\otimes I}$, $G_2(I)=Shv(X^I)$. Here for $\alpha: I\to J$ in $fSet$ the corresponding map $G_1(\alpha): Shv(X)^{\otimes J}\to Shv(X)^{\otimes I}$ is given by the coalgebra structure $(Shv(X), \triangle_*)$, that is, the coproduct. The map $G_2(\alpha) : Shv(X^J) \to Shv(X^I)$ is $\triangle_*$ for $\triangle\colon X^J \to X^I$.

We have the natural transformation $h^\vee$ from G_2 to G_1, it is given on $I \in fSet$ by the functor $h^\vee : Shv(X^I) \to Shv(X)^{\otimes I}$.

Let $\mathcal{C} \in CAlg(\mathrm{DGCat}_{cont})$. This gives a natural transformation of functors from $Tw(fSet)$ to DGCat_{cont}, from

$$(I \to J) \mapsto C^{\otimes I} \otimes Shv(X^J)$$

to

$$(I \to J) \mapsto C^{\otimes I} \otimes Shv(X)^{\otimes J}.$$

Moreover, it is compatible with the right-lax structures on these functors, because h is fully faithful. Passing to the colimit over $Tw(fSet)$, this gives a morphism $\mathcal{C}_{\mathrm{Ran}} \to \underline{\mathrm{coHom}}(\mathcal{C}, Shv(X))$ given by their (11.6). It is actually a map in $CAlg(\mathrm{DGCat}_{cont})$.

1.1.14. In 11.2.3 they consider $\mathrm{QLisse}(X)$ with the $\otimes^!$-monoidal structure. More precisely (probably even without the smoothness assumption of X), they consider the embedding $\mathrm{QLisse}(X) \to Shv(X)$, $E \mapsto E \otimes \omega_X$, so that the $\otimes^!$-monoidal structure on $Shv(X)$ restricts to the $\otimes^*$-symmetric monoidal structure on $\mathrm{QLisse}(X)$. In this case the map

$$Shv(X^J) \to (\mathrm{QLisse}(X)^\vee)^{\otimes J}$$

becomes indeed a natural transformation of functor $(fSet)^{op} \to \mathrm{DGCat}_{cont}$.

1.1.15. For Pp. 11.2.6. They use ([12], Lemma 6.4) and the fact that DGCat_{cont} is naturally an $(\infty, 2)$-category, so $\mathrm{Fun}(fSet, \mathrm{DGCat}_{cont})$ is also an $(\infty, 2)$-category.

We get

$$\mathrm{Fun}_{e,cont}(\mathcal{C}^{X-lisse}, A) \widetilde{\to} \lim_{(I\to J)\in Tw(fSet)^{op}} \mathrm{Fun}_{e,cont}(\mathcal{C}^{\otimes I}, A \otimes (\mathrm{QLisse}(X)^{\otimes J})) \to \lim_{(I\to J)\in Tw(fSet)^{op}} \mathrm{Fun}_{e,cont}(\mathcal{C}^{\otimes I}, A \otimes Shv(X^J)) \widetilde{\to} \mathrm{Fun}_{e,cont}(\mathcal{C}_{\mathrm{Ran}}, A)$$

They also use ([16], 2.2.16, 2.2.17) to conclude. Namely, for each $(I \to J) \in Tw(fSet)$, $\mathrm{Fun}_{e,cont}(\mathcal{C}^{\otimes I}, A \otimes (\mathrm{QLisse}(X)^{\otimes J})) \to \mathrm{Fun}_{e,cont}(\mathcal{C}^{\otimes I}, A \otimes Shv(X^J))$ is fully faithful.

1.1.16. For 11.2.8. My understanding is as follows. Let $B, A \in Alg(\mathrm{DGCat}_{cont})$. Write $A^\otimes, B^\otimes$ for the corresponding functors $\boldsymbol{\Delta}^{op} \to \mathrm{DGCat}_{cont}$. Then

$$\mathrm{Map}_{Alg(\mathrm{DGCat}_{cont})}(B, A) \widetilde{\to} \lim_{([n]\xrightarrow{\alpha}[m])\in Tw(\boldsymbol{\Delta}^{op})^{op}} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(B^\otimes([m]), A^\otimes([n])),$$

here α is a map in $\boldsymbol{\Delta}$. This follows from ([9], 1.3.12). Let now $B_1 \to B_2$ be a map in $Alg(\mathrm{DGCat}_{cont})$ such that for any $A \in \mathrm{DGCat}_{cont})$ dualizable and any $n \geq 0$, $\mathrm{Fun}_{e,cont}(B_2^{\otimes n}, A) \to \mathrm{Fun}_{e,cont}(B_1^n, A)$ is fully faithful. Then for $A \in \mathrm{DGCat}_{cont})$ dualizable the induced map $\mathrm{Map}_{Alg(\mathrm{DGCat}_{cont})}(B_2, A) \to \mathrm{Map}_{Alg(\mathrm{DGCat}_{cont})}(B_1, A)$ is fully faithful map of spaces, that is, a monomorphism.

A similar argument in the commutative case.

In 11.2.8 c) they assume M_2 is dualizable in DGCat_{cont}. For $A \to B$ a map in $Alg(\mathrm{DGCat}_{cont})$ and $M_1, M_2 \in B{-}mod(\mathrm{DGCat}_{cont})$ we have a natural map $\mathrm{Fun}_B(M_1, M_2) \to \mathrm{Fun}_A(M_1, M_2)$. Assume that for any $n \geq 0$,

$$\mathrm{Fun}_{e,cont}(B^{\otimes n} \otimes M_1, M_2) \to \mathrm{Fun}_{e,cont}(A^{\otimes n} \otimes M_1, M_2)$$

is fully faithful. Then $\mathrm{Fun}_B(M_1, M_2) \to \mathrm{Fun}_A(M_1, M_2)$ is fully faithful, as the limit over $[n] \in \boldsymbol{\Delta}$ of the fully faithful functors. This is what happens in our case.

Namely, one first shows that for any $n \geq 0$,

$$\mathrm{Fun}_{e,cont}((\mathcal{C}^{X-lisse})^{\otimes n} \otimes M_1, M_2) \to \mathrm{Fun}_{e,cont}(\mathcal{C}_{\mathrm{Ran}}^{\otimes n} \otimes M_1, M_2)$$

is fully faithful, for this one repeats the argument from their Pp. 11.2.6. Then pass to the totalization. This gives fully faithfulness. This is an isomorphism now by ([16], 2.7.18).

1.1.17. For 11.3.4, 11.3.6. If $F \in Shv(S)^c$ for $S \in \mathrm{Sch}_{ft}$, we have a natural map $F \boxtimes \mathbb{D}(F) \to \vartriangle_* \omega_S$ coming from the composition $F \otimes \mathcal{H}om(F, \omega) \to \omega$, where $\mathcal{H}om(\cdot, \cdot) \in Shv(S)$ denotes the inner hom for the $(Shv(S), \otimes)$-monoidal structure. Here $\vartriangle : S \to S \times S$ is the diagonal. On the other hand, we have a map $\vartriangle_! \mathrm{e}_S \to F \boxtimes \mathbb{D}(F)$, that is, $e_S \to F \otimes^! \mathbb{D}(F)$ in $Shv(S)$. It comes from the characterization

$$\mathcal{H}om_{Shv}(\mathbb{D}(F_1), F_2) \widetilde{\to} \mathrm{R}\Gamma(S, F_1 \otimes^! F_2)$$

of $\mathbb{D}(F_1)$ for $F_1 \in Shv(S)^c, F_2 \in Shv(S)$. Here $\mathcal{H}om_{Shv} \in \mathrm{Vect}$ denotes the relative inner hom for the Vect-action on $Shv(S)$.

Remark 1.1.18. *Let $C_I : I \to \mathrm{DGCat}_{cont}$ be a diagram $i \mapsto \mathcal{C}_i$, where for $i \to j$ the corresponding functor $\phi_{ij} : \mathcal{C}_i \to \mathcal{C}_j$ admits a continuous right adjoint ϕ_{ij}^R. Assume each $\mathcal{C}_i$ dualizable. Assume given a self-diality $\mathcal{C}_i^\vee \widetilde{\to} \mathcal{C}_i$ for each i. Let $C_{I^{op}}^R : I^{op} \to \mathrm{DGCat}_{cont}$ be obtained from C_I by passing to right adjoints. Let $C_{I^{op}}^\vee : I^{op} \to \mathrm{DGCat}_{cont}$ be obtained from C_I by passing to the duals. Assume the functor $(C_{I^{op}}^R)^\vee : I \to \mathrm{DGCat}_{cont}$ obtained from $C_{I^{op}}^R$ by passing to the duals is identified with C_I via the self-dualities on C_i. In particular, for each $i \to j$ in I, the dual to $\phi_{ij}^R : \mathcal{C}_j \to \mathcal{C}_i$ identifies with $\phi_{ij} : \mathcal{C}_i \to \mathcal{C}_j$. Then $D := \operatorname{colim} C_I$ is naturally equipped with a self-duality. This is used in 11.3.9. Moreover, for any i by ([11], ch. I.1, 6.3.6) the diagram commutes*

$$\begin{array}{ccc} \mathrm{Vect} & \stackrel{u}{\to} & D \otimes D \\ \downarrow u_i & & \downarrow ev_i \otimes \mathrm{id} \\ \mathcal{C}_i \otimes \mathcal{C}_i & \stackrel{\mathrm{id} \otimes ins_i}{\to} & \mathcal{C}_i \otimes D, \end{array}$$

where u_i is the unit of the self-duality for $\mathcal{C}_i$, and u is the unit of the self-duality for D. Note that $u \widetilde{\to} \operatorname*{colim}_{i \in I} (ins_i \otimes ins_i)(u_i)$ in D.

1.1.19. For 11.3.8. We use ([16], 4.1.2) and ([11], ch. I.1, 6.3.4).

If $\mathcal{C}$ is a rigid symmetric monoidal category in $CAlg(\mathrm{DGCat}_{cont})$ then for any $\alpha : I \to J$ in $fSet$, let $m : \mathcal{C}^{\otimes I} \to \mathcal{C}^{\otimes J}$ be the product along α. Then $(\alpha^R)^\vee \widetilde{\to} \alpha$ with respect to the canonical self-duality on $\mathcal{C}$, see ([11], ch. I.1, 9.2.6). Moreover, the dual of the counit map $c : \mathcal{C} \otimes \mathcal{C} \to \mathrm{Vect}$ is the unit $u : \mathrm{Vect} \to \mathcal{C} \otimes \mathcal{C}$ via the canonical self-duality on $\mathcal{C}$.

Their isomorphism (11.12) is given in my ([16], 9.2.6). Their claim at the end of 11.4.3 is my ([16], 9.2.37). Their formula (11.14) comes from my Remark 1.1.18 above, namely it gives a commutative square

$$\begin{array}{ccc}\mathrm{Vect} & \stackrel{R_{\mathcal{C},\mathrm{Ran}}}{\to} & \mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}\\ \downarrow u & & \downarrow ins^R_{(I\to J)}\otimes\mathrm{id}\\ (\mathcal{C}^{\otimes I}\otimes Shv(X^J))\otimes(\mathcal{C}^{\otimes I}\otimes Shv(X^J)) & \stackrel{\mathrm{id}\otimes ins_{(I\to J)}}{\to} & (\mathcal{C}^{\otimes I}\otimes Shv(X^J))\otimes\mathcal{C}_{\mathrm{Ran}},\end{array}$$

where u is the unit of the self-duality on $(\mathcal{C}^{\otimes I}\otimes Shv(X^J))$. So, $u\widetilde{\to} R_{\mathcal{C}}^{\boxtimes I}\otimes u_{Shv(X^J)}$, where $u_{Shv(X^J)}$ is the unit of the self-duality on $Shv(X^J)$, and $R_{\mathcal{C}}\in\mathcal{C}\otimes\mathcal{C}$ is the unit of the self-duality on $\mathcal{C}$.

1.1.20. For 11.4.2. The object $R_{\mathcal{C},\mathrm{Ran}}\in\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}$ has a structure of a unital algebra, that is, $R_{\mathcal{C},\mathrm{Ran}}\in Alg(\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}})$, because $R_{\mathcal{C},\mathrm{Ran}}=m^Rm(1\otimes 1)$ is a monad in $\mathrm{Fun}_{\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}}(\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}},\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}})$. This is used later.

1.1.21. For 11.4.6. Let $Y\in\mathrm{Sch}_{ft}$. Then the unit of the canonical self-duality on $Shv(Y)$ is indeed $u_{Shv(Y)}:=h^R(\vartriangle_*\omega_Y)$, where $\vartriangle\colon Y\to Y\times Y$ is the diagonal, and $h: Shv(Y)\otimes Shv(Y)\to Shv(Y\times Y)$ is the exteriour product. Indeed, the counit is given by $Shv(Y)\otimes Shv(Y)\to\mathrm{Vect}$, $(K_1,K_2)\mapsto\mathrm{R}\Gamma(Y,K_1\otimes^! K_2)$. Then the unit is obtained by dualizing the counit map by ([16], 3.1.2.1), as h^R is the dual of h.

Note that for $Z\in\mathrm{Sch}_{ft}$ and $K\in Shv(Z)$ the functor $Shv(Z)\to Shv(Z), L\mapsto L\otimes^! K$ is canonically self-dual with respect to the canonical duality on $Shv(Z)$. So, the dual to the composition $Shv(Y)\otimes Shv(Y)\stackrel{h}{\to} Shv(Y\times Y)\stackrel{\otimes^!\vartriangle_*\omega_Y}{\to} Shv(Y\times Y)\stackrel{\mathrm{R}\Gamma}{\to}\mathrm{Vect}$ is the composition

$$\mathrm{Vect}\stackrel{\otimes\omega_{Y\times Y}}{\to} Shv(Y\times Y)\stackrel{\otimes^!\vartriangle_*\omega_Y}{\to} Shv(Y\times Y)\stackrel{h^R}{\to} Shv(Y)\otimes Shv(Y)$$

If $K\in Shv(Y)$ then we have $u_{Shv(Y)}\otimes^!(\omega_Y\boxtimes K)\widetilde{\to}(\omega_Y\boxtimes K)\otimes^! u_{Shv(Y)}$ canonically. Indeed, we derive this from ([20], Claim in 0.0.7) saying that h^R is a strict morphism of $Shv(Y)$-bimodules. So, one has canonically

$$u_{Shv(Y)}\otimes^!(\omega_Y\boxtimes K)\widetilde{\to} h^R((\vartriangle_*\omega_Y)\otimes^!(\omega_Y\boxtimes K))\widetilde{\to} h^R(\vartriangle_* K)$$

and similarly $(\omega_Y\boxtimes K)\otimes^! u_{Shv(Y)}\widetilde{\to} h^R(\vartriangle_* K)$.

This is used in their Section 11.5.1.

1.1.22. In 11.4.8 they mean the unit map $1_A\to comult(mult(1_A)$ and the counit map $mult(comult(1_A))\to 1_A$ for a rigid symmetric monoidal category A, where $A=\mathcal{C}_{\mathrm{Ran}}$, using the adjunction $mult: A\otimes A\leftrightarrows A: comult$.

1.1.23. The category $Tw(fSet)$ does not seem to satisfy the assumptions of ([8], Lm. 1.3.6).

The natural map $1_{\mathcal{C}_{\mathrm{Ran}}}\to R_{\mathcal{C},\mathrm{Ran}}$ is the natural map $1_{\mathcal{C}_{\mathrm{Ran}}}\widetilde{\to} ins_{(\emptyset\to\emptyset)}(e)\otimes ins_{(\emptyset\to\emptyset)}(e)\to R_{\mathcal{C},\mathrm{Ran}}$.

For (11.16). First, they use the map $h(u_{Shv(X^J)})\to\vartriangle_*\omega_{X^J}$ for $\vartriangle\colon X^J\to X^J\times X^J$, this is $hh^R(\vartriangle_*\omega_{X^J})\to\vartriangle_*\omega_{X^J}$.

1.1.24. For 11.5.1. For $\mathcal{C} \in CAlg(\mathrm{DGCat}_{cont})$ rigid and the unit of the self-duality $R_{\mathcal{C}} \in \mathcal{C}\otimes\mathcal{C}$, the object $R_{\mathcal{C}}^{\boxtimes I} \in \mathcal{C}^{\otimes I}\otimes\mathcal{C}^{\otimes I}$ gives a maps of $\mathcal{C}^{\otimes I}$-bimodules $\mathcal{C}^{\otimes I} \to \mathcal{C}^{\otimes I}\otimes\mathcal{C}^{\otimes I}$.

1.1.25. For 11.6.3. This seems to be the following abstract claim. Let O be a symmetric monoidal presentable category in which the tensor product preserves colimits separately in each variable. Let $A \in CAlg(O)$, B be a unital cocommutative coalgebra in O, which is dualizable, so $B^\vee \in CAlg(O)$.

Let $fSet \to O$, $I \mapsto \mathcal{E}^I$ be a functor sending $\emptyset$ to 1, it plays a role of a commutative algebra, to the exception that $\mathcal{E}^I$ is not necessarily a tensor power. Assume $\mathcal{E}^I$ self-dual in O, so the functor obtained by dualization $fSet^{op} \to O$ is plays a role of a "unital cocommutative coalgebra". Assume given a morphism of functors $fSet \to O$ from the commutative algebra $I \mapsto (B^\vee)^{\otimes I}$ to the functor $I \mapsto \mathcal{E}^I$. Write $\alpha_I : (B^\vee)^{\otimes I} \to \mathcal{E}^I$ for the corresponding morphism in O, let $\alpha_I^\vee : \mathcal{E}^I \to B^{\otimes I}$ be its dual.

Let now $C \in CAlg(O)$ and let $F : C \to A \otimes B^\vee$ be a map in $CAlg(O)$. For each $I \in fSet$ let $F^I : C^{\otimes I} \to A \otimes (B^\vee)^{\otimes I}$ be the composition $C^{\otimes I} \stackrel{F^{\otimes I}}{\to} (A \otimes B^\vee)^{\otimes I} \stackrel{m_I \otimes \mathrm{id}}{\to} A \otimes (B^\vee)^{\otimes I}$, where $m_I : A^{\otimes I} \to A$ is the product. Then F gives rise to a map

$$\underset{(I\to J)\in Tw(fSet)}{\mathrm{colim}} C^{\otimes I} \otimes \mathcal{E}^J \to \underset{(I\to J)\in Tw(fSet)}{\mathrm{colim}} C^{\otimes I} \otimes B^{\otimes J} \to A$$

in O. The composition is given by a compatible system of maps $C^{\otimes I} \otimes \mathcal{E}^J \to A$ for $(I \stackrel{\psi}{\to} J) \in Tw(fSet)$. Given $(I \stackrel{\psi}{\to} J)$, the desired map $C^{\otimes I} \otimes \mathcal{E}^J \to A$ is the composition

$$C^{\otimes I} \otimes \mathcal{E}^J \to C^{\otimes J} \otimes \mathcal{E}^J \to (B \otimes C)^{\otimes J} \to A \tag{3}$$

The first map in (3) comes from the product $C^{\otimes I} \to C^{\otimes J}$ in C, the second one comes from $\alpha_J^\vee : \mathcal{E}^J \to B^{\otimes J}$. The map F gives rise by functoriality to a morphism $\bar{F} : B \otimes C \to B \otimes A$ defined as $\mathrm{id} \otimes \tilde{F} : B \otimes_{B^\vee} (B^\vee \otimes C) \to B \otimes_{B^\vee} (B^\vee \otimes A)$, where $\tilde{F} : B^\vee \otimes C \to B^\vee \otimes A$ is $B^\vee$-linear, and the composition $C \to B^\vee \otimes C \stackrel{\tilde{F}}{\to} B^\vee \otimes A$ is F. Then the third map in (3) is the compositon

$$(B \otimes C)^{\otimes J} \stackrel{\bar{F}^{\otimes J}}{\to} (B \otimes A)^{\otimes J} \stackrel{counit}{\to} A^{\otimes J} \stackrel{m}{\to} A,$$

where $m : A^{\otimes J} \to A$ is the product, and we also used the counit $B \to 1_O$ of the coalgebra B.

They propose a different description of the composition $C^{\otimes J} \otimes \mathcal{E}^J \stackrel{\epsilon}{\to} A$ in (3). Namely, they claim that ϵ is the composition

$$C^{\otimes J} \otimes \mathcal{E}^J \stackrel{F^J \otimes \mathrm{id}}{\to} A \otimes (B^\vee)^{\otimes J} \otimes \mathcal{E}^J \stackrel{\mathrm{id} \otimes \alpha_J \otimes \mathrm{id}}{\to} A \otimes \mathcal{E}^J \otimes \mathcal{E}^J \stackrel{\mathrm{id} \otimes m}{\to} A \otimes \mathcal{E}^J \stackrel{\mathrm{id} \otimes counit}{\to} A$$

Here $m : \mathcal{E}^J \otimes \mathcal{E}^J \to \mathcal{E}^J$ is the product in this "algebra", and $counit : \mathcal{E}^J \to 1_0$ is the "counit" of this "coalgebra". This gives indeed the same answer.

1.1.26. For 11.6.5. Let $\mathcal{Z} \in \mathrm{PreStk}$ with a symmetric monoidal functor $F : \mathcal{C} \to \mathrm{QCoh}(\mathcal{Z}) \otimes QLisse(X)$, so we get a symmetric monoidal functor $\tilde{F} : \mathcal{C}_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathcal{Z})$. Let in addition $M \in \mathcal{C}_{\mathrm{Ran}}{-}mod(\mathrm{DGCat}_{cont})$. The functor $oblv_{\mathrm{Hecke},\mathcal{Z}} : \mathrm{Hecke}(\mathcal{Z}, M)_F \to M \otimes \mathrm{QCoh}(\mathcal{Z})$ is conservative by ([16], 4.0.30).

For 11.6.6. The composition $\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}\stackrel{m}{\to}\mathcal{C}_{\mathrm{Ran}}\stackrel{m^R}{\to}\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}$ is a map of $\mathcal{C}_{\mathrm{Ran}}$-bimodules sending $1\otimes 1$ to $R_{\mathcal{C},\mathrm{Ran}}$. So, it acts on $M\otimes\mathrm{QCoh}(\mathcal{Z})$ via $(\mathrm{id}\otimes\tilde{F})(R_{\mathcal{C},\mathrm{Ran}})$ indeed.

1.1.27. Consider an adjoint pair $l: B_1\leftrightarrows B_2: r$ in $Alg(\mathrm{DGCat}_{cont})$, so l,r are maps of algebras. Assume r is fully faithful, so l is a localization. Then for any $n\geq 0$ the induced map $B_1^{\otimes n}\to B_2^{\otimes n}$ is also a localization. We claim that for any $C\in Alg(\mathrm{DGCat}_{cont})$ the natural map

$$\mathrm{Map}_{Alg(\mathrm{DGCat}_{cont})}(B_2,C)\to\mathrm{Map}_{Alg(\mathrm{DGCat}_{cont})}(B_1,C)$$

is a monomorphism of spaces (and a similar claim for commutative algebras).

Proof: argue as in Section 1.1.16 of this file. Let $B_i^{\otimes}, C^{\otimes}: \boldsymbol{\Delta}^{op}\to\mathrm{DGCat}_{cont}$ be the corresponding functors. For each $m,n\geq 0$ the adjoint pair $B_1^{\otimes}([m])\leftrightarrows B_2^{\otimes}([m])$ gives rise to an adjoint pair

$$\mathcal{L}:\mathrm{Fun}_{e,cont}(B_2^{\otimes}([m]),C^{\otimes}([n]))\leftrightarrows\mathrm{Fun}_{e,cont}(B_1^{\otimes}([m]),C^{\otimes}([n]))$$

in DGCat_{cont} with $\mathcal{L}$ fully faithful. So, in

$$\mathrm{Map}_{Alg(\mathrm{DGCat}_{cont})}(B_1,C)\widetilde{\to}\lim_{([n]\stackrel{\alpha}{\to}[m])\in Tw(\boldsymbol{\Delta}^{op})^{op}}\mathrm{Map}_{\mathrm{DGCat}_{cont}}(B_1^{\otimes}([m]),C^{\otimes}([n])),$$

we replace each term in the limit by a full subspace $\mathrm{Map}_{\mathrm{DGCat}_{cont}}(B_2^{\otimes}([m]),C^{\otimes}([n]))$, hence get a full subspace in the limit.

This is used for De Rham and Betti versions in Version Jan 24, 2022 of this paper, Remark 11.8.3.

1.2. **For Version of March 5, 2022.**

1.2.1. For 1.7.3. Let $H^{acces,c}\subset H$ be the full subcategory of bounded objects, whose all cohomologies lies in $H^c\cap H^{\heartsuit}$. Then $H^{acces,c}$ is a small stable Vect^c-module category, it is closed under the tensor products (under the assumptions of 1.7.2). Indeed, if $h_i\in H^{acces,c}$ then $\mathrm{oblv}(h_1h_2)\widetilde{\to}\mathrm{oblv}(h_1)\,\mathrm{oblv}(h_2)\in\mathrm{Vect}^c$, so each cohomology of h_1h_2 is sent by oblv to a finite-dimensional vector space, so each cohomology of h_1h_2 lies in $H^c\cap H^{\heartsuit}$. Moreover, if $h\in H^c\cap H^{\heartsuit}$ then h is dualizable in H, and $h^{\vee}\in H^{\heartsuit}\cap H^c$. Indeed, since oblv is symmetric monoidal, $\mathrm{oblv}(h^{\vee})$ is the dual of $\mathrm{oblv}(h)$. Since $\mathrm{oblv}(h)\in\mathrm{Vect}^{\heartsuit}\cap\mathrm{Vect}^c$ and h conservative, $h^{\vee}\in H^{\heartsuit}\cap H^c$ by the assumptions of 1.7.2.

Let $h_i\in H$ with h_1 dualizable in H. Then $\mathcal{H}om_H(1,h_1^{\vee}\otimes h_2)\widetilde{\to}\mathcal{H}om_H(h_1,h_2)$. Indeed, for $V\in\mathrm{Vect}$,

$$\begin{aligned}\mathrm{Map}_{\mathrm{Vect}}(V,\mathcal{H}om_H(1,h_1^{\vee}\otimes h_2))&\widetilde{\to}\mathrm{Map}_H(V\otimes 1_H,h_1^{\vee}\otimes h_2)\\&\widetilde{\to}\mathrm{Map}_H(V\otimes h_1,h_2)\widetilde{\to}\mathrm{Map}_{\mathrm{Vect}}(V,\mathcal{H}om_H(h_1,h_2))\end{aligned}$$

So, the assumptions of 1.7.2 imply that for $h_i\in H^c\cap H^{\heartsuit}$ one has $\mathcal{H}om_H(h_1,h_2)\in\mathrm{Vect}^c$, because $h_1^{\vee}\in H^c\cap H^{\heartsuit}$, and $h_1^{\vee}\otimes h_2\in H^c\cap H^{\heartsuit}$ also.

The claim in their Section 1.7.2 in the opposite direction is wrong as stated: it is corrected in version of Apr 3, 2022.

1.2.2. For 12.1.4. Note that $\mathrm{Hecke}(\mathcal{Z}, M) \widetilde{\to} \mathrm{Fun}_{\mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z})}(\mathrm{QCoh}(\mathcal{Z}), M \otimes \mathrm{QCoh}(\mathcal{Z}))$ by adjointness.

Note that for $\mathcal{Z} \in \mathrm{PreStk}$ with a symmetric monoidal functor $F : \mathcal{C} \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X)$ the object $R_{\mathcal{Z}} = R_{\mathcal{Z},F} \in \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z})$ of 12.1.6 satisfies a system of isomorphisms $\mathcal{V} * R_{\mathcal{Z}} \widetilde{\to} R_{\mathcal{Z}} * \tilde{F}(\mathcal{V})$ for $\mathcal{V} \in \mathcal{C}_{\mathrm{Ran}}$. Recall that $R_{\mathcal{Z}} \in \mathrm{Alg}(\mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}))$.

1.2.3. In Section 12.1.11 we need to calculate the following.
Step 1 Let $K \in Shv(X^J)$, $u_{Shv(X^J)} \in Shv(X^J) \otimes Shv(X^J)$ be the unit of the Verdier self-duality on $Shv(X^J)$. Then let us understand by $\omega \boxtimes K$ the corresponding object of $Shv(X^J) \otimes Shv(X^J)$, so $u_{Shv(X^J)} \otimes (\omega \boxtimes K)$ is the product in the commutative algebra $Shv(X^J) \otimes Shv(X^J)$ (with the !-monoidal strucrures on each factor). Then

$$(\mathrm{id} \otimes \mathrm{R}\Gamma(X^J, \cdot))(u_{Shv(X^J)} \otimes (\omega \boxtimes K)) \widetilde{\to} K \tag{4}$$

Indeed, write h for the exteriour product. Both sides being continuous and exact as functors of K, we may and do assume K compact. Then the composition

$$\mathrm{Vect} \overset{\triangle_* \omega_{X^J}}{\to} Shv(X^J \times X^J) \overset{h^R}{\to} Shv(X^J) \otimes Shv(X^J) \overset{\mathrm{id} \otimes (\bullet \otimes^! K)}{\to}$$
$$Shv(X^J) \otimes Shv(X^J) \overset{\mathrm{id} \otimes \mathrm{R}\Gamma(X^J, \bullet)}{\to} Shv(X^J)$$

denoted $\mathcal{R}$ admits the left adjoint $Shv(X^J) \to \mathrm{Vect}$, $M \mapsto \mathrm{R}\Gamma(X^J, M \otimes \mathbb{D}(K))$, because X is proper. Now $\mathcal{H}om(\mathrm{R}\Gamma(X^J, M \otimes \mathbb{D}(K)), e) \widetilde{\to} \mathcal{H}om_{Shv(X^J)}(M, K)$ canonically, where $\mathcal{H}om_{Shv(X^J)}$ denotes the inner hom with respect to the Vect action on $Shv(X^J)$. So, $\mathcal{R}(e) \widetilde{\to} K$, and (4) follows.

Step 2 It suffices to show that for any $(I \overset{\psi}{\to} J) \in Tw(fSet)$, one has

$$ins_{(I \overset{\psi}{\to} J)} \otimes (\tilde{F}\, ins_{(I \overset{\psi}{\to} J)})(R_{\mathcal{C}}^{\boxtimes I} \otimes u_{Shv(X^J)}) \widetilde{\to} ins_{(I \overset{\psi}{\to} J)} \otimes (\mathrm{id} \otimes (F^J \circ mult_{\mathcal{C}}^{\psi}))(R_{\mathcal{C}}^{\boxtimes I})$$

This is now an immediate consequence of Step 1 and the formula for $\tilde{F}$ given in the paper.

1.2.4. Let X be a complete smooth curve. Note that, according to the formula just before Th. 11.10.2, the exteriour product map

$$\mathrm{QCoh}(\mathrm{LocSys}_G^{restr}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}_G^{restr}(X)) \to \mathrm{QCoh}(\mathrm{LocSys}_G^{restr}(X) \times \mathrm{LocSys}_G^{restr}(X))$$

is an equivalence.

1.2.5. For 12.3.1. If $f : \mathcal{Z} \to \mathrm{LocSys}_G^{restr}(X)$ be a morphism, where $\mathcal{Z}$ is a prestack. Then indeed we get a symmetric monoidal functor $F : \mathrm{Rep}(G) \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X)$, because $\mathrm{QLisse}(X)$ is dualizable (as it is compactly generated by Appendix C of version Jan 24, 2022). Here X is a smooth complete curve.

Let $f : \mathcal{Z} \to \mathrm{LocSys}_G^{restr}(X)$ be a map of prestacks. Set for brevity $\mathcal{C} = \mathrm{Rep}(G)$. If M is a $\mathcal{C}_{\mathrm{Ran}}$-module then, since $\mathcal{C}_{\mathrm{Ran}}$ is rigid,

$$\mathrm{Hecke}(\mathcal{Z}, M) \widetilde{\to} \mathcal{C}_{\mathrm{Ran}} \otimes_{\mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}}} (M \otimes \mathrm{QCoh}(\mathcal{Z}))$$

Assume in addition that M is a $\mathrm{QCoh}(\mathrm{LocSys}_G^{restr}(X))$-module (so giving a structure of a $\mathcal{C}_{\mathrm{Ran}}$-module). Then $\mathrm{Hecke}(\mathcal{Z}, M)$ identifies with $M \otimes_{\mathrm{QCoh}(\mathrm{LocSys}_G^{restr}(X))} \mathrm{QCoh}(\mathcal{Z})$. Indeed, the RHS of the above displayed formula becomes

$$(\mathcal{C}_{\mathrm{Ran}} \otimes_{(\mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}})} (A' \otimes A')) \otimes_{A' \otimes A'} (M \otimes \mathrm{QCoh}(\mathcal{Z}))$$

with $A' := \mathrm{QCoh}(\mathrm{LocSys}_G^{restr}(X)$. Now $\mathcal{C}_{\mathrm{Ran}} \otimes_{(\mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}})} (A' \otimes A') \widetilde{\to} A'$ by their Prop. 11.11.6. So,

$$\mathrm{Hecke}(\mathcal{Z}, M) \widetilde{\to} M \otimes_{A'} \mathrm{QCoh}(\mathcal{Z})$$

1.2.6. For 12.4.1. Let X be a smooth projective curve, $\mathcal{C} \in CAlg(\mathrm{DGCat}_{cont})$ is a compactly generated rigid symmetric monoidal category. The functor $Tw(fSet) \to \mathrm{DGCat}_{cont}$, $(I \to J) \mapsto \mathcal{C}^{\otimes I} \otimes Shv(X^J \times Y)$ is still right-lax monoidal, so its colimit is an object of $CAlg(\mathrm{DGCat}_{cont})$.

1.2.7. For 12.4.5. Let $F : \mathcal{C} \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X)$ be a symmetric monoidal functor. We get out of it a symmetric monoidal functor $\mathcal{C}^{\otimes X-lisse} \to \mathrm{QCoh}(\mathcal{Z})$ as in their Section 12.1.3.

Consider now the functor $fSet \to \mathrm{DGCat}_{cont}$, $I \mapsto \mathrm{QLisse}(X)^{\otimes I}$ sending $J' \to J$ to $\alpha : \mathrm{QLisse}(X)^{\otimes J'} \to \mathrm{QLisse}(X)^{\otimes J}$ that fits into the commutative square

$$\begin{array}{ccc} Shv(X^J) & \stackrel{\beta}{\leftarrow} & \mathrm{QLisse}(X)^{\otimes J} \\ \uparrow \vartriangle^! & & \uparrow \alpha \\ Shv(X^{J'}) & \leftarrow & \mathrm{QLisse}(X)^{\otimes J'} \end{array}$$

Here β is the composition $\mathrm{QLisse}(X)^{\otimes J} \to Shv(X)^{\otimes J} \stackrel{\boxtimes}{\to} Shv(X^J)$, where the first map comes from the inclusion $\mathrm{QLisse}(X) \to Shv(X)$, $E \mapsto E \otimes \omega$, here $\mathrm{QLisse}(X)$ is considered with the $*$-monoidal structure. We used $\vartriangle: X^J \to X^{J'}$. Let $\beta^\vee : Shv(X^J) \to (\mathrm{QLisse}(X)^\vee)^{\otimes J}$ be the dual map. We have

$$\mathcal{C}^{\otimes X-lisse} \otimes Shv(Y) \widetilde{\to} \underset{(I \to J) \in Tw(fSet)}{\mathrm{colim}} \mathcal{C}^{\otimes I} \otimes (\mathrm{QLisse}(X)^\vee)^{\otimes J} \otimes Shv(Y)$$

Consider the natural transformation of right-lax functors $Tw(fSet) \to \mathrm{DGCat}_{cont}$ from

$$(I \to J) \mapsto \mathcal{C}^{\otimes I} \otimes Shv(X^J \times Y)$$

to the functor $(I \to J) \mapsto \mathcal{C}^{\otimes I} \otimes (\mathrm{QLisse}(X)^\vee)^{\otimes J} \otimes Shv(Y)$ coming from

$$Shv(X^J \times Y) \to (\mathrm{QLisse}(X)^\vee)^{\otimes J} \otimes Shv(Y) \tag{5}$$

Here (5) is dual to the composition

$$\mathrm{QLisse}(X)^{\otimes J} \otimes Shv(Y) \stackrel{\beta \otimes \mathrm{id}}{\to} Shv(X^J) \otimes Shv(Y) \stackrel{\boxtimes}{\to} Shv(X^J \times Y)$$

Passing to the colimit over $Tw(fSet)$, we get a symmetric monoidal functor $\mathcal{C}_{\mathrm{Ran} \times Y} \to \mathcal{C}^{\otimes X-lisse} \otimes Shv(Y)$. Composing with the symmetric monoidal functor $\mathcal{C}^{\otimes X-lisse} \otimes Shv(Y) \to \mathrm{QCoh}(\mathcal{Z}) \otimes Shv(Y)$ mentioned above, we get the symmetric monoidal functor

$$\tilde{F}_Y : \mathcal{C}_{\mathrm{Ran} \times Y} \to \mathrm{QCoh}(\mathcal{Z}) \otimes Shv(Y)$$

described in their Section 12.4.5. **Question:** is this correct?

Note that $\tilde{F}_Y$ is a morphism of $Shv(Y)$-algebras. By definition

$$\mathrm{Hecke}_Y(\mathcal{Z}, M) = \mathrm{Fun}_{\mathcal{C}_{\mathrm{Ran}\times Y}\otimes_{Shv(Y)}\mathcal{C}_{\mathrm{Ran}\times Y}}(\mathcal{C}_{\mathrm{Ran}\times Y}, M\otimes \mathrm{QCoh}(\mathcal{Z}))$$

Note also that

$$\mathcal{C}_{\mathrm{Ran}\times Y}\otimes_{(\mathcal{C}_{\mathrm{Ran}\times Y}\otimes_{Shv(Y)}\mathcal{C}_{\mathrm{Ran}\times Y})}(\mathcal{C}_{\mathrm{Ran}\times Y}\otimes_{Shv(Y)}(\mathrm{QCoh}(\mathcal{Z})\otimes Shv(Y)))\widetilde{\to}\mathrm{QCoh}(\mathcal{Z})\otimes Shv(Y)$$

Therefore, by adjointness ([16], 9.2.30),

$$\begin{aligned}\mathrm{Hecke}_Y(\mathcal{Z}, M)\widetilde{\to}&\mathrm{Fun}_{\mathcal{C}_{\mathrm{Ran}\times Y}\otimes_{Shv(Y)}(\mathrm{QCoh}(\mathcal{Z})\otimes Shv(Y))}(\mathrm{QCoh}(\mathcal{Z})\otimes Shv(Y), M\otimes\mathrm{QCoh}(\mathcal{Z}))\\ \widetilde{\to}&\mathrm{Fun}_{\mathcal{C}_{\mathrm{Ran}\times Y}\otimes\mathrm{QCoh}(\mathcal{Z})}(\mathrm{QCoh}(\mathcal{Z})\otimes Shv(Y), M\otimes\mathrm{QCoh}(\mathcal{Z}))\end{aligned}$$

The composition $\mathcal{C}_{\mathrm{Ran}}\otimes Shv(Y)\to\mathcal{C}_{\mathrm{Ran}\times Y}\stackrel{\tilde{F}_Y}{\to}\mathrm{QCoh}(\mathcal{Z})\otimes Shv(Y)$ is the map of $Shv(Y)$-algebras $\tilde{F}\otimes\mathrm{id}$, where $\tilde{F}:\mathcal{C}_{\mathrm{Ran}}\to\mathrm{QCoh}(\mathcal{Z})$ is their functor (12.2).

1.2.8. For 12.5.1. This is indeed sufficient for the following reason. The adjoint pair

$$\mathcal{C}_{\mathrm{Ran}\times Y}\otimes\mathrm{QCoh}(\mathcal{Z})\leftrightarrows\mathrm{QCoh}(\mathcal{Z})\otimes Shv(Y): R_{\mathcal{Z},Y}$$

in $\mathcal{C}_{\mathrm{Ran}\times Y}\otimes\mathrm{QCoh}(\mathcal{Z})$-modules with the corresponding monad $R_{\mathcal{Z},Y}$ gives an adjoint pair

$$\mathrm{ind}: M\otimes\mathrm{QCoh}(\mathcal{Z})\leftrightarrows\mathrm{Hecke}_Y(\mathcal{Z}, M) \tag{6}$$

by functoriality (composing with the initial adjoint pair), and the monad obtained in (6) is as desired.

1.2.9. For 12.5.2. The fact that $\Psi:\mathcal{C}_{\mathrm{Ran}\times Y}\to\mathcal{C}_{\mathrm{Ran}}\otimes Shv(Y)$ is right adjoint to $\Phi:\mathcal{C}_{\mathrm{Ran}}\otimes Shv(Y)\to\mathcal{C}_{\mathrm{Ran}\times Y}$ follows from ([16], 9.2.39 last part).

For each J the functor $\boxtimes^R: Shv(X^J\times Y)\to Shv(X^J)\otimes Shv(Y)$ is $Shv(X^J)\otimes Shv(Y)$-linear by ([20], 0.0.7). This is why Ψ is strictly $\mathcal{C}_{\mathrm{Ran}}\otimes Shv(Y)$-linear. Indeed, $\mathcal{C}_{\mathrm{Ran}\times Y}, \mathcal{C}_{\mathrm{Ran}}$ are compactly generated, so it is enough to show that given $(I_1\to J_1)\in Tw(fSet)$, $V_1\in\mathcal{C}^{\otimes I_1}, \mathcal{F}_1\in Shv(X^{J_1}\times Y)$, and $(I_2\to J_2)\in Tw(fSet)$, $V_2\in\mathcal{C}^{I_2}, \mathcal{F}_2\in Shv(X^{J_2})$, $M\in Shv(Y)$ when we act by the pair

$$(ins_{(I_2\to J_2)}(V_2\otimes\mathcal{F}_2), M)$$

on $\Psi(ins_{(I_1\to J_1)}(V_1\otimes\mathcal{F}_1))\widetilde{\to} ins_{(I_1\to J_1)}(V_1\otimes\boxtimes^R(\mathcal{F}_1))$, we get Ψ applied to the result of the action of the pair

$$(ins_{(I_2\to J_2)}(V_2\otimes\mathcal{F}_2), M)$$

on $ins_{(I_1\to J_1)}(V_1\otimes\mathcal{F}_1)$. This works because of ([20], 0.0.31)).

1.2.10. The claim in 12.5.3 follows from the commutativity of the diagram

$$\begin{array}{ccc} Shv(X^J\times Y) & \stackrel{q_*}{\to} & Shv(Y)\\ \downarrow\boxtimes^R & \nearrow \mathrm{R}\Gamma(X^J,\cdot) & \\ Shv(X^J)\otimes Shv(Y) & & \end{array}$$

for $Y\in Sch_{ft}$, here $q: X^J\times Y\to Y$ is the projection.

1.2.11. For 12.5.4. The morphisms $\mathcal{V}_Y * R_{\mathcal{Z},Y} \to R_{\mathcal{Z},Y} \otimes \tilde{F}_Y(\mathcal{V}_Y)$ given by (12.22) are defined as follows for any $\mathcal{V}_Y \in \mathcal{C}_{\mathrm{Ran}\times Y}$. First, we have the adjointness map $\Phi\Psi(\mathcal{V}_Y) \to \mathcal{V}_Y$. Now, $\Phi\Psi(\mathcal{V}_Y) * \Phi(R_{\mathcal{Z}}) \widetilde{\to} \Phi(\Psi(\mathcal{V}_Y) * R_{\mathcal{Z}}) \widetilde{\to} R_{\mathcal{Z},Y} \otimes \tilde{F}_Y(\mathcal{V}_Y)$, here $\Psi(\mathcal{V}_Y) * R_{\mathcal{Z}}$ is of course informal notation (it is understood that we tensor by $Shv(Y)$ the usual action). This gives the map $\mathcal{V}_Y * R_{\mathcal{Z},Y} \to R_{\mathcal{Z},Y} \otimes \tilde{F}_Y(\mathcal{V}_Y)$ denoted (12.22) in their paper.

1.2.12. For 12.5.6. Their adjunction (12.24), that is, $\mathrm{mult}_{\mathrm{QCoh}(\mathcal{Z})} \circ (\tilde{F} \circ \mathrm{id}) : \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}) \leftrightarrows \mathrm{QCoh}(\mathcal{Z}) : R_{\mathcal{Z}}$ is obtained from the adjoint pair $m : \mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}} \leftrightarrows \mathcal{C}_{\mathrm{Ran}} : m^R$ by applying $\otimes_{\mathcal{C}_{\mathrm{Ran}}} \mathrm{QCoh}(\mathcal{Z})$ via the symmetric monoidal functor $\tilde{F} : \mathcal{C}_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathcal{Z})$. Here we use in the LHS the $\mathcal{C}_{\mathrm{Ran}}$-module structure via the product on the second variable. It is understood that the above $R_{\mathcal{Z}}$ is a map of $\mathrm{QCoh}(\mathcal{Z})$-modules.

1.2.13. Their formula (11.21) is a property of the constructible sheaves theory, it is used essentially (!!) in the proof of Prop. 12.5.5.

Proof of their (11.21). Let $Y \in \mathrm{Sch}_{ft}$. Let $h : Shv(Y) \otimes Shv(Y) \to Shv(Y \times Y)$ be the exteriour product. The natural map $hh^R(\vartriangle_* \omega_Y) \to \omega_Y$ yields their map

$$(p_2)_*(u_{Shv(Y)} \otimes^! p_1^! \mathcal{F}) \to (p_2)_*(\vartriangle_* \omega_Y \otimes^! p_1^! \mathcal{F}) \widetilde{\to} \mathcal{F}$$

Let us show this is an isomorphism. We may and do assume $\mathcal{F} \in Shv(Y)^c$. For $K \in Shv(Y)$ we get

$$\mathcal{H}om(K, (p_2)_*(u_{Shv(Y)} \otimes^! p_1^! \mathcal{F})) \widetilde{\to} \mathcal{H}om(p_2^* K, u_{Shv(Y)} \otimes^! p_1^! \mathcal{F}))$$

$$\mathcal{H}om_{Svv(Y\times Y)}(p_2^* K \otimes \mathbb{D}(p_1^! \mathcal{F}), u_{Shv(Y)}) \widetilde{\to} \mathcal{H}om_{Shv(Y)\otimes Shv(Y)}((\mathbb{D}\mathcal{F}) \otimes K, h^R h h^R(\vartriangle_* \omega_Y))$$

We have $h^R h \widetilde{\to} h$, because h is fully faithful, so the above complex identifies with

$$\mathcal{H}om_{Svv(Y\times Y)}((\mathbb{D}\mathcal{F}) \boxtimes K, \vartriangle_* \omega_Y) \widetilde{\to} \mathcal{H}om(p_2^* K, \vartriangle_* \omega_Y \otimes^! (p_1^! \mathcal{F})) \widetilde{\to} \mathcal{H}om(K, (p_2)_*(\vartriangle_* \omega_Y \otimes^! (p_1^! \mathcal{F})))$$

We are done.

1.3. **For version of April 3, 2022.**

1.3.1. If Y is an algebraic stack locally of finite type (a classical one is sufficient, as we are about constructible sheaves theories), then they do not really need to define T^*Y, though this is done somewhere, I think. First, they use the notion of a closed Zarisky subset in T^*Y defined in ([9], A.3.6).

For ([9], A.3.6). Let Y is an algebraic stack locally of finite type, F be a coherent sheaf on Y (placed in coholological degree zero). Then $Tot(F)$ is defined in ([9], A.3.3). My understanding is that a a Zariski-closed subset $Z \subset Tot(F)$ is a compatible family of Zariski closed subsets $Z_S \subset Tot(F \mid_S)$ for any $S \to Y$, where S is an affine scheme of finite type. That is, for $\alpha : S' \to S$ a map of affine schemes of finite type, pick a presentation $Coker(E_1 \to E_0) \widetilde{\to} F \mid_S$ on S, where E_i are locally free sheaves on S. So, $\alpha^* F \widetilde{\to} Coker(\alpha^* E_1 \to \alpha^* E_0)$ on S', and $S' \times_S Tot(E_0) \widetilde{\to} Tot(\alpha^* E_0)$. Our Z_S is a $Tot(E_1)$-invariant closed subset in $Tot(E_0)$. We require that $Z_S \times_S S'$ identifies with $Z_{S'}$ under the above isomorphism.

1.4. **For arxiv version 2.**

1.4.1. For 1.7.1. If $H \in CAlg(\mathrm{DGCat}_{cont})$ is equipped with a t-structure and a symmetric monoidal conservative t-exact functor oblv : $H \to \mathrm{Vect}_e$ then the t-structure is compatible with filtered colimits. It is also right complete: for any $z \in C$ the natural map $\mathrm{colim}_{n\in\mathbb{Z}}\,\tau^{\leq n}z \to z$ is an isomorphism, so it is right complete by ([16], 4.0.10).

