COMMENTS TO: RASKIN, CHIRAL CATEGORIES

for version Sept. 4, 2019

0.0.1. For 4.3.3. $Sets = \mathrm{Ind}(Set_{<\infty})$, and Spc is generated by $Sets$ under geometric realizations (HTT, 5.5.8.14).

Every object $Y \in Sets_{<\infty}$ is projective in Spc in the sense of (HTT, 5.5.8.18), because $\boldsymbol{\Delta}^{op}$ is sifted. Now the canonical inclusion $Sets_{<\infty} \subset \mathrm{Spc}$ extends uniquely by (HTT, 5.5.8.15) to a functor $f : \mathcal{P}_\Sigma(Sets_{<\infty}) \to \mathrm{Spc}$ preserving sifted colimits. We see that the conditions (i),(ii), (iii) of (HTT, 5.5.8.22) are verified, so $f : \mathcal{P}_\Sigma(Sets_{<\infty}) \widetilde{\to} \mathrm{Spc}$ is an equivalence.

The functors $F, F' : \mathrm{Spc} \to \mathrm{Spc}$, $F(Z) = \mathrm{Ran}_Z$ and $F'(Z) = \mathrm{Ran}_{Z,\emptyset}$ clearly commute with sifted colimits indeed. Thus, in the notations of (HTT, 5.5.8.15), these are functors in $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(Sets_{<\infty}), \mathrm{Spc})$. To see that F, F' are LKE of their restriction to $Sets_{<\infty}$, decompose F (resp., F') as

$$\mathcal{P}_\Sigma(Sets_{<\infty}) \overset{i}{\hookrightarrow} \mathcal{P}(Sets_{<\infty}) \overset{b}{\to} \mathrm{Spc},$$

where the first map is the canonical inclusion, and the second is the colimi preserving functor extending $F : Sets_{<\infty} \to \mathrm{Spc}$ (resp., $F' : Sets_{<\infty} \to \mathrm{Spc}$). Now b is the LKE from $Sets_{<\infty}$, and its composition with i is also a LKE from $Sets_{<\infty}$, because i is a right adjoint!

0.0.2. For 2.8. The fact that his Vect coincides with the definition from Dennis and Nick's book follows from (HA, 7.1.1.15-16).

0.0.3. For 4.3.2. Let A be a set. Then $\mathrm{colim}_{I \in fSet^{op}} A^I \widetilde{\to} F(A)$, where $F(A)$ is the set of finite subsets in A. Indeed, for $B \in F(A)$ let $A^I_B = \{f : I \to A \mid Im(f) = B\}$. Then $A^I = \sqcup_{B \in F(A)} A^I_B$. Now

$$\underset{I \in fSet^{op}}{\mathrm{colim}} \sqcup_{B \in F(A)} A^I_B \widetilde{\to} \sqcup_{B \in F(A)} \underset{I \in fSet^{op}}{\mathrm{colim}} A^I_B,$$

and $\mathrm{colim}_{I \in fSet^{op}} A^I_B \widetilde{\to} \underset{I \in I \in fSet^{op}, I \twoheadrightarrow B}{\mathrm{colim}} * \widetilde{\to} *$. We used the fact that the category $(I \in I \in fSet^{op}, I \twoheadrightarrow B)$ has an initial object, so is contractible and the last colimit is $*$.

0.0.4. For 4.5. Once we defined $\mathrm{Ran}^{un}_{\mathcal{G}}$ for $\mathcal{G} \in Set_{<\infty}$, we may extend this definition to $\mathcal{P}_\Sigma(Sets_{<\infty}) \widetilde{\to} \mathrm{Spc}$ by requiring that the functor $\mathrm{Spc} \to 1-\mathcal{C}\mathrm{at}$, $\mathcal{G} \mapsto \mathrm{Ran}^{un}_{\mathcal{G}}$ commutes with sifted colimits using (HTT, 5.5.8.15).

