

COMMENTS TO: RASKIN, CHIRAL CATEGORIES

for version Sept. 4, 2019

0.0.1. For 4.3.3. $\mathit{Sets} = \text{Ind}(\mathit{Set}_{<\infty})$, and Spc is generated by Sets under geometric realizations (HTT, 5.5.8.14).

Every object $Y \in \mathit{Sets}_{<\infty}$ is projective in Spc in the sense of (HTT, 5.5.8.18), because $\mathbf{\Delta}^{op}$ is sifted. Now the canonical inclusion $\mathit{Sets}_{<\infty} \subset \mathit{Spc}$ extends uniquely by (HTT, 5.5.8.15) to a functor $f : \mathcal{P}_{\Sigma}(\mathit{Sets}_{<\infty}) \rightarrow \mathit{Spc}$ preserving sifted colimits. We see that the conditions (i),(ii), (iii) of (HTT, 5.5.8.22) are verified, so $f : \mathcal{P}_{\Sigma}(\mathit{Sets}_{<\infty}) \xrightarrow{\sim} \mathit{Spc}$ is an equivalence.

The functors $F, F' : \mathit{Spc} \rightarrow \mathit{Spc}$, $F(Z) = \text{Ran}_Z$ and $F'(Z) = \text{Ran}_{Z, \emptyset}$ clearly commute with sifted colimits indeed. Thus, in the notations of (HTT, 5.5.8.15), these are functors in $\text{Fun}_{\Sigma}(\mathcal{P}_{\Sigma}(\mathit{Sets}_{<\infty}), \mathit{Spc})$. To see that F, F' are LKE of their restriction to $\mathit{Sets}_{<\infty}$, decompose F (resp., F') as

$$\mathcal{P}_{\Sigma}(\mathit{Sets}_{<\infty}) \xrightarrow{i} \mathcal{P}(\mathit{Sets}_{<\infty}) \xrightarrow{b} \mathit{Spc},$$

where the first map is the canonical inclusion, and the second is the colimi preserving functor extending $F : \mathit{Sets}_{<\infty} \rightarrow \mathit{Spc}$ (resp., $F' : \mathit{Sets}_{<\infty} \rightarrow \mathit{Spc}$). Now b is the LKE from $\mathit{Sets}_{<\infty}$, and its composition with i is also a LKE from $\mathit{Sets}_{<\infty}$, because i is a right adjoint!

0.0.2. For 2.8. The fact that his Vect coincides with the definition from Dennis and Nick's book follows from (HA, 7.1.1.15-16).

0.0.3. For 4.5. Once we defined $\text{Ran}_{\mathcal{G}}^{un}$ for $\mathcal{G} \in \mathit{Set}_{<\infty}$, we may extend this definition to $\mathcal{P}_{\Sigma}(\mathit{Sets}_{<\infty}) \xrightarrow{\sim} \mathit{Spc}$ by requiring that the functor $\mathit{Spc} \rightarrow 1 - \mathit{Cat}$, $\mathcal{G} \mapsto \text{Ran}_{\mathcal{G}}^{un}$ commutes with sifted colimits using (HTT, 5.5.8.15).

For 4.6.1(1) see below. The forgetful functor $0 - \mathit{Cart}/I \rightarrow 1 - \mathit{Cat}$ preserves colimits (see below). For a cartesian fibration in spaces $p : \mathcal{X} \rightarrow I$ any arrow in \mathcal{X} is p -cartesian. So, $'\text{Ran}_{\mathcal{G}}^{un}$ can be defined as the category obtained from $\text{coGroth}(\Psi_{\mathcal{G}})$ inverting all arrows lying over surjective morphisms in $\mathit{Sets}_{<\infty}$.

For 4.9: For $\mathcal{G} \in \mathit{Spc}$, $\text{Ran}_{\mathcal{G}, \emptyset, [0]}^{\rightarrow} = \text{Ran}_{\mathcal{G}, \emptyset}$. Thus, the space underlying the category $'\text{Ran}_{\mathcal{G}}^{un} \xrightarrow{\sim} \text{Ran}_{\mathcal{G}}$ is $\text{Ran}_{\mathcal{G}, \emptyset}$.

For any $n \geq 0$ the category $f\text{Set}_{\emptyset, [n]}^{\rightarrow}$ has a symmetric monoidal structure given by the disjoint union. This provides for any $\mathcal{G} \in \mathit{Spc}$ a structure of a commutative monoid on $\text{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow}$. For example, if $\mathcal{G} \in \mathit{Sets}$ then recall that $\text{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow}$ is the set of collections $\{S_0 \subset \dots \subset S_n \subset \mathcal{G}\}$ with S_i finite. Then the operation sends a pair $(S_0 \subset \dots \subset S_n), (S'_0 \subset \dots \subset S'_n)$ to $(S_0 \cup S'_0 \subset \dots \subset S_n \cup S'_n)$.

0.0.4. For 4.13. Since $\pi_0 : \mathbf{Spc} \rightarrow \mathbf{Sets}$ is a left adjoint, for $\mathcal{G} \in \mathbf{Spc}$ we get isomorphisms $\pi_0 \mathbf{Ran}_{\mathcal{G}} \xrightarrow{\sim} \mathbf{Ran}_{\pi_0(\mathcal{G})}$ and $\pi_0(\mathbf{Ran}_{\mathcal{G}, \emptyset}) \xrightarrow{\sim} \mathbf{Ran}_{\pi_0(\mathcal{G}), \emptyset}$.

For 4.13.2: misprint I think: the space underlying $\mathbf{Ran}_{\mathcal{G}}^{un}$ is $\mathbf{Ran}_{\mathcal{G}, \emptyset}$. However, it is clear how to define $[\mathbf{Ran}_{\mathcal{G}, \emptyset} \times \mathbf{Ran}_{\mathcal{G}, \emptyset}]_{disj}$, and hence we also get $[\mathbf{Ran}_{\mathcal{G}}^{un} \times \mathbf{Ran}_{\mathcal{G}}^{un}]_{disj}$ as a full subcategory restricting objects in the same way.

0.0.5. Let $I \in 1 - \mathbf{Cat}$. Sam claims $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}$ sending $\mathcal{X} \rightarrow I$ to \mathcal{X} preserves colimits. I prove this using ([4], Th. 1.1). Given $F : \mathcal{C}^{op} \rightarrow \mathbf{Spc}$, let $\mathcal{X} \rightarrow \mathcal{C}$ be the cartesian fibration in spaces classifying F . By ([4], Th. 1.1), \mathcal{X} is the oplax colimit of F . That is, $\mathcal{X} \xrightarrow{\sim} \operatorname{colim}_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C}^{op})} \mathcal{C}_{/c} \times F(c')$ in $1 - \mathbf{Cat}$. If now $F \xrightarrow{\sim} \operatorname{colim}_{j \in J} F^j$ in $\mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Spc})$ then we get

$$\mathcal{X} \xrightarrow{\sim} \operatorname{colim}_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C}^{op}), j \in J} \mathcal{C}_{/c} \times F^j(c') \xrightarrow{\sim} \operatorname{colim}_{j \in J} \operatorname{colim}_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C}^{op})} \mathcal{C}_{/c} \times F^j(c') \xrightarrow{\sim} \operatorname{colim}_{j \in J} \mathcal{X}^j,$$

where $\mathcal{X}^j \rightarrow \mathcal{C}$ is the cartesian fibration in spaces classifying F^j . Here the colimit $\operatorname{colim}_{j \in J} \mathcal{X}^j$ is calculated in $1 - \mathbf{Cat}$.

The same argument actually shows that $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}_{/I}$ preserves colimits, hence admits a right adjoint, because these categories are presentable. Recall that the functor $1 - \mathbf{Cat}_{/I} \rightarrow 1 - \mathbf{Cat}$, $(X \rightarrow I) \mapsto X$ preserves colimits, as this is a left adjoint.

Lemma 0.0.6. *Let $I, J \in 1 - \mathbf{Cat}$, $J \subset I$ be a 1-full subcategory with the same class of objects as I . The functor $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}$, $(X \rightarrow I) \mapsto S^{-1}X$ preserves colimits. Here S is the class of those morphisms of X which lie over a morphism of J .*

Proof. For $(X \rightarrow I) \in 1 - \mathbf{Cat}_{/I}$ write $X_J = X \times_I J$. Given $Y \in 1 - \mathbf{Cat}$, one gets

$$\operatorname{Map}_{1 - \mathbf{Cat}}(S^{-1}X, Y) \xrightarrow{\sim} \operatorname{Map}_{1 - \mathbf{Cat}}(X, Y) \times_{\operatorname{Map}_{1 - \mathbf{Cat}}(X_J, Y)} \operatorname{Map}_{1 - \mathbf{Cat}}(X_J, Y^{\mathbf{Spc}})$$

Assume $X \xrightarrow{\sim} \operatorname{colim}_{k \in K} X_k$ in $0 - \mathbf{Cart}_{/I}$. The functor $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}_{/I}$ preserves colimits by the above, so this is also a colimit diagram in $1 - \mathbf{Cat}$, hence $\operatorname{Map}_{1 - \mathbf{Cat}}(X, Y) \xrightarrow{\sim} \lim_{k \in K^{op}} \operatorname{Map}_{1 - \mathbf{Cat}}(X_k, Y)$. The functor $0 - \mathbf{Cart}_{/I} \rightarrow 0 - \mathbf{Cart}_{/J}$, $X' \mapsto X'_J$ preserves colimits, as we may interpret this as $\mathbf{Fun}(I^{op}, \mathbf{Spc}) \rightarrow \mathbf{Fun}(J^{op}, \mathbf{Spc})$. So, $\operatorname{Map}_{1 - \mathbf{Cat}}(X_J, Y) \xrightarrow{\sim} \lim_{k \in K^{op}} \operatorname{Map}_{1 - \mathbf{Cat}}((X_k)_J, Y)$ and

$$\operatorname{Map}_{1 - \mathbf{Cat}}(X_J, Y^{\mathbf{Spc}}) \xrightarrow{\sim} \lim_{k \in K^{op}} \operatorname{Map}_{1 - \mathbf{Cat}}((X_k)_J, Y^{\mathbf{Spc}})$$

Limits commute with limits, so the natural map

$$\operatorname{Map}_{1 - \mathbf{Cat}}(S^{-1}X, Y) \rightarrow \operatorname{Map}_{1 - \mathbf{Cat}}(\operatorname{colim}_{k \in K} S^{-1}(X_k), Y)$$

is an isomorphism. □

The above lemma implies Pp4.6.1(1) of Sam.

Question. What is the right adjoint to the inclusion $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}_{/I}$?

0.0.7. For Sect. 4.14. Since $\mathbf{1-Cat}$ is a 2-category, $\text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat})$ is also a 2-category. This is a general construction associating two a pair of 2-categories, the 2-category of functors between them.

Given $Y \in \text{PreStk}^{lax}$, the object $\text{QCoh}(Y) \in \mathbf{1-Cat}$ is $\mathbf{Map}_{\text{PreStk}^{lax}}(Y, \text{QCoh})$ in the sense of Dennis and Nick book.

Let $i : \text{PreStk} \rightarrow \text{PreStk}^{lax}$ be the inclusion, it has a right adjoint $R : \text{PreStk}^{lax} \rightarrow \text{PreStk}$ sending Y to the prestack $(S \mapsto Y(S)^{\text{Spc}})$ for $S \in \text{Sch}^{aff}$. The functor i also has a left adjoint $\text{PreStk}^{lax} \rightarrow \text{PreStk}$ sending Y to the prestack $(S \mapsto |Y(S)|)$ for $S \in \text{Sch}^{aff}$, here $|Z|$ denotes the inverting of all arrows in the category Z .

The *RKE* of $\text{QCoh}_{\text{PreStk}} : \text{PreStk}^{op} \rightarrow \text{DGCat}_{cont}$ under $\text{PreStk}^{op} \hookrightarrow (\text{PreStk}^{lax})^{op}$ is the composition $(\text{PreStk}^{lax})^{op} \xrightarrow{R^{op}} \text{PreStk}^{op} \rightarrow \text{DGCat}_{cont}$, where the second functor is $\text{QCoh}_{\text{PreStk}}$. It sends Y to $\text{QCoh}(R(Y))$, so this is not the same thing as $\text{QCoh}(Y)$ that Sam means in this paper.

According to ([2], 2.2.1), for $Y \in \text{PreStk}^{lax}$, let $\mathcal{X}_Y \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to $Y : (\text{Sch}^{aff})^{op} \rightarrow \mathbf{1-Cat}$. Let also $\text{Shv}_{\text{Sch}^{aff}} \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to the composition $(\text{Sch}^{aff})^{op} \rightarrow \text{DGCat}_{cont} \rightarrow \mathbf{1-Cat}$, where the first functor is $\text{QCoh}_{\text{Sch}^{aff}}^*$. Then $\text{QCoh}(Y)$ can be defined as the full subcategory of $\text{Fun}_{\text{Sch}^{aff}}(\mathcal{X}_Y, \text{Shv}_{\text{Sch}^{aff}})$ consisting of functors sending cartesian arrows to cartesian arrows.

There is one more way to understand $\text{QCoh}(Y)$ for a lax prestack Y as follows. By ([4], Lemma 6.4), the category $\text{QCoh}(Y)$ is the limit of the composition

$$Tw(\text{Sch}^{aff})^{op} \rightarrow \text{Sch}^{aff} \times (\text{Sch}^{aff})^{op} \xrightarrow{Y^{op} \times \text{QCoh}} (\mathbf{1-Cat})^{op} \times \mathbf{1-Cat} \xrightarrow{\text{Fun}} \mathbf{1-Cat}$$

This allows to view $\text{QCoh}(Y)$ as an object of DGCat_{cont} as follows. Consider the composition

$$Tw(\text{Sch}^{aff})^{op} \rightarrow \text{Sch}^{aff} \times (\text{Sch}^{aff})^{op} \xrightarrow{Y^{op} \times \text{QCoh}} (\mathbf{1-Cat})^{op} \times \text{DGCat}_{cont} \xrightarrow{\text{Fun}} \text{DGCat}_{cont}$$

and take the limit. Since the projection $\text{DGCat}_{cont} \rightarrow \mathbf{1-Cat}$ preserves limits, we are done.

We used the fact that given any $C \in \mathbf{1-Cat}, D \in \text{DGCat}_{cont}, \text{Fun}(C, D) \in \text{DGCat}_{cont}$ naturally (see [11], 9.2.13). Once again, for $Y \in \text{PreStk}^{lax}$,

$$(1) \quad \text{QCoh}(Y) = \lim_{(\alpha: S' \rightarrow S) \in Tw(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S), \text{QCoh}(S'))$$

This implies the following.

Remark 0.0.8. Write $\text{QCoh}_{\text{PreStk}^{lax}}^* : (\text{PreStk}^{lax})^{op} \rightarrow \text{DGCat}_{cont}$ for the functor sending $Y \mapsto \text{QCoh}(Y)$ and $f : Y \rightarrow Y'$ to $f^* : \text{QCoh}(Y') \rightarrow \text{QCoh}(Y)$. This functor preserves limits.

Given a map $\alpha \rightarrow \beta$ in $Tw(\text{Sch}^{aff})$ with $\alpha : S' \rightarrow S, \beta : T' \rightarrow T$, the transition functor $\text{Fun}(Y(T), \text{QCoh}(T')) \rightarrow \text{Fun}(Y(S), \text{QCoh}(S'))$ in (1) admits a right adjoint, however this right adjoint is not continuous in general. The transition functors preserve colimits. For this reason, for a lax prestack Y , $\text{ShvCat}_{/Y}^{naive}$ has colimits, they are computed termwise in $\text{Fun}(Y(S), \text{QCoh}(S'))$ in the above projective system.