I think in the 3rd bullet point of 1.7.2 we have to require $H^{\heartsuit}$ is generated under filtered colimits by $H^{\heartsuit} \cap H^c$.

1.4.2. For 1.7.3. Let $H^{acces,c} \subset H$ be the full subcategory of bounded objects, whose all cohomologies lies in $H^c \cap H^{\heartsuit}$. Then $H^{acces,c}$ is a small stable Vect^c-module category, it is closed under the tensor products (under the assumptions of 1.7.2). Indeed, if $h_i \in H^{acces,c}$ then $\mathrm{oblv}(h_1h_2)\,\tilde{\to}\,\mathrm{oblv}(h_1)\,\mathrm{oblv}(h_2) \in \mathrm{Vect}^c$, so each cohomology of h_1h_2 is sent by oblv to a finite-dimensional vector space, so each cohomology of h_1h_2 lies in $H^c \cap H^{\heartsuit}$. Moreover, if $h \in H^c \cap H^{\heartsuit}$ then h is dualizable in H, and $h^{\vee} \in H^{\heartsuit} \cap H^c$. Indeed, since oblv is symmetric monoidal, $\mathrm{oblv}(h^{\vee})$ is the dual of $\mathrm{oblv}(h)$. Since $\mathrm{oblv}(h) \in \mathrm{Vect}^{\heartsuit} \cap \mathrm{Vect}^c$ and h conservative, $h^{\vee} \in H^{\heartsuit} \cap H^c$ by the assumptions of 1.7.2.

Let $h_i \in H$ with h_1 dualizable in H. Then $\mathcal{H}om_H(1, h_1^{\vee} \otimes h_2)\,\tilde{\to}\,\mathcal{H}om_H(h_1, h_2)$. Indeed, for $V \in \mathrm{Vect}$,

$$\mathrm{Map}_{\mathrm{Vect}}(V, \mathcal{H}om_H(1, h_1^{\vee} \otimes h_2))\,\tilde{\to}\,\mathrm{Map}_H(V \otimes 1_H, h_1^{\vee} \otimes h_2) \\ \tilde{\to}\,\mathrm{Map}_H(V \otimes h_1, h_2)\,\tilde{\to}\,\mathrm{Map}_{\mathrm{Vect}}(V, \mathcal{H}om_H(h_1, h_2))$$

So, the assumptions of 1.7.2 imply that for $h_i \in H^c \cap H^{\heartsuit}$ one has $\mathcal{H}om_H(h_1, h_2) \in \mathrm{Vect}^c$, because $h_1^{\vee} \in H^c \cap H^{\heartsuit}$, and $h_1^{\vee} \otimes h_2 \in H^c \cap H^{\heartsuit}$ also.

1.4.3. For 1.7.6. They use ([11], ch. II.1, Lm. 1.2.4) to see that $H^{acces} \to H$ is t-exact. The map $H^{acces} \to H$ is fully faithful by (HTT, 5.3.5.11).

I think in the 3rd bullet point of 1.7.2 we have to require $H^{\heartsuit}$ is generated under filtered colimits by $H^{\heartsuit} \cap H^c$. Let us then show that for any n, the functor $(H^{access})^{\geq -n} \to H^{\geq -n}$ is an equivalence. We know it is fully faithful, also its image is closed under filtered colimits. So, for each $m \leq n$, $H^{\heartsuit}[m]$ is in the essential image. For each $z \in H^{\geq -n}$ now $z\,\tilde{\to}\,\mathrm{colim}_{m\in\mathbb{Z}}\,\tau^{\leq m}z$ in H by ([16], 4.0.10), because H is right complete. Now $\tau^{\leq m}z \in (H^{access})^{\geq -n}$, and $(H^{access})^{\geq -n}$ is closed under filtered colimits in H. So, $z \in (H^{access})^{\geq -n}$.

1.4.4. For 1.7.10. They assume in this lemma that H is a gentle Tannakian category. If $h \in H^c \cap H^{\heartsuit}$ then for $V \in \mathrm{Vect}^c$ we get

$$\mathrm{Map}_H(h, V \otimes 1)\,\tilde{\to}\,\mathrm{Map}_H(V^{\vee} \otimes h, 1_H)\,\tilde{\to}\,\mathrm{Map}_{\mathrm{Vect}}(V^{\vee}, \mathcal{H}om_H(h, 1_H)) \\ \tilde{\to}\,\mathrm{Map}_{\mathrm{Vect}}(\mathcal{H}om_H(h, 1_H)^{\vee}, V)$$

Both functors $V \mapsto \mathrm{Map}_H(h, V \otimes 1)$, $V \mapsto \mathrm{Map}_{\mathrm{Vect}}(\mathcal{H}om_H(h, 1_H)^{\vee}, V)$ preserve filtered colimits, so this also holds for $V \in \mathrm{Vect}$. We used (HA, 4.6.2.1).

Let $h \in H^{<0}$. Then $\mathcal{H}om_H(h, 1) \in \mathrm{Vect}^{>0}$. Indeed, for $W \in \mathrm{Vect}^{\leq 0}$ we get $\mathrm{Map}_{\mathrm{Vect}}(W, \mathcal{H}om_H(h, 1))\,\tilde{\to}\,\mathrm{Map}_H(W \otimes h, 1) = *$, because $1 \in H^{\heartsuit}$, as oblv is conservative and t-exact.

Let now $h \in (H^{access,c})^{<0}$, in particular h is bounded and each cohomology is in $H^c \cap H^{\heartsuit}$. Then $\mathcal{H}om_H(h,1) \in \mathrm{Vect}^c \cap \mathrm{Vect}^{>0}$, so $\mathcal{H}om_H(h,1)^\vee \in \mathrm{Vect}^{<0}$. This shows that for any n the functor $coinv_H$ sends $H^{\geq -n} \cap H^{\leq 0}$ to $\mathrm{Vect}^{\leq 0}$.

Fort any $z \in H$, $z \widetilde{\to} \lim_n \tau^{\geq -n} z$ in H by ([16], 4.0.10 Remark). Now let $h \in H$. Then for any m, $\tau^{\geq -m}(\lim_n coinv_H(\tau^{\geq -n}h)) \widetilde{\to} \tau^{\geq -m}(coinv_H(\tau^{\geq -m}h))$. Now for $W \in \mathrm{Vect}$,

$$\mathrm{Map}_{\mathrm{Vect}}(\lim_n coinv_H(\tau^{\geq -n}h), W) \widetilde{\to} \lim_{m \in \mathbb{Z}} \mathrm{Map}_{\mathrm{Vect}^{\geq -m}}(\tau^{\geq -m}(\lim_n coinv_H(\tau^{\geq -n}h)), \tau^{\geq -m}W)$$
$$\widetilde{\to} \lim_{m \in \mathbb{Z}} \mathrm{Map}_{\mathrm{Vect}^{\geq -m}}(\tau^{\geq -m}(coinv_H(\tau^{\geq -m}h)), \tau^{\geq -m}W) \widetilde{\to} \lim_{m \in \mathbb{Z}} \mathrm{Map}_{\mathrm{Vect}}(coinv_H(\tau^{\geq -m}h), \tau^{\geq -m}W)$$
$$\widetilde{\to} \lim_{m \in \mathbb{Z}} \mathrm{Map}_H(\tau^{\geq -m}h, (\tau^{\geq -m}W) \otimes 1_H) \widetilde{\to} \mathrm{Map}_H(h, W \otimes 1_H),$$

because H is left complete.

1.4.5. For 14.1.2. The nilpotent cone $\mathfrak{N} \subset \mathfrak{g}$ is the zero fibre of $\mathfrak{g} \to \mathfrak{g}//G$. Here $\mathfrak{g}//G = \mathrm{Spec}\, k[\mathfrak{g}]^G$.

Question What the nilpoitence of $(\mathcal{F}_G, A \in \mathrm{H}^0(X, \mathfrak{g}^\vee_{\mathcal{F}_G} \otimes \Omega))$ means if there is no G-invariant bilinear form on $\mathfrak{g}$?

1.4.6. For 14.1.5. This means that Bun_G is $Nilp$-trancatable in the sense of F.8.6. Recall that for $\mathcal{N} \subset T^*(\mathcal{Y})$ a Zariski-closed subset in a classical algebraic stack $\mathcal{Y}$ locally of finite type with an affine diagonal, $Shv_{\mathcal{N}}(\mathcal{Y})^{constr} = Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^{constr}$.

For 14.1.9. Let $U_i \subset \mathrm{Bun}_G$ for $i \in I$ be as in Thm. 14.1.5, so I is a filtered poset. Then we may consider $\mathrm{colim}_i\, Shv_{Nilp}(\mathcal{U}_i) \cap Shv(\mathcal{U}_i)^c$ in $\mathrm{DGCat}^{non-cocmpl}$ and get

$$\mathrm{Ind}(\mathrm{colim}_i\, Shv_{Nilp}(\mathcal{U}_i) \cap Shv(\mathcal{U}_i)^c) \widetilde{\to} Shv_{Nilp}(\mathrm{Bun}_G)^{access},$$

by my Section 1.4.46. Note also that we get an adjoint pair

$$ren_{\mathrm{Bun}_G, Nilp} : Shv_{Nilp}(\mathrm{Bun}_G)^{access} \leftrightarrows Shv_{Nilp}(\mathrm{Bun}_G)^{ren} : un-ren_{\mathrm{Bun}_G, Nilp}$$

by my Section 1.4.46.

1.4.7. For D.1.1. Let G be reductive group over an algebraically closed field k of characteristic p, $(T \subset B \subset G)$ be a maximal torus and Borel. Then p is called good for G if p is greater than any coefficient of any positive root expressed as a linear combination of simple roots. The bad primes, that is, those which are not good, are as follows: 2 in type B_n, C_n, D_n; 2 and 3 in types G_2, F_4, E_6, E_7; 2,3,5 in type E_8.

Assume first D simple. Then a prime p is called very good for G if it is good for G, and if G is of type A_{n-1} then p does not divide n.

1.4.8. For E.1.6. Let $C \in \mathrm{DGCat}_{cont}$ with a t-structure, $c \in C$. By definition, c has cohomological dimension $\leq n$ iff for any $z \in C^{<-n}$ one has $\mathrm{Hom}_C(c,z) = 0$, here $\mathrm{Hom}_C = \mathrm{H}^0(\mathcal{H}om_C))$, and $\mathcal{H}om_C$ denotes the relative inner hom with respect to Vect action. In other words, this is equivalent to: for any $z \in C^{<-n}$ one has $\mathcal{H}om_C(c,z) \in \mathrm{Vect}^{<0}$.

1.4.9. For E.2.10. This is close to ([11], ch. I.3, Lm. 2.4.5). I think in (a) they meant injective map $c_0 \to c$, and not just a non-zero map.

1.4.10. Attention! In the version of April 3, 2022 a nonstandard definition of a compactly generated t-structure is used!!! (see their E.7.4).

1.4.11. For E.4.4. Let Y be a smooth scheme of finite type. By naive duality they mean the duality $K \mapsto \mathcal{H}om(K, e_Y)$, where $\mathcal{H}om$ is the inner hom with respect to the $\otimes$-monoidal structure.

1.4.12. For E.5.4. Let Y be a smooth scheme of finite type, $\mathcal{N} \subset T^*(Y)$ be a conical Zariski closed subset of $T^*(Y)$. The functor $Shv_{\mathcal{N}}(Y)^{access} \to Shv_{\mathcal{N}}(Y)$ is fully faithful as it composition with $Shv_{\mathcal{N}}(Y) \to Shv(Y)$ is fully faithful. To see that $Shv_{\mathcal{N}}(Y)^{access} \to Shv_{\mathcal{N}}(Y)$ given by (E.9) is t-exact, they use ([11], II.1, Lemma 1.2.4).

To see that $(Shv_{\mathcal{N}}(Y)^{access})^{\geq -n} \to Shv_{\mathcal{N}}(Y)^{\geq -n}$ we use the right completeness of $Shv(Y)$. Namely, for any $F \in Shv(Y)$, $F \widetilde{\to} \operatorname{colim}_n \tau^{\leq n} F$ by ([16], 4.0.10). Since this isomorphism holds in $Shv(Y)$, for any $F \in Shv_{\mathcal{N}}(Y)$ it also holds if the colimit is understood in $Shv_{\mathcal{N}}(Y)$ by the way. Since the essential image of $(Shv_{\mathcal{N}}(Y)^{access})^{\geq -n} \to Shv_{\mathcal{N}}(Y)^{\geq -n}$ contains $Shv_{\mathcal{N}}(Y)^{[-n,m]}$ for $m \geq -n$, the functor $(Shv_{\mathcal{N}}(Y)^{access})^{\geq -n} \to Shv_{\mathcal{N}}(Y)^{\geq -n}$ is indeed an equivalence.

1.4.13. Let us try to formalize this situation. Let $C^{constr} \in \mathrm{DGCat}^{non-cocmpl}$ with C^{constr} small. Assume C^{constr} is equipped with a bounded t-structure and $C = \mathrm{Ind}(C^{constr})$, so $C \in \mathrm{DGCat}_{cont}$. Recall that by ([11], II.1, Lemma 1.2.4), C is equipped with a unique t-structure compatible with filtered colimits and accessible such that the functor $C^{constr} \to C$ is t-exact. Let $(D^{constr})^{\heartsuit} \subset (C^{constr})^{\heartsuit}$ be a full Serre abelian subcategory. Let $D^{constr} \subset C^{constr}$ be the stable subcategory generated by $(D^{constr})^{\heartsuit}$, so D^{constr} is small, set $D = \mathrm{Ind}(D^{constr})$. The inclusion $D^{constr} \subset C^{constr}$ is compatible with the t-structure on C^{constr}, so D^{constr} inherits a t-structure. In turn, D inherits a t-structure from D^{constr}, which is accessible and compatible with filtered colimits. We have a natural functor $D \to C$ obtained by continuous extension of the inclusion $D^{constr} \to C$. The functor $D \to C$ is fully faithful by (HTT, 5.3.5.11) and t-exact by ([11], II.1, Lemma 1.2.4).

By ([16], 9.3.18), the t-structure on C is right complete.

Let ${}_DC \subset C$ be the full subcategory of $c \in C$ such that each cohomology of c lies in $D^{\heartsuit}$. Note that ${}_DC \subset C$ is stable under filtered colimits. Let us show that $D^{\heartsuit} \subset C^{\heartsuit}$ is also a Serre subcategory, that is, closed under extensions and subquotients. Note that for $K \in C^{\heartsuit}$ we have $K \in D^{\heartsuit}$ iff for any $L \in (C^{constr})^{\heartsuit}$ and an injection $L \hookrightarrow K$ we have $L \in (D^{constr})^{\heartsuit}$. This gives that $D^{\heartsuit}$ is closed under subobjects. Let $0 \to K \to L \to M \to 0$ be an exact sequence in $C^{\heartsuit}$ with $K, M \in D^{\heartsuit}$. Then L lies in D. Besides, $D \cap C^{\heartsuit} = D^{\heartsuit}$, so $L \in C^{\heartsuit}$. Similarly, if $0 \to K_1 \to K \to L \to 0$ is an exact sequence in $C^{\heartsuit}$ with $K_1, K \in D^{\heartsuit}$ then $L \in D \cap C^{\heartsuit} = D^{\heartsuit}$. Thus, $D^{\heartsuit} \subset C^{\heartsuit}$ is a Serre subcategory.

This implies that ${}_DC$ is stable under formation of fibres and cofibres of a morphism. So, ${}_DC$ is stable. It is also presentable as the fibre product of presentable categories ${}_DC \widetilde{\to} C \times_{\prod_n C^{\heartsuit}} \prod_n D^{\heartsuit}$. Here the functor $C \to \prod_n C^{\heartsuit}$ is given by taking all the cohomologies. Further,

$$ {}_DC^{\leq 0} \widetilde{\to} C^{\leq 0} \times_{\prod_{n \leq 0} C^{\heartsuit}} \prod_{n \leq 0} D^{\heartsuit}, $$

so ${}_DC^{\leq 0}$ is presentable, so the t-structure on ${}_DC$ is accessible. Thus, ${}_DC \in \mathrm{DGCat}_{cont}$.

We have a (t-exact) full embedding $D \to {}_DC$, because the composition with ${}_DC \to C$ is a full embedding. We claim that for any n, $D^{\geq n} \to ({}_DC)^{\geq n}$ is an equivalence. Indeed, for any $m \geq n$, $D^{[n,m]} \to ({}_DC)^{[n,m]}$ is an equivalence. In adddition, ${}_DC$ is right complete, because C was right complete (we are using here [16], 4.0.10). So, for $K \in C^{\geq n}$ we have $K \widetilde{\to} \mathrm{colim}_m \tau^{\leq m} K$, and each $\tau^{\leq m} K \in D^{\geq n}$. Since $D \to C$ is continuous, $K \in D^{\geq n}$ also.

Assume now that C is left complete. Then ${}_DC$ is also left complete. Indeed, since $C \widetilde{\to} \lim_{n \in \mathbb{Z}^{op}} C^{\geq -n}$, we have a fully faithful embedding ${}_DC \to \lim_{n \in \mathbb{Z}^{op}} ({}_DC)^{\geq -n}$. If now $(x_n)_{n \geq 0} \in \lim_{n \in \mathbb{Z}^{op}} ({}_DC)^{\geq -n}$ is a compatible collection, let $x = \lim_n x_n$ calculated in C. Then for any $n \geq 0$, $\tau^{\geq n} x \widetilde{\to} x_n \in ({}_DC)^{\geq -n}$, so $c \in {}_DC$.

Thus, finally, ${}_DC$ identifies with the left completion of D.

1.4.14. For E.6. Let Y be a scheme if finite type, not necessarily smooth, $\mathcal{N} \subset T^*(Y)$ a closed conical subset. Then the definition of $Shv_{\mathcal{N}}(Y)$ is not given. I accept the following: define $\mathrm{Perv}_{\mathcal{N}}(Y) \subset \mathrm{Perv}(Y)$ as the full subcategory of those $F \in \mathrm{Perv}(Y)$ such that locally in Zariski topology there is a closed embedding $f : Y \to Y'$ with Y' smooth such that $SingSupp(f_*F) \subset (df^*)^{-1}(\mathcal{N}) \subset T^*(Y')$, where $df^* : T^*(Y') \times_{Y'} Y \to T^*(Y)$ is the codifferential.

I assume $\mathrm{Perv}_{\mathcal{N}}(Y) \subset \mathrm{Perv}(Y)$ is a Serre subcategory, so that we define $Shv_{\mathcal{N}}(Y) \subset Shv(Y)$ as the full subcategory of those K such that for any n, $H^n(K) \in \mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(Y))$. We also have

$$Shv_{\mathcal{N}}(Y)^{constr} := Shv_{\mathcal{N}}(Y) \cap Shv(Y)^{constr}$$

and $Shv_{\mathcal{N}}(Y)^{access}$ defined as in their E.5.5.

1.4.15. For E.6.5. It is meant there that $\mathcal{F}_1 \in Shv(Y_1)^{constr}$.

The following holds however and is used in F.6.3. Let $f : S' \to S$ be a smooth morphism of schemes of finite type, $N_S \subset T^*(S)$ a closed conical subset. Let $N_{S'}$ be the image of $N_S \times_S S'$ under the codifferential map $T^*(S) \times_S S' \to T^*(S')$. Then for $F \in Shv_{N_S}(S)$, $f^*F \in Shv_{N_{S'}}(S')$. This is obtained from the property in E.6.5.

1.4.16. For E.7.3. Let $\mathcal{F}_i \in Shv_{\mathcal{N}_i}(Y_i)$, here Y_i is a classical scheme of finite type. Let us show that $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in Shv_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2)$.

Step 1: assume $\mathcal{F}_i \in Shv_{\mathcal{N}_i}(Y_i)^{\leq n_i}$ for some $n_1, n_2 \in \mathbb{Z}$. Then $\mathcal{F}_1 \otimes \mathcal{F}_2 \in (Shv_{\mathcal{N}_1}(Y_1) \otimes Shv_{\mathcal{N}_2}(Y_2))^{\leq n_1+n_2}$. Given $i \leq n_1 + n_2$ let us show that

$$\mathrm{H}^i(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \in \mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2))$$

Pick $r \in \mathbb{Z}$ small enough then if we replace $\mathcal{F}_1$ by $\tau^{\geq r}\mathcal{F}_1$, the above cohomology does not change, in view of their Pp. E.7.2. Similarly, we may replace $\mathcal{F}_2$ by $\tau^{\geq r}\mathcal{F}_2$ without changing this cohomology. For the objects $\mathcal{F}_i \in Shv_{\mathcal{N}_i}(Y_i)^{constr}$ the claim is clear.

Step 2: let $\mathcal{F}_i \in Shv_{\mathcal{N}_i}(Y_i)$. Since $Shv(Y_i)$ is right complete, $\mathcal{F}_i \widetilde{\to} \mathrm{colim}_{m \in \mathbb{Z}} \tau^{\leq m}\mathcal{F}_i$, where the colimit is calculated in $Shv_{\mathcal{N}_i}(Y_i)$. So,

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 \widetilde{\to} \underset{m_1, m_2}{\mathrm{colim}}\, \tau^{\leq m_1}\mathcal{F}_1 \otimes \tau^{\leq m_2}\mathcal{F}_2$$

in $Shv(Y_1 \times Y_2)$. For each m_1, m_2, $\tau^{\leq m_1}\mathcal{F}_1 \otimes \tau^{\leq m_2}\mathcal{F}_2 \in Shv_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2)$ by Step 1, and $Shv_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2) \subset Shv(Y_1 \times Y_2)$ is closed under filtered colimits. We are done.

1.4.17. For E.7.4. Recall that for $C \in \mathrm{DGCat}_{cont}$, $C^{<\infty}, C^c \subset C$ are stable subcategories. Their condition that C is generated under filtered colimits by shifts of the objects of $C^{\le 0} \cap C^c$ is reformulated as follows: C is generated under filtered colimits by the stable subcategory $C^{<\infty} \cap C^c$. Note that $C^{<\infty} \cap C^c \in \mathrm{DGCat}^{non-cocmpl}$, so $\mathrm{Ind}(C^{<\infty} \cap C^c) \in \mathrm{DGCat}_{cont}$, and we have the natural functor $\mathrm{Ind}(C^{<\infty} \cap C^c) \to C$ always extending by continuity the inclusion $C^{<\infty} \cap C^c \hookrightarrow C$. By (HTT, 5.3.5.11), the latter functor is always fully faithful. So, their second condition is that this functor is essentially surjective.

Clearly, if $i : \mathrm{Ind}(C^{<\infty} \cap C^c) \to C$ is an equivalence then the following condition (C) holds: if $\mathcal{H}om_C(c_0, c) = 0$ for any $c_0 \in C^{\le 0} \cap C^c$ then $c = 0$. Conversely, assume (C). It is reformulated as follows: given $c \in C$, if $\mathcal{H}om_C(c_0, c) = 0$ for any $c_0 \in \mathrm{Ind}(C^{<\infty} \cap C^c)$ then $c = 0$. Let i^R be the right adjoint to the inclusion i. So, (C) means that $\mathrm{Ker}(i^R) = 0$. By ([11], I.1, 5.4.5), (C) is equivalent to the fact that i is essentially surjective.

Their E.7.5 is my ([16], 9.3.10). One needs to assume in addition here that $F : C_1 \to C_2$ is continuous.

1.4.18. For E.7.5. This lemma holds more generally for C_i, C equipped with accessible t-structures which are compatible with filtered colimits (by [15], C.4.4.1). Indeed, in this case $C^{\le 0}$ is a Grothendieck prestable category.

1.4.19. For E.8.2. Let Z be an irreducible scheme of finite type and η_Z its generic point. Recall that $Shv(\eta_Z) := \mathrm{colim}_U\, Shv(U)$, where U runs through the category of non-empty open subschemes of Z. The fact that $Shv(\eta_Z)$ is equipped with a t-structure, which is accessible and compatible with filtered colimits is explained in ([16], 9.3.19). Note also that $Shv(\eta_Z) \subset Shv(Z)$ is a full subcategory of sheaves, which are written as j_*F for any open non empty subscheme $j : U \to Z$ and some $F \in Shv(U)$ (the functor $j_* : Shv(U) \to Shv(Z)$ is fully faithful).

Define $\mathrm{IndLisse}(\eta_Z) := \mathrm{colim}_U\, IndLisse(U)$. The fact that $\mathrm{IndLisse}(\eta_Z) \to Shv(\eta_Z)$ is fully faithful follows from ([16], 9.2.47). To see that it is essenially surjective, let $F \in Shv(\eta_Z) \subset Shv(Z)$. Then any compact object of $Shv(\eta_Z)$, by ([6], Lemma 1.9.5) writes as the image $ins_U(F_U)$ of some $F_U \in Shv(U)^c$ in $Shv(\eta_Z)$. There is a non-empty open subscheme $V \subset U$ such that $F_U \mid_V \in Lisse(V)$, so $ins_U(F_U)$ lies in the essential image of $\mathrm{IndLisse}(\eta_Z)$. Since the full subcategory $\mathrm{IndLisse}(\eta_Z) \subset Shv(\eta_Z)$ is closed under filtered colimits, it coincides with $Shv(\eta_Z)$, because $Shv(\eta_Z)$ is compactly generated.

1.4.20. For E.9.2. Here Y is a scheme of finite type. We have indeed such a functor $\mathrm{IndLisse}(X) \otimes Shv_{\mathcal{N}}(Y)^{access} \to Shv_{0\times\mathcal{N}}(X \times Y)^{access}$, which is clearly fully faithful.

Indeed, $Shv_{0\times\mathcal{N}}(X \times Y)^{access} \subset Shv_{0\times\mathcal{N}}(X \times Y) \subset Shv(X \times Y)$ are fully faithful embeddings, and for $E \in Lisse(X), K \in Shv_{\mathcal{N}}(Y)^{constr}$, $E \boxtimes K \in Shv_{0\times\mathcal{N}}(X \times Y)^{constr}$. If D is the smallest stable subcategory of $\mathrm{IndLisse}(X) \otimes Shv_{\mathcal{N}}(Y)^{access}$ containing all objects $E \boxtimes K$ as above then $\mathrm{Ind}(D) \widetilde{\to} \mathrm{IndLisse}(X) \otimes Shv_{\mathcal{N}}(Y)^{access}$ naturally. Since $Shv_{0\times\mathcal{N}}(X \times Y)^{access}$ is a stable subcategory of $Shv(X \times Y)$ and is closed under filtered colimits, we get the desired full embedding.

They appeal to ([9], A.3.8). The following is used inside: Let Y_2 be a smooth scheme of finite type, $i : Y_1 \hookrightarrow Y_2$ be an irreducible closed subscheme, $U \subset Y_1$ a smooth open dense in Y_1. Let F be a nonzero irreducible perverse sheaf on Y_1, which is a shifted local system on U. Then $SingSupp(i_*F)$ containes the conormal to Y_1 at its generic point. Indeed, $SingSupp(F)$ containes the zero section $Y_1 \subset T^*(Y_1)$, and $SingSupp(i_*F) = (di^*)^{-1}(SingSupp(F))$, where $(di)^* : Y_1 \times_{Y_2} T^*(Y_2) \to T^*(Y_1)$ is the codifferential by ([1], Section E.6.5).

For the proof of ([9], A.3.8) in the ind-constructible context, for ([9], Section A.5.6). Here $\mathcal{Y}_1, X$ are smooth irreducible schemes, X is proper. Now F is an irreducible perverse sheaf on $\mathcal{Y}_1 \times X$, which is lisse on $\overset{\circ}{\mathcal{U}} \subset \mathcal{Y}_1 \times X$, where $D' = \mathcal{Y}_1 \times X - \overset{\circ}{\mathcal{U}}$ is a divisor on $\mathcal{Y}_1 \times X$, and $\overset{\circ}{\mathcal{U}}$ is a maximal open on which F is lisse. Moreover, $\mathcal{N} \subset T^*(\mathcal{Y}_1)$ is a closed half-dimensional conical subset and $SingSupp(F) \subset \mathcal{N}' := \mathcal{N} \times \{\text{zero section}\}$. By Beilinson, $SingSupp(F)$ is half-dimensional. If there is an irreducible component of $SingSupp(F)$ which maps dominantly to $\mathcal{Y}_1 \times X$, then it is contained in the zero section of $T^*(\mathcal{Y}_1 \times X)$. So, indeed, for any irreducible component D'_α of D' there is an irreducible component $\mathcal{N}'_\alpha$ of $SingSupp(F)$ such that its image under $T^*(\mathcal{Y}_1 \times X) \to \mathcal{Y}_1 \times X$ contains D'_α and is not contained in the zero section of $T^*(\mathcal{Y}_1 \times X)$. This shows that the image of $\mathcal{N}'_\alpha$ in $\mathcal{Y}_1 \times X$ is D'_α. Any irreducible component of $\mathcal{N}'$ is of the form $\mathcal{N}_\beta \times X$ for some irreducible component $\mathcal{N}_\beta$ of $\mathcal{N}$. So, three is an irreducible component $\mathcal{N}_\alpha$ of $\mathcal{N}$ such that $\mathcal{N}'_\alpha \subset \mathcal{N}_\alpha \times X$. This must be an equality because they are of the same dimension.

In ([9], A.5.7) they use the following. Let $\pi : \mathcal{Z} \to \mathcal{X}$ be a smooth proper morphism of schemes of finite type with $\mathcal{X}$ smooth. Let E be a local system on $\mathcal{Z}$. Then π_*E is lisse, that is, each of its usual cohomology sheaves is lisse (equivalently, each of its perverse cohomology sheaves is a lisse perverse sheaf). Indeed, this follows from ([5], 5.1.2) because E is ULA with respect to π.

We indeed have a map as they indicate

$$\overset{\circ}{\pi}{}^*_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes \overset{\circ}{\pi}{}^*_X(\mathcal{F}_X) \to \overset{\circ}{\mathcal{F}}$$

It comes from a morphism $\mathcal{F}_X \otimes \mathbb{D}(\mathcal{F}_X) \to \omega_X$, which in turn gives $\overset{\circ}{\pi}{}^!_X(\mathcal{F}_X \otimes \mathbb{D}(\mathcal{F}_X)) \to \omega_{\overset{\circ}{\mathcal{Y}} \times X}$. Tensoring with $\overset{\circ}{\mathcal{F}}$, we get a morphism

$$\overset{\circ}{\mathcal{F}} \otimes^! (\overset{\circ}{\pi}{}^*_X(\mathcal{F}_X) \otimes \overset{\circ}{\pi}{}^!_X(\mathbb{D}(\mathcal{F}_X))) \widetilde{\to} \overset{\circ}{\mathcal{F}} \otimes^! (\overset{\circ}{\pi}{}^*_X(\mathcal{F}_X) \otimes \overset{\circ}{\pi}{}^*_X(\mathbb{D}(\mathcal{F}_X)))[2m] \to \overset{\circ}{\mathcal{F}},$$

where $m = \dim \overset{\circ}{\mathcal{Y}}$. Since $\mathcal{F}_X$ is a shifted local system, by remark below, this rewrites as a morphism

$$(\overset{\circ}{\mathcal{F}} \otimes^! \overset{\circ}{\pi}{}^!_X(\mathbb{D}(\mathcal{F}_X))) \otimes \overset{\circ}{\pi}{}^*_X(\mathcal{F}_X) \to \overset{\circ}{\mathcal{F}}$$

We compose the latter with the morphism coming from adjointness

$$\overset{\circ}{\pi}{}^*_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \widetilde{\to} \overset{\circ}{\pi}{}^*_{\mathcal{Y}}(\overset{\circ}{\pi}_{\mathcal{Y}})_*(\overset{\circ}{\mathcal{F}} \otimes^! \overset{\circ}{\pi}{}^!_X(\mathbb{D}(\mathcal{F}_X))) \to (\overset{\circ}{\mathcal{F}} \otimes^! \overset{\circ}{\pi}{}^!_X(\mathbb{D}(\mathcal{F}_X)))$$

This finally gives a morphism

$$\overset{\circ}{\pi}{}^*_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes \overset{\circ}{\pi}{}^*_X(\mathcal{F}_X) \to \overset{\circ}{\mathcal{F}}$$

They claimed that

$$\overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X) \widetilde{\to} \overset{\circ}{\mathcal{F}}_Y \boxtimes \mathcal{F}_X$$

This is wrong as stated, one needs to add shifts. Let $n = \dim X, m = \dim \overset{\circ}{\mathcal{Y}}$, here X and $\mathcal{Y}$ are smooth. So,

$$\overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X) \widetilde{\to} \overset{\circ}{\mathcal{F}}_Y \boxtimes \mathcal{F}_X[2n+2m]$$

You actually meant $\overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes^! \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X) \widetilde{\to} \overset{\circ}{\mathcal{F}}_Y \boxtimes \mathcal{F}_X$. So, (A.18) should also be corrected, I would write it as

$$\overset{\circ}{\pi}{}^*_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes_{\mathcal{E}nd(\mathcal{F}_X)} \overset{\circ}{\pi}{}^*_X(\mathcal{F}_X) \to \overset{\circ}{\mathcal{F}} \tag{7}$$

instead of ! that you used. Here we denoted by $\mathcal{E}nd(\mathcal{F}_X) \in Alg(\text{Vect})$ the inner hom with respect to the Vect-action on $Shv(X)$. The relative tensor product is calculated with respect to the Vect-module structure on $Shv(\overset{\circ}{\mathcal{Y}} \times X)$.

We may also rewrite (7) as

$$\overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes^!_{\mathcal{E}nd(\mathcal{F}_X)} \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X) \to \overset{\circ}{\mathcal{F}}$$

We have to underline here that the tensor product in $Shv(\overset{\circ}{\mathcal{Y}} \times X)$ is taken with respect to the $\otimes^!$-monoidal structure.

Remark: let Z be a scheme of finite type, $K_i \in Shv(Z)^{constr}$, E a local system on Z. Then in the constructible context we get $K_1 \otimes^! (K_2 \otimes E) \widetilde{\to} E \otimes (K_1 \otimes^! K_2)$. $\Box$

Now let $i_y : \operatorname{Spec} k \overset{y}{\to} \overset{\circ}{\mathcal{Y}}$ be the inclusion. Then

$$i^!_y(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \widetilde{\to} \mathrm{R}\Gamma(X, \mathcal{F}_X \otimes^! (\mathbb{D}\mathcal{F}_X)) \widetilde{\to} \mathcal{E}nd(\mathcal{F}_X) \in Alg(\text{Vect})$$

So, for the inclusion $i : y \times X \to \overset{\circ}{\mathcal{Y}} \times X$ we get

$$i^!(\overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes^!_{\mathcal{E}nd(\mathcal{F}_X)} \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X)) \widetilde{\to} (i^! \overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}})) \otimes^!_{\mathcal{E}nd(\mathcal{F}_X)} i^! \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X)) \tag{8}$$

As above $n = \dim X, m = \dim \mathcal{Y}$. Then

$$i^! \overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \widetilde{\to} i^!_y(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes \omega_X \widetilde{\to} \mathcal{E}nd(\mathcal{F}_X) \otimes \omega_X$$

and $i^! \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X) \widetilde{\to} \mathcal{F}_X$. So, (8) identifies with $\mathcal{F}_X$. Now applying $i^!$ to (7) we get the identity indeed.

Both sides of (7) are objects of $\text{IndLisse}(\overset{\circ}{\mathcal{Y}} \times X)$, for the LHS this is because for each $r \geq 0$,

$$\overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes^! \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X) \otimes (\mathcal{E}nd(\mathcal{F}_X))^{\otimes r} \in Lisse(\overset{\circ}{\mathcal{Y}} \times X)$$

So, since after applying $i^!$ we get an isomorphism, the map (7) itself is an isomorphism.

Note also that the tensor product in $\overset{\circ}{\pi}{}^!_{\mathcal{Y}}(\overset{\circ}{\mathcal{F}}_{\mathcal{Y}}) \otimes^!_{\mathcal{E}nd(\mathcal{F}_X)} \overset{\circ}{\pi}{}^!_X(\mathcal{F}_X)$ may be understood as the one in $\text{IndLisse}(\overset{\circ}{\mathcal{Y}} \times X)$ with respect to the $\otimes^!$-monoidal structure.

To finish the proof they use the following property of singular support.

1.4.21. *Claim.* Let $f : Y_1 \to Y_2$ be a proper map of schemes of finite type, $\mathcal{F} \in Shv(Y_1)^{constr}$. Then $SingSupp(f_*\mathcal{F})$ is contained in the image under the projection $T^*(Y_2) \times_{Y_2} Y_1 \to T^*(Y_2)$ of $(df^*)^{-1}(SingSupp(\mathcal{F}))$, where $df^* : T^*(Y_2) \times_{Y_2} Y_1 \to T^*(Y_1)$ is the codifferential.

Proof. Apply [3], Lemma 2.2(ii). □

1.4.22. For E.9.5. Let X be a smooth scheme of finite type, Y be a scheme of finite type. Consider the full subcategory $C \subset \mathrm{Ind}(\mathrm{Perv}(X \times Y))$ consisting of K such that for any inclusion $F \subset K$ in $\mathrm{Ind}(\mathrm{Perv}(X \times Y))$ with $F \in \mathrm{Perv}(X \times Y)$, there is $N \subset T^*(Y)$ closed conical half-dimensional such that $SingSupp(F) \subset \{0\} \times N$.

Claim C is a Serre subcategory.
Proof: it is clearly closed under subobjects. Let now $K \in C$ and $K \to K' \to 0$ be a quotient of K in $\mathrm{Ind}(\mathrm{Perv}(X \times Y))$. Let $F \in \mathrm{Perv}(X \times Y)$ be a subobject of K'. Let $K_1 = \mathrm{Ker}(K \to K')$. Pick I small filtered and write $K_1 \widetilde{\to} \mathrm{colim}_{i \in I} K_1^i$ with $K_1^i \in \mathrm{Perv}(X \times Y)$. Then $K' \widetilde{\to} \mathrm{colim}_{i \in I} K/K_1^i$, where the quotient is taken in $\mathrm{Ind}(\mathrm{Perv}(X \times Y)$. So, the map $F \to K'$ factors through $F \to K/K_1^i$ for some i. For this i let $\tilde{F}$ be the preimage of F under $K \to K/K_1^i$, so we get an exact sequence $0 \to K_1^i \to \tilde{F} \to F \to 0$ in $\mathrm{Perv}(X \times Y)$. Now $SingSupp(F) \subset SingSupp(\tilde{F})$. Thus, C is closed under quotients. It is closed under extensions for the same reason. □

Let say $\mathrm{Perv}_{sm\, X} \subset \mathrm{Perv}(X \times Y)$ be the full subcategory of those $K \in \mathrm{Perv}(X \times Y)$ for which there is a closed conical half-dimensional subset $\mathcal{N} \subset T^*Y$ with $SingSupp(K) \subset 0 \times \mathcal{N}$. Note that $\mathrm{Perv}_{sm\, X} \subset \mathrm{Perv}(X \times Y)$ is a Serre subcategory. By ind-extending, we get a natural map $\mathrm{Ind}(\mathrm{Perv}_{sm\, X}(X \times Y)) \to \mathrm{Ind}(\mathrm{Perv}(X \times Y))$, which is fully faithful and factors through $C \subset \mathrm{Ind}(\mathrm{Perv}(X \times Y))$. It is clear that the obtained map $\mathrm{Ind}(\mathrm{Perv}_{sm\, X}(X \times Y)) \to C$ is an equivalence. In particular, C is presentable.

We now can consider the full subcategory $\bar{C}$ of $Shv(X \times Y)$ of K such that for any n, $\mathrm{H}^n(K) \in C$. This is a stable subcategory of $Shv(X \times Y)$, which is closed under filtered colimits. By construction, $\bar{C}$ is presentable: this is the limit

$$Shv(X \times Y) \times_{\prod_n \mathrm{Ind}(\mathrm{Perv}(X \times Y))} \prod_n C$$

of presentable categories. So, $\bar{C} \in \mathrm{DGCat}_{cont}$.

Since the map $\bar{C} \to Shv(X \times Y)$ is an exact functor preserving filtered colimits, it preserves all colimits.

The fully faithfulness of $\mathrm{QLisse}(X) \otimes Shv(Y) \to Shv(X \times Y)$ is explained in their proof of E.9.5. Let us explain it takes values in $\bar{C}$. If $F_1 \in \mathrm{QLisse}(X), F_2 \in Shv(Y)^{constr}$ then $F_1 \boxtimes F_2 \in \bar{C}$, as for this we may assume F_2 perverse and use Beilison's result ([3], Th. in 1.2) and their (E.12). Now let $\mathcal{E}$ be the smallest stable subcategory $\mathcal{E}$ of $Shv(X \times Y)$ containing such objects $F_1 \boxtimes F_2$. Then $\mathcal{E} \subset \bar{C}$. Now, the objects $F_1 \boxtimes F_1$ with $F_1 \in \mathrm{QLisse}(X), F_2 \in Shv(Y)^{constr}$ generate the essential image of $\mathrm{QLisse}(X) \otimes Shv(Y) \to Shv(X \times Y)$. Since $\bar{C} \to Shv(X \times Y)$ preserves all colimits, we see that the essential image is contained in $\bar{C}$ (by [11], ch. I.1, 5.4.5).

1.4.23. For E.9.9. The following is used in the proof without an explanation. Let $E \in \mathrm{QLisse}(X), \mathcal{F}' \in Shv_{0\times\mathcal{N}}(X\times Y)$. Then $p_X^!(E)\otimes^! \mathcal{F}' \in Shv_{0\times\mathcal{N}}(X\times Y)$. Here $p_X : X\times Y \to X$ is the proejction, and $\mathcal{N}\subset T^*(Y)$ is half-dimensional, X is smooth and proper.

The desired claim follows from my Lemma A.1.3.

1.4.24. For F.2.4, let $\mathcal{Y}$ be a classical quasi-compact algebraic stack locally of finite type. If $f: S\to\mathcal{Y}$ is a smooth cover, where S is an affine classical affine scheme of finite type, we have $Shv(Y)\widetilde{\to} Tot(Shv(S^\bullet))$, where $S^\bullet$ is the Cech nerve of $f: S\to\mathcal{Y}$. For $[n]\in \boldsymbol{\Delta}$ let $f^n : S^n\to\mathcal{Y}$ be the corresponding map, it is smooth. Now for $\mathcal{F},\mathcal{F}'\in Shv(\mathcal{Y})$, $\mathcal{H}om_{Shv(\mathcal{Y})}(\mathcal{F},\mathcal{F}')\widetilde{\to} Tot(\mathcal{H}om_{Shv(S^n)}((f^n)^!\mathcal{F},(f^n)^!\mathcal{F}')$ by ([16], 9.2.49).

Since f^n is smooth, $(f^n)^![-\dim.\mathrm{rel}(f^n)]$ is t-exact for the perverse t-structures.

They use the following: let $C\in\mathrm{DGCat}_{cont}$ with a t-structure and $\mathcal{F}\in C^{\le N},\mathcal{F}'\in C^{\ge m}$. Then $\mathcal{H}om_C(\mathcal{F},\mathcal{F}')\in\mathrm{Vect}^{\ge m-N}$, this follows from ([16], Lm. 9.3.2). For this reason in their notation $\mathcal{H}om_{Shv(S^n)}((f^n)^!\mathcal{F},(f^n)^!\mathcal{F}')\in\mathrm{Vect}^{\ge m-N}$.

Note that F.2.4 does not hold for $\mathcal{Y}$ which are not quasi-compact.

1.4.25. For F.3.2. A system of compact generators of $Shv(\mathcal{Y})$ whose Verdier duals are compact is sufficient, because $Shv(\mathcal{Y})^c$ is idempotent complete.

1.4.26. For F.3.4. Let $Y\in\mathrm{Sch}_{ft}$, G an algebraic group of finite type, $\mathcal{F}'\in Shv(Y/G)$, $\pi_{pt}: pt\to pt/G$ and $q: Y/G\to pt/G$ the natural maps. Let us explain their isomorphism

$$\mathcal{F}'\otimes^! q^!(\pi_{pt})_*(e)\widetilde{\to}\mathcal{F}'\otimes q^*((\pi_{pt})_*e)[2\dim(G)]$$

We have $(\pi_{pt})_*(e)\in Lisse(pt/G)$, so it is dualizable with respect to the $\otimes$-monoidal structure on $Shv(pt/G)$. For $E\in Lisse(S)$, where S is a scheme of finite type, write temporarily $E^\vee\in Lisse(E)$ for its dual with respect to the $\otimes$-monoidal structure, so $E^\vee=\mathcal{H}om(E,e)$, where $\mathcal{H}om$ is the inner hom with respect to the $\otimes$-monoidal structure. One has

$$\mathcal{F}'\otimes^! q^!(\pi_{pt})_*(e)\widetilde{\to}\mathcal{H}om(\mathbb{D}(q^!(\pi_{pt})_*(e)),\mathcal{F}')\widetilde{\to}\mathcal{H}om(q^*(\pi_{pt})_!e,\mathcal{F}')\widetilde{\to}(q^*(\pi_{pt})_!e)^\vee\otimes\mathcal{F}'$$

For any $E\in Lisse(pt/G)$, one has $q^*(E^\vee)\widetilde{\to}(q^*E)^\vee$. We have

$$(\pi_{pt})_*e\widetilde{\to}\mathcal{H}om((\pi_{pt})_!,e[-2\dim(G)])\widetilde{\to}((\pi_{pt})_!)^\vee[-2\dim(G)]$$

So, $(q^*(\pi_{pt})_!e)^\vee\widetilde{\to} q^*(\pi_{pt})_*e[2\dim(G)]$. We are done.

1.4.27. For F.5.1. Let Y be a classical algebraic stack with an affine diagonal, so Y is locally of finite type. (In F.5.1 it is not assumed that Y is Verdier compatible!) Recall that the truncation functors for the perverse t-structure preserve the subcategory $Shv(Y)^{constr}$, so induce a t-structure on it. Besides, the $\otimes$-tensor product makes $Shv(Y)^{constr}$ into a symmetric monoidal category (same for $\otimes^!$). For $F_1\in Shv(Y)^c, F_2\in Shv(Y)^{constr}$ we get $F_1\otimes F_2\in Shv(Y)^c$.

Note that if Y is quasi-compact then any object of $Shv(Y)^{constr}$ is bounded.

Let Y be a quasi-compact classical algebraic stack with an affine diagonal. Then the t-structure on $Shv(Y)^{ren}:=\mathrm{Ind}(Shv(Y)^{constr})$ is compactly generated (even in the stronger sense of their Section E.7.4) by ([11], II.1, Lm. 1.2.4).

The functor $un-ren_Y: Shv(Y)^{ren}\to Shv(Y)$ is t-exact by ([11], II.1, Lm. 1.2.4).

1.4.28. For F.5.2. For any n, $(Shv(Y)^{ren})^{\geq -n} \widetilde{\to} \mathrm{Ind}((Shv(Y)^{constr})^{\geq -n})$ by ([11], II.1, Lm. 1.2.4). Now the functor

$$(Shv(Y)^{ren})^{\geq -n} \to Shv(Y)^{\geq -n}$$

is fully faithful by ([13], 5.3.5.11) and their Pp. F.2.4.

To see it is essentially surjective on hearts, let $F \in Shv(Y)^{\heartsuit}$. Pick I small filtered and $F \widetilde{\to} \mathrm{colim}_{i\in I} F_i$ with $F_i \in Shv(Y)^c$. This exists because $Shv(Y)$ is compacty generated. We have $Shv(Y)^c \subset Shv(Y)^{constr}$, so $F_i \in Shv(Y)^{constr}$. Now the functors $\tau^{\leq 0}, \tau^{\geq 0}$ preserve $Shv(Y)^{constr}$, so $\mathrm{colim}_{i\in I} H^0(F_i) \to F$ is an isomorphism, because the t-structure on $Shv(Y)$ is compatible with filtered colimits.

Remark 1.4.29. *The category $Shv(Y)^c$ is not stable under the functors $\tau^{\leq n}, \tau^{\geq n}$ on $Shv(Y)$. Indeed, take $Y = B(\mathbb{G}_m)$ and $q : \mathrm{Spec}\, k \to B(\mathbb{G}_m)$. Then $q_! e \in Shv(B(\mathbb{G}_m))^c$, however its truncations are not compact.*

1.4.30. For F.5.3. Then fact that $ren_Y : Shv(Y) \leftrightarrows Shv(Y)^{ren} : un{-}ren_Y$ is an adjoint pair follows from ([13], 5.3.5.13). Indeed, $un - ren_Y$ is the restriction

$$\mathrm{Fun}_{e,ex}((Shv(Y)^{constr})^{op}, \mathrm{Vect}) \to \mathrm{Fun}_{e,ex}((Shv(Y)^c)^{op}, \mathrm{Vect})$$

along the exact functor $(Shv(Y)^c)^{op} \to (Shv(Y)^{constr})^{op}$. The subscript e here stands for e-linear.