For 4.6.1(1) see below. The forgetful functor $0-Cart/I \to 1-\mathcal{C}\mathrm{at}$ preserves colimits (see below). For a cartesian fibration in spaces $p : \mathcal{X} \to I$ any arrow in $\mathcal{X}$ is p-cartesian. So, ${}'\mathrm{Ran}^{un}_{\mathcal{G}}$ can be defined as the category obtained from $coGroth(\Psi_{\mathcal{G}})$ inverting all arrows lying over surjective morphisms in $\mathrm{Sets}_{<\infty}$.

Date: April 7, 2024.

For 4.9: For $\mathcal{G} \in \mathrm{Spc}$, $\mathrm{Ran}^{\rightarrow}_{\mathcal{G},\emptyset,[0]} = \mathrm{Ran}_{\mathcal{G},\emptyset}$. Thus, the space underlying the category $''\mathrm{Ran}^{un}_{\mathcal{G}} \widetilde{\to} \mathrm{Ran}_{\mathcal{G}}$ is $\mathrm{Ran}_{\mathcal{G},\emptyset}$.

For any $n \geq 0$ the category $fSet^{\rightarrow}_{\emptyset,[n]}$ has a symmetric monoidal structure given by the disjoint union. This provides for any $\mathcal{G} \in \mathrm{Spc}$ a structure of a commutative monoid on $\mathrm{Ran}^{\rightarrow}_{\mathcal{G},\emptyset,[n]}$ as in ([11], 3.0.75). For example, if $\mathcal{G} \in Sets$ then recall that $\mathrm{Ran}^{\rightarrow}_{\mathcal{G},\emptyset,[n]}$ is the set of collections $\{S_0 \subset \ldots \subset S_n \subset \mathcal{G}\}$ with S_i finite. Then the operation sends a pair $(S_0 \subset \ldots \subset S_n), (S'_0 \subset \ldots \subset S'_n)$ to $(S_0 \cup S'_0 \subset \ldots \subset S_n \cup S'_n)$.

0.0.5. For 4.13. Since $\pi_0 : \mathrm{Spc} \to Sets$ is a left adjoint, for $\mathcal{G} \in \mathrm{Spc}$ we get isomorphisms $\pi_0 \mathrm{Ran}_{\mathcal{G}} \widetilde{\to} \mathrm{Ran}_{\pi_0(\mathcal{G})}$ and $\pi_0(\mathrm{Ran}_{\mathcal{G},\emptyset}) \widetilde{\to} \mathrm{Ran}_{\pi_0(\mathcal{G}),\emptyset}$.

For 4.13.2: misprint I think: the space underlying $\mathrm{Ran}^{un}_{\mathcal{G}}$ is $\mathrm{Ran}_{\mathcal{G},\emptyset}$. However, it is clear how to define $[\mathrm{Ran}_{\mathcal{G},\emptyset} \times \mathrm{Ran}_{\mathcal{G},\emptyset}]_{disj}$, and hence we also get $[\mathrm{Ran}^{un}_{\mathcal{G}} \times \mathrm{Ran}^{un}_{\mathcal{G}}]_{disj}$ as a full subcategory restricting objects in the same way.

0.0.6. Let $I \in 1-\mathcal{C}\mathrm{at}$. Sam claims $0-Cart_{/I} \to 1-\mathcal{C}\mathrm{at}$ sending $\mathcal{X} \to I$ to $\mathcal{X}$ preserves colimits. I prove this using ([4], Th. 1.1). Given $F : \mathcal{C}^{op} \to \mathrm{Spc}$, let $\mathcal{X} \to \mathcal{C}$ be the cartesian fibration in spaces classifying F. By ([4], Th. 1.1), $\mathcal{X}$ is the oplax colimit of F. That is, $\mathcal{X} \widetilde{\to} \underset{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C}^{op})}{\mathrm{colim}} \mathcal{C}_{/c} \times F(c')$ in $1-\mathcal{C}\mathrm{at}$. If now $F \widetilde{\to} \mathrm{colim}_{j\in J} F^j$ in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc})$ then we get

