1. Comments to Krähmer, 'Notes on Koszul algebras'

1.0.1. The bar resolution for bimodules is very nicely described in ([1], Sect.4). There are versions of this resolution for left, right and bimodules and also reduced versions. The version described in ([3], Prop. 12) also appears in ([4], chapter 10).

The proof of ([3], Prop. 12) contains a gap, the exactness of the resolution is not justified. To see the exactness, one constructs a contracting homotopy as follows. Let $C_i = A\underline{\otimes}A_+^{\otimes i}$ for brevity.

We need to show that $\ldots \to C_1 \to C_0 \to k \to 0$ is exact. Let $C_{-1} = k$. We construct the homotopy $s : C_i \to C_{i+1}$ as follows. Let $s_{-1} : k \to A$ be the natural inclusion. Let $s : C_i \to C_{i+1}$ send $a_0 \otimes a_1 \otimes \ldots a_i$ to $1 \otimes (a_0 - \epsilon(a_0)) \otimes a_1 \otimes \ldots \otimes a_i$. Then $ds + sd$ is the identity.

1.0.2. For the definition of the DGA $C^\bullet$ just before Prop. 13. Here $C^\bullet = (C^0 \to C^1 \to C^2 \to \ldots)$ with $C^0 = k, C^i = (A_+^{\otimes i})^*$ is the complex $\mathrm{Hom}_A(C_\bullet^{bar}, k)$. It is understood that C^i is placed in degree i, and in the identification of $C^i \otimes C^j$ with C^{i+j} there is NO Koszul sign rule! Namely, if $\phi \in C^i, \psi \in C^j$ then $\phi \otimes \psi \in C^{i+j}$ is given by $a_1 \otimes \ldots \otimes a_{i+j} \mapsto \phi(a_1 \otimes \ldots \otimes a_i)\psi(a_{i+1} \otimes \ldots \otimes a_{i+j})$. The formula (5) for the differential is correct, and d is a super-derivation, so $C^\bullet$ is a DGA.

For Section 4.5 and below: the notation $(A_+^{\otimes i})_j^*$ means $((A_+^{\otimes i})_j)^*$, here $(A_+^{\otimes i})_j$ is the degree j component of $A_+^{\otimes i}$.

For the proof of Prop.15. The map $d : C^{12} = A_2^* \to (A_1 \otimes A_1)^* = C^{22}$ is a part of the map $d : C^1 \to C^2$, for $\phi \in A_2^*$ it s given by $(d\phi)(a_1 \otimes a_2) = -\phi(a_1 a_2)$ for $a_i \in A_1$. Now let $R \subset A_1 \otimes A_1$ be the subspace defining the quadratic algebra $A = \oplus_{i\geq 0} A_i$. The sequence is exact $0 \to R \to A_1 \otimes A_1 \stackrel{m}{\to} A_2 \to 0$, where m is the product. So, $0 \to A_2^* \to (A_1 \otimes A_1)^* \to R^* \to 0$ is also exact. By construction, $\mathrm{Ext}_A^{22}(k,k) \widetilde{\to} R^*$ is the cokernel of $d : C^{12} \to C^{22}$. We obtain $A_2^* = R^\perp$, so the quadratic relations in $\oplus_{i\geq 0} \mathrm{Ext}_A^{ii}(k,k)$ are precisely $A_2^* = R^\perp$, hence the latter algebra identifies with $A^!$.

We used also the following. The complex $C^\bullet = (C^0 \stackrel{d}{\to} C^1 \stackrel{d}{\to} \ldots)$ starts with $C^0 = k \stackrel{0}{\to} C^1 = A_+^*$, so $\mathrm{Ext}_A^{11}(k,k) = A_1^*$.

1.0.3. For Sect. 4.7. First, we mean $(A^!)_n^* = ((A^!)_n)^*$. Now $\oplus_{n\geq 0}(A^!)_n^*$ is a graded right $A^!$-module with the operation: for $b \in A^!, \phi \in (A^!)_n^*$, $\phi b \in (A^!)_{n-1}^*$ is given by $(\phi b)(c) = \phi(bc)$ for $c \in (A^!)_{n-1}$. Then $K_n(A,k) = A \otimes (A^!)_n^*$ form together a right $A \otimes A^!$-module $\oplus_{n\geq 0} K_n(A,k)$. It is also a left A-module in way commuting with the right $A \otimes A^!$-module structure. Therefore, $(K_\bullet(A,k), d)$ is a chain complex of left A-modules, here $K_n(A,k)$ is placed in degree n.

Let $V = A_1$ and $R \subset A_1 \otimes A_1$ be the subspace of relations, so $(A_1 \otimes A_1)/R = A_2$ then $(A^!)_2^* = R$ canonically. The action map $(A^!)_2^* \otimes (A^!)_1 \to (A^!)_1^* = V$ is the map $\alpha : R \otimes V^* \to V$ corresponding to $R \hookrightarrow V \otimes V$ by adjunction.