0.0.9. For 4.18. In ([1], ch. A.1, 3.2.7) they defined for a pair $\mathbb{S}, \mathbb{T} \in 2 - \mathcal{C}at$ the object $\text{Fun}(\mathbb{S}, \mathbb{T})_{rlax} \in 2 - \mathcal{C}at$. Sam means by $\text{Hom}^{lax}(\mathcal{C}, \mathcal{D})$ the underlying 1-category of left-lax functors $(\text{Fun}(\mathbb{S}, \mathbb{T})_{lax})^{1-\mathcal{C}at}$ (see my file [11], 12.1.44).

Sam claims in 4.18.3 that there is an object, say $2 - CAT_{lax}$ of $2 - \mathcal{C}at$, which extends $2 - \mathcal{C}at$ as follows. The objects of $2 - CAT_{lax}$ are those of $2 - \mathcal{C}at$, and given $C, D \in 2 - \mathcal{C}at$, the mapping category from C to D is $(\text{Fun}(C, D)_{lax})^{1-\mathcal{C}at}$. Let's accept this.

0.0.10. For 4.19. Recall Sam's notation $ShvCat_{/-}$ for the functor

$$ShvCat_{/-} : (\text{Sch}^{aff})^{op} \rightarrow 1 - \mathcal{C}at, S \mapsto ShvCat(S) = \text{QCoh}(S) - mod$$

Let $\mathcal{X}_{Shv} \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to this functor. Given a lax prestack Y , the definition of $ShvCat_{/Y}^{naive}$ is as follows. Let $\mathcal{X}_Y \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to $Y : (\text{Sch}^{aff})^{op} \rightarrow 1 - \mathcal{C}at$. Then $ShvCat_{/Y}^{naive}$ is the full subcategory in $\text{Fun}_{\text{Sch}^{aff}}(\mathcal{X}_Y, \mathcal{X}_{Shv})$ consisting of functors sending cartesian arrows to cartesian arrows. In other words,

$$(2) \quad ShvCat_{/Y}^{naive} = \mathbf{Map}_{\text{PreStk}^{lax}}(Y, ShvCat_{/-}),$$

where we view PreStk^{lax} naturally as a $(\infty, 2)$ -category.

A candidate for his definition of $ShvCat_{/Y}$ for a lax prestack Y is as follows. First, we consider $1 - \mathcal{C}at$ as a 2-category (as in [1], ch. A.1, Sect. 2.4) denoted **1-Cat** in *loc.cit.* So, we have the 2-categories

$$\text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat}), \text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat})_{rlax}, \text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat})_{lax}$$

defined in ([1], ch. A.1, 2.5.4 and 3.2.7).

Question: Consider the corresponding mapping category

$$\mathbf{Map}_{\text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat})_{lax}}(Y, ShvCat_{/-})$$

does it coincide with $ShvCat_{/Y}$ by definition? Sam says yes, but I think this is wrong!

He proposes the following in email of 14 Oct 2019.

Definition 0.0.11. *We have the natural functor $\mathbf{2-Cat} \rightarrow 2 - CAT_{lax}$, here $\mathbf{2-Cat}$ is the 2-category defined in ([1], ch. A.1, Sect. 6). View $ShvCat_{/-}$ as a functor $(\text{Sch}^{aff})^{op} \rightarrow \mathbf{2-Cat}$. It is essential here to use the 2-category structure of $ShvCat(S)$ for $S \in \text{Sch}^{aff}$, see Section 0.2 here. Given $Y \in \text{PreStk}^{lax}$, we may also view Y as the composition $(\text{Sch}^{aff})^{op} \rightarrow 1 - \mathcal{C}at \rightarrow \mathbf{2-Cat} \rightarrow 2 - CAT_{lax}$. Now $ShvCat_{/Y}$ is the category of natural transformations $Y \rightarrow ShvCat_{/-}$ of functors $(\text{Sch}^{aff})^{op} \rightarrow 2 - CAT_{lax}$.*

My understanding is that

$$ShvCat_{/Y} = \mathbf{Map}_{\text{Fun}((\text{Sch}^{aff})^{op}, 2 - CAT_{lax})}(Y, ShvCat_{/-})$$

Using [4], the latter can be rewritten as the limit in $1 - \mathcal{C}at$

$$(3) \quad ShvCat_{/Y} \xrightarrow{\sim} \lim_{(S \rightarrow S') \in Tw(\text{Sch}^{aff})^{op}} (\text{Fun}(Y(S'), ShvCat(S))_{lax})^{1-\mathcal{C}at}$$

taken over $Tw(\text{Sch}^{aff})^{op}$.

Sam claims $ShvCat/Y$ is not presentable, already $DGCat_{cont}$ is not presentable.

It is not clear which 2-categorical enhancement of $ShvCat/Y$ for a lax prestack Y he meant, but there is the following one: write $ShvCat_Y^{2-Cat}$ for the limit in $2-Cat$

$$\lim_{S \rightarrow S'} (\text{Fun}(Y(S'), ShvCat(S))_{lax})$$

taken over $Tw(\text{Sch}^{aff})^{op}$. Here we view $ShvCat(S)$ as an object of $2-Cat$. This is a good idea, because the functor $2-Cat \rightarrow 1-Cat, D \mapsto D^{1-Cat}$ preserves limits.

Note that for $ShvCat_Y^{naive}$ we have similarly to (3)

$$(4) \quad ShvCat_Y^{naive} \xrightarrow{\simeq} \lim_{(S \rightarrow S') \in Tw(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S'), ShvCat(S))$$

taken over $Tw(\text{Sch}^{aff})^{op}$. The category $Tw(\text{Sch}^{aff})$ has no final object! The limit (4) understood as a limit in $1-Cat$ gets the 1-category $ShvCat_Y^{naive}$. It can also be understood as a limit in $2-Cat$, and then gives an object of $2-Cat$ mentioned in his Remark 4.19.2.

0.0.12. The functor $\text{PreStk}^{op} \rightarrow 1-Cat, Y \mapsto ShvCat_Y^{naive}$ factors through the category of cocomplete categories and colimit-preserving functors. However, $ShvCat/Y$ is not presentable already for $Y = \text{Spec } k$. Given $Y \in \text{PreStk}^{lax}$, we get

$$ShvCat_Y^{naive} = \lim_{(c \rightarrow c') \in Tw(\text{Sch}^{aff})^{op}} \text{Fun}(Y(c'), ShvCat(c))$$

If $f : Y_1 \rightrightarrows Y_2 : g$ are maps in $\text{Fun}((\text{Sch}^{aff})^{op}, 1-Cat)$ with morphisms $fg \rightarrow \text{id}$, $\text{id} \rightarrow gf$ making them into an adjoint pair in the 2-category PreStk^{lax} then the functors

$$g^* : ShvCat_Y^{naive} \rightrightarrows ShvCat_{Y_1}^{naive} : f^*$$

also form an adjoint pair by ([2], Lm. 2.2.6). Indeed, apply the formula (2) and functoriality of **Map**.

0.0.13. For 4.19.2. Let Y be a lax prestack. Then QCoh_Y is the sheaf of categories on Y obtained as the pull-back along $Y \rightarrow *$ of Vect . If $\beta, \gamma : S \rightarrow Y$ are two S -points with $S \in \text{Sch}^{aff}$ and $\theta : \beta \rightarrow \gamma$ is a map in $Y(S)$ then the corresponding map $\theta : \beta^* \text{QCoh}_Y \rightarrow \gamma^* \text{QCoh}_Y$ is the identity.

0.0.14. Let $Y \in \text{PreStk}^{lax}$. Let us calculate $ShvCat_Y^{naive}$ assuming in addition that $Y \in \text{PreStk}$.

Lemma 0.0.15. *Let $C \in 1-Cat, Y \in \text{Fun}(C^{op}, 1-Cat), S \in C$. We identify S with its image in $\text{Fun}(C^{op}, 1-Cat)$ under the Yoneda. Then the category of natural transformations $S \rightarrow Y$ is canonically equivalent to $Y(S)$.*

More generally, if $Y_1 \in \mathcal{P}(C)$ then the category of natural transformations $Y_1 \rightarrow Y$ is equivalent to $\lim_{S \rightarrow Y_1} Y_2(S)$, the limit over $(C/Y_1)^{op}$.

Proof. By ([4], Lm. 6.4), the category of natural transformations $Y_1 \rightarrow Y_2$ for $Y_i \in \text{Fun}(C^{op}, 1 - \text{Cat})$ is the limit of the composition

$$Tw(C)^{op} \rightarrow C \times C^{op} \xrightarrow{Y_1^{op} \times Y_2} (1 - \text{Cat})^{op} \times 1 - \text{Cat} \xrightarrow{\text{Fun}} 1 - \text{Cat}$$

So, $\mathbf{Map}_{\text{Fun}(C^{op}, 1 - \text{Cat})}(Y_1, Y_2) \xrightarrow{\sim} \lim_{(c \rightarrow c') \in Tw(C)^{op}} \text{Fun}(Y_1(c'), Y_2(c))$, here $c \rightarrow c'$ is an arrow in C .

By ([4], 6.9) we know that $\mathbf{Map}_{\text{Fun}(C^{op}, 1 - \text{Cat})}(Y_1, Y_2) \xrightarrow{\sim} \text{Fun}_C^{cart}(\mathcal{Y}_1, \mathcal{Y}_2)$, where $\mathcal{Y}_i \rightarrow C$ is the cartesian fibration corresponding to Y_i , and $\text{Fun}_C^{cart}(\mathcal{Y}_1, \mathcal{Y}_2) \subset \text{Fun}_C(\mathcal{Y}_1, \mathcal{Y}_2)$ is the full subcategory of functors sending cartesian arrows to cartesian arrows.

Assume $Y_1 = S$ for $S \in C$. The corresponding mapping category is $\text{Fun}_C(C/S, \mathcal{Y}_2)$, as $C/S \rightarrow C$ is a cartesian fibration in spaces. The functor of evaluation on the final object $S \xrightarrow{\text{id}} S$ of C/S gives a map $\text{Fun}_C(C/S, \mathcal{Y}_2) \rightarrow Y_2(S)$. To get a map in the other direction, we need to construct a functor $Y_2(S) \times C/S \rightarrow \mathcal{Y}_2$ over C . Given $y \in Y_2(S)$ and $c \rightarrow S$ in C/S , we get a morphism $c \rightarrow S$ in C . The desired functor sends this collection to $y' \in \mathcal{Y}_2(c)$, which is the source of a cartesian arrow $y' \rightarrow y$ over $c \rightarrow S$. I hope these two functors are inverse to each other.

More generally, if $Y_1 \in \mathcal{P}(C)$, then $Y_1 \xrightarrow{\sim} \text{colim}_{S \rightarrow Y_1} S$ in $\text{Fun}(C^{op}, 1 - \text{Cat})$ over $\mathcal{C}_{/Y_1}$. Then we should have

$$\mathbf{Map}_{\text{Fun}(C^{op}, 1 - \text{Cat})}(Y_1, Y_2) \xrightarrow{\sim} \lim_{S \rightarrow Y_1} \mathbf{Map}_{\text{Fun}(C^{op}, 1 - \text{Cat})}(S, Y_2) \xrightarrow{\sim} \lim_{S \rightarrow Y_1} Y_2(S)$$

To explain the first isomorphism, more generally, we claim that for any $Y_2 \in \text{Fun}(C^{op}, 1 - \text{Cat})$ the functor $(\text{Fun}(C^{op}, 1 - \text{Cat}))^{op} \rightarrow 1 - \text{Cat}$, $Y_1 \mapsto \mathbf{Map}_{\text{Fun}(C^{op}, 1 - \text{Cat})}(Y_1, Y_2)$ preserves limits. Indeed, if $Y_1 = \text{colim}_{i \in I} Y_1^i$ in $\text{Fun}(C^{op}, 1 - \text{Cat})$ then

$$\mathbf{Map}_{\text{Fun}(C^{op}, 1 - \text{Cat})}(Y_1, Y_2) \xrightarrow{\sim} \lim_{c \rightarrow c'} \text{Fun}(Y_1(c'), Y_2(c)) \xrightarrow{\sim} \lim_{c \rightarrow c'} \lim_{i \in I^{op}} \text{Fun}(Y_1^i(c'), Y_2(c))$$

permuting the two latter limits, one gets the desired isomorphism. \square

Is it true that the functor $\mathcal{D}^{op} \rightarrow 1 - \text{Cat}$, $x \mapsto \mathbf{Map}_{\mathcal{D}}(x, d)$ for a 2-category \mathcal{D} , preserves limits? No in general!!

We see that if $S \in \text{Sch}^{aff}$ then $\text{ShvCat}_{/S}^{naive} \xrightarrow{\sim} (\text{QCoh}(S) - \text{mod})$. More generally, if $Y \in \text{PreStk}$ then $\text{ShvCat}_{/Y}^{naive} \xrightarrow{\sim} \text{ShvCat}(Y)$ by the above lemma.

0.0.16. For 4.20.1. Let $Y \in \text{PreStk}^{lax}$. We have

$$\text{ShvCat}_{/Y^{\text{PreStk}}} \xrightarrow{\sim} \text{ShvCat}(Y^{\text{PreStk}}) \xrightarrow{\sim} \lim_{S \rightarrow S'} \text{Fun}(Y(S)^{\text{Spc}}, \text{ShvCat}(S'))$$

in $1 - \text{Cat}$, the limit over $Tw((\text{Sch}^{aff})^{op})$. To see that $\text{ShvCat}_{/Y} \rightarrow \text{ShvCat}_{/Y^{\text{PreStk}}}$ is conservative, using (3) it would suffice to show that for any $S, S' \in \text{Sch}^{aff}$ the map

$$(\text{Fun}(Y(S), \text{ShvCat}(S'))_{lax})^{1 - \text{Cat}} \rightarrow \text{Fun}(Y(S)^{\text{Spc}}, \text{ShvCat}(S'))$$

is conservative. However, this is not the case (cf. remark below). So, I think $\text{ShvCat}_{/Y} \rightarrow \text{ShvCat}_{/Y^{\text{PreStk}}}$ is not conservative.

Remark 0.0.17. *If $S \in 1 - \mathcal{C}at, T \in 2 - \mathcal{C}at$ then the natural map $\text{Fun}(S, T)_{\text{rlax}}^{1-\mathcal{C}at} \rightarrow \text{Fun}(S^{\text{Spc}}, T)^{1-\mathcal{C}at} = \text{Fun}(S^{\text{Spc}}, T^{1-\mathcal{C}at})$ is not conservative in general. This happens already for $S = [1]$. Namely, if $x, y \in T$, $f, g : x \rightarrow y$ are 1-morphisms in T and $\alpha : f \rightarrow g$ is a 2-morphism in T , which is not an isomorphism, we get a 1-morphism in $\text{Fun}(S, T)_{\text{rlax}}^{1-\mathcal{C}at}$ from f to g , which is not an isomorphism, but its restriction to S^{Spc} is an isomorphism. The notation $\text{Fun}(S, T)_{\text{rlax}} \in 2 - \mathcal{C}at$ is that of ([1], ch. A.1, 3.2.7).*

0.0.18. For 4.23. Let \mathcal{X} be a prestack. My understanding is that for a unital sheaf of categories C on $\text{Ran}_{\mathcal{X}}$, the fusion morphism is a 1-morphism in $\text{ShvCat}/_{\text{Ran}_{\mathcal{X}, \emptyset}^{\rightarrow}}$. Here $\text{Ran}_{\mathcal{X}, \emptyset}^{\rightarrow} \in \text{PreStk}$. The property of being *adj-unital* here means that for any $S \in \text{Sch}^{aff}$ with $f : S \rightarrow \text{Ran}_{\mathcal{X}, \emptyset}^{\rightarrow}$ the corresponding map in $\text{ShvCat}(S)$ admits a continuous right adjoint (it will be automatically a strict functor of $\text{QCoh}(S)$ -modules, as $\text{QCoh}(S)$ is rigid).