$$\mathcal{X} \widetilde{\to} \underset{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C}^{op}), j \in J}{\mathrm{colim}} \mathcal{C}_{/c} \times F^j(c') \widetilde{\to} \underset{j\in J}{\mathrm{colim}} \underset{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C}^{op})}{\mathrm{colim}} \mathcal{C}_{/c} \times F^j(c') \widetilde{\to} \underset{j \in J}{\mathrm{colim}}\, \mathcal{X}^j,$$

where $\mathcal{X}^j \to \mathcal{C}$ is the cartesian fibration in spaces classifying F^j. Here the colimit $\underset{j\in J}{\mathrm{colim}}\, \mathcal{X}^j$ is calculated in $1-\mathcal{C}\mathrm{at}$.

The same argument actually shows that $0-Cart_{/I} \to 1-\mathcal{C}\mathrm{at}_{/I}$ preserves colimits, hence admits a right adjoint, because these categories are presentable. Recall that the functor $1-\mathcal{C}\mathrm{at}_{/I} \to 1-\mathcal{C}\mathrm{at}$, $(X \to I) \mapsto X$ preserves colimits, as this is a left adjoint.

Lemma 0.0.7. *Let $I, J \in 1-\mathcal{C}\mathrm{at}$, $J \subset I$ be a 1-full subcategory with the same class of objects as I. The functor $0-Cart_{/I} \to 1-\mathcal{C}\mathrm{at}$, $(X \to I) \mapsto S^{-1}X$ preserves colimits. Here S is the class of those morphisms of X which lie over a morphism of J.*

Proof. For $(X \to I) \in 1-\mathcal{C}\mathrm{at}_{/I}$ write $X_J = X \times_I J$. Given $Y \in 1-\mathcal{C}\mathrm{at}$, one gets

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(S^{-1}X, Y) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(X, Y) \times_{\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(X_J, Y)} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(X_J, Y^{\mathrm{Spc}})$$

Assume $X \widetilde{\to} \mathrm{colim}_{k\in K} X_k$ in $0-Cart_{/I}$. The functor $0-Cart_{/I} \to 1-\mathcal{C}\mathrm{at}_{/I}$ preserves colimits by the above, so this is also a colimit diagram in $1-\mathcal{C}\mathrm{at}$, hence $\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(X, Y) \widetilde{\to} \lim_{k\in K^{op}} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(X_k, Y)$. The functor $0-Cart_{/I} \to 0-Cart_{/J}$, $X' \mapsto X'_J$ preserves colimits, as we may interprete this as $\mathrm{Fun}(I^{op}, \mathrm{Spc}) \to \mathrm{Fun}(J^{op}, \mathrm{Spc})$. So, $\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(X_J, Y) \widetilde{\to} \lim_{k\in K^{op}} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}((X_k)_J, Y)$ and

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(X_J, Y^{\mathrm{Spc}}) \widetilde{\to} \lim_{k\in K^{op}} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}((X_k)_J, Y^{Spc})$$

Limits commute with limits, so the natural map

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(S^{-1}X, Y) \to \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(\mathrm{colim}_{k\in K} S^{-1}(X_k), Y)$$

is an isomorphism. $\square$

The above lemma implies Pp4.6.1(1) of Sam.

Question. What is the right adjoint to the inclusion $0-Cart_{/I} \to 1-\mathcal{C}\mathrm{at}_{/I}$?

0.0.8. For Sect. 4.14. Since **1-Cat** is a 2-category, $\mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, \mathbf{1\text{-}Cat})$ is also a 2-category. This is a general construction associating two a pair of 2-categories, the 2-category of functors between them.

