1. Comments to my joint paper with Dennis [13], Sept. 11, 2017

1.0.1. The notation DGCat from [13] corresponds to the category denoted DGCat_{cont} in ([15], ch. I.1, Sect. 10). The notation Vect is that of ([15], ch. I.1, Sect. 10.1, so Vect is both left and right complete for its t-structure. This is used, in particular, in Sect. 1.6.1, where it is claimed that $\mathrm{Shv} : (\mathrm{Sch}^{aff})^{op} \to \mathrm{DGCat}^{SymMon}$. Namely, for any $S \in \mathrm{Sch}^{aff}$, the unit of $\mathrm{Shv}(S)$ is the pull-back of the constant sheaf E under $S \to \operatorname{Spec} k$. Now indeed for $f : S_1 \to S_2$ in Sch^{aff} the functor $f^! : Shv(S_2) \to Shv(S_1)$ is symmetric monoidal.

1.0.2. The category $(\mathrm{Sch}^{aff}_{ft})^{op}$ admits finite colimits, so $\mathrm{Ind}((\mathrm{Sch}^{aff}_{ft})^{op}) \widetilde{\to} (\mathrm{Sch}^{aff})^{op}$ is presentable by ([18], 5.5.1.1).

The definition of $Shv : (PreStk)^{op} \to \mathrm{DGCat}$ should indeed be a right Kan extension of $(\mathrm{Sch}^{aff})^{op} \to \mathrm{DGCat}$ under $(\mathrm{Sch}^{aff})^{op} \hookrightarrow \mathcal{P}(\mathrm{Sch}^{aff})^{op}$. That is, if $\mathcal{Y} \in PreStk$ is written as any colimit $\operatorname{colim}_{i\in I} S_i$ in $\mathcal{P}(\mathrm{Sch}^{aff})$, where $S_i \in \mathrm{Sch}^{aff}$ then $Shv(\mathcal{Y}) \widetilde{\to} lim_{i\in I^{op}} Shv(S_i)$ in DGCat.

A prestack given by a functor $F : (\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$ is locally of finite type iff this functor preserves filtered colimits. Then it is completely defined by its restriction to $(\mathrm{Sch}^{aff}_{ft})^{op}$ by (HTT, 5.3.5.10).

1.0.3. For 1.2.1. We may take here S indeed only as a filtered limit of S_α in Sch^{aff}, because the functor $Shv : (\mathrm{Sch}^{aff}_{ft})^{op} \to DGCat$ maybe does not preserve finite colimits. See ([18], 5.5.1.9). Maybe $\mathrm{Shv} : (\mathrm{Sch}^{aff})^{op} \to \mathrm{DGCat}$ does not preserve all colimits.

1.0.4. For 1.2.1. The functor (1.2) inherits a right-lax symmetric monoidal structure by (HA, 4.8.1.10).

1.0.5. If $K, \mathcal{C} \in 1\!-\!\mathcal{C}at$, the relation between $\mathrm{Funct}(K, \mathbb{E}_n^{grp-like}(\mathcal{C}))$ and $\mathbb{E}_n^{grp-like}(\mathrm{Funct}(K, \mathcal{C}))$ is as follows. One has $Mon(Ptd(\mathcal{C})) \widetilde{\to} Mon(\mathcal{C})$ canonically. Clearly,

$$Mon(\mathrm{Fun}(K, \mathcal{C})) \widetilde{\to} \mathrm{Fun}(K, \mathrm{Mon}(\mathcal{C}))$$

So, for any $n \geq 1$, $\mathbb{E}_n(\mathrm{Fun}(K, \mathcal{C})) \widetilde{\to} \mathrm{Fun}(K, \mathbb{E}_n(\mathcal{C}))$. The full subcategory $\mathbb{E}_n^{grp-like}(\mathrm{Fun}(K, \mathcal{C})$ identifies via this isomorphism with $\mathrm{Fun}(K, \mathbb{E}_n^{grp-like}(\mathcal{C}))$ because of ([22], section label{`sec_Nick_equivalence_fiberwise`} and (HA, 5.2.6.2).

If $f : \mathcal{C} \to \mathcal{D}$ is a left-exact functor, and $\mathcal{C}, \mathcal{D}$ admits finite limits then f induces a functor $\mathbb{E}_n(\mathcal{C}) \to \mathbb{E}_n(\mathcal{D})$ for all $n \geq 0$, and also $\mathbb{E}_n^{grp-like}(\mathcal{C}) \to \mathbb{E}_n^{grp-like}(\mathcal{D})$.

1.0.6. For any ∞-topos $\mathcal{C}$, let $\mathcal{C}^0 \subset \mathcal{C}$ be the full subcategory of connected objects. Recall that $\Omega : \mathrm{Ptd}(\mathcal{C}^0) \widetilde{\to} \mathcal{G}\mathrm{rp}(\mathcal{C})$ is an equivalence ([18], 7.2.2.11). The functor $\Omega : \mathrm{Ptd}(\mathcal{C}) \to \mathcal{G}\mathrm{rp}(\mathcal{C})$ has a left adjoint B given by the composition $\mathcal{G}\mathrm{rp}(\mathcal{C}) \widetilde{\to} \mathrm{Ptd}(\mathcal{C}^0) \hookrightarrow \mathrm{Ptd}(\mathcal{C})$. For $G \in \mathcal{G}\mathrm{rp}(\mathcal{C})$ we have canonically $G \widetilde{\to} \Omega B(G)$, because G is a part of the Cech nerve of $* \to B(G)$.

Date: April 3, 2024

1.0.7. For 1.3.4. The category $\mathbb{E}_k^{grp-like}(\mathbf{C})$ is defined for $k \geq 1$. For example, we have a natural functor $\Omega : \mathrm{Mon}(\mathbf{C}) \to \mathcal{G}\mathrm{rp}(\mathrm{Mon}(\mathbf{C})) = \mathbb{E}_2^{grp-like}(\mathbf{C})$. If $\mathbf{C}$ is an ∞-topos, it has a left adjoint $B : \mathbb{E}_2^{grp-like}(\mathbf{C}) \to \mathrm{Mon}(\mathbf{C})$. It takes values in $\mathcal{G}\mathrm{rp}(\mathbf{C})$ because of ([19], Lm. 5.2.6.16). Namely, for $X \in \mathbb{E}_2^{grp-like}(\mathbf{C})$, $B(X)$ is connected, that is, 1-connective because of ([18], 7.2.2.11), now $B(X)$ is grouplike by ([19], Lm. 5.2.6.16).

Assume $\mathbf{C}$ is an ∞-topos. Since the colimits in $\mathbf{C}$ are universal, the forgerful functor $\mathrm{Mon}(\mathbf{C}) \to \mathbf{C}$ preserves sifted colimits by (HA, 3.2.3.2). So, given $G \in \mathcal{G}\mathrm{rp}(\mathrm{Mon}(\mathbf{C}))$, $B(G)$ can be calculated either as a geometric realization of the diagram

$$[\dots G \times G \substack{\to\\ \to\\ \to} G \rightrightarrows *]$$

in $\mathrm{Mon}(\mathcal{C})$ or in $\mathcal{C}$. So, $B(G)$ is connected by ([18], 7.2.2.11).

1.0.8. Explanation for ([13], 1.3.5) coming from ([1], Appendix E). Consider the cocartesian fibration $f : \mathcal{X} \to 1-\mathcal{C}\mathrm{at}$ corresponding to $\mathrm{id} : 1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$. So, $\mathcal{X}$ classifies $C \in 1-\mathcal{C}\mathrm{at}$ and $c \in C$. A morphism in $\mathcal{X}$ from (C, c) to (C', c') is, roughly, a pair (f, g), where $f : C \to C'$ is a functor and $g : f(c) \to c'$ is a morphism in C'. Then $\mathcal{X}$ is a symmetric monoidal category with the product $(C_1, c_1), (C_2, c_2) \mapsto (C_1 \times C_2, c_1 \times c_2)$. The unit of $\mathcal{X}$ is given by $(C = *, *)$. Then f is a monoidal functor.

Write $\mathcal{X}' \subset \mathcal{X}$ for the 1-full subcategory, where we keep all objects, and only morphisms cocartesian over $1-\mathcal{C}\mathrm{at}$. So, $\mathcal{X}' \to \mathcal{X}$ is a cocartesian fibration in spaces. Then $\mathcal{X}'$ is a symmetric monoidal category.

Recall that $\mathrm{Mon}(\mathrm{Spc})$ is a symmetric monoidal category ([15], ch. I.1, 3.3.5). Now we have a symmetric monoidal functor $\mathcal{F} : \mathcal{X}' \to \mathrm{Mon}(\mathrm{Spc})$, $(C, c) \mapsto \mathrm{Map}_C(c, c)$, here Mon is the category of monoids in Spc.

The fact that the above functor is symmetric monoidal is expressed as follows: given $(C, c), (D, d) \in \mathcal{X}$ one has naturally

$$\mathrm{Map}_C(c, c) \times \mathrm{Map}_D(d, d) \widetilde{\to} \mathrm{Map}_{C \times D}((c, d), (c, d)),$$

and $\mathrm{Map}_*(*, *) \widetilde{\to} *$ in Spc.

Now if $A \in \mathrm{Mon}(1-\mathcal{C}\mathrm{at})$ is a monoidal category then $(A, 1_A) \in \mathrm{Mon}(\mathcal{X}')$ with the product $(A \times A, 1 \times 1) \to (A, 1)$ given by the multiplication $m : A \times A \to A$. So, $\mathcal{F}(A, 1_A) = \mathrm{Map}_A(1, 1)$ becomes a monoid in $\mathrm{Mon}(\mathrm{Spc})$. Thus, $\mathrm{Map}_A(1, 1) \in \mathbb{E}_2(\mathrm{Spc})$.

Unwinding the definition, the interiour product on $\mathrm{Map}_A(1, 1)$ is given by the composition in A sending $f_1 : 1 \to 1, f_2 : 1 \to 1$ to $f_1 \circ f_2$. The exteriour product in $\mathrm{Map}_A(1, 1)$ is defined as the composition

$$\mathrm{Map}_A(1, 1) \times \mathrm{Map}_A(1, 1) \widetilde{\to} \mathrm{Map}_{A \times A}(1 \times 1, 1 \times 1) \xrightarrow{m} \mathrm{Map}_A(1, 1),$$

here the first isomorphism is given by the right-lax monoidal structure on $\mathcal{F}$, and the second one is the morphism of Map-spaces for the functor $m : A \times A \to A$. In other words, the exteriour product in $\mathrm{Map}_A(1, 1)$ sends $(f_1, f_2) \in \mathrm{Map}_A(1, 1) \times \mathrm{Map}_A(1, 1)$ to $f_1 \otimes f_2$.

Let $A \in \mathbb{E}_2^{grp-like}(\mathrm{Spc})$, so $B(A) \in \mathcal{G}\mathrm{rp}(\mathrm{Spc})$. A datum of $\tilde{C} \in 1-\mathcal{C}\mathrm{at}$ together with $\tilde{C} \to B^2(A)$ gives a $B(A)$-action on $C := * \times_{B^2(A)} \tilde{C}$. It is given by a morphism $\alpha : B(A) \to \mathrm{Funct}(C, C) =: \mathbf{O}$ in $\mathrm{Mon}(1-\mathcal{C}\mathrm{at})$. In particular, $* \to B(A) \to \mathrm{Funct}(C, C)$ is the identity functor. Passing to the Map-spaces from $1 \in B(A)$ to $1 \in B(A)$ in $B(A)$, the functor α yields a morphism

$$\bar{\alpha} : A \widetilde{\to} \mathrm{Map}_{B(A)}(1,1) \to \mathrm{Map}_{\mathbf{O}}(1_{\mathbf{O}}, 1_{\mathbf{O}})$$

Since $B(A) \in \mathrm{Mon}(1-\mathcal{C}at)$, $\mathrm{Map}_{B(A)}(1,1) \in \mathbb{E}_2(\mathrm{Spc})$ by the above construction, and $\bar{\alpha}$ is a morphism of $\mathbb{E}_2$-objects by functoriality. (Since α is a functor, $\bar{\alpha}$ preserves the compositions, that is, respects the interiour products). In fact, $\bar{\alpha}$ takes values in $\mathrm{Map}_{\mathbf{O}^{\mathrm{Spc}}}(1_{\mathbf{O}}, 1_{\mathbf{O}}) \in \mathbb{E}_2^{grp-like}(\mathrm{Spc})$, which is a full subspace in $\mathrm{Map}_{\mathbf{O}}(1_{\mathbf{O}}, 1_{\mathbf{O}})$.

Remark: if $A, B \in \mathrm{Mon}(1-\mathcal{C}at)$ and $\alpha : A \to B$ is a morphism in $\mathrm{Mon}(1-\mathcal{C}at)$ then the induced map $\mathrm{Map}_A(1_A, 1_A) \to \mathrm{Map}_B(1_B, 1_B)$ is a morphism in $\mathbb{E}_2(\mathrm{Spc})$ by functoriality.

We claim actually here the following. Let $A \in \mathbb{E}_2^{grp-like}(\mathrm{Spc})$, $C \in 1-\mathcal{C}at$, $\mathcal{O} = \mathrm{Funct}(C, C)$. Then to give a morphism $B(A) \to \mathcal{O}$ in $\mathrm{Mon}(1-\mathcal{C}at)$ is the same as to give a morphism $A \to \mathrm{Map}_{\mathcal{O}}(1,1)$ in $\mathbb{E}_2(\mathrm{Spc})$, equivalently, a morphism $A \to \mathrm{Map}_{\mathcal{O}^{\mathrm{Spc}}}(1,1)$ in $\mathbb{E}_2^{grp-like}(\mathrm{Spc})$.

Given $A \to \mathrm{Map}_{\mathcal{O}}(1,1)$ in $\mathbb{E}_2(\mathrm{Spc})$, how to get $B(A) \to \mathcal{O}$ in $\mathrm{Mon}(1-\mathcal{C}at)$? I think as follows. First, $\mathrm{Mon}(1-\mathcal{C}at) \subset 1-\mathcal{C}at$ is stable under small limits, and $\mathrm{Mon}(1-\mathcal{C}at)$ admits all small limits. Besides, $\mathrm{Ptd}(\mathrm{Mon}(1-\mathcal{C}at)) \widetilde{\to} \mathrm{Mon}(1-\mathcal{C}at)$, see HA. Consider the functor $\Omega : \mathrm{Mon}(1-\mathcal{C}at) \to \mathcal{G}rp(\mathrm{Mon}(1-\mathcal{C}at)) = \mathbb{E}_2^{grp-like}(1-\mathcal{C}at)$. The existence of left adjoint to this functor is not clear, as $1-\mathcal{C}at$ is not a topos (are both categories presentable?). Instead, we do the following.

If $\mathcal{O} \in \mathrm{Mon}(1-\mathcal{C}at)$ then $\mathcal{O}^{\mathrm{Spc}} \in \mathrm{Mon}(\mathrm{Spc})$, so $\Omega(\mathcal{O}^{\mathrm{Spc}}) \in \mathbb{E}_2^{grp-like}(\mathrm{Spc})$ and $\mathrm{Map}_{\mathcal{O}^{\mathrm{Spc}}}(1_{\mathcal{O}}, 1_{\mathcal{O}}) \in \mathbb{E}_2^{grp-like}(\mathrm{Spc})$. We have canonically

$$\mathrm{Map}_{\mathcal{O}^{\mathrm{Spc}}}(1_{\mathcal{O}}, 1_{\mathcal{O}}) \widetilde{\to} \Omega(\mathcal{O}^{\mathrm{Spc}})$$

in $\mathbb{E}_2^{grp-like}(\mathrm{Spc})$. By adjointness in $B : \mathbb{E}_2^{grp-like}(\mathrm{Spc}) \leftrightarrows \mathcal{G}rp(\mathrm{Spc}) : \Omega$, it yields a morphism $B(\mathrm{Map}_{\mathcal{O}^{\mathrm{Spc}}}(1_{\mathcal{O}}, 1_{\mathcal{O}})) \to \mathcal{O}^{\mathrm{Spc}}$ in $\mathcal{G}rp(\mathrm{Spc})$. This is also a morphism in $\mathrm{Mon}(1-\mathcal{C}at)$, then compose with $\mathcal{O}^{\mathrm{Spc}} \to \mathcal{O}$.

1.1. If $\mathcal{Y}$ is a prestack, $\mathcal{A}$ is a commutative group object in $PreStk_{/\mathcal{Y}}$ then by

$$\mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}; B^i_{et/\mathcal{Y}}(\mathcal{A}))$$

we mean the mapping space in $PreStk_{/\mathcal{Y}}$. In particular, if A is a torsion abelain group, $\mathrm{Map}(\mathcal{Y}, B^i_{et}(A))$ denotes the mapping space in $PreStk$.

Notation throughout, I think: let $\mathcal{Y}$ be a prestack, $\mathcal{A}$ a group like $\mathbb{E}_n$-object in the category $\mathrm{PreStk}/\mathcal{Y}$. Then we have the functors $B^i : \mathbb{E}_n^{grp-like}(\mathrm{PreStk}/\mathcal{Y}) \to \mathbb{E}_{n-i}^{grp-like}(\mathrm{PreStk}/\mathcal{Y})$ for $0 \le i \le n$ defined as in Section 1.3.4 for the category $\mathcal{C} = \mathrm{PreStk}/\mathcal{Y}$. So, $B^i(\mathcal{A})$ always has this meaning.

1.1.1. Explanation for 1.4.3. Let $\mathrm{Stk} \subset \mathrm{PreStk}$ be the full subcategory of stack for the etale topology. This inclusion is stable under all small limits. Recall that its left adjoint is accessible and left exact (topological localization) functor $L : \mathrm{PreStk} \to \mathrm{Stk}$. By ([18], 7.2.2.5), L induces a functor $\mathcal{G}rp(\mathrm{PreStk}) \to \mathcal{G}rp(\mathrm{Stk})$. If $G \in \mathcal{G}rp(\mathrm{PreStk})$ then $\Omega B_{et}(G) \widetilde{\to} G_{et}$.

Let A be a torsion abelian group, write A_{et} for the sheafification of A on Sch^{aff}. Then $B^i_{et}(A)$ is the i-th delooping of A_{et} in the topos Stk.

We have $\Omega \operatorname{Map}(Y,Z) \widetilde{\to} \operatorname{Map}(Y, \Omega Z)$ for any $Y, Z \in$ PreStk. For $j \le i$ we get $\Omega^j B^i_{et}(A) \widetilde{\to} B^{i-j}_{et}(A)$, and the claim in this case is $\pi_0 \operatorname{Map}(\mathcal{Y}, B^r_{et}(A)) \widetilde{\to} \mathrm{H}^r_{et}(\mathcal{Y}, A)$ for $r \ge 0$.

For $j > i$ we get the following. Recall that for any $\mathcal{D} \in 1-\mathcal{C}\mathrm{at}$, ${}_{\tau \le k}\mathcal{D} \subset \mathcal{D}$ is stable under all limits that exist in $\mathcal{D}$. So, $\Omega(A,1)$ can be calculated in $\tau_{\le 0}\,\mathrm{Spc} = Sets$. We get $\Omega^i(A,1) = *$ for $i > 0$. For this reason, $\Omega^j B^i_{et}(A) \widetilde{\to} *$ is the final object in the category Stk, and the corresponding π_j is zero.

1.2. In fact, the functor $Shv^! : (\mathrm{Sch})^{op} \to 1-\mathcal{C}\mathrm{at}$ takes values in presentable stable cocomplete ∞-categories. Consider the "context of constructible sheaves" as in [8]. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks. By ([8], Cor. 1.4.2), $f_! : Shv^!(\mathcal{Y}_1) \to Shv^!(\mathcal{Y}_2)$ is always defined. Let $\mathcal{Y}$ be a prestack.

Consider the category $\mathrm{Sch}/\mathcal{Y}$. We have a functor $(\mathrm{Sch}/\mathcal{Y})^{op} \to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, $(S \to \mathcal{Y}) \mapsto Shv(S)$, and

$$Shv^!(\mathcal{Y}) = \lim_{S \in (\mathrm{Sch}/\mathcal{Y})^{op}} Shv(S)$$

For each map $\alpha : S_1 \to S_2$ in $\mathrm{Sch}/\mathcal{Y}$ we have the left adjoint $\alpha_! : Shv(S_1) \to Shv(S_2)$ to $\alpha^! : Shv(S_2) \to Shv(S_1)$. Let $\tilde{\mathcal{Y}} \to \mathrm{Sch}/\mathcal{Y}$ be the cartesian fibration corresponding to the above functor $(\mathrm{Sch}/\mathcal{Y})^{op} \to 1-\mathcal{C}\mathrm{at}$. Then it is bicartesian, so we get the functor $Shv_! : \mathrm{Sch}/\mathcal{Y} \to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, $(S \to \mathcal{Y}) \mapsto Shv(S)$.

Let $(\mathrm{Sch}/\mathcal{Y})^{\triangleright}$ be obtained from $\mathrm{Sch}/\mathcal{Y}$ by adjoining a final object. Consider an extension $Shv_!^{\triangleright} : (\mathrm{Sch}/\mathcal{Y})^{\triangleright} \to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ of $Shv_!$, which is a colimit diagram for $Shv_!$. The opposite to $(\mathrm{Sch}/\mathcal{Y})^{\triangleright}$ is the category $((\mathrm{Sch}/\mathcal{Y})^{op})^{\triangleleft}$ obtained by adjoining an initial object to $(\mathrm{Sch}/\mathcal{Y})^{op}$.

Passing to right adjoints in $Shv_!^{\triangleright}$, we get a functor

$$(Shv^!)^{\triangleleft} : ((\mathrm{Sch}/\mathcal{Y})^{op})^{\triangleleft} \to 1-\mathcal{C}\mathrm{at}$$

extending $Shv^!$. By ([15], Ch. I.1, 2.5.7), this is a limit diagram. That is, the corresponding map

$$\operatorname*{colim}_{S \in \mathrm{Sch}/\mathcal{Y}} Shv(S) \to \lim_{S \in \mathrm{Sch}/\mathcal{Y}} Shv(S)$$

is an equivalence (alternatively, use [15], Ch. I.1, 5.3.4). Note that for $\alpha : S_1 \to S_2$ a morphism of schemes both $\alpha_!$ and $\alpha^!$ preserve colimits. So, the limit in the above formula could be taken in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ or $1-\mathcal{C}\mathrm{at}_{Prs}$ or $1-\mathcal{C}\mathrm{at}$ by ([15], 2.5.2(b)).

Let now $A \to Shv^!(\mathcal{Y})$ be a functor, $a \mapsto \mathcal{F}^a$. For each $(S, y) \in \mathrm{Sch}/\mathcal{Y}$, here $y : S \to \mathcal{Y}$ we get the functor $A \to Shv(S)$, $a \mapsto y^!\mathcal{F}^a$. If $\mathcal{F}_y := \lim_{a \in A} y^!\mathcal{F}^a$ exists for any $(S, y) \in \mathrm{Sch}/\mathcal{Y}$, and for any $\alpha : S_1 \to S_2$ in $\mathrm{Sch}/\mathcal{Y}$ the natural map $\alpha^!\mathcal{F}_{y_2} \to \mathcal{F}_{y_1}$ is an isomorphism then $\lim_{a \in A} \mathcal{F}^a$ exists in $Shv^!(\mathcal{Y})$, and the natural map $y^!(\lim_{a \in A} \mathcal{F}^a) \to \mathcal{F}_y$ is an isomorphism by ([15], Ch I.1, 2.6.2). This is ([8], Lemma 1.3.5).

If $\beta : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a morphism of prestacks, $\mathcal{F} \in Shv^!(\mathcal{Y}_1)$ then for $\beta_!\mathcal{F}$ we have some formulas as colimits. For example, let $A \to PreStk$, $a \mapsto S_a$ be a functor that factors through $\mathrm{Sch} \hookrightarrow PreStk$ and $\mathcal{Y}_1 \widetilde{\to} \operatorname{colim}_{a \in A} S_a$ with colimit taken in $PreStk$. Then for the contructible context $Shv^!(\mathcal{Y})\widetilde{\to}\operatorname{colim}_{a\in A} Shv(S_a)$, and $\beta_!\mathcal{F}\widetilde{\to}\operatorname{colim}(\beta\alpha^a)_!(\alpha^a)^!\mathcal{F}$, here $\alpha^a: S_a\to\mathcal{Y}_1$ is the natural map. See ([8], 0.8.5).

1.2.1. Let A be a torsion abelian group. If S is a smooth scheme of dimension n, $Y\subset S$ is closed of codimension ≥ 2 then $\operatorname{Map}(S, B^2_{et}(A))\to\operatorname{Map}(S-Y, B^2_{et}(A))$ is an isomorphism.

1.2.2. For Sect. 1.5.1, just to note. We could consider $A\langle -1\rangle := \lim_n \operatorname{Hom}(\mu_n, A)$ over n invertible in k, where for $n\mid n'$ the transition map $\operatorname{Hom}(\mu_{n'}, A)\to\operatorname{Hom}(\mu_n, A)$ is the composition with the inclusion $\mu_n\hookrightarrow\mu_{n'}$.

In other words, $A\langle -1\rangle$ is defined by the isomorphisms

$$\operatorname{Hom}(B(1), A)\widetilde{\to}\operatorname{Hom}(B, A\langle -1\rangle)$$

functorial in an abelian group B. If A is an N-torsion group then $A\langle -1\rangle$ vanishes, but $A(-1)$ does not. So, this is a different thing!

1.2.3. Let $\mathcal{C}\in 1-\mathcal{C}at$ admit finite products. Write $\operatorname{Mon}(\mathcal{C})$ for the ∞-category of monoids in $\mathcal{C}$, $\operatorname{Mon}^+(\mathcal{C})$ for the category of left modules over a monoid in $\mathcal{C}$, so $\operatorname{Mon}^+(\mathcal{C})\subset\operatorname{Funct}(\boldsymbol{\Delta}^{+,op},\mathcal{C})$ is a full subcategory. We have the forgetfull functor $\operatorname{Mon}^+(\mathcal{C})\to\operatorname{Mon}(\mathcal{C})$. If we are given $\mathcal{A}\in 1-\mathcal{C}at$ and a functor $F:\mathcal{A}^{op}\to\operatorname{Mon}(\mathcal{C})$, we may think of it as a presheaf of monoids in $\mathcal{C}$. Then a lifting of F to a functor $F^+:\mathcal{A}^{op}\to\operatorname{Mon}^+(\mathcal{C})$ can be thought of as a presheaf of left modules over the corresponding presheaf of algebras.

This is used in ([13], 1.6.2). Namely, the functor (1.2) can be seen as a functor $Shv: (\mathrm{Sch}^{aff})^{op}\to\operatorname{Mon}(\mathrm{DGCat})$. We have a projection $\mathrm{Sch}^{aff}/\mathcal{Y}\to\mathrm{Sch}^{aff}$. A presheaf of DG-categories on $\mathrm{Sch}^{aff}/\mathcal{Y}$ is a lifting of the composition

$$(\mathrm{Sch}^{aff}/\mathcal{Y})^{op}\to(\mathrm{Sch}^{aff})^{op}\to\operatorname{Mon}(\mathrm{DGCat})$$

to a functor $(\mathrm{Sch}^{aff}/\mathcal{Y})^{op}\to\operatorname{Mon}^+(\mathrm{DGCat})$.

Let $\mathcal{C}$ be a presheaf of categories over $\mathcal{Y}\in\mathrm{PreStk}$. Note that for a map $f:S_1\to S_2$ in Sch^{aff} and $y_2: S_2\to\mathcal{Y}$ with $y_1=y_2 f$ the diagram commutes

$$\begin{array}{ccc} Shv(S_2)\times\mathcal{C}(S_2,y_2) & \to & \mathcal{C}(S_2,y_2)\\ \downarrow & & \downarrow\\ Shv(S_1)\times\mathcal{C}(S_1,y_1) & \to & \mathcal{C}(S_1,y_1)\end{array}$$

This is the sense of: (1.9) interwines the actions.

Def. of a sheaf of categories in 1.6.6 makes sense, because DGCat contains all colimits (this is the category of modules over some algebra in $1-\mathcal{C}at^{St,cocmpl}_{cont}$).

1.2.4. The category of étale sheaves $\mathrm{Stk}\subset\mathrm{PreStk}$ is a topos, so Stk is presentable, in particular, contains all small colimits and limits ([18], 5.5.2.4). The inclusion $\mathrm{Stk}\hookrightarrow\mathrm{PreStk}$ does not preserve colimits. Indeed, let $S\mapsto S_{shf}$ be the sheafification functor, the left adjoint to the above inclusion. By ([18], 5.2.7.5), given an functor $f:K\to\mathrm{Stk}$, the colimit of f is S_{shf}, where S is the colimit of f in PreStk.

The category Disc(Stk) of discrete objects of Stk is the category of sheaves of sets on Sch^{aff} with respect to the étale topology. Let A be a group objects in Disc(Stk).

We could directly construct $B(A)\in Stk$ using ([18], 7.2.2.12). If A is a commutative group object of $\mathrm{Disc}(Stk)$, we could similarly construct $B^i(A)\in \mathrm{Stk}$ for all $i\geq 0$.

Dennis procedes differently. He considers the topos

$$\mathcal{P}(\mathrm{Sch}^{aff})=\mathrm{Funct}((\mathrm{Sch}^{aff})^{op},\mathrm{Spc})$$

Then $\mathrm{Disc}(\mathcal{P}(\mathrm{Sch}^{aff}))$ is the category of presheaves of sets on Sch^{aff}. Given a group object A in $\mathrm{Disc}(\mathcal{P}(\mathrm{Sch}^{aff}))$, we get the Eilenberg-MacLane object $B(A)\in\mathcal{P}(\mathrm{Sch}^{aff})$ via ([18], 7.2.2.12). Namely, A can be seen as a functor $\mathcal{A}:\boldsymbol{\Delta}^{op}\to\mathcal{P}(\mathrm{Sch}^{aff})$ actually taking values in the full subcategory $\mathrm{Disc}(\mathcal{P}(\mathrm{Sch}^{aff}))\subset\mathcal{P}(\mathrm{Sch}^{aff})$. Then $\mathcal{A}$ extends to a colimit diagram $\mathcal{A}_+:(\boldsymbol{\Delta}_+)^{op}\to\mathcal{P}(\mathrm{Sch}^{aff})$ of its restriction to $\boldsymbol{\Delta}^{op}$. Then $B(A)=\operatorname{colim}_{\boldsymbol{\Delta}^{op}}\mathcal{A}$ calculated in $\mathcal{P}(\mathrm{Sch}^{aff})$.

In general, if $K,S,\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$ and $\mathcal{C}$ admits K-indexed colimits then $\mathrm{Funct}(S,\mathcal{C})$ admit K-indexed colimits, and they are computed pointwise ([18], 5.1.2.3). Therefore, for each $S\in\mathrm{Sch}^{aff}$, we have a functor $\mathcal{A}(S):\boldsymbol{\Delta}^{op}\to\mathrm{Spc}$, and the value $B(A)(S)=\operatorname{colim}_{\boldsymbol{\Delta}^{op}}\mathcal{A}(S)$.

If we assume in addition that A is a sheaf for étale topology then $A\in\mathrm{Disc}(\mathrm{Stk})$, and the functor $\mathcal{A}$ factors as $\boldsymbol{\Delta}^{op}\to Stk\to\mathcal{P}(\mathrm{Sch}^{aff})$. By the above, $B_{et}(A):=\operatorname{colim}_{\boldsymbol{\Delta}^{op}}\mathcal{A}$ in Stk is calculated as the sheafification of the colimit $B(A):=\operatorname{colim}_{\boldsymbol{\Delta}^{op}}\mathcal{A}$ in $\mathcal{P}(\mathrm{Sch}^{aff})$.

We have $B(A)(S)=\operatorname{colim}_{\boldsymbol{\Delta}^{op}}\mathcal{A}(S)$ (colimit in Spc) for any $S\in\mathrm{Sch}^{aff}$. I don't know if the natural map $B(A)\to B_{et}(A)$ is an isomorphism in PreStk.

Now assuming that A is an abelian group object in $\mathrm{Disc}(\mathcal{P}(\mathrm{Sch}^{aff}))$, on the pointed object $*\to B(A)$ we get a structure of a group object in the category $Ptd(\mathrm{PreStk})$. Indeed, recall that for a usual category $\mathcal{C}$, $\mathcal{G}\mathrm{rp}(\mathcal{G}\mathrm{rp}(\mathcal{C}))\widetilde{\to}Ab(\mathcal{C})$, here $\mathcal{G}\mathrm{rp}(\mathcal{C})$ is the category of group objects, $Ab(\mathcal{C})$ is the category of abelian group objects ([18], 7.2.2.12). We get a functor $\mathcal{A}^1:\boldsymbol{\Delta}^{op}\to\mathcal{P}(\mathrm{Sch}^{aff})$ roughly given by a diagram

$$*\leftarrow B(A)\leftleftarrows B(A)\times B(A)\ \substack{\leftarrow\\ \leftarrow\\ \leftarrow}\ \dots$$

Then $B^2(A)=\operatorname{colim}_{\boldsymbol{\Delta}^{op}}\mathcal{A}^1$, the colimit is taken in $\mathcal{P}(\mathrm{Sch}^{aff})$. Then again $*\to B^2(A)$ is a group object in $Ptd(\mathrm{PreStk})$, and we continue the procedure. We get the functor $\mathcal{A}^2:\boldsymbol{\Delta}^{op}\to\mathcal{P}(\mathrm{Sch}^{aff})$ given by a diagram

$$*\leftarrow B^2(A)\leftleftarrows B^2(A)\times B^2(A)\ \substack{\leftarrow\\ \leftarrow\\ \leftarrow}\ \dots$$

and $B^3(A)=\operatorname{colim}_{\boldsymbol{\Delta}^{op}}\mathcal{A}^2$, the colimit is taken in $\mathcal{P}(\mathrm{Sch}^{aff})$. And so on.

1.2.5. Let $f:S\to Z$ be an étale surjective map of schemes. Let $S^\bullet:\boldsymbol{\Delta}^{op}\to\mathrm{Sch}$ be the groupoid underlying the corresponding Cech nerve. Let A be a torsion abelain group, assume the orders of elements in A are prime to the characteristic of k. Let $Y\to Z$ be a A-gerb whose restriction to S is trivial. By definition, $B_{et}(A):(\mathrm{Sch}^{aff})^{op}\to\mathrm{Spc}$ is a group prestack (actually, stack for étale topology).

What is the data on S that allows to recover Y? Our A-gerbe is a map $Z\to B^2_{et}(A)$. So, the answer is given by the sheaf condition: the map $\mathrm{Map}(Z,B^2_{et}(A))\to Tot(\mathcal{Y}(S^\bullet/Z))$ is an isomorphism, where $\mathcal{Y}(S')=\mathrm{Map}(S',B^2_{et}(A))$, and $S^\bullet/Z$ is the Cech nerve of $S\to Z$.

1.2.6. Recall that for usual category $\mathcal{C}$, $Grp(\mathrm{Ab}(\mathcal{C}))\widetilde{\to}\mathrm{Ab}(\mathcal{C})$ canonically. If A is a commutative group object in $Sets$, view A as discrete object in $Grp(\mathrm{\mathcal{C}omGrp}(\mathrm{PreStk}))$. We get a functor $\boldsymbol{\Delta}^{op}\to\mathrm{\mathcal{C}omGrp}(\mathrm{PreStk})$. Take the colimit of the latter functor, we get $B(A)\in\mathrm{\mathcal{C}omGrp}(\mathrm{PreStk})$. Is this the usual way to see that $B(A)$ is a group like object of PreStk?

1.2.7. For A a torsion abelian group and $i\geq 1$, $\Omega B^i_{et}(A)\widetilde{\to}B^{i-1}_{et}(A)$, here the functor $\Omega:\mathrm{Stk}\to\mathrm{Stk}$ is the loop functor in the ∞-topos Stk. This is true for any topos, and is explained in Section 1.0.6.

Why $B^i_{et}(A)$ is an Eilenberg-MacLane object of degree i in Stk? This is because A is an Eilenberg-MacLane object in degree 0 in Stk, now apply ([18], 7.2.2.11) several times.

1.2.8. For 1.4.3. Let A be a finite torsion abelian group. Consider the functor $(\mathrm{Sch}^{aff})^{op}\to\mathrm{PreStk}^{op}\to\mathrm{Spc}$, where the second arrow sends Y to $\mathrm{Map}(Y,B^i_{et}(A))$. Why this functor is the left Kan extension from $(\mathrm{Sch}^{aff}_{ft})^{op}$? Let J be a small filtered category, $p:J\to(\mathrm{Sch}^{aff}_{ft})^{op}$ a diagram $j\mapsto S_j$, whose colimit in $(\mathrm{Sch}^{aff})^{op}$ is S. We have to show that

$$\mathrm{Map}_{\mathrm{PreStk}}(\lim_{j\in J^{op}}S_j,B^i_{et}(A))\widetilde{\to}\mathrm{colim}_{j\in J}\mathrm{Map}(S_j,B^i_{et}(A))$$

1.2.9. Let $1\to A\to H\to G\to 1$ is a central extension of groups in an ∞-topos $\mathcal{X}$, so $A\in ComGrp(\mathcal{X})$. It yields a morphism $G\to B(A)$ in $\mathcal{G}\mathrm{rp}(\mathcal{X})$. Indeed, according to ([13], 1.3.2), such a map is given by a A-torsor on G, namely $H\to G$ is equipped with an ∞-action of A on H such that $H/A\widetilde{\to}G$, hence the desired map $G\to B(A)$.

Actually, $B(A)$ is a commutative group object in $\mathcal{X}$, because A was a commutative group object. Applying B, we get a morphism $B(G)\to B^2(A)$ in $\mathcal{G}\mathrm{rp}(\mathcal{X})$.

This can be used to explain our construction of the gerbe $\mathcal{L}^a$ in ([13], 1.5.2). Namely, the central extension $1\to\mu_n\to\mathbb{G}_m\to\mathbb{G}_m\to 1$ in PreStk yields $B(\mathbb{G}_m)\to B^2(A)$.

Another way to say, the object $\mathcal{L}^{\frac{1}{n}}\in\mathrm{PreStk}/\mathcal{Y}$ defined in 1.5.2 is equipped with an action of $B_{et}(\mu_n)$, and $\mathcal{L}^{\frac{1}{n}}/B_{et}(\mu_n)\widetilde{\to}\mathcal{Y}$, hence the desired map $\mathcal{Y}\to B^2_{et}(\mu_n)$.

1.2.10. For 1.5.4. Here A is a torsion abelian group. Recall that $B^2(A)\in\mathrm{\mathcal{C}omGrp}(\mathrm{Spc})$, $B^2_{et}(A)\in\mathrm{\mathcal{C}omGrp}(\mathrm{PreStk})$. Now for a collection of gerbes $f_i:Y\to B^2_{et}(A)$ we denote by $\otimes f_i:Y\to B^2_{et}(A)$ the composition $\boxtimes f_i:Y\to\prod_i B^2_{et}(A)\to B^2_{et}(A)$, where the last map is the multiplication.

1.2.11. For 1.5.5. If $f:X\to Y$ is a map in Spc, $x\in X, y=f(x)$ and $X_y=X\times_Y y$ then we have a long exact sequence of groups (at the end of pointed sets)

$$\pi_n(X_y,x)\to\pi_n(X,x)\to\pi_n(Y,y)\to\pi_{n-1}(X_y,x)\to\ldots\to\pi_0(X_y)\to\pi_0(X)\to\pi_0(Y)$$

So, for the space $\mathcal{X}:=Ge_A(Y)\times_{Ge_A(Y-Z)}*$ we get the above long exact sequence. It shows essentially that the complex $\mathrm{R}\Gamma(Z,i^!A)$ controls the homotopy groups of $\mathcal{X}$. Namely, we should have $\pi_i(\mathcal{X})\widetilde{\to}\mathrm{H}^{2-i}(Z,i^!A)$ for $0\leq i\leq 2$, by H^j we understand the etale cohomology. So, if $\dim Y=n$ then we need to understand $\mathrm{R}\Gamma_c(Z,A)[2n](n)$ in degrees $[-2,0]$. Since $\dim Z=n-1$, the latter complex is placed in degrees ≤ -2, and its cohomology in degree -2 is $\mathrm{Map}(I,A(1))$. Dualizing, we get $\mathrm{H}^2(Z,i^!A)\widetilde{\to}\mathrm{Map}(I,A(-1))$.

1.2.12. ([13], 1.6.8) is true because the colimit in a topos are universal.

1.2.13. I have to learn the following (to be checked as found on internet): the endomorphisms of the unit object in an $\mathbb{E}_n$-monoidal category $\mathcal{C}$ naturally form an $\mathbb{E}_{n+1}$-monoidal category. These kind of questions seems to be studied in ([19], 5.3).

Let $\mathcal{C} \in 1-\mathcal{C}at$ be symmetric monoidal, $1 \in \mathcal{C}$ be the unit object. Then $\mathrm{Map}_{\mathcal{C}}(1,1)$ is naturally a $\mathbb{E}_2$-object of Spc, that is, lies in $\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Spc})$. Indeed, let $\mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be the corresponding cocartesian fibration. Given $x_i, y_i, z_i \in \mathcal{C}$ for $i = 1, 2$ and maps $x_i \xrightarrow{f_i} y_i \xrightarrow{g_i} z_i$, we get $x_1 \oplus x_2 \in \mathcal{C}^{\otimes}_2$ with a cocartesian arrow $x_1 \oplus x_2 \to x_1 \otimes x_2$ over $\alpha : \langle 2\rangle \to \langle 1\rangle$ active, similarly for y and z. Consider the commutative diagram

$$\begin{array}{ccc}
x_1 \oplus x_2 & \to & x_1 \otimes x_2 \\
\downarrow{\scriptstyle f_1 \oplus f_2} & & \downarrow{\scriptstyle f_1 \otimes f_2} \\
y_1 \oplus y_2 & \to & y_1 \otimes y_2 \\
\downarrow{\scriptstyle g_1 \oplus g_2} & & \downarrow{\scriptstyle g_1 \otimes g_2} \\
z_1 \oplus z_2 & \to & z_1 \otimes z_2,
\end{array}$$

where the horizontal arrows are cocartesian maps in $\mathcal{C}^{\otimes}$ over $\langle 2\rangle \to \langle 1\rangle$. The composition in the left column is $(g_1 f_1) \oplus (g_2 f_2)$, this yields an isomorphism

$$(g_1 f_1) \otimes (g_2 f_2) \widetilde{\to} (g_1 \otimes g_2)(f_1 \otimes f_2)$$

This means that the two operations $\otimes$ and the composition on $\mathrm{Map}_{\mathcal{C}}(1,1)$ are compatible.

1.2.14. Let $\mathcal{Y} \in \mathrm{PreStk}, \mathcal{A}$ be a grouplike $\mathbb{E}_2$-object in $\mathrm{PreStk}/\mathcal{Y}$. Then we have $B^2_{et}(\mathcal{A}) \to \mathcal{Y}$ a pointed object in $\mathrm{PreStk}/\mathcal{Y}$. Let $v : \mathcal{Y} \to B^2_{et}(\mathcal{A})$ be the distinguished point, a map in $\mathrm{PreStk}/\mathcal{Y}$. The square is cartesian

$$\begin{array}{ccc}
B^2_{et}(\mathcal{A}) & \to & B^2_{et}(\mathcal{A}) \times B^2_{et}(\mathcal{A}) \\
\uparrow & & \uparrow{\scriptstyle v \times v} \\
B_{et}(\mathcal{A}) & \to & \mathcal{Y}
\end{array}$$

Indeed, $B^2_{et}(\mathcal{A})$ is obtained from $\mathcal{A}_{et}$ by applying the delooping functor $B : \mathbb{E}_m^{grp-like}(\mathrm{Stk}/\mathcal{Y}) \to \mathbb{E}_{m-1}^{grp-like}(\mathrm{Stk}/\mathcal{Y})$ twice. The delooping for the topos $\mathrm{Stk}/\mathcal{Y}$ of étale sheaves over $\mathcal{Y}$.

This is why an automorphism of the trivial gerb $B^2_{et}(\mathcal{A}) \widetilde{\to} B^2_{et}(\mathcal{A})$ over $\mathcal{Y}$ is an element of the mapping space $\mathrm{Map}_{\mathrm{Map}_{\mathrm{PreStk}/\mathcal{Y}}(\mathcal{Y}, B^2_{et}(\mathcal{A}))}(v, v)$ is ????

1.2.15. For 1.7.1. Let $\mathcal{T} \to \mathcal{Y}$ be a morphism of prestacks, $\mathcal{H}$ a group object in $\mathrm{PreStk}/\mathcal{Y}$ acting on $\mathcal{T}$. Then $\mathcal{T}$ is a $\mathcal{H}$-torsor over $\mathcal{Y}$ by definition here if it comes from a map $\mathcal{Y} \to B_{et}(\mathcal{H})$, so it would be better to call it $\mathcal{H}_{et}$-torsor in etale topology maybe.

For 1.7.3. Let $\mathcal{Y} \in \mathrm{PreStk}$. The commutative group object $B_{et}(E^{*,tors})$ acts on any presheaf of categories on $\mathcal{Y}$, because we have a morphism of groups $B_{et}(E^{*,tors}) \to \mathcal{LS}$, and $\mathcal{LS}$ acts on it.

1.2.16. For 1.7.5. Let $H \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, $\mathcal{E} : H \to \mathcal{LS}$ be a character sheaf on H. So, for any $S \in \mathrm{Sch}^{aff}$, $H(S) \in \mathcal{G}\mathrm{rp}(\mathrm{Spc})$. For each $h \in H(S)$ we are given a rank one local system $\mathcal{E}(S,h)$ on S functorially on (S,h). Let $m : H(S) \times H(S) \to H(S)$ be the product, $1 \in H(S)$ be the unit section. Then we are given isomorphisms $\mathcal{E}(S, m(h_1,h_2)) \tilde{\to} \mathcal{E}(S,h_1) \otimes \mathcal{E}(S,h_2)$ on S, and $\mathcal{E}(S,1) \tilde{\to} E$ on S.

A character sheaf on H can also be seen as a map $B(H) \to B(\mathcal{LS})$ in Ptd(PreStk). Therefore, if H acts on a prestack $\mathcal{Y}$, and $\tilde{\mathcal{Y}} = \mathcal{Y}/H$ fits into $* \times_{B(H)} \tilde{\mathcal{Y}} \tilde{\to} \mathcal{Y}$, we get the composition $\tilde{\mathcal{Y}} \to B(H) \to B(\mathcal{LS})$.

1.2.17. For 1.8.3. If $\mathcal{Y} \in \mathrm{PreStk}$ then $\mathcal{Y}_{dR} \in \mathrm{PreStk}$ is defined by $\mathcal{Y}_{dR}(S) = \mathcal{Y}(S_{red})$ for any $S \in \mathrm{Sch}^{aff}$. We have a canonical map $p : \mathcal{Y} \to \mathcal{Y}_{dR}$. Namely, $S_{red} \hookrightarrow S$ yields $\mathcal{Y}(S) \to \mathcal{Y}(S_{red}) = \mathcal{Y}_{dR}(S)$. Twistings on $\mathcal{Y}$ are the kernel of $\mathrm{Map}(\mathcal{Y}_{dR}, B^2(\mathbb{G}_m)) \to \mathrm{Map}(\mathcal{Y}, B^2(\mathbb{G}_m))$.

By ([16], 6.4.2), the commutative group $Tw(\mathcal{Y}) \in \mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$ of twistings on $\mathcal{Y}$ actually lies in $\infty - PicGrpd_k$, so is a k-module. The example ([16], 6.4.6) produces for a line bundle $\mathcal{L}$ on $\mathcal{Y}$ an element of $T(\mathcal{L}^{\otimes a}) \in Tw(\mathcal{Y})$, hence the forgetful functor $Tw(\mathcal{Y}) \to \mathrm{Ge}_{\mathcal{O}^\times}(\mathcal{Y}_{dR})$ gives the object denoted by $\mathcal{L}^a \in \mathrm{Ge}_{\mathcal{O}^\times}(\mathcal{Y}_{dR})$ in our Sect.1.8.3.

1.2.18. Sect. 2.2.1. The definition of a factorization prestack over Ran is not correct in the cases when Z is not discrete, higher compatibilities are missing (the correct definition is found in Raskin).

Precise definition of a non-unital associative algebra object in a monoidal ∞-category is (Lurie, HA, 5.4.3.3), non-unital commutative algebra objects (Lurie, HA, 5.4.4.1).

I proposed the following definition of a factorization structure on a prestack over Ran, Dennis says it is correct one.

Recall that Lurie denotes by $\mathrm{Surj} \subset \mathcal{F}\mathrm{in}_*$ the subcategory with the same objects, and a morphism $\langle n \rangle \to \langle m \rangle$ is in Surj iff it is surjective. Let $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ be a symmetric monoidal ∞-category. Let $CAlg^{nu}(\mathcal{C}^\otimes) \subset \mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\mathrm{Surj}, \mathcal{C}^\otimes)$ be the full subcategory spanned by functors F sending inert morphisms to inert morphisms in $\mathcal{C}^\otimes$. This is equivalent to requiring that for $i \in I - \{*\}$ the inert map $(* \in I) \to (* \in (*, i))$, $i \mapsto i, j \mapsto *$ for $j \neq i$ is sent by F to a cocartesian arrow over $\mathcal{F}\mathrm{in}_*$.

Let $\mathcal{M}$ be a non-unital commutative algebra object in $C^\otimes$. One has the notion of a subobject of $\mathcal{M}$ in the category $\mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\mathrm{Surj}, C^\otimes)$. This is a map $\mathcal{M}' \to \mathcal{M}$ such that for any $n \geq 0$, $\mathcal{M}'(\langle n \rangle) \subset \mathcal{M}(\langle n \rangle)$ is a subobject. Assume $\mathcal{M}'(\langle 1 \rangle) = \mathcal{M}(\langle 1 \rangle)$. Then $\mathcal{M}'$ is 'stable by the multiplication' automatically, and also stable under the permutations of $I - \{*\}$ for any $(* \in I) \in \mathrm{Surj}$. Note that $\mathcal{M}' \in \mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\mathrm{Surj}, \mathcal{C}^\otimes)$ is not a non-unital algebra itself!

For example, Ran is a non-unital commutative algebra in $PreStk$. Its subobject Ran^{disj} is defined by the property that for any pointed finite set $(* \in I)$, its value on $(* \in I)$ is $(\mathrm{Ran}^{I-*})^{disj}$.

Since Ran^{disj} is a subobject of Ran, it is stable by the multiplication. Besides, $\mathrm{Ran}^{disj}(\langle 1 \rangle) = \mathrm{Ran}(\langle 1 \rangle)$.

Let C be an infinity-category admitting finite limits. Let $\mathcal{M}$ be a non-unital commutative algebra object in C (with its cartesian monoidal structure), let $\mathcal{M}'$ be its subobject. Assume $\mathcal{M}'(\langle 1\rangle) = \mathcal{M}(\langle 1\rangle) =: M$. Let $\alpha : Z^\times \to \mathcal{M}'$ be a map in $\mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\mathrm{Surj}, C^\times)$. Set $Z = Z^\times(\langle 1\rangle)$.

Let $(* \in J) \in \mathcal{F}\mathrm{in}_*$. For $j \in J - \{*\}$ we have the inert map $\rho^j : (* \in J) \to (* \in (*, j))$ in Surj given by $j \mapsto j, k \mapsto *$ for $k \neq j$. It gives the induced map

$$Z^\times(* \in J) \to Z^\times(* \in (*, j)) = Z$$

for each $j \in J - \{*\}$. We want to require that together these maps give rise to an isomorphism

$$Z^\times(* \in J) \widetilde{\to} Z^{J-\{*\}} \times_{\mathcal{M}(*\in J)} \mathcal{M}'(* \in J)$$

In other words, $Z^\times(\langle n\rangle) \widetilde{\to} Z^n \times_{M^n} \mathcal{M}'(\langle n\rangle)$.

Say that a factorization object over $\mathcal{M}'$ is a pair $(Z^\times, \alpha)$ satisfyng the following property. For any $(* \in J) \in$ Surj there is a unique active map $a : J \to \langle 1\rangle$ in Surj, it sends each $j \in J - \{*\}$ to 1. Then $Z^\times(a)$ fits into a diagram

$$\begin{array}{ccc} Z^{J-\{*\}} \times_{M^{J-\{*\}}} \mathcal{M}'(* \in J) & \stackrel{Z^\times(a)}{\to} & Z \\ \downarrow & & \downarrow \\ \mathcal{M}'(* \in J) & \stackrel{\mathcal{M}'(a)}{\to} & \mathcal{M}'(\langle 1\rangle) = M \end{array}$$

We require in addition that for any $(* \in J) \in$ Surj this diagram is pull-back square in $\mathcal{C}$. Compare with the def. from (Raskin, Chiral categories).

Then as far as I understand, the diagram (2.3) is the functoriality of $Z^\times$ for the diagram $I \sqcup * \to J \sqcup * \to \langle 1\rangle$ of active morphisms. I mean you take further $\mathcal{C} = PreStk$ with its cartesian monoidal structure.

When Dennis talks about "compatibilities for higher order compositions" in this subsection, he means compositions of surjections of pointed finite maps $I_1 \to I_2 \to I_3 \to \ldots \to I_r$, where there are more than two maps involved.

1.2.19. In 2.2.2 the definition looks like a linearized (over the sheaf of symmetric monoidal categories Shv on $\mathrm{Sch}^{aff}/\mathcal{Y}$) version of a right-lax monoidal functor.

For 2.2.3. Check that for a diagram $Y \to S \leftarrow Y'$ of prestacks, we have a natural functor $Shv(Y) \otimes_{Shv(S)} Shv(Y') \to Shv(Y \times_S Y')$. This is used in the claim 2.2.3.

If $S \to \mathrm{Ran}^J_{disj}$ is given by the sets $I_j, j \in J$ then for $I = \sqcup I_j$ making the base change in (2.2) by this map we get $S \times_{\mathrm{Ran}^J} Z^J \widetilde{\to} S \times_{\mathrm{Ran}} Z$. Since we have a natural map $S \times_{I,\mathrm{Ran}^J} Z^J \stackrel{\xi}{\to} \prod_J (S \times_{I_j,\mathrm{Ran}} Z)$, it yields

$$\bigotimes_J Shv(S \times_{I_j,\mathrm{Ran}} Z) \to \mathrm{Shv}(S \times_{I,\mathrm{Ran}^J} Z^J) \widetilde{\to} \mathrm{Shv}(S \times_{\mathrm{Ran}} Z)$$

We can further pass to the quotient tensoring over $\mathrm{Shv}(S)$, because we do base change by the diagonal map $S \to S^J$. Everywhere the index like $S \times_{I,\mathrm{Ran}}$ means that the corresponding map $S \to \mathrm{Ran}$ is I. The map ξ is a closed immersion.

1.2.20. For 2.2.6 In the case of $\mathcal{D}$-modules this should be a factorization structure on this sheaf of categories.

1.2.21. For 3.1.3. The map (3.3) has to be an isomorphism of A-gerbes on S.

For 3.1.4. We interprete $\mathcal{P}_G$ as a map $S \times X \to B_{et}(G) \times X$, where the second component is the projection on X.

The displayed formula in 3.3.4 is true for $i = 0, 1$, but wrong for $i = 2$, should be corrected.

Remark: the calculation of homotopy groups (see 3.2.8) shows that the spaces $\mathrm{FactGe}_A(\mathrm{Gr}_G)$ are not isomorphic in local and global case! Do I understand correctly that (3.10) holds for both complete and noncomplete X?