The unit map from Section 4.24 is a 1-morphism in $\text{ShvCat}/_{\text{Ran}_{\mathcal{X}}}$, no lax prestacks here!

0.0.19. For 4.25.1(5). It seems this comes from the following. Let $Y, C \in 1 - \mathcal{C}at$. Assume given functors $f : Y \rightarrow C$ and $g : Y^{\text{Spc}} \rightarrow C$. Write $i : Y^{\text{Spc}} \rightarrow Y$ for the natural functor. Assume given a 1-morphism $\alpha : g \rightarrow f \circ i$ in $\text{Fun}(Y^{\text{Spc}}, C)$ such that for any $y \in Y$, $\alpha(y) : g(y) \rightarrow f(y)$ is a monomorphism in C . Assume also for any 1-morphism $h : y_1 \rightarrow y_2$ in Y the induced map $f(h) : f(y_1) \rightarrow f(y_2)$ in C is such that the composition

$$g(y_1) \xrightarrow{\alpha(y_1)} f(y_1) \xrightarrow{f(h)} f(y_2)$$

factors through $g(y_2) \xrightarrow{\alpha(y_2)} f(y_2)$. Then one gets a functor $g' : Y \rightarrow C$ together with a morphism $g' \rightarrow f$ in $\text{Fun}(Y, C)$ such that the composition $Y^{\text{Spc}} \rightarrow Y \xrightarrow{g'} C$ is g .

0.0.20. For 4.27. Given $\mathcal{X} \in \text{PreStk}$, one first defines $[\text{Ran}_{\mathcal{X}, \emptyset} \times \text{Ran}_{\mathcal{X}, \emptyset}]_{\text{disj}} \in \text{PreStk}$ so that its S -points is

$$\text{colim}_{I, J \in \text{fSet}_{\emptyset}^{\text{op}}} [\mathcal{X}^I \times \mathcal{X}^J]_{\text{disj}}$$

Given prestack \mathcal{X} , the lax prestack $[\text{Ran}_{\mathcal{X}}^{\text{un}} \times \text{Ran}_{\mathcal{X}}^{\text{un}}]_{\text{disj}}$ is defined as follows. For each $S \in \text{Sch}^{aff}$, the space underlying $(\text{Ran}_{\mathcal{X}}^{\text{un}} \times \text{Ran}_{\mathcal{X}}^{\text{un}})(S)$ is $(\text{Ran}_{\mathcal{X}, \emptyset} \times \text{Ran}_{\mathcal{X}, \emptyset})(S)$. We have to precise a full subspace in the latter space, which defines then the corresponding full subcategory. This full subspace is $[\text{Ran}_{\mathcal{X}, \emptyset} \times \text{Ran}_{\mathcal{X}, \emptyset}]_{\text{disj}}(S)$.

0.0.21. The proof of his Lemma 5.10.1. Let $F : \mathcal{J}^{\text{op}} \rightarrow \text{Pr}^L$ be a functor, Pr^L is the notation from HTT, and \mathcal{J} admits fibre products. Let $q : \text{Groth}(F) \rightarrow \mathcal{J}^{\text{op}}$ be the cocartesian fibration corresponding to F . Let $y_1 \leftarrow y \rightarrow y_2$ be a diagram in $\text{Groth}(F)$, let $i_1 \leftarrow i \rightarrow i_2$ be its image in \mathcal{J}^{op} , so $i_1 \rightarrow i \leftarrow i_2$ is the corresponding diagram in \mathcal{J} . Let $\bar{i} = i_1 \times_i i_2$ in \mathcal{J} . By (HTT, 4.3.1.11) there is a diagram

$$(5) \quad \begin{array}{ccc} y_1 & \rightarrow & \bar{y} \\ \uparrow & & \uparrow \\ y & \rightarrow & y_2 \end{array}$$

which is a q -colimit diagram, and lifts the diagram $\bar{i} = i_1 \sqcup_i i_2$ in \mathcal{J}^{op} . Now from (HTT, 4.3.1.5) we see that (5) is a colimit diagram in $Groth(F)$. So, $Groth(F)$ admits pushouts, and q preserves push-outs.

0.0.22. In Sect. 5.13 it is understood that in the last displayed formula the middle diagram is a push-out diagram.

0.0.23. For 5.15. A version with a module would be as follows. Let C be a symmetric monoidal $(\infty, 1)$ -category, $M \in 1 - \mathcal{C}at$ be a C -module. Let $\Phi_C : C \rightarrow 1 - \mathcal{C}at$ be right-lax symmetric monoidal, $\Phi_M : M \rightarrow 1 - \mathcal{C}at$ be a functor such that the map Φ from (C, M) to $(1 - \mathcal{C}at, 1 - \mathcal{C}at)$ is right-lax (in the sense of [1], ch. I.1, 3.5.1), that is, Φ_M is right-lax compatible with actions. Let $f_C : \mathcal{X}_C \rightarrow C, f_M : \mathcal{X}_M \rightarrow M$ be the corresponding cocartesian fibrations. Then \mathcal{X}_C is symmetric monoidal, f_C is symmetric monoidal, and \mathcal{X}_C acts naturally on \mathcal{X}_M . Namely, $(c \in C, y \in \Phi_C(c))$ sends $(m \in M, z \in \Phi_M(m))$ to $(cm \in M, y \circ z \in \Phi_M(cm))$, where $y \circ z$ is the image of (y, z) under the natural map $\Phi_C(c) \times \Phi_M(m) \rightarrow \Phi_M(cm)$. So, the diagram commutes

$$\begin{array}{ccc} \mathcal{X}_C \times \mathcal{X}_M & \rightarrow & \mathcal{X}_M \\ \downarrow & & \downarrow \\ C \times M & \rightarrow & M \end{array}$$

0.0.24. Let Y be a lax prestack. We sometimes want to calculate the limit of the composition $Tw(\mathcal{S}ch^{aff})^{op} \rightarrow (\mathcal{S}ch^{aff})^{op} \rightarrow 1 - \mathcal{C}at$, that is,

$$\lim_{(S \rightarrow S') \in (Tw(\mathcal{S}ch^{aff}))^{op}} Y(S)$$

For any $S \in \mathcal{S}ch^{aff}$, the category $Tw(\mathcal{S}ch^{aff}) \times_{\mathcal{S}ch^{aff}} \mathcal{S}ch_{/S}^{aff}$ has an initial object $S \leftarrow \emptyset \rightarrow *$, hence is contractible. This implies that the RKE of the above composition under $Tw(\mathcal{S}ch^{aff})^{op} \rightarrow (\mathcal{S}ch^{aff})^{op}$ is the same functor $Y : (\mathcal{S}ch^{aff})^{op} \rightarrow 1 - \mathcal{C}at$. So,

$$\lim_{(S \rightarrow S') \in (Tw(\mathcal{S}ch^{aff}))^{op}} Y(S) \xrightarrow{\sim} \lim_{S \in (\mathcal{S}ch^{aff})^{op}} Y(S)$$

For example, let Y be the constant lax prestack with value $C \in 1 - \mathcal{C}at$. Then

$$\mathcal{S}hvCat_{/Y}^{naive} \xrightarrow{\sim} \lim_{S \in (\mathcal{S}ch^{aff})^{op}} \text{Fun}(C, \mathcal{S}hvCat(S)) \xrightarrow{\sim} \text{Fun}(C, \text{DGCat}_{cont})$$

by (4). Indeed, $\mathcal{S}ch^{aff}$ has a final object $\text{Spec } k$. Similarly,

$$\mathcal{S}hvCat_{/Y} \xrightarrow{\sim} \lim_{S \in (\mathcal{S}ch^{aff})^{op}} (\text{Fun}(C, \mathcal{S}hvCat(S))_{llax})^{1-\mathcal{C}at} \xrightarrow{\sim} (\text{Fun}(C, \text{DGCat}_{cont})_{llax})^{1-\mathcal{C}at}$$

We get for the constant lax prestack Y with value $C \in 1 - \mathcal{C}at$, $\text{QCoh}(Y) \xrightarrow{\sim} \text{Fun}(C, \text{Vect})$.

Let Y be the constant prestack with value $C \in 1 - \mathcal{C}at$. Then $\text{QCoh}_Y \in \mathcal{S}hvCat_{/Y}^{naive}$ becomes an object of $\text{Fun}(C, \text{DGCat}_{cont})$, namely the constant functor $C \rightarrow * \rightarrow \text{DGCat}_{cont}$ with value Vect .

If Y is a lax prestack let Y^{inv} be the prestack obtained by termwise inverting all arrows: for $S \in \mathcal{S}ch^{aff}$, $Y^{inv}(S)$ is obtained from $Y(S)$ by inverting all arrows. We have a natural functor $\text{QCoh}(Y^{inv}) \rightarrow \text{QCoh}(Y)$. It is not essentially surjective in general. Already for $Y = [1]$ it is not.

0.0.25. For 5.19. My understanding is that this construction is only used when working with lax prestacks. If we want only to consider $\text{PreStk}_{\text{corr}}^{\text{ShvCat}}$ as a 2-category then already 5.14.1-5.14.2 are sufficient?

0.0.26. For 5.21.1. For each n we have a diagram $S^{\times n} \xleftarrow{m_1} \text{mult}_{n,S} \xrightarrow{m_2} S$, which realizes the n -ary multiplication in $\text{PreStk}_{\text{corr}}^{\text{lax}}$ on S . Then for a multiplicative sheaf of categories Ψ on S we have $m_1^*(\Psi^{\boxtimes n}) \xrightarrow{\sim} m_2^*(\Psi)$ over $\text{mult}_{n,S}$.

0.0.27. The notion of a weakly multiplicative sheaf of categories from 5.20.1 is very interesting in general. For example, let Y be a commutative algebra in PreStk . So, we have the product map $m : Y \times Y \rightarrow Y$ and the unit map $e : * \rightarrow Y$. Then this gives a sheaf of categories $C \in \text{ShvCat}(Y)$ together with maps $C \boxtimes C \rightarrow m^*C$, $\text{Vect} \rightarrow e^*C$ satisfying the compatibilities. What are nontrivial examples in $\text{MultCat}^w(Y)$? What happens for Y an affine scheme?

Note that $\text{MultCat}^w(Y)$ is symmetric monoidal ∞ -category. Example: let $Y = \text{Spec } k$, so $C \in \text{DGCat}_{\text{cont}}$ is equipped with $C \otimes C \rightarrow C$ and $\text{Vect} \rightarrow C$ making it into a commutative algebra in $\text{DGCat}_{\text{cont}}$. My understanding is that

$$\text{MultCat}^w(\text{Spec } k) \xrightarrow{\sim} \text{CAlg}(\text{DGCat}_{\text{cont}})$$

If $f : Y' \rightarrow Y$ is a map in $\text{CAlg}(\text{PreStk})$ and C is a weakly multiplicative sheaf of categories on Y then f^*C is a weakly multiplicative sheaf of categories on Y' , and $f^* : \text{MultCat}^w(Y) \rightarrow \text{MultCat}^w(Y')$ is symmetric monoidal. So, for a commutative algebra C in $\text{DGCat}_{\text{cont}}$ we get for $g : Y \rightarrow \text{Spec } k$ the weakly multiplicative sheaf of categories g^*C .

Given a weakly multiplicative sheaf of categories D on $Y \in \text{CAlg}(\text{PreStk})$, its restriction under $e : \text{Spec } k \rightarrow Y$ is an object of $\text{CAlg}(\text{DGCat}_{\text{cont}})$.

0.0.28. For 5.24. Assume $S \in \text{CAlg}(\text{PreStk}_{\text{corr}}^{\text{lax}})$ and $\Psi, \Psi' \in \text{MultCat}^w(S)$. To understand the difference between $\text{MultCat}^w(S)$ and $\text{MultCat}^{w,\text{lax}}(S)$ note the following. If $\alpha : \Psi \rightarrow \Psi'$ is a morphism in $\text{MultCat}^w(S)$ then we get a commutative diagram in $\text{ShvCat}/\text{mult}_S$ (that is, an explicit invertible 2-morphism in the 2-category $\text{ShvCat}/\text{mult}_S$ is given making the square commute)

$$\begin{array}{ccc} m_1^*(\Psi \boxtimes \Psi) & \xrightarrow{\eta} & m_2^*\Psi \\ \downarrow m_1^*(\alpha \boxtimes \alpha) & & \downarrow m_2^*(\alpha) \\ m_1^*(\Psi' \boxtimes \Psi') & \xrightarrow{\eta'} & m_2^*\Psi' \end{array}$$

Here η, η' are the corresponding structure maps for the weakly multiplicative sheaves of categories Ψ, Ψ' . If α is only a morphism in $\text{MultCat}^{w,\text{lax}}(S)$ then for the above square we are only given a 2-morphism

$$\delta : \eta' \circ m_1^*(\alpha \boxtimes \alpha) \rightarrow m_2^*(\alpha) \circ \eta$$

in $\text{ShvCat}/\text{mult}_S$. So, we here essentially use the 2-category structure of $\text{ShvCat}/\text{mult}_S$.

For example, if $\Psi = \text{QCoh}_S$ then the above map α is given by a section $s \in \Gamma(S, \Psi')$ together with a 1-morphism $\delta : \eta' m_1^*(s \boxtimes s) \rightarrow m_2^*s$ in $\Gamma(\text{mult}_S, m_2^*\Psi')$. If δ is an isomorphism then this is a map in $\text{MultCat}^w(S)$

So, we see that already for $S \in \mathcal{CAlg}(\text{PreStk}_{\text{corr}})$ there is a big difference between $\text{MultCat}^w(S)$ and $\text{MultCat}^{w,\text{lax}}(S)$, as the DG -category $\Gamma(\text{mult}_S, m_2^* \Psi')$ is not a groupoid.

My understanding is that Sam means by morphisms in $\text{MultCat}^{w,\text{lax}}(S)$ the nonunital right-lax maps, that is, the above morphism from Ψ to Ψ' in $\text{MultCat}^{w,\text{lax}}(S)$ contains as a part of data a 2-morphism

$$\eta'_e \rightarrow (e_2^* \alpha) \circ \eta_e$$

in $\text{ShvCat}/\text{unit}_S$ for the diagram

$$\begin{array}{ccc} e_1^* \text{Vect} & \xrightarrow{\eta_\xi} & e_2^* \Psi \\ & \searrow \eta'_e & \downarrow e_2^* \alpha \\ & & e_2^* \Psi' \end{array}$$

Here I used the notations from his Section 5.21, namely $* \xleftarrow{\xi_1} \text{unit}_S \xrightarrow{\xi_2} S$ is the unit map for the algebra S in $\text{PreStk}_{\text{corr}}^{\text{lax}}$.

0.1. For 5.26. The functor $\Gamma(-, -) : \text{Groth}(\text{ShvCat}/_-) \rightarrow \text{DGCat}_{\text{cont}}$ sends $(Y, C \in \text{ShvCat}/_Y)$ to $\Gamma(Y, C)$. A map from (Y', C') to (Y, C) in $\text{Groth}(\text{ShvCat}/_-)$ is a map of lax prestacks $f : Y \rightarrow Y'$ and $f^* C' \rightarrow C$ in $\text{ShvCat}/_Y$. The corresponding map $\Gamma(Y', C') \rightarrow \Gamma(Y, C)$ is the composition $\Gamma(Y', C') \rightarrow \Gamma(Y, f^* C') \rightarrow \Gamma(Y, C)$.