The fact that $(Shv(Y)^{ren}, \otimes)$ is symmetric monoidal follows from ([14], 4.8.1.14). As we have seen in my Section 1.4.27, $Shv(Y)^c$ is a module over $Shv(Y)^{constr}$, so the $\otimes$-tensor product makes $Shv(Y)$ a module over $(Shv(Y)^{ren}, \otimes)$. If Y is Verdier compatible and quasi-compact then $Shv(Y)^c$ is a module over $(Shv(Y)^{constr}, \otimes^!)$, so $Shv(Y)$ becomes a module over $(Shv(Y)^{ren}, \otimes^!)$.

The inclusion $Shv(Y)^c \to Shv(Y)^{constr}$ is a map of $(Shv(Y)^{constr}, \otimes)$-modules. So, $ren_Y : Shv(Y) \to Shv(Y)^{ren}$ is a map of $Shv(Y)^{ren}$-modules. Besides, $un - ren_Y$ is also a map of $Shv(Y)^{ren}$-modules.

1.4.31. *Remark.* let D_0 be a small stable category, a Vect^{fd}-module, $f : C_0 \to D_0$ a stable subcategory and a submodule over Vect^{fd}, so f is exact. Let $L : C = \mathrm{Ind}(C_0) \to D = \mathrm{Ind}(D_0)$ be the ind-extension of f. Note that L is a map in DGCat_{cont}. Then f has the continuous right-adjoint $R : D \to C$ given as the composition with $f^{op} : C_0^{op} \to D_0^{op}$ by ([13], 5.3.5.13). Recall that $R^\vee : C^\vee \to D^\vee$ is obtained as the ind-extension of $f^{op} : C_0^{op} \to D_0^{op}$ by ([11], ch. I.1, 7.3.5). The functor $L^\vee : D^\vee \to C^\vee$ is the restriction along $f : C_0 \to D_0$, and $L^\vee$ is the right adjoint of $R^\vee$.

Assume now given an equivalence $\mathbb{D} : D_0^{op} \to D_0$ yielding $C_0^{op} \widetilde{\to} C_0$ and commuting with f. This gives identifications $D \widetilde{\to} D^\vee$ and $C \widetilde{\to} C^\vee$. Under these identifications, we get $L^\vee \widetilde{\to} R$.

They apply this in F.5.4 in the case of a Verdier-compatible stack Y to see that ren_Y and $un - ren_Y$ are mutually dual.

1.4.32. For F.5.5. Here G is a connected affine algebraic group of finite type. The fact that $e \in C^\cdot(pt/G) - mod$ generates the essential image of $ren_{pt/G} : Shv(pt/G) \to Shv(pt/G)^{ren}$ is obtained as follows, I think. First, we may assume that pt/G is a prestack quotient, as this does not change $Shv(pt/G)$. Then apply their F.4.7(ii'). For an affine scheme of finite type S and a map $S\to pt/G$, it factors as $S\to pt\xrightarrow{\pi} pt/G$. This is why the smallest non-cocomplete DG-subcategory of $Shv(pt/G)$ containing π_*e and closed under taking the direct summands is $Shv(pt/G)^c$. This gives their claim.

Note that for $\pi_{pt}: pt\to pt/G$ we have a natural map $e\to(\pi_{pt})_*e$ in $Shv(B(G))$, passing to the cohomology this gives a structure of a $C^\cdot(pt/G)$-module on e. This is why $ren_{pt}(\pi_{pt})_*e$ corresponds to $e\in C^\cdot(pt/G)-mod$.

Note that $Shv(pt/G)$ is compactly generated by one object π_*e. For example, for $G=\mathbb{G}_m$, $C^\cdot(pt/G)\widetilde{\to}\operatorname{Sym}(e[-2])$, and e is the augmentation module of $\operatorname{Sym}(e[-2])$.

Recall also that $(\pi_{pt})_*e\widetilde{\to}(\pi_{pt})_!e[d]$, where $d=2\dim G_{unip}+\dim G_{red}$, see their F.3.3.

1.4.33. For F.5.7. Bad formulation: $ren_{\mathcal{Y}}$ is left adjoint, not the right adjoint. The same for the second displayed equation.

In F.5.8 the second displayed equation is wrong: ren_Y should be the left adjoint.

The formulation of F.5.7 makes sense, because both $ren_{\mathcal{Y}}$ and $un-ren_{\mathcal{Y}}$ are morphisms of $Shv(Y)^{ren}$-modules, see my Section 1.4.30. Let $\mathcal{Y}=Y/G$, where Y is a scheme of finite type, G is an algebraic group (of finite type).

For the $\otimes$-monoidal structure the functor $Shv(Y)^{ren}\otimes_{Shv(pt/G)^{ren}}Shv(pt/G)\to Shv(Y)$ sends $F\otimes K$ (with $F\in Shv(Y)^{ren}$, $K\in Shv(pt/G)$) to $un-ren_Y(F)\otimes q^*K$ for $q:\mathcal{Y}\to pt/G$.

1.4.34. For F.5.8. Let Y be an algebraic stack locally of finite type with an affine diagonal. If $U\xrightarrow{\alpha}V\hookrightarrow\mathcal{Y}$ are quasi-compact opens then $\alpha^!: Shv(V)^{constr}\to Shv(U)^{constr}$ gives $\alpha^!: Shv(V)^{ren}\to Shv(U)^{ren}$ by the ind-extension. Moreover, the functors $un-ren_V: Shv(V)^{ren}\to Shv(V)$ commute with $\alpha^!$. Passing to the limit, this gives a functor $un-ren_Y: Shv(Y)^{ren}\to Shv(Y)$. Similarly, we get $ren_Y: Shv(Y)\to Shv(Y)^{ren}$ by passing to the limit over the quasi-compact opens.

The fact that $ren_Y: Shv(Y)\leftrightarrows Shv(Y)^{ren}: un-ren_Y$ is a dual pair follows from ([11], ch. I.1, 2.6.4). To get a t-structure on $Shv(Y)^{ren}$ we apply ([11], ch. I.3, 1.5.8) with my explanations from ([16], 10.1.6). In particular, the t-structure on $Shv(Y)^{ren}$ is accessible and compatible with filtered colimits.

The t-structure on $Shv(Y)^{ren}$ is right complete. To see this, by ([16], 4.0.10), it suffices to show that for $K\in Shv(Y)^{ren}$ the natural map $\operatorname{colim}_n\tau^{\le n}K\to K$ is an isomorphism. This is true, because for each quasi-compact open $U\subset Y$, the t-structure on $Shv(U)^{ren}$ is right complete by ([16], 9.3.18).

1.4.35. For F.6.1. Let $\mathcal{Y}$ be an algebraic stack, locally of finite type. Then the cotangent comlplex of $\mathcal{Y}$ is not in general a vector bundle, for example, for Bun_n, it is not a vector bundle, as for $L\in\operatorname{Bun}_n$, $\dim \mathrm{H}^0(X,\mathcal{E}nd(L))$ jumps.

For GL_n we get the following complex on Bun_n. Let $\mathcal{E}$ be the universal vector bundle on $X\times\operatorname{Bun}_n$, $q: X\times\operatorname{Bun}_{\to}\operatorname{Bun}_n$ the projection. Then $q_*(\mathcal{E}nd(\mathcal{E}))[1]$ is the dual of the cotangent complex, as far as I understand.

1.4.36. For F.6.2. Let $\mathcal{Y}$ be a classical algebrais stack locally of finite type, maybe non smooth. By a compatible collection of Zariski-closed subsets $\mathcal{N}_S\subset T^*(S)$ for schemes of finite type equipped with a smooth map $S\to\mathcal{Y}$ they mean such a collection with the property: for $S' \xrightarrow{a} S \to \mathcal{Y}$ with both maps smooth, the image of $\mathcal{N}_S \times_S S'$ under the codifferential $T^*(S) \times_S S' \to T^*(S')$ equals $\mathcal{N}_{S'}$. This is justified by the description of $SingSupp(a^*K)$ via $SingSupp(K)$ given in their Section E.6.5.

1.4.37. For F.6.3. Let $\mathcal{Y}$ be a classical algebrais stack locally of finite type, maybe non smooth, $F \in Shv(\mathcal{Y})^{constr}$. By $SingSupp(F)$ we mean a compatible system of closed subsets in $T^*(S)$ for all $S \in \mathrm{Sch}_{ft}$ equipped with a smooth map $f : S \to \mathcal{Y}$, namely, to (S, f) we associate $SingSupp(f^*F)$. This is a compatible system by their Section E.6.5.

1.4.38. For F.6.4. Let $\mathcal{Y}$ be a classical algebraic stack, locally of finite type, $\mathcal{N} \subset T^*(\mathcal{Y})$ a Zariski-closed subset. Recall that $Shv_{\mathcal{N}}(\mathcal{Y}) = \lim_{S\to\mathcal{Y}} Shv_{\mathcal{N}_S}(S)$, where the limit is over the category of (S, f), where $S \in \mathrm{Sch}_{ft}$ with a smooth morphism $f : S \to \mathcal{Y}$. A moprhism from (S', f') to (S, f) is a smooth morphism $a : S' \to S$ with a datum of $f \circ a \widetilde{\to} f'$. The transition functors are $a^! : Shv_{\mathcal{N}_S}(S) \to Shv_{\mathcal{N}_{S'}}(S')$.

The t-structure on $Shv_{\mathcal{N}}(Y)$ is defined by $Shv_{\mathcal{N}}(Y)^{\le 0} = \lim_{S\xrightarrow{\alpha}\mathcal{Y}} Shv_{\mathcal{N}_S}(S)^{\le -\dim.\mathrm{rel}(\alpha)}$, where for $a : S' \to S$ we use the transition functor $a^!$. This is a t-structure by ([11], ch. I.3, 1.5.8), and $Shv_{\mathcal{N}}(Y)^{>0} = \lim_{S\xrightarrow{\alpha}\mathcal{Y}} Shv_{\mathcal{N}_S}(S)^{>-\dim.\mathrm{rel}(\alpha)}$. It is accessible, compatible with filtered colimits (because the same holds for each $Shv_{\mathcal{N}_S}(S)$) and both left and right complete by *loc.cit.*

Left completeness of $Shv_{\mathcal{N}}(\mathcal{Y})$ follows from the fact that $\lim_n Shv_{\mathcal{N}}(\mathcal{Y})^{\ge n}$ and $\lim_{S\to\mathcal{Y}}$ can be permuted:

$$\lim_n Shv_{\mathcal{N}}(\mathcal{Y})^{\ge -n} \widetilde{\to} \lim_n \lim_{S\xrightarrow{\alpha}\mathcal{Y}} Shv_{\mathcal{N}_S}(S)^{\ge -n-\dim.\mathrm{rel}(\alpha)} \widetilde{\to}$$
$$\lim_{S\xrightarrow{\alpha}\mathcal{Y}} \lim_n Shv_{\mathcal{N}_S}(S)^{\ge -n-\dim.\mathrm{rel}(\alpha)} \widetilde{\to} \lim_{S\xrightarrow{\alpha}\mathcal{Y}} Shv_{\mathcal{N}_S}(S) \widetilde{\to} Shv_{\mathcal{N}}(\mathcal{Y})$$

The right completeness follows from the right colmpleteness of each $Shv_{\mathcal{N}_S}(S)$. Indeed, it suffices (by [16],4.0.10) to show that for $K \in Shv_{\mathcal{N}}(\mathcal{Y})$ the natural map $\mathrm{colim}_n\, \tau^{\le n} K \to K$ is an isomorphism. For this, it suffices to prove that for any $S \xrightarrow{f} \mathcal{Y}$ with $S \in \mathrm{Sch}_{ft}$ and f smooth, the map

$$f^*(\mathrm{colim}_n\, \tau^{\le n} K)[\dim.\mathrm{rel}(f)] \to f^*K[\dim.\mathrm{rel}(f)]$$

is an isomorphism. This is clear, because $f^*[\dim.\mathrm{rel}(f)]$ is t-exact

By definition, $\mathrm{Perv}(\mathcal{Y}) = (Shv(\mathcal{Y})^{constr})^{\heartsuit}$. It is clear that $\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Perv}(\mathcal{Y})$ is a Serre subcategory.

However, for the next their claim, namely the fact that $\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})) \widetilde{\to} Shv_{\mathcal{N}}(\mathcal{Y})^{\heartsuit}$, one needs to assume that $\mathcal{Y}$ is quasi-compact in addition. Indeed, for example for $\mathcal{Y} = \sqcup_{i\in I} \operatorname{Spec} k$ with I infinite set, an object of $\mathrm{Perv}(\mathcal{Y})$ is not always compact in $Shv_{\mathcal{N}}(\mathcal{Y})^{\heartsuit}$. Namely, $Shv(\mathcal{Y}) \widetilde{\to} \prod_{i\in I} \mathrm{Vect}_e$, and a collection $(e)_{i\in I}$ is not compact in $\mathrm{Perv}(\mathcal{Y}) \widetilde{\to} \prod_{i\in I} \mathrm{Vect}_e^{\heartsuit}$.

They did not want to assume throughout Section F.6 that $\mathcal{Y}$ is quasi-compact.

Let us assume now $\mathcal{Y}$ quasi-compact and show that $\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})) \to Shv_{\mathcal{N}}(\mathcal{Y})^{\heartsuit}$ is an equivalence. Let $K \in Shv_{\mathcal{N}}(\mathcal{Y})^{\heartsuit}$. Write $K \widetilde{\to} \mathrm{colim}_{i\in I} K_i$ in $Shv(\mathcal{Y})^{\heartsuit}$, where $K_i \in (Shv(\mathcal{Y})^{constr})^{\heartsuit}$ and I filtered. This is possible by their Section F.5.2. We may and do assume that each map $K_i \to K$ is a monomorphism in $Shv(\mathcal{Y})^{\heartsuit}$. Now for any i, $K_i \in \mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})$ by their chartacterization in Section F.6.4 of $Shv_{\mathcal{N}}(\mathcal{Y})$. This shows that $\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})$ generates $Shv_{\mathcal{N}}(\mathcal{Y})^{\heartsuit}$ under filtered colimits. Now let $F \in \mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})$. Then by their Pp. F.2.4, F is compact in $Shv(\mathcal{Y})^{\geq 0}$, hence also in the full subcategories $Shv_{\mathcal{N}}(\mathcal{Y})^{\heartsuit} \subset Shv(\mathcal{Y})^{\heartsuit}$. Now by ([13], 5.3.5.11), the functor $\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})) \to Shv_{\mathcal{N}}(\mathcal{Y})^{\heartsuit}$ is fully faithful.

1.4.39. For F.6.6. Let us show that their functor $(Shv_{\mathcal{N}}(\mathcal{Y})^{ren})^{\geq -n} \to (Shv_{\mathcal{N}}(\mathcal{Y})^{\geq -n}$ given by (F.12) is an equivalence. Let $F \in (Shv_{\mathcal{N}}(\mathcal{Y})^{constr})^{\geq -n}$. Then by their F.2.4, F is compact in $Shv(\mathcal{Y})^{\geq -n}$, hence also in the full subcategory $(Shv_{\mathcal{N}}(\mathcal{Y})^{\geq -n}$. This implies by ([13], 5.3.5.11) that (F.12) is fully faithful. It is also essentially surjective on hearts because of F.6.4 which gives $\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})) \widetilde{\to} Shv_{\mathcal{N}}(\mathcal{Y})^{\heartsuit}$. This gives the result.

Note that their functor $Shv_{\mathcal{N}}(\mathcal{Y})^{ren} \to Shv_{\mathcal{N}}(\mathcal{Y})$ given by (F.11) is not fully faithful, for example, for $\mathcal{Y} = B(\mathbb{G}_m)$ and $\mathcal{N} = T^*(\mathcal{Y})$.

1.4.40. For F.7.1. Let $f : C \to D$ be a map in DGCat_{cont}, let C, D be equipped with t-structures compatible with filtered colimits, assume f t-exact. Let $D'' \subset D$ be the essential image of f. Let $D' \subset D$ be the full DG-subcategory generated under colimits by D''. Then D' inherits a t-structure? Let $I \to D''$, $i \mapsto K_i$ be a diagram with I small, let $K = \mathrm{colim}_i K_i$ in D, hence also in D'. For each i we have a fibre sequence $\tau^{<n} K_i \to K_i \to \tau^{\geq n} K_i$. Passing to the colimit, we get a fibre sequence $\mathrm{colim}_i \tau^{<n} K_i \to K \to \mathrm{colim}_i \tau^{\geq n} K_i$ in D. Clearly, $\tau^{<n}(K_i), \tau^{\geq n} K_i \in D'$ and $\mathrm{colim}_i \tau^{<n} K_i \in D^{<n}$.

If I is filtered then $\mathrm{colim}_i \tau^{\geq n} K_i \in D^{\geq n}$ and $\tau^{\geq n} K \widetilde{\to} \mathrm{colim}_i \tau^{\geq n} K_i \in D'$. In this case $\tau^{<n} K, \tau^{\geq n} K \in D'$. In general it is not clear.

Assume in addition that for any n, $C^{\geq n} \to D^{\geq n}$ is an equivalence. Then D' is stable under the truncation functors. Indeed, for $K \in D'$, $\tau^{\geq n} K \in D^{\geq n} \subset D'$. We get a fibre sequence $K[-1] \to (\tau^{\geq n} K)[-1] \to \tau^{<n} K$ in D, which shows that $\tau^{<n} K \in D'$. We are done. Note that the t-structure on D' is compatible with filtered colimits.

1.4.41. For F.7.7. Let $\mathcal{Y}$ be a quasi-compact algebraic stack, $\mathcal{N} \subset T^*\mathcal{Y}$ a Zariski-closed subset. The category $Shv_{\mathcal{N}}(\mathcal{Y})^{access}$ is the full cocomplete DG-subcategory of $Shv_{\mathcal{N}}(\mathcal{Y})$ generated by $Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^{constr}$. In particular, $Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c \subset Shv_{\mathcal{N}}(\mathcal{Y})^{access}$ always. Renormalization-adapted means that $Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c$ generates $Shv_{\mathcal{N}}(\mathcal{Y})^{access}$. Note that $Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c \subset Shv_{\mathcal{N}}(\mathcal{Y})^{access}$ is stable under finite colimits, so if it generates $Shv_{\mathcal{N}}(\mathcal{Y})^{access}$ under colimits then it generated $Shv_{\mathcal{N}}(\mathcal{Y})^{access}$ under filtered colimits.

We claim that if $(Y, \mathcal{N})$ is renormalization-adapted then the natural continuous functor

$$\mathrm{Ind}(Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c) \to Shv_{\mathcal{N}}(\mathcal{Y})^{access} \tag{9}$$

is an equivalence. Indeed, it is essentially surjective by definition. To see that it is fully faithful, we show that its composition with fully faithful embeddings $Shv_{\mathcal{N}}(\mathcal{Y})^{access} \hookrightarrow Shv_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow Shv(\mathcal{Y})$ is fully faithful. Indeed, each object of $Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c$ remains compact in $Shv(\mathcal{Y})$, we are done.

The dual pair in F.7.7(III) is obtained from ([13], 5.3.5.13).

For (IV): if $\mathcal{Y}$ is renormalization-adapted and Verdier compatible then

$$\mathbb{D} : (Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(Y)^c)^{op} \widetilde{\to} Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(Y)^c$$

is an equivalence.

1.4.42. For F.7.9. Let Y be a scheme of finite type, G an algebraic group of finite type acting on Y, $\mathcal{N} \subset T^*(Y/G)$ a Zarisk-closed subset. To see that $(Y/G, \mathcal{N})$ is renorlmalization-adapted, it suffices to prove the following. Let $0 \neq F \in Shv_{\mathcal{N}}(Y/G)^{constr}$. We need to find $K \in Shv_{\mathcal{N}}(Y/G) \cap Shv(Y/G)^c$ with a nonzero map $K \to F$. Let $n = \dim G$, $q : Y \to Y/G$ be the natural map, so $q^! = q^*[2n]$. We get

$$\mathcal{H}om_{Shv(Y/G)}(q_! q^* F[2n], F) \widetilde{\to} \mathcal{H}om_{Shv(Y)}(q^* F[2n], q^! F) \neq 0$$

We claim that $q_! q^* F \in Shv_{\mathcal{N}}(Y/G) \cap Shv(Y/G)^c$. It is compact in $Shv(Y/G)$. To see that $q_! q^* F \in Shv_{\mathcal{N}}(Y/G)$ note that $q_! q^* F \widetilde{\to} F \otimes (q_! e)$, and $q_! e$ admits a filtration in $Shv(Y/G)$ with succesive quotients being shifted constant sheaves. Since each successive quotient lies in $Shv_{\mathcal{N}}(Y/G)$, the object itself is also there.

1.4.43. For F.8.2. To get t-structures we have to use ([16], 10.1.6 and below). If $U \overset{j}{\hookrightarrow} V \subset Y$ are quasi-compact opens then the restriction along j is a t-exact functor $Shv_{\mathcal{N}}(V)^{access} \to Shv_{\mathcal{N}}(U)^{access}$, so we may apply ([11], ch. I.3, 1.5.8). We see that the t-structures on both $Shv_{\mathcal{N}}(\mathcal{Y})^{ren}, Shv_{\mathcal{N}}(\mathcal{Y})^{access}$ are accessible and compatible with filtered colimits. By definition, $(Shv_{\mathcal{N}}(\mathcal{Y})^{ren})^{\leq 0} \widetilde{\to} \lim_U (Shv_{\mathcal{N}}(U)^{ren})^{\leq 0}$ and $(Shv_{\mathcal{N}}(\mathcal{Y})^{ren})^{>0} \widetilde{\to} \lim_U (Shv_{\mathcal{N}}(U)^{ren})^{>0}$, where the limit is taken over the poset of quasi-compact opens of the stack Y. Same for access version.

The DG-category $Shv_{\mathcal{N}}(\mathcal{Y})$ is both left and right complete, as we have seen in F.6.4.

To see that $Shv_{\mathcal{N}}(Y)^{access} \to Shv_{\mathcal{N}}(Y)$ is fully faithful, we use the fact that the limit of fully faithful embedding is fully faithful ([16], 2.2.17).

1.4.44. For F.8.7. For an open immersion $j_{12} : U_1 \hookrightarrow U_2$ of quasi-compact algebraic stacks, $(j_{12})_!$ has perverse cohomological amplitude ≤ 0. Besides, $(j_{12})_*$ also has perverse cohomological amplitude $\leq n$ for some n. Indeed, pick a smooth cover $S_2 \to U_2$, where S_2 is a scheme of finite type, let $\bar{j} : S_1 \hookrightarrow S_2$ be the open immersion obtained by base change. Then $\bar{j}_*$ has cohomolocal amplitude $\leq n$ for some n by ([4], 4.2.3). This is used to show that (D) is equivalent to (A).

1.4.45. For F.8.8. In the situation of F.8.6 the adjoint functors $(j_{i_1,i_2})_! : Shv_{\mathcal{N}}(\mathcal{U}_{i_1}) \leftrightarrows Shv_{\mathcal{N}}(\mathcal{U}_{i_2}) : j^*_{i_1,i_2}$ respect the full subcategories $Shv_{\mathcal{N}}(\mathcal{U}_{i_1})^{access}, Shv_{\mathcal{N}}(\mathcal{U}_{i_2})^{access}$, hence induce adjoint functors on those.

To get the adjoint pair $(j_{i_1,i_2})_! : Shv_{\mathcal{N}}(\mathcal{U}_{i_1})^{ren} \leftrightarrows Shv_{\mathcal{N}}(\mathcal{U}_{i_2})^{ren} : j^*_{i_1,i_2}$ we apply ([16], 9.2.53).

1.4.46. For F.8.10. For (a) use ([11], I.1, 7.2.7). Recall that $\mathrm{DGCat}^{non-cocompl}$ admits filtered colimits, so we have $\mathrm{colim}_i\, Shv_{\mathcal{N}}(\mathcal{U}_i)^{constr}$ taken in $\mathrm{DGCat}^{non-cocmpl}$, where the transition functors are $(j_{i_1,i_2})_!$. Then

$$\mathrm{Ind}(\mathrm{colim}_i\, Shv_{\mathcal{N}}(\mathcal{U}_i)^{constr}) \widetilde{\to} \mathrm{colim}_i\, Shv_{\mathcal{N}}(\mathcal{U}_i)^{ren} \widetilde{\to} Shv_{\mathcal{N}}(\mathcal{Y})^{ren} \tag{10}$$

in DGCat_{cont}.

For (b) assume $Shv_{\mathcal{N}}(\mathcal{Y})$ compactly generated. Consider the dual pair $(j_i)^* : Shv_{\mathcal{N}}(\mathcal{Y}) \leftrightarrows Shv_{\mathcal{N}}(\mathcal{U}_i) : (j_i)_*$ in DGCat_{cont}. We see that $(j_i)^*$ preserves compactness. Besides, $(j_i)^*$ is essentially surjective (it is a localization), so $Shv_{\mathcal{N}}(\mathcal{U}_i)$ is compactly generated. The opposite implication is clear.

For (c), recall that $Shv_{\mathcal{N}}(\mathcal{Y})^{acess} \widetilde{\to} \operatorname{colim}_i Shv_{\mathcal{N}}(\mathcal{U}_i)^{access}$ in DGCat_{cont}. Assume first each $(\mathcal{U}_i, \mathcal{N})$ renormalization-adapted. By (9) we get that $Shv_{\mathcal{N}}(\mathcal{Y})^{acess}$ is compactly generated by the union of objects of the form $(j_i)_! K$ for $K \in Shv_{\mathcal{N}}(\mathcal{U}_i) \cap Shv(\mathcal{U}_i)^c$. For such an object we have $(j_i)_! K \in Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c$, because $(j_i)_! : Shv(\mathcal{U}_i) \to Shv(\mathcal{Y})$ has a continuous right adjoint. So, $Shv_{\mathcal{N}}(\mathcal{Y})^{acess}$ is generated under colimits by $Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c$. Moreover, in this case we may consider $\operatorname{colim}_i Shv_{\mathcal{N}}(\mathcal{U}_i) \cap Shv(\mathcal{U}_i)^c$ in $\mathrm{DGCat}^{non-cocmpl}$ and then

$$\mathrm{Ind}(\operatorname{colim}_i Shv_{\mathcal{N}}(\mathcal{U}_i) \cap Shv(\mathcal{U}_i)^c) \widetilde{\to} Shv_{\mathcal{N}}(\mathcal{Y})^{acess},$$

Conversely, assume $(\mathcal{Y}, \mathcal{N})$ remormalization-adapted. Then the objects of $Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c$ are compact in $Shv_{\mathcal{N}}(\mathcal{Y})^{access}$, so the evident functor $\mathrm{Ind}(Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c) \to Shv_{\mathcal{N}}(\mathcal{Y})^{access}$ is fully faithful. It is also essentially surjective by definition (given in F.8.4), so it is an equivalence. For each i, the adjoint pair $(j_i)^* : Shv_{\mathcal{N}}(\mathcal{Y}) \leftrightarrows Shv_{\mathcal{N}}(\mathcal{U}_i) : (j_i)_*$ respects the full subcategories $Shv_{\mathcal{N}}(\mathcal{Y})^{access}, Shv_{\mathcal{N}}(\mathcal{U}_i)^{access}$, hence yield an adjoint pair denoted by the same symbols

$$(j_i)^* : Shv_{\mathcal{N}}(\mathcal{Y})^{access} \leftrightarrows Shv_{\mathcal{N}}(\mathcal{U}_i)^{access} : (j_i)_*,$$

where the right adjoint is fully faithful. So, the left adjoint here is essentially surjective. Now the claim follows from the fact that for $K \in Shv_{\mathcal{N}}(\mathcal{Y}) \cap Shv(\mathcal{Y})^c$ we have $j_i^* K \in Shv_{\mathcal{N}}(\mathcal{U}_i) \cap Shv(\mathcal{U}_i)^c$. These objects compactly generate $Shv_{\mathcal{N}}(\mathcal{U}_i)^{access}$.

In the situation of F.8.6 the adjunction $ren_{\mathcal{Y},\mathcal{N}} : Shv_{\mathcal{N}}(\mathcal{Y})^{access} \leftrightarrows Shv_{\mathcal{N}}(\mathcal{Y})^{ren} : un - ren_{\mathcal{Y},\mathcal{N}}$ is obtained as follows. For each i, we have an adjunction $ren_{\mathcal{U}_i,\mathcal{N}} : Shv_{\mathcal{N}}(\mathcal{U}_i)^{access} \leftrightarrows Shv_{\mathcal{N}}(\mathcal{U}_i)^{ren} : un - ren_{\mathcal{U}_i,\mathcal{N}}$. Moreover, the functors $un - ren_{\mathcal{U}_i,\mathcal{N}}$ are compatible with the !-restrictions transition functors, so give by passing to the limit the functor $un - ren_{\mathcal{Y},\mathcal{N}}$. The functors $ren_{\mathcal{U}_i,\mathcal{N}}$ are compatible with the functors $(j_{i_1,i_2})_!$, so give by passing to the colimit the functor $ren_{\mathcal{Y},\mathcal{N}} : \operatorname{colim}_i Shv_{\mathcal{N}}(\mathcal{U}_i)^{access} \to \operatorname{colim}_i Shv_{\mathcal{N}}(\mathcal{U}_i)^{ren}$. The adjointness of the pair $(ren_{\mathcal{Y},\mathcal{N}}, un - ren_{\mathcal{Y},\mathcal{N}})$ follows now from ([16], 9.2.39).

1.4.47. *Category of relative groupoids* $Grpd_{rel}(\mathcal{C})$. Let $\mathcal{C} \in 1-\mathcal{C}at$ admit finite products. We want to define the category of relative groupoids in $\mathcal{C}$ denoted $Grpd_{rel}(\mathcal{C})$. For a groupoid $\boldsymbol{\Delta}^{op} \to \mathcal{C}$ given by its value $\mathcal{H}$ on $[1]$ say that it is acting on $\mathcal{Y}$, where $\mathcal{Y}$ is its value on $[0]$. An object of our category will be a pair $c \in \mathcal{C}$ and $\mathcal{H} \in \mathrm{Grpd}(\mathcal{C}_{/c})$. A morphism from $(\mathcal{H}, c)$ to $(\mathcal{H}', c')$ should be a map $c' \to c$ in $\mathcal{C}$ and a morphism $\mathcal{H} \times_c c' \to \mathcal{H}'$ in $\mathrm{Grpd}(\mathcal{C}_{/c'})$ inducing an isomorphism $\mathcal{Y} \times_c c' \widetilde{\to} \mathcal{Y}'$ on the values at $[0]$. So, $Grpd_{rel}(\mathcal{C})$ should be equipped with a projection $Grpd_{rel}(\mathcal{C}) \to \mathcal{C}^{op}$ sending $(\mathcal{H}, c)$ to c.

Recall that the functor $\mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$ of evaluation at 1 is a bicartesian fibration by ([16], 2.2.120). For a morphism from $(x_1 \to c_1)$ to $(x_2 \to c_2)$ in $\mathrm{Fun}([1], \mathcal{C})$ this is a cartesian arrow over $c_1 \to c_2$ iff the induced map $x_1 \to c_1 \times_{c_2} x_2$ is an isomorphism.

I propose the following definition. First, there is a functor $\mathcal{F}:\mathcal{C}^{op}\to 1-\mathcal{C}at$ sending c to $\mathrm{Grpd}(\mathcal{C}_{/c})$, and a map $c'\to c$ to the pullback functor $\mathrm{Grpd}(\mathcal{C}_{/c})\to\mathrm{Grpd}(\mathcal{C}_{/c'})$, $\mathcal{H}\mapsto\mathcal{H}\times_c c'$. It is defined as follows.

Let $\mathcal{X}\subset\mathrm{Fun}(\boldsymbol{\Delta}^{op}\times[1],\mathcal{C})\times_{\mathrm{Fun}(\boldsymbol{\Delta}^{op}\times\{1\},\mathcal{C})}\mathcal{C}$ be the following full subcategory. An object of the ambient category is $(X^\bullet\to c)$, where $c\in\mathcal{C}$ and $\mathcal{X}\in\mathrm{Fun}(\boldsymbol{\Delta}^{op},\mathcal{C}_{/c})$. We require that $X^\bullet$ lies in the full subcategory $\mathrm{Grpd}(\mathcal{C}_{/c})\subset\mathrm{Fun}(\boldsymbol{\Delta}^{op},\mathcal{C}_{/c})$, that is, for any $S,S'\subset[n]$ with $S\cap S'=\{s\}$ the diagram

$$\begin{array}{ccc} X(S) & \leftarrow & X([n]) \\ \downarrow & & \downarrow \\ X(\{s\}) & \leftarrow & X(S') \end{array}$$

is cartesian in $\mathcal{C}_{/c}$. My understanding is that $\mathcal{X}\to\mathcal{C}$ sending the above point to c is a cartesian fibration giving the desired functor.

Consider now the category of correspondences $Corr(\mathcal{X})$ equipped with the projection $Corr(\mathcal{X})\to Corr(\mathcal{C})$. An object of $Corr(\mathcal{X})$ is $(X^\bullet\to c)\in\mathcal{X}$. A morphism in $Corr(\mathcal{C})$ from $(X^\bullet\to c)$ to $(X'^\bullet\to c')$ is given by a diagram $X^\bullet\leftarrow Y^\bullet\to X'^\bullet$ in $\mathcal{X}$ over a diagram $c\leftarrow c_Y\to c'$ in $\mathcal{C}$. in Let now $Grpd_{rel}(\mathcal{C})\subset Corr(\mathcal{X})$ be the full subcategory given by the properties:

- the arrow $X^\bullet\leftarrow Y^\bullet$ is cartesian in $\mathcal{X}$ over $c\leftarrow c_Y$ in $\mathcal{C}$, so $Y^\bullet\widetilde{\to}X^\bullet\times_c c'$ in $\mathcal{C}$;
- The map $Y^0\to X'^0$ is an isomorphism in $\mathcal{C}$, hence $c_Y\to c'$ is also an isomorphism in $\mathcal{C}$.

We get the projection $Grpd_{rel}(\mathcal{C})\to\mathcal{C}^{op}$ sending $(X^\bullet\to c)$ to c.

Question Is this a correct definition for the following lemma to hold?

Lemma 1.4.48. *There is a natural functor $Grpd_{rel}(\mathcal{C})\to Alg(Corr(\mathcal{C}))$.*

Proof. We send $(X^\bullet\to c)$ to $X^1\in Alg(Corr(\mathcal{C}))$ naturally. Let now be given a map from $(X^\bullet\to c)$ to $(X'^\bullet\to c')$ in $Grpd_{rel}(\mathcal{C})$, it is realized by a map $c'\to c$ in $\mathcal{C}$ and a morphism $X^\bullet\times_c c'\to X'^\bullet$ in $\mathrm{Grpd}(\mathcal{C}_{/c'})$ inducing an isomorphism $X^0\times_c c'\widetilde{\to}X'^0$.

First, consider the morphism from X^1 to $X^1\times_c c'$ in $\mathcal{C}^{op}\subset Corr(\mathcal{C})$ given by the projection $X^1\times_c c'\to X^1$. It is naturally a morphism in $Alg(Corr(\mathcal{C}))$. Besides, the map $X^1\times_c c'\to X'^1$ in $\mathcal{C}\subset Corr(\mathcal{C})$ is also naturally a morphism in $Alg(Corr(\mathcal{C})$. Their composition is the desired morphism in $Alg(Corr(\mathcal{C}))$. □

1.4.49. *For Hecke actions.* Let $\mathcal{C}\in Alg(1-\mathcal{C}\mathrm{at})$ be a monoidal category, $A\in 1-\mathcal{C}\mathrm{at}$ then on $\mathrm{Fun}(A,\mathcal{C})$ we get a monoidal structure with the pointwise tensor product by ([14], 2.1.3.4) and ([16], 3.0.69) with the property $\mathrm{Fun}(A,Alg(\mathcal{C}))\widetilde{\to}Alg(\mathrm{Fun}(A,\mathcal{C}))$. To be precise, the product of $f_1,f_2\in\mathrm{Fun}(A,\mathcal{C})$ is the functor $A\to A\times A\overset{f_1\times f_2}{\to}\mathcal{C}\times\mathcal{C}\overset{m}{\to}\mathcal{C}$, where m is the product in $\mathcal{C}$, and the first map is diagonal. This is used in ([9], B.2.1).

For ([9], B.2.6). The category $\mathrm{Grpd}(\mathrm{PreStk}_{lft})$ of groupoids in PreStk_{lft} classifies pairs $(\mathcal{H},\mathcal{Y})$, where $\mathcal{Y}\in\mathrm{PreStk}_{lft}$, and $\mathcal{H}$ is a groupoid acting on $\mathcal{Y}$.

If $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$ say that a groupoid $X:\boldsymbol{\Delta}^{op}\to\mathcal{C}$ in $\mathcal{C}$ is over $c\in\mathcal{C}$ if it is an object of $\mathrm{Grpd}(\mathcal{C}_{/c})$.

The correct definition of the category they wanted to denote by Grpd / PreStk /Sch in B.2.6 should use the category of relative groupoids from the previous subsection. Its precise definition is the full subcategory of those objects $(X^\bullet \to Z) \in \mathrm{Grpd}_{rel}(\mathrm{PreStk}_{lft})$ such that $Z \in \mathrm{Sch}$.

In ([9], B.2.6-B.2.9) evident mistake. Consider the category $\mathrm{Grpd}(\mathrm{PreStk}_{lft})$ classifying $\mathcal{H} \in \mathrm{Grpd}(\mathrm{PreStk}_{lft})$ acting on $\mathcal{Y} \in \mathrm{PreStk}_{lft}$ (so, $\mathcal{Y}$ is the value at $[0] \in \boldsymbol{\Delta}$ of our groupoid). Though $\mathcal{H} \in \mathrm{Alg}(Corr(\mathrm{PreStk}_{lft}))$, this does not define a functor $\mathrm{Grpd}(\mathrm{PreStk}_{lft}) \to \mathrm{Alg}(Corr(\mathrm{PreStk}_{lft}))$. Namely, for a map $c : \mathcal{H}' \to \mathcal{H}$ in $\mathrm{Grpd}(\mathrm{PreStk}_{lft})$ over $d : \mathcal{Y}' \to \mathcal{Y}$ the map d is not a map in $\mathrm{Alg}(Corr(\mathrm{PreStk}_{lft}))$ in general. However, assume $\mathcal{H} \in \mathrm{Grpd}(\mathrm{PreStk}_{/Z})$. If $Z' \to Z$ is a map in Sch and $\mathcal{H}'$ is obtained from $\mathcal{H}$ by the base change $Z' \to Z$ then c can be viewed as a map in $(\mathrm{PreStk}_{lft})^{op} \subset Corr(\mathrm{PreStk}_{lft})$, and it is indeed a morphism in $\mathrm{Alg}(Corr(\mathrm{PreStk}_{lft}))$.

From Lemma 1.4.48 we get a natural functor

$$\mathrm{Grpd} / \mathrm{PreStk} / \mathrm{Sch} \to \mathrm{Alg}(Corr(\mathrm{PreStk}_{lft}))$$

Their functor (B.7) from B.2.8 now makes sense. Let $\alpha : I \to J$ be a map in $fSet$. Let $\vartriangle$: $X^J \to X^I$ be the induced map. By Hecke_I we mean the global Hecke stack classifying $(\mathcal{F}_G, \mathcal{F}'_G, \{x_i\})_{i\in I})$, where $\mathcal{F}_G, \mathcal{F}'_G$ are G-torsors on X, $\mathcal{F}_G \widetilde{\to} \mathcal{F}'_G \mid_{S\times X - \cup_i x_i}$ is the isomorphism of G-torsors on the complement to the union of graphs of x_i. Define $h^\leftarrow, h^\to$ by the diagram

$$\mathrm{Bun}_G \times X^I \overset{h^\leftarrow}{\leftarrow} \mathrm{Hecke}_I \overset{h^\to}{\to} \mathrm{Bun}_G \times X^I,$$

where $h^\leftarrow, h^\to$ sends the above point to $(\mathcal{F}_G, (x_i))$ and $(\mathcal{F}'_G, (x_i))$ respectively.

Then in general there is no map $\mathrm{Hecke}_J \to \mathrm{Hecke}_I$ for given α. We only have a correspondence

$$\mathrm{Hecke}_I \overset{a}{\leftarrow} \mathrm{Hecke}_I \times_{X^I} X^J \overset{b}{\to} \mathrm{Hecke}_J \tag{11}$$

Indeed, if $\{x_j\} \in X^J$ and $\{y_i\} \in X^I$ is its image under $\vartriangle$ then the inclusion $S \times X - \cup_j x_j \subset S \times X - \cup_i y_i$ is strict in general.

Note that Hecke_I is a groupoid over $\mathrm{Bun}_G \times X^I$, we have the diagram $\mathrm{pr}, \mathrm{act} : \mathrm{Hecke}_I \to \mathrm{Bun}_G \times X^I$, where pr sends $(\mathcal{F}_G, \mathcal{F}'_G, \{x_i\})$ to $(\mathcal{F}_G, \{x_i\})$ and act sends it to $(\mathcal{F}'_G, \{x_i\})$. So, $\mathrm{Hecke}_I \times_{X^I} X^J$ is a groupoid over $\mathrm{Bun}_G \times X^J$. The above map a can be seen as a map in $\mathrm{PreStk}_{lft}^{op} \subset Corr(\mathrm{PreStk}_{lft})$, and as such it is a morphism in $Alg(Corr(\mathrm{PreStk}_{lft}))$ from Hecke_I to $\mathrm{Hecke}_I \times_{X^I} X^J$. So,

$$a^! : Shv(\mathrm{Hecke}_I) \to Shv(\mathrm{Hecke}_I \times_{X^I} X^J)$$

is monoidal. Now b is a morphism of groupoids acxting on $\mathrm{Bun}_G \times X^J$, so this is a morphism in $Alg(Corr(\mathrm{PreStk}_{lft}))$. So, the composition in (11) is a morphism in $Alg(Corr(\mathrm{PreStk}_{lft}))$. It gives rise to the monoidal functor $b_* a^! : Shv(\mathrm{Hecke}_I) \to Shv(\mathrm{Hecke}_J)$.

1.4.50. *Generalities.* To make things more explicit, consider the composition

$$\mathrm{Grpd} / \mathrm{PreStk} / \mathrm{Sch} \to \mathrm{Alg}(Corr(\mathrm{PreStk}_{lft})) \to \mathrm{DGCat}^{Mon}$$

The first functor send $\mathcal{H}$ acting on $\mathcal{Y}$ over Z to $\mathcal{H} \in \mathrm{Alg}(Corr(\mathrm{PreStk}_{lft}))$ with the multiplication given by the diagram $\mathcal{H} \times \mathcal{H} \leftarrow \mathcal{H} \times_{\mathcal{Y}} \mathcal{H} \stackrel{m}{\to} \mathcal{H}$, and the unit correspondence $\operatorname{Spec} k \leftarrow \mathcal{Y} \to \mathcal{H}$.

Consider two object $(\mathcal{H}, \mathcal{Y}, Z), (\mathcal{H}', \mathcal{Y}', Z') \in \mathrm{Grpd} / \mathrm{PreStk} / \mathrm{Sch}$. By definition, a morphism in $\mathrm{Grpd} / \mathrm{PreStk} / \mathrm{Sch}$ from $\mathcal{H}$ to $\mathcal{H}'$ is a morphism $Z' \to Z$ in Sch (giving rise by base change to $\beta : \mathcal{Y}' \to \mathcal{Y}$ in PreStk_{lft}) and a morphism $\mathcal{H} \times_Z Z' \to \mathcal{H}'$ in $\mathrm{Grpd}((\mathrm{PreStk}_{lft})_{/Z'})$.

Such a datum defines a diagram $\mathcal{H} \stackrel{a}{\leftarrow} \mathcal{H} \times_Z Z' \stackrel{b}{\to} \mathcal{H}'$ in PreStk, here a is the projection. Moreover, let $K \in Shv(\mathcal{H}), F \in Shv(\mathcal{Y})$. Then

$$\beta^!(K * F) \widetilde{\to} (a^! K) * (\beta^! F)$$

Now the action of $Shv(\mathcal{H} \times_{\mathcal{Y}} \mathcal{Y}')$ on $Shv(\mathcal{Y}')$ factors through the monoidal functor $b_* : Shv(\mathcal{H} \times_{\mathcal{Y}} \mathcal{Y}') \to Shv(\mathcal{H}')$. So, we get as desired

$$\beta^!(K * F) \widetilde{\to} (b_* a^! K) * (\beta^! F),$$

where the RHS stands for the action of $Shv(\mathcal{H}')$ on $Shv(\mathcal{Y}')$.

1.4.51. For ([9], B.3.2). An explanation is missing here, as Hecke_I^{loc} is not locally of finite type. First, for $I \in fSet$ we have a group scheme $\mathfrak{L}_I^+(G)$ on X^I whose fibre over x^I is the scheme of maps $\mathcal{D}_{x^I} \to G$. This is a placid group scheme over X^I in the sense of [18] by ([23], Lemma 2.5.1). Consider the stack quotients $X^I / \mathfrak{L}_I^+(G)$. For $S \in \mathrm{Sch}_{ft}^{aff}$, the stack Hecke_I^{loc} has as S-points the collections $x^I \in X^I(S)$, $\mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \widetilde{\to} \mathcal{F}'_G \mid_{\overset{\circ}{\mathcal{D}}_{x^I}}$, where $\mathcal{F}_G, \mathcal{F}'_G$ are G-torsors on $\mathcal{D}_{x^I}$. We have a diagram

$$X^I / \mathfrak{L}_I^+(G) \stackrel{h^{\leftarrow}}{\leftarrow} \mathrm{Hecke}_I^{loc} \stackrel{h^{\to}}{\to} X^I / \mathfrak{L}_I^+(G),$$

where $h^{\leftarrow}$ (resp., $h^{\to}$) sends the above point to $(\mathcal{F}_G, x^I)$ (resp., to $(\mathcal{F}'_G, x^I)$).

We define $Shv(X^I / \mathfrak{L}_I^+(G))$ via our general conventions of ([20], 0.0.40). Similarly, trivializing $\mathcal{F}_G$ say, we may write Hecke_I^{loc} as a quotient of an ind-scheme of ind-finite type by a placid group scheme, so ([20], 0.0.40) also gives a definition of $Shv(\mathrm{Hecke}_I^{loc})$. The monoidal operation on $Shv(\mathrm{Hecke}_I^{loc})$ is defined not as usually! Namely, we have the diagram

$$\mathrm{Hecke}_I^{loc} \times \mathrm{Hecke}_I^{loc} \stackrel{\mathrm{pr}_1 \times \mathrm{pr}_2}{\longleftarrow} \mathrm{Hecke}_I^{loc} \times_{X^I / \mathfrak{L}_I^+(G)} \mathrm{Hecke}_I^{loc} \stackrel{m}{\to} \mathrm{Hecke}_I^{loc}$$

and for $K_i \in Shv(\mathrm{Hecke}_I^{loc})$ we would like to set $K_1 * K_2 \widetilde{\to} m_*(\mathrm{pr}_1 \times \mathrm{pr}_2)^!(K_1 \boxtimes K_2)$. However, the functor $(\mathrm{pr}_1 \times \mathrm{pr}_2)^!$ does not make sense in our formalism! Besides, we would like $Shv(X^I / \mathfrak{L}_I^+(G))$ to be a $Shv(\mathrm{Hecke}_I^{loc})$-module naturally. I imagine we can try to define the convolution instead as $m_*(\mathrm{pr}_1 \times \mathrm{pr}_2)^*(K_1 \boxtimes K_2)$. This is reasonable because m is ind-proper.

Note that $(\mathrm{Hecke}_I^{loc}, X^I / \mathfrak{L}_I^+(G), X^I) \in \mathrm{Grpd} / \mathrm{PreStk} / \mathrm{Sch}$.

1.4.52. For B.3.3. They claim that (B.11) gives a functor $fSet \to Shv(\mathrm{Hecke}_I^{loc})$, $I \mapsto Shv(\mathrm{Hecke}_I^{loc})$. Let us check this. For this Nick suggests that

$$I \mapsto (\mathrm{Hecke}_I^{loc}, X^I/\mathfrak{L}_I^+(G), X^I)$$

extends to a functor $fSet \to \mathrm{Grpd}\,/\,\mathrm{PreStk}\,/\mathrm{Sch}$.