1.2.22. The definition of $\pi_{1,alg}(G)$ in 3.2.5 is correct and taken from ([5], formula (7), p. 5), where it is proved also it is independent of a choice of $\tilde{G}_1$. We always have an exact sequence $1 \to \mu(-1) \to \pi_{1,alg}(G) \to \mathrm{Hom}(\mathbb{G}_m, G_{ab}) \to 1$, where $G_{ab} = G/[G,G]$, and $\mu = \mathrm{Ker}(\tilde{G} \to [G,G])$. Here $\tilde{G}$ is the simply-connected cover of $[G,G]$.

I think this is the usual fundamental group (quotient of Λ by the roots lattice), the complicated definition is to be able enentually to see the action of $\mathrm{Aut}(k)$ maybe? What is it for?

A calculation of $\mathrm{H}^*(B_{et}(G), \bar{\mathbb{Q}}_\ell)$ for G semisimple is done in ([17], Prop. 2.2.5).

1.2.23. For 4.3.1. I think compatibility of $\mathcal{G} \in \mathrm{FactGe}_A(\mathrm{Gr}_T)$ with the group structure on Gr_T means, first, that the morphism $\mathrm{Gr}_T \to B^2_{et}(A) \times \mathrm{Ran}$ is a morphism of group prestacks over Ran, so that the total space $\mathcal{G} \to \mathrm{Gr}_R$ of this gerbe is a group prestack over Ran, and moreover the isomorphisms (2.5) on p. 21, Sect. 2.2.4 are required to be isomorphism of group prestacks over Ran^J_{disj}.

Problem: find a precise rigorous definition here!

1.2.24. For 4.3.4. I think the map $\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A)) \to \mathrm{FactGe}^{com}_A(\mathrm{Gr}_T)$ is analogous to the fact that a $\check{T}$-torsor on X yields an object of $\mathrm{Ext}(\mathrm{Div}(X, \Lambda), \mathbb{G}_m)$ given by ([2], 3.10.7.3).

Namely, commutative factorization A-gerbes on Gr_T give gerbes $\mathcal{G} \to \mathrm{Gr}_T$ such that for any finite set J our isomorphism (2.5) extends to an isomorphism

$$\mathcal{G}^{\boxtimes J} \mid_{\mathrm{Gr}_{T^J}} \widetilde{\to} \mathcal{G} \mid_{\mathrm{Gr}_T \times_{\mathrm{Ran}} \mathrm{Ran}^J}$$

over the whole of Ran^J.

1.2.25. The def of $A(1)$ in 1.5.1 is wrong, it is corrected as follows. For each $n \geq 1$ prime to $char(k)$ let $A_n = \{a \in A \mid a^n = 1\}$, set $A_n(1) = A_n \otimes_{\mathbb{Z}} \mu_n$. Then $A(1) = \mathrm{colim}\, A_n(1)$ with respect to maps $n \mid n'$ for n, n' prime to $char(k)$.

Problem: The definition of the Kummer map from 4.3.4 is not clear.

1.2.26. Formulas in 4.3.9 is a formal consequence of Prop. 4.3.7, proof of 4.3.7 not clear for me.

1.2.27. For 4.4.1. The action of Gr_{T_2} on Gr_{T_1} is free in any sense one can imagine. So, $\mathrm{Gr}_{\Gamma \otimes \mathbb{G}_m}$ can be seen as a stack classifying $(I \in \mathrm{Ran}, \mathcal{P}, \alpha)$, where $\mathcal{P}$ is a $\mathrm{Gr}_{\Gamma \otimes \mathbb{G}_m}$-torsor on X with a trivialization over U_I. Here U_I is the complement of the union of the graphs of points of X given by I. This is clearly a factorization prestack over Ran.

1.2.28. For 4.4.5(a), in the displayed formula Ran should be replaced by X. The object of $\mathrm{FactGe}_A^{com}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$ factorize in a stronger sense, such gerbe gives a morphism of commutative group prestacks $\mathcal{G}\to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}\times_{\mathrm{Ran}}\mathrm{Ran}^J$ over Ran and for any finite set J an isomorphism of commutative group prestacks over Ran^J

$$\mathcal{G}^{\boxtimes J}\widetilde{\to} f^*\mathcal{G},$$

where $f:\mathrm{Gr}^J_{\Gamma\otimes\mathbb{G}_m}\to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}\times_{\mathrm{Ran}}\mathrm{Ran}^J\to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ is the composition.

For 4.4.5(b). The following notation is used here. For abelian groups Γ, A we can define $\mathrm{Quad}(\Gamma, A)$. Namely, this is the space of maps $q:\Gamma\to A$ such that (writing A additively)

1) $\Gamma\times\Gamma\to A$, $(a,b)\mapsto q(a+b)-q(a)-q(b)$ is bilinear;
2) for $n\in\mathbb{Z}, a\in\Gamma$, $q(na)=n^2q(a)$.

In 4.4.5(b), $\mathrm{Hom}(\Gamma, A(-1))_{2-tors}$ denotes the group $\mathrm{Hom}(\Gamma, A(-1)_{2-tors})$ if $char(k)\neq 2$ at least, where $A(-1)_{2-tors}=\{a\in A(-1)\mid 2a=0\}$.

The functoriality that Dennis meant in 4.5.1 is as follows. We may replace Γ by $\Gamma/2\Gamma$, then there is an isomorphism $\Gamma/2\Gamma\widetilde{\to}(\mathbb{Z}/2\mathbb{Z})^K$ for some finite set K, so $\mathrm{Hom}(\Gamma, A_2)=\mathrm{Hom}_{sets}(K, A_2)$. For any finite set K he says he claim one has canonically for a prestack Z

$$\mathrm{Map}(Z, B^2_{et}(\mathrm{Hom}_{sets}(K, A_2)))\widetilde{\to}\mathrm{Hom}(K, \mathrm{Map}(Z, B^2_{et}(A))$$

So, if we have construction for $\mathbb{Z}/2\mathbb{Z}$, we get a contstruction by functoriality for $(\mathbb{Z}/2\mathbb{Z})^K$.

Even better, $\mathrm{Hom}(\Gamma, A_2)\widetilde{\to}\mathrm{Hom}(\Gamma,\mathbb{Z}/2\mathbb{Z})\otimes A_2$. Each map $f\in \mathrm{Hom}(\Gamma,\mathbb{Z}/2\mathbb{Z})$ yields a morphism $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}\to\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$. The construction of $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ is functorial in Γ. That is, if $\Gamma_1\to\Gamma_2$ is a homomorphism, we get a map $\mathrm{Gr}_{\Gamma_1\otimes\mathbb{G}_m}\to\mathrm{Gr}_{\Gamma_2\otimes\mathbb{G}_m}$, and the map of factorization gerbes in the opposite direction.

1.2.29. For 5.1.3. No section $M\to P$ is needed here to get a map $\mathrm{FactGe}_A(\mathrm{Gr}_G)\to\mathrm{FactGe}_A(\mathrm{Gr}_M)$.

1.2.30. For 5.1.4. The meaning of $\mathfrak{p}^!$ is as follows. In 1.2.2 we defined the functor $Shv:(\mathrm{PreStk})^{op}\to\mathrm{DGCat}$. It is understood that for a morphism $\alpha: Z\to Z'$ in PreStk the corresponding morphism $Shv(Z')\to Shv(Z)$ is denoted $\alpha^!$.

1.2.31. For 7.2.2. The action of $\mathfrak{L}(G)$ on Gr_G can be spelled as follows. For $S\in\mathrm{Sch}^{aff}$ and a point $I:S\to\mathrm{Ran}$ we have $\mathcal{D}_I$, $\overset{\circ}{\mathcal{D}}_I$ as in 7.1.2. An S-point of Gr_G over I is given by (cP_G,α), where $\mathcal{P}_G$ is a G-torsor on $\mathcal{D}_I$, $\alpha:\mathcal{P}^0_G\widetilde{\to}\mathcal{P}_G$ over $\overset{\circ}{\mathcal{D}}_G$. An S-point of $\mathfrak{L}(G)$ is a map $\xi:\overset{\circ}{\mathcal{D}}_I\to G$. The action change the trivialization α by ξ.

1.2.32. For 5.2.1. To be precise, let us understand by $\mathrm{detrel}(\mathfrak{g}_{\mathcal{P}_G},\mathfrak{g}_{\mathcal{P}^0_G})$ the line bundle $\det\mathrm{R}\Gamma(X,\mathfrak{g}_{\mathcal{P}_G})\otimes\det\mathrm{R}\Gamma(X,\mathfrak{g}_{\mathcal{P}^0_G})^{-1}$.

For 5.2.4. The ratio of $\det_G|_S$ and $\det_M|_S$ here is $\frac{\det_G}{\det_M}$.

The line

$$K(L):=\frac{\det\mathrm{R}\Gamma(X,E\otimes L)\otimes\det\mathrm{R}\Gamma(X,E^*\otimes L)}{\det\mathrm{R}\Gamma(X,E_0\otimes L)\det\mathrm{R}\Gamma(X,E_0^*\otimes L)}$$

is canonically independent of $L \in \mathrm{Bun}_1$. One sees that $K(L(x)) \widetilde{\to} K(L)$ canonically for $x \in X$. This argument can be also done locally, in the case when X is not complete. This is related to my paper [23].

For 5.3.1. We have $\check{\rho}_{G,M} = \check{\rho}_G - \check{\rho}_M$.

1.2.33. Explanation about $\mathrm{Quad}(\Lambda, A)^W$, where A is a torsion divisible abelian group. Here G is any split reductive.

Note that $\mathrm{Quad}(\Lambda, \mathbb{Z}) \otimes_{\mathbb{Z}} A \widetilde{\to} \mathrm{Quad}(\Lambda, A)$. Let $\kappa_i \in \mathrm{Quad}(\Lambda, \mathbb{Z})^W$ be the Killing form for the i-th connected component of the Dynkin diagram. Let $q_i \in \mathrm{Quad}(\Lambda, \mathbb{Z})^W$ be the corresponding quadratic form, so $q_i(\lambda) = \kappa_i(\lambda, \lambda)/2$ for $\lambda \in \Lambda$. Pick a short coroot α_i for any such i.

For any $q \in \mathrm{Quad}(\Lambda, A)^W$ there are multiples $b_i \in A$ such that $b_i q_i(\alpha_i) = q(\alpha_i)$ in $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$. Let now $R = q - \sum_i b_i q_i$. Let b_R be the bilinear form associated to R, that is, $b_R(\lambda_1, \lambda_2) = R(\lambda_1 + \lambda_2) - R(\lambda_1) - R(\lambda_2)$ for $\lambda_i \in \Lambda$. Let Q be the coroots lattice. Then $2R$ vanishes on Q, and for $\mu \in Q, \lambda \in \Lambda$, $2b_R(\mu, \lambda) = 0$. So, there is $\bar{q} \in \mathrm{Quad}(\pi_{1,alg}(G), A)$ such that $2R$ is the composition $\Lambda \to \pi_{1,alg}(G) \stackrel{\bar{q}}{\to} A$.

An example showing that the map $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A \to \mathrm{Quad}(\Lambda, A)$ is not always surjective: let $A_2 = \{a \in A \mid 2a = 0\}$, we write A additively. A quadratic form $q : \Lambda \to A_2$ such that $q(\alpha) = 0$ for any short coroot α does not always lie in $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$. For example, $G = \mathbb{S}\mathrm{p}_4$, so that $\Lambda = \mathbb{Z}^2$, where we identify in a usual way $\mathrm{Hom}(\mathbb{G}_m, T) \widetilde{\to} \mathbb{Z}$ for a maximal torus $T \subset \mathrm{GL}_2 \subset \mathbb{S}\mathrm{p}_4$. For $c \in A_2$ the quadratic form defined on $(a_1, a_2) \in \mathbb{Z}^2$ by $q(a_1, a_2) = ca_1a_2$ is W-invariant, and is not in $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$.

1.2.34. I claim that the image of

$$\mathrm{Quad}(\pi_{1,alg}(G), A) \to \mathrm{Quad}(\Lambda, A)$$

does not lie in $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$ in general.

Consider an example of $G = (\mathbb{S}\mathrm{pin}_{2n})_{ad}$ with $n \in 4\mathbb{Z}$. In this case $\pi_{1,alg}(G) \widetilde{\to} (\mathbb{Z}/2\mathbb{Z})^2$. We have $\Lambda = \mathbb{Z}^n + \mathbb{Z}\omega$, where $\omega = (\frac{1}{2}, \dots, \frac{1}{2})$, the coroots are $\pm(e_i + e_j), \pm(e_i - e_j)$ for $i \neq j$. Consider the quadratic form $q(x_1, \dots, x_n) = \sum_i x_i^2$ for $x \in \Lambda$. It takes values in $\mathbb{Z}$, we have an isomorphism $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \widetilde{\to} \mathbb{Z}$ sending q to 1. So, elements of $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$ are those of the form $x \mapsto aq(x)$ for $a \in A$, we are writing A additively.

Let $e = (1, 0, \dots, 0)$, so $\{e, \omega\}$ is a base of $\pi_1(G)$ over $\mathbb{Z}/2\mathbb{Z}$. We get $aq(e) = a, aq(\omega) = \frac{n}{4}a$. So, take for example $\bar{q} : \pi_1(G) \to A_2$ linear given by $\bar{q}(e) = 0, \bar{q}(\omega) = c$ for some $c \in A_2$. The restriction of $\bar{q}$ to Λ does not lie in $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$.

This leads to the following contradiction in the paper. By Cor. 4.4.5, any $\bar{q} \in \mathrm{Quad}(\pi_1(G), A)$ can be lifted to an element of $\mathrm{FactGe}_A(\mathrm{Gr}_{\pi_{1,alg}(G) \otimes \mathbb{G}_m})$. Consider its image under

$$\mathrm{FactGe}_A(\mathrm{Gr}_{\pi_{1,alg}(G) \otimes \mathbb{G}_m}) \to \mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A,$$

where the second map is as in Sect. 3.2.9. We get a contradiction. So, either Cor. 4.4.5 is wrong as stated or the calculation of $\mathrm{H}^4_{et}(B(G), A(1))$ from Sect. 3.2.6 is wrong.

1.2.35. The lemma of Reich ([32], Lm. II.7.2) badly explained should be formulated as follows I think.

Let A be a torsion divisible abelian group. For i-th connected component of the Dynkin diagram pick a corresponding short coroot α_i. Let $\kappa_i : \Lambda\otimes\Lambda \to \mathbb{Z}$ be the Killing form for the i-th connected component of Dynkin diagram, and q_i the corresponding quadratic form, so $q_i(\lambda) = \kappa_i(\lambda;\lambda)/2$. Let $Q \subset \Lambda$ be the coroots lattice of G.

Lemma 1.2.36. *Let $q \in \mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$. Let $b_i \in A$ such that $b_i q_i(\alpha_i) = q(\alpha_i)$ in $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$ for each i-th connected component of the Dynkin diagram. Set $R = q - \sum_i b_i q_i$. Let Λ_{ab} be the coweights lattice of $G/[G,G]$. Then there is $\bar{R} \in \mathrm{Quad}(\Lambda_{ab}, A)$ whose restriction to Λ is R.*

Proof. Our q is a linear combination of forms of the form $a\tilde{q}$, where $\tilde{q} \in \mathrm{Quad}(\Lambda, \mathbb{Z})^W$ and $a \in A$. If we prove our claim for q of the form $a\tilde{q}$ then it is also true for a linear combination. So, assume $q = a\tilde{q}$ as above. Pick $r \in \mathbb{N}$ large enough such that there are integers d_i with $r\tilde{q}(\alpha_i) = d_i q_i(\alpha_i)$ for all i. Consider $q_0 = r\tilde{q} - \sum_i d_i q_i \in Quad(\Lambda, \mathbb{Z})^W$. Let b_0 be the bilinear form associated to q_0, that is $b_0(\lambda_1, \lambda_2) = q_0(\lambda_1 + \lambda_2) - q_0(\lambda_1) - q_0(\lambda_2)$ for $\lambda_i \in \Lambda$.

As in ([37], Lemma 1.2), we get $2b_0(\alpha, \lambda) = 2q_0(\alpha)\langle\check{\alpha}, \lambda\rangle = 0$ for any $\lambda \in \Lambda$ and any short coroot α. Since our forms take values in $\mathbb{Z}$, this gives $b_0(\alpha, \lambda) = 0$ for any $\lambda \in \Lambda$ and any short coroot α.

As we have seen in the previous section, $2q_0$ vanishes on Q, and $2b_0(\mu, \lambda) = 0$ for $\mu \in Q, \lambda \in \Lambda$. Let $\tilde{Q} = \{\lambda \in \Lambda \mid \text{there is } m > 0 \text{ with } m\lambda \in Q\}$. Pick $m \in \mathbb{N}$ such that $m\tilde{Q} \subset Q$. We see that $2mb_0(\mu, \lambda) = 0$ for $\mu \in Q, \lambda \in \Lambda$. So, mq_0 descends to a quadratic form $\bar{r} : \Lambda_{ab} \to \mathbb{Z}$. Since A is divisible, we are done. □

Corollary 1.2.37. *The images of the Killing forms κ_i and of $\mathrm{Quad}(\Lambda_{ab}, A)$ generate the subgroup $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$.*

1.2.38. For 6.2.1 The torus $T^\sharp$ is the maximal torus in $G^\sharp$ defined as $\Lambda^\sharp \otimes \mathbb{G}_m$, so $\Lambda^\sharp$ are coweights of $G^\sharp$, and $\check{\Lambda}^\sharp$ are weights of $G^\sharp$.

For 6.2.2. Since the bilinear form corresponding to the gerbe $\mathcal{G}^{T^\sharp}$ vanishes, to show that the quadratic form vanishes on the roots lattice of $(T^\sharp, G^\sharp)$, it suffices indeed to show that the pull-back of $\mathcal{G}^{T^\sharp}$ to $\mathrm{Gr}_{\mathbb{G}_m}$ for any simple coroot $\alpha^\sharp : \mathbb{G}_m \to T$ of $(T^\sharp, G^\sharp)$ is trivialized.

1.2.39. For 6.2.3 .The $\mathbb{Z}/2\mathbb{Z}$-graded factorization line bundle $\det_{\mathbb{G}_m,n}$ has fibre in the global case $\det \mathrm{R}\Gamma(X, L^n) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O}^n)^{-1}$ at $(L, \alpha : L \tilde{\to} \mathcal{O} \mid_{U_I}) \in \mathrm{Gr}_{\mathbb{G}_m}$ over $I \in \mathrm{Ran}$.

For 6.2.4: we should precise here that it suffices to show that $\det_{\mathbb{G}_m,2n}$ admits a canonical $2n$-th root at a factorization line bundle (the corresponding $\mathbb{Z}/2\mathbb{Z}$-grading should be trivial!!).

For 6.2.5: the factorizable line bundles $\det_{\mathbb{G}_m,2n}, \det_{\mathbb{G}_m,1}$ correspond to some θ-data, and the theta datum corresponding to $\det_{\mathbb{G}_m,2n} \otimes (\det_{\mathbb{G}_m,1})^{-4n}$ has trivial $\mathbb{Z}$-valued bilinear form, so is given (according to [2], 3.10.3.1) by some $\check{\mathbb{G}}_m$-torsor, Dennis claims this torsor corresponds to $\Omega_X^{n(2n-1)}$.

Since the precise definition of detrel is not given, it is impossible to verify the 2nd displayed equation in Sect. 6.2.5. That formula is true for one normalization of detrel, not for both!!

Note that we have canonically

$$\det(\mathcal{O}(mx)/\mathcal{O}) \otimes \det(\mathcal{O}(x)/\mathcal{O})^{-m} \widetilde{\to} \Omega_x^{-\frac{m(m-1)}{2}}$$

This calculates $\det_{\mathbb{G}_m,m} \otimes (\det_{\mathbb{G}_m,1})^{-m}$ essentially.

1.2.40. For 6.3.1. It is important that $(\mathcal{G}^{\pi_{1,alg}(G^\sharp)\otimes\mathbb{G}_m})^{com}$ gives a gerbe $\mathrm{Ran} \to B^2_{et}(\mathrm{Hom}(\pi_{1,alg}(G^\sharp), E^{*,tors}))$ over the whole of Ran.

The $Z_H(E)^{tors}$-gerbe $\mathcal{G}_Z$ on X is an element of

$$\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\pi_{1,alg}(G^\sharp), E^{*,tors}))$$

corresponding to $(\mathcal{G}^{\pi_{1,alg}(G^\sharp)\otimes\mathbb{G}_m})^{com}$. Here $Z_H(E)^{tors} = \mathrm{Hom}(\pi_{1,alg}(G^\sharp), E^{*,tors})$.

So, $\mathcal{G}_Z$ gives rise to a $Z_H(E)^{tors}$-gerbe on Ran.

1.2.41. By a symmetric monoidal DG-category in 6.4.1 we mean a commutative algebra object of DGCat.

For 6.4.5: my understanding is that $\mathrm{Fact}(\mathcal{C})_{\mathcal{G}_A}$ and $\mathrm{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$ are prefactorization sheaves of monoidal DG-categories on Ran, we have an equivalence

$$\mathrm{Fact}(\mathcal{C})_{\mathcal{G}_A} \widetilde{\to} \mathrm{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$$

of sheaves of monoidal DG-categories on Ran, but this equivalence is not compatible with the prefactorization structures.

1.2.42. Since T is an abelian group, the factorization isomorphism for Gr_T for a finite set J exends to a morphism of group prestacks over Ran^J

$$h : \mathrm{Gr}_T^J \to \mathrm{Gr}_T \times_{\mathrm{Ran}} \mathrm{Ran}^J$$

sending an S-point $(\mathcal{F}_j, \alpha_j, I_j \in \mathrm{Ran})$, where $\mathcal{F}_j$ is a T-torsor on $S \times_{\mathrm{Ran}} \mathrm{Gr}_T$, $\alpha : \mathcal{F}_j \widetilde{\to} \mathcal{F}^0_T \mid_{X-I_j}$ is a trivialization to $(\otimes_j \mathcal{F}_j, \alpha = \otimes \alpha_j, I = \cup_j I_j)$.

For a multplicative gerbe $\mathcal{G} \in \mathrm{FactGe}^{mult}_A(\mathrm{Gr}_T)$ we get an isomorphism $h^*\mathcal{G} \widetilde{\to} \mathcal{G}^{\boxtimes J}$ over Gr^J_T. However, say if we consider this over $X^2 \to \mathrm{Ran}$, this isomoprhism does not descend to isomorphism of gerbes over $\mathrm{Gr}_T \times_{\mathrm{Ran}} X^{(2)}$, see Sect. 4.2.

1.2.43. Dennis proposed a more general Satake equivalence (on Jan 13, 2018) as follows. Let Γ be a finitely generated abelian group. View $\mathrm{Hom}(\Gamma, \mathbb{G}_m)$ as an algebraic group. Then Satake equivalence for $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ is an equivalence

$$\mathrm{Fact}(\mathrm{Rep}(\mathrm{Hom}(\Gamma, \mathbb{G}_m))) \widetilde{\to} Shv(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$$

in the notations of [13].

1.2.44. To better understand the relation between commutative and multiplicaive A-gerbes on Ran, one may ask the following question. Let Y be a commutative monoid in Sets, Let S be a commutative monoid in ${}_{\tau_{\le 2}}\mathrm{Spc}$.

What can we say about maps of spaces $\mathrm{Map}_{ComMon(\mathrm{Spc})}(Y,S) \to \mathrm{Map}_{Mon(\mathrm{Spc})}(Y,S) \to \mathrm{Map}_{\mathrm{Spc}}(Y,S)$

Are the above morphisms fully faithful embeddings? This would help to think about multiplicativity or commutaivity of a factorization gerbe. We want to apply the above to $S = B^2(A)$. It is not clear that commutativity defines a full subspace.

2. Comments to [13], file version May 25, 2018

2.0.1. One has $\mathrm{Quad}(\Lambda,\mathbb{Z})\otimes A \widetilde{\to} \mathrm{Quad}(\Lambda,A)$. The subgroup $\mathrm{Quad}(\Lambda,\mathbb{Z})^W \subset \mathrm{Quad}(\Lambda,\mathbb{Z})$ is saturated, that is, the cokernel is torsion free. For this reason for any abelian group A the map $\mathrm{Quad}(\Lambda,\mathbb{Z})^W \otimes_{\mathbb{Z}} A \to \mathrm{Quad}(\Lambda,A)$ is injective and takes vales in $\mathrm{Quad}(\Lambda,A)^W$.

2.0.2. For Sect. A.1. Let A be a torsion abelian group, whose elements have orders prime to $char(k)$. We have $\mathrm{H}^2_{et}(B(T),\mathbb{Z}) \widetilde{\to} \check{\Lambda}(-1)$, and $\mathrm{H}^2_{et}(B(T),A) \widetilde{\to} \check{\Lambda}\otimes A(-1)$. So, $\mathrm{H}^4_{et}(B(T),A)$ is the S_2-coinvariants in $\mathrm{H}^2_{et}(B(T),A)\otimes_A \mathrm{H}^2_{et}(B(T),A) \widetilde{\to} \check{\Lambda}\otimes\check{\Lambda}\otimes A(-2)$. Consider the map $\mathrm{Hom}(\Lambda\otimes\Lambda,\mathbb{Z}) \to \mathrm{Quad}(\Lambda,\mathbb{Z})$ sending a bilinear form s to the quadratic form q given by $q(\lambda) = s(\lambda,\lambda)$. This map identifies canonically $\mathrm{Quad}(\Lambda,\mathbb{Z})$ with the S_2-coinvariants of $\check{\Lambda}\otimes\check{\Lambda}$. For this reason we get $\mathrm{H}^4(B(G),A) \widetilde{\to} \mathrm{Quad}(\Lambda,\mathbb{Z})\otimes A(-2)$ in such a way that the coproduct is the above map $\mathrm{Hom}(\Lambda\otimes\Lambda,\mathbb{Z}) \to \mathrm{Quad}(\Lambda,\mathbb{Z})$, $s \mapsto q$.

2.0.3. Any reductive group of semi-simple rank 1 writes as $G_1 \times G_2$, where G_2 is a torus, and $G_1 \widetilde{\to} \mathrm{SL}_2, \mathrm{PSL}_2, \mathrm{GL}_2$. Indeed, just consider possible actions of the simple reflection s on Λ. Let $\Lambda_0 = \mathrm{Ker}\,\check{\alpha}$. The nontrivial case is when $\Lambda_0 \oplus \mathbb{Z}\alpha \subset \Lambda$ is of index 2. Then Λ is generated by $\Lambda_0\oplus\mathbb{Z}\alpha$ and an element $\frac{\alpha+u}{2}$ for some $u\in\Lambda$. If $u/2\in\Lambda$ then we get $\mathrm{PSL}_2\times G_2$. Otherwise, we get $\mathrm{GL}_2\times G_2$, where G_2 is a torus.

Remark 2.0.4. *Consider G simple simply-connected. Then $\mathrm{Quad}(\Lambda,\mathbb{Z})^W \widetilde{\to} \mathbb{Z}$, and there is a distinguished generator q given by the property that $q(\alpha) = 1$ for any short coroot.*

2.0.5. Consider the example of $G = \mathrm{PSL}_n$, Λ is the coroots lattice. In this case $\mathrm{Quad}(\Lambda,\mathbb{Z})^W \widetilde{\to} \mathbb{Z}$ is generated by a quadratic form q_0 such that $q_0(\alpha) = n$ for any coroot.

Lemma 2.0.6. *Assume A a divisible torsion group. Let $q \in \mathrm{Quad}(\Lambda,A)^W_{restr}$. Then there is $q_{\mathbb{Z}} \in \mathrm{Quad}(\Lambda,\mathbb{Z})^W\otimes A$ such that $q - q_{\mathbb{Z}}$ comes from $\mathrm{Quad}(\pi_{1,alg}(G),A)$.*

Proof. For each connected component of the Dynkin diagram let κ_j be the corresponding Killing form for G, so $\kappa_j = \sum_{\check{\alpha}\in\check{R}_j} \check{\alpha}\otimes\check{\alpha} : \Lambda\otimes\Lambda \to \mathbb{Z}$, and $G_{ad} = \prod_j G_j$. Here $\check{R}_j$ is the set of roots of G_j. Let $q_j \in \mathrm{Quad}(\Lambda,\mathbb{Z})^W$ be the quadratic form $q_j(x) = \kappa_j(x,x)/2$. Pick $a_j \in A$ such that for each j, $q(\alpha) = a_j q_j(\alpha)$ for each short coroot of $\check{R}_j$. Set $q_{\mathbb{Z}} = \sum_j a_j q_j$. So, $q(\alpha) = q_{\mathbb{Z}}(\alpha)$ for any short coroot of G. Let $\bar{q} = q - q_{\mathbb{Z}}$, let $\bar{b} : \Lambda\otimes\Lambda \to A$ be the bilinear form associated to $\bar{q}$, that is, $\bar{b}(x_1,x_2) = \bar{q}(x_1+x_2) - \bar{q}(x_1) - \bar{q}(x_2)$. By our assumption, $\bar{q} \in \mathrm{Quad}(\Lambda,A)^W_{restr}$, so

$\bar{b}(\alpha, \lambda) = 0$ for $\lambda \in \Lambda$ and a short coroot α. So, $\bar{b}(\mu, \lambda) = 0$ for $\mu \in \Lambda_{sc}, \lambda \in \Lambda$. Here $\Lambda_{sc} \subset \Lambda$ is the coroots lattice of G. So, for $\lambda \in \Lambda$, $\bar{q}(\lambda)$ depends only on $\lambda + \Lambda_{sc}$. □

2.1. Just to underline: if say $A = E^{\times, tors}$ is the group of torsion elements of order prime to $char(k)$ then $B_{et}(A)$ is a prestack that has a modular interpretation. For a prestack Y, $\mathrm{Map}(Y, B_{et}(A))$ is the space of A-torsors on Y. Question: is it possible to make sense of this without higher category theory?

2.2. For 1.7.1. Let $\mathcal{Y} \in \mathrm{PreStk}$. Via the strengthening for cartesian fibrations, the category $\mathrm{PreStk}\,/\mathcal{Y}$ identifies with the cartesian fibraions in spaces over $\mathrm{Sch}^{aff}/\mathcal{Y}$. Let $H \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk}\,/\mathcal{Y})$, let $\mathcal{X} \to \mathrm{Sch}^{aff}/\mathcal{Y}$ and $\bar{q} : \bar{\mathcal{X}} \to \mathrm{Sch}^{aff}/\mathcal{Y}$ be the cartesian fibration in spaces corresponding to $B(H) \to \mathcal{Y}$ and $B_{et}(H) \to \mathcal{Y}$. We have the natural map $\mathcal{X} \to \bar{\mathcal{X}}$ over $\mathrm{Sch}^{aff}/\mathcal{Y}$. Now given a H-torsor on $\mathcal{Y}$, that is, a section $\mathcal{Y} \to B_{et}(H)$ of the projection $B_{et}(H) \to \mathcal{Y}$, it can be seen as a section $s : \mathrm{Sch}^{aff}/\mathcal{Y} \to \bar{\mathcal{X}}$ of $\bar{q}$. Then $Split(\mathcal{T})$ is defined by the cartesian square

$$\begin{array}{ccc} Split(\mathcal{T}) & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Sch}^{aff}/\mathcal{Y} & \stackrel{s}{\to} & \bar{\mathcal{X}} \end{array}$$

2.3. For 1.8.3. We have $\mathcal{LS}^{1-\dim}(\mathrm{Spec}\, k) = \{$ 1-dimensional local systems within Vect$\}$, this is the space $B(E^\times) \in \mathrm{Spc}$ of E-lines. Therefore, we have $B(E^{\times, tors}) \to B(E^\times) \to \mathcal{LS}^{1-\dim}$.

2.4. For 3.2.8 in the paper. Since

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \widetilde{\to} \mathrm{Map}(B_{et}(G) \times X, B^4_{et}(A(1))) \times_{\mathrm{Map}(X, B^4_{et}(A(1)))} *,$$

we have the corresponding long exact sequence of homotopy groups

$$\ldots \to \pi_1 \to \mathrm{H}^3(B_{et}(G) \times X, A(1)) \to \mathrm{H}^3(X, A(1)) \to \pi_0 \to \mathrm{H}^4(B_{et}(G) \times X, A(1)) \to \mathrm{H}^4(X, A(1)),$$

where $\pi_i = \pi_i(\mathrm{FactGe}_A(\mathrm{Gr}_G))$. It gives the desired calculation.

2.5. Consider a diagram $G_1 \to G_2 \to G_3$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$ such that G_3 is the cofibre of $G_1 \to G_2$. Since $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc}) \widetilde{\to} \mathrm{Sptr}^{\leq 0} \subset \mathrm{Sptr}$ is stable under small colimits, it is cocartesian in Sptr, hence cartesian in Sptr, hence G_1 is a fibre of $G_2 \to G_3$ in the full subcategory $\mathrm{Sptr}^{\leq 0} \widetilde{\to} \mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$.

So, $B_{et}(G_1) \to B_{et}(G_2) \to B_{et}(G_3)$ is a cofiber sequence in $\mathcal{C}\mathrm{omGrp}(\mathrm{PreStk})$, because B_{et} preserves colimits. For any $S \in \mathrm{Sch}^{aff}$, the value of the above sequence on S is a fibre sequence in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$ by the above. Since

$$\mathcal{C}\mathrm{omGrp}(\mathrm{PreStk}) \widetilde{\to} \mathrm{Fun}(\mathrm{Sch}^{aff}, \mathcal{C}\mathrm{omGrp}(\mathrm{Spc})),$$

we see that $B_{et}(G_1)$ is the fibre of $B_{et}(G_2) \to B_{et}(G_3)$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{PreStk})$. Indeed, the limits in functors are computed pointwise.

2.5.1. The above applies in 4.4.4, 4.4.5 of the paper. Namely, given a finitely-generated abelian group Γ, pick a presentation $\Gamma = \Lambda_1/\Lambda_2$, where $\Lambda_2 \subset \Lambda_1$ are lattices. Since our torsion abelian group A is divisible, the sequence is exact $0 \to \mathrm{Hom}(\Gamma, A) \to \mathrm{Hom}(\Lambda_1, A) \to \mathrm{Hom}(\Lambda_2, A) \to 0$. So,

$$B^2_{et}(\mathrm{Hom}(\Gamma, A)) \to B^2_{et}(\mathrm{Hom}(\Lambda_1, A)) \to B^2_{et}(\mathrm{Hom}(\Lambda_2, A)) \tag{1}$$

is a fibre and cofibre sequence in $\mathcal{C}\mathrm{omGrp}(\mathrm{PreStk})$.

The oblivion functor $\mathrm{Fun}(\mathrm{Sch}^{aff}, \mathcal{C}\mathrm{omGrp}(\mathrm{Spc})) \to \mathrm{PreStk}$ preserves small limits, so (1) is a fibre sequence in PreStk also. So, $\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Gamma, A))$ is the fibre of

$$\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda_1, A)) \to \mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda_2, A))$$

in Spc.

If $q \in \mathrm{Quad}(\Gamma, A(-1))$ there is a factorization gerbe in $\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$ with this quadratic form q. Indeed, pick any factorization gerbe $\mathcal{G}$ on Gr_{T_1} with the quadratic form q_1, the restriction of q. Let $\mathcal{G}_2$ be its restriction to Gr_{T_2}. Then $\mathcal{G}_2$ is given by a map $X \to B^2_{et}(\mathrm{Hom}(\Lambda_2, A))$. Note that $\mathrm{H}^2(X, \mathrm{Hom}(\Lambda_1, A)) \to \mathrm{H}^2(X, \mathrm{Hom}(\Lambda_2, A))$ is surjective. So, we may pick $\mathcal{G}' \in \mathrm{FactGe}^{com}_A(\mathrm{Gr}_{T_1})$ whose restriction to Gr_{T_2} is isomorphic to $\mathcal{G}_2$. Then $(\mathcal{G}')^{-1} \otimes \mathcal{G}$ will give rise to a factorization gerbe on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$.

2.6. Let $\mathcal{I}$ be the category of finite nonempty sets, whose morphisms are surjections $I \to J$. We have a functor $\mathcal{I}^{op} \to 1-\mathcal{C}\mathrm{at}$, $I \mapsto \mathrm{Sch}^{aff}/X^I$. If $I \to J$ is a surjection, the functor $\mathrm{Sch}^{aff}/X^J \to \mathrm{Sch}^{aff}/X^I$ is the evident one. Then

$$\mathrm{colim}_{I\in\mathcal{I}^{op}} \mathrm{Sch}^{aff}/X^I \widetilde{\to} \mathrm{Sch}^{aff}/\mathrm{Ran} \;\; ?$$

Here the colimit is taken in $1-\mathcal{C}\mathrm{at}$. This would be true it we considered the colimit in $1-\mathcal{C}\mathrm{at}_{ordn} \subset 1-\mathcal{C}\mathrm{at}$, the full subcategory of ordinary categories. However, the inclusion $1-\mathcal{C}\mathrm{at}_{ordn} \hookrightarrow 1-\mathcal{C}\mathrm{at}$ does not preserve colimits. Since $\mathcal{I}^{op}$ is not filtered, this is not evident.

I wonder if the natural functor

$$\mathrm{Fun}(\mathrm{Sch}^{aff}/\mathrm{Ran}, \mathrm{DGCat}) \to \lim_{I\in\mathcal{I}} \mathrm{Fun}(\mathrm{Sch}^{aff}/X^I, \mathrm{DGCat})$$

is an equivalence, where the limit is calculated in $1-\mathcal{C}\mathrm{at}$.

2.7. If $\mathcal{F}$ is a sheaf of DG-categories on $\mathcal{Y} \in \mathrm{PreStk}$, $\mathcal{C} \in \mathrm{DGCat}$ is it true that $S \mapsto \mathcal{F}(S) \otimes \mathcal{C}$ is a sheaf of DG-categories?

For this we ask the following. Is it true that the tensor product in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ preserves totalizations separately in each variable? The natural functor $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont} \to 1-\mathcal{C}\mathrm{at}$ preserves limits, so the corresponding limit can be calculated in $1-\mathcal{C}\mathrm{at}$. The answer is not clear. Question: does the tensor product in $\mathcal{P}\mathrm{r}^L$ preserves limits separately in each variable? (Maybe some special limits?

2.8. If one wants a more general sheaf theory than the 3 examples in 1.1.2 then one will need the following. For a closed immersion $i : Y \to Y'$ the functor $i_* : \mathrm{Shv}(Y) \to \mathrm{Shv}(Y')$ such that for a cartesian square

$$\begin{array}{ccc} Y & \stackrel{i_Y}{\to} & Y' \\ \uparrow f & & \uparrow f' \\ S & \stackrel{i_S}{\to} & S' \end{array}$$

we have $(f')^! i_{Y*} \widetilde{\to} i_{S*} f^!$. This is needed for the functor $i_{Y!} : \mathrm{Shv}(Y) \to \mathrm{Shv}(Y')$ to be symmetric monoidal. The latter property is used in the construction of $\mathrm{Fact}(\mathcal{C})$ in 6.4.1.

2.9. For 6.4.1. Let I be a finite non-empty set, $f : I \to I'$ a surjection. Then f induces a full embedding $\mathrm{Tw}(I') \subset \mathrm{Tw}(I)$ sending $I' \to J' \to K'$ to $I \stackrel{f'}{\to} J' \to K'$. Here f' is the composition $I \to I' \to J'$.

Let $Q(I)$ be the set of equivalence relations on I. Recall that $Q(I)$ is partially ordered. As in [2], we write $I' \in Q(I)$ for a quotient $I \to I'$ viewed as an equivalence relation on I. We write $I'' \le I'$ iff $I'' \in Q(I')$. Then $Q(I)$ is a lattice. For $I', I'' \in Q(I)$ we have $\inf(I', I'')$. Let now a surjection $f : I \to I'$ be given. We get a functor $Q(I) \to Q(I')$ sending $J \in Q(I)$ to $\inf(J, I') \in Q(I')$.

Define a functor $\xi : \mathrm{Tw}(I) \to \mathrm{Tw}(I')$ sending $I \to J \to K$ to $I' \to J' \to K'$, where $J' = \inf(J, I'), K' = \inf(K, I')$. It sends a morphism

$$\begin{array}{ccccc} I & \to & J_1 & \to & K_1 \\ \| & & \downarrow & & \uparrow \\ I & \to & J_2 & \to & K_2 \end{array} \tag{2}$$

to the induced diagram

$$\begin{array}{ccccc} I' & \to & J'_1 & \to & K'_1 \\ \| & & \downarrow & & \uparrow \\ I' & \to & J'_2 & \to & K'_2 \end{array}$$

Let $\mathcal{F}_I : \mathrm{Tw}(I) \to \mathrm{Shv}(X^I) - mod$ be the functor sending $(I \to J \to K)$ to

$$\mathrm{Shv}(X^K) \otimes \mathcal{C}^{\otimes J}$$

Recall that $\mathrm{Fact}(\mathcal{C})$ associates to $X^I \to \mathrm{Ran}$ the category

$$\mathcal{C}_{X^I} := \underset{(I \to J \to K) \in \mathrm{Tw}(I)}{\mathrm{colim}} \mathrm{Shv}(X^K) \otimes \mathcal{C}^{\otimes J} \in \mathrm{Shv}(X^I) - mod$$

Let now $f : I \to I'$ be a surjection. To the closed immersion $X^{I'} \to X^I$ the sheaf $\mathrm{Fact}(\mathcal{C})$ associates the restriction functor $\mathcal{C}_{X^I} \to \mathcal{C}_{X^{I'}}$ given as follows. For each $(I \to J \to K) \in \mathrm{Tw}(I)$ let $(I' \to J' \to K') \in \mathrm{Tw}(I')$ be its image under ξ. Consider the functor

$$(\triangle^!) \otimes m : \mathrm{Shv}(X^K) \otimes \mathcal{C}^{\otimes J} \to \mathrm{Shv}(X^{K'}) \otimes \mathcal{C}^{\otimes J'}, \tag{3}$$

where $m : \mathcal{C}^{\otimes J} \to \mathcal{C}^{\otimes J'}$ is the product map, and $\triangle : X^{K'} \to X^K$ is the diagonal. Now (3) extends to a morphism of functors $\mathcal{F}_I \to \mathcal{F}_{I'} \circ \xi$ in $\mathrm{Funct}(\mathrm{Tw}(I), \mathrm{Shv}(X^I) - mod)$.

Namely, for any morphism (2) the diagram commutes

$$\begin{array}{ccc}
\mathrm{Shv}(X^{K_1})\otimes \mathcal{C}^{J_1} & \stackrel{\triangle^!\otimes m}{\to} & \mathrm{Shv}(X^{K'_1})\otimes \mathcal{C}^{J'_1}\\
\downarrow m & & \downarrow m\\
\mathrm{Shv}(X^{K_1})\otimes \mathcal{C}^{J_2} & & \mathrm{Shv}(X^{K'_1})\otimes \mathcal{C}^{J'_2}\\
\downarrow \triangle_! & & \downarrow \triangle_!\\
\mathrm{Shv}(X^{K_2})\otimes \mathcal{C}^{J_2} & \stackrel{\triangle^!\otimes m}{\to} & \mathrm{Shv}(X^{K'_2})\otimes \mathcal{C}^{J'_2}
\end{array}$$

It uses the fact that the square is cartesian

$$\begin{array}{ccc}
X^{K_1} & \stackrel{\triangle}{\leftarrow} & X^{K'_1}\\
\downarrow \triangle & & \downarrow \triangle\\
X^{K_2} & \stackrel{\triangle}{\leftarrow} & X^{K'_2}
\end{array}$$

and the base change holds $\triangle^!\triangle_* \widetilde{\to} \triangle_*\triangle^!$.

We get natural functors

$$\mathrm{colim}_{\mathrm{Tw}(I)}\mathcal{F}_I \to \mathrm{colim}_{\mathrm{Tw}(I)}\mathcal{F}_{I'}\circ\xi \to \mathrm{colim}_{\mathrm{Tw}(I')}\mathcal{F}_{I'}$$

This is the desired restriction functor. Now given $S\to X^I$, one may impliment $S\times_{X^I}\cdot$ in the above formulas.

2.10. Kummer theory. For 4.2.4 of final version. Let A be a torsion abelian group, whose elements have orders prime to $char(k)$. Then $(A(-1))(1)\widetilde{\to}A$. The Kummer map $A\times\mathbb{G}_m\to B_{et}(A(1))$ is defined as follows. Replacing A by $A(-1)$, it suffices to define a map $A(-1)\times\mathbb{G}_m\to B_{et}(A)$. We have for each n prime with $char(k)$ the cover $\mathbb{G}_m\to\mathbb{G}_m, x\mapsto x^n$ giving a homomorphism $\mathbb{G}_m\to B_{et}(\mu_n)$ in $\mathcal{C}$omGrp(PreStk). Together they yield a map $\mathbb{G}_m\to\lim_n B_{et}(\mu_n)$, the limit over n prime to $char(k)$. Here if $n\mid m$ then the map $B_{et}(\mu_m)\to B_{et}(\mu_n)$ is induced by the hohmomorphism $\mu_m\to\mu_n, x\mapsto x^{m/n}$. The desired map is the composition $A(-1)\times\mathbb{G}_m\to A(-1)\times\lim_n B_{et}(\mu_n)\to B_{et}(A)$, where the second map is

$$(\mathrm{colim}_m \mathrm{Hom}(\mu_m,A))\times\lim_n B_{et}(\mu_n)\to B_{et}(A)$$

restricted to $\mathrm{Hom}(\mu_m,A)\times\lim_n B_{et}(\mu_n)$ is the composition $\mathrm{Hom}(\mu_m,A)\times\lim_n B_{et}(\mu_n)\to \mathrm{Hom}(\mu_m,A)\times B_{et}(\mu_m)\to B_{et}(A)$, the latter map being the extension of scalars via $f:\mu_m\to A$ of our μ_m-torsor.

My understanding is that the Kummer theory claims that the induced map $A(-1)\to \mathrm{Hom}_{\mathcal{G}\mathrm{rp(PreStk)}}(\mathbb{G}_m, B_{et}(A))$ is an isomorphism. The Kummer theory is: let T be a split torus over our field k. Then the canonical map

$$\mathrm{Hom}(\Lambda, A(-1))\to \mathrm{Hom}_{\mathcal{G}\mathrm{rp(PreStk)}}(T, B_{et}(A))$$

is an isomorphism. It associates to $T\to B_{et}(A)$ the map $\nu:\Lambda\to A(-1)$ such that for $\lambda\in\Lambda$, $\nu(\lambda)$ corresponds to the composition $\mathbb{G}_m\stackrel{\lambda}{\to}T\to B_{et}(A)$.

2.11. For 6.4.1. One needs to assume that that $\mathcal{C}$ is dualizable.

Dennis explained that if $\mathcal{Y} \in \mathrm{PreStk}$, $\mathcal{C} \in \mathrm{DGCat}$ is dualizable then we can garantee that $S \mapsto \mathrm{Shv}(S) \otimes \mathcal{C}$ is a sheaf in $\mathrm{Fun}(\mathrm{Sch}^{aff}/\mathcal{Y}, \mathrm{DGCat})$, not just a presheaf. Moreover, under this assumption, $\mathrm{colim}_{\mathrm{Tw}(I)} \mathrm{Shv}(S \times_{X^I} X^K) \otimes \mathcal{C}^{\otimes J}$ can be rewritten as a limit over $\mathrm{Tw}(I)^{op}$ of the right adjoint functors. For this reason $\mathrm{Fact}(\mathcal{C})$ will be a sheaf.

This works because for any surjection of finite non-empty sets $K \to K'$ the functor $\vartriangle_!\colon \mathrm{Shv}(X^{K'}) \to \mathrm{Shv}(X^K)$ admits a right adjoint.

2.12. Dennis claims the following **suprising thing!** Let A be a torsion abelian group, so $B(A) \in \mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$. Then there could be a nontrivial exact sequence $1 \to B(A) \to G \to \mathbb{Z} \to 1$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$. In other words, this is a fibre sequence in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$, and $\pi_0(G) \to \mathbb{Z}$ is surjective. There could be the situation when G is not isomorphic to $B(A) \times \mathbb{Z}$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$.

He proposes to take $G = B(A) \times \mathbb{Z}$ as an object on $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$ and to introduce a nontrivial commutativity constraint. Namely, define the commutativity constraint by the isomorphism: for $n, m \in \mathbb{Z}$,

$$(n+m, \mathcal{F}^1_A \otimes \mathcal{F}^2_A) = (n, \mathcal{F}^1_A)(m, \mathcal{F}^2_A) \widetilde{\to} (m, \mathcal{F}^2_A)(n, \mathcal{F}^1_A) = (n+m, \mathcal{F}^1_A \otimes \mathcal{F}^2_A)$$

given by multiplication by some $\beta(n,m) : \mathcal{F}^1_A \otimes \mathcal{F}^2_A \widetilde{\to} \mathcal{F}^1_A \otimes \mathcal{F}^2_A$. Here $\beta(n,m) \in A$, and $\mathcal{F}^i_A$ are A-torsors.

A definition of a strictly commutative Picard category (*champs de Picard strictement commutatifs*) is given in (SGA4, Exp. 17, Deligne, Formule de la dualité globale, Sect. 1.4.1). By this definition, to get a strictly commutative Picard category structure on the above G, we must impose the following conditions:

- for $n \in \mathbb{Z}$, $\beta(n,n) = 1$;
- for $n, m \in \mathbb{Z}$, $\beta(n,m)\beta(m,n) = 1$
- hexagon axiom, which in this case says that for $x, y, z \in \mathbb{Z}$,

$$\beta(y,z)\beta(x,z) = \beta(x+y,z)$$

(we write A multiplicatively). So, $\beta : \mathbb{Z} \times \mathbb{Z} \to A$ is bilinear, anti-symmetric and alternating. We see that in our case there is no nontrivial strictly commutative structure on $B(A) \times \mathbb{Z}$.

But there exist nontrivial commutative structures! Under the equivalence

$$\mathcal{C}\mathrm{omGrp}(\mathrm{Spc}) \widetilde{\to} \mathrm{Sptr}^{\leq 0}$$

(we use cohomological idexing conventions), the subcategory of $G \in \mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$ with $\pi_i(G) = 0$ for $i > 1$ becomes $\mathrm{Sptr}^{[-1,0]}$. This is the category of Picard groupoids described in ([21], Sections 2-3). For a free abelian group Λ of finite type and abelian group M, $\mathrm{Ext}^2_{\mathrm{Sptr}}(\Lambda, M) \widetilde{\to} \mathrm{Hom}(\Lambda, M_2)$, where $M_2 \subset M$ is the subgroup of 2-torsion in M.

2.13. The definition of $\mathrm{FactGe}^{mult}_A(\mathrm{Gr}_T)$ and $\mathrm{FactGe}^{com}_A(\mathrm{Gr}_T)$ was not given in the paper. Dennis meant the following definition.

There is the $(\infty,1)$-category $\mathit{FactPreStk}/\mathrm{Ran}$ of factorizable prestacks over Ran.

3. Comments to the 1st joint paper with Dennis: version July 4, 2018 (essentially the same as April 28, 2019)

3.0.1. For 0.4.6. For $C, D \in$ DGCat the tensor product $C \otimes D$ denotes the tensor product over Vect. The isomorphism $(R_1 - \text{ mod }) \otimes (R_2 - \text{ mod }) \widetilde{\to} (R_1 \otimes R_2) -$ mod is a particular case of (ch. 1, Prop. 8.5.4, [15]).

3.0.2. In 1.1.7 if H is a monoidal $(\infty, 1)$-category, by an action of H on $\mathcal{C} \in 1-\mathcal{C}$at we mean a monoidal functor $H \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$. We have a monoidal functor $B(E^*) \to \mathrm{Vect}$ sending a line ℓ to ℓ. Since Vect acts on any $\mathcal{C} \in \mathrm{DGCat}_{cont}$, we get an action of $B(E^*)$ on $\mathcal{C}$.

3.0.3. Recall that $\tau_{\le n}\mathrm{Spc} \subset \mathrm{Spc}$ is stable under filtered colimits (HTT, 5.3.5.6). This is used in 1.2.4: if $\mathcal{F} \in \mathrm{PreStk}_{lft}$ is such that its restriction to $(\mathrm{Sch}^{aff}_{ft})^{op}$ takes values in n-trunctaed spaces then $\mathcal{Y}$ itself is n-truncated.

3.0.4. In 1.2.5 the sheafification functor $L_{et} : \mathrm{PreStk} \to \mathrm{Stk}$ sends n-truncated objects to n-truncated objects, because it is left exact (HTT, 5.5.6.16).

3.0.5. For 1.2.6. The formula $Stk_{lft} := Stk \cap \mathrm{PreStk}_{lft} \subset \mathrm{PreStk}$ from that section is to be compared with 1st displayed formula in ([15], ch. I.2, 2.7.8).

3.0.6. For 1.3.3. Let Y be a prestack. Recall that we have an equivalence

$$F : \mathrm{PreStk}_{/Y} \widetilde{\to} \mathrm{Fun}((\mathrm{Sch}^{aff}_{/Y})^{op}, \mathrm{Spc})$$

Write Stk_Y for the category of objects of $\mathrm{Fun}((\mathrm{Sch}^{aff}_{/Y})^{op}, \mathrm{Spc})$ that satify the descent for the etale topology on the category $\mathrm{Sch}^{aff}_{/Y}$. Clearly, F sends $\mathrm{Stk}_{/Y}$ to the full subcategory Stk_Y. The obtained functor $\mathrm{Stk}_{/Y} \hookrightarrow \mathrm{Stk}_Y$ is fully faitful but not essentially surjective in general. For example, if Y is not a stack, consider the constant functor $f : (\mathrm{Sch}^{aff}_{/Y})^{op} \to \mathrm{Spc}$ with value $*$. Then $F^{-1}(f) \widetilde{\to} Y$, so it is not in $Stk_{/Y}$.

Write $L : \mathrm{Fun}((\mathrm{Sch}^{aff}_{/Y})^{op}, \mathrm{Spc}) \to \mathrm{Stk}_Y$ for the sheafification functor. Let $X \in \mathrm{PreStk}$, X_{et} its sheafification on Sch^{aff}. Is it true that $L(X \times Y)$ identifies with $X_{et} \times Y$? In the main body of the paper we rather use spaces like $\mathrm{Map}(S, B^2_{et}(A))$ without refering to any base prestack Y, that is, we rather use $X_{et} \times Y$ instead of $L(X \times Y)$.

If Z is a truncated prestack (taking values in $\tau_{\le m}\mathrm{Spc}$ for some m then for the etale sheafification $L_{et}(Z)$, the restriction of $L_{et}(Z)$ to $(\mathrm{Sch}^{aff}_{/Y})^{op}$ concides with the sheafification in the etale topology on $\mathrm{Sch}^{aff}_{/Y}$ of the composition $(\mathrm{Sch}^{aff}_{/Y})^{op} \to (\mathrm{Sch}^{aff})^{op} \xrightarrow{Z} \mathrm{Spc}$. This follows from the explicit formula for the sheafification of truncated prestacks ([15], ch. 2, 2.5.2).

In particular, for an abelian group A the restriction of $B^i_{et}(A)$ to $(\mathrm{Sch}^{aff}_{/Y})^{op}$ coincides with $B^i_{et,/Y}(A)$.