Similarly, consider the cocartesian fibration $\mathcal{X} \rightarrow \text{Groth}(\text{ShvCat}/_-)$ corresponding to this functor. Then an object of \mathcal{X} is $(Y \in \text{PreStk}^{\text{lax}}, C \in \text{ShvCat}/_Y, \mathcal{F} \in \Gamma(Y, C))$. Consider a map $(Y_1, C_1) \rightarrow (Y_2, C_2)$ given by $f : Y_2 \rightarrow Y_1, f^* C_1 \rightarrow C_2$. A map from $(Y_1, C_1, \mathcal{F}_1)$ to $(Y_2, C_2, \mathcal{F}_2)$ in \mathcal{X} over it is given in addition by a map $\mathcal{F} \rightarrow \mathcal{F}_2$ in $\Gamma(Y_2, C_2)$. Here \mathcal{F} is the image of $f^* \mathcal{F}_1$ under the induced map $\Gamma(Y_2, f^* C_1) \rightarrow \Gamma(Y_2, C_2)$.

0.1.1. For 5.30. To be clear, $\text{Mult}^{op-w}(\Psi)$ is the category of maps $\text{QCoh}_S \rightarrow \Psi$ in $\text{MultCat}^{op-w,\text{lax}}(S)$. What is the definition of $\text{MultCat}^{op-w,\text{lax}}(S)$? View S as a commutative coalgebra in $\text{PreStk}_{\text{corr}}^{\text{lax}}$ by flipping correspondences. Then $\text{MultCat}^{op-w,\text{lax}}(S)$ is the category of commutative coalgebras in $\text{PreStk}_{\text{corr}}^{\text{lax}, \text{ShvCat}}$ over S , where we allow lax morphisms of commutative coalgebras. However, should we allow "left-lax" or "right-lax" morphisms? According to the next paragraph, we have to allow the following. Given $\alpha : \Psi \rightarrow \Psi'$ in $\text{MultCat}^{op-w,\text{lax}}(S)$, we will have the square

$$\begin{array}{ccc} m_2^* \Psi' & \xrightarrow{\tilde{\eta}'_e} & m_1^*(\Psi' \boxtimes \Psi') \\ \uparrow m_2^* \alpha & & \uparrow m_1^*(\alpha \boxtimes \alpha) \\ m_2^* \Psi & \xrightarrow{\tilde{\eta}_e} & m_1^*(\Psi \boxtimes \Psi) \end{array}$$

This diagram only "lax commutes" in the sense that we are given a 2-morphism $\tilde{\eta}'_e \circ m_2^* \alpha \rightarrow m_1^*(\alpha \boxtimes \alpha) \circ \tilde{\eta}_e$. And similarly for the "counit" maps $\tilde{\eta}_e$.

Suppose the structure maps in $\Psi \in \text{MultCat}^{op-w}(S)$ are given by his (5.29.1), that is,

$$(6) \quad \begin{array}{l} \tilde{\eta}_m : m_2^* \Psi \rightarrow m_1^*(\Psi \boxtimes \Psi) \in \text{ShvCat}/\text{mult}_S \text{ and} \\ \tilde{\eta}_e : e_2^*(\Psi) \rightarrow \text{QCoh}_{\text{unit}_\Psi} \in \text{ShvCat}/\text{unit}_S \end{array}$$

Then an op-weakly multiplicative object in Ψ is an element $s \in \Gamma(S, \Psi)$ together with maps

$$\tilde{\eta}_m m_2^*(s) \rightarrow m_1^*(s \boxtimes s) \in \Gamma(\text{mult}_S, m_1^*(\Psi \boxtimes \Psi))$$

and

$$\tilde{\eta}_e e_2^*(s) \rightarrow k \in \Gamma(\text{unit}_S, \text{QCoh})$$

with the corresponding compatibilities.

0.1.2. In Lemma 5.30.1 there is a mistake: the category $\text{MultCat}^{op-w}(\mathcal{S})$ is not presentable in general, though it probably has colimits and limits. This already happens for $\mathcal{S} = *$, in which case this is the category of commutative coalgebras in DGCat_{cont} . The category $\text{CCoalg}(\text{DGCat}_{cont})$ admits colimits, and the forgetful functor $\text{CCoalg}(\text{DGCat}_{cont}) \rightarrow \text{DGCat}_{cont}$ preserves colimits by (HA, 3.2.2.4).

0.1.3. For 5.35. Assume $\mathcal{T} : (\text{PreStk}^{lax})^{op} \rightarrow 1 - \text{Cat}$ is a right-lax symmetric monoidal functor. Then Sam defined the notion of a multiplicative sheaf on $\mathcal{S} \in \text{CAlg}(\text{PreStk}_{corr}^{lax})$ with values in \mathcal{T} . This is useful for considering other theories of sheaves as in ([2], Sect. 1.1).

0.1.4. For 5.27. For convenience, the definition of a weakly multiplicative object of a weakly multiplicative sheaf of categories: let $S \in \text{CAlg}(\text{PreStk}_{corr}^{lax})$, Ψ be a weakly multiplicative sheaf of categories on S . So, we have the map

$$\eta_m : m_1^*(\Psi \boxtimes \Psi) \rightarrow m_2^*\Psi$$

in $\text{ShvCat}/\text{mult}_S$ and the map

$$\eta_e : e_1^* \text{Vect} \rightarrow e_2^*(\Psi)$$

in $\text{ShvCat}/\text{unit}_S$. A weakly multiplicative object ψ of Ψ (defined in his Section 5.25.1) gives rise to morphisms

$$\eta_m m_1^*(\psi \boxtimes \psi) \rightarrow m_2^*\psi$$

in $\Gamma(\text{mult}_S, m_2^*\Psi)$, and

$$\eta_e(\mathcal{O}_{\text{unit}_S}) \rightarrow e_2^*\psi$$

in $\Gamma(\text{unit}_S, e_2^*\Psi)$ with the corresponding compatibilities.

0.1.5. For 6.3. For $\mathcal{X} \in \text{PreStk}$, $[\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj}$ is a nonunital commutative algebra in PreStk_{corr} via the product

$$\begin{array}{ccc} [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \times [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} & \leftarrow & [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \\ & & \downarrow \\ & & [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \end{array}$$

Here in the middle we have the prestack given by 4 subsets of \mathcal{X} , which are pairwise disjoint. The diagram

$$\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}} \xleftarrow{m_1} [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \xrightarrow{add} \text{Ran}_{\mathcal{X}}$$

is a diagram of maps of nonunital commutative algebras in PreStk_{corr} . So, if C is a chiral category for \mathcal{X} , both $\text{add}^* C$ and $C \boxtimes C$ are multiplicative sheaves of categories on $[\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj}$. My understanding is that the isomorphism

$$m_1^*(C \boxtimes C) \xrightarrow{\sim} \text{add}^* C$$

on $[\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj}$ is an isomorphism of multiplicative sheaves of categories. Here $m_1 : [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \hookrightarrow \text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}$ is the inclusion. Similarly also in the unital case for $\text{Ran}_{\mathcal{X}}^{un}$. Besides, in the unital case the map $i : \text{Spec } k \xrightarrow{\emptyset} \text{Ran}_{\mathcal{X}}^{un}$ is a morphism of commutative algebras in $\text{PreStk}_{corr}^{lax}$, so i^*C is a multiplicative sheaf of categories on $\text{Spec } k$. I think for a unital factorization category C the isomorphism $i^*C \xrightarrow{\sim} \text{Vect}$ is an isomorphism of multiplicative sheaves of categories.

This looks like the claim that $\mathcal{CAlg}(\mathcal{CAlg}(C))$ is isomorphic to C for $C \in 1 - \text{Cat}$. Sam says this is true for any $C \in 1 - \text{Cat}$.

A version of this for unital categories explains the fact that $unit_C$ is a factorization algebra in C . We get isomorphisms $i^*unit_C \xrightarrow{\sim} k$ and

$$\eta_m m_1^*(unit_C \boxtimes unit_C) \xrightarrow{\sim} add^*unit_C$$

If $C, D \in \text{ShvCat}/\text{Ran}_{\mathcal{X}}^{un}$ and $f : C \rightarrow D$ is a map in $\text{ShvCat}/\text{Ran}_{\mathcal{X}}^{un}$ then we get units $unit_C \in \Gamma(\text{Ran}_{\mathcal{X}}, C)$, $unit_D \in \Gamma(\text{Ran}_{\mathcal{X}}, D)$ as in his Section 4.24. Then f yields a natural map $unit_D \rightarrow f(unit_C)$ in $\Gamma(\text{Ran}_{\mathcal{X}}, D)$. Indeed, for $S \in \text{Sch}^{aff}$ and any $g \in \text{Ran}_{\mathcal{X}}(S)$ consider the morphism $\emptyset \xrightarrow{g} g$ in $\text{Ran}_{\mathcal{X}}^{un}(S)$. The description of f in his Section 4.17 gives a 2-morphism $unit_D \rightarrow f \circ unit_C$ for the square below

$$\begin{array}{ccc} \text{QCoh}_S & \xrightarrow{unit_C} & g^*C \\ \downarrow \text{id} & & \downarrow f \\ \text{QCoh}_S & \xrightarrow{unit_D} & g^*D \end{array}$$

The corresponding universal 2-morphism becomes the desired map $unit_D \rightarrow f(unit_C)$ in $\Gamma(\text{Ran}_{\mathcal{X}}, D)$.

Let now D be a unital chiral category for \mathcal{X} , and $s : \text{QCoh} \rightarrow D$ be a morphism in $\text{MultCat}^w(\text{Ran}_{\mathcal{X}}^{un})$ giving a unital factorization algebra s in D . View s just as a map in $\text{ShvCat}/\text{Ran}_{\mathcal{X}}^{un}$. Note that $\mathcal{O}_{\text{Ran}_{\mathcal{X}}} = unit_{\text{QCoh}} \in \Gamma(\text{Ran}_{\mathcal{X}}, \text{QCoh})$. The corresponding map $unit_D \rightarrow f(unit_{\text{QCoh}}) = s$ is *the unit map* for the factorization algebra s as his map (6.3.2). He claims the latter is a map of unital factorization algebras in D .

0.2. For Appendix A. If $Y \in \text{PreStk}$ then $\text{ShvCat}(Y)$ has a natural structure of an $(\infty, 2)$ -category. For $Y \in \text{Sch}^{aff}$ this is done in ([1], ch. I.1, 8.3). Namely, in this case the mapping category is the inner hom by ([1], ch. I.1, 8.3.3). Maybe for any prestack Y , the mapping category should be the inner hom for its symmetric monoidal structure? Not clear.

Given $f : S' \rightarrow S$ in Sch^{aff} , the functor $\text{QCoh}(S) - mod \rightarrow \text{QCoh}(S') - mod$ is a map in $2 - \text{Cat}$, it is clear that we have the corresponding morphisms of the mapping categories (cf. [11], 6.0.6). Sam confirms this, as $2 - \text{Cat}$ can be thought of as the category of $(\infty, 1)$ -categories enriched over $1 - \text{Cat}$. So, the limit $\lim_{S \rightarrow Y} \text{QCoh}(S) - mod$ over $(\text{Sch}_Y^{aff})^{op}$ can be seen as a limit in $2 - \text{Cat}$. Then according to ([11], 13.1.38), we get the mapping category for $C, D \in \text{ShvCat}(Y)$

$$(7) \quad \mathbf{Map}_{\text{ShvCat}(Y)}(C, D) \xrightarrow{\sim} \lim_{S \rightarrow Y} \mathbf{Map}_{\text{ShvCat}(S)}(\Gamma(S, C), \Gamma(S, D))$$

the limit over $(\text{Sch}_{/Y}^{\text{aff}})^{\text{op}}$. Here for $S \in \text{Sch}^{\text{aff}}$, $E_i \in \text{QCoh}(S) - \text{mod}$,

$$\mathbf{Map}_{\text{ShvCat}(S)}(E_1, E_2) = \text{Fun}_{\text{QCoh}(S)}(E_1, E_2)$$

is the inner hom in $\text{ShvCat}(S)$. The limit (7) can be seen as a limit in $\text{DGCat}_{\text{cont}}$, as the projection $\text{DGCat}_{\text{cont}} \rightarrow 1 - \text{Cat}$ preserves limits. So, $\text{ShvCat}(Y)$ is enriched over $\text{DGCat}_{\text{cont}}$. My understanding is that the 2-category $\text{ShvCat}(Y)$ is actually enriched over $\text{QCoh}(Y) - \text{mod}$.

For $C \in \text{ShvCat}(Y)$ we get $\mathbf{Map}_{\text{ShvCat}(Y)}(\text{QCoh}_Y, C) \xrightarrow{\sim} \Gamma(Y, C)$ canonically.

Actually, a stronger claim holds. Namely, for $C, D \in \text{ShvCat}_{/Y}$ there is an inner hom from C to D in the symmetric monoidal category $\text{ShvCat}_{/Y}$. Indeed, for any $S' \rightarrow S \rightarrow Y$ with $S, S' \in \text{Sch}^{\text{aff}}$, there is a canonical isomorphism

$$\text{Fun}_{\text{QCoh}(S)}(\Gamma(S, C), \Gamma(S, D)) \otimes_{\text{QCoh}(S)} \text{QCoh}(S') \xrightarrow{\sim} \text{Fun}_{\text{QCoh}(S')}(\Gamma(S', C), \Gamma(S', D))$$

It holds because $\text{QCoh}(S), \text{QCoh}(S')$ are rigid by ([11], 6.0.6). So, we get the inner hom $\mathcal{H}om(C, D) := \underline{\text{Hom}}_{\text{ShvCat}_{/Y}}(C, D)$ whose sections over S are

$$\text{Fun}_{\text{QCoh}(S)}(\Gamma(S, C), \Gamma(S, D))$$

Now (7) is the category of global sections of $\underline{\text{Hom}}_{\text{ShvCat}_{/Y}}(C, D)$. We may simply say that $\text{ShvCat}_{/Y}$ is a 2-category, because it admits inner homs, which are enriched over $\text{DGCat}_{\text{cont}}$.

More generally, we may view the functor $\text{ShvCat}_{/-} : (\text{Sch}^{\text{aff}})^{\text{op}} \rightarrow 1 - \text{Cat}$, $S \mapsto \text{ShvCat}(S)$ as a functor $\text{ShvCat}_{/-} : (\text{Sch}^{\text{aff}})^{\text{op}} \rightarrow 2 - \text{Cat}$, and consider its RKE as a functor $\text{ShvCat}_{/-} : \text{PreStk}^{\text{op}} \rightarrow 2 - \text{Cat}$. In particular, for any morphism $f : Y' \rightarrow Y$ in PreStk , we get a morphism $f^* : \text{ShvCat}_{/Y} \rightarrow \text{ShvCat}_{/Y'}$ of 2-categories.

Assume $f : X \rightarrow Y$ is 1-affine (cf. A.8). Then the functor $f_* : \text{ShvCat}_{/X} \rightarrow \text{ShvCat}_{/Y}$ from Prop. A.9.1(2) is a morphism in $2 - \text{Cat}$. Indeed, assume first Y affine. Then f_* is the forgetful functor $\text{QCoh}(X) - \text{mod} \rightarrow \text{QCoh}(Y) - \text{mod}$. This is a map in $2 - \text{Cat}$ by ([11], 6.0.6). Now let Y be arbitrary, $C, D \in \text{ShvCat}_{/X}$. We claim that there is a canonical map

$$f_* \mathcal{H}om(C, D) \rightarrow \mathcal{H}om(f_* C, f_* D)$$

in $\text{ShvCat}_{/Y}$. Indeed, from adjointness and the projection formula, we get a natural map

$$f_* C \otimes f_* \mathcal{H}om(C, D) \xrightarrow{\sim} f_*(\mathcal{H}om(C, D) \otimes f^* f_* C) \rightarrow f_*(\mathcal{H}om(C, D) \otimes C) \rightarrow f_* D$$

Finally, passing to the global sections, we get the desired functor between the mapping categories.