For $\alpha : I \to J$ in $fSet$ let $\triangle$: $X^J \to X^I$ be the induced map. For a S-point $(x_j) \in X^J$ let $(y_i) \in X^I$ be its image under $\triangle$. Recall that $\mathcal{D}_{x^J}$ denotes the formal completion of the union of the graphs of x_j in $S \times X$, and $\overset{\circ}{\mathcal{D}}_{x^J} \subset \mathcal{D}_{x^J}$ is the open part obtained by removing the union of the graphs of all x_j. If S is affine then $\mathcal{D}_{x^J}$ is an ind-object of the category Sch^{aff}. We have closed immersions $\mathcal{D}_{y^I} \hookrightarrow \mathcal{D}_{x^J}$ and $\overset{\circ}{\mathcal{D}}_{y^I} \hookrightarrow \overset{\circ}{\mathcal{D}}_{x^J}$. Restricting along these closed immersions gives a morphism $f_\alpha : \mathrm{Hecke}_J^{loc} \to \mathrm{Hecke}_I^{loc}$. Note that $f_\alpha^! : Shv(\mathrm{Hecke}_I^{loc}) \to Shv(\mathrm{Hecke}_J^{loc})$ is not monoidal! Namely, the preimage of the unit section $X^I/\mathfrak{L}_I^+(G) \to \mathrm{Hecke}_I^{loc}$ is much bigger than the unit section of Hecke_J^{loc}.

We have a morphism of group schemes $\mathfrak{L}_J^+(G) \to \triangle^* \mathfrak{L}_I^+(G)$ over X^J, hence maps of stack quotients

$$X^J/\mathfrak{L}_J^+(G) \to X^J/\, \triangle^* \mathfrak{L}_I^+(G) \to X^I/\mathfrak{L}_I^+(G)$$

The composition is given by restricting a G-torsor under $\mathcal{D}_{y^I} \to \mathcal{D}_{x^J}$ essentially.

Mistake I don't think we get this way a functor $fSet \to \mathrm{Grpd}\,/\,\mathrm{PreStk}\,/\mathrm{Sch}$, $I \mapsto (\mathrm{Hecke}_I^{loc}, X^I/\mathfrak{L}_I^+(G), X^I)$, I think this is a mistake.

Maybe it does actually as Nick suggests: consider the prestack $\mathrm{Hecke}_{I,\alpha}^{loc}$ classifying $(x_j) \in X^J$ for which we denote by $(y_i) \in X^I$ its image, G-torsors $\mathcal{F}, \mathcal{F}'$ on $\mathcal{D}_{x^J}$ together with a trivialization $\mathcal{F} \tilde{\to} \mathcal{F}' \mid_{\mathcal{D}_{x^J} - \Gamma_{y^I}}$. Here Γ_{y^I} is the union of graphs of all y_i.

We get a correspondence

$$\mathrm{Hecke}_I^{loc} \overset{a^{loc}}{\leftarrow} \mathrm{Hecke}_{I,\alpha}^{loc} \overset{b^{loc}}{\to} \mathrm{Hecke}_J^{loc},$$

where a^{loc} is given by restricting along $\mathcal{D}_{y^I} \hookrightarrow \mathcal{D}_{x^J}$. The map b^{loc} keeps the G-torsors $\mathcal{F}, \mathcal{F}'$ and restricts the isomorphism between them under the open immersion $\overset{\circ}{\mathcal{D}}_{x^J} \subset \mathcal{D}_{x^J} - \Gamma_{y^I}$.

1.4.53. The categories $Shv(\mathrm{Hecke}_I^{loc})$, $Shv(X^I/\mathfrak{L}_I^+(G))$ are equipped with the perverse t-structures as in ([20], 0.0.40).

For $I \in fSet$ the square is cartesian

$$\begin{array}{ccc} \mathrm{Hecke}_I & \overset{\mathfrak{r}_I}{\to} & \mathrm{Hecke}_I^{loc} \\ \downarrow h^{\leftarrow} & & \downarrow h^{\leftarrow} \\ \mathrm{Bun}_G \times X^I & \to & X^I/\mathfrak{L}_I^+(G), \end{array}$$

where the low horizontal arrow is the restriction along $D_{x^I} \to S \times X$. For this reason the square

$$\begin{array}{ccc} \mathrm{Hecke}_I \times_{\mathrm{Bun}_G \times X^I} \mathrm{Hecke}_I & \overset{m}{\to} & \mathrm{Hecke}_I \\ \downarrow & & \downarrow \mathfrak{r}_I \\ \mathrm{Hecke}_I^{loc} \times_{X^I/\mathfrak{L}_I^+(G)} \mathrm{Hecke}_I^{loc} & \to & \mathrm{Hecke}_I^{loc} \end{array}$$

is also cartesian. Here m is the composition map for the corresponding groupoid.

Mistake Dennis used the functor $\mathfrak{r}_I^!$ in ([9], B.3.3). This functor does not make sense, only $\mathfrak{r}_I^*$ makes sense according to our conventions!

I think one gets $(\mathfrak{r}_I^* K_1) * (\mathfrak{r}_I^* K_2) \widetilde{\to} \mathfrak{r}_I^*(K_1 * K_2)$ for $K_i \in Shv(\mathrm{Hecke}_I^{loc})$. We also have a cartesian square

$$\begin{array}{ccc} X^I \times \mathrm{Bun}_G & \stackrel{u}{\to} & \mathrm{Hecke}_I \\ \downarrow & & \downarrow \\ X^I/\mathfrak{L}_I^+(G) & \stackrel{u}{\to} & \mathrm{Hecke}_I^{loc}, \end{array}$$

where u is the unit section of the corresponding groupoid.

1.4.54. Let G be reductive over k, so $\check{G}$ is reductive over e. Let Λ^+ be the set of dominant coweights of G. Recall that

$$\mathrm{Rep}(\check{G}) \widetilde{\to} \prod_{\lambda \in \Lambda^+} \mathrm{Vect} \widetilde{\to} \underset{\lambda \in \Lambda^+}{\oplus} \mathrm{Vect}$$

by ([16], 9.4.2). Write V^λ for an irreducible $\check{G}$-module over e with h.w. λ. The functor $\mathrm{oblv} : \mathrm{Rep}(\check{G}) \to \mathrm{Vect}$ sends a collection $(W_\lambda) \in \underset{\lambda \in \Lambda^+}{\oplus} \mathrm{Vect}$ to $\oplus_\lambda (V^\lambda \otimes W_\lambda) \in \mathrm{Vect}$. Now $(W_\lambda) \in \mathrm{Rep}(\check{G})^{\le 0}$ iff each $W_\lambda \in \mathrm{Vect}^{\le 0}$. The t-structure on $\mathrm{Rep}(\check{G})$ is accessible, compatible with filtered colimits, left and right complete by ([16], 9.4.2). The t-structure on $\mathrm{Rep}(\check{G})$ is compactly generated in the sense of ([10], 6.3.8). Indeed, for any $C \in \mathrm{DGCat}_{cont}$ compactly generated with an accessible and compatible with filtered colimits t-structure assume that C^c is preserved by the truncation functors. Then the t-structure on C is compactly generated.

For $I \in fSet$, we conclude that $\mathrm{Rep}(\check{G})^{\otimes I}$ is compactly generated with a compactly generated t-structure by ([16], Lemma 9.3.7). In particular, the t-structure on $\mathrm{Rep}(\check{G})^{\otimes I}$ is accessible and compatible with filtered colimits by ([16], Lemma 9.3.5).

Recall that $\mathrm{Rep}(\check{G})$ is rigid, so $\mathrm{Rep}(\check{G})^{\otimes I} \widetilde{\to} \mathrm{Rep}(\check{G}^I)$ by ([11], ch. I.3, 3.4.2). By ([11], I.3, 2.4.3), the natural map

$$\mathrm{D}^+(\mathrm{Rep}(\check{G}^I)^\heartsuit) \to \mathrm{Rep}(\check{G}^I)^+ \qquad (12)$$

is an isomorphism. So, $\mathrm{Rep}(\check{G}^I)$ identifies with the left completion of $\mathrm{D}^+(\mathrm{Rep}(\check{G}^I)^\heartsuit)$, which is $\mathrm{D}(\mathrm{Rep}(\check{G}^I)^\heartsuit)$ in the sense of ([14], 1.3.5.8). Indeed, $\mathrm{Rep}(\check{G}^I)^\heartsuit$ has enough projective objects, so we apply ([14], 1.3.5.24 and 1.3.3.16).

1.4.55. We try to correct the end of the proof of ([9], Pp. B.2.3) as follows. We want to construct a natural transformation τ of functors $fSet \to \mathrm{DGCat}^{Mon}$ from the functor $I \mapsto \mathrm{Rep}(\check{G})^{\otimes I}$ to $I \mapsto Shv(\mathrm{Hecke}_I)$.

As in ([5], 3.2.1), we denote by $* : \mathrm{Sph}_{G^I} \widetilde{\to} \mathrm{Sph}_{G^I}$ the equivalence coming from the swap of Hecke_I^{loc} permuting $\mathcal{F}_G$ and $\mathcal{F}'_G$. We denote by the same symbol the autoequivalence of $\mathrm{Rep}(\check{G}^I)$ obtained via the Satake equivalence $\mathrm{Sph}_{G^I} \widetilde{\to} \mathrm{Rep}(\check{G}^I)$.

For $I \in fSet$ we first define a functor $\mathrm{Rep}(\check{G}^I)^\heartsuit \to Shv(\mathrm{Hecke}_I)$ as follows. Let $\mathrm{Hecke}_{I, \overset{\circ}{X}{}^I} \subset \mathrm{Hecke}_I$ be obtained from Hecke_I by the base change $\overset{\circ}{X}{}^I \to X^I$, this is the complement to all the diagonals. Given $V_i \in \mathrm{Rep}(\check{G})^\heartsuit \cap \mathrm{Rep}(\check{G})^c$ for $i \in I$, we have the perverse sheaf $\overset{\circ}{\mathrm{Loc}}_{\mathrm{Hecke}}(\boxtimes_i V_i)$ on $\mathrm{Hecke}_{I,\overset{\circ}{X}{}^I}$ defined via the usual Satake equivalence, we denote by $\mathrm{Loc}_{\mathrm{Hecke}}(\boxtimes_i V_i)$ its intermediate extension to Hecke_I.

To be precise, the normalization is as follows. Let $\mathrm{Bun}_{G,\overset{\circ}{X}{}^I}$ be the stack classifying $\mathcal{F}_G \in \mathrm{Bun}_G$, $x^I \in \overset{\circ}{X}{}^I$ and a trivialization of $\mathcal{F}_G$ over D_{x^I}. Let $\mathrm{Gr}_{G,\overset{\circ}{X}{}^I}$ be the ind-scheme classifying $(x^I \in \overset{\circ}{X}{}^I, \mathcal{F}_G, \beta)$, where $\mathcal{F}_G$ is a G-torsor on D_{x^I}, and β is its trivialization on $\overset{\circ}{D}_{x^I}$. We have an isomorphism

$$\gamma^{\leftarrow} : \mathrm{Hecke}_I \widetilde{\to} \mathrm{Bun}_{G,\overset{\circ}{X}{}^I} \times^{\mathfrak{L}^+_I(G)} \mathrm{Gr}_{G,\overset{\circ}{X}{}^I},$$

such that the projection $h^{\leftarrow} : \mathrm{Hecke}_I \to \mathrm{Bun}_G \times X^I$ identifies with the projection $\mathrm{Bun}_{G,\overset{\circ}{X}{}^I} \times^{\mathfrak{L}^+_I(G)} \mathrm{Gr}_{G,\overset{\circ}{X}{}^I} \to \mathrm{Bun}_G$. By definition, $\overset{\circ}{\mathrm{Loc}}_{\mathrm{Hecke}}(\boxtimes_i V_i)$ identifies with $\mathrm{IC}(\mathrm{Bun}_G)\widetilde{\boxtimes}\,\mathrm{Loc}_{\boxtimes_i V_i}$. Here $\mathrm{Loc}_{\boxtimes_i V_i}$ is the perverse sheaf on $\mathrm{Gr}_{G,\overset{\circ}{X}{}^I}$ attached to $\boxtimes_i V_i$ via the usual Satake equivalence. Our functor $\tau : \mathrm{Rep}(\check{G}^I)^{\heartsuit} \to Shv(\mathrm{Hecke}_I)$ is given by

$$\boxtimes_i V_i \mapsto \mathrm{Loc}_{\mathrm{Hecke}}(*(\boxtimes_i V_i))[\mid I \mid + \dim \mathrm{Bun}_G],$$

for $V_i \in \mathrm{Rep}(\check{G})^{\heartsuit} \cap \mathrm{Rep}(\check{G})^c$, so the result is a shifted perverse sheaf. The $*$ is added to make this definition compatible with he one from [5], and it is necessary to get a left action, not a right action.

By ind-extension, this gives a functor

$$\tau : \mathrm{Rep}(\check{G}^I)^{\heartsuit} \to Shv(\mathrm{Hecke}_I)^{\heartsuit}[\mid I \mid + \dim \mathrm{Bun}_G],$$

which is an exact functor of abelian categories.

The category $\mathrm{Rep}(\check{G}^I)^{\heartsuit}$ has enough injective objects, so we may consider the stable category $\mathrm{D}^+(\mathrm{Rep}(\check{G}^I)^{\heartsuit})$ defined in a way dual to ([14], 1.3.2.7). Recall that $Shv(\mathrm{Hecke}_I)$ is right complete by ([20], 0.0.10). By the dual version of the universal property ([14], 1.3.3.2), the above functor extends naturally to a left t-exact functor $\mathrm{D}^+(\mathrm{Rep}(\check{G}^I)^{\heartsuit}) \to Shv(\mathrm{Hecke}_I)$, which is also exact. By ([14], 1.3.3.6), the latter functor is t-exact. Since the t-structure on $Shv(\mathrm{Hecke}_I)$ is left complete (by [20], 0.0.28), passing to the left completions and using (12), we get the desired functor $\tau : \mathrm{Rep}(\check{G}^I) \to Shv(\mathrm{Hecke}_I)$. The latter functor is a map in DGCat_{cont} by ([16], 9.3.21).

Let us check that τ is monoidal and functorial in $I \in fSet$. For the diagram

$$\mathrm{Hecke}_I \times \mathrm{Hecke}_I \overset{q}{\leftarrow} \mathrm{Hecke}_I \times_{\mathrm{Bun}_G \times X^I} \mathrm{Hecke}_I \overset{m}{\to} \mathrm{Hecke}_I$$

given $V_i', V_i \in \mathrm{Rep}(\check{G})^{\heartsuit} \cap \mathrm{Rep}(\check{G})^c$ the complex $q^!(\tau(\boxtimes V_i) \boxtimes \tau(\boxtimes V_i'))$ is placed in perverse degree $-\dim \mathrm{Bun}_G - \mid I \mid$. The usual Satake gives

$$m_* q^!(\tau(\boxtimes V_i) \boxtimes \tau(\boxtimes V_i')) \widetilde{\to} \tau(\boxtimes_i (V_i \otimes V_i'))$$

by ([25], Pp. IV.3.4). My understanding is that $\tau(\boxtimes_i V_i)$ is ULA for $\mathrm{Hecke}_I \to X^I$, see [25]. This implies that τ is monoidal.

Let us verify the functoriality of τ on $I \in fSet$. If $\alpha : I \to J$ is a map in $fSet$, recall the diagram (11). From the ULA property we get that $a^! \tau(\boxtimes V_i) \in Shv(\mathrm{Hecke}_I \times_{X^I} X^J)$ is placed in perverse degree $-\dim \mathrm{Bun}_G - \mid J \mid$.

Note that if α is injective then $\vartriangle\colon X^J \to X^I$ is smooth, so $\vartriangle^! [-\dim.\mathrm{rel}(\vartriangle)]$ is t-exact, so in this case the latter claim is evident.

Given $V_i \in \mathrm{Rep}(\check{G})^\heartsuit \cap \mathrm{Rep}(\check{G})^c$, we must show that

$$b_* a^! \tau(\boxtimes_i V_i) \widetilde{\to} \tau(\boxtimes_j W_j), \tag{13}$$

where $W_j = \otimes_{i \in \alpha^{-1}(j)} V_i$.

CASE of $\alpha : I \to J$ injective. In this case $W_j = e$ for $j \notin \alpha(I)$, and the map $b : \mathrm{Hecke}_I \times_{X^I} X^J \to \mathrm{Hecke}_J$ is a closed immersion. The isomorphism (13) is evident in this case.

CASE of $\alpha : I \to J$ surjective. In this case $\vartriangle\colon X^J \to X^I$ and a are closed immersions, and $b : \mathrm{Hecke}_I \times_{X^I} X^J \to \mathrm{Hecke}_J$ is an isomorphism. The isomorphism (13) holds in this case, this is a part of the classical Satake equivalence.

The general case follows formally as a combination of these two.

The natural transformation τ is well-defined on objects and on morphisms, the functoriality on I is explained below. The relation with the factorizable version of Satake is also explained below.

1.4.56. To summarize, given $I \in fSet$, $V_i \in \mathrm{Rep}(\check{G})^\heartsuit \cap \mathrm{Rep}(\check{G})^c$, the action of $\boxtimes_i V_i \in \mathrm{Rep}(\check{G})^{\otimes I}$ on $K \in Shv(\mathrm{Bun}_G \times X^I)$ is the object

$$(h^\to \times s)_*((h^\leftarrow \times s)K \otimes^! \tau(\boxtimes_i V_i)) \in Shv(\mathrm{Bun}_G \times X^I)$$

for the diagram

$$\mathrm{Bun}_G \times X^I \overset{h^\leftarrow \times s}{\leftarrow} \mathrm{Hecke}_I \overset{h^\to \times s}{\to} \mathrm{Bun}_G \times X^I$$

For $\mid I \mid = 1$ this agrees with the functor $\mathrm{H}^\leftarrow_G$ from [5].

1.4.57. Relation with the factorization claimed by Nick: let $\mathcal{C}$ be a monoidal factorization category over Ran. Then for any I let $\mathcal{C}_I$ be the category its global sections over X^I. Then $\mathcal{C}_I \in \mathrm{Alg}(\mathrm{DGCat}_{cont})$, and $\mathcal{C}_I$ depends functorially on $I \in fSet$.

1.5. Question 2: let $\mathcal{A}$ be a Grothendieck abelian e-linear category. Let $D \in CAlg(\mathrm{DGCat}_{cont})$ with an accessible t-structure compatible with filtered colimits. Assume $\mathcal{A}$ is symmetric monoidal, and the monoidal operation on $\mathcal{A}$ is exact separately in each variable and preserving small colimits separately in each variable (and $\mathrm{Vect}^\heartsuit$-linear in each variable). Let $\mathcal{A} \to D^\heartsuit$ be a continuous symmetric monoidal functor, exact functor of abelian categories, and $\mathrm{Vect}^\heartsuit$-linear. Assume D both left and right complete. Let $\mathrm{D}(\mathcal{A})$ be the derived category of $\mathcal{A}$ in the sense of ([14], 1.3.5.8). Do we get monoidal operations on $\mathrm{D}(\mathcal{A})$ and its left completion $\hat{\mathrm{D}}(\mathcal{A})$? Is the functor $\hat{\mathrm{D}}(\mathcal{A}) \to D$ obtained by the universal property symmetric monoidal?

Nick's answer: see appendix C in [15]. We analyze this in details below. This seems the right approach to fill the gap in the proof of ([9], B.2.3).

1.5.1. Recall that an abelian category is Grothendieck if it is presentable and filtered colimits are left exact. By Groth_{ab} Lurie denotes the ∞-category, whose objects are Grothendieck abelian categories, and morphisms are colimit-preserving functors. Then $\mathrm{Groth}_{ab}^{lex} \subset \mathrm{Groth}_{ab}$ is the subcategory, where we restrict the morphisms to those preserving finite limits in addition.

Let $A, B \in \mathcal{P}\mathrm{r}^L$ be Grothendieck abelian categories. Then $A \otimes B \in \mathcal{P}\mathrm{r}^L$ is still a Grothendieck abelian by ([15], C.5.4.16), so Groth_{ab} by ([15], C.5.4.19) inherits a tensor product, and becomes symmetric monoidal.

Nick wanted first to claim that there is a functor $\mathrm{Groth}_{ab}^{lex} \to 1 - \mathcal{C}\mathrm{at}_{cont}^{St,cocmpl}$ sending A to $\hat{\mathrm{D}}(A)$, where $\hat{\mathrm{D}}(A)$ is the left completion of the derived ∞-category $\mathrm{D}(\mathcal{A})$ in the sense of ([14], 1.3.5.8). Moreover, this functor is right-lax monoidal. This is probably nor true as stated and needs a correction. We check this.

1.5.2. Lurie introduces a notion of a Grothendieck prestable ∞-category in ([15], C.1.4.2). For example, if $\mathcal{A}$ in a Grothendieck abelian category then $\mathrm{D}(\mathcal{A})^{\leq 0}$ is a Grothendieck prestable ∞-category by ([15], C.1.4.5), here $\mathrm{D}(\mathcal{A})$ is the derived DG-category of $\mathcal{A}$ in the sense of ([14], 1.3.5.8), note that $\mathrm{D}(\mathcal{A})$ is the category of spectrum objects of $\mathrm{D}(\mathcal{A})^{\leq 0}$.

If $\mathcal{C}$ is a Grothendieck prestable category then $\mathcal{C}$ admits a generator $x \in \mathcal{C}$, by ([15], C.2.1.4). If moreover $\mathcal{C}$ is separated, then $\mathcal{C}$ is generated under colimits by the full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ spanned by the single object x ([15], C.2.1.7). Write $\mathbb{S}\mathrm{p}$ for the category of spectra.

Lurie defines $\mathrm{Groth}_\infty \subset \mathcal{P}\mathrm{r}^L$ as the full subcategory spanned by Grothendieck prestable categories in ([15], C.3.0.5). The important role is played by the category denoted $\mathrm{Groth}_\infty^{lex}$ in ([15], Notation C.3.2.3). Let $\mathcal{P}\mathrm{r}^{St} \subset \mathcal{P}\mathrm{r}^L$ be the full subcategory spanned by stable presentable categories. The first point is that we have a functor $\mathrm{Groth}_\infty^{lex} \to \mathcal{P}\mathrm{r}^{St}$, $C \mapsto Sp(C)$ by ([15], C.3.2.5). Recall that for any Grothendieck prestable ∞-category C the natural map $C \to Sp(C)$ is fully faithful and identifies C with $Sp(C)^{\leq 0}$ with its natural t-structure (by [15], C.1.2.10).

1.5.3. The category Groth_∞ has a symmetric monoidal structure with the unit $\mathbb{S}\mathrm{p}^{\leq 0}$ by ([15], C.4.2.1). The symmetric monoidal structure on Groth_∞ restricts to a symmetric monoidal structure on the subcategory $\mathrm{Groth}_\infty^{lex}$ by ([15], C.4.4.2). Let $\mathrm{Groth}_\infty^{lex,sep}$ denote the full subcategory of $\mathrm{Groth}_\infty^{lex}$, whose objects are separated Grothendieck prestable ∞-categories.

By ([15], C.5.4.5 and C.5.4.10), we have an adjoint pair $\mathrm{Groth}_{ab}^{lex} \leftrightarrows \mathrm{Groth}_\infty^{lex,sep}$, where the left adjoint $\mathcal{A} \mapsto \mathcal{D}(\mathcal{A})^{\leq 0}$ is fully faithful, and the right adjoint sends C to $\tau_{\leq 0} C$.

1.5.4. Denote by $\mathrm{Groth}_\infty^{comp} \subset \mathrm{Groth}_\infty^{sep} \subset \mathrm{Groth}_\infty$ the full subcategories of Groth_∞ spanned by the complete and separated Grothendieck prestable ∞-categories (cf. [15], C.1.2.12). Let $L : \mathrm{Groth}_\infty \to \mathrm{Groth}_\infty^{sep}$ and $L' : \mathrm{Groth}_\infty \to \mathrm{Groth}_\infty^{comp}$ denote the left adjoint to inclusions. Note that L' is the completion functor, and L sends C to C^{sep} by ([15], C.3.6.1). Then by ([15], C.4.6.2), $\mathrm{Groth}_\infty^{comp}$ and $\mathrm{Groth}_\infty^{sep}$ admit essentially unique symmetric monoidal structures for which the localization functors L, L' are symmetric monoidal. So, the inclusions

$$\mathrm{Groth}_\infty^{comp} \subset \mathrm{Groth}_\infty^{sep} \subset \mathrm{Groth}_\infty$$

are right-lax monoidal. The tensor product in $\mathrm{Groth}_\infty^{comp}$ of C and D is $C\hat{\otimes}D$, the completion of $C\otimes D$. The tensor product in $\mathrm{Groth}_\infty^{sep}$ of C and D is $(C\otimes D)^{sep}$.

1.5.5. Recall that we have an adjoint pair $\mathcal{P}\mathrm{r}^L \leftrightarrows \mathcal{P}\mathrm{r}^{St}$, where the left adjoint sends C to $\mathbb{S}\mathrm{p}(C)\widetilde{\to} C\otimes\mathbb{S}\mathrm{p}$, and the right adjoint is the natural inclusion (by [14], 4.8.1.23 and 1.4.4.5). Since $\mathbb{S}\mathrm{p}$ with the sphere spectrum is an idempotent of $\mathcal{P}\mathrm{r}^L$, $\mathcal{P}\mathrm{r}^{\mathbb{S}\mathrm{p}}$ gets a unique symmetric monoidal structure for which the functor $\mathcal{P}\mathrm{r}^L\to\mathcal{P}\mathrm{r}^{St}$, $C\mapsto\mathbb{S}\mathrm{p}(C)$ is symmetric monoidal. So, the inclusion $\mathcal{P}\mathrm{r}^{St}\hookrightarrow\mathcal{P}\mathrm{r}^L$ is right-lax symmetric monoidal.

Any $\mathcal{A}\in\mathrm{Groth}_{ab}$ admits an essentially unique action of $\tau_{\le 0}(\mathbb{S}\mathrm{p}^{\le 0})$ as in ([15], C. 5.4.13), and by ([15], C.5.4.19) we get a symmetric monoidal structure on Groth_{ab}, whose unit is $\tau_{\le 0}(\mathbb{S}\mathrm{p}^{\le 0})$. Moreover, the functor

$$\mathrm{Groth}_\infty\to\mathrm{Groth}_{ab},\ \ C\mapsto\tau_{\le 0}C \tag{14}$$

is symmetric monoidal by ([15], C.5.4.20).

The category $\mathrm{Groth}_{ab}^{lex}$ inherits a symmetric monoidal structure from Groth_{ab}, and the functor (14) restricts to a functor $\mathrm{Groth}_\infty^{lex}\to\mathrm{Groth}_{ab}^{lex}$ (cf. [15], C.5.4.4).

1.5.6. In ([15], C.3.1.3) Lurie defines the ∞-category Groth_∞^+, whose objects are pairs $(C, C^{\le 0})$, where $C\in\mathcal{P}\mathrm{r}^{St}$, and $C^{\le 0}\subset C$ is a core, that is, a full subcategory stable under small colimits and extensions. It is equipped with a cartesian fibration $q:\mathrm{Groth}_\infty^+\to\mathcal{P}\mathrm{r}^{St}$ forgetting $C^{\le 0}$. By ([15], C.3.1.4), we have a full embedding $\mathrm{Groth}_\infty\hookrightarrow\mathrm{Groth}_\infty^+$, $C\mapsto(\mathbb{S}\mathrm{p}(C),\mathbb{S}\mathrm{p}(C)^{\le 0})$.

By ([15], C.4.2.3), Groth_∞^+ is equipped with a symmetric monoidal structure given by the formula:

$$(C, C^{\le 0})\otimes(D, D^{\le 0}) = (C\otimes D, m_!(C_{\ge 0}, D_{\ge 0})),$$

here $m_!(C_{\ge 0}, D_{\ge 0})\subset C\otimes D$ is the smallest full subcategory closed under colimits and extensions and containing $c\boxtimes d$ for $c\in C^{\le 0}, d\in D^{\le 0}$. Then q is symmetric monoidal, and the functor

$$\mathrm{Groth}_\infty\to\mathrm{Groth}_\infty^+,\quad C\mapsto(\mathbb{S}\mathrm{p}(C),\mathbb{S}\mathrm{p}(C)^{\le 0})$$

is symmetric monoidal!

Note that for $C\in\mathrm{Groth}_\infty$ the t-structure on $\mathbb{S}\mathrm{p}(C)$ is compatible with filtered colimits ([15], C.1.4.1). For $C, D\in\mathrm{Groth}_\infty$ then $L\mathrm{Fun}(D, C)\subset L\mathrm{Fun}(\mathbb{S}\mathrm{p}(D),\mathbb{S}\mathrm{p}(C))$ is fully faithful and its image consists of colimit preserving functors, which are right t-exact by ([15], C.3.1.1). Let $L\mathrm{Fun}^{t-ex}(\mathbb{S}\mathrm{p}(D),\mathbb{S}\mathrm{p}(C))\subset L\mathrm{Fun}(\mathbb{S}\mathrm{p}(D),\mathbb{S}\mathrm{p}(C))$ be the full subcagory of functors which are t-exact. It is closed under filtered colimits, because the t-structure on $\mathbb{S}\mathrm{p}(C)$ is compatible with filtered colimits. However, it is not stable under colimits, as for example, for $f\in L\mathrm{Fun}^{t-ex}(\mathbb{S}\mathrm{p}(D),\mathbb{S}\mathrm{p}(C))$, $f[1]$ is not t-exact.

1.5.7. Remark: if $C, D\in\mathrm{Groth}_\infty$ then $C\otimes D$ is generated under colimits by the essential image of the functor $C\times D\to C\otimes D$, $(c,d)\mapsto c\boxtimes d$. So, $\mathbb{S}\mathrm{p}(C\otimes D)^{\le 0}$ is the smallest subcategory of $\mathbb{S}\mathrm{p}(C\otimes D)$ generated under colimits by the essential image of $C\times D\to C\otimes D$, $(c,d)\mapsto c\boxtimes d$, no need to add extensions! See ([15], C.4.2.2-C.4.2.3).

1.5.8. The functor $f : \mathrm{Groth}_{ab}^{lex} \to \mathrm{Groth}_{\infty}^{lex,sep}$, $A \mapsto D(A)^{\leq 0}$ is left-lax symmetric monoidal, because its right adjoint is symmetric monoidal. The left-lax structure on f is not strict: already the natural map $\mathrm{D}(\mathbb{S}\mathrm{p}^{\heartsuit}) \to \mathbb{S}\mathrm{p}^{\leq 0}$ is not an equivalence, as Jacob confirms.

Example: if A is an algebra in $\mathbb{S}\mathrm{p}$ then we have the category $A-mod(\mathbb{S}\mathrm{p})$ of A-modules in $\mathbb{S}\mathrm{p}$. By ([16], 4.0.32), $A-mod(\mathbb{S}\mathrm{p}) \in 1-\mathcal{C}\mathrm{at}_{cont}^{St,cocmpl}$. The t-structure on $\mathbb{S}\mathrm{p}$ is compatible with filtered colimits (by [16], 4.0.66). Recall that $\mathbb{S}\mathrm{p}^{\leq 0} \otimes \mathbb{S}\mathrm{p}^{\leq 0} \widetilde{\to} \mathbb{S}\mathrm{p}^{\leq 0}$, where the tensor product is taken in the sense of $\mathcal{P}\mathrm{r}^L$, see ([15], C.4.1). Assume $A \in \mathbb{S}\mathrm{p}^{\leq 0}$. Then we define the t-structure on $A-mod(\mathbb{S}\mathrm{p})$ so that $A-mod(\mathbb{S}\mathrm{p})^{\leq 0}$ is the preimage of $\mathbb{S}\mathrm{p}^{\leq 0}$ under $\mathrm{oblv} : A-mod(\mathbb{S}\mathrm{p}) \to \mathbb{S}\mathrm{p}$. This is an accessible t-structure by ([14], 1.4.4.11), and $A-mod(\mathbb{S}\mathrm{p})$ is compactly generated by A ([14], 7.1.2.1). We have $\mathcal{M}aps_{A-mod(\mathbb{S}\mathrm{p})}(A, x) \widetilde{\to} \mathrm{oblv}(x)$ in $\mathbb{S}\mathrm{p}$ for $x \in A-mod(\mathbb{S}\mathrm{p})$. Here for $C \in 1-\mathcal{C}\mathrm{at}_{cont}^{St,cocmpl}$ and $c, c' \in C$ we write $\mathcal{M}aps_C(c, c') \in \mathbb{S}\mathrm{p}$ for the relative inner hom. The t-structure on $A-mod(\mathbb{S}\mathrm{p})$ is compactly generated, in the sense that $A-mod(\mathbb{S}\mathrm{p})^{\leq 0}$ is generated under filtered colimits by $A-mod(\mathbb{S}\mathrm{p})^{\leq 0} \cap A-mod(\mathbb{S}\mathrm{p})^c$. Now as in ([16], 9.3.5), the t-structure on $A-mod(\mathbb{S}\mathrm{p})$ is compatible with filtered colimits.

Jacob: let A, A' be algebras in $\mathbb{S}\mathrm{p}^{\heartsuit}$. Then the tensor product of $A-mod(\mathbb{S}\mathrm{p})^{\heartsuit}$ with $A'-mod(\mathbb{S}\mathrm{p})^{\heartsuit}$ in $\mathcal{P}\mathrm{r}^L$ is $A \otimes A' - mod(\mathbb{S}\mathrm{p})^{\heartsuit}$, where the tensor product $A \otimes A'$ is the usual tensor product of abelian groups over $\mathbb{Z}$. However, the tensor product $\mathrm{D}(A-mod(\mathbb{S}\mathrm{p})^{\heartsuit}) \otimes_{\mathrm{D}(\mathbb{Z}-mod(\mathbb{S}\mathrm{p})^{\heartsuit})} \mathrm{D}(A'-mod(\mathbb{S}\mathrm{p})^{\heartsuit})$ will be in general different, here the relative tensor product is taken in $1-\mathcal{C}\mathrm{at}_{cont}^{St,cocmpl}$.

1.5.9. Next simplification: let $\mathrm{Vect}_{\mathbb{Q}}$ be the left completion of the derived DG-category of $\mathbb{Q}$-vector spaces. Then $\mathrm{Vect}_{\mathbb{Q}} \widetilde{\to} \mathbb{Q}-mod(\mathbb{S}\mathrm{p})$ naturally, and $\mathrm{Spc} \to \mathrm{Vect}_{\mathbb{Q}}$ given by $\mathbb{Q}$ is an idempotent in $1-\mathcal{C}\mathrm{at}_{cont}^{St,cocmpl}$ and also in $\mathcal{P}\mathrm{r}^L$, giving the ∞-category

$$\mathrm{DGCat}_{cont,\mathbb{Q}} := \mathrm{Vect}_{\mathbb{Q}} -mod(1-\mathcal{C}\mathrm{at}_{cont}^{St,cocmpl})$$

of DG-categories over $\mathbb{Q}$.

As in ([15], C.4.2.2), we see now that $\mathrm{Spc} \stackrel{\mathbb{Q}}{\to} \mathrm{Vect}_{\mathbb{Q}}^{\leq 0}$ is also an idempotent in $\mathcal{P}\mathrm{r}^L$, so we get a full subcategory

$$\mathrm{Groth}_{\infty,\mathbb{Q}} := \mathrm{Vect}_{\mathbb{Q}}^{\leq 0} -mod(\mathrm{Groth}_{\infty}) \subset \mathrm{Groth}_{\infty}$$

This is the intersection $\mathrm{Vect}_{\mathbb{Q}}^{\leq 0} -mod(\mathcal{P}\mathrm{r}^L) \cap \mathrm{Groth}_{\infty}$ inside $\mathcal{P}\mathrm{r}_L$. We may also define $\mathrm{Groth}_{\infty,\mathbb{Q}}^{lex} \subset \mathrm{Groth}_{\infty,\mathbb{Q}}$ as the subcategory, where we restrict maps to left exact functors.

Similarly, for $n \geq 0$, $\mathrm{Spc} \to \tau_{\leq n}\mathrm{Spc}$ is an idempotent in $\mathcal{P}\mathrm{r}^L$, so $\tau_{\leq n}(\mathrm{Vect}_{\mathbb{Q}}^{\leq 0})$ is also an idempotent in $\mathcal{P}\mathrm{r}^L$, namely the tensor product $(\tau_{\leq n}\mathrm{Spc}) \otimes \mathrm{Vect}_{\mathbb{Q}}^{\leq 0}$ of two idempotents. In particular, $\mathrm{Spc} \stackrel{\mathbb{Q}}{\to} \mathrm{Vect}_{\mathbb{Q}}^{\heartsuit}$ is an idempotent in $\mathcal{P}\mathrm{r}^L$.

Let $\mathrm{Groth}_{ab,\mathbb{Q}} \subset \mathrm{Groth}_{ab}$ be the full subcategory of those $C \in \mathrm{Groth}_{ab}$, which admit an action of $\mathrm{Vect}_{\mathbb{Q}}^{\heartsuit}$.

We get symmetric monoidal structures on $\mathrm{Groth}_{ab,\mathbb{Q}}$ and $\mathrm{Groth}_{\infty,\mathbb{Q}}$ such that both $\mathrm{Groth}_{\infty} \to \mathrm{Groth}_{\infty,\mathbb{Q}}$, $C \mapsto \mathrm{Vect}_{\mathbb{Q}}^{\leq 0} \otimes C$ is symmetric monoidal, and $\mathrm{Groth}_{ab} \to \mathrm{Groth}_{ab,\mathbb{Q}}$,

$C \mapsto \mathrm{Vect}_{\mathbb{Q}}^{\heartsuit} \otimes C$ is symmetric monoidal. Moreover, the inclusions $\mathrm{Groth}_{\infty,\mathbb{Q}} \to \mathrm{Groth}_{\infty}$ and $\mathrm{Groth}_{ab,\mathbb{Q}} \to \mathrm{Groth}_{ab}$ preserve the tensor products, but not the unit objects.

The symmetric monoidal structrure on $\mathrm{Groth}_{ab,\mathbb{Q}}$ restricts to a symmetric monoidal structure on $\mathrm{Groth}_{ab,\mathbb{Q}}^{lex}$, and similarly for $\mathrm{Groth}_{\infty,\mathbb{Q}}^{lex} \subset \mathrm{Groth}_{\infty,\mathbb{Q}}$.

My understanding is that, as above, we have an adjoint pair

$$L_{\mathbb{Q}} : \mathrm{Groth}_{ab,\mathbb{Q}}^{lex} \leftrightarrows \mathrm{Groth}_{\infty,\mathbb{Q}}^{lex,sep} : R_{\mathbb{Q}}$$

where the left adjoint $L_{\mathbb{Q}}$ sends $\mathcal{A}$ to $\mathrm{D}(\mathcal{A})^{\leq 0}$ in the same sense, and the right adjoint $C \mapsto \tau_{\leq 0} C$ is symmetric monoidal.

Claim(Nick) The left-lax structure on $L_{\mathbb{Q}}$ is strict, so it is symmetric monoidal.

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathrm{Groth}_{ab,\mathbb{Q}}$. We want to show that the natural map $\mathrm{D}(\mathcal{A} \otimes \mathcal{B})^{\leq 0} \to \mathrm{D}(\mathcal{A})^{\leq 0} \otimes \mathrm{D}(\mathcal{B})^{\leq 0}$ is an equivalence. Use the Gabriel-Papescu theorem ([15], Theorem C.2.4.1). First, by ([15], C.2.0.12), there are maybe noncommutative rings A, B (in the category of abelian groups) such that $\mathcal{A}$ is a localization (exact in the sense of abelian categories) of $\mathrm{RMod}_A^{\heartsuit}$, and similarly for $\mathrm{RMod}_B^{\heartsuit}$. Here RMod_A is the category of right A-modules in $\mathbb{S}\mathrm{p}$ with its natural t-structure.

Recall that $\mathrm{D}(\mathrm{RMod}_A^{\heartsuit})^-$ is complete by ([14], 1.3.3.16), and we know that $\mathrm{RMod}_A^{\leq 0}$ is complete ([16], 6.1.20). The canonical functor $e : \mathrm{D}(\mathrm{RMod}_A^{\heartsuit})^- \to \mathrm{RMod}_A^{\leq 0}$ is an equivalence? Let $X, Y \in \mathrm{RMod}_A^{\heartsuit}$ with X projective. Then X is a direct summand of a free module, so $\mathrm{Ext}^i_{\mathrm{RMod}_A}(X, Y) = 0$, so that e is an equivalence by ([14], 1.3.3.7).

We have $\mathrm{RMod}_A \otimes \mathrm{RMod}_B \widetilde{\to} \mathrm{RMod}_{A \otimes B}$, and here by $A \otimes B$ we mean the usual tensor product in the category of abelian groups (or $\mathbb{Q}$-vector spaces). So,

$$\mathrm{RMod}_A^{\leq 0} \otimes \mathrm{RMod}_B^{\leq 0} \widetilde{\to} \mathrm{RMod}_{A\otimes B}^{\leq 0}$$

also by ([15], C.4.2.2). Applying the symmetric monoidal functor $R_{\mathbb{Q}}$, we get

$$\mathrm{RMod}_A^{\heartsuit} \otimes \mathrm{RMod}_B^{\heartsuit} \widetilde{\to} \mathrm{RMod}_{A\otimes B}^{\heartsuit}$$

□

1.5.10. We need the following special case only. For $\mathcal{A} \in Groth_{ab}$ assume $\mathcal{A}$ has enough projective objects. Recall that in this case $\mathrm{D}^-(\mathcal{A})$ in the sense of ([14], 1.3.2.7) identifies with $\cup_n \mathrm{D}(\mathcal{A})^{\leq n}$ by ([14], 1.3.5.24), here $\mathrm{D}(\mathcal{A})$ is taken in the sense of ([14], 1.3.5.8).

Let A be a small abelian category, and $\mathcal{A} = \mathrm{Ind}(A)$, assume A has enough projective objects. Then $\mathcal{A}$ is abelian, has enough projective objects by ([13], 1.3.3.13). Write $A_{proj} \subset A$ for the full subcategory of projective obejcts. Note that $\mathcal{A}_{proj} \subset \mathcal{A}$ is closed under finite coproducts, so we have the presentable ∞-category $\mathcal{P}_{\Sigma}(\mathcal{A}_{proj})$ defined in ([13], 5.5.8.8). Moreover, by ([13], 1.3.3.14) we have $\mathrm{D}(\mathcal{A})^{\leq 0} \widetilde{\to} \mathcal{P}_{\Sigma}(A_{proj})$.

Let $\mathcal{K}$ be the collection of finite sets. Let $1-\mathcal{C}\mathrm{at}(\mathcal{K}) \subset 1-\mathcal{C}\mathrm{at}$ be the subcategory, where we restrict objects to small ∞-categories admitting finite coproducts, and morphisms to the functors preserving finite coproducts. For $C \in 1-\mathcal{C}\mathrm{at}(\mathcal{K})$ we get $\mathcal{P}_{\Sigma}(C) \in \mathcal{P}\mathrm{r}^L$ by ([13], 5.5.8.10(1)). Moreover, if $C \to C'$ is a map in $1-\mathcal{C}\mathrm{at}(\mathcal{K})$ then the induced functor $\mathcal{P}_{\Sigma}(C) \to \mathcal{P}_{\Sigma}(C')$ preserve small colimits by ([13], 5.5.8.15 and 5.5.8.10(2)).

Claim: The functor $\mathcal{P}_\Sigma : 1-\mathcal{C}at(\mathcal{K}) \to \mathcal{P}r^L$ is symmetric monoidal, where the source is equipped with the symmetric monoidal structure defined in ([14], 4.8.1.4).

Proof. Recall that $\mathcal{P}_\Sigma(C) = \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(C)$, where $\mathcal{K}$ is the collection of finite sets, and $\mathcal{K}'$ is the collection of small ∞-categories, see ([13], 5.5.8.16). Our claim follows now from ([14], Remark 4.8.1.8).

For example, the unit object of $1-\mathcal{C}at(\mathcal{K})$ is $\mathcal{P}^{\mathcal{K}}_{\emptyset}(*)$, because the functor $1-\mathcal{C}at \to 1-\mathcal{C}at(\mathcal{K}), C \mapsto \mathcal{P}^{\mathcal{K}}_{\emptyset}(C)$ is symmetric monoidal. □

Let now $E \in \mathrm{Groth}_\infty$. Then

$$LFun(\mathrm{D}(\mathcal{A})^{\le 0}, E) \widetilde{\to} \mathrm{Fun}_{\mathcal{K}}(A_{proj}, E) \tag{15}$$

by ([13], 5.3.6.2), here $LFun$ is the category of colimit preserving functors.

Assume in addition $E \in CAlg(\mathrm{Groth}_\infty)$, hence in $\mathcal{P}r^L$, and $A \in CAlg(1-\mathcal{C}at(\mathcal{K}))$. Assume even that the multiplication $A \times A \to A$ is exact in each variable, and restrict to a map $A_{proj} \times A_{proj} \to A_{proj}$, and $1_A \in A_{proj}$, so $A_{proj} \in CAlg(1-\mathcal{C}at(\mathcal{K}))$. By the above Claim, $\mathrm{D}(\mathcal{A})^{\le 0} \in CAlg(\mathcal{P}r^L)$, and actually in Groth_∞. Under the equivalence (15) the symmetric monoidal functors $\mathrm{D}(\mathcal{A})^{\le 0} \to E$ correspond to symmetric monoidal functors $A_{proj} \to E$.

This applies for $A =$ the category of finite-dimensional representations of $\check{G}^I$ over e, where I is a finite set.

1.5.11. *Suggestion of Nick.* Recall that e is an algebraically closed field of characteristic zero. Consider the category, say $\mathcal{C}$. Its objects are presentable abelian e-linear categories $\mathcal{A}$ such that there is a full subcategory $\mathcal{A}_0 \subset \mathcal{A}^c$ generating $\mathcal{A}$ under filtered colimits and $n \in \mathbb{N}$ such that each object of $\mathcal{A}_0$ is of cohomological dimension $\le n$. Morphisms in C are exact functors in the sense of abelian categories, which are e-linear and preserve colimits.

Let us show that if $\mathcal{A} \in \mathcal{C}$ then $\mathcal{A}$ is a Grothendieck abelian category. Let I be small filtered, $I \to \mathrm{Fun}([1], \mathcal{A})$ be a functor $i \mapsto (x_i \stackrel{\alpha_i}{\to} y_i)$, where α_i is a monomrphism. Let $x \to y$ be obtained by passing to colim_I in $\mathcal{A}$. Let $z = \mathrm{Ker}(x \to y)$. It suffices to show that for any $a \in \mathcal{A}_0$, $\mathrm{Hom}(a, z) = 0$. For this it suffices to show that the natural map $\mathrm{Hom}(a, x) \to \mathrm{Hom}(a, y)$ is an isomorphism in Spc. This follows from the fact that $\mathrm{Hom}(a, \cdot)$ preserves filtered colimits and from ([13], 5.3.3.3).

Question 1: My understanding is that we may consider the category $\mathrm{Groth}_{ab,e}$ of Grothendieck abelian categories, which are e-linear. Namely, the preimage of Groth_{ab} under the projection $\mathrm{Vect}_e^{\heartsuit}-mod(\mathcal{P}r^L) \to \mathcal{P}r^L$. Is $\mathrm{Groth}_{ab,e}$ symmetric monoidal with the tensor product over $\mathrm{Vect}_e^{\heartsuit}$? Does the subcategory $\mathrm{Groth}^{lex}_{ab,e} \subset \mathrm{Groth}_{ab,e}$ inherit a symmetric monoidal structure? Does the subcategory $\mathcal{C} \subset \mathrm{Groth}^{lex}_{ab,e}$ inherit a symmetric monoidal structure?

Consider the functor $\mathcal{F} : \mathcal{C} \to \mathrm{DGCat}_{cont}$ sending A to $\mathrm{D}(A)$, where $\mathrm{D}(A)$ is the derived DG-category attached to $\mathcal{A}$ in the sense of ([14], 1.3.5.8). My understanding is that this is indeed a functor (in view of [15], C.5). We do not need left completeness of $\mathrm{D}(A)$, because we will be interested only in t-exact continuous functors from $\mathrm{D}(A)$ to a DG-category left complete in its t-structure. Indeed, for $I \in fSet$, $Shv(\mathrm{Hecke}_I)$

is left complete because of ([20], 0.0.28). We invoke here the fact that the functor L' from Section 1.5.4 is symmetric monoidal.

Question 2: is the above functor $\mathcal{F} : \mathcal{C} \to \mathrm{DGCat}_{cont}$ symmetric monoidal? (maybe at least on the bounded from below parts of our stable categories, this would be sufficient). Namely, for $A, B \in \mathcal{C}$ the natural map $\mathrm{D}(A \otimes_{\mathrm{Vect}^{\heartsuit}} B)^+ \to (D(A) \otimes_{\mathrm{Vect}} D(B))^+$ is an isomorphism? (This would imply that it induces an isomorphism on left completions). How to prove this?