Given $Y \in \mathrm{PreStk}^{lax}$, the object $\mathrm{QCoh}(Y) \in 1-\mathcal{C}\mathrm{at}$ is $\mathbf{Map}_{\mathrm{PreStk}^{lax}}(Y, \mathrm{QCoh})$ in the sense of Dennis and Nick book.

Let $i : \mathrm{PreStk} \to \mathrm{PreStk}^{lax}$ be the inclusion, it has a right adjoint $R : \mathrm{PreStk}^{lax} \to \mathrm{PreStk}$ sending Y to the prestack $(S \mapsto Y(S)^{\mathrm{Spc}})$ for $S \in \mathrm{Sch}^{aff}$. The functor i also has a left adjoint $\mathrm{PreStk}^{lax} \to \mathrm{PreStk}$ sending Y to the prestack $(S \mapsto \mid Y(S) \mid)$ for $S \in \mathrm{Sch}^{aff}$, here $\mid Z \mid$ denotes the inverting of all arrows in the category Z. This is an application of ([11], 2.2.56).

The RKE of $\mathrm{QCoh}_{\mathrm{PreStk}} : \mathrm{PreStk}^{op} \to \mathrm{DGCat}_{cont}$ under $\mathrm{PreStk}^{op} \hookrightarrow (\mathrm{PreStk}^{lax})^{op}$ is the composition $(\mathrm{PreStk}^{lax})^{op} \overset{R^{op}}{\to} \mathrm{PreStk}^{op} \to \mathrm{DGCat}_{cont}$, where the second functor is $\mathrm{QCoh}_{\mathrm{PreStk}}$. It sends Y to $\mathrm{QCoh}(R(Y))$, so this is not the same thing as $\mathrm{QCoh}(Y)$ that Sam means in this paper.

According to ([2], 2.2.1), for $Y \in \mathrm{PreStk}^{lax}$, let $\mathcal{X}_Y \to \mathrm{Sch}^{aff}$ be the cartesian fibration corresponding to $Y : (\mathrm{Sch}^{aff})^{op} \to 1-\mathcal{C}\mathrm{at}$. Let also $Shv_{\mathrm{Sch}^{aff}} \to \mathrm{Sch}^{aff}$ be the cartesian fibration corresponding to the composition $(\mathrm{Sch}^{aff})^{op} \to \mathrm{DGCat}_{cont} \to 1-\mathcal{C}\mathrm{at}$, where the first functor is $\mathrm{QCoh}^*_{\mathrm{Sch}^{aff}}$. Then $\mathrm{QCoh}(Y)$ can be defined as the full subcategory of $\mathrm{Fun}_{\mathrm{Sch}^{aff}}(\mathcal{X}_Y, Shv_{\mathrm{Sch}^{aff}})$ consisting of functors sending cartesian arrows to cartesian arrows.

There is one more way to understand $\mathrm{QCoh}(Y)$ for a lax prestack Y as follows. By ([4], Lemma 6.4), the category $\mathrm{QCoh}(Y)$ is the limit of the composition

$$Tw(\mathrm{Sch}^{aff})^{op} \to \mathrm{Sch}^{aff} \times (\mathrm{Sch}^{aff})^{op} \overset{Y^{op}\times\mathrm{QCoh}}{\to} (1-\mathcal{C}\mathrm{at})^{op} \times 1-\mathcal{C}\mathrm{at} \overset{\mathrm{Fun}}{\to} 1-\mathcal{C}\mathrm{at}$$

This allows to view $\mathrm{QCoh}(Y)$ as an object of DGCat_{cont} as follows. Consider the composition

$$Tw(\mathrm{Sch}^{aff})^{op} \to \mathrm{Sch}^{aff} \times (\mathrm{Sch}^{aff})^{op} \overset{Y^{op}\times\mathrm{QCoh}}{\to} (1-\mathcal{C}\mathrm{at})^{op} \times \mathrm{DGCat}_{cont} \overset{\mathrm{Fun}}{\to} \mathrm{DGCat}_{cont}$$

and take the limit. Since the projection $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}\mathrm{at}$ preserves limits, we are done.