0.2.1. For A.2.3. He uses ([1], ch. I.1, Lm 5.4.3) to see that $C \otimes_{A - \text{mod}} B - \text{mod} \rightarrow C$ is conservative.

0.2.2. For A.3. Let $f : X \rightarrow Y$ be a morphism in PreStk . Then I think in general $\text{coind}_f : \text{ShvCat}(X) \rightarrow \text{ShvCat}(Y)$ does not preserve all colimits. What happens for $f : \bigsqcup_{n \geq 0} * \rightarrow *$?

0.2.3. For A.4.3. If $F : D \rightarrow C$ is a map of A -module categories, this is a map in $\mathcal{A} - \text{mod}(\text{DGCat}_{\text{cont}})$ with $\mathcal{A} = A - \text{mod}$. So, F is continuous. Let $\mathcal{X} \subset D \otimes_{A - \text{mod}} B - \text{mod}$ be the full subcategory of those x such that for any $y \in D \otimes_{A - \text{mod}} B - \text{mod}$ the map $\text{Map}_{D \otimes_{A - \text{mod}} B - \text{mod}}(x, y) \rightarrow \text{Map}_{C \otimes_{A - \text{mod}} B - \text{mod}}(F_B(x), F_B(y))$ is an isomorphism. Then it is easy to see that \mathcal{X} is closed under colimits (since F_B preserves colimits). Since the image of the functor $D \rightarrow D \otimes_{A - \text{mod}} B - \text{mod}$ generates $D \otimes_{A - \text{mod}} B - \text{mod}$ under colimits, it suffices indeed to show that (A.4.1) is an isomorphism.

0.2.4. For A.4.5. The projection $\text{DGCat}_{\text{cont}} \rightarrow 1 - \text{Cat}$ preserves limits (cf.[11]). The limit of the constant family in $1 - \text{Cat}$ over a contractible category is the same constant category. (Contractibility is needed: for example take $\mathcal{J} = * \sqcup *$).

0.2.5. For A.5. If $\text{Spec } A \in \text{Sch}^{\text{aff}}$, $F : C \rightarrow D$ is a morphism of A -linear categories then $F(C) \subset D$ is the smallest cocomplete presentable subcategory containing the objects $F(x), x \in C$. Then $F(C) \in \text{DGCat}_{\text{cont}}$ and is preserved under $A - \text{mod}$ -action on D , so $F(C)$ is an A -linear category.

If $F : C \rightarrow D$ is a morphism in $\text{ShvCat}(Y)$ for some $Y \in \text{PreStk}$ then $F(C) \in \text{ShvCat}(Y)$ is defined as follows. For $S \in \text{Sch}_{/Y}^{\text{aff}}$ we let $\Gamma(S, F(C)) = F(\Gamma(S, C))$. Here we also denoted by $F : \Gamma(S, C) \rightarrow \Gamma(S, D)$ the corresponding S -linear functor, and $F(\Gamma(S, C)) \subset \Gamma(S, D)$ is the presentable stable subcategory generated under colimits by the image of $\Gamma(S, C)$. By A.5.1, this is indeed a sheaf of categories on Y , and $F(C) \subset D$ is locally fully faithful.

0.2.6. For A.6. Recall that for $Y \in \text{PreStk}$, $\text{ShvCat}(Y)$ admits all colimits (computed locally). So, for a map $C \rightarrow D$ in $\text{ShvCat}(Y)$, we get $D/C \in \text{ShvCat}(Y)$. For $S \rightarrow Y$ with $S \in \text{Sch}^{\text{aff}}$, $\Gamma(S, D/C) \xrightarrow{\sim} \Gamma(S, D)/\Gamma(S, C)$, the latter colimit is taken in $\text{QCoh}(S) - \text{mod}$.

0.2.7. Let $S = \text{Spec } A \in \text{Sch}^{\text{aff}}$, $h : C^0 \subset C$ a diagram in $\mathcal{A} - \text{mod}(\text{DGCat}_{\text{cont}})$ with $\mathcal{A} = A - \text{mod}$. In particular, h is continuous. The quotient C/C_0 can also be calculated as a limit, namely let $h^R : C \rightarrow C^0$ be the right adjoint to h . It is a strict functor of \mathcal{A} -module categories, but h^R need not be continuous. Since colimits in $\text{DGCat}_{\text{cont}}$ can be rewritten by passing to right adjoints as limits in DGCat , we get $C/C_0 \xrightarrow{\sim} 0 \times_{C^0} C = \text{Ker}(h^R)$. So, the evaluation $e : C/C_0 \rightarrow C$ is fully faithful, hence conservative. It follows that its left adjoint $p : C \rightarrow C/C^0$ generates C/C^0 under colimits by ([1], ch. I.1, Lm.5.4.3), and p is a localization functor.

Lemma 0.2.8. *Under the above assumptions, the kernel of $p : C \rightarrow C/C^0$ is C^0 .*

Proof. Set $D = \text{Ker } p$, so $C^0 \subset D \subset C$ are full subcategories. By definition, $D = \{x \in C \mid \text{for any } y \in C/C^0, \text{Map}_C(x, y) = *\}$. For $x \in D, y \in C/C^0$ we get $\text{Maps}_{k, C}(x, y) = 0$, because p is Vect-linear. So, $D = \{x \in C \mid \text{for any } y \in C/C^0, \text{Maps}_{k, C}(x, y) = *\}$.

Given $x \in C$, consider the fibre sequence $hh^R(x) \xrightarrow{\alpha} x \xrightarrow{\beta} y$ in C , where α is the natural map. Applying the exact functor h^R to this sequence, we get $h^R(x) \xrightarrow{\text{id}} h^R(x) \rightarrow 0$, so $y \in \text{Ker}(h^R) \xrightarrow{\sim} C/C^0$. We claim that $\beta : x \rightarrow y$ is isomorphic to the natural map $x \rightarrow ep(x)$ in C . Indeed, $p(\beta)$ is an isomorphism, and $p(y) \xrightarrow{\sim} y$. Assume in addition

$x \in D$. Then $0 \xrightarrow{\sim} p(x)$, so the corresponding map $x \rightarrow ep(x)$ is the map $x \rightarrow 0$, hence α is an isomorphism. So, $x \in C^0$. \square

0.2.9. For Lm. A.7.4. The projection $\mathcal{A} - mod(\mathrm{DGCat}_{cont}) \rightarrow \mathrm{DGCat}_{cont}$ preserves colimits for $\mathcal{A} = A - mod$. Let $F : D \rightarrow C$ be a map in DGCat_{cont} , $C/D = \{0\} \sqcup_D C$ in DGCat_{cont} . The diagram $D \rightarrow F(D) \rightarrow C$ shows that we have a natural map $C/D \rightarrow C/\bar{D}$. To check this is an equivalence, for any $E \in \mathrm{DGCat}$ we check that $\mathrm{Map}(C/\bar{D}, E) \rightarrow \mathrm{Map}(C/D, E)$ is an isomorphism, where $\mathrm{Map} = \mathrm{Map}_{\mathrm{DGCat}_{cont}}$. This is the space $\mathrm{Map}(C, E) \times_{\mathrm{Map}(D, E)} \{0\}$, the full subspace in $\mathrm{Map}(C, E)$ of those functors whose restriction to \bar{D} is isomorphic to zero. Indeed, $D \rightarrow F(D)$ generates $F(D)$ under colimits, and our functors are continuous.

0.2.10. For Pp. A.7.3. For each $i \in \mathcal{J}$ we have a full subcategory $\mathrm{Ker}(C_{i_0} \rightarrow C_i) \subset C_{i_0}$. Since \mathcal{J} is filtered, it is contractible. Maybe it can be checked that $\mathrm{colim}_{i \in \mathcal{J}} \mathrm{Ker}(C_{i_0} \rightarrow C_i)$ is a full subcategory in C_{i_0} , but we don't need this because of Lemma A.7.4.

0.2.11. For Th. A.8.1. The category Sch_{aft} of schemes almost of finite type was defined in ([1], ch. I.3, 3.5.1). The claim is for $S \in \mathrm{Sch}_{aft}$, S_{dR} is 1-affine.

In A.8.2 he means schematic morphism.

0.2.12. For A.9. For a morphism of prestacks $f : X \rightarrow Y$ the functor $f_* : \mathrm{ShvCat}(X) \rightarrow \mathrm{ShvCat}(Y)$ does not need to preserve colimits. The reason is that if $S \rightarrow Y$ with $S \in \mathrm{Sch}^{aff}$, $S \times_Y X$ does not need to be affine. Already the functor $\Gamma_X^{enh} : \mathrm{ShvCat}(X) \rightarrow \mathrm{QCoh}(X) - mod$ does not preserve colimits in general.

For A.9.1(3). Let $f : X \rightarrow Y$ be quasi-compact (quasi-separated) schematic morphism in PreStk , $C \in \mathrm{ShvCat}(Y)$. Since f is 1-affine, by A.9.1(2) it satisfies the projection formula. That is, given $S \rightarrow Y$ with $S \in \mathrm{Sch}^{aff}$, $\Gamma(S, C) \rightarrow \Gamma(S \times_Y X, C)$ is the functor $\alpha : \Gamma(S, C) \rightarrow \Gamma(S, C) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S \times_Y X)$. This α admits a right adjoint, because the functor $f^* : \mathrm{QCoh}(S) \rightarrow \mathrm{QCoh}(S \times_Y X)$ admits a right adjoint $f_* : \mathrm{QCoh}(S \times_Y X) \rightarrow \mathrm{QCoh}(S)$, the latter functor is continuous by ([1], ch. I.3, 2.2.2). So, α^R is a continuous functor of $\mathrm{QCoh}(S)$ -module categories. As S varies, they form a map $f_* f^*(C) \rightarrow C$ in $\mathrm{ShvCat}(Y)$.

How do we see $\mathrm{ShvCat}(Y)$ as a 2-category? If $S \in \mathrm{Sch}^{aff}$ corresponding to an algebra $A \in \mathrm{VAlg}(\mathrm{Vect}^{\leq 0})$ then $\mathrm{QCoh}(S) - mod$ is naturally an $(\infty, 2)$ -category, which I think is $(\mathrm{QCoh}(S) - mod_{cont}^{St, cocmpl})^{2-Cat}$ in the notations of ([1], ch. I.1, 8.3.1). Namely, for $C, D \in \mathrm{QCoh}(S) - mod$, the mapping category is $\mathrm{Fun}_A(C, D)$ with $A = \mathrm{QCoh}(S) - mod$, the category of continuous $\mathrm{QCoh}(S) - mod$ -linear functors as in ([1], ch. I.1, 8.2.1).

So, existence of a right adjoint to $C \rightarrow f_* f^*(C)$ means just that for any $S \rightarrow Y$ with $S \in \mathrm{Sch}^{aff}$ the corresponding functor $\Gamma(S, f_* f^*(C)) \rightarrow \Gamma(S, C)$ is continuous (it is automatically $\mathrm{QCoh}(S)$ -linear).

Now if $C, D \in \mathrm{ShvCat}(Y)$, the mapping category from C to D in the corresponding 2-category is

$$\lim_{S \rightarrow Y} \mathrm{Fun}_{\mathrm{QCoh}(S) - mod}(\Gamma(S, C), \Gamma(S, D)),$$

the limit over $(\mathrm{Sch}/Y)^{aff, op}$.

0.2.13. For A.9.2. Let $f : Y \rightarrow Z$ be quasi-compact (quasi-separated) schematic morphism of prestacks, $C \in ShvCat(Z)$. Given a map $g : S' \rightarrow S$ in Sch^{aff}/Z , consider the diagram

$$\begin{array}{ccc} \Gamma(S, C) & \xrightarrow{\alpha_S^R} & \Gamma(S, C) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S \times_Z Y) \\ \downarrow g^* & & \downarrow g'^* \\ \Gamma(S', C) & \xrightarrow{\alpha_{S'}^R} & \Gamma(S', C) \otimes_{\mathrm{QCoh}(S')} \mathrm{QCoh}(S' \times_Z Y) \end{array}$$

the vertical maps being restrictions, and the horizontal ones are as in A.9.1(3). Let $\alpha_C^R, \alpha_{S'}^R$ denote the corresponding right adjoints, they are continuous, and by ([1], ch. I.3, 2.2.2) the base change gives $\alpha_{S'}^R g'^* \xrightarrow{\sim} g^* \alpha_S^R$. So, we may apply ([1], ch. I.1, Lm. 2.6.4) to see that the functor obtained by passing to the limit

$$f_C^* : \Gamma(Z, C) = \lim_{S \rightarrow Z} \Gamma(S, C) \rightarrow \lim_{S \rightarrow Z} \Gamma(S, C) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S \times_Z Y) = \Gamma(Z, f_* f^* C)$$

admits a right adjoint $(f_C)_*$, and for any $S \rightarrow Z$ with $S \in Sch^{aff}$ the diagram commutes

$$\begin{array}{ccc} \Gamma(Z, C) & \xleftarrow{(f_C)_*} & \Gamma(Z, f_* f^* C) \\ \downarrow & & \downarrow \\ \Gamma(S, C) & \xleftarrow{\alpha_S^R} & \Gamma(S \times_Z Y, f^* C) \end{array}$$

Here the vertical arrows are evaluations (restrictions). Since each α_S^R is continuous, $(f_C)_*$ is also continuous, the transition maps in our projective system preserve limits, see ([11], 2.2.69).

0.2.14. For A.10.2. The proof is not clear. Let us show that for the map $p : Y \rightarrow *$ in PreStk^{lax} the functor $p^* : \mathrm{DGCat}_{cont} = ShvCat_{/*}^{naive} \rightarrow ShvCat_{/Y}^{naive}$ has a right adjoint. Clearly, p^* preserves colimits, however, $ShvCat_{/Y}^{naive}$ is not presentable usually, and we can not apply the adjoint functor theorem. The category $Tw(Sch^{aff})$ is contractible, since it has an initial object. Write $ShvCat_{/Y}^{naive}$ as (4). For an object $(S \xrightarrow{\alpha} S') \in Tw(Sch^{aff})$ we have an adjoint pair

$$res : ShvCat(*) \rightleftarrows \mathrm{Fun}(Y(S'), ShvCat(S)) : \gamma_{S', S}$$

Here $\gamma_{S', S}$ is the composition $\mathrm{Fun}(Y(S'), ShvCat(S)) \xrightarrow{\lim} ShvCat(S) \xrightarrow{\Gamma(S, \cdot)} ShvCat(*)$. Passing to the limit over $Tw(Sch^{aff})^{op}$ the functors res in the above yield our functor p^* . Given a map from $(S \xrightarrow{\alpha} S')$ to $(T \xrightarrow{\beta} T')$ in $Tw(Sch^{aff})$, we have a commutative diagram

$$\begin{array}{ccc} ShvCat(*) & \xrightarrow{res} & \mathrm{Fun}(Y(S'), ShvCat(S)) \\ & \searrow_{res} & \uparrow_t \\ & & \mathrm{Fun}(Y(T'), ShvCat(T)), \end{array}$$

the vertical arrows is the transition functor in our diagram indexed by $Tw(Sch^{aff})^{op}$. The functor t admits a right adjoint t^R , which is not continuous in general.