1.6. More on [1] **and** [2]**.**

1.6.1. For ([2], Lemma A.2.6). The argument has evidently be corrected for H which is not connected.

If G is an affine connected algebraic group of finite type, the functor $C^{\cdot}_{\blacktriangle} : Shv(B(G)) \to \mathrm{Vect}$ is completely determined by $C^{\cdot}_{\blacktriangle}(\pi_* e)$, where $\pi : pt \to B(G)$ is the natural map. Indeed, $\pi_* e \widetilde{\to} \pi_! e[d]$ with $d = 2 \dim G_{unip} + \dim G_{red}$, and $\pi_* e$ is a compact generator of $Shv(B(G))$ in our constructible context, see ([1], F.5.5) and my Section 1.4.32. Let $B = C^{\cdot}_c(G, \omega)$, this is an algebra in Vect, and $Shv(B(G)) \widetilde{\to} B-mod(\mathrm{Vect}) =: B-mod$ by Barr-Beck-Lurie. Moreover B is co-commutative coalgebra, because $B^{\vee} \widetilde{\to} C^{\cdot}(G, e)$ is a commutative algebra (because G is a commutative coalgebra in Sch_{ft}). So, B is a co-commutative Hopf algebra. The augmentation of B comes from $G \to pt$, which gives $e \to C^{\cdot}(G)$, and in turn $B \to e$. The object $\underline{e}_{B(G)} \in Shv(B(G))$ corresponds to the auhmentation module $e \in B-mod$. Now $\mathcal{H}om_{Shv(B(G))}(\underline{e}_{B(G)}, \underline{e}_{B(G)}) \widetilde{\to} \mathcal{H}om_{B-mod}(e, e)$. Now ([7], Example 9.1.6) says that the functor $C^{\cdot}_{\blacktriangle} : Shv(B(G)) \to \mathrm{Vect}$ identifies with the functor

$$B-mod \to \mathrm{Vect}, \quad M \mapsto e \otimes_B M[-2\dim(G) + \delta],$$

where $\delta = 0$ if G is unipotent (resp., $\delta = \dim G$ if G is reductive).

This shows that $e \otimes_B e$ is bounded from above and has finite-dimensional cohomologies. Let Y_1, Y_2 be quasi-compact algebraic stacks, which are of the form Z/H, where Z is a scheme of finite type, H is a linear algebraic group of finite type. For this reason for a morphism $f : Y_1 \to Y_2$ and $F \in Shv(Y_1)^{constr}$, $f_{\blacktriangle}(F)$ and $f_*(F)$ has constructible perverse cohomologies: the only nontrivial step in the proof is this claim for the projection $f : Y_2 \times B(G) \to Y_2$, where G is a connected linear algebraic group. In this case we have the projection formula for $F \in Shv(Y_2)^{constr}$, $f_{\blacktriangle}(\omega) \otimes^! F \widetilde{\to} f_{\blacktriangle} f^! F$ by ([20], 0.0.52). By the base change of ([2], A.3.1), $C^{\cdot}_{\blacktriangle}(B(G), e) \otimes \omega_{Y_2} \widetilde{\to} f_{\blacktriangle}(\omega_{Y_2})$. So, the claim follows from the fact that the cohomologies of $C^{\cdot}_{\blacktriangle}(B(G), e)$ are finite-dimensional.

For f_* in the above argument we have to show that for the projection $f : Y_2 \times B(G) \to Y_2$ and $F \in \mathrm{Perv}(Y_2)$, $f_* f^*(F)$ has constructible perverse cohomologies. For this we use the fact that $C^{\cdot}(B(G), e)$ has finite-dimensional cohomologies and argue in the same way. This was implicit in ([2], A.2.7).

1.6.2. For ([2], B.1.4). Let Y_1, Y_2, Z are quasi-compact algebraic stacks (of the form Y/G, where Y is a scheme of finite type, and G is a linear algebraic group of finite type), so Y_i are Verdier compatible. For $Q \in Shv(Y_1 \times Y_2)$ consider their functor

$$\mathrm{id}_Z \boxtimes \mathbf{Q} : Shv(Z \times Y_1) \to Shv(Z \times Y_2), ; F \mapsto (p_{Z,Y_2})_{\blacktriangle}(p^!_{Z,Y_1}(F) \otimes^! p^!_{Y_1,Y_2} Q)$$

for the projections $p_{Z,Y_i} : Z \times Y_1 \times Y_2 \to Z \times Y_i$, $p_{Y_1,Y_2} : Z \times Y_1 \times Y_2 \to Y_1 \times Y_2$. The following is left in *loc.cit.* without an explanation.

Lemma 1.6.3. *Assume $Q \in Shv(Y_1 \times Y_2)^c$. Then $\mathrm{id}_Z \boxtimes \mathbf{Q}$ preserves compact objects.*

Proof. $Shv(Z \times Y_1)^c$ is the idempotent completion of the smallest stable subcategory of $Shv(Z \times Y_1)$ containing q_*K for $q : S \to Z \times Y_1$, where $S \in \mathrm{Sch}_{ft}$ and $K \in Shv(S)^c$. So, it suffices to show that $(\mathrm{id}_Z \boxtimes \mathbf{Q})(q_*K)$ is compact. Let $\bar{q}$ be the composition $S \stackrel{q}{\to} Z \times Y_1 \to Y_1$. For the map $\bar{q} \times \mathrm{id} : S \times Y_2 \to Y_1 \times Y_2$ the object $(\bar{q} \times \mathrm{id})^! Q$ is compact. Indeed, the functor $(\bar{q} \times \mathrm{id})^!$ admits a continuous right adjoint, as we are in the constructible context, see ([20], 0.0.11). So, $\mathrm{pr}_S^!(K) \otimes^! (\bar{q} \times \mathrm{id})^! Q$ is also compact, as the category $Shv(S \times Y_2)^c$ is preserved under the $\otimes^!$-tensor product by objects of $Shv(S \times Y_2)^{constr}$. The result is obtained by applying $(g \times \mathrm{id})_*$ to the above object, where $g : S \to Z$ is the corresponding map, and $g \times \mathrm{id} : S \times Y_2 \to Z \times Y_2$. Since g is schematic, $(g \times \mathrm{id})_*$ preserves compact objects. □

1.6.4. For ([2], B.1.5). Let us be given a system of functors $\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}$ as in *loc.cit.*, let us show that for any $K \in Shv(\mathcal{Z} \times Y_1)$,

$$(\mathrm{id} \boxtimes \mathbf{Q})(K) \widetilde{\to} (p_{\mathcal{Z},Y_2})_\blacktriangle (p^!_{\mathcal{Z},Y_1} K \otimes^! p^!_{Y_1,Y_2} Q), \tag{16}$$

where $Q = (\mathrm{id}_{Y_1} \boxtimes \mathbf{Q})(u_{Y_1})$. Here $u_{Y_1} = \vartriangle_* \omega = \vartriangle_\blacktriangle \omega$ for $\vartriangle : Y_1 \to Y_1 \times Y_1$.

Let $q : \mathcal{Z} \times Y_1 \to \mathcal{Z}$ be the projection. We get a commutative diagram

$$\begin{array}{ccc}
Shv(\mathcal{Z} \times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}}{\to} & Shv(\mathcal{Z} \times Y_2) \\
\uparrow (q \times \mathrm{id})_\blacktriangle & & \uparrow (q \times \mathrm{id})_\blacktriangle \\
Shv(\mathcal{Z} \times Y_1 \times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z} \times Y_1} \boxtimes \mathbf{Q}}{\to} & Shv(\mathcal{Z} \times Y_1 \times Y_2) \\
\uparrow (\mathrm{id} \times \vartriangle \times \mathrm{id})^! & & \uparrow (\mathrm{id} \times \vartriangle \times \mathrm{id})^! \\
Shv(\mathcal{Z} \times Y_1 \times Y_1 \times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z} \times Y_1 \times Y_1} \boxtimes \mathbf{Q}}{\to} & Shv(\mathcal{Z} \times Y_1 \times Y_1 \times Y_2) \\
\uparrow K \boxtimes \cdot & & \uparrow K \boxtimes \cdot \\
Shv(Y_1 \times Y_1) & \stackrel{\mathrm{id}_{Y_1} \boxtimes \mathbf{Q}}{\to} & Shv(Y_1 \times Y_2)
\end{array}$$

Apply the maps of this diagram to the object u_{Y_1}. Its image under the left vertical column is K. The image of Q under the write vertical column is the RHS of (16). We are done.

1.6.5. For ([2], B.1.7). The reason to write $Q_{2,3} * Q_{1,2}$ and not in the opposite order is the definition of composition of functors, it corresponds to $(\mathrm{id}_{\mathcal{Z}} \boxtimes Q_{2,3})(\mathrm{id}_{\mathcal{Z}} \boxtimes Q_{1,2})$.

1.6.6. For ([2], B.2.1). Let Y_1, Y_2 be algebraic stacks as in my Section 1.6.2, $Q \in Shv(Y_1 \times Y_2)$, and $\mathbf{Q}, \mathbf{Q}_{disc} : Shv(Y_1) \to Shv(Y_2)$ the corresponding functors "given by kernel". Then for $K \in Shv(Y_1)^c$, the map $\mathbf{Q}(K) \to \mathbf{Q}_{disc}(K)$ is an isomorphism.

Proof. Let $a : S_1 \to Y_1$ be given with $S_1 \in \mathrm{Sch}_{ft}$ and $F \in Shv(S_1)^c$. It suffices to prove this for $K = a_*F$, as the the idempotent completion of the smallest stable subcategory of $Shv(Y_1)$ containing such objects is $Shv(Y_1)^c$. Let $b : S_2 \to Y_2$ be given with $S_2 \in \mathrm{Sch}_{ft}$. It suffices to establish compatible system of isomorphisms

$b^!\mathbf{Q}(K)\tilde{\to} b^!\mathbf{Q}_{disc}(K)$ for all such b. Let $a\times b: S_1\times S_2\to Y_1\times Y_2$ be the product of the two maps. We get

$$b^!\mathbf{Q}(K)\tilde{\to}(p_2)_\blacktriangle(p_1^!F\otimes^!(a\times b)^!Q)\tilde{\to}(p_2)_*(p_1^!F\otimes^!(a\times b)^!Q)\tilde{\to} b^!\mathbf{Q}_{disc}(K),$$

for the diagram of projections $S_1\stackrel{p_1}{\leftarrow} S_1\times S_2\stackrel{p_2}{\to} S_2$. We used that p_2 is schematic, so $(p_2)_\blacktriangle\tilde{\to}(p_2)_*$. □

If we assume in addition that $Q\in Shv(Y_1\times Y_2)^{constr}$ then for any $K\in Shv(Y_1)^c$ the object $\mathbf{Q}(K)\tilde{\to}\mathbf{Q}_{disc}(K)\in Shv(Y_2)$ is constructible, this is clear from the above proof. In ([2], B.2.1) they forgot the assumption that Q is constructible for the above claim.

If moreover we assume $Q\in Shv(Y_1\times Y_2)^c$ then $\mathbf{Q}\to\mathbf{Q}_{disc}$ is an isomorphism.

Proof. It suffices to show that for any $L\in Shv(Y_2)^c$ the functor $Shv(Y_1)\to$ Vect, $F\mapsto \mathcal{H}om_{Shv(Y_2)}(L,(p_2)_*((p_1^!F)\otimes^! Q))$ is continuous. We have

$$\mathcal{H}om_{Shv(Y_1\times Y_2)}(p_2^*L,(p_1^!F)\otimes^! Q)\tilde{\to}\mathcal{H}om_{Shv(Y_1\times Y_2)}(p_2^*L,\mathcal{H}om(\mathbb{D}Q,p_1^!F))$$
$$\tilde{\to}\mathcal{H}om_{Shv(Y_1\times Y_2)}(p_2^*L\otimes\mathbb{D}(Q),p_1^!F)$$

Since $Y_1\times Y_2$ is Verdier compatible, $\mathbb{D}Q\in Shv(Y_1\times Y_2)^c$, so $p_2^*L\otimes\mathbb{D}(Q)\in Shv(Y_1\times Y_2)^c$ by ([1], F.4.4). □

Since the functor $f^*: Shv(Y_2)\to Shv(Y_1)$ has a cohomological amplitude bounded on the right, f_* has a cohomological amplitude bounded on the left. Now if Q is constructible, there is a constant n such that if $\mathcal{F}\in Shv(Y_1)^{\ge 0}$ then $p_1^!(\mathcal{F})\otimes^! Q$ is in perverse degrees $\ge n$. So, the functor Q_{disc} in this case has the cohomological amplitude bounded on the left.

Let us show that for Q constructible the functor $\mathbf{Q}$ has cohomological amplitude bounded on the right (that is, there is n such that for $F\in Shv(Y_1)^{\le 0}$ we have $\mathbf{Q}(F)\in Shv(Y_2)^{\le n}$). First, the functor $Shv(Y_1)\to Shv(Y_1\times Y_2)$, $F\mapsto (p_1^!F)\otimes^! Q\tilde{\to}\mathcal{H}om(\mathbb{D}(Q),p_1^!F)$ has a bounded cohomological amplitude. Indeed, Q is cohomologically bounded, and $p_1^!$ has bounded cohomoogical amplitude (as ω_{Y_2} is bounded), and the functor $\vartriangle^!$ has a bounded cohomological amplitude for the diagonal map $\vartriangle: Y_2\to Y_2\times Y_2$ by [4]. To finish, apply ([2], A.2.6).

1.6.7. For example, for $Y=B(\mathbb{G}_m)$, we get $C^\cdot_\blacktriangle(B(\mathbb{G}_m),e)\tilde{\to} e\otimes_B e[-1]$, where $B=C^\cdot_c(\mathbb{G}_m,\omega)\tilde{\to} e\oplus e[1]$, see my Section 1.6.1. So, it is not bounded on the left. Compare with ([2], A.28).

1.6.8. If Y is an algebraic stack as in my Section 1.6.2, $F\in Shv(Y)^{\le 0}, K\in Shv(Y)^{\ge 0}$ with F constructible then $\mathcal{H}om(F,K)\in Shv(Y)^{\ge 0}$, see ([4], after 4.2.5).

1.6.9. For ([2], B.3.1). Assume given Y_1,Y_2 as in my Section 1.6.2. Assume given for algebraic stacks $\mathcal{Z}$ as in my Section 1.6.2 a system of functors

$$\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{P}^l: Shv(\mathcal{Z}\times Y_1)\to Shv(\mathcal{Z}\times Y_2)$$

satisfying the compatibilities isomorphisms as in ([2], B.1.5) with $-^!$ replaced by $-^*$, and $-_\blacktriangle$ replaced by $-_!$. Then we may recover an object $P\in Shv(Y_1\times Y_2)$ as in the case of functors given by kernel, see my Section 1.6.4. Namely, let ps-u$_Y$ $=\triangle_!\, e$ for $\triangle\colon Y\to Y\times Y$. Then $(\mathrm{id}_{Y_1}\boxtimes \mathbf{P}^l)(\text{ps-u}_{Y_1})\widetilde{\to} P$.

1.6.10. Correction for ([2], B.3.2). In the RHS of the formula (B.11) one should write $((P^{\sigma}_{1,2})^l\boxtimes \mathrm{id}_{Y_3})(P_{2,3})$.

1.6.11. For ([2], formula (B.12)). This is not evident and should be split into several claims. The first would be as follows, which is a generalization of ([2], B.1.5).

Lemma 1.6.12. *Consider algebraic stacks $\mathcal{Z},\mathcal{Z}',Y_1,Y_2$ as in my Section 1.6.2 and a map $f:\mathcal{Z}'\to\mathcal{Z}$.*
1) The diagram commutes "up to a natural transformation"

$$\begin{array}{ccc} Shv(\mathcal{Z}'\times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z}'}\boxtimes\mathbf{Q}}{\to} & Shv(\mathcal{Z}'\times Y_2)\\ \uparrow (f\times\mathrm{id})^* & & \uparrow (f\times\mathrm{id})^*\\ Shv(\mathcal{Z}\times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q}}{\to} & Shv(\mathcal{Z}\times Y_2)\end{array}$$

namely, there is a natural transformation of functors

$$(f\times\mathrm{id})^*\circ(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q})\to(\mathrm{id}_{\mathcal{Z}'}\boxtimes\mathbf{Q})\circ(f\times\mathrm{id})^*$$

2) The diagram commutes "up to a natural transformation"

$$\begin{array}{ccc} Shv(\mathcal{Z}'\times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z}'}\boxtimes\mathbf{Q}}{\to} & Shv(\mathcal{Z}'\times Y_2)\\ \downarrow (f\times\mathrm{id})_! & & \downarrow (f\times\mathrm{id})_!\\ Shv(\mathcal{Z}\times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q}}{\to} & Shv(\mathcal{Z}\times Y_2)\end{array}$$

namely, there is a natural transformation of functors

$$(f\times\mathrm{id})_!\circ(\mathrm{id}_{\mathcal{Z}'}\boxtimes\mathbf{Q})\to(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q})\circ(f\times\mathrm{id})_!$$

3) Let $p_{\mathcal{Z}}:\mathcal{Z}\times Y_i\to\mathcal{Z}$ denotes the projection. Then there is a natural transformation functorial in $K\in Shv(\mathcal{Z}\times Y_1)$, $M\in Shv(\mathcal{Z})$

$$(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q})(K)\otimes p_{\mathcal{Z}}^*M\to(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q})(K\otimes p_{\mathcal{Z}}^*M)$$

in $Shv(\mathcal{Z}\times Y_2)$.
3') Let $F\in Shv(\mathcal{Z})$. The the diagram canonically commutes

$$\begin{array}{ccc} Shv(\mathcal{Z}\times\mathcal{Z}'\times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z}\times\mathcal{Z}'}\boxtimes\mathbf{Q}}{\to} & Shv(\mathcal{Z}\times\mathcal{Z}'\times Y_2)\\ \uparrow F\boxtimes\cdot & & \uparrow F\boxtimes\cdot\\ Shv(\mathcal{Z}'\times Y_1) & \stackrel{\mathrm{id}_{\mathcal{Z}'}\boxtimes\mathbf{Q}}{\to} & Shv(\mathcal{Z}'\times Y_2)\end{array}$$

Proof. Consider the diagram, where the squares are cartesian

$$\begin{array}{ccccc} \mathcal{Z}'\times Y_1 & \stackrel{\bar p_1}{\leftarrow} & \mathcal{Z}'\times Y_1\times Y_2 & \stackrel{\bar p_2}{\to} & \mathcal{Z}'\times Y_2\\ \downarrow f\times\mathrm{id} & & \downarrow f\times\mathrm{id} & & \downarrow f\times\mathrm{id}\\ \mathcal{Z}\times Y_1 & \stackrel{p_1}{\leftarrow} & \mathcal{Z}\times Y_1\times Y_2 & \stackrel{p_2}{\to} & \mathcal{Z}\times Y_2\end{array} \qquad (17)$$

Let $p:\mathcal{Z}\times Y_1\times Y_2\to Y_1\times Y_2$ be the projection.

1) For $K \in Shv(\mathcal{Z}\times Y_1)$ we get natural maps

$$(f\times \mathrm{id})^*(\mathrm{id}_{\mathcal{Z}}\boxtimes \mathbf{Q})(K)\to (\bar p_2)_{\blacktriangle}(f\times\mathrm{id})^*(p_1^!K\otimes^! p^!Q)$$
$$\to (\bar p_2)_{\blacktriangle}(((f\times\mathrm{id})^*p_1^!K)\otimes^! (f\times\mathrm{id})^!p^!Q)\to (\bar p_2)_{\blacktriangle}((\bar p_1^!(f\times\mathrm{id})^*K)\otimes^! (f\times\mathrm{id})^!p^!Q)$$

Here the first map comes from a natural transformation $(f\times\mathrm{id})^*(p_2)_{\blacktriangle}\to(\bar p_2)_{\blacktriangle}(f\times\mathrm{id})^*$. The second maps comes from the natural transformation

$$(f\times\mathrm{id})^*(K_1\otimes^! K_2)\to (f\times\mathrm{id})^*K_1\otimes^!(f\times\mathrm{id})^!K_2$$

explained in ([20], 0.1.5). Finally, the third map comes from the natural transformation $(f\times\mathrm{id})^*p_1^!\to\bar p_1^!(f\times\mathrm{id})^*$ explained in ([20], 0.1.8).

2) For $K\in Shv(\mathcal{Z}'\times Y_1)$ we get natural maps

$$(f\times\mathrm{id})_!\circ(\mathrm{id}_{\mathcal{Z}'}\boxtimes\mathbf{Q})(K)\to(p_2)_{\blacktriangle}(f\times\mathrm{id})_!(\bar p_1^!K\otimes^!(f\times\mathrm{id})^!p^!Q)$$
$$\to(p_2)_{\blacktriangle}(((f\times\mathrm{id})_!\bar p_1^!K)\otimes^! p^!Q)\to(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q})((f\times\mathrm{id})_!K)$$

Here the first map comes from $(f\times\mathrm{id})_!(\bar p_2)_{\blacktriangle}\to(p_2)_{\blacktriangle}(f\times\mathrm{id})_!$. The second map comes from the natural transformation $(f\times\mathrm{id})_!(K_1\otimes(f\times\mathrm{id})^!K_2)\to(f\times\mathrm{id})_!(K_1)\otimes^! K_2$ from ([20], 0.1.7). The third map comes from the natural morphism $(f\times\mathrm{id})_!\bar p_1^!\to p_1^!(f\times\mathrm{id})_!$, see ([20], Sect. 0.1.8).

3) Our p_i, p are as in (17). We have a natural map

$$((p_2)_{\blacktriangle}(p_1^!K\otimes^! p^!Q))\otimes p_{\mathcal{Z}}^*M\to(p_2)_{\blacktriangle}((p_1^!K\otimes^! p^!Q)\otimes p_2^*p_{\mathcal{Z}}^*M)$$

It comes from the projection formula "up to a natural transformation" for $(p_2)_{\blacktriangle}$ from ([20], 0.1.5). Further, $p_2^*p_{\mathcal{Z}}^*M\,\widetilde{\to}\,p_1^*p_{\mathcal{Z}}^*M$. There is a natural map

$$(p_1^!K\otimes^! p^!Q)\otimes p_1^*p_{\mathcal{Z}}^*M\to p_1^!(K\otimes p_{\mathcal{Z}}^*M)\otimes^! p^!Q$$

constructed in ([20], 0.1.9), we apply the cited result to the map $p_1:\mathcal{Z}\times Y_1\times Y_2\to\mathcal{Z}\times Y_1$. This gives the claim.

3') This follows from ([20], Lemma 0.1.3). This is also ([2], B.1.5). □

Important **Remark**: Lemma 1.6.12 holds also for $\mathbf{Q}$ replaced by $\mathbf{Q}_{disc}$ with a similar proof (the corresponding natural morphisms for $\blacktriangle$-version have a $*$-version also).

1.6.13. For ([2], B.3.4). It requires an explanation, we prepared the previous lemma for this. Let us construct a natural morphism (in their notation (B.12)):

$$(\mathrm{id}_{Y_2}\boxtimes\mathbf{P}^l)\circ(\mathbf{Q}\boxtimes\mathrm{id}_{\tilde Y_1})\to(\mathbf{Q}\boxtimes\mathrm{id}_{\tilde Y_2})\circ(\mathrm{id}_{Y_1}\boxtimes\mathbf{P}^l)$$

Consider the diagram

$$\begin{array}{ccccccc}
\tilde Y_1\times\tilde Y_2\times Y_1 & \stackrel{\bar p_1}{\leftarrow} & \tilde Y_1\times\tilde Y_2\times Y_1\times Y_2 & \stackrel{\bar p_2}{\to} & \tilde Y_1\times\tilde Y_2\times Y_2 & \stackrel{\tau}{\to} \tilde Y_1\times\tilde Y_2\\
\downarrow{\scriptstyle q\times\mathrm{id}} & & \downarrow{\scriptstyle q\times\mathrm{id}} & & \downarrow{\scriptstyle q\times\mathrm{id}} & \\
\tilde Y_1\times Y_1 & \stackrel{p_1}{\leftarrow} & \tilde Y_1\times Y_1\times Y_2 & \stackrel{p_2}{\to} & \tilde Y_1\times Y_2, &
\end{array}$$

where $q:\tilde Y_2\to pt$. Here τ is the projection. Let $K\in Shv(\tilde Y_1\times Y_1)$.

By my Lemma 1.6.12, 1) we get a natural morphism

$$(q\times\mathrm{id})^*(\mathbf{Q}\boxtimes\mathrm{id}_{\tilde Y_1})(K)\to(\mathrm{id}_{\tilde Y_1\times\tilde Y_2}\boxtimes\mathbf{Q})(q\times\mathrm{id})^*K$$

Tensoring by τ^*P, we get morphisms

$$((q\times \mathrm{id})^*(\mathbf{Q}\boxtimes \mathrm{id}_{\tilde{Y}_1})(K))\otimes \tau^*P\to ((\mathrm{id}_{\tilde{Y}_1\times\tilde{Y}_2}\boxtimes\mathbf{Q})(q\times\mathrm{id})^*K)\otimes\tau^*P\to (\mathrm{id}_{\tilde{Y}_1\times\tilde{Y}_2}\boxtimes\mathbf{Q})((q\times\mathrm{id})^*K\otimes\tau^*P),$$

where the second map comes from my Lemma 1.6.12, 3). For the projection $q_2:\tilde{Y}_1\times\tilde{Y}_2\times Y_2\to\tilde{Y}_2\times Y_2$ applying $(q_2)_!$ to the constructed morphism, we get a map

$$(q_2)_!(((q\times\mathrm{id})^*(\mathbf{Q}\boxtimes\mathrm{id}_{\tilde{Y}_1})(K))\otimes\tau^*P)\to (q_2)_!(\mathrm{id}_{\tilde{Y}_1\times\tilde{Y}_2}\boxtimes\mathbf{Q})((q\times\mathrm{id})^*K\otimes\tau^*P)$$

By my Lemma 1.6.12, 2), the latter maps naturally to

$$(\mathrm{id}_{\tilde{Y}_2}\boxtimes\mathbf{Q})((q_2)_!((q\times\mathrm{id})^*K\otimes\tau^*P))\widetilde{\to}(\mathrm{id}_{\tilde{Y}_2}\boxtimes\mathbf{Q})(\mathrm{id}_{Y_1}\boxtimes\mathbf{P}^l)(K)$$

We are done.

1.6.14. For ([2], B.3.4). It is useful to add the following. Let $Y_i,\tilde{Y}_i$ be as in their Section B.3.4, let $Q\in Shv(Y_2\times Y_1)$, $\tilde{P}\in Shv(\tilde{Y}_1\times\tilde{Y}_2)$, $Q'\in Shv(Y_1\times\tilde{Y}_1)$. The map (B.12) gives a morphism

$$\tilde{P}*^l(Q'*Q)\to(\tilde{P}*^l Q')*Q \tag{18}$$

1.6.15. For ([2], B.3.5). This follows from my Remark after the proof of my Lemma 1.6.12.

1.6.16. For ([2], B.4.1). On a separated scheme of finite type S the miraculous functor $\mathrm{Mir}_S: Shv(S)\to Shv(S)$ is given by $K\mapsto e_S\otimes^! K$. The functor $\mathrm{id}^l_S=u^l_S: Shv(S)\to Shv(S)$ is given by $K\mapsto\omega\otimes K$.

If in addition S is smooth of dimension n then it is miraculous. Indeed, Mir_S is given by the kernel $u_S[-2n]$, and the kernel u_S defines the identity functor.

1.6.17. For ([2], B.4.2). For convenience, let Y_1,Y_2 be algebraic stack as in my Section 1.6.2. Then $\mathrm{Mir}_{Y_1}\boxtimes\mathrm{id}_{Y_2}: Shv(Y_1\times Y_2)\to Shv(Y_1\times Y_2)$ for the diagram of projections

$$\begin{array}{ccccc} Y_1\times Y_2 & \stackrel{p_1}{\leftarrow} & Y_1\times Y_1\times Y_2 & \stackrel{p_2}{\to} & Y_1\times Y_2\\ & & \downarrow p & & \\ & & Y_1\times Y_1 & & \end{array} \tag{19}$$

is given by

$$(\mathrm{Mir}_{Y_1}\boxtimes\mathrm{id}_{Y_2})(K)=(p_2)_\blacktriangle(p_1^!K\otimes^! p^!(\text{ps-u}_{Y_1}))$$

Here $\text{ps-u}_{Y_1}=\triangle_! e$ for $\triangle: Y_1\to Y_1\times Y_1$. Here p_i keeps the i-term in the product $Y_1\times Y_1$ and forgets the other term.

The claim in their B.4.2 needs an explanation. Given $Q\in Shv(Y_1\times Y_2)$ let $P=(\mathrm{Mir}_{Y_1}\boxtimes\mathrm{id}_{Y_2})(Q)\in Shv(Y_1\times Y_2)$. Let $F\in Shv(\mathcal{Z}\times Y_1)$. To check their result, we need to establish isomorphisms

$$(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{P}^l)(F)\widetilde{\to}((F^\sigma)^l\boxtimes\mathrm{id}_{Y_2})(\mathrm{id}_{Y_1}\boxtimes Q)(\text{ps-u}_{Y_1})$$

and

$$(\mathrm{id}_{\mathcal{Z}}\boxtimes Q)(F)\widetilde{\to}(\mathrm{id}_{\mathcal{Z}}\boxtimes Q)((F^\sigma)^l\boxtimes\mathrm{id}_{Y_1})(\text{ps-u}_{Y_1})$$

The second isomorphism follows from $((F^\sigma)^l \boxtimes \mathrm{id}_{Y_1})(\text{ps-u}_{Y_1}) \widetilde{\to} F$, which is evident. To get the first one, we get first

$$(\mathrm{Mir}_{Y_1} \boxtimes \mathrm{id}_{Y_2})(Q) = P \widetilde{\to} (\mathrm{id}_{Y_1} \boxtimes Q)(\text{ps-u}_{Y_1}) \tag{20}$$

This is the symmetry in the definition of the functor given by kernel, namely for the diagram (19)

$$P \widetilde{\to} (p_2)_\blacktriangle(p_1^!(Q) \otimes^! p^!(\text{ps-u}_{Y_1})) \quad \text{and} \quad (\mathrm{id}_{Y_1} \boxtimes Q)(\text{ps-u}_{Y_1}) \widetilde{\to} (p_1)_\blacktriangle(p_2^! Q \otimes^! p^!(\text{ps-u}_{Y_1}))$$

The above two objects are identified via the automorphism σ of $Y_1 \times Y_1 \times Y_2$ which swaps two copies of Y_1. Now

$$((F^\sigma)^l \boxtimes \mathrm{id}_{Y_2})(P) \widetilde{\to} (\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{P}^l)(F)$$

by the same symmetry in the definition: for the similar diagram

$$\begin{array}{ccc} \mathcal{Z} \times Y_1 \overset{q_1}{\leftarrow} & \mathcal{Z} \times Y_1 \times Y_2 & \overset{q_2}{\to} \mathcal{Z} \times Y_2 \\ & \downarrow q & \\ & Y_1 \times Y_2 & \end{array}$$

we have

$$(\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{P}^l)(F) \widetilde{\to} (q_2)_!(q_1^* F \otimes q^* P) \widetilde{\to} ((F^\sigma)^l \boxtimes \mathrm{id}_{Y_2})(P)$$

The formula (20) shows, by virtue of my Section 1.6.9, that P is the only object for which we may hope that the system of functors $(\mathrm{id}_{\mathcal{Z}} \boxtimes Q)$ is codefined by the kernel P. This also explains their ([2], Pp. B.4.4).

1.6.18. In the situation of ([2], B.3) let $Q \in Shv(Y_1 \times Y_2)$ and $\mathcal{Z}$, where $Y_i, \mathcal{Z}$ are stacks as in my Section 1.6.2. Let σ denote the isomorphism $Shv(Y_1 \times \mathcal{Z}) \widetilde{\to} Shv(\mathcal{Z} \times Y_1)$ obtained by permuting the terms in the product. The diagram commutes

$$\begin{array}{ccc} Shv(Y_1 \times \mathcal{Z}) & \overset{\mathbf{Q} \boxtimes \mathrm{id}_{\mathcal{Z}}}{\to} & Shv(Y_2 \times \mathcal{Z}) \\ \downarrow \sigma & & \downarrow \sigma \\ Shv(\mathcal{Z} \times Y_1) & \overset{\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}}{\to} & Shv(\mathcal{Z} \times Y_2) \end{array}$$

and similarly for $\mathbf{Q}^l \boxtimes \mathrm{id}_{\mathcal{Z}}$. Let now $P = (\mathrm{Mir}_{Y_1} \boxtimes \mathrm{id}_{Y_2})(Q)$. Applying σ to their natural transformation $\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{P}^l \to \mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}$ of ([2], formula (B.17)), we get a natural transformation

$$\mathbf{P}^l \boxtimes \mathrm{id}_{\mathcal{Z}} \to \mathbf{Q} \boxtimes \mathrm{id}_{\mathcal{Z}}$$

of functors $Shv(Y_1 \times \mathcal{Z}) \to Shv(Y_2 \times \mathcal{Z})$.

1.6.19. For ([2], B.4.6). To get their map (B.19), write $Q_{2,3} * Q_{1,2} \widetilde{\to} (Q_{1,2}^\sigma \boxtimes \mathrm{id}_{Y_3})(Q_{2,3})$ and

$$Q_{2,3} *^l Q_{1,2} \widetilde{\to} ((Q_{1,2}^\sigma)^l \boxtimes \mathrm{id}_{Y_3})(Q_{2,3})$$

by their ([2], Sections B.1.7, B.3.2). Let us simply write $Q_{2,1} = Q_{1,2}^\sigma$. Set

$$P_{1,2} = (\mathrm{id}_{Y_1} \boxtimes \mathrm{Mir}_{Y_2})(Q_{1,2})$$

Note that $(\mathrm{Mir}_{Y_2} \boxtimes \mathrm{id}_{Y_1})(Q_{2,1}) = P_{1,2}^\sigma = P_{2,1}$. The natural transformation $\mathbf{P}_{2,1}^l \boxtimes \mathrm{id}_{Y_3} \to \mathbf{Q}_{2,1} \boxtimes \mathrm{id}_{Y_3}$ of my Section 1.6.18 gives being applied to $Q_{2,3}$ their map (B.19).

1.6.20. For ([2], B.4.8). This follows from $(\mathrm{id}_{Y_1}\boxtimes\mathbf{P}^l)(u_{Y_1})\,\widetilde{\to}\,(\mathrm{id}^l_{Y_1}\boxtimes\mathrm{id}_{Y_2})(P)=Q$ as above.

Their natural transformation (B.20) is the value of the natural transformation

$$(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{P}^l)(F^\sigma\boxtimes\mathrm{id}_{Y_1})\to(F^\sigma\boxtimes\mathrm{id}_{Y_2})(\mathrm{id}_{Y_1}\boxtimes\mathbf{P}^l)$$

(given by their (B.12)) on u_{Y_1}. Indeed, $F\,\widetilde{\to}\,(F^\sigma\boxtimes\mathrm{id}_{Y_1})(u_{Y_1})$ and $Q\,\widetilde{\to}\,(\mathrm{id}_{Y_1}\boxtimes\mathbf{P}^l)(u_{Y_1})$. Besides,

$$(F^\sigma\boxtimes\mathrm{id}_{Y_2})(Q)\,\widetilde{\to}\,(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q})(F)$$

1.6.21. For ([2], B.4.10). Let us explain their natural transformation (B.22). Set $P_{2,1}=P^\sigma_{1,2}$ and $Q_{1,2}=(\mathrm{id}_{Y_1}\boxtimes\mathrm{id}^l_{Y_2})(P_{1,2})$. So, $Q_{2,1}:=Q^\sigma_{1,2}=(\mathrm{id}^l_{Y_2}\boxtimes\mathrm{id}_{Y_1})(P_{2,1})$.

The natural transformation (B.20) gives a natural transformation

$$P^l_{2,1}\boxtimes\mathrm{id}_{Y_3}\to Q_{2,1}\boxtimes\mathrm{id}_{Y_3},$$

which we apply to $P_{2,3}$. Then use their formulas $P_{2,3}*^l P_{1,2}=(P^l_{2,1}\boxtimes\mathrm{id}_{Y_3})(P_{2,3})$ and $P_{2,3}*Q_{1,2}\,\widetilde{\to}\,(Q_{2,1}\boxtimes\mathrm{id}_{Y_3})(P_{2,3})$.

Mistake in the last claim in their Sect. B.4.10. In the last sentence $P^l_{1,2}$ should be replaced by $(P^\sigma_{1,2})^l$.

I think on more way to think about (B.22) is to say that this is the natural map

$$P_{2,3}*^l P_{1,2}=P_{2,3}*^l(u_{Y_2}*P_{1,2})\to P_{2,3}*(u_{Y_2}*^l P_{1,2})$$

1.7. Usual functors.

1.7.1. Let $f:Y_2\to Y_1$ be a morphism of algebraic stacks as in my Section 1.6.2. Then $f^*:Shv(Y_1)\to Shv(Y_2)$ is codefined by the kernel $(\Gamma_f)_!e$ for $\Gamma_f:Y_2\to Y_1\times Y_2$, $\Gamma_f=(f,\mathrm{id})$. The functor $f^!:Shv(Y_1)\to Shv(Y_2)$ is defined by the kernel $(\Gamma_f)_*\omega$.

1.7.2. Let $f:Y_1\to Y_2$ be a morphism of algebraic stacks as in my Section 1.6.2. Then $f_\blacktriangle:Shv(Y_1)\to Shv(Y_2)$ is defined by the kernel $(\Gamma_f)_*\omega$ for $\Gamma_f=(\mathrm{id},f):Y_1\to Y_1\times Y_2$. Besides $f_*:Shv(Y_1)\to Shv(Y_2)$ is of the form Q_{disc} for $Q=(\Gamma_f)_*\omega$.

The functor $f_!:Shv(Y_1)\to Shv(Y_2)$ is codefined by the kernel $(\Gamma_f)_!e$.

Assume in addition f is smooth. Then f^* is both defined and codefined by a kernel. The defining object is $(\Gamma_f)_*\omega[-2\dim.\mathrm{rel}(f)]$. Suppose also that f_* is continuous, so $f_\blacktriangle\,\widetilde{\to}\,f_*$ is an isomorphism (for example, f is schematic). Then f^* admits a right adjoint as a functor defined by kernel. Namely, (f^*,f_*) is an adjoint pair as functors given by kernel. The corresponding pair is $((\Gamma_f)_*\omega[-2\dim.\mathrm{rel}(f)],((\Gamma_f)_*\omega)^\sigma)$.

If $f:Y_1\to Y_2$ is schematic and proper then $f_*=f_!$ is both defined and codefined by a kernel. Moreover, $f_!$ admits a right adjoint as a functor given by kernel by my Proposition 1.8.5. Indeed, for any $\mathcal{Z}$ as in my Section 1.6.2, the functor $\mathrm{id}_{\mathcal{Z}}\boxtimes f_!:Shv(\mathcal{Z}\times Y_1)\to Shv(\mathcal{Z}\times Y_2)$ admits a continuous right adjoint.

Particular cases:
a) if $j:U\to Y$ is an open immersion of algebraic stacks as in my Section 1.6.2 then j^* admits a right adjoint as a functor given by kernel.
b) if $i:Z\to Y$ is a closed immersion of algebraic stacks as in my Section 1.6.2 then $i_!$ admits a right adjoint as a functor given by kernel.

1.8. Adjointness and more.

1.8.1. For ([2], B.5.1). Be carefull, in their Section B.5 the notation $Q_{2,1}$ does not mean $Q^{\sigma}_{1,2}$ that I used before!

The adjunction for a pair $(Q_{1,2}, Q_{2,1})$ means that we are given $un : u_{Y_1} \to Q_{2,1} * Q_{1,2}$, $co : Q_{1,2} * Q_{2,1} \to u_{Y_2}$ such that the compositions

$$Q_{1,2} \stackrel{\mathrm{id}*un}{\to} Q_{1,2} * Q_{2,1} * Q_{1,2} \stackrel{co*\mathrm{id}}{\to} Q_{1,2}$$

and

$$Q_{2,1} \stackrel{un*\mathrm{id}}{\to} Q_{2,1} * Q_{1,2} * Q_{2,1} \stackrel{\mathrm{id}*co}{\to} Q_{2,1}$$

are isomorphic to the identity maps in $Shv(Y_1 \times Y_2)$ and $Shv(Y_2 \times Y_1)$ respectively.

We have in mind that $Q_{1,2}$ corresponds to a functor $\mathbf{Q}_{1,2} : Shv(Y_1) \to Shv(Y_2)$ define by the kernel $Q_{1,2}$. Then if $(Q_{1,2}, Q_{2,1})$ is an adjoint pair in the above sense then $\mathbf{Q}_{1,2} : Shv(Y_1) \leftrightarrows Shv(Y_2) : \mathbf{Q}_{2,1}$ is an adjoint pair, so $\mathbf{Q}_{1,2}$ is a left adjoint.

1.8.2. For ([2], B.5.2). Let $Q_{1,2} \in Shv(Y_1 \times Y_2), Q_{2,1} \in Shv(Y_2 \times Y_1)$. Assume given for any algebraic stack $\mathcal{Z}$ as in my Section 1.6.2 an adjuction datum for a pair $(\mathrm{id}_{\mathcal{Z}} \boxtimes Q_{1,2}, \mathrm{id}_{\mathcal{Z}} \boxtimes Q_{2,1})$ compatible with the isomorphisms from their Section B.1.5. Then one gets the adjunction datum for the pair $(Q_{1,2}, Q_{2,1})$ in the 2-category introduced in their Section B.1.8 as follows. The unit map is the value on u_{Y_1} of the transformation

$$\mathrm{id} \to (\mathrm{id}_{Y_1} \boxtimes Q_{2,1})(\mathrm{id}_{Y_1} \boxtimes Q_{1,2})$$

The counit map is the value on u_{Y_2} of the natural transformation

$$(\mathrm{id}_{Y_2} \boxtimes Q_{1,2})(\mathrm{id}_{Y_2} \boxtimes Q_{2,1}) \to \mathrm{id}_{Y_2 \times Y_2}$$

1.8.3. For ([2], B.6.5). Under the assumptions of this proposition to see that P is constructible, we essentially use the fact that $\mathrm{id}_{\mathcal{Z}} \times Q$ preserve compactness (actually for $\mathcal{Z} = S_1$ an affine scheme together with a smooth covering $S_1 \to Y_1$). Namely, we pick a smooth cover by an affine scheme $f : S_1 \to Y_1$ and check that $(f \times \mathrm{id})^* P \in Shv(S_1 \times Y_2)^c$. We have

$$(f \times \mathrm{id})^* P \widetilde{\to} (\mathrm{id}_{S_1} \boxtimes P^l)((\Gamma_f)_! e),$$

where $\Gamma_f : S_1 \to S_1 \times Y_1$ is the graph of f. Since $(\Gamma_f)_! e \in Shv(S_1 \times Y_1)^c$, our claim follows.

I wonder if the same holds even if $\mathrm{id}_{\mathcal{Z}} \times Q$ is not assumed to preserve compactness.

1.8.4. For ([2], B.6.5) They say that B.6.5 is a partial converse of Theorem B.6.3, but in fact it is a full converse in the following sense:

Proposition 1.8.5. *Let Y_1, Y_2 be algebraic stack as in my Section 1.6.2. Let $Q \in Shv(Y_1 \times Y_2)$. The following conditions are equivalent:*

i) *Q admits a right adjoint as a functor defined by a kernel with the corresponding object $Q^R \in Shv(Y_2 \times Y_1)$.*
ii) *The functor $\mathbf{Q}$ is defined and codefined by a kernel. In addition, for any algebraic stack $\mathcal{Z}$ as in my Section 1.6.2, $\mathrm{id}_{\mathcal{Z}} \boxtimes Q$ preserves compactness.*

Remark 1.8.6. *Moreover, if the equivalent conditions of the above proposition hold then both* Q, Q^R *in i) are constructible, and the codefining object of the functor* $\mathbf{Q}$ *is* $P := \mathbb{D}(Q^R)^\sigma$*. This formula also allows to recover* Q^R *in the case ii). Besides,* Q^R *is safe.*

Proof. Assume i). Then ii) is explained in ([2], B.6.3). We only add the fact that $\mathrm{id}_{\mathcal{Z}} \boxtimes Q$ preserves compactness. This follows from the fact that $(\mathrm{id}_{\mathcal{Z}} \boxtimes Q, \mathrm{id}_{\mathcal{Z}} \boxtimes Q^R)$ is an adjoint pair for any $\mathcal{Z}$, and the functor $\mathrm{id}_{\mathcal{Z}} \boxtimes Q^R$ is continuous by definition. So, ii) holds.

Now assume ii). Let us first show that Q is constructible. This is done as in ([2], B.5.6). Namely, pick a smooth covering $S_1 \to Y_1$ with S_1 an affine scheme of finite type. For the map $\Gamma_f = (\mathrm{id}, f) : S_1 \to S_1 \times Y_1$ we have $(\Gamma_f)_*\omega \in Shv(S_1 \times Y_1)^c$. Now

$$(f \times \mathrm{id})^! Q \widetilde{\to} (\mathrm{id}_{S_1} \boxtimes \mathbf{Q})((\Gamma_f)_*\omega)$$

is compact by our assumption that $\mathrm{id}_{S_1} \boxtimes \mathbf{Q}$ preserves compactness. So, Q is constructible. Now i) follows by applying ([2], B.6.5). □

1.8.7. For ([2], B.6.6). Let $f : B(\mathbb{G}_m) \to pt$ be the map. Then f^* is both defined and codefined by the kernel $Q = e \in Shv(B(\mathbb{G}_m))$, the codefining object is $P = e$. The functor $\mathrm{id}_{\mathcal{Z}} \boxtimes Q$ does not preserve compactness, and Q does not admit a right adjoint as a functor given by kernel (because $Q^R = \omega_{B(\mathbb{G}_m)}$ is not safe).

1.8.8. For ([2], B.7.1). Let Y be an algebraic stack as in my Section 1.6.2. By definition, Y is miraculous if $Q = \text{ps-u}_Y$ admits a right adjoint as a functor given by kernel (with the corresponding object denoted $Q^R \in Shv(Y \times Y)$) and such that $un : u_Y \to Q^R * Q$ and $co : Q * Q^R \to u_Y$ are isomorphisms. In other words, Q is an invertible object in the 2-category defined in ([2], B.1.8).

In this case for any $\mathcal{Z}$ as in my Section 1.6.2, the adjunction data for the pair $(\mathrm{id}_{\mathcal{Z}} \boxtimes Q, \mathrm{id}_{\mathcal{Z}} \boxtimes Q^R)$ are also isomorphisms (by functoriality), so that $\mathrm{id}_{\mathcal{Z}} \boxtimes Q$ and $\mathrm{id}_{\mathcal{Z}} \boxtimes Q^R$ are equivalences inverse to each other.

This is indeed equivalent to the property that for every $\mathcal{Z}$, $\mathrm{id}_{\mathcal{Z}} \boxtimes Q : Shv(\mathcal{Z} \times Y) \to Shv(\mathcal{Z} \times Y)$ is an equivalence. If the latter property holds then the functors $(\mathrm{id}_{\mathcal{Z}} \boxtimes Q)^{-1}$ form a system of functors satisfying he compatibilities of their Section B.1.5, so coming from a uniquely defined object Q^R. Moreover, in this case Q^R and Q are inverses of each other in the 2-category defined in ([2], B.1.8). Note that Q^R is recovered as $Q^R = (\mathrm{id}_Y \times Q)^{-1}(u_Y)$.

1.8.9. For ([2], B.7.2). Recall that $ev_Y : Shv(Y) \otimes Shv(Y) \to \mathrm{Vect}$ is given by $(K, K') \mapsto C^{\cdot}_{\blacktriangle}(Y, K \otimes^! K')$.

1.8.10. For ([2], B.7.3). Here the identifications $(\mathrm{Mir}_Y)^\vee \widetilde{\to} \mathrm{Mir}_Y$ and $(\mathrm{Mir}_Y^{-1})^\vee \widetilde{\to} \mathrm{Mir}_Y^{-1}$ is with respect to hte usual Verdier duality, that is, the above pairing ev_Y, see their Sect. B.1.2.

In B.7.3 they mean that $\mathbb{D}^{\mathrm{Mir}}$ is involutive on $(Shv(Y)^c)^{op}$, as Y is Verdier compatible.