3.0.7. The inclusion $\mathrm{PreStk}_{lft} \subset \mathrm{PreStk}$ is stable under the finite limits because of (HTT, 5.3.4.7) and under all colimits. See also ([15], ch. 2, 1.6.8). In particular, if $F^\bullet : \boldsymbol{\Delta}^{op} \to \mathrm{PreStk}_{lft}$ then $\mid F^\bullet \mid$ is also locally of finite type. In particular, if $G \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, and $G \in \mathrm{PreStk}_{lft}$ then $B(G) \in \mathrm{PreStk}_{lft}$.

Example, if $Z \in \mathrm{Spc}$, we may consider the constant prestack $\underline{Z}$ with value Z. It is locally of finite type. Indeed, for any $Y \in \mathrm{PreStk}$,

$$\mathrm{Map}(Y, \underline{Z}) = \mathrm{Map}_{\mathrm{PreStk}}(Y, \underline{Z}) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(Y(\emptyset), Z)$$

If $S \in \mathrm{Sch}^{aff}$ then $S(\emptyset) = *$, and we get $\mathrm{Map}_{\mathrm{PreStk}}(S, \underline{Z}) \widetilde{\to} Z$. So, if $S = \lim_{i \in I} S_i$ is a filtered limit in Sch^{aff} then $\mathrm{Map}(S, \underline{Z}) \widetilde{\to} \operatorname{colim}_i \mathrm{Map}(S_i, \underline{Z})$, because I is contractible.

In particular, if A is an abelian group then $\underline{A} \in \mathrm{PreStk}_{lft}$, hence $B^i(A) \in \mathrm{PreStk}_{lft}$ for any i. Now $B^i_{et}(A) \in \mathrm{Stk}_{lft}$ by Cor. 1.2.8 from the paper.

3.0.8. Let $K, \mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ and $\mathcal{C}$ admits finite limits. Then for $k \geq 0$,

$$\mathbb{E}_k(\mathrm{Fun}(K, \mathcal{C})) \widetilde{\to} \mathrm{Fun}(K, \mathbb{E}_k(\mathcal{C}))$$

naturally. So, if $X \in \mathbb{E}_k(\mathcal{C})$, $Y \in \mathcal{C}$ then $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is naturally an object of $\mathbb{E}_k(\mathrm{Spc})$. Indeed, the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ induces by applying $\mathbb{E}_k$ a functor $\mathbb{E}_k(\mathcal{C}) \to \mathbb{E}_k(\mathcal{P}(\mathcal{C})) \widetilde{\to} \mathrm{Fun}(\mathcal{C}^{op}, \mathbb{E}_k(\mathrm{Spc}))$, because the Yoneda embedding preserves all limits, which exist in $\mathcal{C}$ by (HTT, 5.1.3.2). The diagram commutes

$$\begin{array}{ccc} \mathbb{E}_k(\mathcal{C}) & \to & \mathrm{Fun}(\mathcal{C}^{op}, \mathbb{E}_k(\mathrm{Spc})) \\ \downarrow & & \downarrow \\ \mathcal{C} & \to & \mathcal{P}(\mathcal{C}), \end{array}$$

where the vertical arrow are the oblivion (forgetful) functors. This is used in Section 1.3.2 of the paper.

If moreover $X \in \mathcal{G}\mathrm{rp}(\mathcal{C})$ then $\mathrm{Map}_{\mathcal{C}}(Y, X) \in \mathcal{G}\mathrm{rp}(\mathrm{Spc})$. Indeed, this follows from ([22], Remark 2.5.18).

3.0.9. The $\mathbb{Z}$-module $\mu_\infty^{pro} = \lim \mu_n$ is flat, bacause it is torsion free. Here the limit is taken over the poset $\mathbb{N}$. If $n \mid m$ then $\mu_m \to \mu_n, x \mapsto x^{m/n}$. For any $n \geq 1$ we have $\mu_\infty^{pro} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \widetilde{\to} \mu_n$. If A is a torsion abelian group then for any $n \geq 1$, $\mu_n \otimes_{\mathbb{Z}/n\mathbb{Z}} A_{n-tors} \widetilde{\to} \mu_\infty^{pro} \otimes_{\mathbb{Z}} A_{n-tors} \subset \mu_\infty^{pro} \otimes_{\mathbb{Z}} A$ is a subgroup. Tensor product commutes with colimits, so $\mu_\infty^{pro} \otimes_{\mathbb{Z}} A \widetilde{\to} \operatorname{colim}_n(\mu_\infty^{pro} \otimes_{\mathbb{Z}} A_{n-tors})$.

In 1.4.1 given $n \mid n' \mid n''$, we identify $\mu_{n'} \otimes_{\mathbb{Z}/n'\mathbb{Z}} A_{n-tors}$ with $\mu_{n''} \otimes_{\mathbb{Z}/n''\mathbb{Z}} A_{n-tors}$ via the map $\mu_{n''} \to \mu_{n'}$, $x \mapsto x^{\frac{n''}{n'}}$. In

$$\operatorname*{colim}_{n \in \mathbb{N}} (\mu_{n'} \otimes_{\mathbb{Z}/n'\mathbb{Z}} A_{n-tors})$$

the transition maps are as follows. Given $n \mid m \mid m'$, we have $\mu_{m'} \otimes_{\mathbb{Z}/m'\mathbb{Z}} A_{n-tors} \hookrightarrow \mu_{m'} \otimes_{\mathbb{Z}/m'\mathbb{Z}} A_{m-tors}$.

3.0.10. For 1.4.2. A generalization of this procedure for T-torsors instead of line bundes. Let $\Lambda_1 \subset \Lambda$ be free abelian groups (subgroup of finite index). Let $T_1 = \Lambda_1 \otimes \mathbb{G}_m, T = \Lambda \otimes \mathbb{G}_m$. The map $T_1 \to T$ is surjective, let K be its kernel. Then $K \widetilde{\to} (\Lambda/\Lambda_1)(1)$ canonically. We have the natural map $T \to B_{et}(K)$ in $ComGrp(\mathrm{Stk})$, hence in turn $B(T) \to B^2_{et}(K)$ in $ComGrp(\mathrm{Stk})$. So, for a homomorphism $a : K \to A$, each T-torsor $\mathcal{F}_T$ on a prestack Y yields a A-gerbe $(\mathcal{F}_T)^a$ via extension of scalars. It is referred to in 4.3.6 of version June 1, 2020.

3.0.11. For 1.5.4: the functor Shv : $(\mathrm{PreStk}_{lft})^{op} \to \mathrm{DGCat}$ preserves small limits by (HTT, 5.1.5.5). Indeed, its opposite $(\mathrm{PreStk}_{lft}) \to \mathrm{DGCat}^{op}$ is the LKE under $\mathrm{Sch}^{aff}_{ft} \hookrightarrow \mathrm{PreStk}_{lft} \widetilde{\to} \mathcal{P}(\mathrm{Sch}^{aff}_{ft})$.

The symmetric monoidal structure on $(\mathrm{Sch}^{aff}_{ft})^{op}$ is cocartesian (HA, 3.2.4.10), so $CAlg((\mathrm{Sch}^{aff}_{ft})^{op}) \widetilde{\to} (\mathrm{Sch}^{aff}_{ft})^{op}$ by (HA, 2.4.3.10). This is why the symmetric monoidal structure on the functor $Shv : (\mathrm{Sch}^{aff}_{ft})^{op} \to \mathrm{DGCat}$ gives rise to a functor $(\mathrm{Sch}^{aff}_{ft})^{op} \to CAlg(\mathrm{DGCat})$.

The category $(\mathrm{PreStk}_{lft})^{op}$ admits finite colimits, we consider it as equipped with the cocartesian symmetric monoidal structure, so $CAlg((\mathrm{PreStk}_{lft})^{op}) \widetilde{\to} (\mathrm{PreStk}_{lft})^{op}$.

Consider the functor $Shv : (\mathrm{PreStk}_{lft})^{op} \to \mathrm{DGCat}$. It inherits a right-lax non-unital symmetric monoidal structure? Nonrigorous explanation: if $Y_1, Y_2 \in \mathrm{PreStk}_{lft}$ then pick presentations $Y_1 \widetilde{\to} \mathrm{colim}_i\, S_1^i$, $Y_2 \widetilde{\to} \mathrm{colim}_j\, S_2^j$ with $S_1^i, S_2^j \in \mathrm{Sch}^{aff}_{ft}$. Then clearly $Y_1 \times Y_2 \widetilde{\to} \mathrm{colim}_{i,j}\, S_1^i \times S_2^j$ in PreStk_{lft}, as PreStk_{lft} is an ∞-topos (colimits are universal). This gives a natural map $\mathrm{Shv}(Y_1) \otimes \mathrm{Shv}(Y_2) \to \mathrm{Shv}(Y_1 \times Y_2) \widetilde{\to} \lim_{i,j} \mathrm{Shv}(S_1^i \times S_2^j)$, because $Shv(Y_1) \widetilde{\to} \lim_i Shv(S_1^i)$ and similarly for Y_2.

Recall that if $f : \mathcal{C}_1 \to \mathcal{C}_2$ is a symmetric monoidal functor between symmetric monoidal ∞-categories then $f^{op} : \mathcal{C}_1^{op} \to \mathcal{C}_2^{op}$ is also symmetric monoidal. So, $Shv : \mathrm{Sch}^{aff}_{ft} \to \mathrm{DGCat}^{op}$ is symmetric monoidal. However, we can not apply now (HA, 4.8.1.10) to its left Kan extension $\mathcal{P}(\mathrm{Sch}^{aff}_{ft}) \to \mathrm{DGCat}^{op}$, as DGCat^{op} does not satisfy the assumptions.

3.0.12. For 1.6.2. The rigorous definition of the ∞-category $ShvCat(Y)$ of sheaves of DG-categories over $Y \in \mathrm{PreStk}$ is given as in ([9], Sect. 1.1.1). Namely, we take the RKE of the functor $(\mathrm{Sch}^{aff}_{ft})^{op} \to 1-\mathcal{C}\mathrm{at}$, $S \mapsto Shv(S)-mod$ with respect to $(\mathrm{Sch}^{aff}_{ft})^{op} \subset (\mathrm{PreStk}_{lft})^{op}$. We see also $ShvCat(Y)$ is a symmetric monoidal ∞-category. Indeed, the forgetful functor $CAlg(1-\mathcal{C}\mathrm{at}) \to 1-\mathcal{C}\mathrm{at}$ preserves limits, so we may first consider the functor $(\mathrm{Sch}^{aff}_{ft})^{op} \to CAlg(1-\mathcal{C}\mathrm{at})$, then take its RKE to $(\mathrm{PreStk}_{lft})^{op}$. The category $ShvCat(Y)$ admits small colimits (proof using my [22], around Lemma 2.2.67).

On the other hand, it is not clear if the category $ShvCat(Y)$ admits small limits, because for a map $f : S \to S'$ in Sch^{aff} the functor $f^! : Shv(S') \to Shv(S)$ does not preserve small limits. It only does preserve them for f proper, Dennis says. Indeed, for f proper, $f^! : Shv(S') \to Shv(S)$ is known to admit a left adjoint. For the sheaf theory of $\mathcal{D}$-modules, $ShvCat(Y)$ admits limits, see my Section 3.7.

We get a functor $\Gamma^{enh} : ShvCat(Y) \to Shv(Y)-mod$ as in [9], it is right-lax symmetric monoidal. It has a left adjoint $\mathrm{Loc}_Y : Shv(Y)-mod \to ShvCat(Y)$ sending C to the sheaf of categories $S \mapsto C \otimes_{Shv(Y)} Shv(S)$. The functor Loc_Y is symmetric monoidal.

If $S \in \mathrm{Sch}^{aff}_{ft}$, $Z \in \mathrm{PreStk}_{lft}$ and $f : Z \to S$ then the functor $cores_f : ShvCat(S) \to ShvCat(Z)$ of restriction has a right adjoint. Indeed, by (Th. 2.6.3, [9]), S_{dR} is 1-affine. Now $cores_f$ is the composition $Shv(S)-mod \to Shv(Z)-mod \stackrel{\mathrm{Loc}_Z}{\to} ShvCat(Z)$. Each functor in this diagram has a right adjoint, so $cores_f$ also has one.

For an arbitary map $f : Y_1 \to Y_2$ in PreStk_{lft} the functor $cores_f : ShvCat(Y_2) \to ShvCat(Y_1)$ probably does not have a right adjoint, not clear.

3.0.13. For 1.6.4. If Y is an ind-scheme of ind-finite type then Y_{dR} is 1-affine. By definition, Y is a filtered colimit of $S_i \in \mathrm{Sch}^{aff}_{ft}$, the transition maps $S_i \to S_j$ being closed immersions. For $Y \in \mathrm{PreStk}$, $D-mod(S)$ is defined as $\mathrm{QCoh}(Y_{dR})$. For any map $f : S_1 \to S_2$ in Sch^{aff}_{ft} and a prestack $\mathcal{Z}$ locally of finite type with $\mathcal{Z} \to S_2$ the functor

$$Shv(S_1) \otimes_{Shv(S_2)} Shv(Z) \to Shv(S_1 \times_{S_2} Z) \tag{4}$$

is an equivalence. Indeed, apply ([9], Lemma 3.2.4) for $(S_1)_{dR} \stackrel{f}{\to} (S_2)_{dR} \stackrel{h}{\leftarrow} Z_{dR}$ and the sheaf $\mathcal{C} = coind_h(\mathrm{QCoh})$ using the fact that $(S_i)_{dR}$ is 1-affine by ([9], Th. 2.6.3).

Remark 3.0.14. *If $S \in \mathrm{Sch}^{aff}$ is not of finite type then we don't know if S_{dR} is 1-affine. For this reason, it is not clear if (4) is an isomorphism. For this reason, we should note that in* [9] *the original definition of a "quasi-coherent sheaf of categories" used the whole category $\mathrm{Sch}^{aff}_{/Y}$ for a prestack Y. In our situation, $ShvCat(Y)$ is defined as $\lim_{(\mathrm{Sch}^{aff}_{ft})_{/Y}} Shv(S)-mod$. Such theory of sheaves seems to be adopted to prestacks locally of finite type. Indeed, for $Y \in \mathrm{PreStk}_{lft}$, $Y = \mathrm{colim}_{S \to Y} S$, the colimit over $(\mathrm{Sch}^{aff}_{ft})_{/Y}$. So, for $Y \in \mathrm{PreStk}_{lft}$,*

$$\lim_{(\mathrm{Sch}^{aff}_{/Y})^{op}} Shv(S)-mod \widetilde{\to} \lim_{((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op}} Shv(S)-mod$$

3.0.15. For 1.6.5. Let $Y \in \mathrm{PreStk}_{lft}$, $\mathcal{C}$ be a sheaf of DG-categories over Y. Let $\bar{\mathcal{C}} : ((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op} \to \mathrm{DGCat}$ be the functor obtained from $\mathcal{C}$ by forgetting the $\mathrm{Shv}(S)$-module structure on each $\mathcal{C}(S, y)$. Then $\bar{\mathcal{C}}$ satisfies the etale descent. Here ([9], Th. 1.5.2) is good but is not sufficient.

The following is true. Let $T \to S$ be an etale surjective map in Sch. Then $T_{dR} \to S_{dR}$ is an etale surjection in PreStk. Indeed, let $S' \in \mathrm{Sch}^{aff}$ and $y : S' \to S_{dR}$ be any map given by $S'_{red} \to S$. Let us show that $S' \times_{S_{dR}} T_{dR}$ is a scheme etale over S'. First, there is an equivalence of categories $\{$schemes etale over $S'\} \widetilde{\to} \{$scheme etale over $S'_{red}\}$ given by $U \mapsto U \times_{S'} S'_{red}$, see ([35], 15.2). So, the etale surjective map $\bar{a} : S'_{red} \times_S T \to S'_{red}$ yields an etale morphism $a : T' \to S'$, where T' is a scheme. The base change of a by $S'_{red} \hookrightarrow S'$ is $\bar{a}$. We claim that

$$S' \times_{S_{dR}} T_{dR} \widetilde{\to} T'$$

over S'. Indeed, for $Z \in \mathrm{Sch}^{aff}$, Z-point of $S' \times_{S_{dR}} T_{dR}$ is a map $Z \to S'$ and a compatible map $Z_{red} \to S'_{red} \times_S T$ over S'_{red}. By ([35], 15.1), this is precisely a datum of a map $Z' \to T'$.

Now by ([9], 1.5.5), $Shv : (\mathrm{Sch}^{aff}_{ft})^{op} \to \mathrm{DGCat}$ satisfies etale descent. **Question**. How this argument extends to any sheaf $\mathcal{C} \in ShvCat(Y)$ for a prestack locally of finite type as in our Sect. 1.6.5?

It seems, in ([16], Sect. 3) it is proved more generally that Shv satisfy fppf descent (namely, crystals satisfy it).

3.0.16. If $Y \in \mathrm{PreStk}_{lft}$ then $ShvCat(Y)$ admits limits? It is not clear. The problem is to check that if $S \to S'$ is a map in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$ then the functor $Shv(S') - mod \to Shv(S) - mod$, $C \mapsto C \otimes_{Shv(S')} Shv(S)$ preserves limits.

It is true that for $S \in \mathrm{Sch}^{aff}_{ft}$, $Shv(S) = \mathrm{QCoh}(S_{dR})$ is dualizable. Indeed, the latter category is compactly generated.

To see this, use ([16], Lemmas 2.2.6) saying that $oblv^l : \mathrm{QCoh}(S_{dR}) \to \mathrm{QCoh}(S)$ is conservative, and ([16], 3.4.7) saying that this $oblv^l$ has a left adjoint. Apply ([15], ch. I.1, 5.4.3) and the fact that $\mathrm{QCoh}(S)$ is compactly generated.

Though we know dualizability of $Shv(S)$, it is not clear if $Shv(S)$ is dualizable as a $Shv(S')$-module, because $Shv(S')$ is not rigid in general. However, we know this for $\mathcal{D}$-modules (Sam Raskin email of 6.02.2020 and Lin Chen).

3.0.17. For 1.6.6. There we may take indeed arbitrary colimits in the formula for $\mathcal{C}(Z)$ because of the following. Let $\mathcal{C} \in 1-\mathcal{C}at$ be small, $\mathcal{D} \in 1-\mathcal{C}at$ be cocomplete, $Y \in \mathcal{P}(\mathcal{C})$ and $\mathcal{C}_{/Y} = \mathcal{C} \times_{\mathcal{P}(\mathcal{C})} \mathcal{P}(\mathcal{C})_{/Y}$. Let $f : \mathcal{C}_{/Y} \to \mathcal{D}$ be a functor, $\bar{f} : \mathcal{P}(\mathcal{C})_{/Y} \to \mathcal{D}$ be the LKE of f along $\mathcal{C}_{/Y} \hookrightarrow \mathcal{P}(\mathcal{C})_{/Y}$. Then $\bar{f}$ preserves colimits (see [22], Lm. 2.2.40).

In Sect. 1.6.6 the assumption $Z \widetilde{\to} \mathrm{colim}_i S_i$ means that $(S_i, y_i) \in \mathrm{Sch}^{aff}_{ft}/_{\mathcal{Y}}$ and the colimit is taken in $(\mathrm{PreStk}_{lft})_{/\mathcal{Y}}$, or what is the same, in $\mathrm{PreStk}_{lft} \widetilde{\to} \mathcal{P}(\mathrm{Sch}^{aff}_{ft})$.

3.0.18. For 1.6.7. The colimits in PreStk are universal. Let $\mathcal{Z} \to \mathcal{Y}$ be a map in PreStk_{lft}. Since $\mathrm{colim}_{(S \to \mathcal{Y}) \in (\mathrm{Sch}^{aff}_{ft})_{/\mathcal{Y}}} S \widetilde{\to} \mathcal{Y}$ in PreStk_{lft} and PreStk, we get

$$\underset{(S \to \mathcal{Y}) \in (\mathrm{Sch}^{aff}_{ft})_{/\mathcal{Y}}}{\mathrm{colim}} S \times_{\mathcal{Y}} \mathcal{Z} \widetilde{\to} \mathcal{Z}$$

So, for $\mathcal{C} = \mathrm{Shv}(\mathcal{Z})_{/\mathcal{Y}}$ we get $\mathcal{C}(\mathcal{Y}) \widetilde{\to} \underset{(S \to \mathcal{Y}) \in ((\mathrm{Sch}^{aff}_{ft})_{/\mathcal{Y}})^{op}}{\lim} \mathrm{Shv}(S \times_{\mathcal{Y}} \mathcal{Z}) \widetilde{\to} \mathrm{Shv}(\mathcal{Z})$ by Sect. 1.5.4 of the paper.

3.0.19. For 1.6.8. The fact that these functors are mutually adjoint is proved as in ([9], 1.3.1), where there is no proof actually. I wrote down the corresponding proof in my file ([24], 0.0.4).

3.0.20. Let $\mathcal{C}$ be a small category, $Y \in \mathcal{P}(\mathcal{C})$. Consider the functor $a : \mathcal{P}(\mathcal{C})_{/Y} \to \mathcal{P}(\mathcal{C}_{/Y})$ sending Z to the presheaf $(c \stackrel{\alpha}{\to} Y) \mapsto Z(c) \times_{Y(c)} \{\alpha\}$. Consider also the functor $b : \mathcal{P}(\mathcal{C}_{/Y}) \to \mathcal{P}(\mathcal{C})_{/Y}$ sending $Z' : (\mathcal{C}_{/Y})^{op} \to \mathrm{Spc}$ to the presheaf given informally by $S \mapsto \{\alpha \in \mathcal{Y}(S), z \in Z'(S, \alpha)\}$. The formal definition: let $\bar{Z}' \to (\mathcal{C}_{/Y})^{op}$ be the cocartesian fibration corresponding to Z'. Then $b(Z')$ is the functor $\mathcal{C}^{op} \to \mathrm{Spc}$ such that the corresponding cocartesian fibration in spaces over $\mathcal{C}^{op}$ is the composition $\bar{Z}' \to (\mathcal{C}_{/Y})^{op} \to \mathcal{C}^{op}$. Then a and b are inverses of each other.

3.0.21. *Category of equivariant sheaves.* Sam explains by email how to define the category of equivariant objects (our definition of the twist of a sheaf of categories by a gerbe in 1.7.2 is not rigorous). If the map $f : X \to Y$ in Sch^{aff} is smooth of relative dimension n then the functor $f^! : Shv(Y) \to Shv(X)$ admits a continuous right adjoint, say $f_*[-2n] : Shv(X) \to Shv(Y)$. Now if $f : X \to Y$ is an affine schematic morphism in PreStk, assume it is smooth of some relative dimension. That is, for any $S \in \mathrm{Sch}^{aff}_{/Y}$, $S \times_Y X \to S$ is a smooth morphism of affine schemes. Then the functor $f^! : Shv(Y) \to Shv(X)$ also admit a continuous right adjoint by ([15], ch. 1, 2.6.4).

More generally, if $I \to \mathrm{Fun}([1], \mathrm{PreStk}_{lft})$ is a functor sending i to $(X_i \xrightarrow{f_i} Z_i)$, assume that each $f_i^! : Shv(Z_i) \to Shv(X_i)$ admits a continuous right adjoint $(f_i)_*[-2n]$. Then let $f : X = \mathrm{colim}\, X_i \to Z = \mathrm{colim}\, Z_i$ be the colimit in PreStk_{lft}. For any $i \to j$ in the index category let $\alpha_{ij} : X_i \to X_j$ and $\beta_{ij} : Z_i \to Z_j$ denote the transition maps, assume the square is cartesian

$$\begin{array}{ccc} X_j & \xrightarrow{f_j} & Z_j \\ \uparrow & & \uparrow \\ X_i & \xrightarrow{f_i} & Z_i \end{array}$$

Then the diagram commutes

$$\begin{array}{ccc} Shv(X_i) & \xleftarrow{\alpha_{ij}^!} & Shv(X_j) \\ \downarrow (f_i)_* & & \downarrow (f_j)_* \\ Shv(Z_i) & \xleftarrow{\beta_{ij}^!} & Shv(Z_j) \end{array}$$

Then $f^! : Shv(Z) \to Shv(X)$ also admits a continuous right adjoint by ([15], ch. 1, 2.6.4). This is because our functors are actually functors out of correspondences (see [15]).

Let G be a group object of Sch^{aff}, it is given by a functor $\mathcal{G} : \boldsymbol{\Delta}^{op} \to \mathrm{Sch}^{aff}$. Assume G is locally of finite type and smooth of dimension n. Then for any map $\alpha : [i] \to [j]$ in $\boldsymbol{\Delta}$ the induced map $\mathcal{G}^\alpha : \mathcal{G}_j \to \mathcal{G}_i$ is such that $(\mathcal{G}^\alpha)^! : Shv(\mathcal{G}_i) \to Shv(\mathcal{G}_j)$ admits a right adjoint. Passing to the right adjoints in the functor $\boldsymbol{\Delta} \xrightarrow{\mathcal{G}} (\mathrm{Sch}^{aff})^{op} \xrightarrow{Shv} \mathrm{DGCat}_{cont}$, we get a functor $\boldsymbol{\Delta}^{op} \to \mathrm{DGCat}_{cont}$. I think this uses the $(\infty, 2)$-category structure on DGCat_{cont}, and the procedure of passing to right adjoint is described in ([15], vol. 1, Appendix: $(\infty, 2)$-categories).

Then incorporating shifts and additionally composing with the corresponding morphisms $Shv(G) \otimes \ldots \otimes Shv(G) \to Shv(G \times \ldots \times G) = Shv(G_m)$ for all $m \geq 0$, we get on $Shv(G)$ a structure of a monoidal DG-category, that is, an algebra object in DGCat. So, the product in $Shv(G)$ is given by $\mathrm{Shv}(G) \otimes Shv(G) \to Shv(G \times G) \xrightarrow{m_*} Shv(G)$. Even if Shv is only right-lax monoidal, this construction works.

A better explanation (similar to the one given in [15], ch. I.3, 2.2.4 for quasi-coherent sheaves, see also ([22], 10.2.5)): consider the 1-full subcategory $\mathrm{PreStk}_{ind-sch} \subset \mathrm{PreStk}_{lft}$, where we restrict 1-morphisms to be ind-schematic. Then we have a well-defined functor

$$Shv_{\mathrm{PreStk}_{ind-sch}} : \mathrm{PreStk}_{ind-sch} \to \mathrm{DGCat}_{cont}$$

sending Y to $Shv(Y)$ and a morphism $f : Y \to Y'$ to $f_* : Shv(Y) \to Shv(Y')$. Moreover, this functor is right-lax symmetric monoidal, so sends algebras to algebras. So, if G is an algebra in $\mathrm{PreStk}_{ind-sch}$, $Shv(G)$ will become a monoidal DG-category with the monoidal convolution structure.

Similarly, if $X \in \mathrm{PreStk}_{lft}$ is equipped with a G-action then for any map $\alpha : [i]^+ \to [j]^+$ in $\boldsymbol{\Delta}^+$ and the corresponding map $T^\alpha : G \times \ldots \times G \times X \to G \times \ldots \times G \times X$ in PreStk_{lft} the correspnding functor $(T^\alpha)^! : Shv(G\times\ldots\times G\times X) \to Shv(G\times\ldots\times G\times X)$ admits a right adjoint. For example, the action map $G \times X \to X$ is smooth, as it is the composition $G \times X \stackrel{\mathrm{act}\times\mathrm{pr}_1}{\to} X \times G \stackrel{\mathrm{pr}_1}{\to} X$. By the same token, we see that $Shv(X)$ is equipped with a left action of $Shv(G)$. The action map is the composition

$$Shv(G) \otimes Shv(X) \to \mathrm{Shv}(G \times X) \stackrel{m_*}{\to} Shv(X)$$

Let now L be a character sheaf on G. Sam says that since L is placed in the heart of the t-structure of $Shv(G)$, the notion of a character sheaf should not involve any coherent-homotopy issues. What is the precise claim?

Our L is a local system on G equipped with $m^*L \widetilde{\to} L\boxtimes L$, where $m^* = m^![-2n]$, $n = \dim G$. For the unit $i : \mathrm{Spec}\, k \to G$ we have a distinguished trivialization $c : i^*L \widetilde{\to} E$. Note that i_*E is the unit of the convolution monodal structure on G. Thus, c yields the counit map $L \to i_*E$ in $Shv(G)$, and L is naturally a coalgebra in $Shv(G)$ for the convolution monoidal structure. Sam proposes to define the category $\mathrm{Shv}(X)^{(G,L)}$ of sheaves on X that are (G, L)-equivariant as $L - comod(Shv(X))$, the category of comodules for this comonad.

In such a way, given a $E^{\times,tors}$-gerbe on $Y \in \mathrm{PreStk}$, one defines $Shv_{\mathcal{G}}(Y)$. Namely, let $\tilde{Y}$ be the total space of this gerbe, so this is a $B_{et}(E^{\times,tors})$-torsor over Y. Equip $Shv(B_{et}(E^{\times,tors}))$ with the convolution monoidal structure, then it acts naturally on $Shv(\tilde{Y})$. Besides, $E \in Shv(B_{et}(E^{\times,tors}))$ is a character sheaf on this stack, and $E^{\times,tors}$ acts on it by the tautological character. So, E is a coalgebra in $Shv(B_{et}(E^{\times,tors}))$ giving rise to a comonad on $Shv(\tilde{Y})$. Then $Shv_{\mathcal{G}}(Y)$ is defined as the category of comodules over this comonade. More general definition of the twist is giving in my Section 3.6.1.

3.0.22. Consider the situation in the previous subsection with $L = E$. Recall that $Shv(X/G) \widetilde{\to} \lim[Shv(X) \rightrightarrows \mathrm{Shv}(G \times X) \substack{\to\\\to\\\to} \ldots]$ taken in DGCat_{cont}. We claim that the natural functor $ev^0 : Shv(X/G) \to \mathrm{Shv}(X)$ is comonadic. Namely, apply ([9], Lemma C.1.9). To check that our co-simplicial category satisfies the ([9], Def. C.1.3), we note that for any n for the map $\mathrm{id}\times\mathrm{act} : (G\times\ldots\times G)\times G\times X \to (G\times\ldots\times G)\times X$ the functor $(\mathrm{id}\times\mathrm{act})^! : Shv((G \times \ldots \times G) \times X) \to Shv((G \times \ldots \times G) \times G \times X)$ admits a right adjoint, and for any map α in $\boldsymbol{\Delta}$ denoting $\alpha^! : Shv(G \times \ldots \times G \times X) \to Shv(G\times\ldots\times G\times X)$ the corresponding map, we have $\alpha^!(\mathrm{id}\times\mathrm{act})_* \widetilde{\to} (\mathrm{id}\times\mathrm{act})_*(\alpha+1)^!$.

The corresponding comonad is the functor $(\mathrm{act})_*\,\mathrm{pr}^* : \mathrm{Shv}(X) \to \mathrm{Shv}(X)$ for $\mathrm{act} : G \times X \to X, \mathrm{pr} : G \times X \to X$ and $n = \dim G$. Here $\mathrm{pr}^* = \mathrm{pr}^![-2n]$. We see that this comonad comes from the fact that the constant sheaf E is a coalgebra in $\mathrm{Shv}(G)$ for the convolution monoidal structure. This justifies the definition of $Shv(X)^{(G,L)}$ from the previous subsection.

3.0.23. $\mathrm{Ran} \in \mathrm{PreStk}_{lft}$, because $\mathrm{PreStk}_{lft} \subset \mathrm{PreStk}$ is stable under all colimits.

3.0.24. The 1-affineness of S_{dR} for $S \in \mathrm{Sch}_{ft}^{aff}$ gives the following. First, if $\mathcal{Z} \in \mathrm{PreStk}_{lft}$, $S \in \mathrm{Sch}_{ft}^{aff}$ then $\mathrm{Shv}(S \times \mathcal{Z}) \widetilde{\to} Shv(S) \otimes Shv(\mathcal{Z})$. Assume now $Shv(\mathcal{Z})$ dualizable. Recall that for $\mathcal{C} \in \mathrm{DGCat}$ dualizable, the functor $\mathrm{DGCat} \to \mathrm{DGCat}$, $D \mapsto D \otimes \mathcal{C}$ commutes with limits by ([22], Lm. 3.1.2). So, if $\mathcal{Z}' \in \mathrm{PreStk}_{lft}$ is written as $\mathcal{Z}' \widetilde{\to} \mathrm{colim}_i S_i$ in PreStk_{lft} with $S_i \in \mathrm{Sch}_{ft}^{aff}$ then $Shv(\mathcal{Z}') \widetilde{\to} \lim_i Shv(S_i)$ and $Shv(\mathcal{Z}') \otimes Shv(\mathcal{Z}) \widetilde{\to} \lim_i(Shv(S_i) \otimes Shv(\mathcal{Z})) \widetilde{\to} \lim_i Shv(S_i \times \mathcal{Z}) \widetilde{\to} \mathrm{Shv}(\mathcal{Z}' \times \mathcal{Z})$, because $\mathcal{Z}' \times \mathcal{Z} \widetilde{\to} \mathrm{colim}_i(S_i \times \mathcal{Z})$ in PreStk_{lft}.

This was used in 2.2.3: if Z is a factorizable prestack such that $\mathrm{Shv}(X^I \times_{\mathrm{Ran}} Z)$ is dualizable for any I then for a surjection of finite nonempty sets $I \to J$ one gets

$$\underset{j\in J}{\otimes} Shv(X^{I_j} \times_{\mathrm{Ran}} Z) \widetilde{\to} Shv(\prod_{j\in J}(X^{I_j} \times_{\mathrm{Ran}} Z))$$

We also used the following consequence of Th. 1.6.9: if $S \in \mathrm{Sch}_{ft}$ non necessarily affine then the functors denoted (1.14) and (1.15) in the paper are equivalences. This is why it suffices to get the equivalence of Section 2.2.3 in the case $S = X^I_{disj,J}$ for $\pi : I \to J$ surjective, this scheme is not necessarily affine! Here $X^I_{disj,J}$ is the scheme of $(x_i) \in X^I$ such that if $\pi(i) \neq \pi(i')$ then $x_i \neq x_{i'}$. There is a misprint in the paper, where the scheme X^I_{disj} is mentioned instead.

We also used the following: given symmetric monoidal DG-categories A_i with $C_i \in A_i - mod$ and a map $\otimes_{i=1}^n A_i \to B$ in $\mathrm{DGCat}_{cont}^{SymMon}$, we have

$$(\otimes_i C_i) \otimes_{(\otimes A_i)} B \widetilde{\to} \otimes_{i,B} (C_i \otimes_{A_i} B)$$

(trivial: extend the scalars first to $\otimes_i A_i$ and then to B). The first isomorphism in the long displayed formula in the paper uses the fact that $Shv(X^{I_j} \times_{\mathrm{Ran}} Z) \otimes_{\mathrm{Shv}(X^{I_j})} Shv(X^I) \widetilde{\to} Shv(X^I \times_{\mathrm{Ran}} Z)$ for the projection $X^I \to X^{I_j}$ by 1.6.4.

3.0.25. For 1.6.9. The reference for [Ga1, Th. 1.5.2] in the paper is a wrong reference, the correct one is [Ga1, Th. 2.6.3].

3.0.26. *Factorization prestacks over* Ran. For 2.2.1. Let $Z \to \mathrm{Ran}_X$ be a map in PreStk. The definition of a factorization structure on Z is not precise. The correct one is given as in [30]. Namely, let PreStk_{corr} be the category of correspondences in prestacks ([30], 4.28). Equip Ran_X with the structure of a non-unital commutative algebra in PreStk_{corr} given by the chiral mutliplication. The chiral product in Ran_X is given by $\mathrm{Ran}_X^2 \leftarrow \mathrm{Ran}^2_{X,disj} \to \mathrm{Ran}_X$. Then $Z \to \mathrm{Ran}_X$ has to be a morphism of non-unital commutative algebras in PreStk_{corr} such that for any nonempty finite set J the induced map

$$Z^J \times_{\mathrm{Ran}_X^J} (\mathrm{Ran}_X^J)_{disj} \to Z \times_{\mathrm{Ran}_X} (\mathrm{Ran}_X^J)_{disj}$$

is an isomorphism.

Similarly, let C be a sheaf of DG-categories over Ran_X (in the sense of ([13], 1.6.2). A precise definition of a factorization structure on C is a non-unital chiral category ([30], Def. 6.2.1).

3.0.27. For 2.2.4. The rigorous definition of a factorization gerbe is as follows. Let Z be a factorization prestack over Ran, A be a torsion abelian group. Since A is a commutative group in Spc, $B^2_{et}(A)$ is a commutative group in PreStk, hence also in PreStk_{corr}. So, $\mathrm{Ran}\times B^2_{et}(A)$ is an object of $CAlg^{nu}(\mathrm{PreStk}_{corr})$, the category of non-unital commutative algebras in PreStk_{corr}. The space of factorization gerbes on Z is the space

$$\mathrm{Map}_{CAlg^{nu}(\mathrm{PreStk}_{corr})}(Z, \mathrm{Ran}\times B^2_{et}(A))\times_{\mathrm{Map}_{CAlg^{nu}(\mathrm{PreStk}_{corr})}(Z,\mathrm{Ran})} *$$

based changed by $\mathrm{Map}_{\mathrm{PreStk}}(Z, \mathrm{Ran}\times B^2_{et}(A))\to \mathrm{Map}_{\mathrm{PreStk}_{corr}}(Z, \mathrm{Ran}\times B^2_{et}(A))$.

However, $\mathrm{Ran}\times B^2_{et}(A)$ is a not a factorization prestack over Ran in our sense!

If Z is 0-truncated, the space of factorizable A-gerbes on Z lies in ${}_{\tau_{\le 2}}\mathrm{Spc}$.

3.0.28. For 2.3.2. If $S\in \mathrm{Sch}^{aff}_{ft}$ it is known that $\mathrm{Shv}(S)\in \mathrm{DGCat}_{cont}$ is dualizable. Now if $Z\widetilde{\to}\operatorname{colim}_{i\in I} Z_i$, where $Z_i\in \mathrm{Sch}^{aff}_{ft}$ and the transition maps $Z_i\to Z_j$ are closed immersions then $Shv(Z)$ is dualizable!

Indeed, for $i\to j$ in I let $h: Z_i\to Z_j$ be the corresponding closed immersion, so $h^!: \mathrm{Shv}(Z_j)\to Shv(Z_i)$ admits a left adjoint $h_!: \mathrm{Shv}(Z_i)\to Shv(Z_j)$ by ([8], 1.5.2). By definition, $Shv(Z)\widetilde{\to}\lim_{i\in I^{op}} Shv(Z_i)$. It also rewrites as $\operatorname{colim}_{i\in I} Shv(Z_i)$ because of ([22], Section 9.2.6), the colimit taken in DGCat_{cont}. Now we may apply ([10], Lm. 2.2.2), which is actually an analog of ([15], ch. 1, Pp. 6.3.4). This shows that $Shv(Z)$ is dualizable.

3.0.29. For 3.1.2, line 3: there the category $\mathrm{Map}_{Ptd(\mathrm{PreStk}_{/X})}(B(G)\times X, B^4_{et}(A(1)))$ does not make sense, it is actually

$$\mathrm{Map}_{\mathrm{PreStk}}(B(G)\times X, B^4_{et}(A(1)))\times_{\mathrm{Map}_{\mathrm{PreStk}}(X, B^4_{et}(A(1)))} * \tag{5}$$

where the distinguished point is the map $X\to *\to B^4_{et}(A(1)))$. In (5) we may replace if needed $B(G)$ by $B_{et}(G)$, because sheafification is a localization functor.

3.0.30. For 3.1.5. More precisely, for $i=3$ or 4 and any element s of $\mathrm{H}^i_{et}(S\times X, A(1))$ or $\mathrm{H}^{i-1}_{et}(U_I, A(1))$ there is an etale cover $S'\to S$ such that the restriction of s to $S'\times X$ (or respectively, U_I for S') vanishes.

For 3.1.6. Note that A_{et} is 0-truncated prestack, so for $Y\in\mathrm{PreStk}$, $\mathrm{H}^0_{et}(Y,A)=\mathrm{Map}(Y, A_{et})$ is a set.

In the version of June 1: a simple idea. If $i: Y\to Z$ is a morphism, $Y=\sqcup_j Y_j$ then $i^!F\widetilde{\to}\oplus i_j^! F$.

In our case the isomorphism $i^! A_{S\times X}(1)[2]\widetilde{\to}\pi^! A_S$ is the isomorphism

$$\mathop{\oplus}_j i_j^! A_{S\times X}(1)[2]\widetilde{\to}\pi^! A_S$$

It is the sum of isomorphisms $i_j^! A_{S\times X}(1)[2]\widetilde{\to}\pi_j^! A_S$, where $\pi_j: \Gamma_j\to S$ is the projection.

3.0.31. For 3.1.11. By relative cohomology here we mean really the abstract definition as on nlab, because $X \to B(G) \times X$ is not a closed immersion. Formula (5) shows this is the relative cohomology of the map $X \to B(G) \times X$ with respect to $* \to B^4_{et}(A(1))$ in the ∞-topos PreStk.

In general, given a map $f : Y \to X$ of prestacks, let $K \to A \to f_*A$ be a distinguished triangle in the derived category of sheaves on X then by $\mathrm{H}^i_{et}(X; Y, A)$ one should mean $\mathrm{H}^i_{et}(X, K)$. If

$$Z \widetilde{\to} \mathrm{Map}(X, B^i_{et}(A)) \times_{\mathrm{Map}(Y, B^i_{et}(A))} *$$

then for $j \le i$ we get $\pi_j(Z) \widetilde{\to} H^{i-j}_{et}(X; Y, A)$.

3.0.32. For 3.2.3. If $\frac{\alpha}{2} \in \Lambda$ then $b(s(\frac{\alpha}{2}), s(\lambda)) = b(\frac{\alpha}{2}, \lambda)$ for the reflexion s corresponding to α. This yields $b(\alpha, \lambda) = \langle \check{\alpha}, \lambda \rangle q(\alpha)$.

The map $\mathrm{Quad}(\Lambda, \mathbb{Z}) \otimes A \to \mathrm{Quad}(\Lambda, A)$ is an isomorphism. First, we check surjectivity. Given $q \in \mathrm{Quad}(\Lambda, A)$ we may first pick a bilinear form $\phi : \Lambda \otimes \Lambda \to A$ such that $\phi(x, y) + \phi(y, x) = b(x, y)$ for any $x, y \in \Lambda$, where b is the bilinear form associated to q. Indeed, if e_i form a base of Λ then $b(e_i, e_i) = 2q(e_i)$. Take ϕ such that for $i < j$, $\phi(e_i, e_j) = b(e_i, e_j)$ and $\phi(e_j, e_i) = 0$. Besides, $\phi(e_i, e_i) = q(e_i)$. So, we may assume $b = 0$. Then $q : \Lambda \to A$ is linear with values in A_{2-tors}. Such quadratic form also writes as $\phi(x, x)$ for a suitable diagonam bilinear form $\phi : \Lambda \otimes \Lambda \to A$. If $\{e_i\}$ is a base of Λ, it gives a base of the free A-module $\mathrm{Quad}(\Lambda, \mathbb{Z}) \otimes A$. Namely, if we write $\check{e}_i$ for the dual base then we have the images of $\check{e}_i \otimes \check{e}_j \otimes 1 \in \check{\Lambda} \otimes \check{\Lambda} \otimes A$ in $\mathrm{Quad}(\Lambda, \mathbb{Z}) \otimes A$ for $i \le j$. This shows injectivity also: a quadratic form on Λ sends $\sum_i x_i e_i$ to

$$\sum_i a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j$$

with $a_i, a_{ij} \in A$.

Note that $\mathrm{Quad}(\Lambda, \mathbb{Z}) \subset \mathrm{Quad}(\Lambda, \mathbb{Z})^W$ is a direct summand. So, $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$ is a direct summand in $\mathrm{Quad}(\Lambda, A)$, and $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A \hookrightarrow \mathrm{Quad}(\Lambda, A)^W_{restr}$ is injective.

For 3.2.4. Assume A divisible. Let us verify that for $q \in \mathrm{Quad}(\Lambda, A)^W_{restr}$ there is $q_{\mathbb{Z}} \in \mathrm{Quad}(\Lambda, \mathbb{Z})^W \otimes_{\mathbb{Z}} A$ such that $q - q_{\mathbb{Z}}$ comes by restriction from a quadratic form on $\pi_1(G)$. Indeed, if $q_i : \Lambda \to \mathbb{Z}$ is the Killing form of i-th connected component of Dynkin, pick $a_i \in A$ such that $a_i q_i(\alpha) = q(\alpha)$ for any short coroot in the i-th connected component of the Dynkin diagram. Let $q_{\mathbb{Z}} = \sum_i a_i q_i$, $q' = q - q_{\mathbb{Z}}$. Let $b' : \Lambda \otimes \Lambda \to A$ be the bilinear form attached to q'. That is,

$$b'(\lambda_1, \lambda_2) = q'(\lambda_1 + \lambda_2) - q'(\lambda_1) - q'(\lambda_2)$$

For any reductive group G, the $\mathbb{Z}$-span of all W-orbits of all short coroots equals the coroots lattice (this is verified separately for any irreducible root system via their classification). So, $b'(\mu, \lambda) = 0$ for any μ in the coroots lattice and $\lambda \in \Lambda$. Thus, b' comes from a bilinear form on $\pi_1(G)$. This also shows that q' is additive on the coroots lattice. Again, since the $\mathbb{Z}$-span of all W-orbits of all short coroots equals the coroots lattice, q' vanishes on the coroots lattice. By the above, for $\lambda \in \Lambda$, μ in the coroots lattice $q'(\lambda + \mu) - q'(\lambda) = b'(\lambda, \mu) = 0$. So, q' descends to a quadratic form $\bar{q} : \pi_1(G) \to A$. We are done.

Note also that $\mathrm{Quad}(\pi_{1,alg}(G)) \subset \mathrm{Quad}(\Lambda, A)^W_{restr}$.

For A.4. We use the fact that any coroot is in the W-orbit of some simple coroot.

Remark 3.0.33. *Consider G simple simply-connected. Then $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \widetilde{\to} \mathbb{Z}$, and there is a distinguished generator q given by the property that $q(\alpha) = 1$ for any short coroot.*

3.0.34. Consider the example of $G = \mathrm{PSL}_n$, Λ is the coroots lattice. In this case $\mathrm{Quad}(\Lambda, \mathbb{Z})^W \widetilde{\to} \mathbb{Z}$ is generated by a quadratic form q_0 such that $q_0(\alpha) = n$ for any coroot.

3.0.35. For A.6. Our q is a sum of expressions of (I) and (II). Recall that

$$\mathcal{C}\mathrm{omGrp}(\mathrm{Spc}) \widetilde{\to} \mathrm{Sptr}^{\leq 0} \subset \mathrm{Sptr}$$

is closed under all colimits. So, $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$ admits all small colimits. We may first define $B(T)/B(T_{sc})$ as the cofibre of $B(T_{sc}) \to B(T)$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$. Then it is also a cofibre in Sptr, hence

$$\begin{array}{ccc} B(T_{sc}) & \to & B(T) \\ \downarrow & & \downarrow \\ pt & \to & B(T)/B(T_{sc}) \end{array} \tag{6}$$

is cartesian in Sptr. So, this square is also cartesian in $\mathrm{Sptr}^{\leq 0} \widetilde{\to} ComGrp(\mathrm{Spc})$.

The oblivion functor $ComGrp(\mathrm{Spc}) \to \mathrm{Spc}$ preserves small limits (Proof: each of the inclusions $ComGrp(\mathrm{Spc}) \subset ComMon(\mathrm{Spc}) \subset \mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathrm{Spc})$ is closed under limits. The evaluation $\mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathrm{Spc}) \to \mathrm{Spc}$ at $\langle 1 \rangle$ preserves limits). So, (6) is also cartesian in Spc.

As for any quotient of some $Z \in \mathrm{Spc}$ by an action of some group $H \in \mathcal{G}\mathrm{rp}(\mathrm{Spc})$, the square is cartesian in Spc

$$\begin{array}{ccc} B(T) & \to & pt \\ \downarrow & & \downarrow \\ B(T)/B(T_{sc}) & \to & B^2(T_{sc}) \end{array}$$

The forgetful functor $\mathrm{Spc}_* \to \mathrm{Spc}$ preserves limits and push-outs.

Consider the $B(T_{sc})$-torsor $q : B(T) \to B(T)/B(T_{sc})$ and the exact triangle on $B(T)/B(T_{sc})$

$$A \to q_* A \to \tau^{\geq 2} \pi_* A$$

The corresponding long exact sequence in cohomology gives

$$\mathrm{H}^2(B(T)/B(T_{sc}), A) \widetilde{\to} \mathrm{Hom}(\pi_1(G), A(-1)), \quad H^i(B(T)/B(T_{sc}), A) = 0 \text{ for } i = 1, 3.$$

We also get an exact sequence $0 \to \mathrm{H}^4(B(T)/B(T_{sc}), A) \to \mathrm{Quad}(\Lambda, A(-2)) \to M$, where M itself fits into an exact sequence $0 \to \mathrm{Hom}(\pi_1(G), A(-1)) \otimes_A \mathrm{Hom}(\Lambda_{sc}, A(-1)) \to M \to \mathrm{Quad}(\Lambda_{sc}, A(-2)) \to 0$. It follows that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^4(B(T)/B(T_{sc}), A) & \widetilde{\to} & \mathrm{Quad}(\pi_1(G), A(-2)) \\ \downarrow & & \downarrow \\ \mathrm{H}^4(B(T), A) & \widetilde{\to} & \mathrm{Quad}(\Lambda, A(-2)), \end{array}$$

where the vertical arrows are natural maps.

Construction of the canonical T^{sc}-torsor over G. Recall that T^{sc} is the maximal torus of the simply-connected cover G^{sc} of $[G,G]$. Pick an exact sequence $1\to T_1\to \tilde{G}\to G\to 1$, where T_1 is a torus, $\tilde{G}$ is reductive and $[\tilde{G},\tilde{G}]$ is simply-connected (see [34], Lemma 7.2.2). Let $\tilde{T}$ be the preimage of T in $\tilde{G}$, so $\tilde{T}$ is a maximal torus in $\tilde{G}$. Our $\tilde{T}$ acts by conjugation on $[\tilde{G},\tilde{G}]$, and $[\tilde{G},\tilde{G}]\to [G,G]$ is the simply-connected cover. Since T_1 is in the center of $\tilde{G}$, it acts trivially on $[\tilde{G},\tilde{G}]$, so T acts on $G^{sc}\widetilde{\to}[\tilde{G},\tilde{G}]$, and this action does not depend on a choice of the above extension $\tilde{G}$.

Now consider the semi-direct product $G^{sc}\rtimes T$. We get a homomorphism $G^{sc}\rtimes T\to G, (g,t)\mapsto \bar{g}t$, where $\bar{g}$ is the image of g in G. This gives an exact sequence $1\to T^{sc}\to G^{sc}\rtimes T\to G\to 1$. This is the canonical T^{sc}-torsor over G. It yields a diagram $B(T^{sc})\to B(G^{sc}\rtimes T)\to B(G)$ in $\mathrm{Ptd}(\mathrm{Spc})$.

We have have the projection homomorphism $\gamma: G^{sc}\rtimes T\to T$. Let us include T^{sc} to $G^{sc}\rtimes T$ by $t\mapsto (t^{-1},\bar{t})$, so γ commutes with the actions of T^{sc} by left translations. Here $t\in T^{sc}$ acts on $t_1\in T$ as $\bar{t}t_1$.

Since T^{sc} is central in $G^{sc}\rtimes T$, $B(T^{sc})$ acts on the left on $B(G^{sc}\rtimes T)$, and $B(G)$ is the quotient of $B(G^{sc}\rtimes T)$ by the left action of $B(T^{sc})$ is PreStk, see ([22], Section 7.2.18).

The map $B(G^{sc}\rtimes T)\to B(T)$ is $B(T^{sc})$-equivariant, so passing to the quotient we get the desired map $B(G)\to B(T)/B(T^{sc})$ (cf. [22], Section 7.2.18).

The calculation of $\mathrm{H}^i(G/B,\bar{\mathbb{Q}}_\ell)$ is done in Proposition 1.3(ii) in [4].

3.0.36. For 3.3.1. We assume A divisible or $\pi_1([G,G])=0$. Let $q:*\to B(G)$ be the trivial torsor. Define M by the distinguished triangle $M\to A(1)\to q_*A(1)$ on $B(G)$. Let $p_X: X\to *$ be the projection. For the map $\mathrm{id}\times p_X: B(G)\times X\to B(G)$ by definition we get

$$\mathrm{H}^{4-i}_{et}(B(G)\times X; X, A(1))\widetilde{\to}\mathrm{H}^{4-i}_{et}(B(G)\times X, (\mathrm{id}\times p_X)^*M)\widetilde{\to}\mathrm{H}^{4-i}_{et}(X, p_X^*(p_{B(G)})_*M),$$

the second isomorphism is by the base change under $p_{B(G)}: B(G)\to *$. The diagram $*\stackrel{q}{\to} B(G)\stackrel{p_{B(G)}}{\to} *$ yields a diagram $A\to (p_{B(G)})_*A\to A$ in the stable category of sheaves of abelian groups on $*$, the composition is $\mathrm{id}: A\to A$. So, A is a retract of $(p_{B(G)})_*A$. Any retract in a stable category splits, so $(p_{B(G)})_*A\widetilde{\to} A\oplus(\tau^{\geq 1}(p_{B(G)})_*A)$.

Applying $(p_{B(G)})_*$ to the fibre sequence $M\to A(1)\to q_*A(1)$, we get a fibre sequence $(p_{B(G)})_*M\to (p_{B(G)})_*A(1)\to A(1)$, so $(p_{B(G)})_*M\widetilde{\to}\tau^{\geq 1}(p_{B(G)})_*A(1)$. For $i\geq 0$ this gives for $K:=\tau^{\leq 4}((p_{B(G)})_*M)$ the isomorphism

$$\mathrm{H}^{4-i}_{et}(B(G)\times X; X, A(1))\widetilde{\to}\mathrm{H}^{4-i}_{et}(X, p_X^*K)$$

Thus, we get an exact sequence

$$0\to \mathrm{H}^2(X,\mathrm{Hom}(\pi_1(G),A))\to \mathrm{H}^4(X,p_X^*K)\widetilde{\to}\mathrm{Quad}(\Lambda, A(-1))^W_{restr}\to 0$$

of abelian groups. The claim that it splits non-canonically. Indeed, if L is a divisible abelian group then L is an injective $\mathbb{Z}$-module. This implies that

$$K\widetilde{\to}\mathrm{Hom}(\pi_1(G),A)[-2]\oplus\mathrm{Quad}(\Lambda,A(-1))^W_{restr}[-4]$$

non-canonically. Namely, the map γ in the triangle

$$K\to \mathrm{Quad}(\Lambda,A(-1))^W_{restr}[-4]\stackrel{\gamma}{\to}\mathrm{Hom}(\pi_1(G),A)[-1]$$

is an element of $\mathrm{Ext}^3(\mathrm{Quad}(\Lambda, A(-1)), \mathrm{Hom}(\pi_1(G), A))$ in the category of abelian groups. The exact sequence $0 \to \Lambda_{sc} \to \Lambda \to \pi_1(G) \to 0$ shows that $\mathrm{Hom}(\pi_1(G), A)$ is quasi-isomorphic in the derived category over $*$ to the complex of injective modules $\mathrm{Hom}(\Lambda, A) \to \mathrm{Hom}(\Lambda_{sc}, A)$. So, $\mathrm{RHom}(\mathrm{Quad}(\Lambda, A(-1))^W_{restr}, \mathrm{Hom}(\pi_1(G), A))$ is the complex

$$\mathrm{Hom}(\mathrm{Quad}(\Lambda, A(-1))^W_{restr}, \mathrm{Hom}(\Lambda, A)) \to \mathrm{Hom}(\mathrm{Quad}(\Lambda, A(-1))^W_{restr}, \mathrm{Hom}(\Lambda_{sc}, A))$$

placed in degrees $0, 1$. Thus, the above Ext^3 vanishes. Actually, Ext^2 also vanishes, so the splitting is canonical. Indeed, the latter map is surjective, because it rewrites as

$$\mathrm{Hom}(? \times \Lambda, A) \to \mathrm{Hom}(? \times \Lambda_{sc}, A)$$

with $? = \mathrm{Quad}(\Lambda, A(-1))^W_{restr}$. Since A is divisible, the latter map is surjective.