Consider the functor $ShvCat_{/Y}^{naive} \rightarrow \mathrm{DGCat}_{cont}$ sending C to

$$(8) \quad \Gamma^{naive}(Y, C) = \lim_{(S \xrightarrow{\alpha} S') \in Tw(Sch^{aff})^{op}} \Gamma(S, \lim_{Y(S')} C_\alpha),$$

where we denoted by C_α the image of C in $\text{Fun}(Y(S'), \text{ShvCat}(S))$. In the above formula $\Gamma(S, \lim_{Y(S')} C_\alpha) \xrightarrow{\sim} \lim_{Y(S')} \Gamma(S, C_\alpha)$. We claim that this is the right adjoint to p^* . Indeed, the mapping spaces in ShvCat/Y^{naive} are written as a similar limit over $\text{Tw}(\text{Sch}^{aff})^{op}$, and we get

$$\begin{aligned} \text{Map}_{\text{ShvCat}/Y^{naive}}(p^*D, C) &\xrightarrow{\sim} \lim_{(S \xrightarrow{\alpha} S')} \text{Map}_{\text{Fun}(Y(S'), \text{ShvCat}(S))}((p^*D)_\alpha, C_\alpha) \xrightarrow{\sim} \\ &\lim_{(S \xrightarrow{\alpha} S')} \text{Map}_{\text{DGCat}_{cont}}(D, \Gamma(S, \lim_{Y(S')} C_\alpha)) \xrightarrow{\sim} \text{Map}_{\text{DGCat}_{cont}}(D, \lim_{(S \xrightarrow{\alpha} S')} \Gamma(S, \lim_{Y(S')} C_\alpha)) \end{aligned}$$

Here $\lim_{(S \xrightarrow{\alpha} S')}$ stands for the limit over $\text{Tw}(\text{Sch}^{aff})^{op}$. We used the fact that there is

a functor $\text{Tw}(\text{Sch}^{aff})^{op} \rightarrow \text{DGCat}_{cont}$ sending α as above to $\Gamma(S, \lim_{Y(S')} C_\alpha)$. Does $\Gamma^{naive}(Y, C)$ coincide with $\mathbf{Map}_{\text{ShvCat}/Y^{naive}}(\text{QCoh}_Y, C)$? Not clear.

Is $\Gamma^{naive}(Y, C)$ enriched over $\text{QCoh}(Y)$ naturally? One gets

$$\begin{aligned} \Gamma^{naive}(Y, \text{QCoh}_Y) &\xrightarrow{\sim} \lim_{(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \lim_{Y(S')} \text{QCoh}(S) \xrightarrow{\sim} \\ &\lim_{(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Fun}(Y^{inv}(S'), \text{QCoh}(S)) \xrightarrow{\sim} \text{QCoh}(Y^{inv}) \end{aligned}$$

Here $Y^{inv} \in \text{PreStk}$ is obtained by inverting all arrows in Y . We have the restriction functor $\text{QCoh}(Y^{inv}) \rightarrow \text{QCoh}(Y)$, which is not an equivalence in general.

We used the fact that for $C, D \in 1 - \text{Cat}$, the limit of the constant functor $\lim_C D$ with value D is $\text{Fun}(C^{inv}, D)$.

By functoriality, $\Gamma^{naive}(Y, C)$ is naturally a module over $\text{QCoh}(Y^{inv}) = \Gamma^{naive}(Y, \text{QCoh})$.

If now $f : Y \rightarrow T$, where $T \in \text{Sch}^{aff}$ then we get the direct image functor $f_* : \text{ShvCat}/Y^{naive} \rightarrow \text{ShvCat}/T$ as the composition

$$\text{ShvCat}/Y^{naive} \rightarrow \text{QCoh}(Y^{inv}) - \text{mod}(\text{DGCat}_{cont}) \rightarrow \text{QCoh}(T) - \text{mod}(\text{DGCat}_{cont})$$

Let now $f : Y \rightarrow Z$ be a map in PreStk^{lax} , $D \in \text{ShvCat}/Z^{naive}$, $C \in \text{ShvCat}/Y^{naive}$. We get

$$\text{Map}_{\text{ShvCat}/Y^{naive}}(f^*D, C) \xrightarrow{\sim} \lim_{(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Map}_{\text{Fun}(Y(S'), \text{ShvCat}(S))}((f^*D)_\alpha, C_\alpha)$$

Given $S' \in \text{Sch}^{aff}$, f yields the functor $Y(S') \rightarrow Z(S')$, so an adjoint pair

$$\text{res}_\alpha : \text{Fun}(Z(S'), \text{ShvCat}(S)) \rightleftarrows \text{Fun}(Y(S'), \text{ShvCat}(S)) : \text{RKE}_\alpha$$

and $\text{res}_\alpha(D_\alpha) = (f^*D)_\alpha$. Thus the above mapping space rewrites as

$$\lim_{(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Map}_{\text{Fun}(Z(S'), \text{ShvCat}(S))}(D_\alpha, \text{RKE}_\alpha C_\alpha)$$

Consider the functor $f_*^{naive} : \text{ShvCat}/Y^{naive} \rightarrow \text{ShvCat}/Z^{naive}$ sending C given by a projective system (C_α) as above to the functor $\text{Tw}(\text{Sch}^{aff})^{op} \rightarrow 1 - \text{Cat}$ given by

$$(9) \quad \alpha \mapsto \text{RKE}_\alpha(C_\alpha) \in \text{Fun}(Z(S'), \text{ShvCat}(S))$$

for $(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})$. Let finally

$$f_*^{naive} C = \lim_{(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})^{op}} RKE_{\alpha}(C_{\alpha})$$

The above shows that

$$\text{Map}_{\text{ShvCat}/Y}^{naive}(f^* D, C) \xrightarrow{\sim} \text{Map}_{\text{ShvCat}/Z}^{naive}(D, f_*^{naive} C)$$

If we are given a map from $(S \xrightarrow{\alpha} S')$ to $(T \xrightarrow{\beta} T')$ in $Tw(\text{Sch}^{aff})$ then write $\delta_{\alpha, \beta} : \text{Fun}(Z(T'), \text{ShvCat}(T)) \rightarrow \text{Fun}(Z(S'), \text{ShvCat}(S))$ for the transition functor in the projective system defining ShvCat/Z^{naive} . Then we only have a natural map

$$(10) \quad \delta_{\alpha, \beta}(RKE_{\beta} C_{\beta}) \rightarrow RKE_{\alpha} C_{\alpha}$$

in $1 - \text{Cat}$. This is what (9) says. The existence of the functor (9) is not rigorously justified. I think if for the above map $\alpha \rightarrow \beta$ in $Tw(\text{Sch}^{aff})$ the induced map $S' \rightarrow T'$ is an isomorphism then (10) is also an isomorphism.

Sam actually claims in A.10.4 that $\Gamma^{naive}(Y, C)$ is also given by a simpler formula:

$$(11) \quad \Gamma^{naive}(Y, C) = \lim_{S \in (\text{Sch}^{aff})^{op}} \lim_{y \in Y(S)} \Gamma(S, y^* C)$$

To be more precise, we consider the cocartesian fibration $Groth(Y) \rightarrow (\text{Sch}^{aff})^{op}$ corresponding to Y . Then the above limit is over $Groth(Y)$.

For $Y \in \text{PreStk}^{lax}$ formulas (8) and (11) give the same result. This follows from the contractibility claim proved in the next subsection.

0.2.15. For A.10.3. For a map $f : A \rightarrow B$ in $1 - \text{Cat}$ he says f is op-cofinal if $f^{op} : A^{op} \rightarrow B^{op}$ is cofinal.

Let Y be a lax prestack, $Groth(Y) \rightarrow (\text{Sch}^{aff})^{op}$ be the cocartesian fibration corresponding to Y . Let $coGroth(Y) \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to Y . Consider the category $Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} coGroth(Y)$. It classifies $(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})^{op}$, and an object $y' \in Y(S')$. We have the map

$$(12) \quad Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} coGroth(Y) \rightarrow Groth(Y)$$

sending (α, y') as above to (S, y) , where y is the composition $S \xrightarrow{\alpha} S' \xrightarrow{y'} Y$. To clarify things, if $(S \xrightarrow{\alpha} S', y' \in Y(S'))$ and $(T \xrightarrow{\beta} T', y'' \in Y(T'))$ are two objects of $Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} coGroth(Y)$ then a map from the second to the first in this category is given by a map $\alpha \rightarrow \beta$ in $Tw(\text{Sch}^{aff})$, and a map $(T', y'') \rightarrow (S', y')$ in $coGroth(Y)$. This means that for the corresponding map say $\mu : T' \rightarrow S'$ in Sch^{aff} we are given a map $y'' \rightarrow y' \mu$ in $Y(T')$. Even the existence of the functor (12) is already interesting.

Is the map (12) op-cofinal? In other words,

$$Tw(\text{Sch}^{aff}) \times_{(\text{Sch}^{aff})^{op}} Groth(Y^{op}) \rightarrow coGroth(Y^{op})$$

is cofinal? We used the fact that $(Groth(Y))^{op} \xrightarrow{\sim} coGroth(Y^{op})$ canonically. Here Y^{op} denotes the lax prestack sending S to $Y(S)^{op}$.

Namely, given $(S, y) \in \text{coGroth}(Y^{op})$, consider the category classifying $(T \xrightarrow{\beta} T') \in Tw(\text{Sch}^{aff}), y' \in Y(T')^{op}$ and a map $(S, y) \xrightarrow{\xi} (T, y'\beta)$ in $\text{coGroth}(Y^{op})$. We have to show it is contractible. The map ξ is simply a map $\tau : S \rightarrow T$ in Sch^{aff} and a morphism $y \rightarrow y'\beta\tau$ in $Y(S)^{op}$. The desired contractibility is not clear (though clear say in the case when Y is constant lax prestack).

Is (12) a cartesian fibration? There seems no reason for that. Let us show that

$$\begin{aligned} (Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} \text{coGroth}(Y)) \times_{\text{Groth}(Y)} \text{Groth}(Y)_{(S,y)/} = \\ (Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} \text{Groth}(Y)_{(S,y)/}) \end{aligned}$$

is contractible, this is what we need. The projection $(Tw(\text{Sch}^{aff})^{op} \rightarrow \text{Sch}^{aff})$ is a cocartesian fibration, and $\text{Groth}(Y)_{(S,y)/}$ is contractible, as it has an initial object.

We have an adjoint pair

$$\text{pr} : (Tw(\text{Sch}^{aff})^{op} \rightleftarrows \text{Sch}^{aff} : u,$$

where pr sends $(S \rightarrow S') \in Tw(\text{Sch}^{aff})$ to S' , and u sends S to $(\emptyset \rightarrow S)$. Moreover, the natural map $\text{pr} \circ u \rightarrow \text{id}$ is an equivalence, so u is fully faithful, pr is a localization. By ([11], 2.7.4), the projection $(Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} \text{Groth}(Y)_{(S,y)/} \rightarrow \text{Groth}(Y)_{(S,y)/})$ admits a right adjoint. Therefore, this projection induces an isomorphism of geometric realizations

$$|(Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} \text{Groth}(Y)_{(S,y)/})| \xrightarrow{\sim} |\text{Groth}(Y)_{(S,y)/}|$$

(by [11], 2.2.106). The RHS is $*$, and we are done!

0.2.16. For Lemma A.11.2 this is ([1], Cor. 3.2.7).

0.2.17. For A.12.1. I think a stronger claim holds: there is a functor $\text{ShvCat}_{/-}^{enh} : \text{PreStk}_{\text{corr}; \text{all}, 1\text{-aff}} \rightarrow 2\text{-Cat}$, whose composition with the forgetful functor $2\text{-Cat} \rightarrow 1\text{-Cat}$ is the one claimed by Sam. Here 2-Cat is the $(\infty, 1)$ -category of $(\infty, 2)$ -categories. This is justified by my Section 0.2.

0.2.18. For A.13.3. here the colimit and limit are calculated in $\text{ShvCat}(Y)$, by tensor product $C \otimes_{\text{QCoh}_Y} \bar{C}$ he means the tensor product in the symmetric monoidal $\text{ShvCat}(Y)$.

For A.13.4(3). This is an analog of ([1], ch. I.1, 6.4.3).

A.13.4(1) clearly has an analog for $1\text{-Cat}_{\text{cont}}^{St, \text{cocompl}}$ instead of $\text{ShvCat}(Y)$.

For A.13.4(2). If the map (A.13.2) $D \otimes_{\text{QCoh}_Y} \bar{C} \rightarrow \lim_{i \in I^{op}} (D \otimes_{\text{QCoh}_Y} C_i^{\vee})$ is an isomorphism for any D then we get as in the proof that for any $D_1, D_2 \in \text{ShvCat}(Y)$ one has $\text{Hom}(D_2, D_1 \otimes_{\text{QCoh}_Y} \bar{C}) \xrightarrow{\sim} \text{Hom}(D_2 \otimes_{\text{QCoh}_Y} C, D_1)$, where by Hom he means mapping spaces in $\text{ShvCat}(Y)$. This shows that C is dualizable.

0.3. For Sect. B.5. For any $[n] \in \mathbf{\Delta}$ we have $\text{Grid}_{[n]} \in 1\text{-Cat}$. This is a functor $\mathbf{\Delta} \rightarrow 1\text{-Cat}$, $[n] \mapsto \text{Grid}_{[n]}$. Namely, if $f : [m] \rightarrow [n]$ is a map in $\mathbf{\Delta}$, and $(i, j) \in \text{Grid}_{[m]}$, that is, $0 \leq i \leq j \leq m$ then $(f(i), f(j)) \in \text{Grid}_{[n]}$. Moreover, it sends horizontal (resp., vertical) arrows to horizontal (resp., vertical).

This is why for a category with directions $(\mathcal{C}, hor, vert)$, $[n] \mapsto \text{Grid}_{[n];hor,vert}(\mathcal{C})$ is indeed a functor $\mathbf{\Delta} \rightarrow \text{Spc}$. Namely, given a map $f : [m] \rightarrow [n]$ in $\mathbf{\Delta}$, we precompose a n -grid $\text{Grid}_{[n]}^{op} \rightarrow \mathcal{C}$ with $\text{Grid}_{[m]}^{op} \rightarrow \text{Grid}_{[n]}^{op}$.

0.4. For 6.7. The natural functor $(fSet_{\emptyset})^{op} \hookrightarrow (Set_{<\infty})^{op}$ is not cofinal. Namely, given $J \in Set_{<\infty}$, the category $fSet_{\emptyset} \times_{Set_{<\infty}} (Set_{<\infty})_{/J}$ is not contractible in general. Namely, the geometric realization of the latter category is the disjoint union of points over all subsets $J_0 \subset J$.

For $\mathcal{G} \in \text{Spc}$ we get a natural map

$$\text{colim}_{I \in (Set_{<\infty})^{op}} \mathcal{G}^I \leftarrow \text{colim}_{I \in (fSet_{\emptyset})^{op}} \mathcal{G}^I = \text{Ran}_{\mathcal{G}, \emptyset}$$

Is it an isomorphism? I hope yes according to his Section 4.8.

In the definition of $fSet_I$ it is assumed that J is finite. The symmetric monoidal category $fSet_{\emptyset}$ act on $fSet_I$. The action map $a : fSet_{\emptyset} \times fSet_I \rightarrow fSet_I$ sends $(J', I \xrightarrow{\alpha} J)$ to $I \rightarrow J \sqcup J'$. So, we get $\mathcal{G}^{J'} \times \mathcal{G}^J \xrightarrow{\sim} \mathcal{G}^{J \sqcup J'}$. By adjointness, we get the canonical map

$$\left(\text{colim}_{J' \in fSet_{\emptyset}^{op}} \mathcal{G}^{J'} \right) \times \left(\text{colim}_{(I \rightarrow J) \in fSet_I^{op}} \mathcal{G}^J \right) \rightarrow \text{colim}_{(I \rightarrow J) \in fSet_I^{op}} \mathcal{G}^J$$

that is, $\text{Ran}_{\mathcal{G}, \emptyset} \times \text{Ran}_{\mathcal{G}, I} \rightarrow \text{Ran}_{\mathcal{G}, I}$. The canonical map $\text{Ran}_{\mathcal{G}, I} \rightarrow \mathcal{G}^I$ is invariant under the action of $\text{Ran}_{\mathcal{X}}^*$.