We used the fact that given any $C \in 1-\mathcal{C}\mathrm{at}$, $D \in \mathrm{DGCat}_{cont}$, $\mathrm{Fun}(C, D) \in \mathrm{DGCat}_{cont}$ naturally (see [11], 9.2.13). Once again, for $Y \in \mathrm{PreStk}^{lax}$,

$$\mathrm{QCoh}(Y) = \lim_{(\alpha: S' \to S) \in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Fun}(Y(S), \mathrm{QCoh}(S')) \tag{1}$$

This implies the following.

Remark 0.0.9. *Write* $\mathrm{QCoh}^*_{\mathrm{PreStk}^{lax}} : (\mathrm{PreStk}^{lax})^{op} \to \mathrm{DGCat}_{cont}$ *for the functor sending* $Y \mapsto \mathrm{QCoh}(Y)$ *and* $f : Y \to Y'$ *to* $f^* : \mathrm{QCoh}(Y') \to \mathrm{QCoh}(Y)$. *This functor preserves limits.*

Given a map $\alpha \to \beta$ in $Tw(\mathrm{Sch}^{aff})$ with $\alpha : S' \to S, \beta : T' \to T$, the transition functor $\mathrm{Fun}(Y(T), \mathrm{QCoh}(T')) \to \mathrm{Fun}(Y(S), \mathrm{QCoh}(S'))$ in (1) admits a right adjoint, however this right adjoint is not continuous in general. The transition functors preserve colimits. For this reason, for a lax prestack Y, $ShvCat^{naive}_{/Y}$ has colimits, they are computed termwise in $\mathrm{Fun}(Y(S), \mathrm{QCoh}(S'))$ in the above projective system.

0.0.10. For 4.18. In ([1], ch. A.1, 3.2.7) they defined for a pair $\mathbb{S}, \mathbb{T} \in 2-\mathcal{C}\mathrm{at}$ the object $\mathrm{Fun}(\mathbb{S}, \mathbb{T})_{rlax} \in 2-\mathcal{C}\mathrm{at}$. Sam means by $\mathrm{Hom}^{lax}(\mathcal{C}, \mathcal{D})$ the underlying 1-category of left-lax functors $(\mathrm{Fun}(\mathbb{S}, \mathbb{T})_{llax})^{1-\mathcal{C}\mathrm{at}}$ (see my file [11], 12.1.44).

Sam claims in 4.18.3 that there is an object, say $2-CAT_{llax}$ of $2-\mathcal{C}\mathrm{at}$, which extends $2-\mathcal{C}\mathrm{at}$ as follows. The objects of $2-CAT_{llax}$ are those of $2-\mathcal{C}\mathrm{at}$, and given $C, D \in 2-\mathcal{C}\mathrm{at}$, the mapping category from C to D is $(\mathrm{Fun}(C, D)_{llax})^{1-\mathcal{C}\mathrm{at}}$. Let's accept this.

0.0.11. For 4.19. Recall Sam's notation $ShvCat_{/-}$ for the functor

$$ShvCat_{/-} : (\mathrm{Sch}^{aff})^{op} \to 1-\mathcal{C}\mathrm{at},\ S \mapsto ShvCat(S) = \mathrm{QCoh}(S)-mod$$

Let $\mathcal{X}_{Shv} \to \mathrm{Sch}^{aff}$ be the cartesian fibration corresponding to this functor. Given a lax prestack Y, the definition of $ShvCat^{naive}_{/Y}$ is as follows. Let $\mathcal{X}_Y \to \mathrm{Sch}^{aff}$ be the cartesian fibration corresponding to $Y : (\mathrm{Sch}^{aff})^{op} \to 1-\mathcal{C}\mathrm{at}$. Then $ShvCat^{naive}_{/Y}$ is the full subcategory in $\mathrm{Fun}_{\mathrm{Sch}^{aff}}(\mathcal{X}_Y, \mathcal{X}_{Shv})$ consisting of functors sending cartesian arrows to cartesian arrows. In other words,

$$ShvCat^{naive}_{/Y} = \mathbf{Map}_{\mathrm{PreStk}^{lax}}(Y, ShvCat_{/-}), \tag{2}$$

where we view PreStk^{lax} naturally as a $(\infty, 2)$-category.