This also shows that $\pi_j \,\mathrm{FactGe}_A(\mathrm{Gr}_G) = 0$ for $j > 2$, because $\mathrm{R}\Gamma_{et}(X, p_X^* K)$ is placed in degrees ≥ 2.

The explanation in 3.3.4 is complicated, but clearly from the above we see that $\mathrm{R}\Gamma(X, p_X^* \mathrm{Hom}(\pi_1(G), A))[-2]$ gives a commutative group in spaces $\mathrm{FactGe}^0_A(\mathrm{Gr}_G)$, which is so a connective spectrum. The above complex should correspond to the connective spectrum $\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\pi_1(G), A))$ somehow by definition, namely

$$\pi_j \,\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\pi_1(G), A)) \,\tilde{\to}\, \mathrm{H}^{2-j}_{et}(X, \mathrm{Hom}(\pi_1(G), A))$$

identifies with $4 - j$-th cohomology group of $\mathrm{R}\Gamma(X, p_X^* \mathrm{Hom}(\pi_1(G), A))[-2]$.

The above calculation shows also that

$$\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(G), B^2_{et}(A(1))) \,\tilde{\to}\, \mathrm{Hom}(\pi_1(G), A)$$

Indeed, $(p_{B(G)})_* M \,\tilde{\to}\, \tau^{\geq 1}(p_{B(G)})_* A(1)$.

3.0.37. For 3.3.4. The equivalence $\mathrm{Mon(PreStk)} \,\tilde{\to}\, \mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, \mathrm{Mon(Spc)})$ restricts to an equivalence

$$\mathcal{G}\mathrm{rp(PreStk)} \,\tilde{\to}\, \mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, \mathcal{G}\mathrm{rp(Spc)})$$

by ([22], Remark 2.5.18). Besides, $\mathbb{E}_0(\mathrm{PreStk}) \,\tilde{\to}\, \mathrm{Fun}(\mathrm{Sch}^{aff})^{op}, \mathrm{Ptd(Spc)})$. Recall that $\mathcal{G}\mathrm{rp}(\mathrm{Spc}_*) \,\tilde{\to}\, \mathcal{G}\mathrm{rp(Spc)}$ by (HTT, 7.2.2.10). Similarly, $\mathcal{G}\mathrm{rp(PreStk)} \,\tilde{\to}\, \mathcal{G}\mathrm{rp(Ptd(PreStk))}$. So if $H \in \mathcal{G}\mathrm{rp(PreStk)}$, $Y \in \mathrm{Ptd(PreStk)}$ then $\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(Y, H)$ is a group in Spc

If A is a commutative group in PreStk, $Y \in \mathrm{Ptd(PreStk)}$ then

$$\Omega \,\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(Y, B^{i+1}_{et}(A)) \,\tilde{\to}\, \mathrm{Map}_{\mathrm{Ptd(PreStk)}}(Y, B^i_{et}(A))$$

in $\mathcal{C}\mathrm{omGrp(Spc)}$ By adjunction, this yields a morphism

$$B(\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(Y, B^i_{et}(A))) \to \mathrm{Map}_{\mathrm{Ptd(PreStk)}}(Y, B^{i+1}_{et}(A)) \tag{7}$$

in $\mathcal{C}\mathrm{omGrp(Spc)}$. If $\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(Y, B^{i+1}_{et}(A))$ is connected, that is, $\mathrm{H}^{i+1}_{et}(Y, A) = 0$ then (7) is an isomorphism.

Question Do I understand correctly that $\mathrm{Map}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B(G) \times X, B^4_{et}(A(1)))$ rewrites as $\mathrm{Map}_{\mathrm{PreStk}}(X, \mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(G), B^4_{et}(A(1)))_{et})$, because $B^4_{et}(A(1))$ is a stack?

We have the natural map

$$B_{et}(\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(G), B^3_{et}(A(1))) \to \mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(G), B^4_{et}(A(1)))_{et}$$

but not to the constant prestack $\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(G), B^4_{et}(A(1)))$, I think. This is why we get indeed a map ?

$$\mathrm{Map}(X, B_{et}(\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(G), B^3_{et}(A(1))) \to \mathrm{Map}_{\mathrm{Ptd(PreStk}_{/X})}(B(G)\times X, B^4_{et}(A(1)))$$

An easy calculation of homotopy groups of $\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\pi_1(G), A)))$ shows that Corollary 3.3.6 is true.

The etale-local triviality claim for $\mathrm{FactGe}^0_A(\mathrm{Gr}_G)$: for the bilinear form b we see that $\mathrm{Ge}_A(X^2)\times_{\mathrm{Ge}_A(X^2-\triangle)} * \widetilde{\to} A$, so it does not change if we replace X by an etale cover. This etale-local triviality is used in 4.3.2.

3.0.38. If $X, Y \in \mathrm{PreStk}$, write $\underline{\mathrm{Map}}(X, Y)$ for the inner hom in PreStk. Note that if $Y \in \mathrm{Stk}$ then $\underline{\mathrm{Map}}(X, Y) \in \mathrm{Stk}$. Let $H \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$ then $B_{et}(H) \widetilde{\to} B_{et}(H_{et})$. Let $Y \in \mathrm{PreStk}$. We claim that there is a natural map in PreStk

$$B_{et}\underline{\mathrm{Map}}(Y, H_{et}) \to \underline{\mathrm{Map}}(Y, B_{et}(H)) \tag{8}$$

Indeed, for $S \in \mathrm{Sch}^{aff}$ one has $\Omega\,\mathrm{Map}(S\times Y, B_{et}(H)) \widetilde{\to} \mathrm{Map}(S\times Y, H_{et})$ in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$. By adjunction, this gives a natural map

$$\mathrm{Map}(S, B\underline{\mathrm{Map}}(Y, H_{et})) \widetilde{\to} B\,\mathrm{Map}(S\times Y, H_{et}) \to \mathrm{Map}(S\times Y, B_{et}(H))$$

in $\mathrm{Ptd(Spc)}$. These maps organize into a morphism of prestacks $B\underline{\mathrm{Map}}(Y, H_{et}) \to \underline{\mathrm{Map}}(Y, B_{et}(H))$. Since the target is a stack, in turn this yields the desired morphism (8). We used the fact that $\mathrm{Map}_{\mathrm{PreStk}}(S, B(H)) \widetilde{\to} B(H(S))$ in Spc, so is connected.

For $X \in \mathrm{PreStk}$ we get a morphism $\mathrm{Map}(X, B_{et}\underline{\mathrm{Map}}(Y, H_{et})) \to \mathrm{Map}(X\times Y, B_{et}(H))$ in Spc. Dennis claims that the image of this map is the full subspace of those maps $X\times Y \to B_{et}(H)$, which are étale-locally trivial along X. By ([15], ch. 2, 2.3.10), $B\underline{\mathrm{Map}}(Y, H_{et}) \to B_{et}\underline{\mathrm{Map}}(Y, H_{et})$ is an etale surjection. So a map $X \to B_{et}\underline{\mathrm{Map}}(Y, H_{et})$ etale-locally over X lifts to a map $X \to *$.

3.0.39. For 4.1.1. Given a surjection $I \to J$ the map $X^J \to X^I$ is the composition $I \to J \to X$.

3.0.40. Recall that $\mathrm{Ran} \widetilde{\to} \mathrm{colim}_{I\in\mathcal{J}^{op}} X^I$ in PreStk, here $\mathcal{J}$ is the category of finite nonempty sets and surjective maps. So, $\mathrm{Shv(Ran)} \widetilde{\to} \lim_{I\in\mathcal{J}} \mathrm{Shv}(X^I)$. On the other hand for a surjection $\phi : I \to J$ of finite nonempty sets the diagonal $d : X^J \to X^I$ the functor $d^! : \mathrm{Shv}(X^I) \to \mathrm{Shv}(X^J)$ admits a left adjoint $d_! : Shv(X^J) \to Shv(X^I)$. So, by ([22], 9.2.6), $\mathrm{Shv(Ran)} \widetilde{\to} \mathrm{colim}_{I\in\mathcal{J}^{op}} \mathrm{Shv}(X^I)$.

More generally, this holds for pseudo-proper prestacks in the sense of ([8], 1.5.1). Let us check formally the proof of ([8], 1.5.4). A map $f : Y_1 \to Y_2$ is PreStk is preudo-proper if for any $S \in \mathrm{Sch}$, $Y_1 \times_{Y_2} S$ is a pseudo-proper prestack over S. Consier the functor $f^! : \lim_{S\to Y_2} Shv(S) \widetilde{\to} Shv(Y_2) \to Shv(Y_1) \widetilde{\to} \lim_{S\to Y_2} Shv(Y_1\times_{Y_2} S)$, here the limit is over $(\mathrm{Sch}^{aff}_{/Y_2})^{op}$. It is obtained by passing to this limit in the system of functors $f^!_S : Shv(S) \to Shv(Y_1\times_{Y_2} S)$ for $f_S : Y_1\times_{Y_2} S \to S$. Each $f^!_S$ admits a left adjoint

$(f_S)_!$ by ([8], 1.5.2). If $S \to S' \to Y_2$ is a map in $\mathrm{Sch}^{aff}_{/Y_2}$, for the diagram

$$\begin{array}{ccc} Y_1 \times_{Y_2} S' & \stackrel{f_{S'}}{\to} & S' \\ \uparrow g_Y & & \uparrow g \\ Y_1 \times_{Y_2} S & \stackrel{f_S}{\to} & S \end{array}$$

the natural transformation $(f_S)_! g_Y^! \to g^!(f_S)_!$ is an equivalence by ([8], 1.5.2). Passing to the limit over $(\mathrm{Sch}^{aff}_{/Y_2})^{op}$, we get the functor $h := \lim_{S \to Y_2} (f_S)_! : \mathrm{Shv}(Y_1) \to \mathrm{Shv}(Y_2)$. It is left adjoint to $f^!$ because of ([22], Lemma 2.4.1). Now given a cartesian diagram of prestacks

$$\begin{array}{ccc} Y_1 & \stackrel{f}{\to} & Y_2 \\ \uparrow g_1 & & \uparrow g_2 \\ Y_1' & \stackrel{f'}{\to} & Y_2' \end{array}$$

with f pseudo-proper, we want to check that the natural transformation $\epsilon : f'_! g_1^! \to g_2^! f_!$ is an isomorphism. We get for each $S \to Y_2$ in $\mathrm{Sch}^{aff}_{/Y_2}$ the base changed diagram $Y_{1,S} \stackrel{f_S}{\to} S \stackrel{g_{2,S}}{\leftarrow} Y'_{2,S}$ with $Y'_{1,S} \times_S Y'_{2,S} \widetilde{\to} Y'_{1,S}$, and a natural transformation

$$\epsilon_S : (f'_S)_! g^!_{1,S} \to g^!_{2,S}(f_S)_!$$

of functors $Shv(Y_{1,S}) \to Shv(Y'_{2,S})$. If we show that ϵ_S is an isomorphism then passing to the limit over $S \in (\mathrm{Sch}^{aff}_{/Y_2})^{op}$, we will conclude that ϵ is an isomorphism. Thus, we may and assume $Y_2 \in \mathrm{Sch}^{aff}$. Similarly, now for each $S \to Y_2'$ in $\mathrm{Sch}^{aff}_{/Y_2'}$ let $f'_S : Y'_{1,S} \to Y'_{2,S}$ be the base change of f'. For the diagram

$$\begin{array}{ccc} Y_1 & \stackrel{f}{\to} & Y_2 \\ \uparrow g_{1,S} & & \uparrow g_{2,S} \\ Y'_{1,S} & \stackrel{f'_S}{\to} & Y'_{2,S} \end{array}$$

we know that the transformation $(f'_S)! g^!_{1,S} \to g^!_{2,S} f_!$ is an isomorphism by ([8], 1.5.1). I think passing to the limit over $S \in (\mathrm{Sch}^{aff}_{/Y_2'})^{op}$, we may conclude that ϵ is an isomorphism.

3.0.41. If $X \in \mathrm{PreStk}$ is a pseudo-scheme ([8], 7.4.1) then the diagonal map $X \to X \times X$ is pseudo-proper. Indeed, if $X \widetilde{\to} \mathrm{colim}_{a \in A} Z_a$, where $Z_a \in \mathrm{Sch}$, and the transition maps $Z_{a_1} \to Z_{a_2}$ are proper then for any $S \in \mathrm{Sch}^{aff}$ and a map $h : S \to X \times X$ there is $a, b \in A$ such that h factors through $h : Z_a \times Z_b \to X \times X$, and the claim follows from the fact that $Z_a \to X$ is pseudo-proper by ([8], 7.4.2). Indeed, $X \times_{X \times X} Z_a \times Z_b \widetilde{\to} Z_a \times_X Z_b$. The morphism $Z_a \times_X Z_b \to Z_a \times Z_b$ comes in the sense of ([8], Remark 7.4.4) from a morphism in PreStk_{proper} by LKE, hence is pseudo-proper.

3.0.42. For ([8], Pp 7.4.2). Let us show that $\mathrm{Sch}_{proper} \to \mathrm{Sch}$ preserves finite limits. Because of (HTT, 5.3.2.9), it suffices to show that this map is left exact in the sense of (HTT, 5.3.2.1). To this end, it suffices by (HTT, 5.3.2.5) to show that for any $S \in \mathrm{Sch}$ the category $(\mathrm{Sch}_{proper} \times_{\mathrm{Sch}} \mathrm{Sch}_{S/})^{op}$ is filtered. This is true, because the category $\mathrm{Sch}_{proper} \times_{\mathrm{Sch}} \mathrm{Sch}_{S/}$ admits pullbacks.

Now it remains to prove ([8], Lm 7.4.3). In the case when C' admits finite limits, this is nothing but (HTT, 6.1.5.2). For $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, $Pro(\mathcal{C})$ is defined as $(\mathrm{Ind}(\mathcal{C}^{op}))^{op}$. By (HTT, 5.3.5.14), the Yoneda embedding $\mathcal{C} \to Pro(\mathcal{C})$ preserve all finite limits which exist in $\mathcal{C}$. As formulated, I don't understand the proof of ([8], Lm 7.4.3). However, consider a little different claim, namely assume $F : C' \to C$ left exact in the sense of (HTT, 5.3.2.1). Then by (HTT, 5.3.2.5), for $c \in C$, $C'^{op} \times_{C^{op}} (C^{op})_{/c}$ is filtered. Let now $\Phi \in \mathrm{Fun}((C')^{op}, \mathrm{Spc})$ and $c \in C$. It suffices to show that the functor $\Phi \mapsto LKE(\Phi)(c)$ preserves finite limits. One has $LKE(\Phi)(c) \widetilde{\to} \mathrm{colim}_{c \to F(c')} \Phi(c')$, the colimit in Spc over the filtered category $C'^{op} \times_{C^{op}} (C^{op})_{/c}$. The claim follows now from (HTT, 5.3.3.3).

For ([8], Remark 7.4.4). It is essential that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms in Sch, gf, g are proper then f is automatically proper. For this reason any pseudo-proper morphism $Y_1 \to Y_2$ in PreStk, where Y_i are pseudo-schemes, comes from a morphism in PreStk_{proper}.

3.0.43. From ([8], Lemma 7.4.7) it follows that for a proper morphism of separated schemes $f : X \to Y$, the functor $f_! : Shv(X) \to Shv(Y)$ preserves limits in the context of constructible sheaves, so has a left adjoint. It should be f^* for $f_* = f_!$. In the context of $\mathcal{D}$-modules this is not true, $f_!$ in general does not preserve limits!

3.0.44. By ([8], 7.4.11), if $f : S' \to S$ is a morphism of separated schemes of finite type, which is surjective on k-points, then $f^! : \mathrm{Shv}(S) \to \mathrm{Shv}(S')$ is conservative. There this is claimed without a proof. For a Zariski cover Lin Chen has proposed a proof in his email.

It should be true that if $a : F \to F'$ is a map in $\mathrm{Shv}(S)$ such that for any field-valued point $i : s \to S$, $i^! F \to i^! F'$ is an isomorphism then a is an isomorphism (see emails of Sam).

3.0.45. The following holds for the sheaf theory: let $f : X \to Y$ be a proper morphism of separated schemes of finite type. The commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}(Y) \otimes \mathrm{Shv}(X) & \to & \mathrm{Shv}(Y \times X) \\ \uparrow \scriptstyle{\mathrm{id} \times f^!} & & \uparrow \scriptstyle{(\mathrm{id} \times f)^!} \\ \mathrm{Shv}(Y) \otimes \mathrm{Shv}(Y) & \to & \mathrm{Shv}(Y \times Y) \end{array}$$

coming from the right-lax structure on Shv gives rise to a natural transformation $(\mathrm{id} \times f)_!(F \boxtimes H) \to F \boxtimes (f_! H)$ functorial in $F \in \mathrm{Shv}(Y), H \in \mathrm{Shv}(X)$. This map is an isomorphism, because $Shv_{\mathrm{PreStk}_{ind-sch}}$ is right-lax symmetric monoidal, see Section 3.0.21 of this file. This is not really explained in [8], though Dennis refers to this as the base change.

I think ([8], Lm 7.2.3 in the constructible context) should be also an axiom for the sheaf theory.

More generally, if $f : X_1 \to X_2, g : Y_1 \to Y_2$ are pseudo-proper maps of prestacks, $K \in \mathrm{Shv}(X_1), H \in \mathrm{Shv}(Y_1)$ then $(f_! K) \boxtimes (g_! H) \widetilde{\to} (f \times g)_!(K \boxtimes H)$.

3.0.46. Under the conventions of [8], one has the projection formula for a pseudo-proper map $f : X \to Y$. Let $F \in \mathrm{Shv}(X), G \in \mathrm{Shv}(Y)$. Denote by $\otimes : \mathrm{Shv}(Y) \times \mathrm{Shv}(Y) \to \mathrm{Shv}(Y)$ the functor sending G_1, G_2 to $\mathrm{diag}^!(G_1 \boxtimes G_2)$, this is the "pointwise symmetric monoidal structure" on $\mathrm{Shv}(Y)$. Then

$$(f_! F) \otimes G \widetilde{\to} f_!(F \otimes (f^! G))$$

in $\mathrm{Shv}(Y)$.

Proof. Since $f \times \mathrm{id} : X \times Y \to Y \times Y$ is pseudo-proper, $(f_! F) \boxtimes G \widetilde{\to} (f \times \mathrm{id})_!(F \boxtimes G)$. Write $\mathrm{Gr}_f : X \to X \times Y$ for the graph of f. By the base change for the pseudo-proper map $f \times \mathrm{id}$ ([8], Cor.1.5.4), one gets for $\mathrm{diag} : Y \to Y \times Y$

$$(f_! F) \otimes G \widetilde{\to} \mathrm{diag}^!((f_! F) \boxtimes G) \widetilde{\to} \mathrm{diag}^!(f \times \mathrm{id})_!(F \boxtimes G) \widetilde{\to} f_!(\mathrm{Gr}_f)^!(F \boxtimes G) \widetilde{\to}$$
$$f_! \,\mathrm{diag}^!(F \boxtimes (f^! G)) \widetilde{\to} f_!(F \otimes (f^! G)),$$

because the composition $X \overset{\mathrm{diag}}{\to} X \times X \overset{\mathrm{id} \times f}{\to} X \times Y$ is Gr_f. □

3.0.47. For ([8], 7.1). If $f_i : X_i \to Y_i$ is a map in PreStk then the diagram commutes

$$\begin{array}{ccc} \mathrm{Shv}(X_1) \otimes \mathrm{Shv}(X_2) & \to & \mathrm{Shv}(X_1 \times X_2) \\ \uparrow {\scriptstyle f_1^! \otimes f_2^!} & & \uparrow {\scriptstyle (f_1 \times f_2)^!} \\ \mathrm{Shv}(Y_1) \otimes \mathrm{Shv}(Y_2) & \to & \mathrm{Shv}(Y_1 \times Y_2) \end{array}$$

in particular, $\omega_{X_1} \boxtimes \omega_{X_2} \widetilde{\to} \omega_{X_1 \times X_2}$. Here $\omega_{\mathcal{X}}$ is the dualizing sheaf on X.

3.0.48. For ([8], Lm. 7.4.9) a strengthened version: write $\mathcal{Y} \widetilde{\to} \mathrm{colim}_a Z_a$, where $Z_a \in \mathrm{Sch}$, and the transition maps $Z_{a_1} \to Z_{a_2}$ are proper. If $\mathcal{Y}$ is a pseudo-scheme with a finitary diagonal then for any a, b one may write $Z_a \times_{\mathcal{Y}} Z_b \widetilde{\to} \mathrm{colim}_{i \in I} Z^i_{a,b}$ in PreStk, where $Z^i_{a,b}$ is a scheme proper over both Z_a and Z_b, and the indexing category I is finite.

Indeed, both projections $Z_a \leftarrow Z_a \times_{\mathcal{Y}} Z_b \to Z_b$ are pseudo-proper. Pick a presentation $Z_a \times_{\mathcal{Y}} Z_b \widetilde{\to} \mathrm{colim}_{i \in I} Z^i_{a,b}$ in PreStk, where $Z^i_{a,b}$ is a scheme proper over Z_b for any i. Since $Z_a \times_{\mathcal{Y}} Z_b \to Z_a$ is pseudo-proper, for any i the composition $Z^i_{a,b} \to Z_a \times_{\mathcal{Y}} Z_b \to Z_a$ is pseudo-proper by ([8], 7.4.2). Finally, use the following consequence of ([8], end of proof of Corollary 7.5.6): if $h : S_1 \to S_2$ is a pseudo-proper morphism between schemes (recall that schemes are assumed separated) then h is proper, see also ([8], Remark 7.4.4). So, for each i, $Z^i_{a,b} \to Z_a$ is proper.

3.0.49. If $f : Y_1 \to Y_2$ is an etale morphism of prestacks then $d : Y_1 \to Y_1 \times_{Y_2} Y_1$ is affine schematic and pseudo-proper, so $d_!$ exists. Besides, for any $S \in \mathrm{Sch}$, $Y_1(S) \to (Y_1 \times_{Y_2} Y_1)(S)$ is a monomorphism of spaces. So, $d_!$ is fully faithful, that is, $\mathrm{id} \to d^! d_!$ is an isomorphism ([8], 7.4.11).

3.0.50. For 4.1.2 and 4.1.4. First, for finite nonempty sets I, J, $X^I \times_{\mathrm{Ran}} X^J$ is the prestack, whose S-points are pairs of morphisms $S \to X^I, S \to X^J$ such that the corresponding subsets of $\mathrm{Map}(S, X)$ coincide (they are quotient sets of a set of $\mid J \mid$-elements and of a set of $\mid I \mid$-elements. It is described in ([8], 8.1.2) as $\underset{I \twoheadrightarrow K \twoheadleftarrow J}{\mathrm{colim}} X^K$.

Let $\mathcal{J}$ be the category whose objects are (I, λ^I), where I is a nonempty finite set, $\lambda^I : I \to \Lambda$ is a map. A morphism $(J, \lambda^J) \to (I, \lambda^I)$ in $\mathcal{J}$ is a surjection $\phi : I \to J$ such that $\lambda_j = \sum_{i \in \phi^{-1}(j)} \lambda_i$ for all j. Recall that $\mathrm{Gr}_{T,comb} = \mathrm{colim}_{(I,\lambda^I) \in \mathcal{J}} X^I$ in PreStk. So, $\mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^J \widetilde{\to} \mathrm{colim}_{(I,\lambda^I) \in \mathcal{J}} (X^I \times_{\mathrm{Ran}} X^J)$.

For a finite non-empty set J consider the category $\mathcal{J}_J$, whose objects are triples $(I, \lambda^I, J \overset{\pi}{\to} I)$, where π is a surjection, and $\lambda^I : I \to \Lambda$ is a map. A morphism from $(I, \lambda^I, J \to I)$ to $(I', \lambda^{I'}, J \overset{\pi'}{\to} I')$ is a surjection $\phi : I' \to I$ compatible with surjections from J such that $\lambda_i = \sum_{i' \in \phi^{-1}(i)} \lambda_{i'}$. We have a map

$$\underset{(I,\lambda^I, J \to I) \in \mathcal{J}_J}{\mathrm{colim}} X^I \to \mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^J \tag{9}$$

Namely, for $(I, \lambda^I, J \to I) \in \mathcal{J}_J$ we get the map $X^I \to \mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^J$, where the projection on $\mathrm{Gr}_{T,comb}$ is the natural map, and the projection $X^I \to X^J$ comes from $J \to I$. The map (9) is an isomorphism in PreStk, I think.

Indeed, one has

$$\mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^J \widetilde{\to} \underset{(I,\lambda^I), I \to K \leftarrow J}{\mathrm{colim}} X^K$$

Here the colimit is over the diagram, whose objects are collections $(I, \lambda^I, I \to K \leftarrow J)$, the maps being surjective. A morphism from $(I', \lambda^{I'}, I' \to K' \leftarrow J)$ to $(I, \lambda^I, I \to K \leftarrow J)$ is a pair of surjections $I \to I'$ and $K \to K'$ such that the diagram commutes

$$\begin{array}{ccccc} I & \to & K & \leftarrow J \\ \downarrow \phi & & \downarrow & \swarrow \\ I' & \to & K' & \end{array}$$

and $\lambda_{i'} = \sum_{\phi(i) = i'} \lambda_i$. This diagram maps naturally to $\mathcal{J}_J$ sending the above point to $(J \to K, \lambda^K)$, where λ_K is the direct image of λ^I along $I \to K$. We first calculate the LKE along this projection. This is easy, and produces precisely the colimit in the LHS of (9).

For each $\lambda \in \Lambda$ consider the object $a_\lambda = (*, \lambda) \in \mathcal{J}$, let $\mathcal{J}_{a_\lambda /}$ be the corresponding undercategory. Then the geometric realization of $\mathcal{J}_{a_\lambda /}$ is $*$, because it has an initial object. So, $\underset{(I,\lambda^I) \in \mathcal{J}}{\mathrm{colim}} * \widetilde{\to} \Lambda$ in Spc. Recall also that for any $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, $\mid \mathcal{C} \mid \widetilde{\to} \mid \mathcal{C}^{op} \mid$.

The prestack $\mathrm{Gr}_T \times_{\mathrm{Ran}} X^J$ writes as $\mathrm{colim}_{J \to K} \mathrm{Gr}_{T, X^K}$ over the category opposite to the category of surjections $J \to K$, where K is a finite non-empty set. Here we denoted by Gr_{T,X^K} the prestack classifying a point $x^K \in X^K$, a T-torsor $\mathcal{F}$ on X together with a trivialization $\beta : \mathcal{F} \widetilde{\to} \mathcal{F}^0$ over $X - x^K$. The map $\mathrm{Gr}_{T,X^K} \to X^K$ is pseudo-proper.

The map

$$\mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^J \to \mathrm{Gr}_T \times_{\mathrm{Ran}} X^J \tag{10}$$

is pseudo-proper and surjective on k-points. It is finitary pseudo-proper. Indeed, pick a base $\{\check{e}_i\}$ in $\check{\Lambda}$. For $N \geq 0$ let $\mathrm{Gr}_{T,X^I,N} \subset \mathrm{Gr}_{T,X^I}$ be the closed subscheme classifying $(x^I \in X^I, \mathcal{F}, \beta)$ such that for any i one has

$$V^{\check{e}_i}_{\mathcal{F}^0_T}(-Nx^I) \subset V^{\check{e}_i}_{\mathcal{F}_T} \subset V^{\check{e}_i}_{\mathcal{F}^0_T}(Nx^I),$$

where $V^{\check{e}_i}$ is the 1-dimensional T-module with weight $\check{e}_i$. Then $\mathrm{Gr}_{T,X^I} = \underset{N}{\mathrm{colim}}\, \mathrm{Gr}_{T,X^I,N}$. For $N \geq 0$ and a surjection $J \to I$, the base change of (10) by the natural map $\mathrm{Gr}_{T,X^I,N} \to \mathrm{Gr}_T \times_{\mathrm{Ran}} X^J$ is written as a finite colimit of proper schemes.

To check that (10) is a monomorphism of prestacks, it is easier to check that the diagonal map $Y_1 \to Y_1 \times_{Y_2} Y_1$ is an isomorphism, where $Y_1 \to Y_2$ is the map (10). Indeed, if S is say an affine scheme of finite type, an S-point of $Y_1 \times_{Y_2} Y_1$ comes from a collection: $(I_1, I_2, \lambda^{I_1}, \lambda^{I_2}, \pi_1 : J \to I_1, \pi_2 : J \to I_2, x^{I_1} \in X^{I_1}(S), x^{I_2} \in X^{I_2}(S)$ over the same point $x^J \in X^J(S)$ and an isomorphism $\mathcal{F}^0_T(\sum_{j\in J} \lambda^1_{\pi_1(j)}) \widetilde{\to} \mathcal{F}^0_T(\sum_{j\in J} \lambda^2_{\pi_2(j)})$ over $S \times X$, whose restriction to the complement of x^J is the identity. We see that the diagonal map $Y_1 \to Y_1 \times_{Y_2} Y_1$ is an isomorphism. This is to apply ([8], 7.4.11(d)). This gives the claim from 4.1.2 in our joint paper: the natural map $Shv_{\mathcal{G}}(\mathrm{Gr}_T)_{/\mathrm{Ran}} \to Shv_{\mathcal{G}}(\mathrm{Gr}_{T,comb})_{/\mathrm{Ran}}$ is an isomorphism of sheaves of categories.

The isomorphism (9) also gives the fact from 4.1.4 of the paper that

$$Shv_{\mathcal{G}}(\mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^J) \widetilde{\to} \lim_{(I,\lambda^I,J\to I)\in \mathcal{J}_J^{op}} Shv_{\mathcal{G}_{\lambda^I}}(X^I)$$

The latter also rewrites as

$$\underset{(I,\lambda^I,J\to I)\in\mathcal{J}_J}{\mathrm{colim}} Shv_{\mathcal{G}_{\lambda^I}}(X^I),$$

because for each morphism from $(I, \lambda^I, J \to I)$ to $(I', \lambda^{I'}, J \to I')$ in $\mathcal{J}_J$ and the corresponding closed immersion $h : X^I \to X^{I'}$ the functor $h^! : Shv_{\mathcal{G}_{\lambda^{I'}}}(X^{I'}) \to Shv_{\mathcal{G}_{\lambda^I}}(X^I)$ admits a left adjoint $h_! : Shv_{\mathcal{G}_{\lambda^I}}(X^I) \to Shv_{\mathcal{G}_{\lambda^{I'}}}(X^{I'})$ as in my Section 3.0.28.

3.0.51. The factorization structure on $\mathrm{Gr}_{T,comb}$ is as follows. Let $\phi : J \to J'$ be a surjection of finite nonempty sets. Let $X^J_{\phi,disj}$ be as in (18). We construct an isomorphism

$$\mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^J_{\phi,disj} \widetilde{\to} (\prod_{j'\in J'} \mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^{J_{j'}}) \times_{X^J} X^J_{\phi,disj}$$

as follows. The LHS is

$$(\underset{(I,\lambda^I,J\to I)\in\mathcal{J}_J}{\mathrm{colim}} X^I) \times_{X^J} X^J_{\phi,disj}$$

By Lemma 3.3.1 of this file, $X^I \times_{X^J} X^J_{\phi,disj}$ is empty unless ϕ factors as $J \to I \overset{\phi'}{\to} J'$, and then $X^I \times_{X^J} X^J_{\phi,disj} \widetilde{\to} X^I_{\phi',disj}$. So, the index category becomes $\prod_{j'\in J'} \mathcal{J}_{J_{j'}}$. We get

$$\underset{(I_{j'},\, \lambda^{I_{j'}},\, J_{j'}\to I_{j'})\in\mathcal{J}_{J_{j'}}}{\mathrm{colim}} X^{I_{j'}} \widetilde{\to} \mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^{J_{j'}}$$

and the claim easily follows.

3.0.52. For 4.1.5. The identification of θ-data with factorization $\mathbb{Z}/2\mathbb{Z}$-line bundles on $\mathrm{Gr}_{T,comb}$ is as follows. A datum of a factorization line bundle on $\mathrm{Gr}_{T,comp}$ gives for each finite nonempty set I with $\lambda^I : I \to \Lambda$ a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle L^{λ^I} on X^I. For a surjection $\phi : I \to J$ we have an isomorphism

$$L^{\lambda^I} \mid_{X^I_{\phi,disj}} \widetilde{\to} (\underset{j\in J}{\otimes} L^{\lambda^{I_j}}) \mid_{X^I_{\phi,disj}}$$

The corresponding θ-datum is a collection λ^γ, here $\lambda^\gamma = L^{\lambda^I}$ for $I = *$ and $\lambda^I : * \to \Lambda$ given by γ. For a pair $\gamma_1, \gamma_2 \in \Lambda$ the isomorphism $L^{\gamma_1,\gamma_2} \mid_{X^2|\triangle} \widetilde{\to} L^{\gamma_1} \boxtimes L^{\gamma_2} \mid_{X^2|\triangle}$ extends to an isomorphism

$$L^{\gamma_1,\gamma_2} \widetilde{\to} L^{\gamma_1} \boxtimes L^{\gamma_2}(-\kappa(\gamma_1,\gamma_2)\ \triangle)$$

over X^2 for a suitable symmetric bilinear form $\kappa : \Lambda \otimes \Lambda \to \mathbb{Z}$. Resticting to $\triangle$, this gives the isomorphisms $c^{\gamma_1,\gamma_2} : \lambda^{\gamma_1+\gamma_2} \widetilde{\to} \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \otimes \Omega^{\kappa(\gamma_1,\gamma_2)}$ on X.

Consider the sheaf denoted by $\mathrm{Div}(X,\Lambda)$ in ([2], 3.10.7). We get a morphism $\mathrm{Gr}_{T,comb} \to \mathrm{Div}(X,\Gamma)$. What is the relation between $\mathrm{Div}(X,\Gamma)$ and Gr_T? They are not the same. Given a T-torsor $\mathcal{F}$ on $S \times X$ trivialized away Γ_I for some $I \in \mathrm{Ran}(S)$, we get a relative Cartier divisor on $S \times X$ proper over S. Namely, for each $\check{\lambda} \in \check{\Lambda}$ the corresponding line bundle $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}}$ with its trivialization over $S \times X - \Gamma_I$ is a relative Cartier divisor. So, if we pick a base of Λ, we get a point of $\mathrm{Div}(X,\Gamma)$. This gives a map $\mathrm{Gr}_T \to \mathrm{Div}(X,\Gamma)$, which is not an isomorphism (already at the level of k-points). For example, for $x \neq y \in X, \lambda \in \Lambda$ consider the k-point $(I, \mathcal{O}(\lambda y), \mathcal{O}(\lambda y) \widetilde{\to} \mathcal{O} \mid_{X-x-y}) \in \mathrm{Gr}_T$ with $I = \{x, y\} \subset X$. We may also consider the k-point $(y, \mathcal{O}(\lambda y), \mathcal{O}(\lambda y) \widetilde{\to} \mathcal{O} \mid_{X-y}) \in \mathrm{Gr}_T$. Their images in $\mathrm{Div}(X,\Lambda)$ are the same, but these are different points of Gr_T.

Dennis claims that the map $\mathrm{Gr}_T \to \mathrm{Div}(X,\Gamma)$ induces an isomorphism between any factorizable structures on both prestacks. More generally, for G an algebraic group, one has the version $GRAS_G$ of the affine grassmanian defined in ([3], 4.3.14). Namely, for $S \in \mathrm{Sch}^{aff}$ its S-points is $\mathrm{colim}_U\, \mathcal{C}_U$, here the colimit is taken over (the opposite) of the category of open subsets $U \subset X \times S$ such that the fibre of U over any point of S is nonempty. We denoted by $\mathcal{C}_U$ the groupoid of G-torsors on $X \times S$ with a trivialization over U (in fact, $\mathcal{C}_U$ is a set, so the above colimit is also a set). This $GRAS_G$ is not a factorization prestack in the sense of our paper, but one may define for example the notion of a factorizable line bundle on $GRAS_G$.

Probably for G reductive, the natural map $f : \mathrm{Gr}_G \to GRAS_G$ induces an isomorphism of any factorizable structures on both prestacks. Though $GRAS_G$ is not a factorization prestack over Ran, one defines factorizable structures on it naturally. For example, $\mathrm{FactPic}(GRAS_G)$ is the groipoid classifying a line bundle L on $GRAS_G$ and a factorization structure on f^*L. Are the fibres of f contractible?

There is a subtlety in the definition of a $\mathbb{Z}/2\mathbb{Z}$-graded factorization line bundle on a factorization prestack. It is crucial to require a suitable sign for the commutativity constraint. For example, in the definition of the θ-datum in the commutativity constraint it is crucial to require the sign: $c^{\gamma_1,\gamma_2} = (-1)^{\kappa(\gamma_1,\gamma_2)} c^{\gamma_2,\gamma_1}\sigma$ in ([2], 3.10.3(ii)).

If we do not require the sign, the following would be a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle $\mathfrak{L}$ on $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m,comb}$. Write $k[-1]$ for the k-vector space k placed in degree one as $\mathbb{Z}/2\mathbb{Z}$-graded. We define a the $\mathbb{Z}/2\mathbb{Z}$-graded line bundle $\mathfrak{L}$ on $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m,comb}$ so that its restriction to X^I for $(I, \lambda^I : I \to \mathbb{Z}/2\mathbb{Z})$ is $(k[-1])^{\otimes\lambda}$ with $\lambda = \sum_{i\in I}\lambda_i$. It would have the following factorization structure as a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle. Given a surjection $\phi : I \to J$ of finite nonempty sets, one has a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\underset{j\in J}{\otimes}\big(\underset{i\in I_j}{\otimes}(k[-1])^{\otimes\lambda_j}\big)\,\widetilde{\to}\,(k[-1])^{\otimes\lambda},$$

where $\lambda_j = \sum_{i\in I_j}\lambda_i$. We view the latter as the corresponding factorization isomorphism over $X^I_{\phi,disj}$. However, the sign for the commutativity constraint is not correct!

3.0.53. For 4.2.4. The map (4.8) given by $A(-1)\times\mathbb{G}_m \to B_{et}(A)$ is bilinear. The linearity with respect to the second variable is the multiplicativity of the A-torsor χ_a on $\mathbb{G}_m$. For its definition: if $n \mid n'$ then we have the $\mu_{n'}$-torsor $\mathbb{G}_m \overset{x\mapsto x^{n'}}{\to} \mathbb{G}_m$ over $\mathbb{G}_m$. Its extension of scalars under $\mu_{n'} \to \mu_n$, $x \mapsto x^{n'/n}$ is canonically the torsor $\mathbb{G}_m \overset{x\mapsto x^{n}}{\to} \mathbb{G}_m$. For this reason the maps $\mathrm{Hom}(\mu_n, A)\times\mathbb{G}_m \to B_{et}(A)$ are compatible, so yield the morphism (4.7). The corresponding map $A(-1) \to Tors_A(\mathbb{G}_m), a \mapsto \chi_a$ is a group homomorphism.

3.0.54. For 4.2.7, version June 1. Let $\gamma_{\lambda_1,\lambda_2} : \mathcal{G}_{\lambda_1,\lambda_2}\,\widetilde{\to}\,\mathcal{G}_{\lambda_1}\boxtimes\mathcal{G}_{\lambda_2}\otimes\mathcal{O}(\triangle)^{b(\lambda_1,\lambda_2)}$ be the isomorphism as in 4.2.1. Then for $\lambda_1 \neq \lambda_2$ the canonical commutativity datum for the diagram (4.14) does not give something additional, as the two isomorphisms in the horizontal lines of (4.14) are not the same. The isomoprhisms $\sigma^*\gamma_{\lambda_2,\lambda_1}$ and $\gamma_{\lambda_1,\lambda_2}$ are identified in (4.6), but they are not the same, so this is just and abstract A-torsor on X with a trivialization of its square.

3.0.55. For 4.2.8. The relation $q(\lambda+\mu) = q(\lambda) + q(\mu) + b(\lambda,\mu)$ is proved as in ([32], II.3.4). Namely, consider $\mathcal{G}_{\lambda,\mu,\lambda,\mu}$ on X^4. The factorization isomorphism becomes

$$\begin{aligned}(11)\quad \mathcal{G}_{\lambda,\mu,\lambda,\mu}\,\widetilde{\to}\,(\mathcal{G}_\lambda\boxtimes\mathcal{G}_\mu\boxtimes\mathcal{G}_\lambda\boxtimes\mathcal{G}_\mu)\otimes\mathcal{O}(\triangle_{12})^{b(\lambda,\mu)}\otimes\mathcal{O}(\triangle_{34})^{b(\lambda,\mu)}\otimes\\ \mathcal{O}(\triangle_{23})^{b(\lambda,\mu)}\otimes\mathcal{O}(\triangle_{14})^{b(\lambda,\mu)}\otimes\mathcal{O}(\triangle_{13})^{2q(\lambda)}\otimes\mathcal{O}(\triangle_{24})^{2q(\mu)}\end{aligned}$$

Reich restricts it to $\triangle_{12}\cap\triangle_{34}$, this gives

$$(12)\quad \mathcal{G}_{\lambda+\mu,\lambda+\mu}\,\widetilde{\to}\,((\mathcal{G}_\lambda\otimes\mathcal{G}_\mu)\boxtimes(\mathcal{G}_\lambda\otimes\mathcal{G}_\mu))\otimes(\mathcal{T}^{b(\lambda,\mu)}\boxtimes\mathcal{T}^{b(\lambda,\mu)})\otimes\mathcal{O}(\triangle)^{2b(\lambda,\mu)+2q(\lambda)+2q(\mu)}$$

Here $\mathcal{T} = \Omega^{-1}$ on X. The factorization isomorphism $\mathcal{G}_{\lambda,\mu}\,\widetilde{\to}\,(\mathcal{G}_\lambda\boxtimes\mathcal{G}_\mu)\otimes\mathcal{O}(\triangle)^{b(\lambda,\mu)}$ on X^2 restricts to the diagonal as

$$\mathcal{G}_{\lambda+\mu}\,\widetilde{\to}\,(\mathcal{G}_\lambda\otimes\mathcal{G}_\mu)\otimes\mathcal{T}^{b(\lambda,\mu)}$$

So, (12) becomes

$$\mathcal{G}_{\lambda+\mu,\lambda+\mu}\,\widetilde{\to}\,(\mathcal{G}_{\lambda+\mu}\boxtimes\mathcal{G}_{\lambda+\mu})\otimes\mathcal{O}(\triangle)^{2b(\lambda,\mu)+2q(\lambda)+2q(\mu)}$$

on X^2. On the other hand, we have the factorization isomorphism for $(\lambda+\mu,\lambda+\mu)$ given by

$$\mathcal{G}_{\lambda+\mu,\lambda+\mu}\,\widetilde{\to}\,(\mathcal{G}_{\lambda+\mu}\boxtimes\mathcal{G}_{\lambda+\mu})\otimes\mathcal{O}(\triangle)^{2q(\lambda+\mu)}$$

on X^2. This gives the equality $2q(\lambda+\mu) = 2b(\lambda,\mu) + 2q(\lambda) + 2q(\mu)$. Consider now the permutation $\tau = (13)(24) \in S_4$. Its action on X^4 preserves the closed subscheme $\triangle_{12}\cap\triangle_{34}\,\widetilde{\to}\,X^2$, and gives the nontrivial permutation σ on it. Our τ preserves $\triangle_{13}$

and $\triangle_{24}$. The isomorphism (11) is τ-equivariant. This allows to extract square root of the above equality.

Recall ([22], 7.2.20). It shows that a base of Λ indexed by a finite set I yields an isomorphism $B^2_{et}(\mathrm{Hom}(\Lambda, A)) \widetilde{\to} \prod_I B^2_{et}(A)$.

3.0.56. In Sect. 4.2.10, in the diagram (4.14) the commutativity datum is the identity one, because the quadratic form vanishes, not only the bilinear form (this quadratic form precisely is given by this commutativity datum).

3.0.57. For 4.3.13 version June 1, 2020. The exact sequence of constructibel sheaves $0 \to \iota^* A_1 \to (s_1)_* A_1 \oplus (s_2)_* A_1 \to (s_{1,2})_* A_1 \to 0$ yields a distinguished triangle

$$(s_{1,2})_* A_1(-2)[-4] \to (s_1)_* A_1(-1)[-2] \oplus (s_2)_* A_1(-1)[-2] \to \iota^! A_1$$

by passing to the Verdier dual.

3.0.58. For 4.3.1. For a notion of a factorization group prestack over Ran. Let Z be a factorization prestack over Ran, that is, we are given a map $Z \to \mathrm{Ran}$ in PreStk lifted to a morphism in $CAlg^{nu}(\mathrm{PreStk}_{corr})$. Moreover, we assume that for any finite nonempty set J the induced map $Z^J \times_{\mathrm{Ran}^J} \mathrm{Ran}^J_d \to Z \times_{\mathrm{Ran}} \mathrm{Ran}^J_d$ is an isomorphism. To provide a structure of a factorization group prestack on Z means to lift it to an object of

$$\mathcal{G}\mathrm{rp}((CAlg^{nu}(\mathrm{PreStk}_{corr}))_{/\,\mathrm{Ran}}) \times_{\mathcal{G}\mathrm{rp}((\mathrm{PreStk}_{corr})_{/\,\mathrm{Ran}})} \mathcal{G}\mathrm{rp}(\mathrm{PreStk}_{/\,\mathrm{Ran}}) \tag{13}$$

In other words, product $m : Z \times_{\mathrm{Ran}} Z \to Z$ should be a map of factorization prestacks over Ran, and similarly for the unit $u : \mathrm{Ran} \to Z$ over Ran.

A good way to say is as follows I think. Let $\mathrm{FactPreStk}_{/\,\mathrm{Ran}}$ be the ∞-category of factorization prestacks over Ran. It afmits products. So, we may consider the category $\mathcal{G}\mathrm{rp}(\mathrm{FactPreStk}_{/\,\mathrm{Ran}})$ of groups in this category.

Our Gr_T is such a factorization group prestack over Ran. Let $H \in \mathcal{C}\mathrm{om}\mathrm{Grp}(\mathrm{Spc})$ then H is a commutative group in PreStk, hence also in PreStk_{corr}. So, $\mathrm{Ran} \times H \in CAlg^{nu}(\mathrm{PreStk}_{corr})$. The product for $\mathrm{Ran} \times H$ is given by the diagram

$$\mathrm{Ran}^2 \times H^2 \leftarrow \mathrm{Ran}^2_d \times H^2 \overset{\mathrm{add} \times m_H}{\to} \mathrm{Ran} \times H$$

Moreover $\mathrm{Ran} \times H \in \mathcal{G}\mathrm{rp}(CAlg^{nu}(\mathrm{PreStk}_{corr})_{/\,\mathrm{Ran}})$. A map

$$Z \to \mathrm{Ran} \times H$$

in $\mathrm{PreStk}_{/\,\mathrm{Ran}}$ lifted to a morphism in $CAlg^{nu}(\mathrm{PreStk}_{corr})_{/\,\mathrm{Ran}}$ should be called multiplicative if it is a morphism in

$$\mathcal{G}\mathrm{rp}(CAlg^{nu}(\mathrm{PreStk}_{corr})_{/\,\mathrm{Ran}})$$

In particular, such a morphism yields morphism in $\mathcal{G}\mathrm{rp}(\mathrm{PreStk}_{/\,\mathrm{Ran}})$. Note that $\mathrm{Ran} \times H$ is not a factorization prestack in our sense.

Taking $H = B^2_{et}(A)$, we get a definition of a $\mathrm{FactGe}^{mult}_A(Z)$.

Question. I think the proof of Pp. 4.3.2 in the paper is correct, but not very clear, because not sufficiently conceptual. Can you give a conceptual proof? I have as a model a claim like this: if $C \in 1-\mathcal{C}\mathrm{at}$ then $ComGrp(ComGrp(C)) \widetilde{\to} ComGrp(C)$ canonically. Maybe it would become clearer if formulated more generally?

The idea of your proof is that for Z a factorization group prestack the desired isomorphisms for $(z,1)\in (Z\times_{\mathrm{Ran}} Z)\times_{\mathrm{Ran}} \mathrm{Ran}^2_{disj}$ over $(x_1,x_2)\in \mathrm{Ran}^2_{disj}$ corresponding to factorization and to the multiplication by 1 are the same.

Formally, for a finite nonempty set J the diagram commutes

$$\begin{array}{ccc} Z\times_{\mathrm{Ran}}\mathrm{Ran}^J_d & \widetilde{\to} & Z^J\times_{\mathrm{Ran}^J}\mathrm{Ran}^J_d\\ \uparrow m & & \uparrow m^J\\ (Z\times_{\mathrm{Ran}} Z)\times_{\mathrm{Ran}}\mathrm{Ran}^J_d & \widetilde{\to} & (Z\times_{\mathrm{Ran}} Z)^J\times_{\mathrm{Ran}^J}\mathrm{Ran}^J_d \end{array}$$

and we apply this for $J=\{1,2\}$ and the point of $(Z\times_{\mathrm{Ran}} Z)\times_{\mathrm{Ran}}\mathrm{Ran}^J_d$ over $(x_1,x_2)\in \mathrm{Ran}^2_d$ given by $(z_1,1)$ at x_1 and $(1,z_2)$ at x_2. Here m is the multiplication on Z.

3.0.59. For the definition of a multiplicative gerbe from the previous section. If $Z\in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, A is a torsion abelian group then multiplicative A-gerbes on Z are defined as $\mathrm{Map}_{\mathcal{G}\mathrm{rp}(\mathrm{PreStk})}(Z, B^2_{et}(A))\widetilde{\to}\mathrm{Map}_{\mathrm{Ptd}(\mathrm{PreStk})}(B(Z), B^3_{et}(A))$.

3.0.60. For 4.3.3. Actually we need $A=E^{\times,tors}$, the group of torsion elements in E^* of order coprime to $char(k)$. Since E is of characteristic zero alg. closed, $\{\pm 1\}=\mu_2\subset E$ canonically indeed.

3.0.61. For 4.3.4. For $k\geq 1$ I think the definition of $\mathrm{FactGe}^{\mathbb{E}_k}_A(\mathrm{Gr}_T)$ can be given as in my Section 3.0.58 replacing (13) by

$$\mathbb{E}^{grp-like}_k((CAlg^{nu}(\mathrm{PreStk}_{corr}))_{/\,\mathrm{Ran}})\times_{\mathbb{E}^{grp-like}_k((\mathrm{PreStk}_{corr})_{/\,\mathrm{Ran}})}\mathbb{E}^{grp-like}_k(\mathrm{PreStk}_{/\,\mathrm{Ran}})$$

Its description is proposed in Remark 4.3.5 of the paper.

3.0.62. For Remark 4.6.7. $\mathrm{Map}_{\mathcal{G}\mathrm{rp}(\mathrm{Spc})}(\Lambda, B^2(A))$ classifies central extensions $\mathcal{C}$ of Λ by $B(A)$, see ([22], 7.2.18).

We have $\mathrm{Map}_{\mathbb{E}_2(\mathrm{Spc})}(\Lambda, B^2(A))\widetilde{\to}\mathrm{Map}_{Ptd(\mathrm{Spc})}(B^2(\Lambda), B^4(\Lambda))$ by adjointness. To lift an object $\mathcal{C}$ of $\mathrm{Map}_{\mathcal{G}\mathrm{rp}(\mathrm{Spc})}(\Lambda, B^2(A))$ to an object of $\mathrm{Map}_{\mathbb{E}_2(\mathrm{Spc})}(\Lambda, B^2(A))$ means to provide a braiding on the monoidal category $\mathcal{C}$, see [19].

Dennis says here that

$$\pi_2\,\mathrm{Map}_{Ptd(\mathrm{Spc})}(B^2(\Lambda), B^4(A))\widetilde{\to}\pi_0\Omega^2\,\mathrm{Map}_{Ptd(\mathrm{Spc})}(B^2(\Lambda), B^4(A))$$

Further, $\Omega\,\mathrm{Map}_{Ptd(\mathrm{Spc})}(X,Y)\widetilde{\to}\mathrm{Map}_{Ptd(\mathrm{Spc})}(X,\Omega Y)$ for any $X,Y\in\mathrm{Ptd}(\mathrm{Spc})$. So, the above group identifies with $\pi_0\,\mathrm{Map}_{\mathbb{E}_2(\mathrm{Spc})}(\Lambda, A)=\mathrm{Hom}_{Ab}(\Lambda, A)$.

My understanding is that Dennis claims that $\pi_0(\mathrm{Map}_{\mathbb{E}_2(\mathrm{Spc})}(\Lambda, B^2(A))\widetilde{\to} Quad(\Lambda, A)$, this is the set of isomorphism classes of such braided monoidal categories $\mathcal{C}$, see ([22], 7.3) for that. Moreover, $\pi_0(\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A))\widetilde{\to}\mathrm{Hom}(\Lambda, A_{2-tors})$, this corresponds to symmetric monoidal categories. Note also that

$$\pi_2(\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)))\widetilde{\to}\pi_0\,\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, A)=\mathrm{Hom}(\Lambda, A)$$

This gives a canonical map

$$B^2(\mathrm{Hom}(\Lambda, A)\to \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \tag{14}$$

in $\mathbb{E}_\infty(\mathrm{Spc})$ by adjointness. It is used in Section 4.3.7 of the paper. Besides,

$$\pi_1(\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)))=0$$

3.0.63. For 4.3.7. The fact that $\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \widetilde{\to} FactGe_A^{mult}(\mathrm{Gr}_T)$ is obtained as follows. The description of $\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X))$ can be given as in Sect. 4.2.10 of the paper with the following change: for $\lambda \in \Lambda$, we are given $\mathcal{G}^\lambda \in \mathrm{Ge}_A(X)$. For $\lambda_i \in \Lambda$ we are given an isomorphism $\mathcal{G}^{\lambda_1+\lambda_2} \widetilde{\to} \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2}$ on X associative in the natural sense. We are also given a datum of commutativity for the square

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \to & \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_2+\lambda_1} & \to & \mathcal{G}^{\lambda_2} \otimes \mathcal{G}^{\lambda_1} \end{array}$$

satisfying the hewagon axiom. Moreover, the square of the commutativity constraint is the identity. However, we do not require any more that the datum of the commutativity for (4.14) in the paper is the identity one!

This datum of commutativity gives precisely a map $\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \to \mathrm{Hom}(\Lambda, A_{2-tors})$. As in (4.11) of the paper, we get a fibre sequence in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$

$$\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A))) \to \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \to \mathrm{Hom}(\Lambda, A_{2-tors})$$

See also my Section 3.0.62.