The functor $fSet_I \rightarrow fSet_{\emptyset}$, $(I \rightarrow J) \mapsto J$ yields by functoriality a morphism $\text{Ran}_{\mathcal{G}, I} \rightarrow \text{Ran}_{\mathcal{G}}$, which is $\text{Ran}_{\mathcal{G}}^*$ -equivariant.

The $\text{Ran}_{\mathcal{G}}^{ch}$ -module structure on $\text{Ran}_{\mathcal{G}, I}$ is given (via his example 6.6.3) by the correspondence

$$\text{Ran}_{\mathcal{G}} \times \text{Ran}_{\mathcal{G}, I} \leftarrow (\text{Ran}_{\mathcal{G}} \times \text{Ran}_{\mathcal{G}, I})_{disj} \rightarrow \text{Ran}_{\mathcal{G}, I}$$

0.4.1. If $\mathcal{X} \in \text{PreStk}$, C is a chiral category for \mathcal{X} , and $\mathcal{Z} \in \text{PreStk}$ is a $\text{Ran}_{\mathcal{X}}^{ch}$ -module in PreStk_{corr} , let $M \in \text{ShvCat}_{/\mathcal{Z}}$. Then a structure of a chiral C -module category on M over \mathcal{Z} means that $(\mathcal{Z}, M) \in \text{PreStk}_{corr}^{ShvCat}$ is equipped with a structure of a $(\text{Ran}_{\mathcal{X}}^{ch}, C)$ -module in the symmetric monoidal category $\text{PreStk}_{corr}^{ShvCat}$, and moreover the action map

$$\text{act}_1^*(C \boxtimes M) \rightarrow \text{act}_2^*(M)$$

is an isomorphism in $\text{ShvCat}_{/act_M}$. Here the diagram

$$\text{Ran}_{\mathcal{X}} \times \mathcal{Z} \xleftarrow{act_1} act_M \xrightarrow{act_2} M$$

defined the action map as in his 5.6.

0.4.2. For 6.10.4. I think $\text{Ran}_{\mathcal{X}}^{un, \rightarrow}, \text{Ran}_{\mathcal{X}, I}^{un}$ are not defined explicitly in the paper. For example, $\text{Ran}_{\mathcal{X}}^{un, \rightarrow}$ should be defined as follows I think: for \mathcal{X} a set consider the set $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ classifying pairs of finite subsets $S \subset T \subset \mathcal{X}$, here S, T could be empty. Now consider $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ as a partially ordered set, where $(S \subset T) \leq (S' \subset T')$ if $S \subset S'$ and $T \subset T'$. Then $\text{Ran}_{\mathcal{X}}^{un, \rightarrow}$ is the category associated to this poset. Then extend this definition to any $\mathcal{X} \in \text{Spc}$ by requiring that $\text{Spc} \rightarrow 1 - \text{Cat}$, $\mathcal{X} \mapsto \text{Ran}_{\mathcal{X}}^{un, \rightarrow}$ commutes with sifted colimits.

For a set \mathcal{X} , $\text{Ran}_{\mathcal{X},I}$ should be the category associated to the following poset: its objects are (α, S) , where $S \subset \mathcal{X}$ is a finite subset, and $\alpha : I \rightarrow S$ is any map. Then $(\alpha, S) \leq (\alpha', S')$ iff $S \subset S'$ and the composition $I \xrightarrow{\alpha} S \rightarrow S'$ equals α' . Then probably we can extend this definition to any \mathcal{X} requiring that $\mathcal{X} \mapsto \text{Ran}_{\mathcal{X},I}$ commutes with sifted colimits.

We can also proceed as in 4.8. Namely, define the category $f\text{Set}_{I,[n]}^{\rightarrow}$ (generalizing $f\text{Set}_{\emptyset,[n]}^{\rightarrow}$ from 4.8). Its objects are diagrams $I \xrightarrow{\alpha_I} I_0 \rightarrow I_1 \rightarrow \dots \rightarrow I_n$ with I_i maybe empty finite set. The morphisms are diagrams

$$\begin{array}{ccccccc} I_0 & \rightarrow & I_1 & \rightarrow & \dots & \rightarrow & I_n \\ \downarrow & & \downarrow & & & & \downarrow \\ J_0 & \rightarrow & J_1 & \rightarrow & \dots & \rightarrow & J_n, \end{array}$$

where the vertical maps are surjections. It is required that the above diagram is compatible with maps $I \xrightarrow{\alpha_I} I_0$ and $I \xrightarrow{\alpha_I} J_0$. Then proceed as in 4.8 to define $\text{Ran}_{\mathcal{X},I}^{un} \in 1 - \text{Cat}$.

0.4.3. For 6.13.1, $\text{Ran}_{\mathcal{X},I,\emptyset}$ is never defined? I think this is $\text{Ran}_{\mathcal{X},I}$, where we added a disjoint point corresponding to empty set.

0.4.4. For 6.15. He starts with a commutative algebra in $\text{PreStk}_{\text{corr}}$, but an assumption is missing: he needs $m_2 : \text{mult}_S \rightarrow S$ and $e_2 : \text{unit}_S \rightarrow S$ to be 1-affine.

Under these assumptions, we get a symmetric monoidal convolution structure on $\text{ShvCat}_{/S}$ as he explains. Actually, $\text{ShvCat}_{/S}$ is a 2-category, and he claims implicitly that $\text{ShvCat}_{/S}$ is a symmetric monoidal 2-category. This means in particular, that the convolution $* : \text{ShvCat}_{/S} \times \text{ShvCat}_{/S} \rightarrow \text{ShvCat}_{/S}$ is a morphism in $2 - \text{Cat}$.

For 6.15.2. If $C \in \text{MultCat}^{op-w}(S)$ then we have maps $m_2^*C \rightarrow m_1^*(C \boxtimes C)$ in $\text{ShvCat}_{\text{mult}_S}$ and $e_2^*C \rightarrow \text{QCoh}_{\text{unit}_S}$ in $\text{ShvCat}_{\text{unit}_S}$. A map from C to D in $\text{MultCat}^{op-w}(S)$ gives a morphism $\alpha : C \rightarrow D$ in $\text{ShvCat}_{/S}$ in particular such that the diagram commutes (an explicit invertible 2-morphism is given)

$$\begin{array}{ccc} m_2^*C & \rightarrow & m_1^*(C \boxtimes C) \\ \downarrow m_2^*\alpha & & \downarrow \\ m_2^*D & \rightarrow & m_1^*(D \boxtimes D) \end{array}$$

Applying m_2^* , we get a diagram

$$(13) \quad \begin{array}{ccccc} C & \rightarrow & m_{2*}m_2^*C & \rightarrow & m_{2*}m_1^*(C \boxtimes C) \\ \downarrow \alpha & & \downarrow m_{2*}m_2^*\alpha & & \downarrow \\ D & \rightarrow & m_{2*}m_2^*D & \rightarrow & m_{2*}m_1^*(D \boxtimes D), \end{array}$$

Similarly for unit. This gives a coalgebra structure on C in the symmetric monoidal category $(\text{ShvCat}_{/S}, *)$.

Question. My understanding is that in (13) the outer square is commutative (that is, an explicit invertible 2-morphism in the 2-category $\text{ShvCat}_{/S}$ is given). Is this true? This would mean probably that the corresponding map from C to D in $\text{ShvCat}_{/S}$ is a true morphism of commutative algebras, and not a lax morphism. However, in the formulation of Prop. 6.15.2, ComCoalg^{lax} appears. Maybe, $\text{ComCoalg}((\text{ShvCat}_{/S}, *))$ is meant in the formulation of Prop. 6.15.2? Or maybe you meant $\text{MultCat}^{op-w,lax}(S)$ instead?

0.4.5. For 6.16. Let $S \in \mathcal{CAlg}(\text{PreStk}_{\text{corr}})$ with e_2, m_2 1-affine. The unit in $(\text{ShvCat}/_S, *)$ is $e_{2*} \text{QCoh}_{\text{unit}_S}$. If $C \in \text{MultCat}^{op-w, r.adj}(S)$ then the maps $\tilde{\eta}_m : m_2^* C \rightarrow m_1^*(C \boxtimes C)$ and $\tilde{\eta}_e : e_2^* C \rightarrow \text{QCoh}_{\text{unit}_S}$ admit right adjoints in $\text{ShvCat}_{\text{mult}_S}$ and in $\text{ShvCat}_{\text{unit}_S}$ respectively. For $\tilde{\eta}_m$ this means that for any $T \rightarrow \text{mult}_S$ with $T \in \text{Sch}^{aff}$ the corresponding right adjoint over T is continuous (similarly for $\tilde{\eta}_e$). So, the map

$$m_{2*} \tilde{\eta}_m : m_{2*} m_2^* C \rightarrow m_{2*} m_1^*(C \boxtimes C)$$

admits a right adjoint in $\text{ShvCat}/_S$. Assume m_2, e_2 are quasi-compact quasi-separated schematic morphisms. Then $C \rightarrow m_{2*} m_2^* C$ admits a right adjoint in $\text{ShvCat}/_S$ by his A.9.1(3). So, the composition $C \rightarrow m_{2*} m_1^*(C \boxtimes C) = C * C$ admits a right adjoint in $\text{ShvCat}/_S$. Similarly, $C \rightarrow e_{2*} \text{QCoh}_{\text{unit}_S}$ has a right adjoint.

0.4.6. For 6.17. For clarity, if $\Psi \in \text{MultCat}^{op-w, r.adj}(S)$ then first we may see Ψ as an object of $\text{ComCoalg}^{r.adj}((\text{ShvCat}/_S, *))$ via his Section 6.16, hence as an object of $\mathcal{CAlg}^{l.adj}((\text{ShvCat}/_S, *))$ via his (6.16.1). So, Ψ is a commutative algebra in $(\text{ShvCat}/_S, *)$, hence $\Gamma(S, \Psi)$ inherits a symmetric monoidal structure, which he still denotes by $*$.

Explicitly, view the maps (6) as

$$\begin{aligned} \tilde{\eta}_m : \Psi &\rightarrow \Psi * \Psi \in \text{ShvCat}/_S \text{ and} \\ \tilde{\eta}_e : \Psi &\rightarrow e_{2*} \text{QCoh}_{\text{unit}_S} \in \text{ShvCat}/_S \end{aligned}$$

They define the corresponding object of $\text{ComCoalg}^{r.adj}((\text{ShvCat}/_S, *))$. Let μ_m, μ_e be the functors right adjoint to $\tilde{\eta}_m$ and $\tilde{\eta}_e$ respectively. So,

$$(\Psi, \mu_m, \mu_e) \in \mathcal{CAlg}^{l.adj}((\text{ShvCat}/_S, *))$$

The symmetric monoidal structure $(\Gamma(S, \Psi), *)$ is obtained by passing to global sections in the latter.

0.4.7. In his Prop. 6.17.1 in the RHS one has to replace $\text{ComAlg}(\Gamma(S, \Psi), *)$ by

$$\text{ComCoalg}(\Gamma(S, \Psi), *)$$

0.4.8. For 6.19. Explicitly, if $C \in \text{Cat}^{ch}(X_{dR})$ the non-unital commutative algebra structure on $\Gamma(\text{Ran}_{X_{dR}}, C)$ in $\text{DGCat}_{\text{cont}}$ is as follows. For the diagram

$$\text{Ran}_{X_{dR}} \times \text{Ran}_{X_{dR}} \xleftarrow{m_1} (\text{Ran}_{X_{dR}} \times \text{Ran}_{X_{dR}})_{\text{disj}} \xrightarrow{m_2} \text{Ran}_{X_{dR}}$$

we have isomorphism $\eta_m : m_1^*(C \boxtimes C) \xrightarrow{\sim} m_2^* C$. We consider $\eta_m^{-1} : m_2^* C \rightarrow m_1^*(C \boxtimes C)$, hence a morphism $\eta_m^{-1} : C \rightarrow m_{2*} m_1^*(C \boxtimes C) = C * C$, so C is a non-unital commutative coalgebra in $(\text{ShvCat}_{\text{Ran}_{X_{dR}}}, *)$. Since m_2 is quasi-compact quasi-separated schematic morphism, the map $\eta_m^{-1} : C \rightarrow C * C$ admits a right adjoint $\mu : C * C \rightarrow C$ by his Section 6.16, which makes C a non-unital commutative algebra in $(\text{ShvCat}_{\text{Ran}_{X_{dR}}}, *)$. In turn,

$\Gamma(\text{Ran}_{X_{dR}}, C)$ inherits a non-unital symmetric monoidal structure with product \otimes^{ch} .

If $s \in \Gamma(\text{Ran}_{X_{dR}}, C)$ then $s \otimes^{ch} s = \mu(m_1^*(s \boxtimes s)) \in \Gamma(\text{Ran}_{X_{dR}}, C)$. If $s \in \text{ComCoalg}^{ch}(C)$, we get a map

$$s \rightarrow \mu(m_1^*(s \boxtimes s)) \in \Gamma(\text{Ran}_{X_{dR}}, C)$$

For 6.20. Given a factorization category C on X_{dR} , I don't see what is the map

$$\Gamma(X_{dR}, C |_{X_{dR}}) \rightarrow \Gamma(\text{Ran}_{X_{dR}}, C)$$

This is like extension by zero for \mathcal{D} -modules.

0.4.9. For 7.2.1. In the definition of a commutative weak chiral category he means it is a weakly multiplicative sheaf of categories on $\text{Ran}_{\mathcal{X}}^*$. In fact, the map $\text{id} : \text{Ran}_{\mathcal{X}}^{ch} \rightarrow \text{Ran}_{\mathcal{X}}^*$ is a morphism in $\mathcal{CAlg}(\text{PreStk}_{corr})$, so that his Section 5.34 applies, id^* is the desired functor.

0.4.10. For 7.7. If $S \in \text{Sch}^{aff}$ then $\mathcal{Z}(S)$ is the cocartesian fibration corresponding to $\text{Sets}_{<\infty} \rightarrow 1 - \text{Cat}$, $I \mapsto \text{Ran}_{\mathcal{X}, disj}^{un, I}(S)$.

For 7.8. There is a correction needed in the second displayed formula, one has actually

$$\mathit{ShvCat}_{\text{Sets}_{<\infty}}^{naive} = \text{Fun}(\text{Sets}_{<\infty}, \text{DGCat}_{cont})$$

as for any constant lax prestack, see my Section 0.0.24. The functor $\mathcal{CAlg}(\text{DGCat}_{cont}) \rightarrow \mathit{ShvCat}_{\text{Sets}_{<\infty}}^{naive}$, $I \mapsto D^{\otimes I}$ is fine.

0.4.11. For 7.12. For a surjection $p : I \rightarrow J$ of finite sets, the subscheme $U(p) \subset X_{dR}^I$ seems not defined. I think this is the scheme of collections $(x_i)_{i \in I}$ such that if $p(i) \neq p(i')$ then $(x_i, x_{i'}) \in X_{disj}^2$.