1.8.11. For ([2], B.7.4). If Y is miraculous, then $\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y : Shv(\mathcal{Z}\times Y)\to Shv(\mathcal{Z}\times Y)$ preserves constructibility for any $\mathcal{Z}$ as above. They use their Section B.2.4 to get this. We get $\mathrm{Mir}_Y^{-1}=\mathrm{id}_Y^l$ by their Th. B.6.3. The functor id_Y^l is also defined and codefined by kernel, and for the same reason preserves constructibility (it admits a left adjoint as a functor defined by kernel, so is safe by their B.6.3). So, $\mathrm{Mir}_Y : Shv(Y)^{constr}\to Shv(Y)^{constr}$ is an equivalence inducing an equivalence $\mathrm{Mir}_Y : Shv(Y)^c \widetilde{\to} Shv(Y)^c$ on compact objects.

As a corollary, $\mathbb{D}^{\mathrm{Mir}} : (Shv(Y)^{constr})^{op}\to Shv(Y)^{constr}$ is also an equivalence.

1.8.12. For ([2], B.8.2). Let $Q\in Shv(Y_1\times Y_2)$ be constructible, assume it admits a right adjoint $Q^R\in Shv(Y_2\times Y_1)$ as a functor given by kernel. Then

$$\mathbb{D}(Q^R)\widetilde{\to}((\mathrm{Mir}_{Y_1}\boxtimes \mathrm{id}_{Y_2})(Q))^\sigma$$

by their Cor. B.6.8, and Q^R is constructible. So, $Q^R\widetilde{\to}(\mathbb{D}((\mathrm{Mir}_{Y_1}\boxtimes \mathrm{id}_{Y_2})(Q)))^\sigma$. Now

$$\mathbb{D}((\mathrm{Mir}_{Y_1}\boxtimes \mathrm{id}_{Y_2})(Q))\widetilde{\to}(u^l_{Y_1}\boxtimes \mathrm{id}_{Y_2})(\mathbb{D}Q),$$

because $(\mathrm{Mir}_{Y_1}\boxtimes \mathrm{id}_{Y_2})(Q)$ is constructible. Recall that $\mathrm{id}^l_{Y_1}=u^l_{Y_1}$ by definition in their Sect. B.4.7. So indeed $'Q^R=Q^R$.

1.8.13. For ([2], B.8.4). Their map (B.27) comes from the natural morphism $e_{Y_1\times Y_2}\to \mathcal{H}om(Q,Q)\widetilde{\to} Q\otimes^! (\mathbb{D}Q)$ on $Y_1\times Y_2$.

They follow (B.28) by the map

$$(\mathrm{id}_{Y_1}\boxtimes \mathrm{id}^l_{Y_1})(Q^\sigma_{disc}\boxtimes \mathrm{id}_{Y_1})(\mathbb{D}Q^\sigma)\to (Q^\sigma_{disc}\boxtimes \mathrm{id}_{Y_1})(\mathrm{id}_{Y_2}\boxtimes u^l_{Y_1})(\mathbb{D}Q^\sigma)$$

given by (B.14). In the above formula the RHS becomes by definition

$$(Q^\sigma_{disc}\boxtimes \mathrm{id}_{Y_1})('Q^R)\widetilde{\to}'Q^R *_{disc} Q$$

The meaning of the map (B.27) is as follows. Note that $\mathrm{id}_{Y_1}\boxtimes Q^\sigma_{disc}$ has a left adjoint, which is given by $\mathrm{id}_{Y_1}\boxtimes(\mathbb{D}Q)^l$ according to their Section B.6.1. The unit of this adjunction

$$\mathrm{id}_{Y_1\times Y_1}\to (\mathrm{id}_{Y_1}\boxtimes Q^\sigma_{disc})(\mathrm{id}_{Y_1}\boxtimes(\mathbb{D}Q)^l)$$

applied to $\mathrm{ps\text{-}u}_{Y_1}$ is the map $\mathrm{ps\text{-}u}_{Y_1}\to (\mathbb{D}Q)^\sigma *_{disc} Q$ given by (B.27).

Now they try to do "the best possible" as the case when the family of functors $(\mathrm{id}_{\mathcal{Z}}\boxtimes(\mathbb{D}Q)^l)$ would be defined by kernel. In this case the corresponding kernel would be given by $(\mathrm{id}^l_{Y_1}\boxtimes \mathrm{id}_{Y_2})(\mathbb{D}Q)=('Q^R)^\sigma$. We have natural transformations as in their (B.20)

$$\mathrm{id}_{\mathcal{Z}}\boxtimes(\mathbb{D}Q)^l\to \mathrm{id}_{\mathcal{Z}}\boxtimes('Q^R)^\sigma$$

And for Q to have a right adjoint as a functor given by kernel one needs the latter transformation to be an isomorphism.

1.8.14. In ([2], B.8.5) their map (B.29) comes from the evident morphism $Q\otimes(\mathbb{D}Q)\to\omega$ on $Y_1\times Y_2$.

The meaning of their (B.29) is as follows. We have an adjoint pair $((\mathbb{D}Q)^l, Q^\sigma_{disc})$. Applying the counit map $(\mathrm{id}_{Y_2}\boxtimes(\mathbb{D}Q)^l)(\mathrm{id}_{Y_2}\boxtimes Q^\sigma_{disc})\to \mathrm{id}_{Y_2\times Y_2}$ to the object u_{Y_2}, we get a morphism $(\mathbb{D}Q)*^l Q^\sigma\to u_{Y_2}$, which is the same as their $Q*^l(\mathbb{D}Q)^\sigma\to u_{Y_2}$ given by (B.29).

As for the unit, now they "do the best possible" as the case when the family of functors $(\mathrm{id}_{\mathcal{Z}}\boxtimes(\mathbb{D}Q)^l)$ would be defined by kernel.

1.8.15. For ([2], B.9.2). Assume Q admits a right adjoint. To get (iii) it suffices to show that $(\mathbb{D}Q)^l$ is defined by a kernel (according to their Section B.4.10). We know that $((Q^R)^\sigma, Q^\sigma)$ are adjoints as functors defined by kernels by their Section B.5.4. So, by their Theorem B.6.3, for $\mathbb{D}Q=(\mathbb{D}Q^\sigma)^\sigma$ the functor $(\mathbb{D}Q)^l$ is indeed defined by a kernel.

1.8.16. For ([2], B.9.3). If we assume (iii) then $(\mathbb{D}Q)^l$ is defined by a kernel. The corresponding kernel is $('Q^R)^\sigma$.

Let $f:\tilde Y_2\to Y_2$ be a smooth cover, where $\tilde Y_2$ is a scheme. They denote by $'\tilde Q^R$ the object defined by the same formula as $'Q^R$, where one replaces Q by $\tilde Q$, that is, $'\tilde Q^R=(\mathrm{id}_{\tilde Y_2}\boxtimes\mathrm{id}^l_{Y_1})((\mathbb{D}\tilde Q)^\sigma)$, where $\tilde Q=(\mathrm{id}\times f)^*Q\in Shv(Y_1\times\tilde Y_2)$.

A simple calculation shows that the composition $(\mathrm{id}_{\mathcal{Z}}\boxtimes{}'Q^R)(\mathrm{id}_{\mathcal{Z}}\boxtimes f_*)$ is given by the kernel $(f\times\mathrm{id})^!('Q^R)\in Shv(\tilde Y_2\times Y_1)$. The latter identifies indeed with $'\tilde Q^R$.

1.8.17. For ([2], B.9.3). They use the following. Let $f:S\to Y$ be a smooth surjective morphism of algebraic stacks as in my Section 1.6.2, which is schematic. So, $f_*: Shv(Y_1)\to Shv(Y_2)$ is continuous. Let us check that any $F'\in Shv(Y_2)$ writes as a totalization of objects in the essential image of f_*.

Since f is a smooth cover, $Shv(Y)\widetilde{\to}\operatorname{Tot}Shv(S^\bullet)$, where $S^\bullet$ is the Cech nerve of f. For $n\ge 0$ let $f^n:S^n\to Y$ be the corresponding map. As in Section 1.4.24, for $F',F\in Shv(Y)$ we have $\mathcal{H}om_{Shv(Y)}(F,F')\widetilde{\to}\operatorname{Tot}\mathcal{H}om_{Shv(S^n)}((f^n)^!F,(f^n)^!F')$. Since f^n is smooth, $(f^n)^!=(f^n)^*[2\dim.\mathrm{rel}(f^n)]$. So,

$$\mathcal{H}om_{Shv(Y)}(F,F')\widetilde{\to}\operatorname{Tot}\mathcal{H}om_{Shv(Y)}(F,(f_n)_*(f^n)^!F'[-2\dim.\mathrm{rel}(f^n)])\widetilde{\to}$$
$$\operatorname{Tot}\mathcal{H}om_{Shv(Y)}(F,(f_n)_*(f^n)^*F')\widetilde{\to}\mathcal{H}om_{Shv(Y)}(F,\operatorname{Tot}_n(f_n)_*(f^n)^*F')$$

We used the fact that we have a functor $\boldsymbol{\Delta}\to Shv(Y),[n]\mapsto(f_n)_*(f^n)^*F'$. This shows that the natural map $F'\to\operatorname{Tot}_n(f_n)_*(f^n)^*F'$ is an isomorphism.

1.8.18. For ([2], B.9.6). It is useful to note that for any algebraic stack Y as in my Section 1.6.2, one has $(\mathrm{id}_Y\boxtimes\mathrm{id}^l_Y)(\text{ps-u}_Y)\widetilde{\to}u_{Y_1}$. That is, the identity functor is both defined and codefined by a kernel.

A way to understand their diagfam would be to wtrite down a more general one. Namely, assume given objects Q,A,B which are "composable", in the sense that $Q*A$ and $A*B$ are defined. Recall that we have canonical functorial morphisms

$$Q*^l(A*B)\to(Q*^l A)*B \tag{21}$$

for such a triple. Recall also that we have an involution $Q \mapsto Q^\sigma$ with the property that $(Q * A)^\sigma = A^\sigma * Q^\sigma$ and $(Q *^l A)^\sigma = A^\sigma *^l Q^\sigma$. Conjugating (21) by σ, we obtain functorial maps

$$(Q * A) *^l B \to Q * (A *^l B) \tag{22}$$

The morphisms (21), (22) yield a diagram

$$\begin{array}{ccc} Q *^l (A * B) & \to & Q * (u *^l (A * B)) \\ \downarrow & & \downarrow \\ (Q *^l A) * B & & Q * (u *^l A) * B) \\ \downarrow & & \downarrow \text{id} \\ Q * (u *^l A) * B & = & Q * (u *^l A) * B \end{array}$$

which actually commutes!

Question: what this property for a pair of monoidal structures given by $*^l$ and $*$ means, maybe there is a name for this? Here u is the unit for the $*$-monoidal structure. I suggest something in the next subsection.

1.8.19. Consider a scheme of finite type S. We equip $Shv(S \times S)$ with two monoidal structures: the first is $(A, B) \mapsto A * B$, the second is $(A, B) \mapsto A *^l B$. We denote then $(Shv(S \times S), *)$ and $(Shv(S \times S), *^l)$.

For $B \in Shv(S \times S)$ let $r_B : Shv(S \times S) \to Shv(S \times S)$ be the functor $A \mapsto A * B$. If we view $Shv(S \times S)$ as a left module over $(Shv(S \times S), *^l)$ (given by the multiplication on the left) then r_B is naturally a right-lax functor of $(Shv(S \times S), *^l)$-module categories. This encodes the maps (21). Let $l_B : Shv(S \times S) \to Shv(S \times S)$ be the functor $A \mapsto B * A$. If we view $Shv(S \times S)$ as a right module over $(Shv(S \times S), *^l)$ (given by the multiplication on the right) then for $Q \in Shv(S \times S)$ the functor l_Q is a right-lax functor of $(Shv(S \times S), *^l)^{op}$-module categories. This encodes the maps (22). The two structures are swapped by σ. There is some strange compatibility between the two structures given by the latter diagram (to be clarified).

1.8.20. For any morphism $f : Y_1 \to Y_2$ of algebraic stacks as in my Section 1.6.2, we have an adjoint pair $(f_\blacktriangle, (f_!)^\vee)$. Moreover, if f is schematic then $(f^!, (f^*)^\vee)$ is an adjoint pair.

1.8.21. Let Y_1, Y_2 be algebraic stacks as in my Section 1.6.2, $Q \in Shv(Y_1 \times Y_2)$ gives the functor $\mathbf{Q} : Shv(Y_1) \to Shv(Y_2)$ defined by the kernel Q. We identify $Shv(Y_i)^\vee \widetilde{\to} Shv(Y_i)$ as in their Section A.4.1. Then $\mathbf{Q}^\vee : Shv(Y_2) \to Shv(Y_1)$ is defined by the kernel Q^σ.

Let now $I \to Shv(Y_1 \times Y_2)$ be a diagram, $i \mapsto Q_i$, let $\mathbf{Q}_i : Shv(Y_1) \to Shv(Y_2)$ be the correspoinding functor. Then for $K \in Shv(Y_1)$ we have $\operatorname{colim}_i \mathbf{Q}_i(K) \widetilde{\to} \mathbf{Q}(K) \in Shv(Y_2)$, where $\mathbf{Q}$ is the functor given by the kernel $\operatorname{colim}_i Q_i$. And the same for finite limits. Moreover, the same holds for the functors codefined by kernels.

1.8.22. *Remark.* Let $\mathcal{C}^0 \subset Shv(Y_1 \times Y_2)$ be the full subcategory of those Q for which $\mathbf{Q}$ is defined and codefined by kernel. Let $\mathcal{C} \subset \mathcal{C}^0$ be the full subcategory of those Q for which the functor $\mathbf{Q}$ admits a right adjoint as a functor given by kernel. Then

i) $\mathcal{C}^0 \subset Shv(Y_1 \times Y_2)$ is a stable subcategory, closed under colimits (and finite limits).

ii) $\mathcal{C} \subset \mathcal{C}^0$ is a stable subcategory

Proof. i) It suffices to show $\mathcal{C}^0$ is closed under the formation of limits/colimits in $Shv(Y_1 \times Y_2)$. We give a proof for colimits, it is similar for the limits. Let $\mathcal{Z}$ be a stack as in in my Section 1.6.2 and $K \in Shv(\mathcal{Z} \times Y_1)$. Let $I \to \mathcal{C}^0, i \mapsto Q_i$ be a diagram and $Q = \operatorname{colim}_i Q_i$ calculated in $Shv(Y_1 \times Y_2)$. Let $P_i = (\mathrm{Mir}_{Y_1} \boxtimes \mathrm{id}_{Y_2})(Q_i) \in Shv(Y_1 \times Y_2)$ and $P = (\mathrm{Mir}_{Y_1} \boxtimes \mathrm{id}_{Y_2})(Q)$, so $P \widetilde{\to} \operatorname{colim} P_i$ in $Shv(Y_1 \times Y_2)$. By ([2], B.4.4), it suffices to show that the natural map

$$(\mathrm{id}_{\mathcal{Z}} \boxtimes P^l)(K) \to (\mathrm{id}_{\mathcal{Z}} \boxtimes Q)(K)$$

is an isomorphism. The desired isomorphism is obtained by passing to the colimit over $i \in I$ in the corresponding isomorphisms $(\mathrm{id}_{\mathcal{Z}} \boxtimes P_i^l)(K) \to (\mathrm{id}_{\mathcal{Z}} \boxtimes Q_i)(K)$, taking into account the previous subsection.

ii) Clearly, $\mathcal{C} \subset \mathcal{C}^0$ is closed under translations. By ([14], 1.1.3.3) it suffices to show $\mathcal{C} \subset \mathcal{C}^0$ is closed under cofibers. Let $I \to \mathcal{C}, i \mapsto Q_i$ be a finite diagram and $Q = \operatorname{colim}_i Q_i$ in $Shv(Y_1 \times Y_2)$. It suffices to show $Q \in \mathcal{C}$. We apply my Proposition 1.8.5. Let $\mathcal{Z}$ be a stack as in in my Section 1.6.2 and $K \in Shv(\mathcal{Z} \times Y_1)^c$. It suffices to show that $\operatorname{colim}_{i \in I}(\mathrm{id}_{\mathcal{Z}} \boxtimes Q_i)(K) \in Shv(\mathcal{Z} \times Y_2)^c$. However, $Shv(\mathcal{Z} \times Y_2)^c \subset Shv(\mathcal{Z} \boxtimes Y_2)$ is a stable subcategory by ([16], 4.2.2), it is also closed under finite colimits by (HTT, 5.3.4.15). □

We write $\mathcal{C}(Y_1 \times Y_2)$ and $\mathcal{C}^0(Y_1 \times Y_2)$ if we need to express the dependence on Y_i.

1.8.23. In the situation of my Remark 1.8.22 assume $i : Y_2 \hookrightarrow Y_2'$ is a closed immersion of algebraic stacks as in my Section 1.6.2. Let $K \in Shv(Y_1 \times Y_2)$. Then $Q \in \mathcal{C}^0(Y_1 \times Y_2)$ iff $(\mathrm{id}_{Y_1} \boxtimes i_!)(Q) \in \mathcal{C}^0(Y_1 \times Y_2')$ and similarly for $\mathcal{C}^0$ replaced by $\mathcal{C}$.

Proof. 1) Let $P = (\mathrm{Mir}_{Y_1} \boxtimes \mathrm{id}_{Y_2})(Q)$ and $Q' = (\mathrm{id}_{Y_1} \boxtimes i_!)(Q)$. Set $P' := (\mathrm{id}_{Y_1} \boxtimes i_!)(P)$ then $(\mathrm{Mir}_{Y_1} \boxtimes \mathrm{id}_{Y_2'})(Q') \widetilde{\to} P'$. Assume $i_! \circ \mathbf{Q}$ is both defined and codefined by a kernel. Let us check that $\mathbf{Q}$ is codefined by a kernel. Note that $i_! \circ \mathbf{Q} \widetilde{\to} \mathbf{Q}'$.

We check that for $\mathcal{Z}$ as in my Section 1.6.2 the natural map $\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{P}^l \to \mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}$ is an isomorphism. It suffices to show that the composition

$$(\mathrm{id}_{\mathcal{Z}} \boxtimes i_!)(\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{P}^l) \to (\mathrm{id}_{\mathcal{Z}} \boxtimes i_!)(\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q})$$

is an isomorphism. We have $(\mathrm{id}_{\mathcal{Z}} \boxtimes i_!)(\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}) \widetilde{\to} (\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}')$ and

$$(\mathrm{id}_{\mathcal{Z}} \boxtimes i_!)(\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{P}^l) \widetilde{\to} \mathrm{id}_{\mathcal{Z}} \boxtimes (\mathbf{P}')^l,$$

Now the map $\mathrm{id}_{\mathcal{Z}} \boxtimes (\mathbf{P}')^l \to \mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}'$ is an isomorphism by our assumption.

2) Recall that $i_!$ admits a right adjoint as a functor given by kernel by my Section 1.7.2. Assume $i_! \circ \mathbf{Q} \widetilde{\to} \mathbf{Q}'$ admits a right adjoint as a functor given by kernel. Let us check the same for $\mathbf{Q}$ itself. By 1) we know already that $\mathbf{Q}$ is both defined and codefined by a kernel. By my Prop. 1.8.5, it suffices to check that for $\mathcal{Z}$ as in my

Section 1.6.2 the functor $\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q} : Shv(\mathcal{Z} \times Y_1) \to Shv(\mathcal{Z} \times Y_2)$ preserves compactness. If $K \in Shv(\mathcal{Z} \times Y_1)^c$ then $(\mathrm{id}_{\mathcal{Z}} \boxtimes i_!)(\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q})(K) \in Shv(\mathcal{Z} \times Y_2')^c$. Now $(\mathrm{id} \times i)^!$ preserves compactness, as it is schematic. We are done. □

1.9. Around ULA property.

1.9.1. For ([2], B.10.2). Let $f : Y \to S$ be a morphism, where S is a separated scheme of finite type, and Y is an algebraic stack as in my Section 1.6.2. Let $\mathcal{F} \in Shv(Y)$. Let $\Gamma_f : S \to S \times Y$ be the graph of f, this is a closed immersion. Set $P = (\mathrm{Mir}_S \boxtimes \mathrm{id}_Y)((\Gamma_f)_*\mathcal{F})$, so $P \widetilde{\to} (\Gamma_f)_*((f^! e) \otimes^! \mathcal{F})$. The functor $P^l : Shv(S) \to Shv(Y)$ is given by

$$\mathcal{G} \mapsto ((f^! e) \otimes^! \mathcal{F}) \otimes f^*\mathcal{G}$$

Write $F : Shv(S) \to Shv(Y)$ for the functor given by the kernel $Q := (\Gamma_f)_*\mathcal{F}$.

By their B.4.4, we have a natural transformation $\mathrm{id}_{\mathcal{Z}} \boxtimes P^l \to \mathrm{id}_{\mathcal{Z}} \boxtimes Q$ for any $\mathcal{Z}$ as in my Section 1.6.2. The functor $F : Shv(S) \to Shv(Y)$, $\mathcal{G} \mapsto \mathcal{F} \otimes^! f^!\mathcal{G}$ is codefined by a kernel iff this natural transformation is an isomorphism. In this case the codefining object is P by their B.4.4.

The functor $\mathrm{id}_{\mathcal{Z}} \boxtimes F : Shv(\mathcal{Z} \times S) \to Shv(\mathcal{Z} \times Y)$ is

$$\mathcal{G} \mapsto p_Y^! \mathcal{F} \otimes^! (\mathrm{id} \times f)^! \mathcal{G}$$

for the diagram $Y \stackrel{p_Y}{\leftarrow} \mathcal{Z} \times Y \stackrel{\mathrm{id} \times f}{\to} \mathcal{Z} \times S$. The functor $\mathrm{id}_{\mathcal{Z}} \boxtimes P^l : Shv(\mathcal{Z} \times S) \to Shv(\mathcal{Z} \times Y)$ is

$$\mathcal{G} \mapsto p_Y^*((f^! e) \otimes^! \mathcal{F}) \otimes (\mathrm{id} \times f)^* \mathcal{G}$$

If S is smooth of dimension d then $e_S = \omega[-2d]$, and the above natural transformation writes for $\mathcal{G} \in Shv(\mathcal{Z} \times S)$

$$p_Y^*(\mathcal{F}) \otimes (\mathrm{id} \times f)^* \mathcal{G}[-d] \to p_Y^! \mathcal{F} \otimes^! (\mathrm{id} \times f)^! \mathcal{G}[d]$$

In $SGA4\frac{1}{2}$ the notion of local acyclicity was formulated even in the case when the base S is not smooth (in [5] it was reformulated for morphisms as above with S smooth). Some modified version of ULA property appeared in ([17], Definition 4.8.2). There is also the abstract ULA property of Raskin from ([22], B.5.1).

Example: Assume in the above situation that both S and f are smooth and $\mathcal{F} = e_Y$. Then $\mathcal{F}$ is ULA with respect to $f : Y \to S$ in the sense of ([2], B.10.2).

1.9.2. For $\mathcal{F} \in Shv(Y)$ as in the previous subsection write $\mathcal{H}om_{Y,S}(\mathcal{F}, ?) : Shv(Y) \to Shv(S)$ for the relative inner hom with respect to the $Shv(S)$-action on $Shv(Y)$. It is understood that $\mathcal{G} \in Shv(S)$ acts on $M \in Shv(Y)$ as $M \otimes^! f^!\mathcal{G}$. This inner hom always exists, because $Shv(Y)$ is presentable.

Assume now that F is codefined by the kernel P as above. Then for $K \in Shv(Y), \mathcal{G} \in Shv(S)$ we get

$$\mathcal{H}om_{Shv(S)}(\mathcal{G}, \mathcal{H}om_{Y,S}(\mathcal{F}, K)) \widetilde{\to} \mathcal{H}om_{Shv(Y)}(\mathcal{F} \otimes^! f^!\mathcal{G}, K) \widetilde{\to}$$
$$\mathcal{H}om_{Shv(Y)}(((f^! e) \otimes^! \mathcal{F}) \otimes f^*\mathcal{G}, K) \widetilde{\to} \mathcal{H}om_{Shv(Y)}(f^*\mathcal{G}, \mathcal{H}om((f^! e) \otimes^! \mathcal{F}, K)) \widetilde{\to}$$
$$\mathcal{H}om_{Shv(S)}(\mathcal{G}, f_*\mathcal{H}om((f^! e) \otimes^! \mathcal{F}, K))$$

Here $\mathcal{H}om$ means relative inner hom in $(Shv(Y), \otimes)$. So,

$$\mathcal{H}om_{Y,S}(\mathcal{F}, K) \widetilde{\to} f_*\mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K) \in Shv(S)$$

functorially on $K \in Shv(Y)$. Let us check if the functor $K \mapsto f_*\mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K)$ is $Shv(S)$-linear. It will not always be the case!!!

For $M \in Shv(S)$ we always have a natural map

$$M \otimes^! \mathcal{H}om_{Y,S}(\mathcal{F}, K) \to \mathcal{H}om_{Y,S}(\mathcal{F}, K \otimes^! f^!M) \tag{23}$$

We want to check if it is an isomorphism. We have

$$M \otimes^! f_*\mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K) \widetilde{\to} f_*(f^!M \otimes^! \mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K)$$

Moreover, there is a natural map

$$f^!M \otimes^! \mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K) \to \mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K \otimes^! f^!M) \tag{24}$$

It suffices to show it is an isomorphism. It is evidently the case for $\mathcal{F} \in Shv(Y)^{constr}$, as in this case $(f^!e) \otimes^! \mathcal{F} \in Shv(Y)^{constr}$, and

$$\mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K) \widetilde{\to} \mathbb{D}((f^!e) \otimes^! \mathcal{F}) \otimes^! K$$

In the case $M \in Shv(S)^c$ we get

$$f^!M \otimes^! \mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K) \widetilde{\to} \mathcal{H}om(\mathbb{D}(f^!M), \mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K)) \widetilde{\to}$$
$$\mathcal{H}om(\mathbb{D}(f^!M) \otimes ((f^!e) \otimes^! \mathcal{F}), K) \widetilde{\to} \mathcal{H}om((f^!e) \otimes^! \mathcal{F}, \mathcal{H}om(\mathbb{D}(f^!M), K)) \widetilde{\to} \mathcal{H}om((f^!e) \otimes^! \mathcal{F}, K \otimes^! f^!M)$$

We are done in this case also.

However, in general neither (24) nor (23) is an isomorphism.

Example Take $f = \mathrm{id} : S \to S$ and $S = \operatorname{Spec} k$. In this case $M, \mathcal{F}, K \in Vect$ and the map (24) becomes

$$M \otimes \mathcal{H}om(\mathcal{F}, K) \to \mathcal{H}om(\mathcal{F}, K \otimes M)$$

If say $M, \mathcal{F}$ are infinite-dimensional vector spaces placed in degree zero and $K = e$ then the above map is not surjective. On the other hand, any $\mathcal{F} \in \mathrm{Vect}$ is ULA in the sense of ([2], Definition B.10.2) with respect to $\mathrm{id} : \operatorname{Spec} k \to \operatorname{Spec} k$. **Warning!** So, in this case though F is ULA in the sense of ([2], Definition B.10.2), it is not necessarily ULA in the sense of ([22], B.5.1).

1.9.3. Assume now $f : Y \to S$ as in Section 1.9.1 and $\mathcal{F} \in Shv(Y)^{constr}$ and assume that the functor $F : Shv(S) \to Shv(Y)$ defined by the kernel $Q = (\Gamma_f)_*\mathcal{F}$ is also codefined by the above kernel P. We have seen in the previous subsection that the functor $Shv_{Y,S} : Shv(Y) \to Shv(S)$ is $Shv(S)$-linear. Is it continuous?

If f is schematic then this is evidently the case.

In general it is not continuous! For example, take $S = pt$, $Y = B(\mathbb{G}_m)$ and $\mathcal{F} = e_Y$. Then the functor $Shv_{Y,S} : Shv(Y) \to \mathrm{Vect}$ becomes the functor $K \mapsto \mathrm{R}\Gamma(B(\mathbb{G}_m), K)$, and we know it is not continuous. So, $e_{B(\mathbb{G}_m)}$ is not ULA with respect to $f : B(\mathbb{G}_m) \to pt$ in the sense of ([22], B.5.1). On the other case it is ULA in the sense of ([2], B.10.2) as we have seen above.

Sam: it is only meaningful to compare his definition of ULA with that of ([2], B.10.2) for $\mathcal{F} \in Shv(Y)^c$. In this case indeed, the above functor $Shv(Y) \to Shv(S)$, $K \mapsto \mathcal{H}om_{Y,S}(\mathcal{F}, K)$ is continuous, because it is of the form Q^R_{disc} for

$$Q^R = ((\Gamma_f)_*(f^*\omega \otimes \mathbb{D}\mathcal{F}))^\sigma \in Shv(Y \times S)^c$$

So, Q^R_{disc} is continuous by ([2], B.2.3).

There is a discussion of the abstract ULA property (in a situation of a dualizable category) in ([9], appendix D).

Sam: in the constructible context probably if $\mathcal{F} \in Shv(Y)^c$ and the above $Shv_{Y,S} : Shv(Y) \to Shv(S)$ is $Shv(S)$-linear and continuous, this does not imply that $\mathcal{F}$ is ULA in the sense of ([2], B.10.2) (though it does for $\mathcal{D}$-modules (as we know from ([9], appendix D).

1.9.4. In the situation of my Section 1.9.1 assume $\mathcal{F} \in Shv(Y)^{constr}$. Let us make precise the condition that the functor $F : Shv(S) \to Shv(Y), \mathcal{G} \mapsto \mathcal{F} \otimes^! f^!\mathcal{G}$ admits a right adjoint as a functor given by kernel. Let

$$'Q^R = ((\Gamma_f)_*(f^*\omega \otimes \mathbb{D}\mathcal{F}))^\sigma \in Shv(Y \times S)$$

Their map $u_S \to' Q^R * Q$ given by ([2], (B.26)) becomes a morphism

$$\mu : \omega_S \to f_\blacktriangle(((f^*\omega_S) \otimes \mathbb{D}\mathcal{F}) \otimes^! \mathcal{F})$$

on S. Set $\mathcal{F}^\vee := f^*\omega_S \otimes \mathbb{D}\mathcal{F}$. This becomes a map $\mu : \omega_S \to f_\blacktriangle(\mathcal{F} \otimes^! \mathcal{F}^\vee)$. The counit of a would-be adjunction becomes a map $Q * {}'Q^R \to u_Y$, that is, a morphism $\epsilon : \bar{\vartriangle}_*(\mathcal{F} \boxtimes_S \mathcal{F}^\vee) \to \vartriangle_* \omega_Y$, here $\vartriangle: Y \to Y \times Y$ is the diagonal, and $\bar{\vartriangle} : Y \times_S Y \to Y \times Y$ is the natural map. The notation $K_1 \boxtimes_S K_2$ means $q_1^! K_1 \otimes^! q_2^! K_2$ for the projections $q_i : Y \times_S Y \to Y$. We may view ϵ as a map $\mathcal{F} \boxtimes_S \mathcal{F}^\vee \to \bar{\vartriangle}^! \vartriangle_* \omega_Y$.

Note that if $Y \in \mathrm{Sch}_{ft}$ then the property that Q admits a right adjoint as a functor given by kernel becomes precisely the property ([9], D.4.2(i)), which was formulated in *loc.cit.* for $\mathcal{F}$ compact.

The property (iii) of ([9], D.4.2) is maybe strictly weaker then the fact that Q admits a right adjoint as a functor given by kernel: the implication (iii)⇒(ii) is not justified in ([9], D.4.2).

Remark Let $\mathcal{F} \in Shv(Y)^{constr}$. If $\mathcal{F}$ is ULA over S in the sense of ([2], B.10.1) then for any base change $g : \tilde{S} \to S$, where S' is a separated scheme of finite type, for the base changed map $\tilde{f} : \tilde{Y} \to \tilde{S}$, $\tilde{\mathcal{F}} = g_Y^! \mathcal{F}$ is also ULA with respect to $\tilde{f}$. Here $g_Y : \tilde{Y} \to Y$ is the induced map.

Proof: let $h : Y' \to Y$ be a smooth surjective map, where $Y' \in \mathrm{Sch}_{ft}$. Then the same holds for the base changed map $\tilde{Y}' \to \tilde{Y}$. So, by the next subsection, it suffices to show that the !-restriction of $\tilde{\mathcal{F}}$ to $\tilde{Y}'$ is ULA over S. Let $\mathcal{F}'$ be the !-restriction of $\mathcal{F}$ to Y'. Then $\mathcal{F}' \in Shv(Y')^c$ is ULA over S, so the functor $Shv(S) \to Shv(Y'), \mathcal{G} \mapsto h^!\mathcal{F} \otimes^! h^! f^! \mathcal{G}$ admits a right adjoint as a functor given by kernel by ([2], B.10.4). The !-restriction under $\tilde{Y}' \to Y'$ will produce a functor, which also has a right adjoint as a functor given by kernel. So, again by ([2], B.10.4) we see that $\tilde{\mathcal{F}}'$ is ULA over S. We are done. □

1.9.5. Let us show that the ULA condition from ([2], B.10.2) is local in the smooth topology on Y. Let $g : Y' \to Y$ be a smooth surjective morphism of stacks as in my Section 1.6.2, $\mathcal{F} \in Shv(Y)$. Then $g^*[2\dim.\mathrm{rel}(g)] \widetilde{\to} g^!$ is both defined and codefined by a kernel.

Now $\mathcal{F}$ is ULA with respect to f in the sense of ([2], B.10.2) iff $f^!\mathcal{F}$ is ULA with respect to $f \circ g$. Indeed, we compose the functor $Shv(S) \to Shv(Y)$, $\mathcal{G} \mapsto \mathcal{F} \otimes^! f^!\mathcal{G}$ with $g^!$. The composition of funtors defined by a kernel (resp., codefined by a kernel) is defined by a kernel (resp., codefined by a kernel).

1.9.6. For ([2], B.10.4). Assume $\mathcal{F} \in Shv(Y)^c$ and $\mathcal{F}$ is ULA w.r.t. $f : Y \to S$ in the sense of their Def. B.10.2. So, $F : Shv(S) \to Shv(Y)$, $\mathcal{G} \mapsto \mathcal{F} \otimes^! f^!\mathcal{G}$ is codefined by $P = (\Gamma_f)_*((f^!e) \otimes^! \mathcal{F})$. Let $Q^R = (\mathbb{D}P)^\sigma$. Then $Q^R \in Shv(Y \times S)^c$, because $(f^!e) \otimes^! \mathcal{F} \in Shv(Y)^c$ by ([2], A.2.2). So, the functors $\mathrm{id}_{\mathcal{Z}} \boxtimes Q^R_{disc}$ are continuous by ([2], B.2.3), hence $\mathrm{id}_{\mathcal{Z}} \boxtimes F$ preserve compactness. Now by ([2], B.6.5), F admits a right adjoint as a functor defined by kernel.

Conversely, if F admits a right adjoint as a functor defined by kernel then F is defined and codefined by a kernel by ([2], B.6.3).

1.9.7. For ([2], B.10.5). Let $f : Y \to S$ be a map as in their Def. B.10.2, and $F : Shv(S) \to Shv(Y)$ be the functor given by the kernel $Q = (\Gamma_f)_*(\mathcal{F})$ for $\mathcal{F} \in Shv(Y)^{constr}$ and $\Gamma_f : Y \to S \times Y$ the graph of f. Let us write down explicitly their map (B.30). We get ${}'Q^R \widetilde{\to} \mathbb{D}(P)^\sigma \widetilde{\to} ((\Gamma_f)_*(f^*\omega \otimes \mathbb{D}\mathcal{F}))^\sigma$. Let ${}'\triangle \colon Y \times_S Y \to Y \times Y$ be the natural map, this is a closed immersion.

We get

$$Q * {}'Q^R \widetilde{\to} ({}'\triangle)_*({}'\triangle^!)((f^*\omega \otimes \mathbb{D}\mathcal{F}) \boxtimes \mathcal{F}) \in Shv(Y \times Y)$$

Now $Q *^l (\mathbb{D}Q)^\sigma \widetilde{\to} ({}'\triangle)_*({}'\triangle^*)((\mathbb{D}\mathcal{F}) \boxtimes \mathcal{F})$. So, their map (B.30) becomes the push-out $({}'\triangle)_*$ of the morphism

$$({}'\triangle^*)((\mathbb{D}\mathcal{F}) \boxtimes \mathcal{F}) \to ({}'\triangle^!)(((f^*\omega_S) \otimes \mathbb{D}\mathcal{F}) \boxtimes \mathcal{F}) \tag{25}$$

So, in their formula in Remark B.10.5 there is a **mistake**: f^*e_S should be replaced by $f^*\omega_S$.

Assume the map (25) is an isomorphism. Let $h : Y' \to Y$ be a smooth cover of the relative dimension d, where Y' is a scheme of finite type. Consider teh commutative diagram

$$\begin{array}{ccc} Y \times_S Y & \stackrel{{}'\triangle}{\to} & Y \times Y \\ \uparrow \tau & & \uparrow h\times h \\ Y' \times_S Y' & \stackrel{{}''\triangle}{\to} & Y' \times Y', \end{array}$$

where $\tau = h \times_S h$. Note that $\tau^! = \tau^*[4d]$. Applying τ^* to (25) we get an isomorphism

$$\begin{aligned} {}''\triangle^* \, (h^*(\mathbb{D}\mathcal{F}) \boxtimes h^*\mathcal{F})[-2d] \widetilde{\to} \tau^!({}'\triangle^!)(((f^*\omega_S) \otimes \mathbb{D}\mathcal{F}) \boxtimes \mathcal{F})[-4d] \widetilde{\to} \\ {}''\triangle^! \, (((fh)^*\omega_S \otimes \mathbb{D}(h^*\mathcal{F})) \boxtimes h^*\mathcal{F})[-2d] \end{aligned}$$

Indeed, $h^! \widetilde{\to} h^*[2d]$. We see that the version of (25) for $h^*\mathcal{F}$ is an isomorphism. We conclude by their B.8.8 that Q admits a right adjoint as a functor given by kernel.

Since $h^*\mathcal{F} \in Shv(Y')^c$, this gives that $h^*\mathcal{F}$ is ULA with respect to $hf : Y' \to S$ by their B.10.4, so $\mathcal{F}$ is also ULA over S.

1.9.8. For ([2], B.10.10). We amy assume $\mathcal{Z} = S$ is a scheme of finite type. Let $\mathcal{G} \in Shv_{\mathcal{N}\times T^*(Y)}(S \times Y)$. We want to show that $(\mathrm{id}_S \boxtimes F)(\mathcal{G}) \in Shv_{\mathcal{N}}(S)$. Let $S \stackrel{h}{\leftarrow} U \stackrel{t}{\to} S'$ be a $\mathcal{N}$-transversal test pair, so U, S' are smooth. We may and do assume S' separated. We want to show that $h^*(\mathrm{id}_S \boxtimes F)(\mathcal{G})$ is ULA with respect to t. We have

$$h^*(\mathrm{id}_S \boxtimes F)(\mathcal{G}) \widetilde{\to} (\mathrm{id}_U \boxtimes F)((h \times \mathrm{id})^*\mathcal{G})$$

by the version of their B.1.5 for functors codefined by kernels. Here $h\times\mathrm{id} : U\times Y \to S\times Y$. By my Section A.1.5, $SingSupp((h\times\mathrm{id})^*\mathcal{G}) \subset (h\times\mathrm{id})^\circ(\mathcal{N}\times T^*(Y)) = h^\circ(\mathcal{N})\times T^*(Y)$. Here $(h \times \mathrm{id})^\circ$ is the notation of Beilinson [3]. The composition $U \times Y \to U \stackrel{t}{\to} S'$ is $h^\circ(\mathcal{N}) \times T^*(Y)$-transversal. Now $S \times Y \stackrel{h\times\mathrm{id}}{\leftarrow} U \times Y \to S'$ is a $\mathcal{N} \times T^*(Y)$-transversal pair, so $(h \times \mathrm{id})^*\mathcal{G}$ is ULA over S'.

So, indeed, it suffices to show that given $K \in Shv(U\times Y)$ which is ULA with respect to the composition $U \times Y \to U \stackrel{t}{\to} S'$ then $(\mathrm{id}_U \boxtimes F)(K)$ is ULA over S'. This indeed follows from their B.10.7.

1.10. For [9], Appendix D.

1.10.1. For D.1.4. First, in the defining formula for $m^{\vee,C}$ one may replace if needed Map $\in$ Spc by $\mathcal{H}om \in$ Vect.

Let $C \in CAlg(\mathrm{DGCat}_{cont})$ and $M \in C-mod(\mathrm{DGCat}_{cont})$ be dualizable as a C-module. Let $m \in M, m^{\vee,C} \in M^{\vee,C}$ be equipped with an isomorphism

$$\mathrm{Maps}_M(c \otimes m, m') \widetilde{\to} \mathrm{Map}_C(c, counit^C_M(m' \otimes_C m^{\vee,C}))$$

To this data one associates $\mu : 1_C \to counit^C_M(m\otimes_C m^{\vee,C})$ corresponding to $\mathrm{id} : m \to m$ in the LHS. Now $\mathrm{Fun}_C(M, M) \widetilde{\to} M^{\vee,C} \otimes_C M$ naturally, and under this isomorphism $unit^C_M(1)$ corresponds to $\mathrm{id} : M \to M$. Taking in the above formula $c = counit^C_M(m'\otimes_C m^{\vee,C})$, we get a morphism

$$counit^C_M(m' \otimes_C m^{\vee,C}) \otimes m \to m'$$

in M functorial in $m' \in M$. This is a morphism in $\mathrm{Fun}_C(M, M) \widetilde{\to} M^{\vee,C} \otimes_C M$ from the functor $m^{\vee,C} \otimes_C m$ to the identity functor of M. In other words, this is a map $\epsilon : m^{\vee,C} \otimes_C m \to unit^C_M(1_C)$.

Now ϵ gives for any $m' \in M$ a morphism $\mathrm{Maps}_C(c, counit^C_M(m' \otimes_C m^{\vee,C})) \to \mathrm{Maps}_M(c \otimes m, m')$. It sends $\alpha : c \to counit^C_M(m' \otimes_C m^{\vee,C}))$ to the composition

$$c \otimes m \stackrel{\alpha\otimes\mathrm{id}}{\to} counit^C_M(m' \otimes_C m^{\vee,C})) \otimes m \stackrel{\epsilon}{\to} m'$$

Our μ gives for $m' \in M$ a morphism

$$\mathrm{Maps}_M(c \otimes m, m') \to \mathrm{Maps}_C(c, counit^C_M(m' \otimes_C m^{\vee,C}))$$

It sends $\beta : c \otimes m \to m'$ to the composition

$$c \stackrel{\mu}{\to} c\otimes counit^C_M(m\otimes_C m^{\vee,C}) \widetilde{\to} counit^C_M(c\otimes m\otimes_C m^{\vee,C}) \stackrel{counit^C_M(\beta\otimes\mathrm{id})}{\to} counit^C_M(m'\otimes_C m^{\vee,C})$$

Their property that (D.2) and its analog for $m^{\vee,C}$ are isomorphisms is a way to say that these two arrows are inverse to each other. The analog of (D.2) for $m^{\vee,C}$ says that the composition

(26)
$$m^{\vee,C} \stackrel{\mathrm{id}\otimes\mu}{\to} m^{\vee,C}\otimes_C counit^C_M(m\otimes_C m^{\vee,C}) = (\mathrm{id}\otimes counit^C_M)(m^{\vee,C}\otimes_C m\otimes_C m^{\vee,C})$$
$$\stackrel{\epsilon\otimes\mathrm{id}}{\to} (\mathrm{id}\otimes counit^C_M)(unit^C_M(1_C)\otimes m^{\vee,C})$$

is isomorphic to the identity map. In these axioms m and $m^{\vee,C}$ appear symmerically. This is why if m is ULA then $m^{\vee,C}\in M^{\vee,C}$ is also ULA and its dual is m.

The second axiom is obtained as follows. We start for any $m'\in M$ with the identity morphism $counit^C_M(m'\otimes_C m^{\vee,C})\to counit^C_M(m'\otimes_C m^{\vee,C})$. Then send it to the LHS of the formula, and then further to the RHS. The result should be again the identity map as above. This is equivalent to requiring that the composition (26) is the identity. Indeed, the latter property is equivalent to the fact that for any $m'\in M$ applying $counit^C_M(m'\otimes_C \bullet)$ to (26) one gets the identity.

1.10.2. In the situation of D.1.4, for $m\in M$ the functor $\underline{\mathrm{Hom}}(m,\bullet): M\to C$ is right-lax functor of C-module categories.

So, if C is rigid, it is automatically a strict functor.

1.10.3. For their D.2.1. For $c\in C^c$ and $V\in \mathrm{Vect}^{fd}$ one has $\mathbb{D}_C(V\otimes c)\widetilde{\to} V^\vee\otimes\mathbb{D}_C(c)$ canonically.

1.10.4. For their D.2.4. Let $C\in CAlg(\mathrm{DGCat}_{cont})$ with C compactly generated such that $1_C\in C^c$. Let $M\in\mathrm{DGCat}_{cont}$ be compactly generated. We assume the existence of $\tilde{1}_C\in C^c$ as in their D.2.1, hence an equivalence $\mathbb{D}_C: (C^c)^{op}\widetilde{\to} C^c$. By definition, for $c_1\in C^c, c_2\in C$ one has

$$\mathcal{H}om_C(\mathbb{D}_C(c_1), c_2)\widetilde{\to}\mathcal{H}om_C(\tilde{1}_C, c_1\otimes c_2)$$

Let $M^\vee$ be the dual of M in DGCat_{cont}, and $<,>: M\times M^\vee\to\mathrm{Vect}$ the tautological pairing.

We explain that given $m\in M, m'\in M$, there is $\langle m,m'\rangle_C\in C$ such that for any $c\in C^c$ one has $\mathcal{H}om_C(c,\langle m,m'\rangle_C)=\langle\mathbb{D}_C(c)\otimes m, m'\rangle$. This is done as in ([16], 9.2.3, a version for Spc replaced by Vect).

Note also that $\mathbb{D}_M: (M^c)^{op}\widetilde{\to}(M^\vee)^c$ is characterised by the property: for $m_1,m_2\in M^c$ one has $\langle\mathbb{D}_M(m_1),m_2\rangle=\mathcal{H}om_M(m_1,m_2)$.

1.10.5. For their D.2.7. We may add in the formulation that if the conditions (i) (equivalently, (ii) or (iii)) hold then $\tilde{1}_C\otimes m$ and $m^{\vee,C}$ are compact.

Let us explain the implication (iii)⇒(ii). If $\tilde{1}_C\otimes m\in M^c$ and $c\in C^c$ then

$$\mathcal{H}om_M(\tilde{1}_C\otimes m,\mathbb{D}_C(c)\otimes m')\widetilde{\to}\langle\mathbb{D}_M(\tilde{1}_C\otimes m),\mathbb{D}_C(c)\otimes m'\rangle\widetilde{\to}\mathcal{H}om_C(c,\langle\mathbb{D}_M(\tilde{1}_C\otimes m),m'\rangle_C)$$

So, in this case the functor $M\to C, m'\mapsto\langle\mathbb{D}_M(\tilde{1}_C\otimes m),m'\rangle_C$ preserves colimits and is C-linear.

Explanation for (i)⇒(iii). We get $counit^C_M(m'\otimes_C m^{\vee,C})\widetilde{\to} < m', m^{\vee,C}\rangle_C$. By construction, the functor $M\to C, m'\mapsto\langle m',m^{\vee,C}\rangle_C$ is continuous and C-linear.

For $m' \in M$ we get $\mathcal{H}om_M(\tilde{1}_C \otimes m, m') \tilde{\to} \langle m', m^{\vee,C}\rangle$. Since the functor $m' \mapsto \langle m', m^{\vee,C}\rangle$ is continuous, we see that $\tilde{1}_C \otimes m \in M^c$ and

$$\langle m', m^{\vee,C}\rangle \tilde{\to} \langle \mathbb{D}_M(\tilde{1}_C \otimes m), m'\rangle$$

for any $m' \in M$. This gives $\mathbb{D}_M(\tilde{1}_C \otimes m) \tilde{\to} m^{\vee,C}$, and $m^{\vee,C} \in M^c$.

1.10.6. For their D.2.8. This is correct. If $M_0 \in \mathrm{DGCat}_{cont}$ is compactly generated then identify $(C \otimes M_0)^\vee \tilde{\to} M_0^\vee \otimes C$ via the self-duality on C given by their D.2.1. Then the tautological pairing $<,>: (C \otimes M_0) \times C \otimes M_0^\vee \to \mathrm{Vect}$ becomes:

$$(c \otimes m_0, c' \otimes m_0') \mapsto < m_0, m_0' > \otimes \mathcal{H}om_C(\tilde{1}_C, c \otimes c')$$

This implies the claim.