The complex $\ldots \to A \otimes (A^!)_2^* \to A \otimes (A^!)_1^* \to A$ becomes

$$\ldots \to A \otimes R \to A \otimes V \stackrel{m}{\to} A,$$

here m is just the product map, and the map $A \otimes R \to A$ is the composition $A \otimes R \stackrel{coev}{\to} A \otimes R \otimes (V \otimes V^*) \to A$, the second one is given by $a \otimes r \otimes v \otimes v^* \mapsto (av) \otimes \alpha(r \otimes v^*)$.

1.0.4. For the complex $B_\bullet$ defined in Sect. 4.6 after Cor. 6. Since C^{ij} is nonzero only for $j \geq i$, $B_n = \oplus_{j-i=n} C^{ij}$ is nonzero for $n \geq 0$. The differential now becomes $d(B_n) \subset B_{n-1}$, so $B_\bullet = (\ldots \to B_1 \to B_0)$ is a chain complex, and a DGA. We get $B_0 = \oplus_{i \geq 0} C^{i,i}$, where $C^{i,i} = A_1^* \otimes \ldots \otimes A_1^*$ (i factors), $B_1 = \oplus_{i \geq 0} C^{i,i+1}$, where

$$C^{i,i+1} = \oplus_{k=1}^{i} \, (A_1^*)^{k-1} \otimes A_2^* \otimes (A_1^*)^{i-k}$$

The differential on B is given by the general construction of reshuffling from the next subsection.

1.0.5. *Generality.* Let $C_\bullet = (C^0 \xrightarrow{d} C^1 \xrightarrow{d} \ldots)$ be a DGA, the product map $C \otimes C \to C$ is denoted $\phi \otimes \psi \mapsto \phi\psi$. Assume $C^i = \oplus_{j \geq i} C^{i,j}$, and $d(C^{i,j}) \subset C^{i+1,j}$ for any i, j. So, for $\phi \in C^i, \psi \in C^j$,

$$d(\phi\psi) = (d\phi)\psi + (-1)^i \phi(d\psi)$$

Set $B_n = \oplus_{i-j=n} C^{i,j}$. Let the differential in $B_\bullet$ be exactly the same as above, so $B_\bullet = (\ldots \to B_{-1} \to B_0)$ is a cochain complex placed in negative degrees. Define the new product map $B \otimes B \to B$ as follows. If $\phi \in C^{i,j}, \psi \in C^{r,s}$ we let

$$m'(\phi \otimes \psi) = (-1)^{jr} \phi\psi$$

Lemma 1.0.6. *The map $m' : B \otimes B \to B$ is a morphism of chain complexes, so B is a DGA.*

Proof. Let $\phi \in C^{i,j}, \psi \in C^{r,s}$. We get $d(\phi \otimes \psi) = (d\phi) \otimes \psi + (-1)^{i-j} \phi \otimes d\psi$ in $B \otimes B$ (according to the definition from [2]). Further,

$$d(m'(\phi \otimes \psi)) = (-1)^{jr}((d\phi)\psi + (-1)^i \phi d\psi) = m' d(\phi \otimes \psi)$$

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We can if needed think that B_{-n} is placed in degree n, and B is a chain complex, this is the point of view from [3].

1.0.7. The complex $B_{bar}^\bullet$ after Example 12 is defined as follows. Recall that $C_i^{bar} = A \underline{\otimes} A_+^{\otimes i}$. We set

$$B_{bar}^n = \bigoplus_{\substack{j_0 + \ldots + j_i = i+n \\ i \geq 0, j_0 \geq 0, j_1, \ldots, j_i \geq 1}} A_{j_0} \otimes \ldots \otimes A_{j_i}$$

Let $B_{bar}^{n,j}$ be its direct summand given by the condition $j_0 + \ldots + j_i = j$. The differential of C^{bar} gives by the same formula a differential on $B_{bar}^\bullet$, and $d(B_{bar}^{n,j}) \subset B_{bar}^{n+1,j}$. This is a complex $(B_{bar}^0 \to B_{bar}^1 \to \ldots)$. As far as I undestand, this complex has only H^0, and its $H^0 \tilde{\to} k$.

Main result (Prop. 18). Let $A = \oplus_{i \geq 0} A_i$ be a graded unital associative noetherian algebra, $A_0 = k$ a field, $\dim A_i < 0$. Then A is Koszul iff $K(A, k)$ is quasi-isomorphic to k.

References

[1] V. Ginzburg, Lectures on noncommutative geometry, arxix
[2] Keller, On differential graded categories
[3] U. Krähmer, Notes on Koszul algebras, http://www.maths.gla.ac.uk/ ukraehmer/connected.pdf
[4] S. MacLane, Homology, Springer, 1963