0.4.12. The construction of $\text{Loc}_{X_{dR}}(D)$ for $D \in \mathcal{CAlg}(\text{DGCat}_{cont})$ from 7.6-7.13 is based on the following: The category $\text{Sets}_{<\infty}$ is symmetric monoidal under $(J_1, J_2) \mapsto J_1 \sqcup J_2$. Consider the functor $\text{Sets}_{<\infty} \rightarrow \text{DGCat}_{cont}$, $I \mapsto D^{\otimes I}$. If $I \rightarrow J$ is any map then the corresponding map $D^{\otimes I} \rightarrow D^{\otimes J}$ is given by the products along I_j for each $j \in J$. We view it as an object \mathcal{F} of $\mathit{ShvCat}_{\text{Sets}_{<\infty}}^{naive}$. Then for the above map $u : \text{Sets}_{<\infty} \times \text{Sets}_{<\infty} \rightarrow \text{Sets}_{<\infty}$ we get $u^*\mathcal{F} \xrightarrow{\sim} \mathcal{F} \boxtimes \mathcal{F}$, and for the empty set $e : * \rightarrow \text{Sets}_{<\infty}$ we have $e^*\mathcal{F} \xrightarrow{\sim} \text{Vect}$. So, $\text{Sets}_{<\infty}$ is naturally an object of $\mathcal{CAlg}(1 - \text{Cat})$, and \mathcal{F} is a multiplicative sheaf of categories on $\text{Sets}_{<\infty}$.

It is this structure that yields a multiplicative structure on $\text{Loc}_{X_{dR}}(D)$. So, let $\mathcal{C} \in \mathcal{CAlg}(1 - \text{Cat})$ and $F \in \text{Fun}(\mathcal{C}, \text{DGCat}_{cont})$ giving an object $\mathcal{F} \in \mathit{ShcCat}_{\mathcal{C}}^{naive}$, where \mathcal{C} is viewed as a constant lax prestack. For which F the sheaf \mathcal{F} is multiplicative? This is equivalent to $F : \mathcal{C} \rightarrow \text{DGCat}_{cont}$ being a symmetric monoidal functor.

0.4.13. For 7.15.5. If $f : Y \rightarrow Y'$ is a proper morphism between schemes of finite type then we have an adjoint pair $f_! : D(Y) \rightleftharpoons D(Y') : f^!$, where $f^!$ is the functor

$$f^* : \text{QCoh}(Y'_{dR}) \rightarrow \text{QCoh}(Y_{dR}) = D(Y),$$

and moreover $f_!$ is a strict morphism of $D(Y')$ -modules ([2], 1.5.2). Recall that Y'_{dR} is 1-affine by ([3], Th. 2.6.3), so $\mathit{ShvCat}(Y'_{dR}) = D(Y) - \text{mod}(\text{DGCat}_{cont})$. Let $C \in \mathit{ShvCat}(Y'_{dR})$, which we view as a $D(Y')$ -module. Let us explain that $f^* : \Gamma(Y'_{dR}, C) \rightarrow \Gamma(Y_{dR}, f^*C)$ has a left adjoint denoted $f_{*, dR, C}$.

This is a particular case of the following general observation. Let $h^! : B \rightarrow A$ be a map in $\mathcal{CAlg}(\text{DGCat}_{cont})$ admitting a left adjoint $h_! : A \rightarrow B$, so $h_!$ is a map in DGCat_{cont} . Assume the left-lax monoidal structure on the functor $h_!$ of B -modules is strict, so $h_!$ is a morphism in $B - \text{mod}(\text{DGCat}_{cont})$, and the adjoint pair $h_! : A \rightleftharpoons B : h^!$

takes place in $B - \text{mod}(\text{DGCat}_{\text{cont}})$. Now if $C \in B - \text{mod}(\text{DGCat}_{\text{cont}})$ then we get the adjoint pair

$$h_{C!} : C \otimes_B A \rightleftarrows C : h_C^!$$

in $B - \text{mod}(\text{DGCat}_{\text{cont}})$.

In our case $f^!$ yields a morphism $f^* : C \rightarrow C \otimes_{D(Y')} D(Y)$ in $D(Y') - \text{mod}$. So, we get the left adjoint $f_{*,dR,C} : C \otimes_{D(Y')} D(Y) \rightarrow C$ to f^* .

For this reason if now Y is a pseudo-indscheme in the sense of Def. 7.15.1 then for $C \in \text{ShvCat}/_Y$ we get $\Gamma(Y_{dR}, C) \xrightarrow{\sim} \text{colim}_{j \in J} \Gamma(Y_{j,dR}, \psi_j^* C)$, the colimit taken in $\text{DGCat}_{\text{cont}}$. Here $Y = \text{colim}_{j \in J} Y_j$, and $\psi_j : Y_j \rightarrow Y$ is the natural map.

The morphism $f_{*,dR,C}$ in his Pp-Construction 7.15.5 exists I think only if $f : Y \rightarrow Z$ is pseudo-indproper.

0.5. More about lax prestacks. Using appendix, for $\mathcal{C} \in 1 - \text{Cat}$ and a functor $G : \mathcal{C}^{op} \rightarrow 1 - \text{Cat}$ one has

$$(14) \quad \text{oplaxlim } G = \lim_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})^{op}} \text{Fun}(\mathcal{C}_{c'}, G(c'))$$

Here $c' \rightarrow c$ is a morphism in \mathcal{C} . We used the identification $\text{Tw}(\mathcal{C}) \xrightarrow{\sim} \text{Tw}(\mathcal{C}^{op})$. We use this in the following result.

Let $\mathcal{C} \in 1 - \text{Cat}$ and $Y : \mathcal{C} \rightarrow \text{PreStk}^{lax}$ be a functor, which we also view as $Y : \mathcal{C} \times (\text{Sch}^{aff})^{op} \rightarrow 1 - \text{Cat}$. For $c \in \mathcal{C}$ let $Y_c \in \text{PreStk}^{lax}$ be its value on c . For $S \in \text{Sch}^{aff}$ let $Y_S : \mathcal{C} \rightarrow 1 - \text{Cat}$ be the restriction of Y , where we fix the corresponding variable.

Define the *relative lax colimit* of Y over $(\text{Sch}^{aff})^{op}$ as $\bar{Y} \in \text{PreStk}^{lax}$ given by

$$(15) \quad \bar{Y}(S) = \text{laxcolim } Y_S = \lim_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})} \mathcal{C}_{c'} \times Y(S, c')$$

for $S \in \text{Sch}^{aff}$, here the colimit is taken in $1 - \text{Cat}$.

Lemma 0.5.1. *Under the assumptions of Section 0.5 we have canonical equivalences*

1) $\text{QCoh}(\bar{Y}) \xrightarrow{\sim} \text{oplaxlim } G$ in $\text{DGCat}_{\text{cont}}$, where G is the composition

$$\mathcal{C}^{op} \xrightarrow{Y^{op}} (\text{PreStk}^{lax})^{op} \xrightarrow{\text{QCoh}} \text{DGCat}_{\text{cont}}$$

2) $\text{ShvCat}/_{\bar{Y}}^{\text{naive}} \xrightarrow{\sim} \text{oplaxlim } G'$, where G' is the composition

$$\mathcal{C}^{op} \xrightarrow{Y^{op}} (\text{PreStk}^{lax})^{op} \xrightarrow{\text{ShvCat}/_{-}^{\text{naive}}} \text{DGCat}_{\text{cont}}$$

Proof. 1) For $S \in \text{Sch}^{aff}$ using (15) and (1), we get

$$\begin{aligned} \text{QCoh}(\bar{Y}) &\xrightarrow{\sim} \lim_{(S' \rightarrow S) \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Fun}(\bar{Y}(S), \text{QCoh}(S')) \xrightarrow{\sim} \\ &\lim_{(S' \rightarrow S) \in \text{Tw}(\text{Sch}^{aff})^{op}} \lim_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})^{op}} \text{Fun}(\mathcal{C}_{c'}, \text{Fun}(Y(S, c'), \text{QCoh}(S'))) \xrightarrow{\sim} \\ &\lim_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})^{op}} \text{Fun}(\mathcal{C}_{c'}, \lim_{(S' \rightarrow S) \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S, c'), \text{QCoh}(S'))) \xrightarrow{\sim} \\ &\lim_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})^{op}} \text{Fun}(\mathcal{C}_{c'}, \text{QCoh}(Y_{c'})) \end{aligned}$$

By (14), we are done.

2) same argument. \square

So, informally we may say that $\text{QCoh} : (\text{PreStk}^{\text{lax}})^{\text{op}} \rightarrow \text{DGCat}_{\text{cont}}$ not only preserves limits. It actually transforms relative lax colimits in $\text{PreStk}^{\text{lax}}$ (over $(\text{Sch}^{\text{aff}})^{\text{op}}$) into oplax limits.

The above does not seem to work for $\text{ShvCat}_{/_}$. Namely, For $T \in 2 - \text{Cat}$ the functor $(1 - \text{Cat})^{\text{op}} \rightarrow 1 - \text{Cat}$, $V \mapsto (\text{Fun}(V, T)_{\text{llax}})^{1 - \text{Cat}}$ preserves limits by ([11], 13.1.45). However, for $A, B \in 1 - \text{Cat}$, $T \in 2 - \text{Cat}$, $\text{Fun}(A \times B, T)_{\text{llax}}^{1 - \text{Cat}}$ does not seem to express as functors from A to $\text{Fun}(B, T)_{\text{llax}}^{1 - \text{Cat}}$ or something like that. For that, given $X \in 1 - \text{Cat}$, we have to better understand $X \otimes (A \times B) \in 2 - \text{Cat}$.

1. APPENDIX: LAX COLIMITS

1.0.1. One has the notion of lax limit and colimit from [4]. For $\mathcal{C} \in 1 - \text{Cat}$ they denote by $Tw(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{\text{op}}$ the cartesian fibration in spaces associated to the functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Spc}$, $(c', c) \mapsto \text{Map}_{\mathcal{C}}(c', c)$. So, an object of $Tw(\mathcal{C})$ is an arrow $\alpha : c' \rightarrow c$ in \mathcal{C} . A map from $\alpha : c' \rightarrow c$ to $\beta : d' \rightarrow d$ in $Tw(\mathcal{C})$ gives rise to maps $c' \rightarrow d'$ and $d \rightarrow c$ in \mathcal{C} such that the diagram commutes

$$\begin{array}{ccc} c' & \xrightarrow{\alpha} & c \\ \downarrow & & \uparrow \\ d' & \xrightarrow{\beta} & d \end{array}$$

Another definition of $Tw(\mathcal{C})$ is given in ([12], Appendix B).

For $\mathcal{C} \in 1 - \text{Cat}$ the projections $\text{Fun}([1], \mathcal{C}) \rightarrow \mathcal{C}$, $\text{Fun}([1], \mathcal{C}) \rightarrow \mathcal{C}$ on the source and target respectively are cartesian and cocartesian fibrations. The corresponding strengthenings are the functors $\mathcal{C}_{-/} : \mathcal{C}^{\text{op}} \rightarrow 1 - \text{Cat}$, $x \mapsto \mathcal{C}_{x/}$ and $\mathcal{C}_{/_} : \mathcal{C} \rightarrow 1 - \text{Cat}$, $x \mapsto \mathcal{C}_{/x}$.

Given a functor $F : \mathcal{C} \rightarrow 1 - \text{Cat}$, its lax colimit is

$$\text{colim}_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C})} \mathcal{C}_{c/} \times F(c')$$

and its oplax colimit is

$$\text{colim}_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C})} (\mathcal{C}_{c/})^{\text{op}} \times F(c')$$

The lax limit of F is

$$\lim_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C})^{\text{op}}} \text{Fun}(\mathcal{C}_{/c'}, F(c))$$

The oplax limit of F is

$$\lim_{(c' \rightarrow c) \in Tw(\mathcal{C})^{\text{op}}} \text{Fun}((\mathcal{C}_{/c'})^{\text{op}}, F(c))$$

There is a natural map $\xi : \lim F \rightarrow \text{laxlim} F$. Indeed, there is a natural morphism of functors from the composition $Tw(\mathcal{C})^{\text{op}} \rightarrow \mathcal{C} \xrightarrow{F} 1 - \text{Cat}$ to the functor $Tw(\mathcal{C})^{\text{op}} \rightarrow 1 - \text{Cat}$ sending $\alpha : c' \rightarrow c$ to $\text{Fun}(\mathcal{C}_{/c'}, F(c))$. It comes from the fact that for $\alpha : c' \rightarrow c$ in $Tw(\mathcal{C})$, one has the natural map $\text{const} : F(c) \rightarrow \text{Fun}(\mathcal{C}_{/c'}, F(c))$ functorial in $\alpha \in Tw(\mathcal{C})^{\text{op}}$. Here const is given by constant functors.

Note that for any $\alpha : c' \rightarrow c$ in $Tw(\mathcal{C})$ the above map $const : F(c) \rightarrow \text{Fun}(\mathcal{C}_{/c'}, F(c))$ is fully faithful. Indeed, $L : \mathcal{C}_{/c'} \rightarrow *$ is left adjoint to $R : * \rightarrow \mathcal{C}_{/c'}$, here $R(*)$ is the final object of $\mathcal{C}_{/c'}$. Now by ([11], Lemma 2.2.17), ξ is fully faithful.

Example: if $\mathcal{C} = [1]$, that is, the category with objects $0, 1$ and a unique nontrivial arrow $0 \rightarrow 1$ then given $F : [1] \rightarrow 1 - \text{Cat}$, the lax limit of F is as follows. Let $f : F(0) \rightarrow F(1)$ be the corresponding map in $1 - \text{Cat}$. Then $\text{laxlim} F = F(0) \times_{F(1)} \text{Fun}([1], F(1))$, where the map $\text{Fun}([1], F(1)) \rightarrow F(1)$ keeps the source of an arrow.

1.0.2. If $F : \mathcal{C}^{op} \rightarrow 1 - \text{Cat}$ is a map in $1 - \text{Cat}$, let $\mathcal{X} \rightarrow \mathcal{C}$ be the corresponding cartesian fibration. Then $\text{Func}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ is the oplax limit of F by ([4], 7.1). Recall that $\lim F$ is the category $\text{Func}_{\mathcal{C}}^{cart}(\mathcal{C}, \mathcal{X})$, the full subcategory of those functors in $\text{Func}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ that send cartesian arrows to cartesian arrows ([11], 2.2.67).

1.0.3. For $\mathcal{C} \in 1 - \text{Cat}$ we have an isomorphism $Tw(\mathcal{C}) \xrightarrow{\sim} Tw(\mathcal{C}^{op})$ compatible with the projections to $\mathcal{C} \times \mathcal{C}^{op}$. Indeed, for $c, c' \in \mathcal{C}$, $\text{Map}_{\mathcal{C}}(c', c) \xrightarrow{\sim} \text{Map}_{\mathcal{C}^{op}}(c, c')$.

REFERENCES

- [1] D. Gaitsgory, N. Rozenblyum, A study in derived algebraic geometry, book version on Dennis homepage
- [2] D. Gaitsgory, The Atiyah-Bott formula for the cohomology of the moduli space of bundles on a curve, arxiv version 2 (24 June 2019)
- [3] D. Gaitsgory, Sheaves of categories and the notion of 1-affineness, arxiv
- [4] David Gepner, Rune Haugseng, Thomas Nikolaus, Lax colimits and free fibrations in ∞ -categories, Documenta Mathematica 22 (2017), 12251266
- [5] J. Lurie, Higher topos theory
- [6] J. Lurie, Higher algebra
- [7] J. Lurie, Derived Algebraic Geometry I: Stable ∞ -Categories
- [8] J. Lurie, Derived Algebraic Geometry II: Noncommutative Algebra, arxiv
- [9] J. Lurie, Derived Algebraic Geometry III: Commutative Algebra, arXiv:math/0703204
- [10] J. Lurie, $(\infty, 2)$ -Categories and the Goodwillie Calculus I, arXiv: 0905.0462
- [11] S. Lysenko, Comments to Gaitsgory Lurie Tamagawa, my homepage
- [12] S. Raskin, Chiral categories, his homepage, version 4 Sept. 2019