1.10.7. For their Pp. D.4.2. Here $f : Z \to Y$ is a morphism of schemes of finite type, so $\tilde{f}^! : Shv(\tilde{Y}) \to Shv(\tilde{Z})$ has a continuous right adjoint. So, for $\mathcal{F}_{\tilde{Y}} \in Shv(\tilde{Y})^c$ in (iii), $\tilde{f}^!(\mathbb{D}(\mathcal{F}_{\tilde{Y}})) \in Shv(\tilde{Z})^c$, and $\tilde{f}^!(\mathbb{D}(\mathcal{F}_{\tilde{Y}})) \otimes^! \tilde{\mathcal{F}}' \tilde{\to} \mathcal{H}om(\tilde{f}^*\mathcal{F}_{\tilde{Y}}, \tilde{\mathcal{F}}')$. Here $\mathcal{H}om$ is the inner hom for $(Shv(\tilde{Z}), \otimes)$. So, (D.12) says that the canonical map

$$\tilde{f}^*\mathcal{F}_{\tilde{Y}} \otimes (\tilde{f}^!(e_{\tilde{Y}}) \otimes^! \tilde{\mathcal{F}}) \to \tilde{f}^!(\mathcal{F}_{\tilde{Y}}) \otimes^! \tilde{\mathcal{F}}$$

defined in ([20], 0.1.9) is an isomorphism.

In the 2nd line of the proof, (D.9) should be replaced by (D.12). Indeed, the RHS of (D.12) writes

$$\mathcal{H}om_{Shv(\tilde{Z})}(\tilde{f}^*\mathcal{F}_{\tilde{Y}} \otimes (\tilde{f}^!(e_{\tilde{Y}}) \otimes^! \tilde{\mathcal{F}}), \tilde{\mathcal{F}}') \tilde{\to} \mathcal{H}om_{Shv(\tilde{Z})}(\tilde{f}^*\mathcal{F}_{\tilde{Y}}, \mathcal{H}om(\tilde{f}^!(e_{\tilde{Y}}) \otimes^! \tilde{\mathcal{F}}, \tilde{\mathcal{F}}'))$$
$$\tilde{\to} \mathcal{H}om_{Shv(\tilde{Y})}(\mathcal{F}_{\tilde{Y}}, \tilde{f}_*(\tilde{\mathcal{F}}' \otimes^! \mathbb{D}(\tilde{f}^!(e_{\tilde{Y}}) \otimes^! \tilde{\mathcal{F}})))$$

Question: Why in the RHS of (D.14) one gets the same thing as in the LHS? This boils down to establishing an isomorphism

$$\vartriangle^* (p_1^!(e_Z) \otimes^! p_2^!\mathcal{F}) \tilde{\to} f^!(e_Y) \otimes^! \mathcal{F},$$

in $Shv(Z)$, where $\vartriangle: Z \to Z \times_Y Z$ is the diagonal, and $p_i : Z \times_Y Z \to Z$ are the projections. Not clear where it comes from.

1.11. For [2], Appendix C.

1.11.1. For their C.1.1. If U_i are quasi-compact algebraic stacks (of the form Z/G, where Z is a scheme of finite type, G is an affine algebraic group of finite type according to their conventions), $j : U_1 \to U_2$ is an open immersion. If j is cotruncative then $j_! : Shv(U_1) \to Shv(U_2)$ is both defined and codefined by a kernel. So, the compatibilities of their B.1.5 hold for it, as well as the version of B.1.5 with $-!$ replaced by $-*$ and $-_\blacktriangle$ replaced by $-_!$.

Note that j is cotruncative iff j^* has a left adjoint as a functor given by kernel. Let $i : \mathcal{Z} \hookrightarrow U_2$ be a closed substack whose complement is U_1.

Claim. U_1 is cotruncative in U_2 iff the functor $i^!$ admits a right adjoint as a functor given by kernel. This is also equivalent to i_* admits a left adjoint as a functor given by kernel (write i^* for this left adjoint given by kernel).

Proof. The proof is inspired by ([6], Prop. 3.1.2).
1) By my Section 1.8.23, $i^!$ admits a right adjoint as a functor given by kernel iff the same holds for $i_!i^!$.

If j_* admits a right adjoint as a functor given by kernel then so does j_*j^* by my Section 1.7.2. For $K \in Shv(U_2)$ we have a functorial fibre sequence $i_!i^!K \to K \to j_*j^*K$ in $Shv(U_2)$. By my Remark 1.8.22, we see that $i_!i^!$ admits a right adjoint as a functor given by kernel, hence the same for $i^!$.

Conversely, assume $i^!$ admits a right adjoint as a functor given by kernel. Recall that $i_!$ admits a right adjoint as a functor given by kernel by my Section 1.7.2. Again by my Remark 1.8.22, we see that j_*j^* admits a right adjoint as a functor given by kernel, let $\bar{Q} \in Shv(U_2 \times U_2)$ be the object defining this right adjoint by kernel. By ([2], B.6.3), j_*j^* is defined and codefined by a kernel. Now $j_* = (j_*j^*) \circ j_!$ is codefined by a kernel as a composition of functors codefined by kernels. So, j_* is defined and codefined by kernel. By my Prop. 1.8.5, it suffices to show for any $\mathcal{Z}$ as above the functor $(\mathrm{id}_{\mathcal{Z}} \boxtimes j_*) : Shv(\mathcal{Z} \times U_1) \to Shv(\mathcal{Z} \times U_2)$ preserves compactness. We have $(\mathrm{id}_{\mathcal{Z}} \boxtimes j_*) = (\mathrm{id}_{\mathcal{Z}} \boxtimes j)_* = (\mathrm{id}_{\mathcal{Z}} \boxtimes j_*j^*)(\mathrm{id}_{\mathcal{Z}} \boxtimes j_!)$. Since both $\mathrm{id}_{\mathcal{Z}} \boxtimes j_*j^*$ and $\mathrm{id}_{\mathcal{Z}} \boxtimes j_!$ preserve compactness, we are done.

2) The fact that $(i^!, Q)$ is an adjoint pair as functors given by kernel is equivalent to the fact that (Q^σ, i_*) is an adjoint pair as functors given by kernel by ([2], B.5.4) and my Section 1.7.2. □

In the situation of the above claim write $i_?$ for the functor given by kernel and right adjoint to $i^!$. Similarly, the right adjoint to j_* given by kernel is denoted $j^?$. Using ([2], A.4.4) we see that $i_?$ is the dual to i^*, and $j_!$ is the dual to $j^?$. Once again,

$$i_! = (i^*)^\vee, \quad j_! = (j^?)^\vee$$

1.11.2. To summarize, consider the situation of the previous subsection, $i : \mathcal{Z} \to U_2$ is a closed immersion, and $j : U_1 \hookrightarrow U_2$ is the complementary open. The following conditions are equivalent:

- j_* admits a right adjoint as a functor given by kernel;
- $i^!$ admits a right adjoint as a functor given by kernel;
- i_* admits a left adjoint as a functor given by kernel;
- j^* admits a left adjoint as a functor given by kernel

Under the above conditions we may say that $j : U_1 \hookrightarrow U_2$ is *cotruncative*, and $i : \mathcal{Z} \hookrightarrow U_2$ is *trancative*, following [6]. We arrange these functors into sequences: functors between $Shv(\mathcal{Z})$ and $Shv(U_2)$

$$i^*, i_*, i^!, i_?$$

and functors between $Shv(U_1)$ and $Shv(U_2)$

$$j_!, j^*, j_*, j^?$$

As in *loc.cit.*, the natural maps $i^!i_? \to \mathrm{id}$ and $\mathrm{id} \to j^*j_!$ are isomorphism of functors given by kernels.

1.11.3. Consider a locally closed embedding $i : \mathcal{Z} \to Y$ of algebraic stacks as in my Section 1.6.2. Following *loc.cit.*, call $\mathcal{Z}$ *truncative* in Y if $i^!$ admits a right adjoint $i_?$ as a functor given by kernel.

Lemma 1.11.4. *Let $Z \xrightarrow{i'} Y' \xrightarrow{j} Y$ be a diagram of stacks as in my Section 1.6.2, where i' is a closed embedding, and j is an open embedding. Then i' is truncative iff $j \circ i'$ is truncative.*

Proof. The composition of truncative morphisms is truncative. This gives the "only if" direction. Assume now $j \circ i'$ is truncative. Note that $i'^!$ is defined by a kernel. The functor $i'^!$ identifies with $(ji')^! \circ j_!$. Since $j \circ i'$ admits a right adjoint as a functor given by kernel, it is codefined by a kernel, and $j_!$ is codefined by a kernel, hence $i'^!$ is codefined by a kernel. It remains to show that for any $\mathcal{Z}$ as in my Section 1.6.2, $\mathrm{id}_{\mathcal{Z}} \boxtimes i'^!$ preserves compactness. The latter functor is $(\mathrm{id}_{\mathcal{Z}} \times i')^!$. It preserves compactness by my Section 1.8.20. □

1.11.5. Consider a truncative locally closed embedding $i : X \to Y$ of algebraic stacks as in my Section 1.6.2. If $\mathcal{Z}$ is another algebraic stack as in my Section 1.6.2 then $\mathcal{Z} \times X \to \mathcal{Z} \times Y$ is also truncative.

Proof. Let $\mathbf{Q}^R : Shv(Y) \to Shv(X)$ be the functor given by kernel Q^R, which is the right adjoint to $i^!$ as a functor given by kernel. Then for any $\mathcal{Z}'$ as in my Section 1.6.2 we have an adjoint pair $(\mathrm{id}_{\mathcal{Z}'} \boxtimes i^!, \mathrm{id}_{\mathcal{Z}'} \boxtimes \mathbf{Q}^R)$ compatible with the isomorphisms of ([2], B.1.5). So, the same holds for the system of functors $(\mathrm{id}_{\mathcal{Z}' \times \mathcal{Z}} \boxtimes i^!, \mathrm{id}_{\mathcal{Z}' \times \mathcal{Z}} \boxtimes \mathbf{Q}^R)$. By ([2], B.5.2), we are done. □

1.11.6. Let $f : Y_1 \to Y_2$ be a schematic morphism of stacks as in my Section 1.6.2. Let $\Gamma_f : Y_1 \to Y_1 \times Y_2$ be its graph. Recall that $f^! : Shv(Y_2) \to Shv(Y_1)$ is defined by the kernel $Q := (\sigma\Gamma_f)_*\omega \in Shv(Y_2 \times Y_1)$. Let $'Q^R$ be constructed out of Q by their formula (B.25) in their Section B.8.1. So, if Q admits a right adjoint given by kernel, it is given by $'Q^R$. One gets $'Q^R \widetilde{\to} (f \times \mathrm{id}_{Y_2})^* u_{Y_2}$, here $u_{Y_2} = \vartriangle_* \omega$ for $\vartriangle : Y_2 \to Y_2 \times Y_2$. We get

$$'Q^R * Q \widetilde{\to} (f \times \mathrm{id})_{\blacktriangle} (f \times \mathrm{id})^* u_{Y_2} \in Shv(Y_2 \times Y_2)$$

Their map (B.26) becomes $u_{Y_2} \to (f \times \mathrm{id})_{\blacktriangle} (f \times \mathrm{id})^* u_{Y_2}$, this is the map coming from the adjunction simply. We have canonically.

$$Q * {'Q^R} \widetilde{\to} (\mathrm{id} \times f)^! (f \times \mathrm{id})^* u_{Y_2}$$

Now $Q *^l (\mathbb{D}Q)^\sigma \widetilde{\to} (f \times \mathrm{id}_{Y_1})^* ((\sigma\Gamma_f)_*\omega) \widetilde{\to} (f \times \mathrm{id}_{Y_1})^* (\mathrm{id}_{Y_2} \times f)^! u_{Y_2}$ canonically. So, their map (B.30) becomes

$$(f \times \mathrm{id}_{Y_1})^* (\mathrm{id}_{Y_2} \times f)^! u_{Y_2} \to (\mathrm{id} \times f)^! (f \times \mathrm{id})^* u_{Y_2} \tag{27}$$

It comes from the canonical natural transformation

$$(f \times \mathrm{id}_{Y_1})^* (\mathrm{id}_{Y_2} \times f)^! \to (\mathrm{id} \times f)^! (f \times \mathrm{id})^*$$

for the cartesian square

$$\begin{array}{ccc} Y_1\times Y_1 & \stackrel{\mathrm{id}\times f}{\to} & Y_1\times Y_2\\ \downarrow f\times\mathrm{id} & & \downarrow f\times\mathrm{id}\\ Y_2\times Y_1 & \stackrel{\mathrm{id}\times f}{\to} & Y_2\times Y_2 \end{array}$$

Let now $h_2 : Y'_2 \to Y_2$ an etale schematic morphism, define f', h_1 by the cartesian square

$$\begin{array}{ccc} Y'_1 & \stackrel{f'}{\to} & Y'_2\\ \downarrow h_1 & & \downarrow h_2\\ Y_1 & \stackrel{f}{\to} & Y_2 \end{array}$$

Claim 1. If $f^!$ admits a right adjoint as a functor given by kernel then so does $f'^!$.

Proof. By ([2], B.8.8), it suffices to show that the version of the map (27) for f replaced by f' is an isomorphism. Applying $(h_1\times h_1)^! = (h_1\times h_1)^*$ to the isomorphism (27), one gets an isomorphism

$$(f'\times\mathrm{id})^*(\mathrm{id}\times f')^!(h_2\times h_2)^! u_{Y_2} \widetilde{\to} (\mathrm{id}\times f')^!(f'\times\mathrm{id})^*(h_2\times h_2)^! u_{Y_2}$$

Since Y'_2 is a connected component of $Y'_2\times_{Y_2} Y'_2$, the object $u_{Y'_2}$ is a direct summand of $(h_2\times h_2)^! u_{Y_2}$. So, the version of the map (27) for f replaced by f' is an isomorphism. □

Let $P = (\mathrm{Mir}_{Y_2}\boxtimes\mathrm{id}_{Y_1})(Q)$ then

$$P\widetilde{\to}\sigma^!(f\times\mathrm{id}_{Y_2})^!(\text{ps-u}_{Y_2}) = (\mathrm{id}\times f)^!\,\text{ps-u}_{Y_2}\in Shv(Y_2\times Y_1)$$

Then $\mathbf{Q}$ is codefined by a kernel iff for any $\mathcal{Z}$ as in my Section 1.6.2, the map $\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{P}^l\to\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q}$ is an isomorphism.

Claim 2. Let $\alpha : U\to Y_1$ be a smooth surjective schematic morphism of some relative dimension d. Then $f^!$ admits a right adjoint as a functor given by kernel iff $(f\alpha)^!$ admits a right adjoint as a functor given by kernel.

Proof. We have $\alpha^!\widetilde{\to}\alpha^*[2d]$. The functors $\alpha^*, \alpha^!$ admit right adjoints as functors given by kernel by my Section 1.7.2. This gives the 'only if" direction.

Now assume $(f\alpha)^!$ admits a right adjoint as a functor given by kernel. By ([2], B.8.8) the map

$$(f\alpha\times\mathrm{id})^*(\mathrm{id}\times f\alpha)^! u_{Y_2}\to(\mathrm{id}\times f\alpha)^!(f\alpha\times\mathrm{id})^* u_{Y_2}$$

is an isomorphism. Since $(\mathrm{id}\times\alpha)^! = (\mathrm{id}\times\alpha)^*[2d]$, the above map rewrites as

$$(\alpha\times\alpha)^*(f\times\mathrm{id})^*(\mathrm{id}\times f)^! u_{Y_2}[2d]\to(\alpha\times\alpha)^*(\mathrm{id}\times f)^!(f\times\mathrm{id})^* u_{Y_2}[2d]$$

It is obtained by applying $(\alpha\times\alpha)^*[2d]$ to the map (27). Since $(\alpha\times\alpha)^*$ is conservative, (27) is an isomorphism, so $f^!$ asdmits a right adjoint as a functor given by kernel by ([2], B.8.8). □

The following is an analog of ([6], 3.6.4).

Corollary 1.11.7. *Let $i : \mathcal{Z} \to Y$ be a locally closed substack, $f : \tilde{Y} \to Y$ be a smooth schematic morphism, where all the stacks are as in my Section 1.6.2. Let $\tilde{\mathcal{Z}} \subset \mathcal{Z} \times_Y \tilde{Y}$ be an open substack such that the resulting morphism $f' : \tilde{\mathcal{Z}} \to \mathcal{Z}$ is surjective. If the locally closed embedding $\tilde{i} : \tilde{\mathcal{Z}} \to \tilde{Y}$ is truncative then $\mathcal{Z}$ is truncative in Y.*

Proof. By Claim 2 of my Section 1.11.6, it suffices to show that $(i \circ f')^!$ admits a right adjoint as a functor given by kernel. We have $\tilde{i}^! f^! \widetilde{\to} (i \circ f')^!$. Since f is smooth and schematic, $f^!$ admits a right adjoint given by kernel. Our claim follows. □

The following is immediate from the previous corollary.

Corollary 1.11.8. *Let $i : \mathcal{Z} \to Y$ be a locally closed substack, all the stacks are as in my Section 1.6.2. Suppose that any $z \in \mathcal{Z}$ has a Zariski open neighbourhood $U \subset Y$ such that $\mathcal{Z} \cap U \hookrightarrow U$ is truncative. Then $\mathcal{Z}$ is truncative in Y.* □

1.11.9. (analog of [6], 3.6.10). Let Y be as in my Section 1.6.2. Let $f : \tilde{Y} \to Y$ be a locally closed embedding. If a locally closed substack $i : \mathcal{Z} \to Y$ is truncative then the same holds for $\tilde{i} : \mathcal{Z} \times_Y \tilde{Y} \to \tilde{Y}$.

Proof. Any locally closed embedding writes as a composition of a closed embedding followed by an open embedding. If f is an open embedding, our claim follows from Claim 1 in my Section 1.11.6.

So, we may and do assume f is a closed embedding. Write $\tilde{f} : \mathcal{Z} \times_Y \tilde{Y} \to \mathcal{Z}$ for the projection. By my Section 1.8.23, it suffices to show that $\tilde{f}_! \tilde{i}^!$ admits a right adjoint as a functor given by kernel. We have $\tilde{f}_* \tilde{i}^! \widetilde{\to} f_* i^!$, and the functor f_* admits a right adjoint as a functor given by kernel by my Section 1.7.2. We are done. □

1.11.10. For ([2], C.1.2). Assume $\mathcal{Z}_1, \mathcal{Z}_2 \subset Y$ are locally closed truncative substacks of Y, where all the stacks are as in my Section 1.6.2. Then the same holds for $\mathcal{Z}_1 \cap \mathcal{Z}_2$.

Proof. By my Section 1.11.9, $\mathcal{Z}_1 \cap \mathcal{Z}_2 \subset \mathcal{Z}_1$ is truncative. Since $\mathcal{Z}_1 \subset Y$ is truncative, the composition of these embeddings is also truncative. □

Let $U_i \subset Y$ be the complement to $\mathcal{Z}_i$. The above claim says that if both U_i are cotruncative in Y then $U_1 \cup U_2$ is also cotruncative.

1.11.11. For ([2], C.1.5). Let $j : U \hookrightarrow Y$ be a quasi-compact open substack. Then indeed the property that $j_!$ sends $Shv_{\mathcal{N}}(U)$ to $Shv_{\mathcal{N}}(Y)$ is equivalent to the fact that j_* sends $Shv_{\mathcal{N}}(U)$ to $Shv_{\mathcal{N}}(Y)$ by ([1], F.8.7).

1.11.12. For ([2], C.2.2). Recall we are in the constructible context. Assume Y is an algebraic stack locally of finite type. Then by definition, $Shv(Y)_{co} \widetilde{\to} \operatorname{colim}_* Shv(\mathcal{U})$, where the colimit is taken over the poset of quasi-compact open substacks $\mathcal{U} \subset Y$.

For each such $j : \mathcal{U} \to Y$, the functor $j_{*,co}$ is fully faithful by ([16], Remark in 9.2.7). Indeed, for each inclusion $j_{12} : \mathcal{U}_1 \subset \mathcal{U}_2$ of quasi-compact opens, $(j_{12})_*$ is fully faithful and has a continuous right adjoint $((j_{12})_!)^\vee$.

Besides, $Shv(\mathcal{U})_{co}$ is compactly generated by ([6], 1.9.4). For each quasi-compact open substack $j : \mathcal{U} \to Y$ and $F \in Shv(\mathcal{U})^c$, $j_{*,co}(F) \in Shv(\mathcal{U})^c_{co}$, and such objects generate $Shv(Y)_{co}$. Moreover, by ([6], 1.9.5) each compact object of $Shv(Y)_{co}$ is of the form $j_{*,co}(F)$ for $j : \mathcal{U} \to Y$ quasi-compact open and $F \in Shv(\mathcal{U})^c$.

1.11.13. For ([2], C.2.3). Let Y be an algebraic stack locally of finite type. For any quasi-compact open $j : \mathcal{U} \hookrightarrow Y$ we let $Shv(Y)$ act on $Shv(\mathcal{U})$ so that $F \in Shv(Y)$ sends $K \in Shv(\mathcal{U})$ to $K \otimes^! j^! F$. If $j_{12} : \mathcal{U}_1 \hookrightarrow \mathcal{U}_2$ is an open immersion of quasi-compact substacks then $((j_{12})_* K) \otimes^! j_2^! F \widetilde{\to} (j_{12})_*(K \otimes^! j_1^! F)$ canonically for $K \in Shv(\mathcal{U}_1)$, $F \in Shv(Y)$ and $j_i : \mathcal{U}_i \to Y$. So, the colimit $\operatorname{colim}_* Shv(\mathcal{U}) \widetilde{\to} Shv(Y)_{co}$ may be understood in the category $Shv(Y)-mod(\mathrm{DGCat}_{cont})$, because $\mathrm{oblv} : Shv(Y)-mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves colimits (by [16], 3.0.53).

With this definition for a quasi)-compact open $j : U \subset Y$ we have

$$j_{*,co}(F) \otimes^! K \widetilde{\to} j_{*,co}(F \otimes^! j^! K)$$

for $F \in Shv(U), K \in Shv(Y)$.

By ([16], 9.2.47), the functor $\mathrm{id}_Y^{naive} : Shv(Y)_{co} \to Shv(Y)$ is fully faithful.

Consider full subcategory $C \subset Shv(Y)$ consisting of objects of the form $j_*(K)$ for $j : U \hookrightarrow Y$ a quasi-compact open substack and $K \in Shv(U)^c$. By the above, C is precisely the image of $Shv(Y)^c_{co}$ under $\mathrm{id}_Y^{naive} : Shv(Y)_{co} \hookrightarrow Shv(Y)$. Since the canonical map $\mathrm{Ind}(C) \to Shv(Y)_{co}$ is an equivalence, we conclude that the ind-extension of the inclusion $C \subset Shv(Y)$ defines a fully faithful functor $\mathrm{Ind}(C) \to Shv(Y)$, which factors as $\mathrm{Ind}(C) \widetilde{\to} Shv(Y)_{co} \stackrel{\mathrm{id}_Y^{naive}}{\to} Shv(Y)$.

1.11.14. For ([2], C.2.4). Let Y be an algebraic stack locally of finite type and $\mathcal{N}$-truncatable. Then for each $\mathcal{N}$-cotruncative quasi-compact open $j : U \to Y$, the functor $j_{*,co} : Shv_{\mathcal{N}}(U) \to Shv_{\mathcal{N}}(Y)_{co}$ is fully faithful? Consider a pair of $\mathcal{N}$-cotruncative quasi-compact opens $j_{12} : U_1 \hookrightarrow U_2 \subset Y$.

Question: the functor $(j_{12})_* : Shv_{\mathcal{N}}(U_1) \to Shv_{\mathcal{N}}(U_2)$ has a continiuous right adjoint ??? This would imply that $j_{*,co} : Shv_{\mathcal{N}}(U) \to Shv_{\mathcal{N}}(Y)_{co}$ is fully faithful.

Let us show that $Shv_{\mathcal{N}}(Y)_{co} \to Shv(Y)_{co}$ is fully faithful. For each pair of $\mathcal{N}$-cotruncative opens $j_{12} : U_1 \hookrightarrow U_2 \subset Y$ the functor $(j_{12})_* : Shv_{\mathcal{N}}(U_1) \to Shv_{\mathcal{N}}(U_2)$ has a maybe discontinuous right adjoint $h_{21} : Shv_{\mathcal{N}}(U_2) \to Shv_{\mathcal{N}}(U_1)$, so

$$Shv_{\mathcal{N}}(Y)_{co} \widetilde{\to} \lim_{U} Shv_{\mathcal{N}}(U),$$

where the transition functors are h_{21}, and the limit is taken in DGCat. Similarly, $Shv(Y)_{co} \widetilde{\to} \lim_U Shv(U)$ taken in DGCat_{cont}, hence also in DGCat. Now for each $\mathcal{N}$-cotruncative quasi-compact $U \subset Y$, $Shv_{\mathcal{N}}(U) \to Shv(U)$ is fully faithful. Passing to the limit, we see that the desired functor is also fully faithful.

1.11.15. For ([2], C.3.3-C.3.4). They do not assume Y truncative. Their formula for the pairing $Shv(Y) \otimes Shv(Y)_{co} \to \mathrm{Vect}$ is correct. Let us explain how it implies the desired formula for the induced equivalence $(Shv(Y)^c)^{op} \widetilde{\to} (Shv(Y)_{co})^c$. Let $j : U \subset Y$ be an open immersion with U quasi-compact. We check that for $\mathcal{F}_U \in Shv(U)^c, K \in Shv(Y)_{co}$ one has canonically

$$\mathcal{H}om_{Shv(Y)_{co}}(j_{*,co}(\mathbb{D}\mathcal{F}_U), K) \widetilde{\to} C_{\blacktriangle}(Y, (j_! \mathcal{F}_U) \otimes^! K) \tag{28}$$

Here in the RHS by $C_{\blacktriangle}(Y, \cdot)$ we mean the corresponding functor $Shv(Y)_{co} \to \mathrm{Vect}$. Both parts being continuous functors in $K \in Shv(Y)_{co}$, we may assume K compact.

So, we pick an open quasi-compact substack $j': V \to Y$ and $L \in Shv(V)^c$ so that $K = j'_{*,co}(L)$. Let $U \overset{j_U}{\leftarrow} U \cap V \overset{j_V}{\hookrightarrow} V$ be the open immersions. The RHS of (28) becomes

$$C_{\blacktriangle}(V, L \otimes^! (j_V)_! j_U^* \mathcal{F}_U) \widetilde{\to} \mathcal{H}om_{Shv(V)}((j_V)_* j_U^* \mathbb{D}\mathcal{F}_U, L)$$

In view of my Section 1.11.13, it remains to obtain an isomorphism

$$\mathcal{H}om_{Shv(Y)}(j_*(\mathbb{D}\mathcal{F}_U), j'_* L) \widetilde{\to} \mathcal{H}om_{Shv(V)}((j_V)_* j_U^* \mathbb{D}\mathcal{F}_U, L)$$

The latter is clear.

1.11.16. Let $j : U_1 \hookrightarrow U_2$ be an open immersion of quasi-compact algebraic stacks. Let $\mathcal{N} \subset T^*(U_2)$ be a closed conical subset, its restriction to U_1 is still denoted by $\mathcal{N}$ by abuse of notations. Assume $(U_i, \mathcal{N})$ is duality-adapted in the sense of ([2], A.5.4) for $i = 1, 2$. So, the usual Verdier duality yields equivalences $Shv_{\mathcal{N}}(U_i)^{\vee} \widetilde{\to} Shv_{\mathcal{N}}(U_i)$. Assume $j_!$ (equivalently, j_*) sends $Shv_{\mathcal{N}}(U_1)$ to $Shv_{\mathcal{N}}(U_2)$. Then the dual of the functor $j^! : Shv_{\mathcal{N}}(U_2) \to Shv_{\mathcal{N}}(U_1)$ is $j_* : Shv_{\mathcal{N}}(U_1) \to Shv_{\mathcal{N}}(U_2)$.

In this case the adjoint pair $j_! : Shv_{\mathcal{N}}(U_1) \leftrightarrows Shv_{\mathcal{N}}(U_2) : j^!$ gives by passing to the duals an adjoint pair

$$j_* : Shv_{\mathcal{N}}(U_1) \leftrightarrows Shv_{\mathcal{N}}(U_2) : (j_!)^{\vee}$$

In particular, j_* has a continuous right adjoint.

Application: assume Y is an algebraic stack locally of finite type, $\mathcal{N} \subset T^*(Y)$ is a closed conical subset, and Y is $\mathcal{N}$-truncatable in the sense of ([1], F.8.6). Pick a filtered collection of quasi-compact opens $j_i : U_i \to Y$ such that $(j_i)_!$ sends $Shv_{\mathcal{N}}(U_i)$ to $Shv_{\mathcal{N}}(Y)$. Assume each $(U_i, \mathcal{N})$ is duality-adapted in the sense of ([2], A.5.4). Then we get $Shv_{\mathcal{N}}(Y)_{co} \widetilde{\to} \operatorname{colim}_i^* Shv_{\mathcal{N}}(U_i)$, where the transition functors are the $*$-direct images. Since for each pair i_1, i_2 and an open immersion $j_{i_1,i_2} : U_{i_1} \hookrightarrow U_{i_2}$ the functor $(j_{i_1,i_2})_* : Shv_{\mathcal{N}}(U_{i_1}) \to Shv_{\mathcal{N}}(U_{i_2})$ has a continuous right adjoint, by ([16], Remark in 9.2.7), each $(j_i)_{*,co} : Shv_{\mathcal{N}}(U_i) \to Shv_{\mathcal{N}}(Y)_{co}$ is fully faithful.

1.11.17. Let Y, Y' be algebraic stacks locally of finite type (whose all quasi-compact open substacks are as in my Section 1.6.2). Let $f : Y' \to Y$ be a morphism of finite type. Then we have natural functors that I denote $f^!_{co} : Shv(Y)_{co} \to Shv(Y')_{co}$ and $f_{\blacktriangle,co} : Shv(Y')_{co} \to Shv(Y)_{co}$. In fact, $f_{\blacktriangle,co}$ is defined even if f is not of finite type. Namely, for a cofinal diagram of quasi-compact opens $U_i \subset Y$, $i \in I$ with I filtered small, let $U'_i = f^{-1}(U_i)$. Then U'_i is quasi-compact open in Y', and $\cup_i U'_i = Y'$. We have a morphism of diagrams

$$i \mapsto (Shv(U_i) \overset{(f_i)^!}{\to} Shv(U'_i))$$

in DGCat_{cont}, where $f_i : U'_i \to U_i$ is the restriction of f. Here for $\alpha : U_i \hookrightarrow U_j$ the transition functors are the $*$-direct images with respect to α and α' respectively, where $\alpha' : U'_i \to U'_j$ is obtained by base change. Passing to $\operatorname{colim}_{i \in I}$, we get the functor $f^!_{co}$. In fact, $f^!_{co}$ is naturally a morphism of $Shv(Y)$-modules.

Assume first f of finite type. We also have a morphism of diagrams

$$i \mapsto (Shv(U'_i) \overset{(f_i)_{\blacktriangle}}{\to} Shv(U_i))$$

with the same transition functors. Passing to the colimit over I, we get the functor $f_{\blacktriangle,co} : Shv(Y')_{co} \to Shv(Y)_{co}$. Let now f be arbitary. Then we first define the functors $Shv(U'_i) \stackrel{(f_i)_\blacktriangle}{\to} Shv(U_i)$ as follows. They come from a compatible system of functors $(f_i^V)_\blacktriangle : Shv(V) \to Shv(U_i)$ for quasi-compact opens $V \subset U'_i$, where f_i^V is the composition $V \hookrightarrow U'_i \stackrel{f_i}{\to} U_i$.

Note that if $f : Y' \to Y$ is an open immersion of a quasi-compact open then $f_{\blacktriangle,co} : Shv(Y') \to Shv(Y)$ has the same meaning as $f_{*,co}$ in ([2], C.2.2).

If $j : Y' \to Y$ is an open immersion we also write $j^*_{co} = j^!_{co}$. Moreover, in this case we get an adjoint pair $j^*_{co} : Shv(Y)_{co} \leftrightarrows Shv(Y')_{co} : j_{\blacktriangle,co}$ by ([16], 9.2.39). Indeed, it is obatined by passing to the colimit over $i \in I$ in the adjoint pair $j_i^* : Shv(U_i) \leftrightarrows Shv(U'_i) : (j_i)_*$. Here $j_i : U'_i \to U_i$ is an open immersion.

We can get more formalism for the co-category by applying the Verider duality to the usual formalism and using their equivalence $Shv(Y)^\vee \widetilde{\to} Shv(Y)_{co}$ from ([2], C.3.3).

Applying ([2], A.4.4) we get the following. The Verdier dual of the above functor $f_{\blacktriangle,co} : Shv(Y')_{co} \to Shv(Y)_{co}$ is $f^! : Shv(Y) \to Shv(Y')$ even if f is not of finite type. Assume f of finite type. Then the Verdier dual of

$$f^!_{co} : Shv(Y)_{co} \to Shv(Y')_{co}$$

is $f_\blacktriangle : Shv(Y') \to Shv(Y)$. Here $f_\blacktriangle$ is obtained by passing to the limit over i (with the transition functors given by the usual restriction) in the diagram $(f_i)_\blacktriangle : Shv(U'_i) \to Shv(U_i)$.

Remark The functor $\mathrm{id}_Y^{naive} : Shv(Y)_{co} \to Shv(Y)$ is Verdier self-dual.

Proof. We have to establish for $F, F' \in Shv(Y)_{co}$ an isomorphism

$$C_\blacktriangle(Y, \mathrm{id}_Y^{naive}(F') \otimes^! F) \widetilde{\to} C_\blacktriangle(Y, \mathrm{id}_Y^{naive}(F) \otimes^! F')$$

Let $j : U \hookrightarrow Y$ and $j' : U' \hookrightarrow Y$ be two quasi-compact opens, $K \in Shv(U), K' \in Shv(U')$. It suffices to establish the above isomorphism for $F = j_{*,co}(K)$, $F' = j'_{*,co}(K')$ in a way compatible with the transition functors in the corresponding direct diagram. The desired isomorphism becomes

$$C_\blacktriangle(Y, j'_*(K') \otimes^! j_{*,co}(K)) \widetilde{\to} C_\blacktriangle(U, K \otimes^! j^! j'_* K') \widetilde{\to}$$
$$C_\blacktriangle(U', K' \otimes^! (j')^! j_* K) \widetilde{\to} C_\blacktriangle(Y, j_*(K) \otimes^! j_{*,co}(K'))$$

The isomorphism in the middle comes from the fact that both sides identify with $C_\blacktriangle(U \cap U', \bar{j}^! K' \otimes^! (\bar{j}')^! K)$. Here $U \stackrel{\bar{j}'}{\leftarrow} U \cap U' \stackrel{\bar{j}}{\to} U'$ are the corresponding open immersions. □

Note also that id_Y^{naive} is the limit over the quasi-compact opens $j : U \subset Y$ of the restriction functors $j^*_{co} : Shv(Y)_{co} \to Shv(U)$, as $\lim^*_{U \subset Y} Shv(U) \widetilde{\to} Shv(Y)$.

For any $f : Y' \to Y$ as above (f may be of infinite type) the functor $f_{\blacktriangle,co} : Shv(Y')_{co} \to Shv(Y)_{co}$ satisfies **the projection formula**: for $F \in Shv(Y')_{co}$, $K \in Shv(Y)$ one has canonically

$$f_{\blacktriangle,co}(F) \otimes^! K \widetilde{\to} f_{\blacktriangle,co}(F \otimes^! f^! K)$$

Proof. Since $Shv(Y')_{co}$ is compactly generated and both sides are continuous in F, we may assume F compact. Then there is a quasi-compact open $j' : U' \subset Y'$ and $F' \in Shv(U')^c$ such that $F \widetilde{\to} j'_{*,co}(F')$. Pick a quasi-compact open $j : U \subset Y$ such that the map $f \circ j'$ factors as $U' \stackrel{\bar{f}}{\to} U \stackrel{j}{\hookrightarrow} Y$. By definition, $f_{\blacktriangle,co}(F) \widetilde{\to} j_{*,co}\bar{f}_{\blacktriangle}(F)$. The claim follows from the projection formula for $\bar{f}_{\blacktriangle}$. □

Assume $f : Y' \to Y$ of finite type in addition. Then f satisfies **the second projection formula**: for $M \in Shv(Y'), L \in Shv(Y)_{co}$ one has canonically

$$f_{\blacktriangle}(M) \otimes^! L \widetilde{\to} f_{\blacktriangle,co}(M \otimes^! f^!_{co}(L))$$

Proof. Both sides are continuous in L, and $Shv(Y)_{co}$ is compactly generated, so we may assume L compact. So, there is a quasi-compact open substack $j : U \hookrightarrow Y$ and $L_U \in Shv(U)^c$ such that $L = j_{*,co}(L_U)$. We get

$$f_{\blacktriangle}(M) \otimes^! L \widetilde{\to} j_{*,co}(L_U \otimes^! j^! f_{\blacktriangle}(M)) \widetilde{\to} j_{*,co}(L_U \otimes^! (f_U)_{\blacktriangle}(j'^! M)),$$

where $j' : U' \hookrightarrow Y'$ is obtained by base change, and $f_U : U' \to U$ is the restriction of f. On the other hand, $f^!_{co}(L) \widetilde{\to} j'_{*,co}(f^!_U L_U)$. So,

$$M \otimes^! f^!_{co}(L) \widetilde{\to} j'_{*,co}((f^!_U L_U) \otimes^! (j')^! M)$$

Now $f_{\blacktriangle,co}(M \otimes^! f^!_{co}(L)) \widetilde{\to} j_{*,co}(f_U)_{\blacktriangle}((f^!_U L_U) \otimes^! (j')^! M)$. So, the claim follows from the usual projection formula:

$$(f_U)_{\blacktriangle}((f^!_U L_U) \otimes^! (j')^! M) \widetilde{\to} L_U \otimes^! (f_U)_{\blacktriangle}(j')^! M$$

□

1.11.18. Consider a cartesian square

$$\begin{array}{ccc} Y' & \stackrel{f_Y}{\to} & Y \\ \uparrow g' & & \uparrow g \\ Z' & \stackrel{f_Z}{\to} & Z \end{array}$$

of algebraic stacks locally of finite type. Assume g of finite type. Then one has canonically $g^!_{co}(f_Y)_{\blacktriangle,co} \widetilde{\to} (f_Z)_{\blacktriangle,co}(g')^!_{co}$. This is obtained from the usual $(-_{\blacktriangle}, -^!)$- base change by Verdier by dualization.

1.11.19. Let $f : Y' \to Y$ be a morphism of algebraic stacks as in my Section 1.11.17, assume f of finite type. Then the diagram commutes canonically

$$\begin{array}{ccc} Shv(Y) & \stackrel{f^!}{\to} & Shv(Y') \\ \uparrow \mathrm{id}^{naive}_Y & & \uparrow \mathrm{id}^{naive}_{Y'} \\ Shv(Y)_{co} & \stackrel{f^!_{co}}{\to} & Shv(Y')_{co} \end{array}$$

In addition, the diagram commutes canonically

$$\begin{array}{ccc} Shv(Y') & \stackrel{f_{\blacktriangle}}{\to} & Shv(Y) \\ \uparrow \mathrm{id}^{naive}_Y & & \uparrow \mathrm{id}^{naive}_{Y'} \\ Shv(Y')_{co} & \stackrel{f_{\blacktriangle,co}}{\to} & Shv(Y)_{co} \end{array}$$

1.11.20. For ([2], C.4.2). If $\mathcal{Z}, Y$ are algebraic stacks, the functor $\mathrm{id}_{\mathcal{Z}}\boxtimes\mathrm{id}_Y^{naive}: Shv(\mathcal{Z}\times Y)_{co}\to Shv(\mathcal{Z}\times Y)_{co_{\mathcal{Z}}}$ is defined as follows. For each quasi-compact open $j: U_Y\subset Y$ we have the restriction $(\mathrm{id}\times j)^*_{co}: Shv(\mathcal{Z}\times Y)_{co}\to Shv(\mathcal{Z}\times U_Y)_{co}$. For a pair of quasi-compact opens $j_U: U'_Y\subset U_Y$ we have $(\mathrm{id}\times j_U)^*_{co}(\mathrm{id}\times j)^*_{co}\widetilde{\to}(\mathrm{id}\times j')_{co}$, where $j'=j\circ j_U$. This gives the desired functor

$$Shv(\mathcal{Z}\times Y)_{co}\to\lim_{U_Y\subset Y}{}^{*}\, Shv(\mathcal{Z}\times U_Y)_{co}$$

The functor

$$\mathrm{id}_{\mathcal{Z}}^{naive}\boxtimes\mathrm{id}_Y: Shv(\mathcal{Z}\times Y)_{co_{\mathcal{Z}}}\to Shv(\mathcal{Z}\times Y)$$

is defined as the limit over the quasi-compact opens $U_Y\subset Y$ of the functors $\mathrm{id}^{naive}_{\mathcal{Z}\times U_Y}: Shv(\mathcal{Z}\times U_Y)_{co}\to Shv(\mathcal{Z}\times U_Y)$. Since this is a limit of fully fauthful functors, $\mathrm{id}_{\mathcal{Z}}^{naive}\boxtimes\mathrm{id}_Y$ is fully faithful.

Since $\mathrm{id}^{naive}_{\mathcal{Z}\times Y}: Shv(\mathcal{Z}\times Y)_{co}\to Shv(\mathcal{Z}\times Y)$ is fully faithful, we see that $\mathrm{id}_{\mathcal{Z}}\boxtimes\mathrm{id}_Y^{naive}$ is also fully faithful.

1.11.21. For ([2], C.4.3). Assume Y truncatable. Let $j_Y: U_{1,Y}\to U_{2,Y}$ be a cotruncative open immersion of quasi-compact open substacks of Y. Then we have an adjoint pair $(\mathrm{id}\times j_Y)_{\blacktriangle}: Shv(\mathcal{Z}\times U_{1,Y})\leftrightarrows Shv(\mathcal{Z}\times U_{2,Y}): (\mathrm{id}\times j_Y)^?$ in DGCat_{cont}. Dualizing, it gives an adjoint pair

$$((\mathrm{id}\times j_Y)^?)^\vee: Shv(\mathcal{Z}\times U_{1,Y})_{co}\leftrightarrows Shv(\mathcal{Z}\times U_{2,Y})_{co}: (\mathrm{id}\times j_Y)^*_{co} \tag{29}$$

So, we may pass to left adjoints in the diagram $\lim^*_{U_Y\subset Y} Shv(\mathcal{Z}\times U_Y)_{co}\widetilde{\to} Shv(\mathcal{Z}\times Y)_{co_{\mathcal{Z}}}$ and get

$$Shv(\mathcal{Z}\times Y)_{co_{\mathcal{Z}}}\widetilde{\to}\operatorname*{colim}_{U_Y\subset Y} Shv(\mathcal{Z}\times U_Y)_{co} \tag{30}$$

Let $U_{1,\mathcal{Z}}\overset{j_{\mathcal{Z}}}{\hookrightarrow} U_{2,\mathcal{Z}}\subset\mathcal{Z}$ be quasi-compact opens. Let us explain that their displayed diagram

$$\begin{array}{ccc} Shv(U_{1,Y}\times U_{1,\mathcal{Z}}) & \overset{(j_Y\times\mathrm{id})_!}{\to} & Shv(U_{2,Y}\times U_{1,\mathcal{Z}})\\ \downarrow (\mathrm{id}\times j_{\mathcal{Z}})_* & & \downarrow (\mathrm{id}\times j_{\mathcal{Z}})_*\\ Shv(U_{1,Y}\times U_{2,\mathcal{Z}}) & \overset{(j_Y\times\mathrm{id})_!}{\to} & Shv(U_{2,Y}\times U_{2,\mathcal{Z}})\end{array} \tag{31}$$

commutes. After Verdier dualization, it is enough to show that the following diagram commutes

$$\begin{array}{ccc} Shv(U_{1,Y}\times U_{1,\mathcal{Z}}) & \overset{(j_Y\times\mathrm{id})^?}{\leftarrow} & Shv(U_{2,Y}\times U_{1,\mathcal{Z}})\\ \uparrow (\mathrm{id}\times j_{\mathcal{Z}})^! & & \uparrow (\mathrm{id}\times j_{\mathcal{Z}})^!\\ Shv(U_{1,Y}\times U_{2,\mathcal{Z}}) & \overset{(j_Y\times\mathrm{id})^?}{\leftarrow} & Shv(U_{2,Y}\times U_{2,\mathcal{Z}})\end{array}$$

This is true, because $(j_Y\times\mathrm{id})^?$ is a functor given by kernel, this is precisely the 2nd compatibility property in ([2], B.1.5).

We may pass to the colimit (with respect to the $*$-direct images) over $U_{\mathcal{Z}}$ in the diagram $(j_Y\times\mathrm{id})_!: Shv(U_{1,Y}\times U_{\mathcal{Z}})\leftrightarrows Shv(U_{2,Y}\times U_{\mathcal{Z}}): (j_Y\times\mathrm{id})^*$. By ([16], 9.2.39) this gives precisely the adjoint pair (29). So, the RHS of (30) rewrites as

$$\operatorname*{colim_!}_{U_Y\subset Y}\ \operatorname*{colim_*}_{U_{\mathcal{Z}}\subset\mathcal{Z}} Shv(U_{\mathcal{Z}}\times U_Y)$$

It would be reasonable to denote the left adjoint in (29) by $(\mathrm{id}\times j_Y)_{!,co}$.

Remark Recall that Y is assumed truncatable. We have an action of $Shv(\mathcal{Z}\times Y)$ on $Shv(\mathcal{Z}\times Y)_{co_{\mathcal{Z}}}$. Namely, we let $F\in Shv(\mathcal{Z}\times Y)$ act on $K\in Shv(U_{\mathcal{Z}}\times Y)$ as $K\otimes^! (j\times\mathrm{id})^! F$ for $j: U_{\mathcal{Z}}\hookrightarrow \mathcal{Z}$. This is compatible with the $*$-transition functors in $\underset{U_{\mathcal{Z}}\subset\mathcal{Z}}{\mathrm{colim}_*}\, Shv(U_{\mathcal{Z}}\times Y)$.

1.11.22. ([2], C.4.3) more. Assume Y truncatable. Then the Verdier duality gives a canonical equivalence

$$(Shv(\mathcal{Z}\times Y)_{co_{\mathcal{Z}}})^{\vee}\,\widetilde{\to}\, Shv(\mathcal{Z}\times Y)_{co_Y}$$

Proof. Apply ([11], ch. I.1, 6.3.4) to the diagram $Shv(\mathcal{Z}\times Y)_{co_{\mathcal{Z}}}\,\widetilde{\to}\,\underset{U_{\mathcal{Z}}\subset\mathcal{Z}}{\mathrm{colim}_*}\, Shv(U_{\mathcal{Z}}\times Y)$, where $U_{\mathcal{Z}}$ runs through the quasi–compact open substacks of $\mathcal{Z}$. For a pair of such opens $j: U'_{\mathcal{Z}}\hookrightarrow U_{\mathcal{Z}}$ the functor $(j\times\mathrm{id})_*: Shv(U'_{\mathcal{Z}}\times Y)\to Shv(U_{\mathcal{Z}}\times Y)$ admits a continuous right adjoint. The Verdier dual of $(j\times\mathrm{id})_*$ is the functor $(j\times\mathrm{id})^!_{co}: Shv(U_{\mathcal{Z}}\times Y)_{co}\to Shv(U'_{\mathcal{Z}}\times Y)_{co}$, and $Shv(\mathcal{Z}\times Y)_{co_Y}\,\widetilde{\to}\,\underset{U_{\mathcal{Z}}\subset\mathcal{Z}}{\lim{}^*} Shv(U_{\mathcal{Z}}\times Y)_{co}$. □

1.11.23. For ([2], C.4.4). Let Y_1, Y_2 be a pair of truncatable algebraic stacks, $\mathcal{Z}$ an algebraic stack locally of finite type. By definition *functors defined by kernel* are the functors in their Section C.4.4.