To explain his formula

$$\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \widetilde{\to} \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \times^{B^2(\mathrm{Hom}(\Lambda, A))} \mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A)))$$

note the following. First, $\pi_2 \mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A))) \widetilde{\to} \mathrm{Hom}(\Lambda, A)$, as X is connected. By adjointness, this gives a morphism $B^2(\mathrm{Hom}(\Lambda, A)) \to \mathrm{Map}_{\mathrm{PreStk}}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A)))$ in $\mathrm{Ptd}(Spc)$. We also have the map (14) above, which together give a diagonal action of $B^2(\mathrm{Hom}(\Lambda, A))$ on

$$\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \times \mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A)))$$

We have also $\pi_1 \mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A))) \widetilde{\to} \mathrm{H}^1_{et}(X, \mathrm{Hom}(\Lambda, A))$ and

$$\pi_0 \mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A))) \widetilde{\to} \mathrm{H}^2_{et}(X, \mathrm{Hom}(\Lambda, A))$$

So, at the level of homotopy groups this seems to give the correct result, same homotopy groups as for $\mathrm{FactGe}_A^{mult}(\mathrm{Gr}_T)$.

3.0.64. It should be noted I think in the paper that the notion of a Hecke eigen-sheaf could be spelled as in the paper "On the de Jong conjecture" instead of complicated definition using sheaves of categories!

3.0.65. For 4.4.1. Dennis uses the "topology of finite surjective maps", no precise definition given!

Lemma 3.0.66. *Let $Y \xrightarrow{f} Z \xleftarrow{g} Z'$ be a diagram in Sch^{aff}_{ft}. Let $f' : Y' \to Z'$ be obtained from f by the base change g. Assume both f', g are finite morphisms surjective on k-points. Then f is also finite surjective on k-points.*

Proof. (Alain Genestier) Write $Y = \mathrm{Spec}\, B$, let $A \to A'$ be the homomorphism of k-algebras corresponding to g. Let $B' = A' \otimes_A B$, let I be the kernel of $h : B \to B'$. Since B' is a finite B-module, each element of I is nilpotent. Since B is noetherian, there is $n > 0$ such that $I^n = 0$. Let B_0 be the image of h. Since B' is a finite A-module,

A is noetherian, we conclude that B_0 is a finite A-module. For any i, the B_0-module I^i/I^{i+1} is of finite type, so I^i/I^{i+1} is also a finite type A-module. Thus I is a finite type A-module. We are done. □

We equip Sch^{aff}_{ft} with a collection of coverings, where a covering of $S \in \mathrm{Sch}^{aff}_{ft}$ is a finite collection of maps $f_i : S_i \to S$ such that f_i is finite and the map $\sqcup_i S_i \to S$ is surjective on k-points. The axioms of ([36], Definition 6.2) are verified, so we get a site. Thus for $\mathrm{PreStk}_{lft} = \mathrm{Fun}(\mathrm{Sch}^{aff}_{ft}, \mathrm{Spc})$ we get the corresponding localization.

Dennis proposed the following. Call a morphism $Y_1 \to Y_2$ in PreStk_{lft} ind-finite if for $S \to Y_2$ with $S \in \mathrm{Sch}^{aff}_{ft}$, $Y_1 \times_{Y_2} S$ can be written as a filtered colimit $\mathrm{colim}_{i \in I} Z_i$, where each Z_i is a scheme finite over S.

Remark 3.0.67. *Let $f : Y_1 \to Y_2$ be an ind-finite morphism in PreStk_{lft} inducing a surjection on k-points $Y_1(k) \to Y_2(k)$. Then it is a surjection in the topology of finite surjective maps.*

Proof. Let $S = \mathrm{Spec}\, A \in \mathrm{Sch}^{aff}_{ft}$ with a section $S \to Y_2$. Write $S \times_{Y_2} Y_1 \widetilde{\to} \mathrm{colim}\, Z_i$ with Z_i a scheme finite over S. Let $S_i \subset S$ be the schematic image of $Z_i \times_{Y_2} S \to S$. Then we get an inductive system $\{S_i\}_{i \in I}$ such that for the corresponding system of their ideals $I_i \subset A$ any maximal ideal $\mathfrak{m} \subset A$ contains some I_i. Then there is $i \in I$ such that $Z_i \times_{Y_2} S \to S$ is surjective on k-points. It is also finite, so we can localize in our topology using the cover $Z_i \times_{Y_2} S \to S$. We get the desired lifting $Z_i \times_{Y_2} S \to Y_1$ of $S \to Y_2$. So, f is a surjectiion in this topology. □

3.0.68. *Combinatorial Grassmanian.* For a finitely generated abelian group Γ we may define $\mathrm{Gr}_{\Gamma \otimes \mathbb{G}_m, comb}$ similarly to the case of a torus. Namely, consider the index category $\mathcal{C}$ whose objects are pairs (I, λ^I) with I a finite non-empty set, $\lambda^I : I \to \Gamma$. Write λ_i for the value of λ^I on i. A map from (J, λ^J) to (I, λ^I) in $\mathcal{C}$ is a surjection $\phi : I \to J$ such that $\lambda_j = \sum_{\phi(i)=j} \lambda_i$. Set $\mathrm{Gr}_{\Gamma \otimes \mathbb{G}_m, comb} = \underset{(I,\lambda^I) \in \mathcal{C}}{\mathrm{colim}}\, X^I$.

If $\Gamma = \Lambda_1/\Lambda_2$, we get a diagram $\mathrm{Gr}_{T_2, comb} \to \mathrm{Gr}_{T_1, comb} \to \mathrm{Gr}_{\Gamma \otimes \mathbb{G}_m, comb}$, hence a map $\mathrm{Gr}_{T_1, comb} / \mathrm{Gr}_{T_2, comb} \to \mathrm{Gr}_{\Gamma \otimes \mathbb{G}_m, comb}$. Probably, the latter map is an isomorphism after sheafification in the topology of finite surjective maps. Why?? This would imply that the sheafifications of $\mathrm{Gr}_{\Gamma \otimes \mathbb{G}_m, comb}$ and of $\mathrm{Gr}_{T_1} / \mathrm{Gr}_{T_2}$ in this topology are isomorphic.

3.0.69. For 4.4.5. If $b_1(\Lambda_2, -) = 0$ then we get $b : \Gamma \times \Gamma \to A(-1)$. If in addition $q_1 \mid_{\Lambda_2} = 0$ then we get the quadratic form $q : \Gamma \to A(-1)$ given by $q(\lambda \mod \Lambda_2) = q_1(\lambda)$ for $\lambda \in \Lambda_1$.

Hopefully a proof of 4.4.5 could be obtained as follows. Recall the isomorphism $\mathrm{Map}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B_{et}(T_i) \times X, B^4_{et}(A(1)) \times X) \widetilde{\to} \mathrm{FactGe}_A(\mathrm{Gr}_{T_i})$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$ for $T_i = \Lambda_i \otimes \mathbb{G}_m$. We assume $\Gamma = \Lambda_1/\Lambda_2$. Consider the map

$$\mathrm{Map}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B_{et}(T_1) \times X, B^4_{et}(A(1)) \times X) \to \mathrm{Map}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B_{et}(T_2) \times X, B^4_{et}(A(1)) \times X)$$

given by restricting along $B_{et}(T_2) \to B_{et}(T_1)$. Does Dennis claim that the fibre of the latter map identifies canonically with $\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma \otimes \mathbb{G}_m})$? I think no, because the kernel of $\mathrm{Quad}(\Lambda_1, A(-1)) \to \mathrm{Quad}(\Lambda_2, A(-1))$ is too big: for q in the kernel the bilinear form $b(\Lambda_2, -)$ does not becessary vanish.

3.0.70. For 4.4.6. In the last displayed formula if that section one needs to replace $B^2_{et}(\mathrm{Hom}(\Gamma, A))$ by $B^2(\mathrm{Hom}(\Gamma, A))$.

If $\Gamma = \Lambda_1/\Lambda_2$, where $0 \to \Lambda_2 \to \Lambda_1 \to \Gamma \to 0$ is an exact sequence in abelian groups, Λ_i are lattices then $\Lambda_1 \to \Lambda_2 \to \Gamma$ is a fibre sequence in Sptr, hence a cofibre sequence in Sptr. Now $\mathrm{Sptr}^{\leq 0} \widetilde{\to} \mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$ is stable under small colimits, so Γ is a cofibre of $\Lambda_1 \to \Lambda_2$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$. So, $\mathrm{Map}_{\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})}(\Gamma, B^2(A))$ is the fibre of the map $\mathrm{Map}_{\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})}(\Lambda_1, B^2(A)) \to \mathrm{Map}_{\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})}(\Lambda_2, B^2(A))$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$, and also in Spc.

Similarly, the fibre of the natural map

$$\mathrm{Map}_{\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})}(\Lambda_1, \mathrm{Ge}_A(X)) \to \mathrm{Map}_{\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})}(\Lambda_2, \mathrm{Ge}_A(X))$$

in $\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})$ is $\mathrm{Map}_{\mathcal{C}\mathrm{omGrp}(\mathrm{Spc})}(\Gamma, \mathrm{Ge}_A(X))$.

To be clear: if Γ is torsion free then the assumption that A is divisible in 4.4.6(e,f) is not needed according to Sect. 3.3 of the paper.

3.1. For 4.4.7. Pick a presentation $1 \to T_2 \to \tilde{G}_2 \to G \to 1$, where T_2 is a torus, and $[\tilde{G}_2, \tilde{G}_2]$ is simply-connected, set $T_1 = \tilde{G}_2/[\tilde{G}_2, \tilde{G}_2]$. We get the maps $\mathrm{Gr}_{T_2} \to \mathrm{Gr}_{\tilde{G}_2} \to \mathrm{Gr}_{T_1} \to \mathrm{Gr}_{\pi_1(G)\otimes\mathbb{G}_m}$, and $\mathrm{Gr}_{\pi_1(G)\otimes\mathbb{G}_m} \widetilde{\to} \mathrm{Gr}_{T_1} / \mathrm{Gr}_{T_2}$. Actually, T_2 is central in $\tilde{G}_2$, so Gr_{T_2} acts on $\mathrm{Gr}_{\tilde{G}_2}$, and we get a map of quotients $\mathrm{Gr}_{\tilde{G}_2} / \mathrm{Gr}_{T_2} \to \mathrm{Gr}_{T_1} / \mathrm{Gr}_{T_2}$. The natural map $\mathrm{Gr}_{\tilde{G}_2} / \mathrm{Gr}_{T_2} \to \mathrm{Gr}_G$ is a monomorphism of prestacks. Yifei claims that the map $\mathrm{Gr}_{\tilde{G}_2} \to \mathrm{Gr}_G$ is surjective in any topology including finite surjective maps as coverings. This would imply that $f : \mathrm{Gr}_{\tilde{G}_2} / \mathrm{Gr}_{T_2} \to \mathrm{Gr}_G$ becomes an isomorphism after the sheafification in this topology.

Note that f is surjective on k-points. I think it is pseudo-proper. Is it true that after any base change $S \to \mathrm{Ran}$ with $S \in \mathrm{Sch}^{aff}_{ft}$ it becomes finitary pseudo-proper? This looks plausible. Then we would apply ([8], Lemma 7.4.11(d)). Dennis will treat this question in a new version.

Question. If $Z \to \mathrm{Ran}$ is a factorization prestack, $Z \in \mathrm{PreStk}_{lft}$, consider the sheafification Z' of Z in the topology of finite surjective maps. Why Z' is still a factorization prestack? This is not clear at all!

It is not clear if Ran is a sheaf in this topology. We could in principle consider the sheafification on the category of $(\mathrm{Sch}^{aff}_{ft})_{/\mathrm{Ran}}$, but even then it is not clear why a sheafification of a factorization prestack is still a factorization prestack. This will be changed in a new version.

Remark: $\mathbb{A}^1$ is not a sheaf on Sch^{aff}_{ft} in the topology of finite surjective maps.

3.1.1. For 4.5. We are mostly interested in the case $A = E^{\times, tors}$, the group of torsion elements in E^* of orders coprime to $char(k)$. If $check(k) = 2$ we get $A_{2-tors} = 0$, otherwise $A_{2-tors} = \mathbb{Z}/2\mathbb{Z}$. In the case of $char(k) = 2$ there is no problem of splitting of multiplicative gerbes.

Dennis claims that $\pi_0 \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A)) \widetilde{\to} \mathrm{Hom}(\Gamma, A_{2-tors})$, see ([22], 7.3). Indeed, if $\Gamma = \Lambda_1/\Lambda_2$, we get that $\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))$ is the fibre of

$$\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda_1, B^2(A)) \to \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda_2, B^2(A))$$

in $\mathbb{E}_\infty(\mathrm{Spc})$. This allows to calculate the homotopy groups of $\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))$, since we know the answer for Γ a lattice. We get

$$\pi_2 \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A)) \widetilde{\to} \mathrm{Hom}(\Gamma, A)$$

Now $\pi_1 \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A)) = 0$, because this is a cokernel of $\mathrm{Hom}(\Lambda_1, A) \to \mathrm{Hom}(\Lambda_2, A)$, and A is divisible.

3.1.2. For 4.8.2. Misprint: he meant $\pi_0(\mathcal{C}) = \pi_1(\mathcal{C}) = \mathbb{Z}/2\mathbb{Z}$. This $\mathcal{C} = \mathbb{Z}/2\mathbb{Z} \times B(\mathbb{Z}/2\mathbb{Z})$ is equipped with the braiding $b'(\lambda, \mu) : c^\lambda \otimes c^\mu \to c^\mu \otimes c^\lambda$ for any $\lambda, \mu \in \mathbb{Z}/2\mathbb{Z}$. Here b' is a bilinear form on $\mathbb{Z}/2\mathbb{Z}$ with values in $\mathbb{Z}/2\mathbb{Z}$ given by $b'(1,1) = 1$. Then the square of the brading $c^\lambda \otimes c^\mu \to c^\mu \otimes c^\lambda \to c^\lambda \otimes c^\mu$ is the identity, and the quadratic form $q(x) = b'(x,x)$ in $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}) = \mathrm{Quad}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is the identity map $q = \mathrm{id}$.

By functoriality he means the following. We have a morphism $\mathrm{Hom}_{Ab}(\Gamma, \mathbb{Z}/2\mathbb{Z}) \times \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\mathbb{Z}/2\mathbb{Z}, B^2(A)) \to \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))$ given by composing with $\Gamma \to \mathbb{Z}/2\mathbb{Z}$. It is bilinear. So, our distinguished element of $\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\mathbb{Z}/2\mathbb{Z}, B^2(A))$ by restriction gives a map $\mathrm{Hom}_{Ab}(\Gamma, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))$ in $\mathbb{E}_\infty(\mathrm{Spc})$, whose composition with the projection to $\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ is the identity.

3.1.3. For 4.5.3. The gerbe $\mathcal{G}^\epsilon$ is defined for $\epsilon \in \mathrm{Hom}(\Gamma, A_{2-tors})$ under the assumption that $A_{2-tors} \subset \mathbb{Z}/2\mathbb{Z}$ imposed in 4.5!

3.1.4. For 4.5.6. The determinant line bundle $\det_{\mathbb{G}_m, St}$ is not defined on $\mathrm{Gr}_{\mathbb{G}_m}$ explicitly however it is uniquely recovered from what is written in that section. Namely, $\det_{\mathbb{G}_m, St}$ is the line bundle sending $(I, L, \beta : L \widetilde{\to} \mathcal{O} \mid_{S \times X - \Gamma_I}) \in \mathrm{Gr}_{\mathbb{G}_m}$ to $\det \mathrm{R}\Gamma(X, L) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{-1}$. In the local setting this is

$$\det(L : \mathcal{O}) = \frac{\det(L/L')}{\det(\mathcal{O}/L')}$$

for any $L' \subset L \cap \mathcal{O}$.

Similarly, $\det_{\mathrm{SL}_2, St}$ is the line bundle sending $(I, L, \beta : L \widetilde{\to} \mathcal{O}^2 \mid_{S \times X - \Gamma_I}) \in \mathrm{Gr}_{\mathrm{SL}_2}$ to $\det(L : \mathcal{O}^2)$.

3.1.5. In the last paragraph of Sect. 4.6.1 it is affirmed that $(\mathcal{L}^{\otimes 2})^{\frac{1}{2}}$ identifies canonically with $\mathcal{G}^{\epsilon_{taut}} \mid_Z$. I think this is correct but not completely clear. In particular this implies that the factorization line bundle $\mathcal{L}^{\otimes 2}$ is not trivial!

3.1.6. For 4.6.3. Let $\mathcal{C}$ be a sheaf of categories on $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{G}_m}$. By a "factorization structure on $\mathcal{C}$ compatible with the factorization structure on $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{G}_m}$ we mean a multiplicative sheaf of categories over $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{G}_m} \in CAlg^{nu}(\mathrm{PreStk}_{corr})$ in the sense of [30]. So, given $S \in \mathrm{Sch}^{aff}_{ft}$ and a map

$$s : S \to \mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{G}_m} \times_{\mathrm{Ran}} \mathrm{Ran}^J_d \widetilde{\to} \mathrm{Gr}^J_{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{G}_m} \times_{\mathrm{Ran}^J} \mathrm{Ran}^J_d,$$

which is a collection $s_j : S \to \mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{G}_m} \times_{\mathrm{Ran}} \mathrm{Ran}^J_d \widetilde{\to} \mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{G}_m}$ for $j \in J$, we have an equivalence

$$\underset{j \in J, \mathrm{Shv}(S)}{\otimes} \mathcal{C}(S, s_j) \widetilde{\to} \mathcal{C}(S, s)$$

functorial in (S, s).

3.1.7. For 4.6.5. The definition of the endofunctor $c \mapsto c[d]$ given by (4.23) was explained to me by Dennis: he meant the cohomological shift $c \mapsto c[d]$ over the connected component given by d.

3.1.8. For 5.1.4. About a formalism: if $f : Y_1 \to Y_2$ is an ind-schematic morphism in PreStk_{lft}, why $f_* : \mathrm{Shv}(Y_1) \to \mathrm{Shv}(Y_2)$ exists and how it is defined? Dennis says this is done in [13] in the case when $f : Y_1 \to Y_2$ is a morphism of schemes. The general case: it suffices to define the direct image for $S \times_{Y_2} Y_1 \to S$ for any $S \to Y_2$, $S \in \mathrm{Sch}^{aff}_{ft}$. Write $S \times_{Y_2} Y_1 \widetilde{\to} \mathrm{colim}_i\, Z_i$ so that Z_i is a scheme and $h : Z_i \to Z_j$ is a closed immersion. Then $Shv(S \times_{Y_2} Y_1) \widetilde{\to} \mathrm{colim}_i\, \mathrm{Shv}(Z_i)$ as in ([8], 1.5.2). The desired functor comes from a compatible system of functors $(q_i)_* : Shv(Z_i) \to Shv(S)$ for $q_i : Z_i \to S$.

3.1.9. For 5.2.1. To be precise, let us understand by $\mathrm{detrel}(\mathfrak{g}_{\mathcal{P}_G}, \mathfrak{g}_{\mathcal{P}^0_G})$ the line bundle $\det \mathrm{R}\Gamma(X, \mathfrak{g}_{\mathcal{P}_G}) \otimes \det \mathrm{R}\Gamma(X, \mathfrak{g}_{\mathcal{P}^0_G})^{-1}$.

3.1.10. For 5.2.2. We have $\check{\rho}_{G,M} = \check{\rho}_G - \check{\rho}_M$. Here $\check{\rho}_G$ is the half sum of positive roots of G.

For 5.2.4. The line

$$K(L) := \frac{\det \mathrm{R}\Gamma(X, E \otimes L) \otimes \det \mathrm{R}\Gamma(X, E^* \otimes L)}{\det \mathrm{R}\Gamma(X, E_0 \otimes L) \det \mathrm{R}\Gamma(X, E_0^* \otimes L)}$$

is canonically independent of $L \in \mathrm{Bun}_1$. One sees that $K(L(x)) \widetilde{\to} K(L)$ canonically for $x \in X$. This argument can be also done locally, in the case when X is not complete. This is related to my paper [23].

The factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle $\det_{\mathfrak{g}}$ on Gr_G comes from a theta-datum such that corresponding symmetric bilinear form on Λ is the Killing form $\kappa_{G,Kil} = \sum_{\check{\alpha}} \check{\alpha} \otimes \check{\alpha}$, the sum over all roots. So, the factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle $\det_{\mathfrak{n}(P)}$ corresponds to the bilinear form $\frac{1}{2}(\kappa_{G,Kil} - \kappa_{M,Kil})$.

In general, if $\mathcal{L}$ is a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on Gr_T corresponding to a symmetric bilinear form $\kappa : \Lambda \otimes \Lambda \to \mathbb{Z}$ when viewed as a θ-datum (cf. 4.1.5 of the paper) then $\mathcal{L}^{\otimes 2}$ is a factorization line bundle, and the factorization μ_2-gerbe $(\mathcal{L}^{\otimes 2})^{\frac{1}{2}}$ corresponds to the quadratic form $q : \Lambda \to \mathbb{Z}/2\mathbb{Z}$, where $q(x) = \kappa(x,x) \mod 2$. So, $(\mathcal{L}^{\otimes 2})^{\frac{1}{2}}$ is a multiplicative factorization gerbe.

In 5.2.3 the following calculation is used: set $q(x) = \frac{1}{2}\kappa_{G,Kil}(x,x) \mod 2 \in \mathbb{Z}/2\mathbb{Z}$ for $x \in \Lambda$. Then $q(x) = \langle 2\check{\rho}, x\rangle \mod 2$.

3.1.11. For 6.2.3. The $\mathbb{Z}/2\mathbb{Z}$-graded factorization line bundle $\det_{\mathbb{G}_m,n}$ has fibre in the global case $\det \mathrm{R}\Gamma(X, L^n) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O}^n)^{-1}$ at $(L, \alpha : L \widetilde{\to} \mathcal{O} \mid_{U_I}) \in \mathrm{Gr}_{\mathbb{G}_m}$ over $I \in \mathrm{Ran}$.

The factorization line bundle $\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1}$ on $\mathrm{Gr}_{\mathbb{G}_m}$ corresponds when viewed as a θ-datum, to the symmetric bilinear form $b : \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}, b(x,y) = 2xy$. So, the quadratic form $q : \Lambda \to A(-1)$ with $\Lambda = \mathbb{Z}$ corresponding to the factorization gerbe $(\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^a$ is given by $q(1) = a$.

Let $n = ord(a)$. Then the pull-back of the factorization line bundle $\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1}$ under $\mathbb{G}_m \to \mathbb{G}_m, x \mapsto x^n$ identifies with $(\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^{n^2}$ canonically by ([23], Lm. 5.7). In particular, the restriction of the factorization gerbe $(\det_{\mathbb{G}_m,1}\otimes\det_{\mathbb{G}_m,-1})^a$ under this map is canonically trivialized.

3.2. Let us think about the question: is there $\mathcal{K}\in\mathcal{C}\mathrm{omGrp}(\mathrm{PreStk})$ such that $\mathrm{Gr}_{\pi_1(G)\otimes\mathbb{G}_m}$ identifies with the prestack classifying $I\in\mathrm{Ran}(S)$, a map $S\times X\to\mathcal{K}$ together with a trivialization of the composition $U_I\to S\times X\to\mathcal{K}$?

Write $\mathrm{Stk}\subset\mathrm{PreStk}$ for the full subcategory of stacks in etale topology. Pick an exact sequence $1\to T_2\to\tilde{G}\to G\to 1$ with $[\tilde{G},\tilde{G}]$ simply-conneted. Write $B(G)$ for the corresponding colimit in PreStk. Clearly, $B_{et}(T_2)$ acts on $B_{et}(\tilde{G})\in\mathrm{Ptd}(\mathrm{Stk})$ on the left. Since our extension is central, the corresponding map $G\to B_{et}(T_2)$ is a morphism in $\mathcal{G}\mathrm{rp}(Stk)$, hence induces after applying B a morphism $B_{et}(G)\to B^2_{et}(T_2)$ in $\mathrm{Ptd}(Stk)$. The fibre of this morphism is $B_{et}(\tilde{G})$, so $B_{et}(G)$ is the quotient of $B_{et}(\tilde{G})$ by the action of $B_{et}(T_2)$ in the ∞-topos Stk. Let $T_1=\tilde{G}/[\tilde{G},\tilde{G}]$.

Let $(B(T_1)/B(T_2))_c$ be the cofibre of $B(T_2)\to B(T_1)$ in $\mathcal{C}\mathrm{omGrp}(\mathrm{PreStk})$. We have a full subcategory $\mathcal{C}\mathrm{omGrp}(\mathrm{PreStk})\subset\mathrm{Fun}(\mathrm{Sch}^{aff},\mathrm{Sptr})$, this is also a cofibre in the stable category $\mathrm{Fun}(\mathrm{Sch}^{aff},\mathrm{Sptr})$, because $\mathrm{Sptr}^{\le 0}\subset\mathrm{Sptr}$ is stable under all colimits. So, $B(T_2)$ is the fibre of $B(T_1)\to(B(T_1)/B(T_2))_c$ in $\mathrm{Fun}(\mathrm{Sch}^{aff},\mathrm{Sptr})$, hence also in its full subcategory $\mathcal{C}\mathrm{omGrp}(\mathrm{PreStk})$.

Write $B_{et}(T_1)/B_{et}(T_2)$ for the quotient of $B_{et}(T_1)$ by $B_{et}(T_2)$ in the ∞-topos Stk. We have a natural map $B_{et}(T_1)/B_{et}(T_2)\to B^2_{et}(T_2)$ whose fibre is $B_{et}(T_1)$. The map $\tilde{G}\to T_1$ gives a morphism of quotients

$$B_{et}(G)\widetilde{\to}B_{et}(\tilde{G})/B_{et}(T_2)\to B_{et}(T_1)/B_{et}(T_2) \tag{15}$$

in Ptd(Stk). How $B_{et}(T_1)/B_{et}(T_2)$ depends on a choice of $\tilde{G}$?

Consider the case of $\pi_1(G)$ finite. Then the kernel of $T_2\to T_1$ is $K:=\pi_1(G)(1)$. The fibre sequence $1\to K\to T_2\to T_1\to 1$ gives a map $B_{et}(T_1)\to B^2_{et}(K)$, whose fibre is $B_{et}(T_2)$. This means that $B_{et}(T_1)$ is the quotient of $B_{et}(T_2)$ by $B_{et}(K)$ in Stk (cf. more generally [22], 7.2.18). Considering now the natural map $B_{et}(T_2)\to B_{et}(T_2)/B_{et}(K)$ and taking its quotient by the action of $B_{et}(T_2)$, we should get $*\to B^2_{et}(K)$. So, I hope that $B_{et}(T_1)/B_{et}(T_2)$ identifies with $B^2_{et}(K)$ in $Ptd(\mathrm{Stk})$. I think this can also be checked calculating the homotopy groups of $B_{et}(T_1)/B_{et}(T_2)$ using the fibre sequence $B_{et}(T_1)\to B_{et}(T_1)/_{et}B(T_2)\to B^2_{et}(T_2)$ in Stk.

So, (15) is a canonical morphism $B_{et}(G)\to B^2_{et}(K)$ in Stk.

Let Y_K be the prestack (locally of finite type) over Ran sending $S\in\mathrm{Sch}^{aff}_{ft}$ to $I\in\mathrm{Ran}(S)$, a map $X\times S\to B^2_{et}(K)$ together with a trivialization of the composition $U_I\to X\times S\to B^2_{et}(K)$. This is a factorization prestack over Ran, we have a natural map $\mathrm{Gr}_G\to Y_K$ of factorization prestacks over Ran. In 3.1.6 of the paper we constructed a map of prestacks $Y_K\to K(-1)_{et}$ under the assumption that K is of order coprime to $char(k)$.

Let us interprete $\mathrm{Gr}_{T_1}/\mathrm{Gr}_{T_1}$ as the quotient in PreStk_{lft}. My understanding is that the natural map $\mathrm{Gr}_{T_1}/\mathrm{Gr}_{T_2}\to Y_K$ is an isomorphism. Is this correct?

More generally, remove the assumption that $\pi_1(G)$ is finite. Then I think that $B_{et}(T_1)/B_{et}(T_2)\in\mathrm{Ptd}(\mathrm{Stk})$ is independent of our choice of $\tilde{G}$. Indeed, let $1\to T_2'\to\tilde{G}'\to G\to 1$ be another exact sequence with T_2' a torus, and $[\tilde{G}',\tilde{G}']$ simply connected.

Then we may argue as in ([33], 7.2.5). Namely, let $\tilde{G}'' = \tilde{G}\times_G \tilde{G}'$. The projection $\tilde{G}''\to\tilde{G}$ fits into an exact sequence $1\to T_2'\to\tilde{G}''\to\tilde{G}\ \to 1$. Moreover, the exact sequence splits

$$1\to T_2'\to\tilde{G}''_{ab}\to\tilde{G}_{ab}\to 1,$$

where $\tilde{G}_{ab}$ stands for the abelinization of G_{ab}. We have the natural map

$$B_{et}(\tilde{G}''_{ab})/B_{et}(T_2\times T_2')\to B_{et}(T_1)/B_{et}(T_2)$$

The fact that this map is an isomorphism follows from the fact that

$$B_{et}(T_1\times T_2)\widetilde{\to} B_{et}(T_1)\times B_{et}(T_2)$$

Let Y be the prestack sending S to $I\in\mathrm{Ran}(S)$, a map $X\times S\to B_{et}(T_1)/B_{et}(T_2)$ together with a trivialization of the composition $U_I\to X\times S\to B_{et}(T_1)/B_{et}(T_2)$. Then Y is a factorization prestack, and we get a natural map $\mathrm{Gr}_G\to Y$ over Ran.

Question. It seems that $(\mathrm{Gr}_{T_1}/\mathrm{Gr}_{T_2})_{et}\widetilde{\to} Y$ in general. Is this correct? Here $(\mathrm{Gr}_{T_1}/\mathrm{Gr}_{T_2})_{et}$ is the sheafification in the etale topology.

Maybe then we can take Y for $\mathrm{Gr}_{\pi_1(G)\times\mathbb{G}_m}$?

3.2.1. For 7.4. To describe the multiplicative A-torsors on T, we have to analyse $\mathrm{Map}_{\mathcal{G}rp(\mathrm{PreStk})}(T, B_{et}(A))\widetilde{\to}\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(T), B^2_{et}(A))$. This is the relative cohomology $\mathrm{Map}_{\mathrm{PreStk}}(B(T), B^2_{et}(A))\times_{\mathrm{Map}_{\mathrm{PreStk}}(*, B^2_{et}(A))} *$. Let $q: *\to B(T)$ be the natural map in PreStk. Define K by the fibre sequence $K\to A\to q_*A$ in the corresponding stable category of sheaves on $B(T)$. The corresponding long exact sequence in cohomology gives $0\to \mathrm{H}^2_{et}(B(T), K)\to\mathrm{H}^2_{et}(B(T), A)\to 0$ is an isomorphism, so $\mathrm{H}^2_{et}(B(T), K)\widetilde{\to}\mathrm{Hom}(\Lambda, A(-1))$ by Th. 3.2.6 of our paper. So,

$$\pi_0\,\mathrm{Map}_{\mathcal{G}rp(\mathrm{PreStk})}(T, B_{et}(A))\widetilde{\to}\mathrm{Hom}(\Lambda, A(-1))$$

If $\mathcal{G}$ is an A-gerbe over $*$, to provide its descent datum under the map $*\to B(T)$ means essentially to provide a point of $\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B(T), B^2_{et}(A))$. Indeed, we may assume our gerbe on $*$ trivial. The corresponding multiplicative A-torsor on T is obtained as follows: we have $\Omega B(T)\widetilde{\to} T$. So, for $h: T\to *$ we get an automorphism of $h^*\mathcal{G}$, which is given by a A-torsor on T.

3.2.2. For 7.5.1. The quotient by $\mathfrak{L}^+(G)$ in (7.10) is understood as the quotient in the topos of prestacks sheafified in etale topology. The prestack Z^n sends $S\in\mathrm{Sch}^{aff}$ to the collection: $F_0,\ldots,F_n$, where F_i is a G-torsor on $S\times X$, $I\in\mathrm{Ran}(S)$, and $\alpha_i: F_{i-1}\widetilde{\to} F_i\mid_{U_I}$ is an isomorphism. The simplicial structure comes from the fact that for any finite nonempty linearly ordered set I we may similarly define Z^I sending S to: $F_i, i\in I$, $I\in\mathrm{Ran}(S)$ and isomorphisms $\alpha_i F_{i-1}\widetilde{\to} F_i\mid_{U_I}$ for $i\in I$ different from the initial element. Here $i-1$ is the element preceding i.

The factorization structure on Z^n can be obtained using the Beauville-Laszlo theorem: Z^n sends $S\in\mathrm{Sch}^{aff}$ to the collection: $I\in\mathrm{Ran}(S)$; G-torsors $F_0,\ldots,F_n$ on D_I together with isomorphisms $F_0\widetilde{\to} F_1\widetilde{\to}\ldots\widetilde{\to} F_n$ over $\overset{\circ}{D}_I$. If $\{I_j\}_{j\in J}\in\mathrm{Ran}^I_{disj}$ then $D_I=\sqcup_j D_{I_j}$, so the above data factorize. So, $Z^\bullet$ is a simplicial object in

$CAlg^{nu}(\mathrm{PreStk}_{corr})$. Then $\mathcal{Z} :=\mid Z^\bullet \mid$ is a colimit in $\mathrm{PreStk}_{/\mathrm{Ran}}$, and Dennis claims that the corresponding maps

$$\mathcal{Z}^J \times_{\mathrm{Ran}^J} \mathrm{Ran}^J_{disj} \to \mathcal{Z} \times_{\mathrm{Ran}} \mathrm{Ran}^J_{disj}$$

are still isomorphisms for all finite nonempty sets J, so that $\mathcal{Z}$ is a factorization prestack. I think the category of factorization prestacks over Ran admits all small colimits.

Dennis says that if $\mathcal{G}$ is a factorization A-gerbe on Gr_G then $\mathcal{G}^{\tilde{\boxtimes} n}$ constructed in Sect. 7.3 over $\widetilde{\mathrm{Gr}}^n_G \times_{\mathrm{Ran}^n} \mathrm{Ran}$ descend to a gerbe on Z^n, and can be see as a map $Z^n \to B^2_{et}(A)^{\times n}$. This gives a morphism of simplicial prestacks, and passing to the geometric realizations (shifified in etale topology), one gets a morphism $\mid Z^\bullet \mid \to B^3_{et}(A)$.

For 7.5.2. My understanding is that, more generally, let $H \subset G$ be a subgroup in a group maybe in some ∞-topos $\mathcal{C}$. Then the Cech nerve of $B(H) \to B(G)$ is obtained as follows. For $n \geq 0$ it sends $[n]$ to $G\backslash((G/H)^{\times(n+1)})$, where G acts diagonally on the product $(G/H)^{\times(n+1)}$. The latter identifies also with $H\backslash((G/H)^n)$, where H acts diagonally.

The identification of Z^n with the prestack $\mathfrak{L}(G)\backslash(\mathrm{Gr}_G^{\times(n+1)})$ is as follows. For a point $(F_0, \dots, F_n, \alpha_i)$ of Z^n as above, pick trivializations δ_i of $F_0, \dots, F_n$ over the disk D_x. Then our datum becomes a collection $\gamma_1, \dots, \gamma_n \in G(F)$, where F is the field of fraction of $\mathcal{O}_x$. Namely, γ_i is the induced isomorphism $F^0 \stackrel{\delta_0^{-1}}{\to} F_0 \to F_1 \to \dots \to F_i \stackrel{\delta_i}{\to} F^0$ over $\mathrm{Spec}\, F$. This gives a point of $\mathfrak{L}^+(G)\backslash(\mathrm{Gr}^n_G)$.

Dennis claims that, according to (HTT, 6.2.3.4), the map $B_{et}(\mathfrak{L}^+(G)) \to B_{et}(\mathfrak{L}(G))$ is a (-1)-truncated object of $\mathrm{PreStk}_{/B_{et}(\mathfrak{L}(G))}$. This is equivalent to saying that the diagonal morphism

$$B_{et}(\mathfrak{L}^+(G)) \to B_{et}(\mathfrak{L}^+(G)) \times_{B_{et}(\mathfrak{L}(G))} B_{et}(\mathfrak{L}^+(G))$$

is an isomorphism in PreStk.

The space of multiplicative gerbes on $\mathfrak{L}(G)$ with a multiplicative trivialzation of their restriction to $\mathfrak{L}^+(G)$ is (the one in the LHS of formula (7.3) in the paper)

$$\begin{aligned}\mathrm{Map}_{\mathcal{G}\mathrm{rp(PreStk)}}(\mathfrak{L}(G), B^2_{et}(A)) \times_{\mathrm{Map}_{\mathcal{G}\mathrm{rp(PreStk)}}(\mathfrak{L}^+(G), B^2_{et}(A))} * \widetilde{\to} \\ \mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B_{et}(\mathfrak{L}(G)), B^3_{et}(A)) \times_{\mathrm{Map}_{\mathrm{Ptd(PreStk)}}(B_{et}(\mathfrak{L}^+(G)), B^3_{et}(A))} *\end{aligned}$$

So, we produced an object of this space out of a factorization gerbe $\mathcal{G}$ on Gr_G.

3.3. About $\mathrm{Fact}(\mathcal{C})$**.** For 8.1.4. For a finite nonempty set I, the notation $Tw(I)$ here is not standard. Write $fSets$ for the category whose objects are finite noempty sets, and morphisms are surjections. For $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ write $\mathcal{T}w(\mathcal{C})$ for the twisted arrows category (cf. [25], Appendix). Then $\mathcal{T}w(fSets) \times_{fSets} fSets_{I/} \widetilde{\to} \mathrm{Tw}(I)$.

For $(I \to J \to K) \in Tw(I)$, the 2nd displayed formula in 8.1.5 means $\underset{k\in K}{\boxtimes} \mathcal{C}_X^{\otimes J_k}$ rather. That is, in $\underset{k\in K}{\otimes} \mathcal{C}_X^{\otimes J_k}$ we make[1] base change by $\underset{k\in K}{\otimes} Shv(X) \to Shv(X^K)$.

[1]For $\mathcal{D}$-modules this is not necessary, as the corresponding map is an equivalence, but we want a consruction working for other sheaf theories also.

We use everywhere the fact that in any sheaf theory an ind-scheme of ind-finite type is 1-affine (this is proved by Lin Chen in his email).

The colimit of the functor (8.6) can be understood also in $Shv(X^I)-mod$ instead of DGCat_{cont}, the projection $Shv(X^I)-mod \to \mathrm{DGCat}_{cont}$ preserves colimits. For a map (16) in $Tw(I)$ the diagram commutes

$$\begin{array}{ccc} \underset{k\in K_1}{\boxtimes} \mathcal{C}_X^{\otimes (J_1)_k} & \stackrel{m}{\to} & \underset{k\in K_1}{\boxtimes} \mathcal{C}_X^{\otimes (J_2)_k} \\ \downarrow & & \downarrow \\ \underset{k\in K_2}{\boxtimes} \mathcal{C}_X^{\otimes (J_1)_k} & \stackrel{m}{\to} & \underset{k\in K_2}{\boxtimes} \mathcal{C}_X^{\otimes (J_2)_k}, \end{array}$$

where the vertical arrows are direct image functors for $X^{K_1} \hookrightarrow X^{K_2}$.

For 8.1.6. To check that the construction of 8.1.2-8.1.5 produces a factorization sheaf of symmetric monoidal categories on Ran, we do the following.

Let I be a finite non-empty set, $f : I \to I'$ a surjection. Then f induces a full embedding $\mathrm{Tw}(I') \subset \mathrm{Tw}(I)$ sending $I' \to J' \to K'$ to $I \stackrel{f'}{\to} J' \to K'$. Here f' is the composition $I \to I' \to J'$.

Let $Q(I)$ be the set of equivalence relations on I. Recall that $Q(I)$ is partially ordered. As in [2], we write $I' \in Q(I)$ for a quotient $I \to I'$ viewed as an equivalence relation on I. We write $I'' \le I'$ iff $I'' \in Q(I')$. Then $Q(I)$ is a lattice. For $I', I'' \in Q(I)$ we have $\inf(I', I'')$. Let now a surjection $f : I \to I'$ be given. We get a functor $Q(I) \to Q(I')$ sending $J \in Q(I)$ to $\inf(J, I') \in Q(I')$. It can be seen as a push-out in the category of finite sets.

If $J_2 \le J_1$ in $Q(I)$ then for $J'_i = \inf(J_i, I')$ we get $X^{J_2} \times_{X^{J_1}} X^{J'_1} \widetilde{\to} X^{J'_2}$.

Define a functor $\xi : \mathrm{Tw}(I) \to \mathrm{Tw}(I')$ sending $I \to J \to K$ to $I' \to J' \to K'$, where $J' = \inf(J, I'), K' = \inf(K, I')$. It sends a morphism

(16)
$$\begin{array}{ccccc} I & \to & J_1 & \to & K_1 \\ \| & & \downarrow & & \uparrow \\ I & \to & J_2 & \to & K_2 \end{array}$$

to the induced diagram

$$\begin{array}{ccccc} I' & \to & J'_1 & \to & K'_1 \\ \| & & \downarrow & & \uparrow \\ I' & \to & J'_2 & \to & K'_2 \end{array}$$

Let $\mathcal{F}_I : \mathrm{Tw}(I) \to \mathrm{Shv}(X^I) - mod$ be the functor sending $(I \to J \to K)$ to

$$\underset{k\in K}{\boxtimes} \mathcal{C}_X^{\otimes J_k}$$

(the latter category is actually an object of $Shv(X^K) - mod$). By definition, $\mathrm{Fact}(\mathcal{C})$ associates to $X^I \to \mathrm{Ran}$ the category

$$\mathcal{C}_{X^I} := \underset{(I\to J\to K)\in \mathrm{Tw}(I)}{\mathrm{colim}} \big(\underset{k\in K}{\boxtimes} \mathcal{C}_X^{\otimes J_k}\big)$$

the colimit taken in $\mathrm{Shv}(X^I) - mod$.

Let now $f : I \to I'$ be a surjection. To the closed immersion $X^{I'} \to X^I$ the sheaf $\mathrm{Fact}(\mathcal{C})$ associates the restriction functor $\mathcal{C}_{X^I} \to \mathcal{C}_{X^{I'}}$ given as follows. For each

$(I\to J\to K)\in \mathrm{Tw}(I)$ let $(I'\to J'\to K')\in \mathrm{Tw}(I')$ be its image under ξ. Consider the functor

$$\underset{k\in K}{\boxtimes}\, \mathcal{C}_X^{\otimes J_k}\to \underset{k'\in K'}{\boxtimes}\, \mathcal{C}_X^{\otimes J'_{k'}}, \tag{17}$$

given as the composition

$$\underset{k\in K}{\boxtimes}\, \mathcal{C}_X^{\otimes J_k}\to (\underset{k\in K}{\boxtimes}\, \mathcal{C}_X^{\otimes J_k})\otimes_{\mathrm{Shv}(X^K)} Shv(X^{K'})\,\widetilde{\to} \underset{k'\in K'}{\boxtimes}\, \mathcal{C}_X^{\otimes J_{k'}}\to \underset{k'\in K'}{\boxtimes}\, \mathcal{C}_X^{\otimes J'_{k'}}$$

where the second map is the product in $\mathcal{C}$ along the natural maps $J_{k'}\to J'_{k'}$ for any $k'\in K'$. We also used the closed immersion $X^{K'}\to X^K$. Now (17) extends to a morphism $\mathcal{F}_I\to \mathcal{F}_{I'}\circ\xi$ in $\mathrm{Funct}(\mathrm{Tw}(I), \mathrm{Shv}(X^I)-mod)$. Namely, for any morphism (16) the diagram commutes

$$\begin{array}{ccc} \underset{k\in K_1}{\boxtimes}\, \mathcal{C}_X^{\otimes (J_1)_k} & \to & \underset{k\in K'_1}{\boxtimes}\, \mathcal{C}_X^{\otimes (J'_1)_k}\\ \downarrow & & \downarrow\\ \underset{k\in K_2}{\boxtimes}\, \mathcal{C}_X^{\otimes (J_2)_k} & \to & \underset{k\in K'_2}{\boxtimes}\, \mathcal{C}_X^{\otimes (J'_2)_k}\end{array}$$

It uses the fact that the square is cartesian

$$\begin{array}{ccc} X^{K_1} & \stackrel{\vartriangle}{\leftarrow} & X^{K'_1}\\ \downarrow \vartriangle & & \downarrow \vartriangle\\ X^{K_2} & \stackrel{\vartriangle}{\leftarrow} & X^{K'_2}\end{array}$$

and the base change holds $\vartriangle^!\vartriangle_*\,\widetilde{\to}\,\vartriangle_*\vartriangle^!$. Here $K'_1=inf(K_1, K'_2)$.

We get natural functors

$$\mathrm{colim}_{\mathrm{Tw}(I)}\,\mathcal{F}_I\to \mathrm{colim}_{\mathrm{Tw}(I)}\,\mathcal{F}_{I'}\circ\xi\to \mathrm{colim}_{\mathrm{Tw}(I')}\,\mathcal{F}_{I'}$$

This is the desired restriction functor. Given $S\to \mathrm{Ran}$ with $S\in \mathrm{Sch}^{aff}_{ft}$, it factors through X^I for some I finite nonempty set.

Example: if $I=*$ then $\mathrm{Fact}(\mathcal{C})(X)=\mathcal{C}_X$. If $I=\{1,2\}$ then $\mathrm{Fact}(\mathcal{C})(X^I)$ is the colimit of the diagram $\mathcal{C}_X\boxtimes\mathcal{C}_X\leftarrow \mathcal{C}_X^{\otimes 2}\to\mathcal{C}_X$, so factorizes over X^I-X.

Let us show that $\mathcal{C}_{X^I}\otimes_{\mathrm{Shv}(X^I)} Shv(X^{I'})\to \mathcal{C}_{X^{I'}}$ is an isomorphism. Denote by $Tw(I)^f\subset Tw(I)$ the full subcategory of $(I\to J\to K)$ such that $K\in Q(I')$. The embedding $Tw(I)^f\subset Tw(I)$ has a right adjoint $\beta: Tw(I)\to Tw(I)^f$ sending $(I\to J\to K)$ to $(I\to J\to K')$ with $K'=inf(I', K)$. We have

$$\mathcal{C}_{X^I}\otimes_{\mathrm{Shv}(X^I)} Shv(X^{I'})\,\widetilde{\to} \underset{(I\to J\to K)\in Tw(I)}{\mathrm{colim}}(\underset{k'\in K'}{\boxtimes}\, \mathcal{C}_X^{\otimes J_{k'}}),$$

here $(I\to J\to K')=\beta(I\to J\to K)$. The expression under the colimit is the composition

$$Tw(I)\stackrel{\beta}{\to} Tw(I)^f\stackrel{\mathcal{F}_I^f}{\to} Shv(X^{I'})-mod,$$

wher $\mathcal{F}^f_I: Tw(I)^f\to Shv(X^{I'})-mod$ is the restriction of $\mathcal{F}_I$ to this full subcategory. So, we first calculate the LKE under $\beta: Tw(I)\to Tw(I)^f$ of $\mathcal{F}^f_I\circ\beta$. By ([15], ch. I.1, 2.2.3), β is cofinal, so the above colimit identifies with

$$\underset{(I\to J\to K)\in Tw(I)^f}{\operatorname{colim}}(\underset{k\in K}{\boxtimes}\mathcal{C}_X^{\otimes J_k}).$$

Consider now the full embedding $Tw(I')\subset Tw(I)^f$. It has a left adjoint $\xi^f: Tw(I)^f\to Tw(I')$. Here ξ^f is the restriction of ξ. So, the full embedding $Tw(I')\subset Tw(I)^f$ is cofinal. We so rewrite the above colimit as

$$\underset{(I'\to J'\to K')\in Tw(I')}{\operatorname{colim}}(\underset{k'\in K'}{\boxtimes}\mathcal{C}_X^{\otimes J'_{k'}})\widetilde{\to}\mathcal{C}_{X^{I'}}$$

To prove the factorization property, we use the following lemma from ([26], 1.3.35). Recall that for a surjection $\phi: I\to I'$ of finite nonempty sets we write

$$X^I_{\phi,disj}=\{(x_i)\in X^I\mid \text{if}\,\phi(i)\neq\phi(i')\,\text{then}\,x_i\neq x_{i'}\}\tag{18}$$

Lemma 3.3.1. *Let $I'\overset{\phi}{\leftarrow} I\to K$ be a diagram of surjection of finite nonempty sets. Then $X^I_{\phi,disj}\times_{X^I}X^K$ is empty unless $I'\in Q(K)$, that is, ϕ decomposes as $I\to K\overset{\phi'}{\to} I'$. In the latter case the square is cartesian*

$$\begin{array}{ccc} X^I_{\phi,disj} & \hookrightarrow & X^I\\ \uparrow & & \uparrow\vartriangle\\ X^K_{\phi',disj} & \hookrightarrow & X^K, \end{array}$$

where $\vartriangle$ is the diagonal.

Given a surjection $\phi: I\to I'$, we want to establish an isomorphism

$$\mathcal{C}_{X^I}\mid_{X^I_{\phi,disj}}\widetilde{\to}(\underset{i'\in I'}{\boxtimes}\mathcal{C}_{X^{I_{i'}}})\mid_{X^I_{\phi,disj}}\tag{19}$$

Write $Tw(I)_\phi$ for the full subcategory of $Tw(I)$ spanned by objects $(I\to J\to K)$ such that $I'\in Q(K)$. We have the equivalence $Tw(I)_\phi\widetilde{\to}\prod_{i'\in I'}Tw(I_{i'})$ sending $(I\to J\to K)$ to the collection $(I_{i'}\to J_{i'}\to K_{i'})\in Tw(I_{i'})$ for $i'\in I'$, the corresponding fibres over i'.

The base change by $Shv(X^I)\to Shv(X^I_{\phi,disj})$ commutes with colimits, so the LHS of (19) is

$$\underset{(I\to J\to K)\in Tw(I)}{\operatorname{colim}}((\underset{k\in K}{\boxtimes}\mathcal{C}_X^{\otimes J_k})\otimes_{\mathrm{Shv}(X^I)}Shv(X^I_{\phi,disj}))$$

By my Lemma 3.3.1, the above colimit rewrites as the colimit over $Tw(I)_\phi$. For $(I\to J\to K)\in Tw(I)_\phi$ we get

$$(\underset{k\in K}{\boxtimes}\mathcal{C}_X^{\otimes J_k})\otimes_{\mathrm{Shv}(X^I)}Shv(X^I_{\phi,disj})\widetilde{\to}(\underset{i'\in I'}{\boxtimes}(\underset{k\in K_{i'}}{\boxtimes}\mathcal{C}_X^{\otimes J_k}))\otimes_{\mathrm{Shv}(X^I)}Shv(X^I_{\phi,disj})$$

Since

$$\underset{(I_{i'}\to J_{i'}\to K_{i'})\in Tw(I_{i'})}{\operatorname{colim}}(\underset{k\in K_{i'}}{\boxtimes}\mathcal{C}_X^{\otimes J_k})\widetilde{\to}\mathcal{C}_{X^{I_{i'}}},$$

passing to the colimit we get the desired isomorphism.

An alternative construction of $\mathrm{Fact}(\mathcal{C})$ is given in ([31], 6.6). Dennis says the definition from our joint paper is better, because it is more general. I think the advantage of defining $\mathrm{Fact}(\mathcal{C})(X^I)$ as a colimit is that for any morphism $f: Y\to Y'$ in PreStk the restriction functor $f^!: ShvCat_{/Y'}\to ShvCat_{/Y}$ for any theory of sheaves will preserve colimits. For limits this is not clear, because for a morphism $S \to S'$ in Sch^{aff}_{ft} it is not clear in general if $Shv(S)$ is dualizable as an object of $Shv(S') - mod$. Even the existence of limits in $ShvCat_{/Y}$ is not clear for this reason in general. (However, if Y is 1-affine then $ShvCat_{/Y}$ has limits).

The structure of a commutative chiral category on our $\mathrm{Fact}(\mathcal{C})$ is as follows. Given finite nonempty sets I_1, I_2 let $I = I_1 \sqcup I_2$. Consider the functor $\alpha : Tw(I_1) \times Tw(I_2) \to Tw(I)$ sending a pair $(I_1 \to J_1 \to K_1), (I_2 \to J_2 \to K_2)$ to $(I \to J \to K)$ with $J = J_1 \sqcup J_2, K = K_1 \sqcup K_2$ given by the coproduct. Note that α is fully faithful. For an object of $Tw(I_1) \times Tw(I_2)$ whose image under α is $(I \to J \to K)$ we have an isomorphism

$$(\underset{k\in K_1}{\boxtimes} \mathcal{C}_X^{\otimes J_{1,k}}) \boxtimes (\underset{k\in K_2}{\boxtimes} \mathcal{C}_X^{\otimes J_{2,k}}) \widetilde{\to} \underset{k\in K}{\boxtimes} \mathcal{C}_X^{\otimes J_k} \tag{20}$$

It extends naturally to an isomorphism of functors $\mathcal{F}_{I_1} \boxtimes \mathcal{F}_{I_2} \widetilde{\to} \mathcal{F}_I \circ \alpha$ in $\mathrm{Fun}(Tw(I_1) \times Tw(I_2), Shv(X^I) - mod)$. Passing to colimits over $Tw(I_1) \times Tw(I_2)$ (using the fact that for a morphism of commutative algebras $A \to B$ in DGCat_{cont} the functor $A-$ mod $(\mathrm{DGCat}_{cont}) \to B-$ mod (DGCat_{cont}), $M \mapsto M \otimes_A B$ commutes with colimits) we get a morphism

$$\mathrm{Fact}(\mathcal{C})(X^{I_1}) \boxtimes \mathrm{Fact}(\mathcal{C})(X^{I_2}) \to \underset{Tw(I_1)\times Tw(I_2)}{\mathrm{colim}} \mathcal{F}_I \circ \alpha \to \underset{Tw(I)}{\mathrm{colim}} \mathcal{F}_I = \mathrm{Fact}(\mathcal{C})(X^I) \tag{21}$$

in $Shv(X^I) - mod$. Let us check it becomes an isomorphism after the base change by $Shv(X^I) \to Shv((X^{I_1} \times X^{I_2})_d)$. Here $(X^{I_1} \times X^{I_2})_d \subset X^I$ is the open subscheme given by the property that if $i_1 \in I_1, i_2 \in I_2$ then $(x_{i_1}, x_{i_2}) \in X^2 - X$.

For an object $(I \to J \to K) \in Tw(I)$, $X^K \times_{X^I} (X^{I_1} \times X^{I_2})_d$ is empty unless $(I \to J \to K)$ lies in the full subcategory $Tw(I_1) \times Tw(I_2)$. So, (21) becomes an isomorphism over $(X^{I_1} \times X^{I_2})_d$.