1) Let $Q\in Shv(Y_1\times Y_2)$. For a pair of quasi-compact opens $U_{\mathcal{Z}}\subset\mathcal{Z}, U_1\subset Y_1$ consider the diagram

$$\begin{array}{ccccc} U_{\mathcal{Z}}\times U_1 & \stackrel{p_1}{\leftarrow} & U_{\mathcal{Z}}\times U_1\times Y_2 & \stackrel{p_2}{\to} & U_{\mathcal{Z}}\times Y_2\\ && \downarrow p && \\ && Y_1\times Y_2 && \end{array}$$

We get a functor $Shv(U_{\mathcal{Z}}\times U_1)\to Shv(U_{\mathcal{Z}}\times Y_2)$, $K\mapsto (p_2)_{\blacktriangle}(p_1^!K\otimes^! p^!Q)$. If now $U_{\mathcal{Z}}\subset U'_{\mathcal{Z}}, U_1\subset U'_1$ are open immersions of quasi-compact opens, then the above functor is compatible with the $*$-direct images with respect to $U_{\mathcal{Z}}\times U_1\hookrightarrow U'_{\mathcal{Z}}\times U'_1$ on the source, and with respect to $U_{\mathcal{Z}}\times Y_2\hookrightarrow U'_{\mathcal{Z}}\times Y_2$ on the target. Passing to the colimit, we get a functor

$$Shv(\mathcal{Z}\times Y_1)_{co}\,\widetilde{\to}\,\underset{U_{\mathcal{Z}},U_1}{\mathrm{colim}_*}\, Shv(U_{\mathcal{Z}}\times U_1)\to \underset{U_{\mathcal{Z}}}{\mathrm{colim}_*}\, Shv(U_{\mathcal{Z}}\times Y_2)\,\widetilde{\to}\, Shv(\mathcal{Z}\times Y_2)_{co_{\mathcal{Z}}}$$

denoted $\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q}$.

If we pass to the colimit only over U_1 with the transition functors being $*$-direct images, one gets the functor

$$f_{\mathcal{Z}}: Shv(U_{\mathcal{Z}}\times Y_1)_{co}\to Shv(U_{\mathcal{Z}}\times Y_2) \tag{32}$$

These functors are compatible with the $*$-restrictions along the open immersions $U_{\mathcal{Z}}\subset U'_{\mathcal{Z}}$. So, passing to the limit over $U_{\mathcal{Z}}$, we get the functor

$$Shv(\mathcal{Z}\times Y_1)_{co_{Y_1}}\,\widetilde{\to}\,\underset{U_{\mathcal{Z}}\subset\mathcal{Z}}{\lim{}^*}\, Shv(U_{\mathcal{Z}}\times Y_1)_{co}\to \underset{U_{\mathcal{Z}}\subset\mathcal{Z}}{\lim{}^*}\, Shv(U_{\mathcal{Z}}\times Y_2)\,\widetilde{\to}\, Shv(\mathcal{Z}\times Y_2)$$

also denoted $\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q}$. (The latter functor makes sense even if Y_1, Y_2 are not assumed truncatable).

2) Let $Q \in Shv(Y_1 \times Y_2)_{co}$. Let $U_{\mathcal{Z}} \subset \mathcal{Z}$ be a quasi-compact open. First, we define a functor $g_{U_{\mathcal{Z}}} : Shv(U_{\mathcal{Z}} \times Y_1) \to Shv(U_{\mathcal{Z}} \times Y_2)_{co}$ as follows. Consider the diagram

$$\begin{array}{ccccc} U_{\mathcal{Z}} \times Y_1 & \stackrel{p_1}{\leftarrow} & U_{\mathcal{Z}} \times Y_1 \times Y_2 & \stackrel{p_2}{\to} & U_{\mathcal{Z}} \times Y_2 \\ & & \downarrow p & & \\ & & Y_1 \times Y_2 & & \end{array}$$

We have the functors $Shv(Y_1 \times Y_2)_{co} \stackrel{p^!_{co}}{\to} Shv(U_{\mathcal{Z}} \times Y_1 \times Y_2)_{co} \stackrel{(p_2)_{\blacktriangle,co}}{\to} Shv(U_{\mathcal{Z}} \times Y_2)_{co}$ defined in my Section 1.11.17. Set

$$g_{U_{\mathcal{Z}}}(K) = (p_2)_{\blacktriangle,co}(p_1^!(K) \otimes^! p^!_{co}(Q))$$

Here we used the action of $Shv(U_{\mathcal{Z}} \times Y_1 \times Y_2)$ on $Shv(U_{\mathcal{Z}} \times Y_1 \times Y_2)_{co}$ discussed in my Section 1.11.13.

If $j : U'_{\mathcal{Z}} \hookrightarrow U_{\mathcal{Z}}$ is an open substack, the diagram commutes

$$\begin{array}{ccc} Shv(U_{\mathcal{Z}} \times Y_1) & \stackrel{g_{U_{\mathcal{Z}}}}{\to} & Shv(U_{\mathcal{Z}} \times Y_2)_{co} \\ \downarrow (j\times \mathrm{id})^! & & \downarrow (j\times \mathrm{id})^!_{co} \\ Shv(U'_{\mathcal{Z}} \times Y_1) & \stackrel{g_{U'_{\mathcal{Z}}}}{\to} & Shv(U_{\mathcal{Z}} \times Y_2)_{co} \end{array}$$

Passing to the limit over quasi-compact opens $U_{\mathcal{Z}} \subset \mathcal{Z}$, the functors $g_{U_{\mathcal{Z}}}$ give the functor

$$Shv(\mathcal{Z} \times Y_1) \widetilde{\to} \lim_{U_{\mathcal{Z}} \subset \mathcal{Z}}{}^! \, Shv(U_{\mathcal{Z}} \times Y_1) \to \lim_{U_{\mathcal{Z}} \subset \mathcal{Z}}{}^! \, Shv(U_{\mathcal{Z}} \times Y_2)_{co} \widetilde{\to} Shv(\mathcal{Z} \times Y_2)_{co_{Y_2}}$$

denoted $\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}$.

Besides, if $j : U'_{\mathcal{Z}} \hookrightarrow U_{\mathcal{Z}}$ is an open substack, the diagram commutes

$$\begin{array}{ccc} Shv(U_{\mathcal{Z}} \times Y_1) & \stackrel{g_{U_{\mathcal{Z}}}}{\to} & Shv(U_{\mathcal{Z}} \times Y_2)_{co} \\ \uparrow (j\times \mathrm{id})_{\blacktriangle} & & \uparrow (j\times \mathrm{id})_{\blacktriangle,co} \\ Shv(U'_{\mathcal{Z}} \times Y_1) & \stackrel{g_{U'_{\mathcal{Z}}}}{\to} & Shv(U_{\mathcal{Z}} \times Y_2)_{co} \end{array}$$

Passing to the colimit over $U_{\mathcal{Z}} \subset \mathcal{Z}$, one gets the functor

$$Shv(\mathcal{Z} \times Y_1)_{co_{\mathcal{Z}}} \widetilde{\to} \operatorname*{colim}_{U_{\mathcal{Z}} \subset \mathcal{Z}}{}_* \, Shv(U_{\mathcal{Z}} \times Y_1) \to \operatorname*{colim}_{U_{\mathcal{Z}} \subset \mathcal{Z}}{}_* \, Shv(U_{\mathcal{Z}} \times Y_2)_{co} \widetilde{\to} Shv(\mathcal{Z} \times Y_2)_{co}$$

also denoted $\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}$.

3) Let $Q \in Shv(Y_1 \times Y_2)_{co_{Y_1}} \widetilde{\to} \lim_{U_2 \subset Y_2}{}^* \, Shv(Y_1 \times U_2)_{co}$, where U_2 runs through the quasi-compact opens of Y_2. Write Q_{U_2} the image of Q in $Shv(Y_1 \times U_2)_{co}$. For quasi-compact opens $U_{\mathcal{Z}} \stackrel{j_{\mathcal{Z}}}{\hookrightarrow} \mathcal{Z}, U_2 \subset Y_2$ consider the diagram

$$\begin{array}{ccccc} U_{\mathcal{Z}} \times Y_1 & \stackrel{p_1}{\leftarrow} & U_{\mathcal{Z}} \times Y_1 \times U_2 & \stackrel{p_2}{\to} & U_{\mathcal{Z}} \times U_2 \\ & & \downarrow p & & \\ & & Y_1 \times U_2 & & \end{array}$$

Define the functor $h : Shv(U_{\mathcal{Z}} \times Y_1) \to Shv(U_{\mathcal{Z}} \times U_2)$ by

$$h(K) = (p_2)_{\blacktriangle,co}(p_1^!(j_{\mathcal{Z}} \times \mathrm{id})^! K \otimes^! p^!_{co}(Q_{U_2}))$$

Let $j: U'_{\mathcal{Z}}\hookrightarrow U_{\mathcal{Z}}$ and $j_2: U'_2\subset U_2$ be open immersions. Then the diagram canonically commute

$$\begin{array}{ccc} Shv(U_{\mathcal{Z}}\times Y_1) & \stackrel{h}{\to} & Shv(U_{\mathcal{Z}}\times U_2)\\ \downarrow (j\times\mathrm{id})^! & & \downarrow (j\times j_2)^!\\ Shv(U'_{\mathcal{Z}}\times Y_1) & \stackrel{h}{\to} & Shv(U'_{\mathcal{Z}}\times U'_2)\end{array}$$

We get a morphism of diagrams indexed by pairs of quasi-compact opens $U_{\mathcal{Z}}, U_2$. Passing to the limit (and using that the index category is filtered, hence contractible), we get the functor

$$Shv(\mathcal{Z}\times Y_1)\widetilde{\to}\lim_{U_{\mathcal{Z}}\subset\mathcal{Z}}{}^* Shv(U_{\mathcal{Z}}\times Y_1)\stackrel{\lim h}{\to}\lim_{U_{\mathcal{Z}},U_2}{}^* Shv(U_{\mathcal{Z}}\times U_2)\widetilde{\to} Shv(\mathcal{Z}\times Y_2)$$

denoted $\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q}$. This does not uses the fact that Y_1, Y_2 are truncatable.

Let $j: U'_{\mathcal{Z}}\hookrightarrow U_{\mathcal{Z}}$ be an open immersion of quasi-compact opens in $\mathcal{Z}$. Then the diagram commutes

$$\begin{array}{ccc} Shv(U_{\mathcal{Z}}\times Y_1) & \stackrel{h}{\to} & Shv(U_{\mathcal{Z}}\times U_2)\\ \uparrow (j\times\mathrm{id})_* & & \uparrow (j\times\mathrm{id})_*\\ Shv(U'_{\mathcal{Z}}\times Y_1) & \stackrel{h}{\to} & Shv(U'_{\mathcal{Z}}\times U_2)\end{array}$$

Passing to the colimit over $U_{\mathcal{Z}}$ with the transition functors being the $*$-direct images, we get the functor

$$Shv(\mathcal{Z}\times Y_1)_{co_{\mathcal{Z}}}\widetilde{\to}\underset{U_{\mathcal{Z}}\subset\mathcal{Z}}{\mathrm{colim}_*}\, Shv(U_{\mathcal{Z}}\times Y_1)\to\underset{U_{\mathcal{Z}}\subset\mathcal{Z}}{\mathrm{colim}_*}\, Shv(U_{\mathcal{Z}}\times U_2)\widetilde{\to} Shv(\mathcal{Z}\times U_2)_{co},$$

here we used the fact that Y_1 is trancatable. These functors are compatible with the $*, co$-restriction functors along $\mathcal{Z}\times U'_2\subset\mathcal{Z}\times U_2$. Passing to the limit over U_2, we get the functor

$$Shv(\mathcal{Z}\times Y_1)_{co_{\mathcal{Z}}}\to\lim_{U_2\subset Y_2}{}^* Shv(\mathcal{Z}\times U_2)_{co}\widetilde{\to} Shv(\mathcal{Z}\times Y_2)_{co_{\mathcal{Z}}}$$

still denoted $\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q}$.

4) Let $Q\in Shv(Y_1\times Y_2)_{co_{Y_2}}\widetilde{\to}\lim_{U_1}{}^* Shv(U_1\times Y_2)_{co}$. Using the fact that Y_1 is truncatable, we may rewrite

$$Shv(Y_1\times Y_2)_{co_{Y_2}}\widetilde{\to}\underset{U_2\subset Y_2}{\mathrm{colim}_*}\, Shv(Y_1\times U_2)$$

We identify $Shv(Y_1\times Y_2)_{co_{Y_2}}$ with the essential image of the fully faithful functor $\mathrm{id}_{Y_1}\boxtimes\mathrm{id}^{naive}_{Y_2}: Shv(Y_1\times Y_2)_{co_{Y_2}}\hookrightarrow Shv(Y_1\times Y_2)$.

For a quasi-compact open $U_{\mathcal{Z}}\subset\mathcal{Z}$ recall the functor $f_{\mathcal{Z}}$ given by (32). Our assumption that $Q\in Shv(Y_1\times Y_2)_{co_{Y_2}}$ together with my Section 1.11.19 garantees that (32) actually takes values in a full subcategory $Shv(U_{\mathcal{Z}}\times Y_2)_{co}\stackrel{\mathrm{id}^{naive}}{\hookrightarrow} Shv(U_{\mathcal{Z}}\times Y_2)$. The so obtained functors

$$f_{\mathcal{Z}}: Shv(U_{\mathcal{Z}}\times Y_1)_{co}\to Shv(U_{\mathcal{Z}}\times Y_2)_{co}$$

are compatible with the $*$-restrictions along the open immersions $U_{\mathcal{Z}}\subset U'_{\mathcal{Z}}$. So, passing to limit over $U_{\mathcal{Z}}$, we get the functor

$$Shv(\mathcal{Z}\times Y_1)_{co_{Y_1}}\widetilde{\to}\lim_{U_{\mathcal{Z}}\subset\mathcal{Z}}{}^* Shv(U_{\mathcal{Z}}\times Y_1)_{co}\to\lim_{U_{\mathcal{Z}}\subset\mathcal{Z}}{}^* Shv(U_{\mathcal{Z}}\times Y_2)_{co}\widetilde{\to} Shv(\mathcal{Z}\times Y_2)_{co_{Y_2}}$$

denoted $\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q}$.

To get their last assertion here we claim that the functor $\mathrm{id}_{\mathcal{Z}} \boxtimes \mathbf{Q} : Shv(\mathcal{Z} \times Y_1)_{co} \to Shv(\mathcal{Z} \times Y_2)_{co_{\mathcal{Z}}}$ defined in 1) for $Q \in Shv(Y_1 \times Y_2)$ actually takes values in the full subcategory $Shv(Y_1 \times Y_2)_{co} \subset Shv(Y_1 \times Y_2)$ provided that

$$Q \in Shv(Y_1 \times Y_2)_{co_{Y_2}} \subset Shv(Y_1 \times Y_2)$$

The claim follows by passing to the colimit as in 1) over $(U_{\mathcal{Z}}, U_1)$, because

$$\underset{U_{\mathcal{Z}} \subset \mathcal{Z}}{\mathrm{colim}_*}\, Shv(U_{\mathcal{Z}} \times Y_2)_{co} \widetilde{\to} Shv(\mathcal{Z} \times Y_2)_{co}$$

1.11.24. ([2], C.4.4) more. Since Y_1, Y_2 are truncatable, all the DG-categories appearing in their C.4.4 are dualizable. For a functor given by kernel Q (in all the versions appearing in C.4.4) the Verdier dual functor is also given by kernel Q^σ. This gives a good consistency check of the claims of their C.4.4 (using my Section 1.11.22).

By definition, if for example $Q \in Shv(Y_1 \times Y_2)_{co_{Y_1}}$ then $Q^\sigma \in Shv(Y_2 \times Y_1)_{co_{Y_1}}$ and so on.

1.11.25. For ([2], C.4.5). Let $\mathcal{F} \in Shv(Y)$. We show that $(\triangle_Y)_*(\mathcal{F})_{co_1} \in Shv(Y \times Y)_{co_1}$ is well-defined. Let $U' \overset{j_U}{\hookrightarrow} U \overset{j}{\hookrightarrow} Y$ be open immersions, where U, U' are quasi-compact. Let $j' = j \circ j_U$. Let $\nu : U \to Y \times U$ be the graph of j, and similarly for $\nu' : U' \to Y \times U'$. The base change of ν by $\mathrm{id} \times j_U : Y \times U' \to Y \times U$ is the map ν'. Now from my Section 1.11.18 we get

$$(\mathrm{id} \times j_U)^!_{co}\, \nu_{\blacktriangle,co}(j^*\mathcal{F}) \widetilde{\to} \nu'_{\blacktriangle,co}\, (j_U)^!_{co}(j^*\mathcal{F}) \widetilde{\to} \nu'_{\blacktriangle,co}(j'^*\mathcal{F})$$

because $(j_U)^!_{co} = j_U^! = j_U^*$. So, $(\triangle_Y)_*(\mathcal{F})_{co_1}$ is well-defined, similarly for $(\triangle_Y)_*(\mathcal{F})_{co_2}$.

If $\mathcal{F} \in Shv(Y)_{co}$ then by $(\triangle_Y)_*(\mathcal{F}) \in Shv(Y \times Y)_{co}$ they mean what I denoted above by $(\triangle_Y)_{\blacktriangle,co}(\mathcal{F})$.

1.11.26. For ([2], C.4.7). Let Y be an algebraic stack locally of finite type. The miraculous functor $\mathrm{Mir}_Y : Shv(Y)_{co} \to Shv(Y)$ is Verdier self-dual (see my Section 1.11.24).

1.11.27. For ([2], C.4.8). Their formula (C.4) makes sense, because $j_!$ is defined by a kernel, not just codefined by a kernel.

We check their formula (C.4). Let $j : U \to Y$ be a cotruncative open embedding, assume first both U, Y are algebraic stacks of finite type. Then $j_!$ is defined by a kernel by my Section 1.11.1, we are using here ([2], B.6.3). Recall that $j_!$ is codefined by the kernel $(\Gamma_j)_! e$, so it is defined by the kernel $Q := (\mathrm{id}^l_U \boxtimes \mathrm{id}_Y)((\Gamma_j)_! e)$ by their B.4.8, here $\Gamma_j : U \to U \times Y$ is the graph of j. One checks that $Q \widetilde{\to} (\mathrm{id} \times j)_!(u_U)$, where $u_U = (\triangle_U)_*\omega$ for $\triangle_U \colon U \to U \times U$.

The functor $j_* : Shv(U) \to Shv(Y)$ is defined by the kernel $(\Gamma_j)_*\omega$, and Mir_Y is defined by the kernel $\text{ps-u}_Y = (\triangle_Y)_! e$ for $\triangle_Y \colon Y \to Y \times Y$. This gives that $\mathrm{Mir}_Y \circ j_*$ is defined by the kernel $(\Gamma_j)_! e$.

Now we compose the functors given by kernel $j_! \circ \mathrm{Mir}_U$, the result is defined by the kernel $Q * (\text{ps-u}_U) \widetilde{\to} (\mathrm{id}_U \boxtimes Q)(\text{ps-u}_U)$, here we used their formula (B.5). Now again use the fact that Q is codefined by the kernel, so that $\mathrm{id}_U \boxtimes Q \widetilde{\to} \mathrm{id}_U \boxtimes j_!$ as functors $Shv(U \times U) \to Shv(U \times Y)$. One checks that $\mathrm{id}_U \boxtimes j_! = (\mathrm{id}_U \times j)_!$. This gives

$$(\mathrm{id}_U \boxtimes Q)(\text{ps-u}_U) \widetilde{\to} (\mathrm{id}_U \times j)_!(\triangle_U)_! e \widetilde{\to} (\Gamma_j)_! e$$

as desired! Note that we obtained their formula only under the assumption that $j_!$ is defined by a kernel. We did not need the full strength of the property that j_* admits a right adjoint as a functor given by kernel.

Let now $j : Y' \hookrightarrow Y$ be an open immersion of algebraic stacks (maybe of infinite type), which is cotruncative. To check their formula (C.4) in this case, note that the question is local in Zariski topology: it suffices to show that for any quasi-compact open $U \subset Y$ for the contruncative open immersion $U \cap Y' \hookrightarrow U$ we have the corresponding formula.

1.11.28. For ([2], C.4.9). Let Y_1, Y_2 be truncatable and $j : U_2 \hookrightarrow Y_2$ be a contruncative open, where U_2 is quasi-compact, let $P_2 \in Shv(Y_1 \times U_2)$ and $P = (\mathrm{id} \times j)_*(P_2) \in Shv(Y_1 \times Y_2)$. Let now $\mathcal{Z}$ be an algebraic stack locally of finite type (as we consider, that is, locally as in my Section 1.6.2). Let $U_{\mathcal{Z}} \subset \mathcal{Z}$ be a quasi-compact open. Consider the functor $\mathrm{id}_{U_{\mathcal{Z}}} \boxtimes P^l : Shv(U_{\mathcal{Z}} \times Y_1) \to Shv(U_{\mathcal{Z}} \times Y_2)$.

First, we claim that it takes values in the full subcategory $Shv(U_{\mathcal{Z}} \times Y_2)_{co}$. Indeed, the functor j_* is codefined by a kernel, by assumption. Write $\tilde{P} \in Shv(U_2 \times Y_2)$ for the object that codefines j_*. So, $P \widetilde{\to} (\mathrm{id}_{Y_1} \boxtimes \tilde{P}^l)(P_2)$. Let $K \in Shv(U_{\mathcal{Z}} \times Y_1)$. By their Section B.3.2,

$$\begin{aligned}(\mathrm{id}_{U_{\mathcal{Z}}} \boxtimes P^l)(K) \widetilde{\to} ((K^\sigma)^l \boxtimes \mathrm{id})(P) \widetilde{\to} ((K^\sigma)^l \boxtimes \mathrm{id})(\mathrm{id}_{Y_1} \boxtimes \tilde{P}^l)(P_2) \\ \widetilde{\to} (\mathrm{id}_{Y_1} \boxtimes \tilde{P}^l)((K^\sigma)^l \boxtimes \mathrm{id})(P_2)\end{aligned}$$

Thus, we obtained a functor

$$\mathrm{id}_{U_{\mathcal{Z}}} \boxtimes P^l : Shv(U_{\mathcal{Z}} \times Y_1) \to Shv(U_{\mathcal{Z}} \times Y_2)_{co} \tag{33}$$

The so obtained functors are compatible with the $*$-direct images of P_2 under open immersions $U_2 \hookrightarrow U_2'$ of quasi-compact cotruncated opens of Y_2. Passing to the colimit over U_2, we see that for any $P \in \underset{U_2 \subset Y_2}{\mathrm{colim}_*}\, Shv(Y_1 \times U_2) \widetilde{\to} Shv(Y_1 \times Y_2)_{co_{Y_2}}$ we get the functor (33).

By definition, $Shv(\mathcal{Z} \times Y_2)_{co_{Y_2}} \widetilde{\to} \lim^*_{U_{\mathcal{Z}}} Shv(U_{\mathcal{Z}} \times Y_2)_{co}$. The functors (33) are compatible with the $*$-restrictions along the open immersions $U_{\mathcal{Z}} \hookrightarrow U'_{\mathcal{Z}}$ for quasi-compact opens of $\mathcal{Z}$. Passing to the limit, one gets the functor

$$Shv(\mathcal{Z} \times Y_1) \widetilde{\to} \lim_{U_{\mathcal{Z}} \subset \mathcal{Z}}{}^* Shv(U_{\mathcal{Z}} \times Y_1) \to \lim_{U_{\mathcal{Z}} \subset \mathcal{Z}}{}^* Shv(U_{\mathcal{Z}} \times Y_2)_{co} \widetilde{\to} Shv(\mathcal{Z} \times Y_2)_{co_{Y_2}}$$

still denoted $\mathrm{id}_{\mathcal{Z}} \boxtimes P^l$.

1.11.29. For ([2], C.5.2). Here Y is assumed locally of finite type and truncatable. Let us explain that Y is miraculous iff each cotruncative quasi-compact open substack $U \subset Y$ is miraculous.

Assume each such U is miraculous. It suffices to show that for any quasi-compact stack $\mathcal{Z}$ the functor $\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y : Shv(\mathcal{Z}\times Y)_{co}\to Shv(\mathcal{Z}\times Y)$ is an equivalence. For any quasi-compact cotruncative open $j: U\hookrightarrow Y$ we have

$$(\mathrm{id}_{\mathcal{Z}}\boxtimes j_!)(\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_U)\widetilde{\to}(\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y)(\mathrm{id}_{\mathcal{Z}}\boxtimes j_{*,co})$$

by their (C.4), and $(\mathrm{id}_{\mathcal{Z}}\boxtimes j_{*,co}) : Shv(\mathcal{Z}\times U)\to Shv(\mathcal{Z}\times Y)_{co}$ is fully faithful (see my Section 1.11.12). So, the restriction of $\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y$ to the full subcategory $Shv(\mathcal{Z}\times U)$ is fully faithful. By ([16], 9.2.47) this implies that $\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y$ is fully faithful. Write $j^?$ for the right adjoint to $j_{*,co}$.By Verdier duality, their (C.4) gives $\mathrm{Mir}_U\circ j^?\widetilde{\to} j^*\circ \mathrm{Mir}_Y$.

To see that $\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y$ is essentially surjective, let $K\in Shv(\mathcal{Z}\times Y)$. For each quasi-compact open $j: U\hookrightarrow Y$ let

$$F_U=(\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_U^{-1})(\mathrm{id}_{\mathcal{Z}}\boxtimes j^*)(K)$$

Recall that $Shv(\mathcal{Z}\times Y)_{co}\widetilde{\to}\lim_{U\subset Y}{}^? Shv(\mathcal{Z}\times U)$. For an open immersion $j_U: U\hookrightarrow U'$ of quasi-compact contruncative opens substacks of Y, we have canonically $j_U^? F_{U'}\widetilde{\to} F_U$, so the collection (F_U) defines an object $F\in Shv(\mathcal{Z}\times Y)_{co}$ with $j^!_{co}(F)\widetilde{\to} F_U$ for each contruncative quasi-compact open U. Then formally we get $(\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y)(F)\widetilde{\to} K$.

Conversely, assume Y is miraculous. Then again, for any quasi-compact stack $\mathcal{Z}$, the fully faithfulness and essential surjectivity of Mir_U follows from their (C.4). For the essential surjectivity, let $K\in Shv(\mathcal{Z}\times U)$ and $F=(\mathrm{id}_{\mathcal{Z}}\boxtimes j^?)(\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y^{-1})(\mathrm{id}_{\mathcal{Z}}\boxtimes j_!)(K)$. Then $(\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_U)(F)\widetilde{\to} K$.

1.11.30. For ([2], C.5.3). Since Y is miraculous, for any quasi-compact open $U_{\mathcal{Z}}\subset \mathcal{Z}$ the functor $\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y : Shv(U_{\mathcal{Z}}\times Y)_{co}\to Shc(U_{\mathcal{Z}}\times Y)$ is an equivalence. They are compatible with the restrictions for the open immersions $U'_{\mathcal{Z}}\subset U_{\mathcal{Z}}$. Passing to the limit over $U_{\mathcal{Z}}$, we get an equivalence $\mathrm{id}_{\mathcal{Z}}\boxtimes \mathrm{Mir}_Y : Shv(\mathcal{Z}\times Y)_{co_Y}\widetilde{\to} Shv(\mathcal{Z}\times Y)$. Dualizing, we get the second equivalence.

1.11.31. For ([2], 0.3.4). Let Y be a connected separated scheme of finite type, $Q\in Shv(Y)$. When does Q admits a right adjoint as a functor given by kernel? By their B.5.6, we need $Q\in Shv(Y)^{constr}$ for this, which we assume. Note that in this case $Q\in Shv(Y)^c$, so by their B.1.4 for any quasi-compact algebraic stack $\mathcal{Z}$ (as in my Section 1.6.2), the functor $\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q}$ preserves compactness. So, by my Proposition 1.8.5, Q admits a right adjoint as a functor given by kernel iff Q is codefined by a kernel.

Set $'Q^R=\mathrm{id}^l_Y(\mathbb{D}Q)$, we get $'Q^R\widetilde{\to}\omega\otimes(\mathbb{D}Q)$. By Verdier duality, $(Q,'Q^R)$ is a dual pair in the corresponding 2-category iff $(('Q^R)^\sigma, Q^\sigma)$ is an adjoint pair in the same 2-category. Here Q^σ defines by the kernel the functor $Shv(pt)\to Shv(Y), V\mapsto V\otimes Q$.

Let $P=\mathrm{Mir}_Y(Q)$. One gets immediately $P\widetilde{\to} e\otimes^! Q\widetilde{\to}\mathcal{H}om(\omega_Y, Q)$. To be precise, by $\mathcal{H}om(\cdot,\cdot)\in Y$ we mean the inner hom for $(Shv(Y),\otimes)$. Then Q is codefined by a kernel iff for any quasi-compact stack $\mathcal{Z}$ as above, $\mathrm{id}_{\mathcal{Z}}\boxtimes P^l\to \mathrm{id}_{\mathcal{Z}}\boxtimes Q$ is an isomorphism by their B.4.4. The morphism of functors $\mathbf{P}^l\to\mathbf{Q}$ in $\mathrm{Fun}_{e,cont}(Shv(Y),\mathrm{Vect})$ comes from the natural morphism $F\otimes(e_Y\otimes^! Q)\to F\otimes^! Q$ for $F\in Shv(Y)$ defined in ([20], 0.1.10). Namely, it is the composition

$$C_c(Y, F\otimes(e_Y\otimes^! Q))\to C_*(Y, F\otimes(e_Y\otimes^! Q))\to C_*(Y, F\otimes^! Q)$$

Note that $\mathrm{Mir}_Y : Shv(Y)\to Shv(Y)$ sends K to $e\otimes^! K$.

Assume that $e[2d]\widetilde{\to}\omega$, for example, Y is smooth of dimension d. More generally, this holds for example, for $Y=\mathbb{A}^n/S_2$, where the nontrivial element of S_2 acts as multiplication by -1. So $\mathrm{Mir}_Y(K)\widetilde{\to}K[-2d]$, and Mir_Y is an equivalence. Assume in addition Y proper and $Q\in\mathrm{Lisse}(Y)$. Let us show that the functor $\mathbf{Q}$ is indeed codefined by a kernel. Let $\mathcal{Z}$ be a quasi-compact algebraic stack. For $K\in Shv(\mathcal{Z}\times Y)$, $M\in Shv(\mathcal{Z})$ we get for the diagram of projections $\mathcal{Z}\overset{p_1}{\leftarrow}\mathcal{Z}\times Y\overset{p_2}{\to}Y$

$$\mathcal{H}om((\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q})(K),M)\widetilde{\to}\mathcal{H}om(K\otimes^! p_2^!Q,p_1^!M) \tag{34}$$

Write $Q^\vee=\mathcal{H}om(Q,e)$, this is the dual of Q in $(\mathrm{Lisse}(Y),\otimes)$. Then $Q\widetilde{\to}\mathcal{H}om(Q^\vee,e)$, and $\mathbb{D}Q\widetilde{\to}Q^\vee[2d]$. Now

$$K\otimes p_2^!Q\widetilde{\to}\mathcal{H}om(p_2^*(\mathbb{D}Q),K)\widetilde{\to}\mathcal{H}om(p_2^*(Q^\vee)[2d],K)\widetilde{\to}p_2^*Q\otimes K[-2d]$$

We used that $p_2^*(Q^\vee)\widetilde{\to}(p_2^*Q)^\vee$. By ([20], 0.0.7), the functor $K\mapsto p_2^*Q\otimes K[-2d]$ has a continuousn right adjoint given by $\mathcal{K}\mapsto\mathcal{K}\otimes^!\mathbb{D}(p_2^*Q)[2d]$. So, (34) identifies with

$$\mathcal{H}om(K,p_1^!M\otimes^! p_2^!(\mathbb{D}Q)[2d])\widetilde{\to}\mathcal{H}om(K,(\mathrm{id}_{\mathcal{Z}}\boxtimes{}'Q^R)(M))$$

So, $(\mathrm{id}_{\mathcal{Z}}\boxtimes\mathbf{Q},\mathrm{id}_{\mathcal{Z}}\boxtimes{}'Q^R)$ is a dual pair for any $\mathcal{Z}$, so that Q admits a right adjoint as a functor given by kernel.

Appendix A. On singular support

A.0.1. *Question.* What is the relation between the abstract ULA property of Sam with the notion of singular support?

A.0.2. Let X be a smooth scheme of finite type, $F\in Shv(X)^{constr}$ in the constructible context. Let E be a local system on X. Let us show that $SingSupp(F)=SingSupp(F\otimes E)$.

Let $C\subset T^*X$ be a closed conical subset such that F is micro-supported on C in the sense of [3]. Let $X\overset{h}{\leftarrow}U\overset{f}{\to}Y$ be a test pair, which is C-transversal, so h^*F is ULA with respect to Y. Then $h^*(F\otimes E)$ is ULA with respect to Y, because the ULA property is local in smooth topology of the source. So, the pair (h,f) is $F\otimes E$-acyclic. So, $F\otimes E$ is also micro-supported on C. Recall that Beilinson write $\mathcal{C}(F)$ for the set of conical closed subsets $C'\subset T^*(X)$ such that F is micro-supported on C'. On the other hand if $h^*(F\otimes E)$ is ULA with respect to $f:U\to Y$ then h^*F is also ULA with respect to f as a direct summand of $h^*(F\otimes E\otimes E^*)$. So, $\mathcal{C}(F)=\mathcal{C}(F\otimes E)$ and $SingSupp(F)=SingSupp(F\otimes E)$.

For any closed conical subset $\mathcal{N}\subset T^*(X)$ we get an action of $(Lisse(X),\otimes)$ on $Shv_{\mathcal{N}}(X)$ such that $E\in\mathrm{Lisse}(X)$ sends K to $K\otimes E$.

A.0.3. Let us now X a be scheme of finite type, not necessarily smooth, we want to check the same property. Let E be a local system on X, $F\in Shv(X)^{constr}$. Let us show that $SingSupp(F)=SingSupp(F\otimes E)$.

Assume X' is a smooth scheme of finite type and $j:X\to X'$ is a closed embedding. By definition, it suffices to show that $SingSupp(j_*F)=SingSupp(j_*(F\otimes E))$.

Let $C\subset T^*X'$ be a closed conical subset and $X'\overset{h'}{\leftarrow}U'\overset{f}{\to}Y$ be a test pair, which is C-transversal. Let $j_U:U\to U'$ be obtained from j by the base change $h':U'\to X'$.

Let $h : U \to X$ be the projection. By ([5], property 3 in 5.1.2), $(j_U)_* h^* F$ is ULA with respect to f iff h^*F is ULA with respect to $f \circ j_U$. Now h^*F is ULA with respect to $f \circ j_U$ iff $h^*(F \otimes E)$ is ULA with respect to $f \circ j_U$. So, $\mathcal{C}(j_*F) = \mathcal{C}(j_*(F \otimes E))$, and we are done.

Remark A.0.4. *Recall that* $\mathrm{QLisse}(X)$ *is equipped with the t-structures inheritied from the usual (not perverse) t-structure of* $Shv(X)$*, and* $Shv(X)$ *is considered with the perverse t-structure. If* $K \in Shv(X)^{\le 0}$*,* $V \in \mathrm{QLisse}(X)^{\le 0}$ *then* $K \otimes V \in Shv(X)^{\le 0}$*.*

Proof. We characterize $Shv(X)^{\le 0} \subset Shv(X)$ as the full subcategory of $K' \in Shv(X)$ such that for any point $i : x \to X$ the object i^*K' is placed in usual cohomological degrees $\le -\dim x$, here x is not necessarily a closed point. Since i^*V is placed in usual degrees ≤ 0, and i^*K is placed in usual degrees $\le -\dim x$, $i^*(K \otimes V)$ is placed in usual degrees $\le -\dim x$. □

A.1. Let $p : Z \to X$ be a morphism of schemes of finite type with X smooth. Let $E \in Lisse(X)$ and $K \in Shv(Z)$. Then we get

$$K \otimes^! p^! E \widetilde{\to} p^* E \otimes K[-2\dim X]$$

The same holds for $E \in \mathrm{IndLisse}(X)$.
Proof:

$$K \otimes^! p^! E \widetilde{\to} \mathcal{H}om(\mathbb{D}(p^! E), K) \widetilde{\to} \mathcal{H}om(p^*(E^\vee[2\dim X], K) \widetilde{\to} p^* E \otimes K[-2\dim X]$$

where $\mathcal{H}om$ denotes the inner hom for the monoidal category $(Shv(Z), \otimes)$, and $E^\vee = \mathcal{H}om(E, e_X)$ is the naive duality on X. □

In particular, if $K \in Shv(Z)^{constr}$ then $SingSupp(K \otimes^! p^! E) = SingSupp(K)$ by the above.

Lemma A.1.1. *Let* $K \in Shv(Z)^{\ge 0}, E \in \mathrm{QLisse}(X)^{\ge 0}$ *then* $K \otimes^! p^! E \in Shv(Z)^{\ge 0}$*.*

Proof. For $K' \in Shv(Z)$ the condition that $K' \in Shv(Z)^{\ge 0}$ means that for any maybe non clolsed point $i : z \to Z$, $i^! K'$ is placed in usual degrees $\ge -\dim z$. For such a point $i : z \to Z$ let $\bar{i} : \bar{z} \to X$ be the image of z in X. Then $\bar{i}^! E$ is placed in usual degrees $\ge 2\dim X - 2\dim \bar{z}$. So, $i^!(K \otimes^! p^! E) \widetilde{\to} (i^! K) \otimes (\bar{i}^! E)$ is placed in usual degrees $\ge -\dim z + 2\dim X - 2\dim \bar{z} \ge -\dim z$, because $\dim \bar{z} \le \dim X$. So, $K \otimes p^! E$ is placed in perverse degrees ≥ 0. □

Lemma A.1.2. *Let* $K \in Shv(Z)^{\le 0}$*. Then the functor* $\mathrm{QLisse}(E)^- \to Shv(Z)$*,* $E \mapsto K \otimes^! p^! E$ *commutes with Postnikov towers. That is, the natural map* $K \otimes^! p^! E \to \lim_n (K \otimes^! p^! (\tau^{\ge -n} E))$ *is an isomorphism in* $Shv(Z)$*.*

Proof. The idea is due to Sam, see his ([24], Lemma 3.12.1). We consider $\mathrm{QLisse}(X)$ equipped with the usual t-structure. First, if $E \in \mathrm{QLisse}(X)^{\le r}$ and is bounded then

$$K \otimes^! p^! E \widetilde{\to} p^* E \otimes K[-2\dim X] \in Shv(Z)^{\le r + 2\dim X} \tag{35}$$

For each $m \in \mathbb{Z}$,

$$\tau^{\ge -m}(K \otimes^! p^! E) \widetilde{\to} \lim_n \tau^{\ge -m}(K \otimes^! p^! (\tau^{\ge -n} E)),$$

because the functor $n \mapsto \tau^{\geq -m}(K \otimes^! p^!(\tau^{\geq -n}E)$ stabilizes as n goes to $+\infty$, here we used (35). Now

$$K \otimes^! p^! E \widetilde{\to} \lim_m \tau^{\geq -m}(K \otimes^! p^! E) \widetilde{\to} \lim_m \lim_n \tau^{\geq -m}(K \otimes^! p^!(\tau^{\geq -n}E)$$
$$\widetilde{\to} \lim_n \lim_m \tau^{\geq -m}(K \otimes^! p^!(\tau^{\geq -n}E) \widetilde{\to} \lim_n (K \otimes^! p^!(\tau^{\geq -n}E))$$

□

Lemma A.1.3. *Let $\mathcal{N} \subset T^*Z$ be a closed conical subset. Then for $K \in Shv_{\mathcal{N}}(Z)$, $E \in QLisse(X)$ we have $K \otimes^! p^! E \in Shv_{\mathcal{N}}(Z)$*

Proof. Since $K \widetilde{\to} \operatorname{colim}_n \tau^{\leq n} K$ and $E = \operatorname{colim}_n \tau^{\leq n} E$, we may and do assume $K \in Shv_{\mathcal{N}}(Z)^{\leq 0}, E \in \mathrm{QLisse}(X)^{\leq 0}$. Recall that we consider $\mathrm{QLisse}(X)$ with the usual t-structure.

It suffices to show that for any m, $\tau^{\geq -m}(K \otimes^! p^! E) \in Shv_{\mathcal{N}}(Z)$. Given such m, for n large enough we have

$$\tau^{\geq -m}(K \otimes^! p^! E) \widetilde{\to} \tau^{\geq m}(K \otimes^! p^!(\tau^{\geq -n}E))$$

as in the proof of Lemma A.1.2. Since $K \otimes^! p^!(\tau^{\geq -n}E) \in Shv_{\mathcal{N}}(Z)$ by Section A.0.2, we are done. □

We may now consider $(\mathrm{QLisse}(X), \otimes^!)$, and the above lemma shows that this symmetric monoidal category acts on $Shv_{\mathcal{N}}(Z)$, so that $E \in \mathrm{QLisse}(X)$ sends $K \in Shv_{\mathcal{N}}(Z)$ to $K \otimes^! p^! E$.

Lemma A.1.4. *Let X, Y be schemes of finite type, $K \in Shv(X)^-$. Then the functor $Shv(Y) \to Shv(X \times Y)$, $F \mapsto K \boxtimes F$ preserves Postniklov towers. That is, the natural map $K \boxtimes F \to \lim_n (K \boxtimes \tau^{\geq -n} F)$ is an isomorphism in $Shv(X \times Y)$.*

Proof. Note that if $K \in Shv(X)^{\leq a}$ and $L \in Shv(Y)^{\leq b}$ then $X \boxtimes Y \in Shv(X \times Y)^{\leq a+b}$ by ([1], Pp. E.7.2).

Let $m \in \mathbb{Z}$. Consider the functor $n \in \mathbb{Z} \mapsto \tau^{\geq -m}(K \boxtimes \tau^{\geq -n} F) \in Shv(X \times Y)$. By the above remark, it stabilizes for n large enough. So,

$$\lim_n \tau^{\geq -m}(K \boxtimes \tau^{\geq -n} F) \widetilde{\to} \tau^{\geq -m}(K \boxtimes F)$$

Thus, $\lim_n (K \boxtimes \tau^{\geq -n} F) \widetilde{\to} \lim_m \lim_n \tau^{\geq -m}(K \boxtimes \tau^{\geq -n} F) \widetilde{\to} K \boxtimes F$. □

A.1.5. ([3], Lemma 2.2) is important and says: let X be a smooth variaty, $\phi : U \to X$ a moprhism in Sch_{ft}, $F \in Shv(X)^{constr}$. If $\phi : U \to X$ is C-transversal with $SS(F) \subset C$ then $SS(\phi^* F) \subset \phi^\circ(C)$.

Appendix B. Langlands functoriality

B.0.1. We try to derive a correct setting for the geometric Langlands functoriality on the DG-level as an application of [1]. Let H, G be connected reductive groups over k, Let $\kappa : \check{G} \to \check{H}$ be a morphism of dual groups over e.

B.0.2. Work in the constructible setting. For a map $f : \mathcal{Y} \to \mathcal{Y}'$ of classical algebraic stacks (with affine diagonals), we denote by $f_* : Shv(\mathcal{Y}) \to Shv(\mathcal{Y}')$ the right adjoint to f^* as in ([2], A.1.8). It is not necessarily continuous, but satisfies the base change with respect to the !-pullbacks by *loc.cit.*

Recall that $fSet$ denotes the category of finite sets. Consider the functors $h_G, h_H : fSet \to \mathrm{DGCat}_{cont}$, $h_G(I) = Shv(\mathrm{Bun}_G \times X^I)$ and $h_H(I) = Shv(\mathrm{Bun}_H \times X^I)$. For a map $\alpha : I \to J$ in $fSet$ the corresponding transition maps are given by the !-pullback with respect to $\mathrm{Bun}_G \times X^J \to \mathrm{Bun}_G \times X^I$ and $\mathrm{Bun}_H \times X^J \to \mathrm{Bun}_H \times X^I$ respectively. We need to have a natural transformation $\mathcal{L}$ from h_G to h_H.

B.0.3. let $\mathcal{C}_H : fSet \to Alg(\mathrm{DGCat}_{cont})$ be the functor sending I to $\mathrm{Rep}(\check{H})^{\otimes I} \otimes Shv(X^I)$. For a morphism $\alpha : I \to J$ the transition map $\mathcal{C}_H(I) \to \mathcal{C}_H(J)$ is the morphism

$$\mathrm{Rep}(\check{H})^{\otimes I} \otimes Shv(X^I) \stackrel{m\otimes\vartriangle^!}{\to} \mathrm{Rep}(\check{H})^{\otimes J} \otimes Shv(X^J),$$

where m is the multiplication along α, and $\vartriangle: X^J \to X^I$ is induced by α. Similarly we have

$$\mathcal{C}_G : fSet \to Alg(\mathrm{DGCat}_{cont}), \quad I \mapsto \mathrm{Rep}(\check{G})^{\otimes I} \otimes Shv(X^I)$$

We view them as objects of $Alg(\mathrm{Fun}(fSet, \mathrm{DGCat}_{cont}))$. Restriction along κ defines a morphism

$$\mathrm{Res}^{\kappa} : \mathcal{C}_H \to \mathcal{C}_G \tag{36}$$

in $Alg(\mathrm{Fun}(fSet, \mathrm{DGCat}_{cont}))$.

Recall that $h_G \in \mathrm{Fun}(fSet, \mathrm{DGCat}_{cont})$ is a module over $\mathcal{C}_G$ via the action constructed in ([9], Pp. B.2.3), similarly, $h_H \in \mathrm{Fun}(fSet, \mathrm{DGCat}_{cont})$ is a module over $\mathcal{C}_H$. By restriction along (36), we may view h_G as a module over h_H. We write

$$\mathrm{act}_G : \mathrm{Rep}(\check{G})^{\otimes I} \otimes Shv(\mathrm{Bun}_G \times X^I) \to Shv(\mathrm{Bun}_G \times X^I)$$

for the corresponding action map, and similarly for act_H.

B.0.4. *Setting A) for the functoriality.* We want a (maybe discontinuous) natural transformation $\mathcal{L} : h_G \to h_H$ to be a morphism of $\mathcal{C}_H$-modules.

When talking about possibily discontinuious natural transformation, we mean of course that for each $I \in fSet$ the functor $\mathcal{L}_I : Shv(\mathrm{Bun}_G \times X^I) \to Shv(\mathrm{Bun}_H \times X^I)$ should be a morphism in DGCat, the notation of ([11], ch. I.1, 10.3.4).

B.0.5. Our first trial is as follows. Let $\mathcal{M} \in Shv(\mathrm{Bun}_G \times \mathrm{Bun}_H)$. It defines a natural transformation $\mathcal{L} : h_G \to h_H$ sending I to the functor

$$\mathcal{L}_I : Shv(\mathrm{Bun}_G \times X^I) \to Shv(\mathrm{Bun}_H \times X^I), \quad \mathcal{L}_I(K) = (p_H)_*(p_G^! K \otimes^! q^! \mathcal{M})$$

for the diagram of projections

$$\begin{array}{ccccc} \mathrm{Bun}_G \times X^I & \stackrel{p_G}{\leftarrow} & \mathrm{Bun}_G \times \mathrm{Bun}_H \times X^I & \stackrel{p_H}{\to} & \mathrm{Bun}_H \times X^I \\ & & \downarrow q & & \\ & & \mathrm{Bun}_G \times \mathrm{Bun}_H & & \end{array}$$

B.0.6. Note that for a (maybe discontinuous) morphism $f : D \to D'$ in DGCat and $C \in \mathrm{DGCat}_{cont}$ dualizable, we have a morphism $\mathrm{id} \otimes f : C \otimes D \to C \otimes D'$ defined as the composition $\mathrm{Fun}_{e,cont}(C^\vee, D) \to \mathrm{Fun}_{e,cont}(C^\vee, D')$ with f.

So, for each $I \in fSet$ we get morphisms

$$\mathrm{id} \otimes \mathcal{L}_I : \mathrm{Rep}(\check{H})^{\otimes I} \otimes Shv(\mathrm{Bun}_G \times X^I) \to \mathrm{Rep}(\check{H})^{\otimes I} \otimes Shv(\mathrm{Bun}_H \times X^I)$$

The requirement of *Setting A)* contains, in particular, the commutativity datum for the diagram

$$\begin{array}{ccc} \mathrm{Rep}(\check{H})^{\otimes I} \otimes Shv(\mathrm{Bun}_G \times X^I) & \overset{\mathrm{act}_G \circ (\mathrm{Res}^\kappa \otimes \mathrm{id})}{\to} & Shv(\mathrm{Bun}_G \times X^I) \\ \downarrow \mathrm{id} \otimes \mathcal{L}_I & & \downarrow \mathcal{L}_I \\ \mathrm{Rep}(\check{H})^{\otimes I} \otimes Shv(\mathrm{Bun}_H \times X^I) & \overset{\mathrm{act}_H}{\to} & Shv(\mathrm{Bun}_H \times X^I) \end{array} \tag{37}$$

in a way functorial in $I \in fSet$.

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