Let now $I_1 \to I_1', I_2 \to I_2'$ be maps in $fSet$. Then (21) fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{X^{I_1}} \boxtimes \mathcal{C}_{X^{I_2}} & \stackrel{(21)}{\to} & \mathcal{C}_{X^{I_1 \sqcup I_2}} \\ \downarrow & & \downarrow \\ \mathcal{C}_{X^{I_1'}} \boxtimes \mathcal{C}_{X^{I_2'}} & \stackrel{(21)}{\to} & \mathcal{C}_{X^{I_1' \sqcup I_2'}}, \end{array} \tag{22}$$

where the vertical maps are !-restrictions along the closed immersions $X^{I_1'} \hookrightarrow X^{I_1}, X^{I_2'} \hookrightarrow X^{I_2}$ and $X^{I_1' \sqcup I_2'} \hookrightarrow X^{I_1 \sqcup I_2}$. Passing to the limit over $I_1, I_2 \in fSet \times fSet$, the above diagram yield the functor

$$\beta : \Gamma(\mathrm{Ran}, \mathrm{Fact}(\mathcal{C})) \boxtimes \Gamma(\mathrm{Ran}, \mathrm{Fact}(\mathcal{C})) \to \Gamma(\mathrm{Ran} \times \mathrm{Ran}, u^* \mathrm{Fact}(\mathcal{C})) \tag{23}$$

for the sum map $u : \mathrm{Ran} \times \mathrm{Ran} \to \mathrm{Ran}$.

3.3.2. *Question.* How does one gets a unital commutative chiral category structure on $\mathrm{Fact}(\mathcal{C})$ (similar to ([31], 6.6))?

3.3.3. As in ([31], 6.6), we get the following. For any finite nonempty set I, $\mathcal{F}_I$: $Tw(I) \to CAlg^{nu}(Shv(X^I) - mod)$, however, we take the colimit rather of the composition $Tw(I) \to CAlg^{nu}(Shv(X^I) - mod) \to Shv(X^I) - mod$. The structure on $\mathcal{C}_{X^I}$ of a sheaf of symmetric monoidal DG-categories on X^I is not clear, has to be precised. The category $Tw(I)$ has an object $(I \to I \to I)$. So, we get the morphism $\mathrm{Loc} : \underset{i\in I}{\boxtimes} \mathcal{C}_X \to \mathcal{C}_{X^I}$ of sheaves of (symmetric monoidal?) DG-categories on X^I.

The morphisms Loc are evidently compatible with surjections $I \to I'$. That is, the diagram commutes

$$\begin{array}{ccc} \underset{i\in I}{\boxtimes} \mathcal{C}_X & \to & \mathcal{C}_{X^I} \\ \downarrow & & \downarrow \\ \underset{i\in I'}{\boxtimes} \mathcal{C}_X & \to & \mathcal{C}_{X^{I'}} \end{array}$$

Here the right vertical arrow comes from the isomorphism $\mathcal{C}_{X^I} \otimes_{Shv(X^I)} Shv(X^{I'}) \widetilde{\to} \mathcal{C}_{X^{I'}}$, and the left vertical arrow comes from the !-retsriction to $X^{I'}$ and the corresponding product map along $I \to I'$.

3.3.4. For 8.1.7. The construction of the non-unital symmetric monoidal structure on $\mathrm{Fact}(\mathcal{C})(\mathrm{Ran})$ is as in ([30], Sect. 7.17). This uses ([30], Pp. 7.15.5), which is formulated only for D-modules, but holds for any sheaf theory. Namely, if $f : Y \to Z$ is a map of pseudo-indschemes in the sense of ([30], 7.15.1), $\mathcal{C}$ is a sheaf of categories on Z then there is a canonical morphism $\Gamma(Y, C \mid_Y) \to \Gamma(Z, C)$, see ([25], 0.4.13) and Section 3.7.10 of this file, see also ([24], Section 2) for sheaves on categories for any sheaf theory.

The product in $\mathrm{Fact}(\mathcal{C})(\mathrm{Ran})$ is given by the diagram

$$\Gamma(\mathrm{Ran}, \mathrm{Fact}(\mathcal{C})) \otimes \Gamma(\mathrm{Ran}, \mathrm{Fact}(\mathcal{C})) \to \Gamma(\mathrm{Ran} \times \mathrm{Ran}, \mathrm{Fact}(\mathcal{C}) \boxtimes \mathrm{Fact}(\mathcal{C})) \to$$
$$\Gamma(\mathrm{Ran} \times \mathrm{Ran}, u^* \mathrm{Fact}(\mathcal{C})) \overset{u_{*,\mathrm{Fact}(\mathcal{C})}}{\to} \Gamma(\mathrm{Ran}, \mathrm{Fact}(\mathcal{C}))$$

Here $u : \mathrm{Ran} \times \mathrm{Ran} \to \mathrm{Ran}$ is the multiplication, $u_{*,\mathrm{Fact}(\mathcal{C})}$ is the left adjoint to the restriction map $\Gamma(\mathrm{Ran}, \mathrm{Fact}(\mathcal{C})) \to \Gamma(\mathrm{Ran} \times \mathrm{Ran}, u^* \mathrm{Fact}(\mathcal{C}))$. Since u is pseudo-indproper morphism of pseudo-indschemes in the sense of ([30], 7.15.1), $u_{*,\mathrm{Fact}(\mathcal{C})}$ exists by Section 3.7.10 of this file.

Since $\mathrm{Ran} \widetilde{\to} \mathrm{colim}_{I\in fSet^{op}} X^I$, for any sheaf of categories E on Ran,

$$\Gamma(\mathrm{Ran}, E) \widetilde{\to} \lim_{I\in fSet} \Gamma(X^I, E),$$

and we may pass to left adjoint in this diagram. So, $\Gamma(\mathrm{Ran}, E) \widetilde{\to} \mathrm{colim}_{I\in fSet^{op}} \Gamma(X^I, E)$. In the latter colimit for a map $I \to J$ in $fSet$ let $a : X^J \to X^I$ be the corresponding closed immersion. Then the transition map $\Gamma(X^J, E) \to \Gamma(X^I, E)$ is $a_{*,E}$.

3.4. Let us again be as in 8.1.4. We want to compare the definition of $\mathrm{Fact}(\mathcal{C})$ from 8.1.4 with the one from ([31], 6.8). Let $\mathcal{C} \in CAlg(\mathrm{DGCat}_{cont})$. Work with any of the 4 sheaf theories from [14]. We take $\mathcal{C} \otimes \mathrm{Shv}(X)$ as our sheaf of categories over X and apply Dennis' construction of $\mathrm{Fact}(\mathcal{C})$.

In Section 3.4 we assume that $\mathcal{C}$ is compactly generated, and the product $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ admits a continuous right adjoint.

Lemma 3.4.1. *Let I be a finite nonempty set. For any $(I \to J \to K) \in Tw(I)$ the natural functor $\mathcal{C}^{\otimes J} \otimes Shv(X^K) \to \mathcal{C}_{X^I}$ admits a continuous right adjoint, which is a strict morphism of $Shv(X^I)$-modules. One may pass to right adjoints in $\mathcal{F}_{I,C}$ and get a functor $\mathcal{F}^R_{I,C} : Tw(I)^{op} \to Shv(I) - mod$. Then $C_{X^I} \widetilde{\to} \lim\limits_{Tw(I)^{op}} \mathcal{F}^R_{I,C}$ naturally.*

Let $\mathcal{F}^\vee_{I,C} : Tw(I)^{op} \to Shv(X^I) - mod$ be obtained from $\mathcal{F}_{I,C}$ by passing to the duals. Then $\mathcal{C}_{X^I}$ is dualizable as a $Shv(X^I)$-module, and its dual is $(\mathcal{C}_{X^I})^\vee \widetilde{\to} \lim\limits_{Tw(I)^{op}} \mathcal{F}^\vee_{I,C}$.

Proof. For any $(I \to J \to K) \in Tw(I)$, $\mathcal{C}^{\otimes J} \otimes Shv(X^K)$ is compactly generated, hence dualizable in DGCat_{cont}. Indeed, $\mathrm{Shv}(S)$ is compactly generated for any $S \in \mathrm{Sch}^{aff}_{ft}$, and Vect is rigid, so we applied ([15], ch. I.1, 8.7.4).

Note that $Shv(X^K)$ is dualizable as a $Shv(X^I)$-module (see my Section 3.7.1 below). The functor $\mathrm{DGCat}_{cont} \to Shv(X^I) - mod$, $D \mapsto D \otimes Shv(X^I)$ is symmetric monoidal, so sends dualizable objects to dualizable. So, $C \otimes Shv(X^I)$ is dualizable in $Shv(X^I) - mod$. The product of dualizable objects is dualizable, so $\mathcal{C}^{\otimes J} \otimes Shv(X^K)$ is dualizable in $Shv(X^I) - mod$.

Since the product $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ admits a continuous right adjoint, for any $J \to K$ the product $\mathcal{C}^J \to \mathcal{C}^K$ admits a continuous right adjoint also by ([22], 4.1.6). We claim now that any morphism in $Tw(I)$ is sent by $\mathcal{F}_I$ to the functor $\mathcal{C}^{\otimes J} \otimes Shv(X^K) \to C^{\otimes J'} \otimes Shv(X^{K'})$ admitting a continuous right adjoint, which is moreover $Shv(X^I)$-linear (not just right-lax). Indeed, for a surjection $K' \to K$ and the corresponding diagonal $\delta : X^K \to X^{K'}$ the functor $\delta_!$ admits a continuous right adjoint $\delta^!$, which is $Shv(X^I)$-linear. Write $\mathcal{F}^R_{I,C} : Tw(I)^{op} \to Shv(X^I) - mod$ for the functor obtained from $\mathcal{F}_{I,C}$ by passing to right adjoints. We get $C_{X^I} \widetilde{\to} \lim\limits_{Tw(I)^{op}} \mathcal{F}^R_{I,C}$.

Now proceed as in ([15], ch. I.1, 6.3.4) replacing only $1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ by $Shv(X^I) - mod$. We used the fact that the projection $Shv(X^I) - mod \to \mathrm{DGCat}_{cont}$ preserves colimits and limits. $\square$

Let us also construct a functor ζ from Dennis version to Sam's version of $\mathrm{Fact}(\mathcal{C})$. So, $\mathcal{C}_X = \mathcal{C} \otimes Shv(X)$. Sam's definition is

$$\bar{\mathcal{C}}_{X^I} = \lim_{(I \xrightarrow{p} J \to K) \in Tw(I)^{op}} Shv(X^I_{p,disj}) \otimes \mathcal{C}^{\otimes K}$$

His transition map attaches for the diagram (16) the functor

$$Shv(X^I_{p_2,disj}) \otimes \mathcal{C}^{\otimes K_2} \to Shv(X^I_{p_1,disj}) \otimes \mathcal{C}^{\otimes K_1},$$

which is the tensor product of the product $\mathcal{C}^{\otimes K_2} \to \mathcal{C}^{\otimes K_1}$ along $K_2 \to K_1$ and the restriction along the open immersion $X^I_{p_1,disj} \subset X^I_{p_2,disj}$. Denote by $\bar{\mathcal{F}}_I : Tw(I)^{op} \to Shv(X^I) - mod$ the above diagram defining $\bar{\mathcal{C}}_{X^I}$. We write $\bar{\mathcal{F}}_{I,C}$ if we need to express the dependence on C.

For any $C \in CAlg(\mathrm{DGCat}_{cont})$ the functor $\zeta : \mathcal{C}_{X^I} \to \bar{\mathcal{C}}_{X^I}$ is defined as follows. Pick $(I \xrightarrow{p} J \to K) \in Tw(I)$. We define a compatible system of morphisms $\mathcal{C}_{X^I} \to Shv(X^I_{p,disj}) \otimes \mathcal{C}^{\otimes K}$ as follows. Given $(I \to J_1 \to K_1) \in Tw(I)$, $X^I_{p,disj} \times_{X^I} X^{K_1}$ is empty unless $J \in Q(K_1)$. The map

$$\mathrm{Shv}(X^{K_1}) \otimes \mathcal{C}^{J_1} \to Shv(X^I_{p,disj}) \otimes \mathcal{C}^{\otimes K} \tag{24}$$

vanishes unless $J \in Q(K_1)$. In the latter case we get a diagram $I \to J_1 \to K_1 \to J \to K$, hence a map $\mathcal{C}^{\otimes J_1} \to \mathcal{C}^{\otimes K}$ given by the product along $J_1 \to K$. Then (24) is the composition

$$\mathrm{Shv}(X^{K_1}) \otimes \mathcal{C}^{J_1} \to \mathrm{Shv}(X^{K_1}) \otimes \mathcal{C}^{K} \to Shv(X^I_{p,disj}) \otimes \mathcal{C}^{\otimes K},$$

where the second map is the restriction (followed by the direct image under closed immersion). These maps are compatible, so yield the desired functor $\mathcal{C}_{X^I} \to Shv(X^I_{p,disj}) \otimes \mathcal{C}^{\otimes K}$. The latter functors are compatible, hence yield $\zeta : \mathcal{C}_{X^I} \to \bar{\mathcal{C}}_{X^I}$.

By construction, $\bar{\mathcal{C}}_{X^I} \in CAlg(Shv(X^I) - mod)$, and for each $(I \to J_1 \to K_1) \in Tw(I)$ the corresponding map $\mathrm{Shv}(X^{K_1}) \otimes \mathcal{C}^{J_1} \to \bar{\mathcal{C}}_{X^I}$ is a map in $CAlg^{nu}(Shv(X^I) - mod)$. So, $\zeta : \mathcal{C}_{X^I} \to \bar{\mathcal{C}}_{X^I}$ is $Shv(X^I)$-linear.

Lemma 3.4.2. *Recall that C is compactly generated and $m : C^{\otimes 2} \to C$ admits a continuous right adjoint. Then*
i) the functor $\zeta : \mathcal{C}_{X^I} \to \bar{\mathcal{C}}_{X^I}$ is an equivalence.
ii) for each $(I \to J \to K) \in Tw(I)$ the projection $\bar{C}_{X^I} \to Shv(X^I_{p,d}) \otimes \mathcal{C}^{\otimes K}$ admits a continuous $Shv(X^I)$-linear right adjoint.

Proof. For $S \in \mathrm{Sch}_{ft}$, $E \in Shv(S) - mod, x, x' \in E$ write $\underline{\mathcal{H}om}_E(x, x') \in Shv(S)$ for the relative inner hom for the $Shv(S)$-action.

Our C_{X^I} is ULA over $Shv(X^I)$ by Section 3.4.6. The functor $\mathrm{Loc} : Shv(X^I) \otimes C^{\otimes I} \to C_{X^I}$ was defined in Section 3.3.3. We first prove i).

Step 1 We claim that $\zeta : \mathcal{C}_{X^I} \to \bar{\mathcal{C}}_{X^I}$ admits a $Shv(X^I)$-linear continuous right adjoint. Using Lemma 3.5.2 and Proposition 3.7.7, it suffices to show that if $c \in (C^{\otimes I})^c$ then $\zeta(\mathrm{Loc}(c \otimes \omega)) \in \bar{\mathcal{C}}_{X^I}$ is ULA over $Shv(X^I)$. Indeed, the objects of the form $c \otimes \mathcal{K}$ with $\mathcal{K} \in Shv(X^I)^c$, $c \in (C^{\otimes I})^c$ generate $C^{\otimes I} \otimes Shv(X^I)$. Let $c \in (C^{\otimes I})^c$.

By ([22], 2.4.7), if $L \in \bar{\mathcal{C}}_{X^I}$ is such that for any $(I \xrightarrow{p} J \to K) \in Tw(I)$, the image of L in $Shv(X^I_{p,d}) \otimes \mathcal{C}^{\otimes K}$ is compact then L is compact in $\bar{\mathcal{C}}_{X^I}$, because $Tw(I)$ is finite. For $\mathcal{K} \in Shv(X^I)^c$ the image of $\zeta(\mathrm{Loc}(c \otimes \mathcal{K}))$ in each $Shv(X^I_{p,d}) \otimes \mathcal{C}^{\times K}$ is compact, so $\zeta(\mathrm{Loc}(c \otimes \mathcal{K})) \in (\bar{\mathcal{C}}_{X^I})^c$. This shows that ζ admits a continuous right adjoint ζ^R.

Let $L \in Shv(X^I), M \in \bar{\mathcal{C}}_{X^I}$. We must show that the natural map

$$L \otimes^! \underline{\mathcal{H}om}_{\bar{\mathcal{C}}_{X^I}}(\zeta(\mathrm{Loc}(c \otimes \omega)), M) \to \underline{\mathcal{H}om}_{\bar{\mathcal{C}}_{X^I}}(\zeta(\mathrm{Loc}(c \otimes \omega), L \otimes M) \tag{25}$$

is an isomorphism in $Shv(X^I)$. For $\Sigma = (I \xrightarrow{p} J \to K) \in Tw(I)$ write M_Σ for the projection of M to $Shv(X^I_{p,d}) \otimes C^{\otimes K}$, write f_Σ for the composition

$$Shv(X^I) \otimes C^{\otimes I} \xrightarrow{\mathrm{Loc}} \mathcal{C}_{X^I} \xrightarrow{\zeta} \bar{C}_{X^I} \to Shv(X^I_{p,d}) \otimes C^{\otimes K}$$

One has

$$\underline{\mathcal{H}om}_{\bar{\mathcal{C}}_{X^I}}(\zeta(\mathrm{Loc}(c \otimes \omega)), M) \widetilde{\to} \lim_{(I \xrightarrow{p} J \to K) \in Tw(I)^{op}} \underline{\mathcal{H}om}_{Shv(X^I_{p,d}) \otimes C^{\otimes K}}(f_\Sigma(c \otimes \omega), M_\Sigma)$$

in $Shv(X^I)$. Clearly, f_Σ has a $Shv(X^I)$-linear continuous right adjoint f_Σ^R, and

$$\underline{\mathcal{H}om}_{Shv(X^I_{p,d})\otimes C^{\otimes K}}(f_\Sigma(c\otimes\omega), M_\Sigma)\widetilde{\to}\underline{\mathcal{H}om}_{Shv(X^I)\otimes C^{\otimes I}}(c\otimes\omega, f_\Sigma^R(M_\Sigma))$$

The key point is that the functor $Shv(X^I)\to Shv(X^I)$, $\cdot\mapsto L\otimes^!\cdot$ commutes with finite limits, as this functor is exact. So, the LHS of (25) identifies with

$$\lim_{(I\xrightarrow{p}J\to K)\in Tw(I)^{op}} L\otimes^! \underline{\mathcal{H}om}_{Shv(X^I)\otimes C^{\otimes I}}(c\otimes\omega, f_\Sigma^R(M_\Sigma))$$

Since $c\otimes\omega\in Shv(X^I)\otimes C^{\otimes I}$ is ULA over $Shv(X^I)$, the latter limit becomes

$$\lim_{(I\xrightarrow{p}J\to K)\in Tw(I)^{op}} \underline{\mathcal{H}om}_{Shv(X^I)\otimes C^{\otimes I}}(c\otimes\omega, L\otimes f_\Sigma^R(M_\Sigma))\widetilde{\to}$$
$$\lim_{(I\xrightarrow{p}J\to K)\in Tw(I)^{op}} \underline{\mathcal{H}om}_{Shv(X^I)\otimes C^{\otimes I}}(c\otimes\omega, f_\Sigma^R(L\otimes M_\Sigma))\widetilde{\to}$$
$$\lim_{(I\xrightarrow{p}J\to K)\in Tw(I)^{op}} \underline{\mathcal{H}om}_{Shv(X^I_{p,d})\otimes C^{\otimes K}}(f_\Sigma(c\otimes\omega), L\otimes M_\Sigma)\widetilde{\to}\underline{\mathcal{H}om}_{\bar{\mathcal{C}}_{X^I}}(\zeta(\mathrm{Loc}(c\otimes\omega)), L\otimes M)$$

Step 2 Let $U\subset X^I$ be the complement to the main diagonal $X\hookrightarrow X^I$. By Proposition 3.7.8, it suffices to show now that ζ becomes an isomorphism after applying $\cdot\otimes_{Shv(X^I)} Shv(X)$ and $\cdot\otimes_{Shv(X^I)} Shv(U)$. But both properties follow from factorization. For the open part, we use here that the union of $X^I_{p,d}$ for $p: I\to J$ in $fSet$ with $\mid J\mid > 1$ is U. We also use the following claim. If $\nu: B\to B'$ is a map in $Shv(U)-mod$, which becomes an equvalence after Zariski localization then ν is an equivalence. So, i) is proved.

ii) For any $(I\to J_1\to K_1)\in Tw(I)$ the functor (24) admits a continuous $Shv(X^I)$-linear right adjoint. Recall that each transition functor in the diagram $\mathcal{F}_{I,C}$ admits also a $Shv(X^I)$-linear continuous right adjoint. Passing to the right adjoints in $Shv(X^I)-mod$, we get a canonical map $Shv(X^I_{p,d})\otimes\mathcal{C}^{\otimes K}\to \lim\limits_{Tw(I)^{op}} \mathcal{F}^R_{I,C}\widetilde{\to} C_{X^I}$ in $Shv(X^I)-mod$. By ([22], 9.2.6), this is the desired $Shv(X)$-linear continuous right adjoint to the projection $\bar{C}_{X^I}\to Shv(X^I_{p,d})\otimes\mathcal{C}^{\otimes K}$. $\square$

Note that we may pass to right adjoints in the functor $\bar{\mathcal{F}}_{I,C}: Tw(I)^{op}\to Shv(X^I)-mod$ and get a functor denoted $(\bar{\mathcal{F}}_{I,C})^R: Tw(I)\to Shv(X^I)-mod$. Moreover, by the above lemma we may pass to right adjoints in the limit diagram ${}^{\triangleleft}(Tw(I)^{op})\to Shv(X^I)-mod$ of the functor $\bar{\mathcal{F}}_{I,C}$, this produces a functor denoted $(\bar{\mathcal{F}}_{I,C})^{R,\triangleright}: Tw(I)^{\triangleright}\to Shv(X^I)-mod$, whose value on the final object is $\bar{C}_{X^I}$. In other words, we constructed a map in $Shv(X^I)-mod$

$$\operatorname*{colim}_{Tw(I)}(\bar{\mathcal{F}}_{I,C})^R\to\bar{C}_{X^I}. \tag{26}$$

3.4.3. *Question.* Is the map (26) an equivalence?

3.4.4. We hope that for any $C \in CAlg(\mathrm{DGCat}_{cont})$, $\mathcal{C}_{X^I}$ can be lifted naturally to an object of $CAlg(Shv(X^I) - mod)$ such that for each $(I \to J_1 \to K_1) \in Tw(I)$ the corresponding map $\mathrm{Shv}(X^{K_1}) \otimes \mathcal{C}^{J_1} \to \mathcal{C}_{X^I}$ is symmetric monoidal. (The definition of the symmetric monoidal structure on the $Shv(X^I)$-module $\mathcal{C}_{X^I}$ is not clear in general, as $Tw(I)$ is not sifted, so $CAlg(Shv(X^I) - mod) \to Shv(X^I) - mod$ does not preserve the $Tw(I)$-indexed colimits maybe.

Note however that under the assumptions of Section 3.4, we have indeed $C_{X_I} \widetilde{\to} \bar{C}_{X^I} \in CAlg(Shv(X^I) - mod)$.

Remark 3.4.5. *Assume in the situation of Lemma 3.4.1 in addition that $1 \in \mathcal{C}$ is compact. Since X^K is smooth, the unit object $\omega \in Shv(X^K)$ is compact. Indeed, the functor $\Gamma(X^K, -) : Shv(X^K) \to Shv(*)$ is continuous. Thus, $1 \otimes \omega \in (\mathcal{C} \otimes Shv(X))^c$. So, the image of $1 \otimes \omega$ under the natural map $\mathcal{C} \otimes Shv(X) \to \mathcal{C}_{X^I}$ (corresponding to $(I \to * \to *) \in Tw(I)$) is compact by Lemma 3.4.1. So, the unit of $\mathcal{C}_{X^I}$ is compact.*

3.4.6. As in [31], we want to show that $\mathcal{C}_{X^I}$ is ULA in the sense of Section 3.7.6 below. Let $c \in \mathcal{C}^{\otimes I}$ be compact. Then $c \otimes \omega \in Shv(X^I) \otimes \mathcal{C}^I$ is ULA. Here ω is the unit object of $Shv(S)$ for $S \in \mathrm{Sch}^{aff}_{ft}$. Indeed, we have an adjoint pair $L : \mathrm{Vect} \rightleftarrows \mathcal{C}^{\otimes I} : R$, where $L(K) = K \otimes c$. Tensoring by $Shv(X^I)$, we get an adjoint pair $\bar{L} : Shv(X^I) \rightleftarrows \mathcal{C}^{\otimes I} \otimes Shv(X^I) : \bar{R}$. Since $\bar{R}$ is continuous and $Shv(X^I)$-linear, $c \otimes \omega$ is ULA.

Recall the functor $\mathrm{Loc} : \underset{i \in I}{\boxtimes} \mathcal{C}_X \to \mathcal{C}_{X^I}$ of Section 3.3.3 above. By Lemma 3.4.1, Loc admits a continuous right adjoint, which is $\mathrm{Shv}(X^I)$-linear. If $c \in \mathcal{C}^{\otimes I}$ is compact then $\mathrm{Loc}(c \otimes \omega) \in \mathcal{C}_{X^I}$ is ULA by Proposition 3.7.7 below. Indeed, $\mathcal{C}^{\otimes I} \otimes Shv(X^I) \overset{\mathrm{Loc}}{\to} \mathcal{C}_{X^I}$ admits a continuous right adjoint, which is $Shv(X^I)$-linear.

By Lemma 3.5.2 below, the essential image of $\mathrm{Loc} : \underset{i \in I}{\boxtimes} \mathcal{C}_X \to \mathcal{C}_{X^I}$ generates $\mathcal{C}_{X^I}$ under colimits. We also check below in Lemma 3.5.12 that $\mathcal{C}_{X^I}$ is ULA over $Shv(X^I)$.

Concretely, if $c \in C^{\otimes I}$ is compact then $\mathrm{Loc}(c \otimes \omega) \in C_{X^I}$ is ULA. Since $C^{\otimes I} \otimes Shv(X^I)$ is compactly generated by objects of the form $c \otimes z$, $c \in (C^{\otimes I})^c$, $z \in Shv(X^I)^c$, this shows that C_{X^I} is ULA over $Shv(X^I)$ using Lemma 3.5.2.

3.4.7. If $\mathcal{C} = \mathrm{Vect}$ then $\mathcal{C}_{X^I} \widetilde{\to} Shv(X^I)$ in $Shv(X^I) - mod$. Indeed, as in the proof of Lemma 3.5.2 we see that (28) is an isomorphism in this case. In turn, ${}^0Tw(I)^{op}$ has an initial object $(I \overset{\mathrm{id}}{\to} I \overset{id}{\to} I)$, so the limit (28) in this case becomes the value at $(I \overset{\mathrm{id}}{\to} I \overset{id}{\to} I)$, that is, $Shv(X^I)$.

3.4.8. Let Γ be an affine algebraic group of finite type. Let $\mathcal{C} = \mathrm{Rep}(\Gamma) = \mathrm{QCoh}(B(\Gamma))$. It is known to be rigid. We have a conservative forgetful functor $\mathcal{C} \to \mathrm{Vect}$. The functoriality of Fact yields a conservative functor $Oblv_{X^I} : \mathcal{C}_{X^I} \to \mathrm{Fact}(\mathrm{Vect})(X^I) = \mathrm{Shv}(X^I)$, here we use the definition of C_{X^I} as a colimit.

Write $\mathcal{F}_{I,C}$ for the functor $\mathcal{F}_I$ is we want to underline the dependence on the category C. Write $\mathcal{F}^R_{I,C} : Tw(I)^{op} \to \mathrm{DGCat}_{cont}$ for the functor obtained from $\mathcal{F}_I$ by passing to the right adjoints, then we do not have a map of functors $\mathcal{F}^R_{I,C} \to \mathcal{F}^R_{I,\mathrm{Vect}}$ for the forgetfull functor $\mathrm{oblv} : \mathcal{C} \to \mathrm{Vect}$. That is, to get $Oblv_{X^I}$ we can not use the definition $C_{X^I} \widetilde{\to} \lim \mathcal{F}^R_{I,C}$, we consider the colimits instead.

As in ([31], 6.22.1) we derive from Proposition 3.7.7 that $Oblv_{X^I}$ has a $Shv(X^I)$-linear right adjoint $Av^w_{X^I,*} : Shv(X^I) \to \mathcal{C}_{X^I}$????

3.4.9. For $I \in fSet$ the union of $X^I_{p,disj}$ for $p : I \to J$ in $fSet$ with $\mid J \mid> 1$ equals $X^I - X$. For $\Sigma = (I \to J \to K) \in Tw(I)$ we have a morphism

$$\begin{array}{ccccc} I & \overset{p}{\to} & J & \to & K \\ \uparrow \mathrm{id} & & \uparrow & & \downarrow \\ I & \overset{p}{\to} & J & \to & * \end{array}$$

functorial in Σ. The corresponding functor $Shv(X^I_{p,d}) \otimes C^{\otimes K} \overset{\mathrm{id}\otimes m}{\to} Shv(X^I_{p,d}) \otimes C$ is functorial in Σ.

Let $ins : fSet_{I/} \subset Tw(I)$ be the full subcategory of objects of the form $(I \to J \to *)$. We get an adjoint pair $fSet_{I/} \leftrightarrows Tw(I) : \tau$, where $\tau(I \to J \to K) = (I \to J \to *)$.

Lemma 3.4.10. *One has* $\lim\limits_{(I\to J\to K)\in Tw(I)^{op}} Shv(X^I_{p,d}) \widetilde{\to} Shv(X^I)$, *these are the sections over* X^I *of the factorization category* Shv.

Proof. Consider the functor

$$\eta : (fSet_{I/})^{op} \to \mathrm{DGCat}_{cont}, \quad (I \overset{p}{\to} J \to *) \mapsto Shv(X^I_{p,d}),$$

where the transition functors are restrictions. Its RKE along the inclusion $(fSet_{I/})^{op} \hookrightarrow Tw(I)^{op}$ is $\eta \circ \tau^{op}$ by ([22], 2.2.39). So,

$$\lim_{(I\to J\to K)\in Tw(I)^{op}} Shv(X^I_{p,d}) \widetilde{\to} \lim_{(I\to J\to *)\in (fSet_{I/})^{op}} Shv(X^I_{p,d})$$

The category $fSet_{I/}$ has the final object $(I \to *)$, so the latter limit identifies with the value at $(I \to * \to *)$. $\square$

Since C is assumed dualizable, we have

$$Shv(X^I) \otimes C \widetilde{\to} \lim_{(I\to J\to K)\in Tw(I)^{op}} (Shv(X^I_{p,d}) \otimes C)$$

Passing to the limit over $\Sigma \in Tw(I)^{op}$, the above gives a functor $\bar{C}_{X^I} \to Shv(X^I) \otimes C$. For $U = X^I - X$ tensoring by $Shv(U)$, we get a morphism $\bar{C}_{X^I} \otimes_{Shv(X^I)} Shv(U) \to Shv(U) \otimes C$. The square is cartesian

$$\begin{array}{ccc} \bar{C}_{X^I} & \to & \bar{C}_{X^I} \otimes_{Shv(X^I)} Shv(U) \\ \downarrow & & \downarrow \\ Shv(X^I) \otimes C & \to & Shv(U) \otimes C, \end{array}$$

where the horizontal arrows are restrictions. This is "a way to do induction" over $\mid I \mid$.

3.4.11. For a reductive group G such that the set of irreducible representations of $\check{G}$ is infinite the functor $m : \mathrm{Rep}(\check{G})^{\otimes 2} \to \mathrm{Rep}(\check{G})$ does not have a left adjoint. So, one should not hope to be able to pass to the left adjoints in the diagram defining $\bar{C}_{X^I}$

$$\lim_{(I\to J\to K)\in Tw(I)^{op}} Shv(X^I_{p,d}) \otimes C^{\otimes K} \widetilde{\to} \bar{C}_{X^I}$$

3.4.12. Recall that $C \in CAlg(\mathrm{DGCat}_{cont})$ is assumed compacty generated, and we assume that $m : C^{\otimes 2} \to C$ admits a continuous right adjoint.

Let $\Sigma = (I \overset{p}{\to} J \to K) \in Tw(I)$. By Section 3.7.1, $Shv(X^I_{p,d})$ is self-dual in $Shv(X^I) - mod$. Besides, $C^{\otimes K} \otimes Shv(X^I)$ is dualizable in $Shv(X^I) - mod$, so their tensor product $Shv(X^I_{p,d}) \otimes C^{\otimes K}$ is dualizable in $Shv(X^I) - mod$, and its dual is

$$Shv(X^I_{p,d}) \otimes (C^\vee)^{\otimes K}$$

Let $D \in Shv(X^I) - mod$. Then we get an equivalence

$$\mathrm{Fun}_{Shv(X^I)}(Shv(X^I_{p,d}) \otimes (C^\vee)^{\otimes K}, D) \widetilde{\to} C^{\otimes K} \otimes Shv(X^I_{p,d}) \otimes_{Shv(X^I)} D$$

Our purpose is to understand the limit

$$\lim_{(I \to J \to K) \in Tw(I)^{op}} C^{\otimes K} \otimes Shv(X^I_{p,d}) \otimes_{Shv(X^I)} D,$$

where this diagram is obtained from the one defining $\bar{C}_{X^I}$ by applying $\cdot \otimes_{Shv(X^I)} D$. We rewrite it as

$$\lim_{(I \to J \to K) \in Tw(I)^{op}} \mathrm{Fun}_{Shv(X^I)}(Shv(X^I_{p,d}) \otimes (C^\vee)^{\otimes K}, D) \tag{27}$$

Denote by $\bar{\mathcal{F}}^\vee_{I,C} : Tw(I) \to Shv(X^I) - mod$ the diagram obtained from $\bar{\mathcal{F}}_{I,C}$ by passing to the duals. The diagram (27) is obtained by functoriality from $\bar{\mathcal{F}}^\vee_{I,C}$ by applying $\mathrm{Fun}_{Shv(X^I)}(\cdot, D)$. So, the limit (27) identifies with

$$\mathrm{Fun}_{Shv(X^I)}(\operatorname*{colim}_{Tw(I)} \bar{\mathcal{F}}^\vee_{I,C}, D)$$

3.4.13. *Question.* The equivalence $C_{X^I} \widetilde{\to} \lim_{Tw(I)^{op}} \bar{\mathcal{F}}_{I,C}$ of Lemma 3.4.2 yields by passing to the duals a morphism $\operatorname*{colim}_{Tw(I)} \bar{\mathcal{F}}^\vee_{I,C} \to (C_{X^I})^\vee$, where the colimit is calculated in $Shv(X^I) - mod$. Is the latter an equivalence?

Remark 3.4.14. *Assume in addition C rigid in* DGCat_{cont}*. Then in the case of $\mathcal{D}$-modules the answer to Question 3.4.13 is yes, this follows directly from (*[31]*, Lemma 6.18.1).*

3.5. Additional results about $\mathrm{Fact}(\mathcal{C})$.

3.5.1. We work here with any of our 4 sheaf theories. The theory of sheaves of categories in this context is developed in [24]. Let $C \in CAlg(ShvCat(X))$. Write $\mathcal{F}_{I,C}$ for the functor $\mathcal{F}_I : Tw(I) \to Shv(X^I) - mod$ if we need to underline the dependence on the category $C(X)$.

The category $Tw(I)$ has an object $(I \to I \to I)$. So, for $I \in fSet$ we get the morphism $\mathrm{Loc} : \boxtimes_{i \in I} \mathcal{C}(X) \to \mathcal{C}_{X^I} = \mathrm{Fact}(\mathcal{C})(X^I)$.

Lemma 3.5.2. *The functor* $\mathrm{Loc} : \boxtimes_{i \in I} \mathcal{C}(X) \to \mathcal{C}_{X^I}$ *generates* $\mathcal{C}_{X^I}$ *under colimits.*

Proof. It suffices to show, by ([32], ch. I.1, 5.4.3) that the right adjoint $\mathrm{Loc}^R : \mathcal{C}_{X^I} \to \boxtimes_{i \in I} \mathcal{C}(X)$ is conservative. Denote by $\mathcal{F}_I : \mathrm{Tw}(I) \to Shv(X^I) - mod$ our functor sending $(I \to J \to K)$ to $\boxtimes_{k \in K} C^{\otimes J_k}(X)$, so $\mathcal{C}_{X^I} := \operatorname{colim} \mathcal{F}_I$. Denote by $j : {}^0\mathrm{Tw}(I) \subset \mathrm{Tw}(I)$

the full subcategory spanned by objects of the form $(I \xrightarrow{p} J \to K)$, where p is an isomorphism. We have an adjoint pair

$$j : {}^0Tw(I) \leftrightarrows Tw(I) : j^R,$$

where $j^R(I \xrightarrow{p} J \xrightarrow{q} K) = (I \xrightarrow{\mathrm{id}} I \xrightarrow{qp} K)$.

We may write

$$\mathcal{C}_{X^I} \widetilde{\to} \lim_{(I\to J\to K)\in Tw(I)^{op}} \underset{k\in K}{\boxtimes} C^{\otimes J_k}(X)$$

the limit of the functor $\mathcal{F}_I^R$ obtained from $\mathcal{F}_I$ by passing to right adjoints. Restricting to the full subcategory ${}^0Tw(I)^{op}$ the functor $\mathcal{F}_I^R$, we get a morphism

$$\mathcal{C}_{X^I} \to \lim_{(I\to J\to K)\in {}^0Tw(I)^{op}} \mathcal{F}_I^R \tag{28}$$

Let $\bar{\mathcal{F}}_I^R$ denote the RKE of $\mathcal{F}_I^R \circ j^{op}$ under ${}^0Tw(I)^{op} \to Tw(I)^{op}$. By ([22], 2.2.39),

$$\bar{\mathcal{F}}_I^R = \mathcal{F}_I^R \circ j^{op} \circ (j^R)^{op}.$$

The map of functors $\mathcal{F}_I^R \to \bar{\mathcal{F}}_I^R$ evaluated at an object $(I \xrightarrow{p} J \xrightarrow{q} K) = \Sigma \in Tw(I)^{op}$ becomes

$$\underset{k\in K}{\boxtimes} C^{\otimes J_k}(X) \to \underset{k\in K}{\boxtimes} C^{\otimes I_k}(X).$$

It is conservative, as its left adjoint is surjective. So, passing to the limit over $Tw(I)^{op}$, we conclude by ([22], Cor. 2.5.3) that (28) is conservative.

The category ${}^0Tw(I)^{op}$ has an initial object $(I \xrightarrow{\mathrm{id}} I \xrightarrow{id} I)$. So,

$$\lim_{(I\to J\to K)\in {}^0Tw(I)^{op}} \mathcal{F}_I^R \widetilde{\to} \underset{i\in I}{\boxtimes} C(X)$$

Thus, Loc^R is conservative. □

From this lemma it follows that there could be at most a unique symmetric monoidal structure on $\mathcal{C}_{X^I}$ for which Loc is symmetric monoidal. Add the proof that it exists indeed. I assume moreover that for any $(I \to J \to K) \in Tw(I)$ the corresponding functor $\underset{k\in K}{\boxtimes} \mathcal{C}^{\otimes J_k}(X) \to \mathcal{C}_{X^I}$ is symmetric monoidal.

3.5.3. *Factorization algebras in* Fact(C). Let in addition $A \in CAlg(C(X))$. We want to analyse the construction of the corresponding commutative factorization algebra in Fact(C).

For $J \in fSet$ write $* : C^{\otimes J}(X) \to C(X)$ for the product map, so we get the product map $A^{*J} \to A$ in $C(X)$ for A, here $A^{*J} \in C(X)$ is the image of $A^{\otimes J}(X)$ under $*$. Now given a map $\phi : J \to J' \in fSet$, for the product map $m_\phi : C^{\otimes J}(X) \to C^{\otimes J'}(X)$ we get the product map $m_\phi(A^{\otimes J}) \to A^{\otimes J'}$ in $C^{\otimes J'}(X)$ for the algebra A.

We define the functor $\mathcal{F}_{I,A} : Tw(I) \to \mathcal{C}_{X^I}$ as follows. We will write $\mathcal{F}_{I,A}^C = \mathcal{F}_{I,A}$ if we need to express its dependence on C. The functor $\mathcal{F}_{I,A}$ sends $(I \to J \to K)$ to the image under $\mathcal{F}_I(I \to J \to K) \to \mathcal{C}_{X^I}$ of the object

$$\underset{k\in K}{\boxtimes} A^{\otimes J_k} \in \underset{k\in K}{\boxtimes} \mathcal{C}^{\otimes J_k}(X) = \mathcal{F}_I(I \to J \to K)$$

Now for a map in $Tw(I)$ given by (2) we get a morphism in $\underset{k\in K_2}{\boxtimes}\mathcal{C}^{\otimes (J_2)_k}(X)$ and hence in C_{X^I}

$$\mathcal{F}_{I,A}(I\to J_1\to K_1)\to \mathcal{F}_{I,A}(I\to J_2\to K_2)$$

as follows. First, for the diagram (defining the transition functor for $\mathcal{F}_I$)

$$\underset{k\in K_1}{\boxtimes}\mathcal{C}^{\otimes (J_1)_k}(X)\stackrel{\alpha}{\to}\underset{k\in K_1}{\boxtimes}\mathcal{C}^{\otimes (J_2)_k}(X)\stackrel{\beta}{\to}\underset{k\in K_2}{\boxtimes}\mathcal{C}^{\otimes (J_2)_k}(X)$$

we get natural product map $\alpha(\underset{k\in K_1}{\boxtimes}A^{\otimes (J_1)_k})\to \underset{k\in K_1}{\boxtimes}A^{\otimes (J_2)_k}$ in $\underset{k\in K_1}{\boxtimes}\mathcal{C}^{\otimes (J_2)_k}(X)$ for the algebra A. Further, for $\vartriangle\colon X^{K_1}\hookrightarrow X^{K_2}$ we have

$$\vartriangle^!\,(\underset{k\in K_2}{\boxtimes}A^{\otimes (J_2)_k})\widetilde{\to}\underset{k\in K_1}{\boxtimes}A^{\otimes (J_2)_k}$$

in $\underset{k\in K_1}{\boxtimes}\mathcal{C}^{\otimes (J_2)_k}(X)$. So, we compose the previous product map with

$$\vartriangle_!\,(\underset{k\in K_1}{\boxtimes}A^{\otimes (J_2)_k})\widetilde{\to}\vartriangle_!\vartriangle^!\,(\underset{k\in K_2}{\boxtimes}A^{\otimes (J_2)_k})\to \underset{k\in K_2}{\boxtimes}A^{\otimes (J_2)_k}$$

Finally, $A_{X^I}\in C_{X^I}$ is defined as $\underset{(I\to J\to K)\in Tw(I)}{\operatorname{colim}}\mathcal{F}_{I,A}$ in C_{X^I}. That is,

$$A_{X^I}\widetilde{\to}\underset{(I\to J\to K)\in Tw(I)}{\operatorname{colim}}\underset{k\in K}{\boxtimes}A^{\otimes J_k}$$

taken in C_{X^I}.

3.5.4. Let us check that this defines indeed an object of Fact(C)(Ran). That is, for a surjection $I\to I'$ in $fSet$, the restriction functor $C_{X^I}\to C_{X^{I'}}$ defined in Section 3.3 sends A_{X^I} to $A_{X^{I'}}$.

We argue as and use the notations of Section 3.3. First, the image of A_{X^I} under $C_{X^I}\to C_{X^{I'}}$ writes as

$$\underset{(I\to J\to K)\in Tw(I)}{\operatorname{colim}}\underset{k\in K'}{\boxtimes}A^{\otimes J_k}$$

taken in $C_{X^{I'}}$, where $K'=\inf(K,I')$. So, this is the colimit of the composition

$$Tw(I)\stackrel{\beta}{\to}Tw(I)^f\stackrel{\mathcal{F}^f_{I,A}}{\to}\mathcal{C}_{X^{I'}},$$

where $\mathcal{F}^f_{I,A}$ is the restriction of $\mathcal{F}_{I,A}$ to the full subcategory $Tw(I)^f\subset Tw(I)$ composed with the natural map $\mathcal{C}_{X^I}\to\mathcal{C}_{X^{I'}}$. Since β is cofinal, the above colimit rewrites as

$$\underset{(I\to J\to K')\in Tw(I)^f}{\operatorname{colim}}(\underset{k\in K'}{\boxtimes}A^{\otimes J_k})$$

taken in $C_{X^{I'}}$. Since $Tw(I')\to Tw(I)^f$ is cofinal, the above colimit rewrites as

$$\underset{(I'\to J'\to K')\in Tw(I')}{\operatorname{colim}}(\underset{k\in K'}{\boxtimes}A^{\otimes J'_k})$$

taken in $C_{X^{I'}}$, hence identifies with $A_{X^{I'}}$. We obtained an object Fact(A) of Fact(C)(Ran).

3.5.5. Given $\phi : I \to J$ a map in $fSet$, arguing as after Lemma 3.3.1, one gets an isomorphism

$$A_{X^I} \mid_{X^I_{\phi,disj}} \widetilde{\to} (\underset{i' \in I'}{\boxtimes} A_{X^{I_{i'}}}) \mid_{X^I_{\phi,disj}}$$

in $\mathcal{C}_{X^I} \otimes_{Shv(X^I)} Shv(X^I_{\phi,disj})$, where we use the equivalence (19) to see both sides in the same category.

3.5.6. Recall that in Section 3.3.4 we equipped $\Gamma(\mathrm{Ran}, \mathrm{Fact}(C))$ with a structure of an object of $CAlg^{nu}(\mathrm{DGCat}_{cont})$. Write $\star$ for the monoidal operation on $\mathrm{Fact}(C)(\mathrm{Ran})$. Let us lift $\mathrm{Fact}(A) \in \mathrm{Fact}(C)(\mathrm{Ran})$ to a non-unital commutative algebra in $\mathrm{Fact}(C)(\mathrm{Ran})$.

Let $I_1, I_2 \in fSet$ with $I = I_1 \sqcup I_2$. Let

$$(I_1 \to J_1 \to K_1) \in Tw(I_1), (I_2 \to J_2 \to K_2) \in Tw(I_2)$$

Recall the functor $\alpha : Tw(I_1) \times Tw(I_2) \to Tw(I)$, let $(I \to J \to K)$ is the image of this pair under α. Under the equivalence (20) one gets an isomorphism

$$(\underset{k \in K_1}{\boxtimes} A^{\otimes (J_1)_k}) \boxtimes (\underset{k \in K_2}{\boxtimes} A^{\otimes (J_2)_k}) \widetilde{\to} \underset{k \in K}{\boxtimes} A^{\otimes J_k}$$

in $\underset{k \in K}{\boxtimes} C^{\otimes J_k}(X)$, hence also in $C_{X^{I_1}} \boxtimes C_{X^{I_2}}$. Passing to the colimit over $Tw(I_1) \times Tw(I_2)$ in $C_{X^{I_1}} \boxtimes C_{X^{I_2}}$, we get in

$$A_{X^{I_1}} \boxtimes A_{X^{I_2}} \widetilde{\to} \underset{\substack{(I_1 \to J_1 \to K_1) \in Tw(I_1) \\ (I_2 \to J_2 \to K_2) \in Tw(I_2)}}{\mathrm{colim}} \underset{k \in K}{\boxtimes} A^{\otimes J_k},$$

where $K = K_1 \sqcup K_2$. Applying further the natural functor $C_{X^{I_1}} \boxtimes C_{X^{I_2}} \to C_{X^I}$, we get a natural map in $\mathcal{C}_{X^I}$

$$\beta_{I_1,I_2} : A_{X^{I_1}} \boxtimes A_{X^{I_2}} \to A_{X^I} \tag{29}$$

Now if $I_1 \to I'_1, I_2 \to I'_2$ are maps in $fSet$, using the commuttaive diagram (22) we !-restrict (29) under $X^{I'} \hookrightarrow X^I$, where $I' = I'_1 \sqcup I'_2$, and get the same morphism

$$\beta_{I'_1,I'_2} : A_{X^{I'_1}} \boxtimes A_{X^{I'_2}} \to A_{X^{I'}}$$

Passing to the limit over $I, I' \in fSet \times fSet$, this gives a map

$$\beta(\mathrm{Fact}(A) \boxtimes \mathrm{Fact}(A)) \to u^! \, \mathrm{Fact}(A)$$

in $\Gamma(\mathrm{Ran} \times \mathrm{Ran}, u^* \, \mathrm{Fact}(C))$, here β is the morphism (23). We have denoted here by $u^! : \Gamma(\mathrm{Ran}, \mathrm{Fact}(C)) \to \Gamma(\mathrm{Ran} \times \mathrm{Ran}, u^* \, \mathrm{Fact}(C))$ the natural "shriek-pullback" for sections. Recall from ([24], 2.0.2) that this $u^!$ has a left adjoint

$$u_{!,\mathrm{Fact}(C)} : \Gamma(\mathrm{Ran} \times \mathrm{Ran}, u^* \, \mathrm{Fact}(C)) \to \Gamma(\mathrm{Ran}, \mathrm{Fact}(C)),$$

because u is a pseudo-indproper morphism of pseudo-indschemes. By definition of the monoindal structure on $\Gamma(\mathrm{Ran}, \mathrm{Fact}(C))$, this gives a map

$$\mathrm{Fact}(A) \star \mathrm{Fact}(A) \to \mathrm{Fact}(A)$$

This is the product on $\mathrm{Fact}(A)$ in $(\Gamma(\mathrm{Ran}, \mathrm{Fact}(C)), \star)$.

3.5.7. From now on for the rest of Section 3.5 we assume that $\mathcal{C}(X)$ is compactly generated, dualizable as a $Shv(X)$-module, the functor $C^{\otimes 2}(X) \overset{m}{\to} \mathcal{C}(X)$ admits a continuous right adjoint, which is $Shv(X)$-linear, and $Shv(X) \overset{1_{\mathcal{C}}}{\to} C(X)$ admits a $Shv(X)$-linear continuous right adjoint. Recall that $\mathcal{C}^{\otimes J}(X)$ denotes the J-tensor power of $\mathcal{C}(X)$ in $Shv(X) - mod$.

Recall that for $\mathcal{D}$-module dualizability of $\mathcal{C}(X)$ as a $Shv(X)$-module is equivalent to its dualizability as a plein object of DGCat_{cont}. This is maybe not true in the constructible context.

Lemma 3.5.8. *Let I be a finite nonempty set. Then* $\mathrm{Fact}(\mathcal{C})(X^I)$ *is dualizable as a* $Shv(X^I)$*-module. Besides, for any* $(I \to J \to K) \in Tw(I)$ *the natural functor*

$$\underset{k\in K}{\boxtimes} \mathcal{C}^{\otimes J_k}(X) \to \mathcal{C}_{X^I} \tag{30}$$

admits a continuous right adjoint, which is a strict morphism of $Shv(X^I)$*-modules.*

Proof. **Step 1** For any $(I \to J \to K) \in Tw(I)$, $\underset{k\in K}{\boxtimes} C^{\otimes J_k}(X)$ is dualizable in $Shv(X^I)-mod$. Indeed, $\boxtimes_{i\in J}\mathcal{C}(X)$ is dualizable as a $Shv(X^J)$-module, as the functor

$$\prod_J Shv(X) - mod \to Shv(X^J) - mod$$

of exteriour product is symmetric monoidal. Now the extensions of scalars functor $Shv(X^J) - mod \to Shv(X^K) - mod$ with respect to $\vartriangle^!$: $Shv(X^J) \to Shv(X^K)$ is symmetric monodal. So, $\underset{k\in K}{\boxtimes} C^{\otimes J_k}(X)$ is dualizable in $Shv(X^K) - mod$. Finally, applying ([22], 9.2.32) for the colocalization $Shv(X^K) \leftrightarrows Shv(X^I)$, we conclude that $\underset{k\in K}{\boxtimes} C^{\otimes J_k}(X)$ is dualizable in $Shv(X^I) - mod$.

Step 2 Consider a morphism in $Tw(I)$ given by (2). We claim that in the diagram

$$\underset{k\in K_1}{\boxtimes} \mathcal{C}^{\otimes (J_1)_k}(X) \to \underset{k\in K_1}{\boxtimes} \mathcal{C}^{\otimes (J_2)_k}(X) \to \underset{k\in K_2}{\boxtimes} \mathcal{C}^{\otimes (J_2)_k}(X)$$

both maps admit continuous right adjoints, which are $Shv(X^I)$-linear. For the first map we first check that it is $Shv(X^{K_1})$-linear using ([22], 4.1.6), and apply the functor of direct image $Shv(X^{K_1}) - mod \to Shv(X^I) - mod$. For the second map we use the fact that for any $M \in Shv(X^{K_2}) - mod$, we have an adjoint pair

$$\vartriangle_!\colon M \otimes_{Shv(X^{K_2})} Shv(X^{K_1}) \leftrightarrows M : \vartriangle^!$$

in $Shv(K^2) - mod$, which is also an adjoint pair in $Shv(X^I) - mod$.

So, we get the functor $\mathcal{F}_I : Tw(I) \to Shv(X^I) - mod$, sending $(I \to J \to K)$ to $\underset{k\in K}{\boxtimes} C^{\otimes J_k}(X)$, and we may pass to right adjoints here and get $\mathcal{F}_I^R : Tw(I)^{op} \to Shv(X^I) - mod$. Recall that the functor $\mathrm{oblv} : Shv(X^I) - mod \to \mathrm{DGCat}_{cont}$ preserves limits and colimits, so we may understand $\lim \mathcal{F}_I^R$ either in DGCat_{cont} or in $Shv(X^I) - mod$. Recall that $\mathrm{colim}\, \mathcal{F}_I \widetilde{\to} \lim \mathcal{F}_I^R$, where the limit is understood in DGCat_{cont}, the claim about the right adjoint to (30) follows. To get the dualizability of $\mathrm{Fact}(\mathcal{C})(X^I)$ is $Shv(X^I) - mod$ we may apply ([22], 3.1.10). ☐

3.5.9. The right adjoint $m^R : C(X) \to C^{\otimes 2}(X)$ of m together with $1_C^R : C(X) \to Shv(X)$ defines on $C(X)$ the structure of a cocommutative coalgebra in $Shv(X)-mod$. Write $C^\vee(X)$ for the dual of $C(X)$ in $Shv(X)-mod$. Passing to the duals, $C^\vee(X)$ becomes a commutative algebra in $Shv(X)-mod$ with the product $(m^R)^\vee : (C^\vee)^{\otimes 2}(X) \to C^\vee(X)$ and unit $(1_C^R)^\vee : Shv(X) \to C^\vee(X)$.

Our our assumptions, the map $C(X) \mapsto C^\vee(X)$ is an involution. It interacts nicely with the construction of $\mathrm{Fact}(C)$, we discuss this in the next subsection.

3.5.10. Under our assumptions, for $\Sigma = (I \to J \to K) \in Tw(I)$, the dual of $\underset{k\in K}{\boxtimes} C^{\otimes J_k}(X)$ in $Shv(X^I)-mod$ is $\underset{k\in K}{\boxtimes} (C^\vee)^{\otimes J_k}(X)$. From Lemma 3.5.8 we conclude that the dual of C_{X^I} in $Shv(X^I)-mod$ writes as

$$(C_{X^I})^\vee \widetilde{\to} \lim_{(I\to J\to K\in Tw(I)^{op}} \underset{k\in K}{\boxtimes} (C^\vee)^{\otimes J_k}(X) \tag{31}$$

(limit taken in $Shv(X^I)-mod$). For a map (16) in $Tw(I)$ the transition map in the latter limit is

$$\underset{k\in K_2}{\boxtimes} (C^\vee)^{\otimes (J_2)_k}(X) \overset{\vartriangle^!}{\to} \underset{k\in K_1}{\boxtimes} (C^\vee)^{\otimes (J_2)_k}(X) \overset{m^\vee}{\to} \underset{k\in K_1}{\boxtimes} (C^\vee)^{\otimes (J_1)_k}(X)$$

for $\vartriangle: X^{K_1} \to X^{K_2}$, and $m^\vee$ is the dual to the product map m in $Shv(X^I)-mod$.

We may pass to the left adjoints in $Shv(X^I)-mod$ in the diagram (31), and get

$$(C_{X^I})^\vee \widetilde{\to} \operatorname*{colim}_{(I\to J\to K\in Tw(I)} \underset{k\in K}{\boxtimes} (C^\vee)^{\otimes J_k}(X)$$

The corresponding diagram is nothing but the functor $\mathcal{F}_{I,C^\vee}$. We conclude that

$$(C_{X^I})^\vee \widetilde{\to} (C^\vee)_{X^I}$$

naturally.

Note that for $D \in Shv(X^I)-mod$ one has

$$\mathrm{Fun}_{Shv(X^I)}(\mathcal{C}_{X^I}, D) \widetilde{\to} (\mathcal{C}_{X^I})^\vee \otimes_{Shv(X^I)} D \widetilde{\to} (\mathcal{C}^\vee)_{X^I} \otimes_{Shv(X^I)} D$$

3.5.11. Though we don't know how to define the symmetric monoidal structure on $\mathcal{C}_{X^I}$, for $(I \to * \to *) \in Tw(I)$ the corresponding functor $\mathcal{C}(X) \to \mathcal{C}_{X^I}$ has to be symmetric monoidal. Since $1_C \in \mathcal{C}(X)^c$, the unit of $\mathcal{C}_{X^I}$ has also to be compact by Lemma 3.5.8.

Lemma 3.5.12. *Assume that $\mathcal{C}(X)$ is ULA over $Shv(X)$. Then for any $I \in fSet$, $\mathcal{C}_{X^I}$ is ULA over $Shv(X^I)$ in the sense of Definition 3.7.6. In particular, $\mathcal{C}_{X^I}$ is compactly generated.*

Proof. **Step 1** Recall that our notation $\underset{i\in I}{\boxtimes} C(X)$ actually means $(C(X)^{\boxtimes I}) \otimes_{(Shv(X)^{\boxtimes I})} Shv(X^I)$. Let us show that the latter category is compactly generated by objects of the form

$$(\underset{i\in I}{\boxtimes} c_i) \otimes_{(Shv(X)^{\boxtimes I})} z \tag{32}$$

with $c_i \in C(X)$ ULA over $Shv(X)$, and $z \in Shv(X^I)^c$. In the case of $\mathcal{D}$-modules, $Shv(X)^{\boxtimes I} \to Shv(X^I)$ is an equivalence, and there is nothing to prove. Assume now we are in the constructible context.

In this case for any $S \in \mathrm{Sch}_{ft}$, $\otimes^! : Shv(S) \otimes Shv(S) \to Shv(S)$ has a continuous right adjoint. Note that if $c_i \in \mathcal{C}(X)$ is ULA over $Shv(X)$ then $\boxtimes_{i\in I} c_i$ is ULA over $Shv(X)^{\boxtimes I}$. So, for any $z \in Shv(X^I)^c$, (32) is compact in $(C(X)^{\boxtimes I}) \otimes_{(Shv(X)^{\boxtimes I})} Shv(X^I)$ by Remark 3.7.3.

Let $\mathcal{D} \subset (C(X)^{\boxtimes I}) \otimes_{(Shv(X)^{\boxtimes I})} Shv(X^I)$ be a full embedding in DGCat_{cont} such that $\mathcal{D}$ contains all the objects of the form (32). Then it contains all the objects $c' \otimes_{(Shv(X)^{\boxtimes I})} z$ for $c' \in C(X)^{\boxtimes I}, z \in Shv(X^I)$ by ([15], I.1, 7.4.2). Applying in addition ([15], I.1, 8.2.6), we see that $\mathcal{D} = (C(X)^{\boxtimes I}) \otimes_{(Shv(X)^{\boxtimes I})} Shv(X^I)$.

Step 2 If $c_i \in \mathcal{C}(X)$ are ULA over $Shv(X)$ then $(\underset{i\in I}{\boxtimes} c_i) \otimes_{(Shv(X)^{\boxtimes I})} \omega$ is ULA over $Shv(X^I)$.

Indeed, consider the adjoint pair $Shv(X)^{\boxtimes I} \leftrightarrows \mathcal{C}(X)^{\boxtimes I}$ in $Shv(X)^{\boxtimes I} - mod$, where the left adjoint is the multimlication by $\boxtimes_i c_i$. Tensoring with $Shv(X^I)$ over $Shv(X)^{\boxtimes I}$, we get the desired adjoint pair in $Shv(X^I)$.

Step 3 By Lemma 3.5.2, the essential image of $\mathrm{Loc} : \underset{i\in I}{\boxtimes} C(X) \to C_{X^I}$ generates C_{X^I} under colimits. Now if $c_i \in \mathcal{C}(X)$ are ULA over $Shv(X)$, $\boxtimes_i c_i \in \underset{i\in I}{\boxtimes} C(X)$ is ULA over $Shv(X^I)$. By Proposition 3.7.7 and Lemma 3.5.8, $\mathrm{Loc}(\boxtimes_i c_i)$ is ULA over $Shv(X^I)$.

By Lemma 3.5.8, if $c_i \in \mathcal{C}(X)$ are ULA over $Shv(X)$, $z \in Shv(X^I)^c$ then

$$\mathrm{Loc}((\underset{i\in I}{\boxtimes} c_i) \otimes_{(Shv(X)^{\boxtimes I})} z)$$

is compact in $\mathcal{C}_{X^I}$, and these objects generate $\mathcal{C}_{X^I}$ by Lemma 3.5.2. □

3.5.13. Assume in addition that we are given an adjoint pair $O : \mathcal{C}(X) \leftrightarrows Shv(X) : O^R$ in $Shv(X) - mod$, where O is conservative, comonadic, and a map in $CAlg(Shv(X) - mod)$. The comonad $OO^R : Shv(X) \to Shv(X)$ is $Shv(X)$-linear, so is given by some coalgebra $\mathcal{O}_C \in Shv(X)$.

The map O is a morphism in $CAlg(ShvCat(X))$, hence we may apply the construction of Fact to this map. For any $I \in fSet, \Sigma = (I \to J \to K) \in Tw(I)$ we get an adjoint pair

$$O_\Sigma : \underset{k\in K}{\boxtimes} C^{\otimes J_k}(X) \leftrightarrows Shv(X^K) : O_\Sigma^R$$

in $Shv(X^I) - mod$, where O_Σ is obtained from O by the functoriality of the construction of $\mathcal{F}_I$. Since all the involved categories are dualizable, by ([22], 9.2.67), for any $I \in fSet, \Sigma = (I \to J \to K) \in Tw(I)$ the functor O_Σ is conservative. The comonad $O_\Sigma O_\Sigma^R$ on $Shv(X^K)$ is given by tensoring with $\underset{k\in K}{\boxtimes} \mathcal{O}_C^{\otimes J_k} \in Shv(X^K)$. The map $\vartriangle_!$: $Shv(X^K) \to Shv(X^I)$ is left-lax monoidal, so sends coalgebras to coalgebras. So, we may think of $\underset{k\in K}{\boxtimes} \mathcal{O}_C^{\otimes J_k}$ as a coalgebra in $Shv(X^I)$. Since for $\vartriangle$: $X^K \to X^I$ the functor $\vartriangle_*$: $Shv(X^K) \to Shv(X^I)$ is fully faithful, we have

$$\underset{k\in K}{\boxtimes} \mathcal{O}_C^{\otimes J_k} - comod(Shv(X^K)) \widetilde{\to} \underset{k\in K}{\boxtimes} \mathcal{O}_C^{\otimes J_k} - comod(Shv(X^I))$$

canonically.

Assume that O_Σ is comonadic for any $I \in fSet, \Sigma \in Tw(I)$. This is the case in the main example below in Section 3.5.19 by ([31], 6.23.2). In particular, for any $J \in fSet$,

$$\mathcal{C}^{\otimes J}(X) \tilde{\to} \mathcal{O}_C^{\otimes J} - comod(Shv(X))$$

We assume in addition that $\mathcal{O}_C \in coAlg(Shv(X))$ is lifted $\mathcal{O}_C \in CAlg(coAlg(Shv(X))$, and the structure of an object of $CAlg(Shv(X))$ on $C(X)$ comes now from this bialgebra structure on $\mathcal{O}_C$.

Namely, the unit $1_{\mathcal{O}_C} : \omega_X \to \mathcal{O}_C$ is a map in $coAlg(Shv(X)$, so gives the extensions of scalars map

$$Shv(X) \tilde{\to} \omega_X - comod(Shv(X)) \to \mathcal{O}_C - comod(Shv(X)) \tilde{\to} C(X),$$

which is the unit of $C(X)$. For $J \in fSet$, the product map $\mathcal{O}_C^{\otimes J} \to \mathcal{O}_C$ in $coAlg(Shv(X))$ gives via extension of scalars the morphism

$$C^{\otimes J}(X) \tilde{\to} \mathcal{O}_C^{\otimes J} - comod(Shv(X)) \to \mathcal{O}_C - comod(Shv(X)) \tilde{\to} C(X)$$

which is the product for $C(X)$ along $J \to *$.

3.5.14. Passing to the colimit over $Tw(I)$, O_Σ yields a functor denoted $O_I : \mathcal{C}_{X^I} \to Shv(X^I)$ in $Shv(X^I) - mod$. By ([22], 9.2.39), O_I admits a continuous right adjoint O_I^R obtianed from O_Σ^R by passing to the limit over $Tw(I)^{op}$. We obtained an adjoint pair

$$O_I : \mathcal{C}_{X^I} \leftrightarrows Shv(X^I) : O_I^R$$

in $Shv(X^I){-}mod$. The corresponding comonad is given by some coalgebra in $Shv(X^I)$.

Is it true that O_I is comonadic? Why is it conservative? In our main example, the functor $\mathcal{F}_I^R$ is not compatible with O.

Note that if $I \to I'$ is a map in $fSet$ then applying $\otimes_{Shv(X^I)} Shv(X^{I'})$ to the above adjoint pair, one gets canonically the adjoint pair

$$O_{I'} : \mathcal{C}_{X^{I'}} \leftrightarrows Shv(X^{I'}) : O_{I'}^R$$

For O_I this follows from the functoriality of Fact, so for the right adjoint it is automatic. This means that we get after passing to the limit over $I \in fSet$ the adjoint pair

$$O_{\mathrm{Ran}} : \mathrm{Fact}(C)(\mathrm{Ran}) \leftrightarrows Shv(\mathrm{Ran}) : O_{\mathrm{Ran}}^R$$

by ([15], I.1, 2.6.4). This is an adjoint pair in $Shv(\mathrm{Ran}) - mod$, where $Shv(\mathrm{Ran})$ s equipped with the $\otimes^!$ pointwise symmetric monoidal structure.

3.5.15. To a morphism (16) in $Tw(I)$ we attach the composition

$$\boxtimes_{k \in K_1} \mathcal{O}_C^{\otimes (J_1)_k} \overset{m}{\to} \boxtimes_{k \in K_1} \mathcal{O}_C^{\otimes (J_2)_k} \overset{\vartriangle_*}{\to} \boxtimes_{k \in K_2} \mathcal{O}_C^{\otimes (J_2)_k}$$

in $coAlg(Shv(X^{K_2}))$, hence also in $coAlg(Shv(X^I))$. This defines the functor

$$\mathcal{F}_{I,\mathcal{O}_G}^{coAlg} : Tw(I) \to coAlg(Shv(X^I))$$

whose underlying functor $\mathcal{F}_{I,\mathcal{O}_G} : Tw(I) \to Shv(X^I)$ is as in Section 3.5.3 for the factorization category $Shv(\mathrm{Ran})$.

The forgetful functor $coAlg(Shv(X^I)) \to Shv(X^I)$ preserves colimits, so

$$(\mathcal{O}_C)_{X^I} = \underset{(I\to J\to K)\in Tw(I)}{\mathrm{colim}} \mathcal{F}_{I,\mathcal{O}_G}$$

can be understood in $Shv(X^I)$ or equivalently in $coAlg(Shv(X^I))$. Now $\mathcal{F}_{I,C} : Tw(I) \to Shv(X^I) - mod$ is obtained from the functor $\mathcal{F}^{coAlg}_{I,\mathcal{O}_G}$ passing to comodules in $Shv(X^I)$, that is, the equivalence

$$\underset{k\in K}{\boxtimes} C^{\otimes J_k}(X) \widetilde{\to} \underset{k\in K}{\boxtimes} \mathcal{O}_C^{\otimes J_k} - comod(Shv(X^I))$$

becomees functorial in $\Sigma \in Tw(I)$, where on the RHS we use the functor $\mathcal{F}^{coAlg}_{I,\mathcal{O}_G}$.

Passing to the colimit over $Tw(I)$ this gives a canonical diagram

$$\begin{array}{ccc} C_{X^I} & \to & (\mathcal{O}_C)_{X^I} - comod(Shv(X^I)) \\ \downarrow O_I & \swarrow \text{oblv} & \\ Shv(X^I) & & \end{array}$$

Is it an equivalence?

3.5.16. The counit map $\mathcal{O}_C \to \omega_X$ in $CAlg(Shv(X))$ by functoriality of the construction of factorization algebras in $Shv(\mathrm{Ran})$ gives a morphism in

$$\mathrm{Fun}(Tw(I), Shv(X^I) - mod)$$

from $\mathcal{F}_{I,\mathcal{O}_C}$ to $\mathcal{F}_{I,\omega_X}$. Namely, for $\Sigma = (I \to J \to K) \in Tw(I)$ the map $\underset{k\in K}{\boxtimes} \mathcal{O}_C^{\otimes J_k} \to \omega_{X^K}$ is functorial in $\Sigma \in Tw(I)$. Passing to the colimit over $\Sigma \in Tw(I)$, this gives a map in $Shv(X^I)$

$$(\mathcal{O}_C)_{X^I} \to \omega_{X^I}. \tag{33}$$

It is compatible with factorizations, and gives as I varies in $fSet$ the map $\mathrm{Fact}(\mathcal{O}_C) \to \mathrm{Fact}(\omega_X) \widetilde{\to} \omega_{\mathrm{Ran}}$ in $Shv(\mathrm{Ran})$.

3.5.17. In fact, in our situation $\mathcal{O}_C \in CAlg(C(X))$. For $J \in fSet$ the product for $J \to *$ is given as follows. The J-th tensor power of $\mathcal{O}_C$ in the symmetric monoidal category $C(X)$ is $\mathcal{O}_C^{\otimes J}$ (where the tensor power is taken in $(Shv(X), \otimes^!)$) with the $\mathcal{O}_C$-comodule structure given by $\mathcal{O}_C^{\otimes J} \stackrel{m_J}{\to} \mathcal{O}_C$. Here m_J is the product on $\mathcal{O}_C$ as an object of $CAlg(Shv(X))$. Then m_J itself becomes the desired product map.

The unit of the symmetric monoidal category $C(X)$ is $\omega_X \in \mathcal{O}_C - comod(Shv(X))$, on which the $\mathcal{O}_C$-comodule structure is given by the map $1_{\mathcal{O}_C} : \omega_X \to \mathcal{O}_C$ in $coAlg(Shv(X))$.

The unit of $\mathcal{O}_C$ as a commutative algebra in $C(X)$ is the morphism $1_{\mathcal{O}_C} : \omega_X \to \mathcal{O}_C$ in $\mathcal{O}_C - comod(Shv(X))$.

3.5.18. So, we may apply the construction of the factorization algebra to $\mathcal{O}_C \in CAlg(C(X))$. In other words, we may think of the colimit of the functor $\mathcal{F}^{Shv(X)}_{I,\mathcal{O}_C} : Tw(I) \to Shv(X^I)$ as the image under $O_I : C_{X^I} \to Shv(X^I)$ of $\underset{Tw(I)}{\mathrm{colim}}\, \mathcal{F}^C_{I,\mathcal{O}_C}$.

By abuse of notations, we sometimes write $(\mathcal{O}_C)_{X^I} \in \mathcal{C}_{X^I}$. Now (33) is actually a morphism $O_I((\mathcal{O}_C)_{X^I}) \to \omega_{X^I}$. By adjointness, it gives the morphism in $\mathcal{C}_{X^I}$

$$(\mathcal{O}_C)_{X^I} \to O_I^R(\omega_{X^I}) \tag{34}$$

compatible with factorizations and base changes under $\vartriangle^!$ for $\vartriangle: X^{I'} \to X^I$, where $I \to I'$ is a map in $fSet$. Here we used the observation from Section 3.5.14 that the formation of O_I^R commutes with functors $\vartriangle^!$.

Since for $I = *$ the map (34) is an isomorphism, we conclude that for any I it is also an isomorphism.

We conclude that the comonad $O_I O_I^R : Shv(X^I) \to Shv(X^I)$ is given by $(\mathcal{O}_C)_{X^I} \in coAlg(Shv(X^I))$.

How to prove that $O_I : C_{X^I} \to Shv(X^I)$ is comonadic???????

3.5.19. *Example.* Classically, for $\mathcal{C}(X) = \mathrm{Rep}(\check{G}) \otimes Shv(X)$, where $\check{G}$ is a reductive group over e, the map $* \xrightarrow{1} \check{G}$ yields the dual pair $O : \mathrm{Rep}(\check{G}) \otimes Shv(X) \leftrightarrows Shv(X) : O^R$ in $Shv(X)-mod$, while the left adjoint to O does not exist (when the set of irreducible representations of $\check{G}$ is infinite). All the assumptions of Section 3.5.13 are satisfied.

3.6. For 8.2. It is understood that $\mathcal{C}$ is a commutative algebra in DGCat_{cont}.

3.6.1. *Twist of a sheaf of categories by a gerbe.* For 8.2.2. There we need a general definition of the twist. Let $\mathcal{C} \in \mathrm{DGCat}_{cont}$, A be a torsion abelian group with a given monoidal functor $B(A) \to \mathrm{Fun}_{k,cont}(\mathcal{C}, \mathcal{C})$. That is, A acts on $\mathcal{C}$ by automorphisms of the identity functor. (For example, if $\mathcal{C} \in CAlg(\mathrm{DGCat}_{cont})$ then we have a version where the input datum is a monoidal functor $B(A) \to \mathrm{Fun}^{\otimes}_{k,cont}(\mathcal{C}, \mathcal{C})$, the latter category denotes the category of k-linear continuous symmetric monoidal functors from $\mathcal{C}$ to itself).

Let a A-gerbe $\mathcal{G} : Y \to B^2_{et}(A)$ be given. We want to construct a sheaf of DG-categories (resp., a sheaf of symmetric monoidal DG-categories) $\mathcal{C}_{\mathcal{G}}$ on Y. Recall that $ShvCat_{/Y}$ satisfies the etale descent (for any theory of sheaves). Pick $f : Y' \to Y$ an etale surjection and a trivialization of our gerbe over Y'. Then we get the Cech nerve $Y'^{\bullet}/Y$ of $Y' \to Y$ with $Y'^n/Y = Y' \times_Y \times \ldots \times_Y Y'$, the product taken $n+1$ times. The natural map $ShvCat_{/Y} \to Tot(ShvCat_{Y'^{\bullet}/Y})$ is an isomorphism in $1-\mathcal{C}\mathrm{at}$. We construct the corresponding object of $Tot(ShvCat_{Y'^{\bullet}/Y})$ as follows. As an object of $ShvCat_{Y'^n/Y}$ this is the constant sheaf $\mathcal{C} \otimes Shv(Y'^n/Y)$. Over $Y' \times_Y Y' = Y'^1/Y$ we get a A-torsor $\mathcal{F}$ giving an automorphism of the trivial A-gerbe over $Y' \times_Y Y'$. Over $Y' \times_Y Y' \times_Y Y'$ we get an identification $\mathrm{pr}_{23}^* \mathcal{F} \circ \mathrm{pr}_{12}^* \mathcal{F} \widetilde{\to} \mathrm{pr}_{13}^* \mathcal{F}$ of the automorphisms of the trivial A-gerbe.

Over $Y' \times_Y Y'$ we get an automorphism

$$\bar{\mathcal{F}} : \mathcal{C} \otimes Shv(Y' \times_Y Y') \widetilde{\to} \mathcal{C} \otimes Shv(Y' \times_Y Y')$$

of this sheaf of categories given as the composition $Y' \times_Y Y' \xrightarrow{\mathcal{F}} B(A) \to \mathrm{Fun}_{k,cont}(\mathcal{C}, \mathcal{C})$. Over Y'^2/Y we then get the commutativity datum for the diagram

$$\begin{array}{ccc} \mathcal{C} \otimes Shv(Y'^2/Y) & \overset{\mathrm{pr}_{12}^* \bar{\mathcal{F}}}{\to} & \mathcal{C} \otimes Shv(Y'^2/Y) \\ & \searrow \mathrm{pr}_{13}^* \bar{\mathcal{F}} & \downarrow \mathrm{pr}_{23}^* \bar{\mathcal{F}} \\ & & \mathcal{C} \otimes Shv(Y'^2/Y) \end{array}$$

and so on, which together define the desired object of $Tot(ShvCat_{Y'^{\bullet}/Y})$.

More precisely, we are given as an input a commutative diagram

$$\begin{array}{ccc} Y & \stackrel{\mathcal{G}}{\to} & B^2_{et}(A)\\ \uparrow & & \uparrow\\ Y' & \to & pt\end{array}$$

Passing to the Cech nerves, we get a morphism of groupoids in PreStk, $Y'^{\bullet}/Y \to pt^{\bullet}/B^2_{et}(A)$. Here $pt^n/B^2_{et}(A) \widetilde{\to} B_{et}(A)\times\ldots\times B_{et}(A)$, the product taken n times for $n\ge 0$. In this sense $\mathcal{F}$ extends to a what could be called a multiplicative A-torsor on the groupoid $Y'^{\bullet}/Y$. Now $\bar{\mathcal{F}}$ is a morphism of groupoids from $Y'^{\bullet}/Y$ to the groupoid in DGCat_{cont} corresponding to $\mathrm{Fun}_{k,cont}(\mathcal{C},\mathcal{C})$.

So, $\bar{\mathcal{F}}$ looks like an algebra in the monoidal category $\mathrm{Fun}_{k,cont}(\mathcal{C},\mathcal{C})$ with the difference that the symplicial object $[n]\mapsto \mathrm{Fun}_{k,cont}(\mathcal{C},\mathcal{C})^{\otimes n}$ defining this monoidal category is replaced by the simplicial object $[n]\mapsto \mathrm{Fun}_{k,cont}(\mathcal{C},\mathcal{C})^{\otimes n}\otimes Shv(Y'^n/Y)$. It seems the desired category $\mathcal{C}_{\mathcal{G}}$ is the category of $\bar{\mathcal{F}}$-*algebras* in $\mathcal{C}\otimes Shv(Y')$. One should still give a sense to this notion similarly to the notion of a module over an algebra in the $(\infty,1)$-category setting. (To be improved later).

3.6.2. For 8.3.1. For a group H, $Z(H)$ acts on $\mathrm{Rep}(H)$ by the automorphisms of the identity functor (viewed as a symmetric monoidal category). This means that 1) for $z\in Z(H), V_i\in \mathrm{Rep}(H)$ letting z_i be the action of z in V_i, the action of z on $V_1\otimes V_2$ is z_1z_2; 2) the action of $z\in Z(H)$ on the trivial representation is trivial.

3.6.3. For 8.3.3. The sheaf $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_G)_{\mathrm{Ran}}$ was defined in 2.4.2.

Where the symmetric monoidal structure on the sheaf of categories $Shv_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}}$ comes from?

Let $Y\in \mathrm{PreStk}_{lft}$, $Z\in \mathcal{G}\mathrm{rp}((\mathrm{PreStk}_{lft})_{/Y})$ and $\mathcal{G}$ be a multiplicative A-gerbe on Z, that is, given by an element in $\mathrm{Map}_{\mathcal{G}\mathrm{rp}((\mathrm{PreStk}_{lft})_{/Y})}(Z, B^2_{et}(A)\times Y)$. Then we can consider the sheaf of categories $Shv_{\mathcal{G}}(Z)_{/Y}$ over Y sending $S\to Y$ to $\mathrm{Shv}_{\mathcal{G}}(S\times_Y Z)$. We need some assumptions to get the convolution monoidal structure on this sheaf of DG-categories. Assume for example that $f: Z\to Y$ is ind-schematic, so that $f_*: Shv(Z)\to Shv(Y)$ is defined, see my Section 3.1.8. Let $m: Z\times_Y Z\to Z$ be the product map, then it is automatically ind-schematic, to that $m_*:\mathrm{Shv}(S\times_Y(Z\times_Y Z))\to Shv(S\times_Y Z)$ exists for $S\in(\mathrm{Sch}^{aff}_{ft})_{/Y}$. The usual convolution product is the composition

$$\mathrm{Shv}(S\times_Y Z)\otimes_{Shv(S)} Shv(S\times_Y Z)\to \mathrm{Shv}(S\times_Y(Z\times_Y Z))\stackrel{m_*}{\to} Shv(S\times_Y Z)$$

Twisting by $\mathcal{G}$, we get the desired convolution morphism

$$\mathrm{Shv}_{\mathcal{G}}(S\times_Y Z)\otimes_{Shv(S)} Shv_{\mathcal{G}}(S\times_Y Z)\to \mathrm{Shv}_{\mathcal{G}}(S\times_Y(Z\times_Y Z))\stackrel{m_*}{\to} Shv_{\mathcal{G}}(S\times_Y Z)$$

For clarity, $Shv_{\mathcal{G}}(S\times_Y Z)$ is naturally a $Shv(S\times_Y Z)$-module, hence a $Shv(S)$-module.

Assume also the unit map $u: Y\to Z$ ind-schematic, so $u_*: Shv(S)\to Shv(S\times_Y Z)$ exists. By assumption, $u^*\mathcal{G}$ is trivialized over Y. So, we get the morphism $u_*: \mathrm{Shv}(S)\to Shv_{\mathcal{G}}(S\times_Y Z)$ for $S\to Y$ with $S\in\mathrm{Sch}^{aff}_{ft}$.

If in addition $Z\in \mathcal{C}\mathrm{omGrp}((\mathrm{PreStk}_{lft})_{/Y})$ and

$$\mathcal{G}\in \mathrm{Map}_{\mathcal{C}\mathrm{omGrp}((\mathrm{PreStk}_{lft})_{/Y})}(Z, B^2_{et}(A)\times Y)$$

then I think $Shv_{\mathcal{G}}(Z)_{/Y}$ will be a sheaf of symmetric monoidal DG-categories over Y.

Remark 3.6.4. *We used without a definition the notion of a sheaf of (symmetric) monoidal* DG*-categories on some* $Y \in \mathrm{PreStk}_{lft}$*. The definition is that it is an object of* $CAlg(ShvCat_{/Y})$*, where* $ShvCat_{/Y}$ *is considered as a symmetric monoidal category with termwise tensor product: if* $C, C' \in ShvCat_{/Y}$ *then* $C \otimes_{Shv_{/Y}} C'$ *is the sheaf of categories whose sections over* $S \to Y$ *are* $C(S) \otimes_{Shv(S)} C'(S)$.

3.6.5. For 8.3.3 more. Let us check the equivalence (8.11) in the case when the gerbe $\mathcal{G}_T$ is trivial. Note that $\mathrm{Rep}(\check{T}) \widetilde{\to} \oplus_{\lambda \in \Lambda} \mathrm{Vect} \widetilde{\to} \prod_{\lambda \in \Lambda} \mathrm{Vect}$, where on the corresponding piece Vect the group $\check{T}$ acts by λ.

For I a finite nonempty set, the sheaf $Shv(\mathrm{Gr}_T)_{/\,\mathrm{Ran}}$ associates to X^I the category

$$Shv(\mathrm{Gr}_{T,comb} \times_{\mathrm{Ran}} X^I) \widetilde{\to} \underset{(K, \lambda^K, I \to K) \in \mathcal{J}_I}{\mathrm{colim}} Shv(X^K) \tag{35}$$

as we have seen in my Section 3.0.50.

By definition, the sheaf of categories $\mathrm{Fact}(\mathrm{Rep}(\check{T}))$ associates to X^I the following category. Consider the category $Tw(I)_\Lambda$, whose objects are collections: $(I \to J \to K) \in Tw(I)$ and a map $\lambda^J : J \to \Lambda$. A morphism from $(I \to J \to K, \lambda^J)$ to $(I \to J' \to K', \lambda^{J'})$ is a morphism from $(I \to J \to K)$ to $(I \to J' \to K')$ in $Tw(I)$ as in 8.1.4 with the surjection $J \to J'$ denoted ϕ such that for each $j' \in J'$ one has $\lambda_{j'} = \sum_{\phi(j)=j'} \lambda_j$. Then the value of $\mathrm{Fact}(\mathrm{Rep}(\check{T}))$ on X^I is

$$\underset{(I \to J \to K, \lambda^J) \in Tw(I)_\Lambda}{\mathrm{colim}} Shv(X^K) \tag{36}$$

Indeed, we may rewrite $\underset{k \in K}{\boxtimes} \mathrm{Rep}(\check{T})_X^{\otimes J_k}$ as $\underset{\lambda_J : J \to \Lambda}{\oplus} Shv(X^K)$.

We have the functor $Tw(I)_\Lambda \to \mathcal{J}_I$ sending $(I \to J \xrightarrow{\nu} K, \lambda^J)$ to $(I \to K, \lambda^K)$, where λ^K is the direct image of λ^J under ν. So, let's calculate (36) in two steps: first take the LKE along this functor and then colimit over $\mathcal{J}_I$.

Given an object, say $a = (I \to K, \lambda^K) \in \mathcal{J}_I$, we claim that $Tw(I)_\Lambda \times_{\mathcal{J}_I} (\mathcal{J}_I)_{/a}$ is contractible.

An object of the latter category is given by a diagram $(I \to J' \xrightarrow{\nu} K', \lambda^{J'}) \in Tw(I)_\Lambda$ and a surjection $\phi : K \to K'$ compatible with surjections $I \to K, I \to K'$ such that $\phi_! \lambda^K = \nu_! \lambda^{J'}$.

Consider first the full subcategory $\mathcal{Y} \subset Tw(I)_\Lambda \times_{\mathcal{J}_I} (\mathcal{J}_I)_{/a}$ consisting of those objects for which K' has only one element. Then the inclusion $\mathcal{Y} \subset Tw(I)_\Lambda \times_{\mathcal{J}_I} (\mathcal{J}_I)_{/a}$ is not cofinal, however it induces an isomorphism of geometric realizations. Indeed, this functor admits an adjoint $Tw(I)_\Lambda \times_{\mathcal{J}_I} (\mathcal{J}_I)_{/a} \to \mathcal{Y}$ sending a point $(I \to J' \xrightarrow{\nu} K' \xleftarrow{\phi} K, \lambda^{J'})$ to $(I \to J' \to * \leftarrow K, \lambda^{J'})$. Besides, $\mathcal{Y}$ has a final object. The final object of $\mathcal{Y}$ is of course $(I \to * \to *, \lambda) \in Tw(I)_\Lambda$, where $\lambda = \sum_{k \in K} \lambda_k$. We have proved the contractibility of $Tw(I)_\Lambda \times_{\mathcal{J}_I} (\mathcal{J}_I)_{/a}$

So, the LKE in question produces precisely the colimit (35). The equivalence (8.11) for $\mathcal{G}_T$ trivial follows.

3.6.6. For 8.4.1. The definition of t-structure on $\mathrm{Rep}(H)_{\mathcal{G}_Z}(X)$ should be as follows I think. Let $S \in \mathrm{Sch}_{ft}^{aff}$, $S \to X$ be an etale morphism such that the gerbe $\mathcal{G}_Z$ becomes trivial on S. Any trivialization of $\mathcal{G}_Z$ over S gives a functor $\mathrm{Rep}(H)_{\mathcal{G}_Z}(X) \to \mathrm{Rep}(H) \otimes Shv(S)$. Consider the forgetful functor $\mathrm{Rep}(H) \to \mathrm{Vect}$ given by restriction to $\{1\} \subset H$. Composing with $\mathrm{Rep}(H) \otimes Shv(S) \to \mathrm{Vect} \otimes Shv(S) \widetilde{\to} Shv(S)$, we get a functor $r_S : \mathrm{Rep}(H)_{\mathcal{G}_Z}(X) \to Shv(S)$. The t-structure on $\mathrm{Rep}(H)_{\mathcal{G}_Z}(X)$ is such that r_S is t-exact for the perverse t-structure on $Shv(S)$.

In the definition of a twisted local system we have to require that the functor $\mathrm{Rep}(H)_{\mathcal{G}_Z}(X) \to Shv(X)$ is $Shv(X)$-linear, that is, comes from a morphism of sheaves of categories $\mathrm{Rep}(H)_{\mathcal{G}_Z} \to Shv_{/X}$ over X. This is also used in 8.4.3 for the functoriality of the construction.

3.6.7. My impression is that one of the advantages of the framework from the book [15] is as follows. Consider $B^i_{et}(E^{\times,tors})$ in the classical algebraic geometry setting this would be a stack over $\mathrm{Spec}\, E$, but we view it as an object of $\mathrm{PreStk} = \mathrm{Fun}(\mathrm{Sch}^{aff}, \mathrm{Spc})$, where Sch^{aff} are over k. For any $Y \in \mathrm{PreStk}$ we may consider $\mathrm{Map}(Y, B^i_{et}(E^{\times,tors}))$. I mean that was the following problem in the classical setting. For example, for $\bar{\mathbb{Q}}_\ell$-sheaves given a scheme Y over a field k, we were not able to view a $\bar{\mathbb{Q}}_\ell^{\times,tors}$-gerbe on Y as a geometric object. More precisely, for a finite abelian group say H viewed as a group scheme over k, we can consider a H-gerbe $\tilde{Y} \to Y$. But to get the desired category of $\bar{\mathbb{Q}}_\ell$-sheaves on $\tilde{Y}$, we need a character $H \to \bar{\mathbb{Q}}_\ell^*$.

More basically, an abelian group H directly is an object of PreStk, a constant prestack, while in the classical setting we need first to realize it as an algebraic group over $\mathrm{Spec}\, k$ to get the corresponding geometric object.

3.6.8. For 9.1.1. We may view the gerbe $\mathcal{G}^G \otimes \det_{\mathfrak{g}}^{\frac{1}{2}}$ as a gerbe over the quotient $\mathfrak{L}^+(G)\backslash \mathrm{Gr}_G$. This quotient is a factorization prestack over Ran, and this gerbe is a factorization gerbe over the factorization prestack $\mathfrak{L}^+(G)\backslash \mathrm{Gr}_G$. So, by 2.2.6 we get a factorization sheaf of categories over Ran.

The monoidal structure on $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ is obtained formally as follows. Consider the map $\widetilde{\mathrm{Gr}}_G^2 \to \mathrm{Gr}_G \times_{\mathrm{Ran}} \mathrm{Ran}^2$ given by (7.6). Restricting to the diagonal under $\mathrm{Ran} \to \mathrm{Ran}^2$, we get a map $f : \widetilde{\mathrm{Gr}}_G^2 \times_{\mathrm{Ran}^2} \mathrm{Ran} \to \mathrm{Gr}_G$ over Ran. Further $f^*\mathcal{G}^G \widetilde{\to} \mathcal{G}^G \widetilde{\boxtimes} \mathcal{G}^G$ as in 7.3.4. The gerbe $\mathcal{G}^G \otimes \det_{\mathfrak{g}}^{\frac{1}{2}}$ satisfies the same property, because $f^* \det_{\mathfrak{g}} \widetilde{\to} \det_{\mathfrak{g}} \widetilde{\boxtimes} \det_{\mathfrak{g}}$. The desired convolution is the direct image $f_!$, here f is ind-proper so $f_!$ is defined by ([8], 1.5.2).

3.7. Generalities about sheaves theories. Let us take $\mathrm{Shv}(S) = \mathrm{D}-mod(S)$ for $S \in \mathrm{Sch}_{ft}^{aff}$ as the sheaf theory. Sam claims then $Shv(S)$ is not rigid, however, the following property holds. Let $\mathcal{C} \in Shv(S) - mod(\mathrm{DGCat}_{cont})$. Then $\mathcal{C}$ is dualizable as an object of $Shv(S) - mod$ iff it is dualizable as an object of DGCat_{cont}. This is a non evident result!

Proof. 1) $Shv(S)$ is dualizable in $Shv(S \times S) - mod$. Indeed, $Shv(S)$ is a retract of $Shv(S \times S)$.

2) Let $C, D \in Shv(S)-mod$. Then by ([22], Section 9.2.45) one has

$$\mathrm{Fun}_{Shv(S\times S)}(Shv(S), \mathrm{Fun}_{k,cont}(C,D)) \widetilde{\to} \mathrm{Fun}_{Shv(S)}(C,D)$$

Besides, by ([22], Section 6.1.17), $Shv(S)\otimes_{Shv(S)\otimes Shv(S)}(C\otimes D)\widetilde{\to} C\otimes_{Shv(S)} D$.
3) Assume $C, D\in Shv(S)-mod$ and C is dualizable in DGCat_{cont}. Then we get

$$\mathrm{Fun}_{Shv(S)}(C,D)\widetilde{\to}\mathrm{Fun}_{Shv(S)\otimes Shv(S)}(Shv(S), \mathrm{Fun}_{k,cont}(C,D))\widetilde{\to}$$
$$Shv(S)\otimes_{Shv(S)\otimes Shv(S)} C^{\vee}\otimes D\widetilde{\to} C^{\vee}\otimes_{Shv(S)} D$$

This implies that C is dualizable in $Shv(S)-mod$.
4) If C is dualizable in $Shv(S)-mod$ then $C\otimes_{Shv(S)}\mathrm{QCoh}(S)$ is dualizable in $\mathrm{QCoh}(S)-mod$. Since $\mathrm{QCoh}(S)$ is rigid, $C\otimes_{Shv(S)}\mathrm{QCoh}(S)$ is dualizable in DGCat_{cont}. The functor $oblv : C\to C\otimes_{Shv(S)}\mathrm{QCoh}(S)$ is monadic, so C is dualizable as well. My understanding here is as follows: there is a monad $\mathcal{A}$ acting on $\mathrm{QCoh}(S)$ such that $\mathcal{A}-mod(\mathrm{QCoh}(S))\widetilde{\to} Shv(S)$. Then $C\widetilde{\to}(C\otimes_{Shv(S)}\mathrm{QCoh}(S))\otimes_{\mathrm{QCoh}(S)}\mathcal{A}-mod(\mathrm{QCoh}(S))$. Since both $\mathcal{A}-mod(\mathrm{QCoh}(S))$ and $C\otimes_{Shv(S)}\mathrm{QCoh}(S)$ are dualizable in $\mathrm{QCoh}(S)-mod$, their tensor product is also dualizable in $\mathrm{QCoh}(S)-mod$, hence dualizable in DGCat_{cont}. I have not checked the fact that $Shv(S)\widetilde{\to}\mathcal{A}-mod(\mathrm{QCoh}(S))$. The dualizability of $\mathcal{A}-mod(O)$ in O-modules for some $O\in Alg(\mathrm{DGCat}_{cont})$ is in ([10], 4.7.1). $\square$

Recall that for any $S\in \mathrm{Sch}^{aff}_{ft}$, $Shv(S)$ is dualizable. So, for a morphism $f: S'\to S$ in Sch^{aff}_{ft}, the functor $Shv(S)-mod\to Shv(S')-mod$, $\mathcal{E}\mapsto \mathcal{E}\otimes_{Shv(S)}Shv(S')$ preserves limits for $\mathcal{D}$-modules. For this reason, for any $Y\in \mathrm{PreStk}_{lft}$, $ShvCat_{/Y}$ admits small limits for $\mathcal{D}$-modules.

3.7.1. Consider a closed immersion $f: Y\hookrightarrow X$ of schemes. Then, for any theory of sheaves, $f_!: Shv(Y)\to Shv(X)$ is fully faithful by ([8], 1.5.2 and 7.4.11), and actually a retract of $Shv(X)$. Note that $Shv(X)$ is dualizable in $Shv(X)-mod$. The assumptions of ([22], 3.1.10) are satisfied, because $Shv(X)-mod$ admits small colimits, and the tensor product preserves small colimits separately in each variable. So, $Shv(Y)$ is dualizable in $Shv(X)-mod$, and is self-dual in $Shv(X)-mod$.

We especially need this for closed immersions $X^J\hookrightarrow X^I$ corresponding to surjections of finite nonempty sets $I\to J$ for establishing factorizable Satake.

If $j: U\hookrightarrow X$ is an open immersion, $X\in Sch_{ft}$ then $Shv(U)$ is a retract of $Shv(X)$. So, by ([22], 3.1.10), $Shv(U)$ is dualizable in $Shv(X)-mod$ and is actually self-dual in $Shv(X)-mod$.

3.7.2. Consider any of our 4 sheaf theories. Let $S\in\mathrm{Sch}^{aff}_{ft}$, $C\in Shv(S)-mod$, here we view $Shv(S)$ with the $\otimes^!$-symmetric monoidal structure. Recall that $Shv(S)$ is compactly generated in any sheaf theory. As in ([31], B.5.1), Sam proposes the following.
Definition. (Sam Raskin) An object $c\in C$ is ULA iff the functor $\mathcal{H}om_C(c,-): C\to Shv(S)$ is continuous and $Shv(S)$-linear. Here $\mathcal{H}om_C$ denotes the inner hom with

respect to the monodal category $Shv(S)$. Since $Shv(S)$ is presentable, this inner hom automatically exists. Moreover, for any $x \in C$, $K \in Shv(S)$ we have a canonical map

$$K \otimes \mathcal{H}om_C(c, x) \to \mathcal{H}om_C(c, x \otimes K) \tag{37}$$

Indeed, it comes from the natural morphism $K \otimes \mathcal{H}om_C(c, x) \otimes c \to x \otimes K$ (the latter comes from $\mathcal{H}om_C(c, x) \otimes c \to x$). The $Shv(S)$-linearity means that (37) is an isomorphism for any $K \in Shv(S)$.

Remark 3.7.3. *Let $C \in Shv(S) - mod$, $c \in C$. If c is ULA then for any $M \in Shv(S) - mod$ and $m \in M^c$, the product $c \boxtimes_{Shv(S)} m$ is compact in $C \otimes_{Shv(S)} M$.*

Proof. as in ([31], B.5.1). By assumption, the functor $Shv(S) \to C$, $K \mapsto K \otimes c$ has a continuous right adjoint. □

Remark 3.7.4. *If $L : \mathcal{C} \to \mathcal{D}$ is a $Shv(S)$-linear continuous functor admitting a $Shv(S)$-linear continuous right adjoint then L sends ULA objects to ULA objects.*

Lemma 3.7.5. *Let $j : U \hookrightarrow S$ be an open immersion, $S \in \mathrm{Sch}_{ft}$, $C \in Shv(S) - mod$, $F \in C$ be ULA over S. Then for any $G \in Shv(U)$, $j_!(G) \otimes^! F \widetilde{\to} j_!(G \otimes^! F)$. In particular, $j_!(F)$ is defined for the partially defined left adjoint $j_! : C_U := C \otimes_{Shv(S)} Shv(U) \to C$ to $j^!$.*

Proof. First, without any ULA assumptions, for any $F', K \in C$,

$$j_*(\omega_U \otimes \mathcal{H}om_C(F', K)) \widetilde{\to} j_* j^! \mathcal{H}om_C(F', K) \widetilde{\to} (j_*\omega_U) \otimes^! \mathcal{H}om_C(F', K)$$

in C.

Since F s ULA, we get in addition $(j_*\omega_U) \otimes \mathcal{H}om_C(F', K) \widetilde{\to} \mathcal{H}om_C(F, (j_*\omega_U) \otimes^! K)$. Now for any $\tilde{F} \in C$,

$$\begin{aligned}\mathrm{Hom}_C(j_! G \otimes^! F, \tilde{F}) &\widetilde{\to} \mathcal{H}om_{Shv(S)}(j_! G, \mathcal{H}om_C(F, \tilde{F})) \widetilde{\to} \mathcal{H}om_{Shv(U)}(G, j^! \mathcal{H}om_C(F, \tilde{F})) \\ &\widetilde{\to} \mathcal{H}om_{Shv(U)}(G, \mathcal{H}om_{C_U}(j^! F, j^! \tilde{F})) \widetilde{\to} \mathcal{H}om_{C_U}(G \otimes^! j^! F, j^! \tilde{F})\end{aligned}$$

as desired. □

Definition 3.7.6. *(Sam Raskin) Let $S \in \mathrm{Sch}_{ft}$, $C \in Shv(S) - mod$. Say that C is ULA if it is compactly generated as a $Shv(S)$-module category by ULA objects. That is, C is generated by objects of the form $c \otimes m$ with $c \in C$ ULA and $m \in Shv(S)^c$.*

Write $\mathcal{C}^{ULA} \subset \mathcal{C}$ for the full subcategory of ULA objects. For any sheaf theory $\mathcal{C}^{ULA} \subset \mathcal{C}^c$. Indeed, for $\mathcal{D}$-modules this is ([31], B.4.2), and in the constructible context this follows from the fact that ω_S is compact, see below. Moreover, If C is ULA over $Shv(S)$ then for any $c \in C^{ULA}, K \in Shv(S)^c$, $K \otimes^! c \in C^c$.

Proposition 3.7.7. *Let $\mathcal{C} \in Shv(S) - mod$ be ULA and $F : \mathcal{C} \to \mathcal{D}$ be a map in $Shv(S) - mod$. Then F has a $Shv(S)$-linear continuous right adjoint iff $F(\mathcal{C}^{ULA}) \subset \mathcal{D}^{ULA}$.*

More generally, assume $C_0 \subset \mathcal{C}^{ULA}$ is a full subcategory such that the objects of the form $c \otimes_{Shv(S)} F$ for $c \in C_0, F \in Shv(S)^c$ generate $\mathcal{C}$. If $F(C_0) \subset \mathcal{D}^{ULA}$ then F has a $Shv(S)$-linear continuous right adjoint.

Proof. For $\mathcal{D}$-modules this is ([31], B.7.1), and the proof of *loc.cit.* holds for constructible context also. □

Proposition 3.7.8. *Let $j : U \hookrightarrow S$ be an open subscheme, the compement to the closed immersion $i : Z \to S$. Let $f : C \to D$ be a morphism in $Shv(S) - mod$, assume C is ULA over $Shv(S)$. Then f is an isomorphism iff f preserves ULA objects and induces equivalences*

$$C \otimes_{Shv(S)} Shv(U) \to D \otimes_{Shv(S)} Shv(U), \;\; C \otimes_{Shv(S)} Shv(Z) \to D \otimes_{Shv(S)} Shv(Z)$$

Instead of preservation of ULA objects, it suffices to require that f admits a $Shv(S)$-linear continuous right adjoint.

Proof. as in ([31], B.8.1) □

3.7.9. Consider a sheaf theory from [14] in the constructible context. Let $S \in \mathrm{Sch}_{ft}$ be separated. Sam claims the Verdier duality gives an equivalence $Shv(S)^c \widetilde{\to} (Shv(S)^c)^{op}$, hence by passing to Ind, an equivalence $\mathrm{Shv}(S) \widetilde{\to} Shv(S)^\vee$. What about D-module case?

Consider the diagonal map $\delta : S \times S \to S$. The functor $\delta^! : Shv(S \times S) \to Shv(S)$ preserves compact objects. Indeed, it identifies with $\mathbb{D}\delta^*\mathbb{D}$, so it suffices to show that δ^* preserves compact objects. This is true, because it has a continuous right adjoint $\delta_* = \delta_!$.

(In the D-module case Lin and Sam claim that $\delta^!$ does not preserve compact objects. CHECK!!!)

Consider the tensor product functor $m : Shv(S) \otimes Shv(S) \to Shv(S)$, $K_1 \boxtimes K_2 \mapsto \delta^!(K_1 \boxtimes K_2)$. If $K_i \in Shv(S)^c$ then $\delta^!(K_1 \boxtimes K_2)$ is compact. Since $Shv(S) \otimes Shv(S)$ is generated by compact objects of the form $K_1 \boxtimes K_2$ with $K_i \in Shv(S)^c$, we obtain by ([22], 4.2.3) that the right adjoint m^R to m is continuous.

The failure of rigidity of $Shv(S)$ in the constructible context comes from the fact that certain compact objects are not dualizable. Example of Lin Chen: let S be a smooth scheme of finite type, $x \in S, j : S - x \hookrightarrow S$. Let δ_x denotes the delta sheaf supported at x. It is not dualizable. Indeed, assume it is dualizable, let $M = (\delta_x)^\vee$. Then for $F, N \in Shv(S)$ we get $\mathrm{Map}(F \otimes^! \delta_x, N) \widetilde{\to} \mathrm{Map}(F, N \otimes^! M)$, where $\mathrm{Map}_{Shv(S)} = \mathrm{Map}$. Taking $y \in S$ closed with $y \neq x$ and $F = \delta_y$, we get by base change for proper morphisms $\delta_y \otimes^! \delta_x = 0$, so $\mathrm{Map}_{\mathrm{Vect}}(k, i_y^!(N \otimes^! M)) = *$. We could similarly take $\delta_y[n]$ for any $n \in \mathbb{Z}$, which shows that $i_y^!(N \otimes^! M) = 0$ (see [22], 9.2). On the other hand, take $F = j_!\omega_{S-x}$ and $N = \omega_S$, here ω_S is the dualizing complex of S. Then $\mathrm{Map}(j_!\omega_{S-x}) \otimes^! \delta_x, \omega_S) \widetilde{\to} \mathrm{Map}(\Omega_{S-x}, j^! M)$ is nontrivial. Indeed, $\mathbb{D}(j_!\omega_{S-x}) \otimes^! \delta_x) \widetilde{\to} \vartriangle^* (j_*\bar{\mathbb{Q}}_\ell \boxtimes (i_x)_!\bar{\mathbb{Q}}_\ell)$ is nonzero. Here we denoted by $\bar{\mathbb{Q}}_\ell$ the corresponding "constant sheaves", that is, $\mathbb{D}\omega$.

In the constructible context (at least) for $S \in \mathrm{Sch}_{ft}$, the dualizing sheaf $\omega_S \in Shv(S)$ is compact. Indeed, the functor $Shv(S) \to \mathrm{Vect}$, $M \mapsto \mathrm{R}\Gamma_c(S, \omega_S \otimes \mathbb{D}(M))$ is continuous. We have $\mathrm{RHom}(\omega_S, M) \widetilde{\to} \mathbb{D}\,\mathrm{R}\Gamma_c(S, \omega_S \otimes \mathbb{D}(M))$.

3.7.10. If $f : Y \to Z$ is a morphism in PreStk_{lft} and $C \in ShvCat(Z)$ then for any sheaf theory there is a natural functor $\Gamma(Z, C) \to \Gamma(Y, f^*C)$. Indeed, $\Gamma(Z, C)$ is the value on Z of the functor $((\mathrm{PreStk}_{lft})_{/Z})^{op} \to \mathrm{DGCat}_{cont}$, which is the RKE of $((\mathrm{Sch}^{aff}_{ft})_{/Z})^{op} \to$

DGCat$_{cont}$, $S \mapsto \Gamma(S, C)$. Since this RKE is a functor, it yields the desired functor. It may happen that the right adjoint to the restriction $ShvCat(Z) \to ShvCat(Y)$ does not exist, I think.

Assume Y, Z are pseudo-indschemes, and $f : Y \to Z$ is pseudo-indproper in the sense of ([30], 7.15.1). Let $C \in ShvCat(Z)$. Then the restriction functor $\Gamma(Z, C) \to \Gamma(Y, f^*C)$ admits a left adjoint $f_{*,C} : \Gamma(Y, f^*C) \to \Gamma(Z, C)$. Same proof as in ([30], 7.15.5). Namely, let $Z = \operatorname{colim}_{j\in J} Z_j$, where the transition maps $\alpha : Z_j \to Z_{j'}$ are proper, and each $Z_i \in \mathrm{Sch}_{ft}$ (separated). Recall that Z_i is 1-affine, and we have the adjoint pair $\alpha_! : Shv(Z_j) \rightleftarrows Shv(Z_{j'}) : \alpha^!$ in $Shv(Z_{j'}) - mod$. Tensoring this adjoint pair by $\Gamma(Z_{j'}, C)$, we get an adjoint pair $\alpha_{!,C} : \Gamma(Z_j, C) \rightleftarrows \Gamma(Z_{j'}, C) : \alpha^{!,C}$. Assume now $I \to J$ is a diagram, and $Y = \operatorname{colim}_{i\in I} Y_i$, here Y_i is a separated scheme of finite type, and the transition maps $Y_i \to Y_{i'}$ are proper. Then $\Gamma(Y, C) \widetilde{\to} \operatorname{colim}_{i\in I} \Gamma(Y_i, C)$. The desired functor $f_{*,C}$ is obtained from the compatible system of functors $\beta_{!,C} : \Gamma(Y_i, C) \to \Gamma(Z_{j(i)}, C)$. Here the corresponding morphism $\beta : Y_i \to Z_{j(i)}$ is proper.

Compare with ([22], 9.2.21).

3.7.11. If we have a cartesian square

$$\begin{array}{ccc} X & \overset{f}{\leftarrow} & Y \\ \downarrow g & & \downarrow h \\ X' & \overset{t}{\leftarrow} & Y' \end{array}$$

then it can not be true that $f^! g^* \widetilde{\to} t^! h^*$. For example take $t = g : \operatorname{Spec} k \overset{0}{\to} \mathbb{A}^1$. Then $g^!$ is different from g^*. Here we have taken the fibre product in the sense of non-derived algebraic geometry (but the derived geometry does not cure this).

3.8. More for version June 1, 2020.

3.8.1. In Th. A.3.3 the quotient $\mathrm{Gr}_{T_1} / \mathrm{Gr}_{T_2}$ is understood in the topos of prestacks, using the fact that $\mathrm{Gr}_{T_1} \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$.

3.8.2. For A.3.6. Let $S \in \mathrm{Sch}^{aff}_{ft}$. If $I \in \mathrm{Ran}(S)$ and $\mathcal{G}$ is a μ_n-gerbe on $S \times X$ (with a trivilization over U_I then localizing in etale topology of S), there is a line bundle $\mathcal{L}$ on $S \times X$ and an isomorphism $\mathcal{L}^{\frac{1}{n}} \widetilde{\to} \mathcal{G}$ over $S \times X$. Indeed, for $\mathrm{pr} : X \times S \to S$, we have $\mathrm{pr}_* \mu_n \widetilde{\to} \mu_n \oplus \mathrm{H}^1(X, \mu_n)[-1] \oplus \mathrm{H}^2(X, \mu_n)[-2]$. Localizing in the etale topology of S, our class in $\mathrm{H}^2_{et}(S \times X)$ comes from an element of $\mathrm{H}^2(X, \mu_n)$. However, the map $\mathrm{H}^1(X, \mathcal{O}^*) \to \mathrm{H}^2(X, \mu_n) \widetilde{\to} \mathbb{Z}/n\mathbb{Z}$ coming from the Kummer sequence $1 \to \mu_n \to \mathcal{O}^* \to \mathcal{O}^* \to 1$ is surjective: if L is a line bundle of degree 1 on X then $\mathcal{L}^{\frac{1}{n}}$ equals $1 \in \mathbb{Z}/n\mathbb{Z}$.

3.8.3. For A.3.6. If Γ is a finite abelian group of order coprime to $char(k)$ and $S \in \mathrm{Sch}_{ft}$ is smooth and separated then S-points of $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ is the set: $I \in \mathrm{Ran}(S)$ and a map $I \to \Gamma$. More generally, the same holds for S irreducible if for $i \neq j \in I$, $\Gamma_i \cap \Gamma_j$ is of dimension $< \dim S$.

As in Sect. A.3.1 of the paper, present our S-point of $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ by an element of H^0 of $C^\bullet(\Gamma_I, \pi^!(\Gamma))$, here $\pi : \Gamma_I \to S$ is the projection. Note that $\pi^!(\Gamma)$ is placed in cohomological degrees ≥ 0. Consider the map $s : \sqcup_{i\in I} S \to \Gamma_I$ whose i-th component is the natural map $\Gamma_i \to \Gamma_I$. We have a natural map

$$s_!(\Gamma) = s_! s^! \pi^!(\Gamma) \to \pi^!(\Gamma)$$

Localizing S if necessary in the topology of finite surjective maps, assume S reduced irreducible. Let also assume that for $i \neq j \in I$, $\Gamma_i \cap \Gamma_j$ is of dimension $< \dim S$. We claim that the obtained map $\eta : s_!(\Gamma) \to \tau^{\leq 0}\pi^!(\Gamma)$ is an isomorphism.

Indeed, the usual constructible sheaf $\tau^{\leq 0}\pi^!(\Gamma)$ has no subsheaves supported on closed subschemes of dimension $< \dim S$, because for such a sheaf F we have $\mathrm{Hom}(\pi_! F, \Gamma) = 0$. This means that this sheaf is the nonderived $*$-extension of its restriction to Γ_I with all the intersections $\Gamma_i \cap \Gamma_j$ removed (for $i \neq j$). This gives $\mathrm{Map}(S, \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \widetilde{\to} \mathrm{Map}(I, \Gamma)$ in this case.

3.8.4. Let Γ be a finitely generated abelian group of order coprime to $char(k)$. As in Section 3.1 of the paper, one constructs a map

$$\mathrm{Map}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B_{et}(\Gamma\otimes\mathbb{G}_m)\times X, B^4_{et}(A(1))\times X) \to \mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$$

Is it an isomorphism? What are the homotopy groups of the LHS? See below.

3.9. More on Appendix A.

3.9.1. Maybe add the following in Appendix A?

Let Γ be a finitely generated abelian group of order coprime to $char(k)$. Define $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m,comb}$ similarly to the case of a torus. Namely, consider the index category $\mathcal{C}$ whose objects are pairs (I, λ^I) with I a finite non-empty set, $\lambda^I : I \to \Gamma$. Write λ_i for the value of λ^I on i. A map from (J, λ^J) to (I, λ^I) in $\mathcal{C}$ is a surjection $\phi : I \to J$ such that $\lambda_j = \sum_{\phi(i)=j} \lambda_i$. Set $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m,comb} = \underset{(I,\lambda^I)\in\mathcal{C}}{\mathrm{colim}}\, X^I$.

Pick a section of

$$\Gamma \to \Gamma/\Gamma^{tors} \tag{38}$$

We get a decomposition $\Gamma \widetilde{\to} \Gamma^{free}\times\Gamma^{tors}$. So,

$$\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m} \widetilde{\to} \mathrm{Gr}_{\Gamma^{free}\otimes\mathbb{G}_m}\times \mathrm{Gr}_{\Gamma^{tors}\otimes\mathbb{G}_m}$$

There is a natural map

$$\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m,comb} \to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m} \tag{39}$$

Namely, for $(I,\lambda^I)\in\mathcal{C}$ our λ^I is a pair $\lambda^{I,free} : I \to \Gamma^{free}$, $\lambda^{I,tors} : I \to \Gamma^{tors}$. We have the evident map

$$\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m,comb} \to \mathrm{Gr}_{\Gamma^{free}\otimes\mathbb{G}_m,comb}\times \mathrm{Gr}_{\Gamma^{tors}\otimes\mathbb{G}_m,comb}$$

We have already constructed the map $\mathrm{Gr}_{\Gamma^{free}\otimes\mathbb{G}_m,comb} \to \mathrm{Gr}_{\Gamma^{free}\otimes\mathbb{G}_m}$ in the paper.

The map

$$\mathrm{Gr}_{\Gamma^{tors}\otimes\mathbb{G}_m,comb} \to \mathrm{Gr}_{\Gamma^{tors}\otimes\mathbb{G}_m} \tag{40}$$

is constructed as follows. For each $(I, \lambda^I : I \to \Gamma^{tors})$ apply Lemma 1.4.5 of the paper, which shows that λ^I gives a point of

$$\mathrm{Ge}_{\Gamma^{tors}(1)}(X^I\times X)\times_{\mathrm{Ge}_{\Gamma^{tors}(1)}(U_I)} *$$

These maps as $(I, \lambda^I : I \to \Gamma^{tors})$ vary define the desired map (40). Composing the above, one gets the map (39). I think this construction does not depend on a choice of a section of (38).

The map (39) is compatible with the factorization structures. Moreover, (39) is a map of factorization group prestacks over Ran. The map (39) is a monomorphism of prestacks.

From Section 3.8.3 of this file we see that (39) is surjective after sheafification in the topology of finite surjective maps. For $i \geq 0$, $B^i_{et}(A)$ is a sheaf for the topology generated by finite surjective maps. This implies the following.

Proposition 3.9.2. *The map (39) becomes an isomorphism after sheafification in the topology of finite surjective maps.* $\square$

Remark 3.9.3. *We can also inverse the logic now and derive Theorem A.3.3 of the paper from the surjectivity, after sheafification in the topology of finite surjective maps, of the map* $\mathrm{Gr}_{T_1,comb} \to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$. *This would avoid Th. A.3.7 completely! This would simplify the proof, I think.*

3.9.4. As in Section 4.1.3 of the paper, we obtain an exactly similar combinatorial description of $\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m,comb})$:

For a finite set I and a map $\lambda^I : I \to \Gamma$ we specify a gerbe $\mathcal{G}^{\lambda^I}$ on X^I. For a surjection of finite sets $\phi : I \to J$ such that $\lambda_j = \sum_{\phi(i)=j} \lambda_i$, we specify an isomorphism $\nu : (\triangle_\phi)^* \mathcal{G}^{\lambda^I} \widetilde{\to} \mathcal{G}^{\lambda^J}$. These isomorphism are equipped with the compatibility data for composition of surjections of finite sets. We are also given factorization data for $\phi : I \to J$ compatible with compositions of surjections of finite sets, and compatible with maps ν.

The claim from Section 4.1.4 of the paper also extends to the case of $\Gamma\otimes\mathbb{G}_m$ I think. The consruction of $q : \Gamma \to A(-1)$ from Section 4.2 of the paper extends to this case as is.

This would help to understand Corollary 4.7.5 of the paper, whose proof was omited.

I think now the content of Sect. 4.3-4.4 of the papers extends to the case of Λ replaced by any Γ.

One more thing, we may define $\Theta(\Gamma)$ as in Section 4.5.1 of the paper for any finitely generated abelian group Γ. Let

$$\Theta^0(\Gamma) \widetilde{\to} \mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$$

be the fibre of the projection $\Theta(\Gamma) \to \mathrm{Quad}(\Gamma, A(-1))$. We then get

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \widetilde{\to} \mathrm{Map}_{\mathrm{PreStk}}(X, B^2_{et}(\mathrm{Hom}(\Gamma, A)),$$

this is also claimed in Cor. 4.7.9 of the paper.

3.9.5. For Remark 4.7.7. He means by Ab the following. Consider the category of chain complexes of abelian groups as a DG-category over $\mathbb{Z}$ first, to which we apply the construction of a DG-nerve in the sense of ([19], 1.3.1.6), which is an ∞-category by ([19], 1.3.1.10). This is Ab.

Dennis claims that $\mathrm{Map}_{Ab}(\Gamma, B^2(A))$ has homotopy groups only in degrees $1, 2$. Recall that for Λ a free abelian group of finite type $\mathrm{Map}_{Ab}(\Lambda; B^2(A)) \widetilde{\to} B^2(\mathrm{Hom}(\Lambda, A))$.

We have $\pi_2 \operatorname{Map}_{Ab}(\Gamma, B^2(A)) \widetilde{\to} \operatorname{Map}_{Ab}(\Gamma, A) = \operatorname{Hom}(\Gamma, A)$, and

$$\pi_1 \operatorname{Map}_{Ab}(\Gamma, B^2(A)) \widetilde{\to} \pi_0 \operatorname{Map}_{Ab}(\Gamma, B(A)) \widetilde{\to} \operatorname{Ext}^1(\Gamma, A),$$

the Ext calculated in the category of abelian groups.

If $0 \to \Lambda_2 \to \Lambda_1 \to \Gamma \to 0$ is an exact sequence in abelian groups, Λ_i are free if finite type then $\operatorname{Map}_{Ab}(\Gamma, B^2(A)) \to \operatorname{Map}_{Ab}(\Lambda_1, B^2(A)) \to \operatorname{Map}_{Ab}(\Lambda_2, B^2(A))$ is a fibre sequence. The long exact sequence of π_i then shows that $\pi_0 \operatorname{Map}_{Ab}(\Gamma, B^2(A)) = 0$, because $\operatorname{Map}_{Ab}(\Lambda_i, B^2(A)) \widetilde{\to} B^2(\operatorname{Hom}(\Lambda_i, B^2(A)))$.

3.9.6. For 4.5.2: if A, D are abelian groups then view $B(A) \times D$ as a monoidal category. To provide a braiding on it is equivalent to giving a bilinear form $b : D \times D \to A$. This braided monoidal category is then symmetric iff b takes values in A_{2-tors}.

3.9.7. For Γ a finitely generated abelian group let $\operatorname{Bun}_{\Gamma\otimes\mathbb{G}_m}$ be the stack sending $S \in \operatorname{Sch}^{aff}_{ft}$ to $\operatorname{Map}(S \times X, B_{et}(\Gamma \otimes \mathbb{G}_m)$.

Assume Γ finite. Then there is a natural map $\operatorname{Map}(S, \operatorname{Bun}_{\Gamma\otimes\mathbb{G}_m}) \to \operatorname{Map}(S, \Gamma_{et})$. Namely, $B_{et}(\Gamma\otimes\mathbb{G}_m) \widetilde{\to} B^2_{et}(\Gamma(1))$. Since $\mathrm{H}^2(X, \Gamma(1)) \widetilde{\to} \Gamma$, we get a morphism as above. For $\gamma \in \Gamma$ write $\operatorname{Bun}^{\gamma}_{\Gamma\otimes\mathbb{G}_m}$ for the substack given by requiring that $S \to \Gamma_{et}$ equals γ. We have the projection $\operatorname{Gr}_{\Gamma\otimes\mathbb{G}_m} \to \operatorname{Bun}_{\Gamma\otimes\mathbb{G}_m}$. Let $\operatorname{Gr}^{\gamma}_{\Gamma\otimes\mathbb{G}_m}$ be the preimage of $\operatorname{Bun}^{\gamma}_{\Gamma\otimes\mathbb{G}_m}$.

3.10. For 4.9.1. If $S \in \operatorname{Sch}^{aff}_{ft}$ with $S \to \operatorname{Gr}^a_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$ for some $a \in \mathbb{Z}/2\mathbb{Z}$, assume the composition $S \to \operatorname{Gr}^a_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m} \to \operatorname{Ran}$ is lifted to $\operatorname{Ran}^J_{disj}$. Suppose for $j \in J$ the j-th map $S \to \operatorname{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$ coming from the factorization takes values in $\operatorname{Gr}^{a_j}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$ for $a_i \in \mathbb{Z}/2\mathbb{Z}$. So, $a = \sum_j a_j$. Let $\mathcal{G}$ be the trivial μ_2-gerbe. What is the factorization isomorphism $\mathcal{G} \widetilde{\to} \mathcal{G}^{\boxtimes J}$ over

$$(\prod_{j\in J} \operatorname{Gr}^{a_j}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}) \times_{\operatorname{Ran}^J} \operatorname{Ran}^J_{disj}$$

It is given by some μ_2-torsor. What is this torsor?

My impression is that this is just the torsor sending a finite set J to the set of orders of J up to an even permutation.

3.10.1. For 7.1.2. Recall that $\operatorname{Ind}(\operatorname{Sch}^{aff}) \subset \operatorname{PreStk}$ is a full subcategory by definition from HTT.

3.10.2. Write $\operatorname{Grpd}(\mathcal{C})$ for the category of groipoids in an ∞-category $\mathcal{C}$. A multiplicative A-gerbe on $Z \in \operatorname{Grpd}(\operatorname{PreStk})$ is an element in $\operatorname{Map}_{\operatorname{Grpd}(\operatorname{PreStk})}(Z, B^2_{et}(A))$.

3.10.3. For 7.3.3. The multiplicativity of this gerbe is obtained as follows. We have the composition map $m : \operatorname{Hecke}^{loc}_G \times_{\mathfrak{L}^+(G)\backslash \operatorname{Ran}} \operatorname{Hecke}^{loc}_G \to \operatorname{Hecke}^{loc}_G$. We want to construct an isomorphism $m^{\mathcal{G}} \widetilde{\to} \mathcal{G}\boxtimes\mathcal{G}$. A point of the LHS is a collection $\mathcal{F}^i : \mathcal{D}_I \to B_{et}(G)$ for $i = 1, 2, 3$ and isomorphisms $\mathcal{F}^i \widetilde{\to} \mathcal{F}^{i+1} \mid_{\overset{\circ}{\mathcal{D}}_I}$. The section of $C^{\bullet}_{et}(\Gamma_I, \hat{\iota}^! A(1))$ corresponding to $(\mathcal{F}^1, \mathcal{F}^3)$ is the sum of the sections corresponding to $(\mathcal{F}^1, \mathcal{F}^2)$ and $(\mathcal{F}^2, \mathcal{F}^3)$. The compatibilty with the factorization follows from the corresponding decomposition $\Gamma_I = \sqcup \Gamma_{I_i}$ when I is decomposed into I_i.

3.10.4. For the proof of 7.3.5. If $S\in \mathrm{Sch}^{aff}$, $I\in \mathrm{Ran}(S)$ then $\Gamma_I\to S$ is flat. Indeed, consider first the case of $S=X^I$ for a finite set I, let $D\subset X^I\times X$ be the union of $\triangle_i$, here $\triangle_i$ is the locus, where i-th coordinate coincides with the last one. Then D is an effective Cartier divisor, hence is flat over X^I. The general case is obtained by the base change under $S\to X^I$.

3.10.5. For 7.5.1, first claim: To understand the structure of $\mathfrak{L}^+(T)_X$-equivariance on an A-gerbe $\mathcal{G}_X$ on $\Lambda\times X$, he means

$$\mathrm{Map}_{\mathcal{G}\mathrm{rp}(\mathrm{PreStk}_{/X})}(\mathfrak{L}^+(T)_X, B_{et}(A)\times X)\widetilde{\to}\mathrm{Map}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(X/\mathfrak{L}^+(T)_X, B^2_{et}(A)\times X)$$

The LHS gives a multiplicative A-torsor on $\mathfrak{L}^+(T)_X$.

3.11. Ideas from Sam, twistings.

3.11.1. If A is a finite group then $\mathrm{Shv}(B(A))\widetilde{\to}\mathrm{QCoh}(B(A))$ in our case, where the sheaf theory is $\mathcal{D}$-modules. Indeed, $B(A)\widetilde{\to}\mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}} A^n$, hence

$$\mathrm{Shv}(B(A))\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{Shv}(A^n)$$

Now for a finite union of points $Y=\sqcup_{i\in I}\mathrm{Spec}\, k$, $\mathrm{Shv}(Y)\widetilde{\to}\prod_{i\in I}\mathrm{Vect}\widetilde{\to}\mathrm{QCoh}(Y)$. In turn, $\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{QCoh}(A^n)\widetilde{\to}\mathrm{QCoh}(B(A))$.

His idea is that $B(A)$ and $B_{et}(A)$ should be 1-affine for any sheaf theory. (In the setting of quasi-coherent sheaves this is [9, Thereom 2.2.2, Remark 2.5.2]). Indeed, since $ShvCat:(\mathrm{PreStk}_{lft})^{op}\to 1-\mathcal{C}\mathrm{at}$ preserves limits, we have

$$ShvCat(B(A))\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}} ShvCat(A^n)$$

If the sheaf theory is $\mathcal{D}$-modules then, since $ShvCat(A^n)\widetilde{\to}\mathrm{QCoh}(A^n)-mod$, $ShvCat(B(A))$ identifies with the same category in the setting of quasi-coherent sheaves. However, in the latter case we know that $B(A)$ is 1-affine, so $ShvCat(B(A))\widetilde{\to}\mathrm{QCoh}(B(A))-mod(\mathrm{DGCat}_{cont})$. Thus, $B(A)$ is 1-affine in this case.

For other sheaf theory we get $ShvCat(B(A))\widetilde{\to}Rep(A)-mod(\mathrm{DGCat}_{cont})$, where now the field of coefficients is E, maybe different from k. Here $Rep(A)=\mathrm{QCoh}(B(A))$ with coefficients in E.

Recall also that $\mathrm{QCoh}(B(A))\widetilde{\to}\mathrm{QCoh}(B_{et}(A))$ by ([15], I.3, 1.3.8).

3.11.2. Let A be a finite abelian group. For the trivial torsor $q:\mathrm{Spec}\, k\to B^2(A)$ consider the induced restriction map $cores_q: ShvCat(B^2(A))\to ShvCat(\mathrm{Spec}\, k)=\mathrm{DGCat}_{cont}$. We want to check it is comonadic and calculate the corresponding comonade.

The functor $ShvCat:(\mathrm{PreStk}_{lft})^{op}\to 1-\mathcal{C}\mathrm{at}$ preserves limits. Since

$$B^2(A)\widetilde{\to}\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, B(A)^n,$$

we get

$$ShvCat(B^2(A))\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}} ShvCat(B(A)^n)\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{QCoh}(B(A)^n)-mod \tag{41}$$

Write $ShvCat_{qcoh}(Y)$ for the category of sheaves of categories on a prestack Y in the quasi-coherent setting. We conclude that $ShvCat(B^2(A))\widetilde{\to}ShvCat_{qcoh}(B^2(A))$

naturally. In the setting of quasi-coherent sheaves we know that $cores_q$ admits a right adjoint $coind_q$, hence the same holds for any sheaf theory. Note that q is 1-affine in the sense of ([30], A.8), because $B(A)$ is 1-affine. So, $coind_q$ preserves small colimits and is a morphism of $ShvCat(B^2(A))$-module categories, that is, satisfies the projection formula by ([30], Pp. A.9.1(2)). Since $B(A)$ is 1-affine, from ([9], Lemma 3.2.4) we get $\mathrm{QCoh}(B(A)^n)\widetilde{\to}\mathrm{QCoh}(B(A))^{\otimes n}$.

Consider the cosimplicial category

$$\mathrm{DGCat}_{cont} \rightrightarrows \mathrm{QCoh}(B(A)) - mod \substack{\to\\ \to\\ \to} \mathrm{QCoh}(B(A)^2) - mod \ldots],$$

given by (41). It suffices to check that this cosimplicial category satisfies the comonadic Beck-Chevalley condition ([9], Def. C.1.2). For each $i \geq 0$ consider the projection $pr : B(A)^{i+1} \to B^i(A)$ forgetting the last factor. We must check the corresponding functor $\mathrm{pr}^* : ShvCat_{qcoh}(B(A)^i) \to ShvCat_{qcoh}(B(A)^{i+1})$ admits a right adjoint pr_*. This follows from ([30], Lm. A.9.1). For every map $\alpha : [j] \to [i]$ in $\boldsymbol{\Delta}$ let $\alpha + 1 : [j+1] \to [i+1]$ be the map given by α on $\{0, \ldots, j\}$ and sending $j+1$ to $i+1$. Write $T^\alpha : ShvCat(B(A)^j) \to ShvCat(B(A)^i)$ for the corresponding transition functor in the above cosimplicial category. We must check that the natural transformation in the diagram

$$\begin{array}{ccc} ShvCat(B(A)^i) & \stackrel{T^\alpha}{\leftarrow} & ShvCat(B(A)^j) \\ \uparrow \mathrm{pr}_* & & \uparrow \mathrm{pr}_* \\ ShvCat(B(A)^{i+1}) & \stackrel{T^{\alpha+1}}{\leftarrow} & ShvCat(B(A)^{j+1}) \end{array}$$

is an isomorphism. In other words, for the corresponding diagram

$$\begin{array}{ccc} B(A)^j & \stackrel{q_\alpha}{\leftarrow} & B(A)^i \\ \uparrow \mathrm{pr} & & \uparrow \mathrm{pr} \\ B(A)^{j+1} & \stackrel{q_{\alpha+1}}{\leftarrow} & B(A)^{i+1} \end{array}$$

we have to show that $(q_\alpha)^* \mathrm{pr}_* \widetilde{\to} \mathrm{pr}_*(q_{\alpha+1})^*$. We have denoted by q_α the corresponding transition morphism in the simplicial object given by the group $B(A)$. This base change follows from ([30], A.9.1(1)). Thus, $cores_q$ is comonadic. The corresponding comonade, by ([9], Lm. C.1.9), is isomorphic, as a plain endo-functor of DGCat_{cont}, to the functor $C \mapsto C \otimes \mathrm{QCoh}(B(A))$.

(Does it also satisfy the monadic Beck-Chevalley condition?)

Now use the fact that $\mathrm{QCoh}(B(A))$ is rigid (in the sense of [9], D.1.1). Consider the product map $m : B(A) \times B(A) \to B(A)$. Since $\mathrm{QCoh}(B(A)^i)$ is rigid for $i = 1, 2$, we may apply ([15], I.3, 3.4.4), it says that $m_* : \mathrm{QCoh}(B(A)^2) \to \mathrm{QCoh}(B(A))$ is continuous, and $m_* \widetilde{\to} (m^*)^\vee$. So, $\mathrm{QCoh}(B(A)) \in CoAlg(\mathrm{DGCat}_{cont})$ identifies with the dual of $\mathrm{QCoh}(B(A))$, where the algebra structure on $\mathrm{QCoh}(B(A))$ is given by the convolution $m_* : \mathrm{QCoh}(B(A)^2) \to \mathrm{QCoh}(A)$. Applying now ([22], 3.2.1-3.2.2), we obtain an equivalence

$$ShvCat(B^2(A)) \widetilde{\to} \mathrm{QCoh}(B(A)) - mod(\mathrm{DGCat}_{cont}), \tag{42}$$

where we use the convolution monoidal structure on $\mathrm{QCoh}(B(A))$.

3.11.3. *Twist of a category by a gerbe.* For a finite abelian group A as above let $\mathcal{C} \in \mathrm{DGCat}_{cont}$ be equipped with a monoidal functor $\tau : B(A) \to \mathrm{Fun}_{k,cont}(\mathcal{C}, \mathcal{C})$. That is, A acts on $\mathcal{C}$ by automorphisms of the identity functor. (For example, if $\mathcal{C} \in CAlg(\mathrm{DGCat}_{cont})$ then we have a version where the input datum is a monoidal functor $\tau : B(A) \to \mathrm{Fun}^{\otimes}_{k,cont}(\mathcal{C}, \mathcal{C})$, the latter category denotes the category of k-linear continuous symmetric monoidal functors from $\mathcal{C}$ to itself).

Since DGCat_{cont} is cocomplete, it is tensored over Spc in the terminology of Lurie, in this sense we have the tensor product $B(A) \otimes \mathrm{Vect} \in \mathrm{DGCat}_{cont}$. This is the colimit of $B(A) \to * \overset{\mathrm{Vect}}{\to} \mathrm{DGCat}_{cont}$. By the universal property of the colimit, τ extends to a map $\bar{\tau} : B(A) \otimes \mathrm{Vect} \to \mathrm{Fun}_{k,cont}(\mathcal{C}, \mathcal{C})$ in $Alg(\mathrm{DGCat}_{cont})$. In turn, $B(A) \otimes \mathrm{Vect}$ as an object of $Alg(\mathrm{DGCat}_{cont})$ identifies with $\mathrm{QCoh}(B(A))$ with the convolution monoidal structure (cf. [22], 9.2.20). So, our $\mathcal{C}$ becomes an object of (42). For any of the 4 sheaf theories, the functor $(\mathrm{PreStk}_{lft})^{op} \to 1-\mathrm{Cat}$, $Y \mapsto ShvCat(Y)$ satisfies etale descent, so we get an object of

$$ShvCat(B^2_{et}(A)) \widetilde{\to} ShvCat(B^2(A))$$

Now given $Y \in \mathrm{PreStk}$ with a map $\mathcal{G} : Y \to B^2_{et}(A)$, we pull back the corresponding sheaf of categories and get the twisted sheaf of categories $\mathcal{C}_{\mathcal{G}}$ on Y.

3.11.4. Explanations from Dennis email of 1.06.2020.

Consider a factorization gerbe $\mathcal{G}^G \in \mathrm{FactGe}_A(\mathrm{Gr}_G)$. The associated dual metaplectic data (without the critical twist) in two particular cases.

i) If we start with $\mathcal{G}^G$ trivial then $H = \check{G}$, $\epsilon = 0$, $\mathcal{Z}$ trivial.

ii) If $\mathcal{G}^G = (\det_{\mathfrak{g}})^{\frac{1}{2}}$ then we get $H = \check{G}$, $\epsilon = (2\rho)(-1) \in Z_H(E)$ for $2\rho : \mathbb{G}_m \to Z_H$, and $\mathcal{G}_Z$ is the extension of scalar via $\epsilon : \mathbb{Z}/2\mathbb{Z} \to Z_H$ of the gerbe of square roots of Ω_X.

This answer is obtained via the procedure of Section 6 of the paper without any critical twist (the latter happens in Section 5 of the paper).

3.11.5. For C.1.2. Let $b \in Bil(\Lambda, A)$ be given by a matrix (b_{ij}) in a base $\{e_i\}$ of Λ, that is, $b(e_i, e_j) = b_{ij}$. Then b is alternating iff $b_{ii} = 0$ and $b_{ij} = -b_{ji}$ for $i < j$.

Recall that $d_1 : Bil(\Lambda, A) \to Bil(\Lambda, A)$ sends b to b' with $b'(\lambda, \mu) = b(\lambda, \mu) - b(\mu, \lambda)$. Then $\mathrm{Ker}(Alt(\Lambda, A) \overset{d_1}{\to} Bil(\Lambda, A)) = Alt(\Lambda, A_{2-tors})$. Any $b \in Alt(\Lambda, A_{2-tors})$ writes as $b(\lambda, \mu) = q(\lambda + \mu) - q(\lambda) - q(\mu)$ for suitable $q \in \mathrm{Quad}(\Lambda, A_{2-tors})$. Indeed, for $q(x) = x_i x_j$ we get $q(x + y) - q(x) - q(y) = x_i y_j + y_i x_j$.

The kernel of $Bin(\Lambda, A) \to Quad(\Lambda, A)$ is $Alt(\Lambda, A)$. Is the map $d_1 : Alt(\Lambda, A) \to Alt(\Lambda, A)$ surjective? Yes, because A is divisible: At the level of matrices, d_1 sends (b_{ij}) to the matrix with ij-term $b_{ij} - b_{ji}$. So, if b is alternating then the matrix of $d_1(b)$ has ij-th term $2b_{ij}$. Since A is divisible, this map is surjective.

3.11.6. For C.4.2, for clarity. For any $b' \in Bilin(\Lambda, A(-1))$ we get a theta datum $\Theta_{b'}$. It attaches to λ the gerbe $\mathcal{G}^{\lambda} = (\omega_X^{-1})^{q(\lambda)}$ for $q(\lambda) = b'(\lambda, \lambda)$ and isomorphisms

$$c_{\lambda_1, \lambda_2} : \mathcal{G}^{\lambda_1 + \lambda_2} \widetilde{\to} \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \otimes (\omega_X^{-1})^{b(\lambda_1, \lambda_2)}$$

Given $b'' \in Bilin(\Lambda, A(-1))$, we get an isomorphism $\phi_{b''} : \Theta_{b'} \widetilde{\to} \Theta_{b' + d_1(b'')}$ given on $\mathcal{G}^{\lambda}$ by $(-1)^{b''(\lambda, \lambda)}$. Now given $q'' \in \mathrm{Quad}(\Lambda, A(-1))$, we get a 2-morphism $\phi_{b''} \to \phi_{b'' + d_2(q'')}$

in $\Theta(\Lambda)$. This 2-morphism is essentially a trivialization, for each $\lambda \in \Lambda$, of the A-torsor $(-1)^{d_2(q'')(\lambda)}$, which squares to the identity. This trivialization, as we have seen in Section 4.2.5 of the paper, is a datum of $c \in A(-1)$ with $2c = d_2(q'')$. Our c is then $q''(\lambda)$.

3.11.7. For C.6. To see that the cohomology in degree -1 of $\tilde{\mathcal{D}}(\Lambda)_G$ is trivial, we have to show that $M \stackrel{d_2}{\to} Alt(\Lambda, A_{2-tors})$ is surjective, where $M = \mathrm{Ker}(\mathrm{Quad}(\Lambda, A_{2-tors}) \to \prod_{i\in I} A_{2-tors})$. Given $b' \in Alt(\Lambda, A_{2-tors})$, let $q \in \mathrm{Quad}(\Lambda, A_{2-tors})$ be any such that $d_2(q) = b'$. We may correct it by an element $\gamma \in \mathrm{Hom}(\Lambda, A_{2-tors})$ with prescribed values on α_i. However, any map $\oplus_i \mathbb{Z}\alpha_i \to A_{2-tors}$ extends to a map $\Lambda \to A$. Why the latter is in values with A_{2-tors}?

3.12. Recall that the functor $f : \mathrm{Spc} \to \mathrm{DGCat}_{cont}$, $X \mapsto X \otimes \mathrm{Vect}$ is symmetric monoidal and preserves colimits ([22], 9.2.20).

Let A be a finite abelian group. Let us show that $B(A) \otimes \mathrm{Vect} \widetilde{\to} \mathrm{QCoh}(B(A))$. Since $B(A) \widetilde{\to} \mathrm{colim}_{[n]\in \boldsymbol{\Delta}^{op}} A^n$ in Spc, and f preserves colimits, we get

$$B(A) \otimes \mathrm{Vect} \widetilde{\to} \underset{[n]\in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, A^n \otimes \mathrm{Vect} \widetilde{\to} \underset{[n]\in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, \mathrm{QCoh}(A^n)$$

Here for a morphism $\beta : [m] \to [n]$ in $\boldsymbol{\Delta}$ and the corresponding morphism $\bar{\beta} : A^n \to A^m$ of finite sets, the corresponding functor $\mathrm{QCoh}(A^n) \to \mathrm{QCoh}(A^m)$ is $\bar{\beta}_*$. It has the right adjoint $\bar{\beta}^*$. We may pass to the right adjoints in the functor $\boldsymbol{\Delta}^{op} \to \mathrm{DGCat}_{cont}, [n] \mapsto \mathrm{QCoh}(A^n)$, and thus we get a functor $\boldsymbol{\Delta} \to \mathrm{DGCat}_{cont}, [n] \mapsto \mathrm{QCoh}(A^n)$. For a morphism $\beta : [m] \to [n]$ in $\boldsymbol{\Delta}$ the corresponding transition functor is $\bar{\beta}^* : \mathrm{QCoh}(A^m) \to \mathrm{QCoh}(A^n)$. Now applying ([22], 9.2.6), we get $\lim_{[n]\in\boldsymbol{\Delta}} \mathrm{QCoh}(A^n) \widetilde{\to} \mathrm{QCoh}(B(A))$. We are done.

Since $B(A) \in CAlg(\mathrm{Spc})$, $B(A) \otimes \mathrm{Vect} \in CAlg(\mathrm{DGCat}_{cont})$. We claim that this symmetric monoidal structure on $B(A)\otimes\mathrm{Vect}$ corresponds to the convolution symmetric monoidal structure on $\mathrm{QCoh}(B(A))$. Indeed, recall first that, by ([15], I.3, 3.4.4), $m_* : \mathrm{QCoh}(B(A)^2) \to \mathrm{QCoh}(B(A))$ is continuous. Note that for any $[n] \in \boldsymbol{\Delta}$ and the corresponding map $\gamma : A^n \to B(A)$ the functor $\gamma^* : \mathrm{QCoh}(B(A)) \to \mathrm{QCoh}(A^n)$ admits a left adjoint, which is actually given by γ_*. For any $[n] \in \boldsymbol{\Delta}$ we have a commutative diagram

$$\begin{array}{ccc} (A\times A)^n & \stackrel{\gamma}{\to} & B(A\times A) \\ \downarrow h_n & & \downarrow m \\ A^n & \stackrel{\gamma'}{\to} & B(A), \end{array}$$

where $m : B(A) \times B(A) \to B(A)$ is the product map, and h_n is induced by the product in A. We see that passing to the colimit over $[n] \in \boldsymbol{\Delta}^{op}$ in the functors $(h_n)_* : \mathrm{QCoh}((A\times A)^n) \to \mathrm{QCoh}(A^n)$, we get the functor $m_* : \mathrm{QCoh}(B(A)\times B(A)) \to \mathrm{QCoh}(B(A))$. We are done.

Let $\mathcal{C} \in CAlg(\mathrm{Spc})$ be the symmetric monoidal groupoid defined in Sect. 4.8.2 of the paper. Then $\mathcal{C} \otimes \mathrm{Vect} \in CAlg(\mathrm{DGCat}_{cont})$. Since $\mathcal{C} \widetilde{\to} \sqcup_{\mathbb{Z}/2\mathbb{Z}} B(\mathbb{Z}/2\mathbb{Z})$, we get $\mathcal{C} \otimes \mathrm{Vect} \widetilde{\to} \sqcup_{\mathbb{Z}/2\mathbb{Z}} B(\mathbb{Z}/2\mathbb{Z}) \otimes \mathrm{Vect}$, let refer this coproduct as grading by $\mathbb{Z}/2\mathbb{Z}$. We also get a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{C} \otimes \mathrm{Vect}$ by the automorphisms of the identity functor by functoriality. It is given by a map $B(\mathbb{Z}/2\mathbb{Z}) \to \mathrm{Fun}_{k,cont}(\mathcal{C}, \mathcal{C})$. Let $\mathrm{Vect}^{\epsilon} \subset \mathcal{C} \otimes \mathrm{Vect}$

be the full subcategory of those objects, on which the parity concides with the values of the $\mathbb{Z}/2\mathbb{Z}$-action by the automorphisms of the identity functor. We should refer to it as the DG-*category of super-vector spaces*. It inherits a symmetric monoidal structure from $\mathcal{C}\otimes\mathrm{Vect}$.

Let now $\mathcal{D}\in CAlg(\mathrm{DGCat}_{cont})$ equipped with a monoidal functor $B(\mathbb{Z}/2\mathbb{Z})\to \mathrm{Fun}^{\otimes}_{k,cont}(\mathcal{D},\mathcal{D})$. We simply denote by ϵ the corresponding automorphism of the identity functor of $\mathcal{D}$. The object $\mathcal{D}^{\epsilon}\in CAlg(\mathrm{DGCat}_{cont})$ defined in Sect. 8.2.4 of the paper is, in fact, the category of even objects in $\mathrm{Vect}^{\epsilon}\otimes_{\mathrm{Vect}}\mathcal{D}\in CAlg(\mathrm{DGCat}_{cont})$. Here we view both Vect^{ϵ} and $\mathcal{D}$ as $\mathbb{Z}/2\mathbb{Z}$-graded, where the grading on $\mathcal{D}$ is given by the action of ϵ.

3.12.1. For 9.5.1. Let us explain the monoidal structure on $Shv(\mathrm{Gr}_G)^{\mathfrak{L}^+(G)}$, without any gerbes on Gr_G. We have the following analog of the convolution diagram from [28].

Let $\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G$ be the prestack whose S-point is a collection $I_1,I_2\in\mathrm{Ran}(S)$, G-torsors $\mathcal{F}^1,\mathcal{F}$ on $S\times X$ with isomorphisms $\nu_1:\mathcal{F}^0\widetilde{\to}\mathcal{F}^1\mid_{X\times S-\Gamma_{I_1}}$ and $\eta:\mathcal{F}^1\widetilde{\to}\mathcal{F}\mid_{X\times S-\Gamma_{I_2}}$.

Let $C_{G,X}$ be the prestack whose S-point is a collections $I_1,I_2\in\mathrm{Ran}(S)$, G-torsors $\mathcal{F}^i$ on $S\times X$ with isomorphisms $\nu_i:\mathcal{F}^0\widetilde{\to}\mathcal{F}^i\mid_{X\times S-\Gamma_{I_i}}$ and a trivialization $\mu_1:\mathcal{F}^0\widetilde{\to}\mathcal{F}^1\mid_{D_{I_2}}$. We get a diagram

$$\mathrm{Gr}_G\times\mathrm{Gr}_G\stackrel{p}{\leftarrow}C_{G,X}\stackrel{q}{\to}\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G\stackrel{m}{\to}\mathrm{Gr}_G\times_{\mathrm{Ran}}(\mathrm{Ran}\times\mathrm{Ran})\stackrel{\mathrm{id}\times u}{\to}\mathrm{Gr}_G,$$

where p forgets μ_1, so keeps $((\mathcal{F}^1,\nu_1,I_1),(\mathcal{F}^2,\nu_2,I_2))\in\mathrm{Gr}_G\times\mathrm{Gr}_G$. The map q is given by the property that $\mathcal{F}$ is obtained by gluing of $\mathcal{F}^1_{X\times S-\Gamma_{I_2}}$ and of $\mathcal{F}^2\mid_{D_{I_2}}$ via

$$\nu_2\mu_1^{-1}:\mathcal{F}^1\widetilde{\to}\mathcal{F}^2\mid_{\overset{\circ}{D}_{I_2}}$$

The map q is a torsor under the group scheme on $\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G$, which is the pull-back of $\mathfrak{L}^+(G)$ under $\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G\to\mathrm{Ran}$ sending a point as above to I_2. We may take the quotient of p under a suitable action of $\mathfrak{L}^+(G)$, and get a morphism $\bar{p}:\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G\to\mathrm{Gr}_G\times(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)$. So, we get a diagram

$$\mathrm{Gr}_G\times(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)\stackrel{\bar{p}}{\leftarrow}\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G\stackrel{m}{\to}\mathrm{Gr}_G\times_{\mathrm{Ran}}(\mathrm{Ran}\times\mathrm{Ran})\stackrel{\mathrm{id}\times u}{\to}\mathrm{Gr}_G$$

Now write $\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G$ as the prestack whose S-points are $I_1,I_2\in\mathrm{Ran}(S)$, G-torsors $\mathcal{F}^1,\mathcal{F}$ on $D_{I_1\cup I_2}$ with isomorphisms $\nu_1:\mathcal{F}^0\widetilde{\to}\mathcal{F}^1\mid_{D_{I_1\cup I_2}-\Gamma_{I_1}}$ and $\eta:\mathcal{F}^1\widetilde{\to}\mathcal{F}\mid_{D_{I_1\cup I_2}-\Gamma_{I_2}}$. This allows to conclude that $\mathrm{Map}(D_{I_1\cup I_2},G)$ acts on $(\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G)(S)$. Moreover m is equivariant with respect to the actions of $\mathfrak{L}^+(G)$ pulled back under $u:\mathrm{Ran}\times\mathrm{Ran}\to\mathrm{Ran}$.

We have a natural map $\xi:\mathrm{Map}(D_{I_1\cup I_2},G)\to\mathrm{Map}(D_{I_1},G)$ given by composing with $D_{I_1}\to D_{I_1\cup I_2}$. Consider the pull-back of the group scheme $\mathfrak{L}^+(G)$ under $\mathrm{Gr}_G\times(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)\to\mathrm{Ran}\times\mathrm{Ran}\stackrel{u}{\to}\mathrm{Ran}$. So, it maps naturally to the pull-back of $\mathfrak{L}^+(G)$ under $\mathrm{Gr}_G\times(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)\stackrel{\mathrm{pr}_1}{\to}\mathrm{Gr}_G\to\mathrm{Ran}$. The map $\bar{p}$ is equivariant under the actions of $\mathrm{Map}(D_{I_1\cup I_2},G)$, where on the target it acts through the above homomorphism ξ. Taking the quotients, we get a diagram

$$(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)\times(\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G)\stackrel{\tilde{p}}{\leftarrow}\mathfrak{L}^+(G)\backslash(\mathrm{Gr}_G\tilde{\times}\mathrm{Gr}_G)\stackrel{\tilde{m}}{\to}\mathfrak{L}^+(G)\backslash\mathrm{Gr}_G$$

Here we used the action of $\mathfrak{L}^+(G)$ on $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ described above. Now the monoidal operation on $Shv(\mathfrak{L}^+(G)\backslash \mathrm{Gr}_G)$ is given by $(K_1, K_2) \mapsto \tilde{m}_! \tilde{p}^*(K_1 \boxtimes K_2)$. The functor $\tilde{m}_!$ makes sense, because the map $\tilde{m}$ is pseudo-proper.

Question. How to justify the existence of the functor $\tilde{p}^*$?

The definition of the category $Shv(\mathfrak{L}^+(G)\backslash \mathrm{Gr}_G)$ and the correspnding convention is as in ([27], 0.0.40). In 9.5.1 he meant a version of this definition with gerbes incorporated.

3.12.2. *Hecke action of* $Shv(\mathfrak{L}^+(G)\backslash \mathrm{Gr}_G)$ *on* $Shv(\mathrm{Bun}_G)$. Recall the stack Hecke^{loc}_G from Section 7.3.1 of the paper, it classifies $I \in \mathrm{Ran}$, G-torsors $\mathcal{F}_G, \mathcal{F}'_G$ on D_I and an isomorphism $\mathcal{F}_G \tilde{\to} \mathcal{F}'_G \mid_{\overset{\circ}{D}_I}$. We have $\mathrm{Hecke}^{loc}_G \tilde{\to} \mathfrak{L}^+(G)\backslash \mathrm{Gr}_G$, where the quotient is understood in the stack sense (etale sheafification of the prestack quotient). We have the involution of Hecke^{loc}_G given swapping $\mathcal{F}_G$ and $\mathcal{F}'_G$. We denote by $* : Shv(\mathrm{Hecke}^{loc}_G) \to Shv(\mathrm{Hecke}^{loc}_G)$ the induced involution.

Now we may define the Hecke functors as in ([6], Section 3.2.4). Let $\mathcal{G} \to \mathrm{Bun}$ be the prestack classifying $I \in \mathrm{Ran}, \mathcal{F}_G \in \mathrm{Bun}_G$ and an isomorphism $\mathcal{F}^0_G \tilde{\to} \mathcal{F}_G \mid_{D_I}$.

Let $\mathrm{Hecke}(G)_{Ran}$ be the global Hecke stack classifying $I \in \mathrm{Ran}$, G-torsors $\mathcal{F}_G, \mathcal{F}'_G$ on X, and an isomorphism $\beta : \mathcal{F}_G \tilde{\to} \mathcal{F}'_G \mid_{X-\Gamma_I}$. Let $h^{\leftarrow}, h^{\rightarrow} : \mathrm{Hecke}(G)_{\mathrm{Ran}} \to \mathrm{Bun}_G$ be the map sending the above point to $\mathcal{F}$, $\mathcal{F}'$ respectively.

We have isomorphisms $\mathrm{id}^l, \mathrm{id}^r : \mathrm{Hecke}(G)_{\mathrm{Ran}} \tilde{\to} (\mathrm{Gr}_G \times_{\mathrm{Ran}} \mathcal{G})/\mathfrak{L}^+(G)$ such that the projection of the RHS to Bun_G corresponds to $h^{\leftarrow}, h^{\rightarrow}$ respectively. This gives a diagram

$$\mathrm{Hecke}^{loc}_G \overset{\mathrm{pr}_1}{\leftarrow} (\mathrm{Gr}_G \times_{\mathrm{Ran}} \mathcal{G})/\mathfrak{L}^+(G) \overset{\mathrm{pr}_2}{\to} \mathrm{Bun}_G$$

We set for $\mathcal{S} \in Shv(\mathrm{Hecke}^{loc}_G), K \in Shv(\mathrm{Bun}_G)$,

$$(\mathcal{S} \boxtimes K)^l = (\mathrm{id}^l)^!(\mathrm{pr}_1 \times \mathrm{pr}_2)^*(\mathcal{S} \boxtimes K) \quad \text{and} \quad (\mathcal{S} \boxtimes K)^r = (\mathrm{id}^r)^!(\mathrm{pr}_1 \times \mathrm{pr}_2)^*(\mathcal{S} \boxtimes K)$$

The map $\mathrm{pr}_1 \times \mathrm{pr}_2$ is a torsor under the placid group scheme $\mathfrak{L}^+(G)$, so the functor $(\mathrm{pr}_1 \times \mathrm{pr}_2)^*$ is defined as in ([27], 0.0.36).

Now define Hecke functors $\mathrm{H}^{\rightarrow}_G, \mathrm{H}^{\leftarrow}_G : Shv(\mathrm{Hecke}^{loc}_G) \times Shv(\mathrm{Bun}_G) \to Shv(\mathrm{Bun}_G)$ by

$$\mathrm{H}^{\leftarrow}_G(\mathcal{S}, K) = h^{\leftarrow}_!(*\mathcal{S} \boxtimes K)^r \quad \text{and} \quad \mathrm{H}^{\rightarrow}_G(\mathcal{S}, K) = h^{\rightarrow}_!(\mathcal{S} \boxtimes K)^l$$

My understanding is that this defines a left and right action of $Shv(\mathrm{Hecke}^{loc}_G)$ with the above monoidal structure on $Shv(\mathrm{Bun}_G)$.

3.13. Category of Hecke eigen-sheaves. Dennis says the definition from [11] is not a good one for objects $K \in \mathrm{D}(\mathrm{Bun}_G)$ which are not in the heart of a t-structure!!

The following idea is from ([12], Section 4.4.2). Let $\mathrm{Hecke}(G)_{\mathrm{Ran}}$ be the Ran version of the Hecke stack. Its S-point is a finite subset $I \subset \mathrm{Map}(S, X)$, which is a S-point of Ran, two G-torsors $\mathcal{F}, \mathcal{F}'$ on $S \times X$ and an isomorphism $\mathcal{F} \tilde{\to} \mathcal{F}' \mid_{S\times X-\Gamma_I}$, here Γ_I is the union of the graphs of maps $S \to X$ given by I. Let $h^{\leftarrow}, h^{\rightarrow} : \mathrm{Hecke}(G)_{\mathrm{Ran}} \to \mathrm{Bun}_G$ be the map sending the above point to $\mathcal{F}$, $\mathcal{F}'$ respectively. Let $U_I = S \times X - \Gamma_I$

We get diagrams

$$\begin{array}{ccc} \mathrm{Hecke}(G)_{\mathrm{Ran}} \underset{h^{\to},\mathrm{Bun}_G,h^{\leftarrow}}{\times} \mathrm{Hecke}(G)_{\mathrm{Ran}} & \stackrel{b}{\to} & \mathrm{Hecke}(G)_{\mathrm{Ran}} \times \mathrm{Hecke}(G)_{\mathrm{Ran}} \\ \downarrow a & & \\ \mathrm{Hecke}(G)_{\mathrm{Ran}} \times_{\mathrm{Ran}} (\mathrm{Ran}\times\mathrm{Ran}) & & \\ \downarrow \mathrm{id}\times u & & \\ \mathrm{Hecke}(G)_{\mathrm{Ran}} & & \end{array}$$

Here $u : \mathrm{Ran}\times\mathrm{Ran} \to \mathrm{Ran}$ is the product map. The map a sends

$$(I, J \in \mathrm{Ran}, \mathcal{F}, \mathcal{F}', \mathcal{F}'', \beta : \mathcal{F} \widetilde{\to} \mathcal{F}' \mid_{U_I}, \gamma : \mathcal{F}' \widetilde{\to} \mathcal{F}'' \mid_{U_J})$$

to $(\mathcal{F}, \mathcal{F}'', \gamma\beta : \mathcal{F} \widetilde{\to} \mathcal{F}'' \mid_{U_{I\cup J}})$.

The maps $h^{\leftarrow} \times \mathrm{supp}, h^{\to} \times \mathrm{supp} : \mathrm{Hecke}(G)_{\mathrm{Ran}} \to \mathrm{Bun}_G \times \mathrm{Ran}$ and u are pseudo-proper in the sense of ([8], 1.5), so the functors $(\mathrm{id}\times u)_!, a_!$ are defined between the corresponding categories of sheaves by ([8], 1.5.2).

He claims $Shv(\mathrm{Hecke}(G)_{\mathrm{Ran}})$ has a non-unital monoidal structure with the product given by $(K, K') \mapsto (\mathrm{id}\times u)_! a_! b^!(K \boxtimes K')$.

Similarly, we have the diagram

$$\begin{array}{ccc} \mathrm{Hecke}(G)_{\mathrm{Ran}} & \stackrel{\mathrm{id}\times h^{\to}}{\to} & \mathrm{Hecke}(G)_{\mathrm{Ran}} \times \mathrm{Bun}_G \\ \downarrow h^{\leftarrow} & & \\ \mathrm{Bun}_G & & \end{array}$$

He proposes to define a left $Shv(\mathrm{Hecke}(G)_{\mathrm{Ran}})$-module structure on $Shv(\mathrm{Bun}_G)$ via the action map $Shv(\mathrm{Hecke}(G)_{\mathrm{Ran}}) \otimes Shv(\mathrm{Bun}_G) \to Shv(\mathrm{Bun}_G)$ sending (K, F) to $h^{\leftarrow}_!(\mathrm{id}\times h^{\to})^!(K \boxtimes F)$. Since $\mathrm{Ran} \to \mathrm{Spec}\, k$ is pseudo-proper, the functor $h^{\leftarrow}_!$ makes sense.

We see that $\mathrm{Hecke}(G)_{\mathrm{Ran}}$ has a structure of a groupoid acting on Bun_G. Besides, $\mathrm{Hecke}(G)_{\mathrm{Ran}}$ has a structure of a non-unital associative algebra in PreStk_{corr}. This is why applying Shv, one gets a non-unital monoidal category.

We may also consider the (non-integral) Hecke functors defined as follows. For the diagram

$$\begin{array}{ccc} \mathrm{Hecke}(G)_{\mathrm{Ran}} & \stackrel{\mathrm{id}\times h^{\to}}{\to} & \mathrm{Hecke}(G)_{\mathrm{Ran}} \times \mathrm{Bun}_G \\ \downarrow \mathrm{supp}\times h^{\leftarrow} & & \\ \mathrm{Ran}\times\mathrm{Bun}_G & & \end{array}$$

we could consider the functor $\mathrm{H} : Shv(\mathrm{Hecke}(G)_{\mathrm{Ran}}) \times Shv(\mathrm{Bun}_G) \to Shv(\mathrm{Ran}\times\mathrm{Bun}_G)$ given by $\mathrm{H}(K, F) = (\mathrm{supp}\times h^{\leftarrow})_!(\mathrm{id}\times h^{\to})^!(K \boxtimes F)$.

Question What is the compatibility of H with the symmetric monoidal structure on $\mathrm{Rep}(\check{G})$?

3.14. For version June 7, 2021.

3.14.1. For 4.5.7. Here A is assumed divisible (and the of its elements are coprime to $char(k)$). Recall that $\mathcal{I}$ is the set of vertices of the Dynkin diagram. We have an exact sequence of abelian groups $0 \to \mathrm{Hom}(\pi_1(G), A) \to \mathrm{Hom}(\Lambda, A) \to \prod_{i\in\mathcal{I}} A \to 0$, where the second map is given by evaliation on simple coroots. This gives a map $\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Lambda, A)) \to \mathrm{Map}(X, B^2_{et}(\prod_{i\in \mathcal{I}} A) \widetilde{\to} \prod_i \mathrm{Ge}_A(X)$ in $ComGrp(\mathrm{Spc})$, whose fibre in $ComGrp(\mathrm{Spc})$ is $\mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\pi_1(G), A)))$.

3.14.2. For A.3. By definition, $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ is the prestack over Ran whose S-points are $I \in \mathrm{Ran}(S)$, and a map $S \times X \to B_{et}(\Gamma \otimes \mathbb{G}_m)$ together with a trivialization ofg its restriction to $U_I \subset S \times X$. Here U_I is the complement of $\cup_i \Gamma_i$, here Γ_i is the graph of i-th map $S \to X$.

3.14.3. In Remark 4.6.9 and elsewhere we denote by Ab the derived DG-category of abelian groups. In 4.6.7 Dennis mentions instead the ∞-category of chain complexes of abelian groups, but he actually means the derived DG-category . In other words, let $\mathcal{A}b$ be the usual category of abelian groups. Then it is a Grothendieck abelian category, so we may consider $\mathrm{D}(\mathcal{A}b)$ in the sense of ([19], 1.3.5.8). We have the canonical functor $\mathrm{Ab} \to \mathrm{Sptr}^{\le 0}$ given by the universal property of derived DG-categories ([19], 1.3.3.2). I think it coincides with the Dold-Kan functor used in Remark 4.6.9.

3.14.4. For 4.6.8. We consider $\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A))$ as a connected spectrum, by this we mean the inner hom I think in $\mathrm{Sptr}^{\le 0} \widetilde{\to} \mathbb{E}_\infty(\mathrm{Spc})$.

3.14.5. For Cor. 4.7.6. Let Γ be a finitely denerated abelian group whose torsion part is of order prime to $char(k)$, let A be a divisible abelian group. To summarize, we have a fibre sequence

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{Quad}(\Gamma, A(-1))$$

in $ComGrp(\mathrm{Spc})$. Moreover, we have $\mathrm{Map}_{Ab}(\Gamma, B^2(A)) \widetilde{\to} B^2(\mathrm{Hom}(\Gamma, A))$ by Remark 4.6.9 of the paper, because $\mathrm{Ext}^1_{Ab}(\Gamma, A) = 0$, and

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \widetilde{\to} \mathrm{Map}(X, B^2_{et}(\mathrm{Hom}(\Gamma, A)))$$

is an isomorphism now by Remark 4.7.7 of the paper.

3.14.6. For 4.8.1. Here A is divisible I think. Here $\mathrm{Map}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A))$ classifies $\mathcal{C} \in CAlg(\mathrm{Spc})$, which are usual groupoids with $\pi_0(\mathcal{C}) \widetilde{\to} \Gamma$ as a commutative monoid, and the group of automorphisms of an object is A.

3.14.7. For 9.1.1. For a theory of sheaves, which are not $\mathcal{D}$-modules, the formula $S \mapsto Shv(S \times_{Ran} Z)$ does not in general define a sheaf of categories for a factorization prestack Z over Ran.

To define a sheaf of categories over Ran for any sheaf theory, not that $ShvCat : (\mathrm{PreStk}_{lft})^{op} \to 1 - \mathcal{C}\mathrm{at}$ preserves limits. Since $\mathrm{Ran} \widetilde{\to} \mathrm{colim}_I X^I$ over the category of non empty finite sets and surjections, we get $ShvCat(\mathrm{Ran}) \widetilde{\to} \lim_I ShvCat(X^I)$. Besides, $ShvCat(X^I) \widetilde{\to} Shv(X^I) - mod(\mathrm{DGCat}_{cont})$ for any of our 4 sheaf theories.

So, for any sheaf theory we understand $\mathrm{Sph}_{\mathcal{G}^G}(G)$ as a compatible family of objects of $Shv(X^I) - mod$ for all non empty finite sets I.

3.14.8. For 9.2 line 2. I think there is a mistake there, namely, $\mathcal{G}^G \otimes \det_{\mathfrak{g}}^{\frac{1}{2}}$ should be replaced by $\mathcal{G}^G$. Otherwise, no critical shift would be needed in the formulation of Satake.

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