A candidate for his definition of $ShvCat_{/Y}$ for a lax prestack Y is as follows. First, we consider $1-\mathcal{C}\mathrm{at}$ as a 2-category (as in [1], ch. A.1, Sect. 2.4) denoted **1-Cat** in *loc.cit.* So, we have the 2-categories

$$\mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, \mathbf{1\text{-}Cat}),\ \mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, \mathbf{1\text{-}Cat})_{rlax},\ \mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, \mathbf{1\text{-}Cat})_{llax}$$

defined in ([1], ch. A.1, 2.5.4 and 3.2.7).

Question: Consider the corresponding mapping category

$$\mathbf{Map}_{\mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, \mathbf{1\text{-}Cat})_{llax}}(Y, ShvCat_{/-})$$

does it coincide with $ShvCat_{/Y}$ by definition? Sam says yes, but I think this is wrong!

He proposes the following in email of 14 Oct 2019.

Definition 0.0.12. *We have the natural functor* **2-Cat** $\to 2-CAT_{llax}$*, here* **2-Cat** *is the 2-category defined in ([1], ch. A.1, Sect. 6). View* $ShvCat_{/-}$ *as a functor* $(\mathrm{Sch}^{aff})^{op} \to$ **2-Cat**. *It is essential here to use the 2-category structure of* $ShvCat(S)$ *for* $S \in \mathrm{Sch}^{aff}$*, see Section 0.2 here. Given* $Y \in \mathrm{PreStk}^{lax}$*, we may also view* Y *as the composition* $(\mathrm{Sch}^{aff})^{op} \to 1-\mathcal{C}\mathrm{at} \to$ **2-Cat** $\to 2-CAT_{llax}$. *Now* $ShvCat_{/Y}$ *is the category of natural transformations* $Y \to ShvCat_{/-}$ *of functors* $(\mathrm{Sch}^{aff})^{op} \to 2-CAT_{llax}$.

My understanding is that

$$ShvCat_{/Y} = \mathbf{Map}_{\mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, 2-CAT_{llax})}(Y, ShvCat_{/-})$$

Using [4], the latter can be rewritten as the limit in $1-\mathcal{C}\mathrm{at}$

$$ShvCat_{/Y} \widetilde{\to} \lim_{(S\to S')\in Tw(\mathrm{Sch}^{aff})^{op}} (\mathrm{Fun}(Y(S'), ShvCat(S))_{llax})^{1-\mathcal{C}\mathrm{at}} \tag{3}$$

taken over $Tw(\mathrm{Sch}^{aff})^{op}$.

Sam claims $ShvCat_{/Y}$ is not presentable, already DGCat_{cont} is not presentable.

It is not clear which 2-categorical enhancement of $ShvCat_{/Y}$ for a lax prestack Y he meant, but there is the following one: write $ShvCat_{/Y}^{2-\mathcal{C}\mathrm{at}}$ for the limit in $2-\mathcal{C}\mathrm{at}$

$$\lim_{S\to S'} (\mathrm{Fun}(Y(S'), ShvCat(S))_{llax})$$

taken over $Tw(\mathrm{Sch}^{aff})^{op}$. Here we view $ShvCat(S)$ as an object of $2-\mathcal{C}\mathrm{at}$. This is a good idea, because the functor $2-\mathcal{C}\mathrm{at}\to 1-\mathcal{C}\mathrm{at}, D\mapsto D^{1-\mathcal{C}\mathrm{at}}$ preserves limits.

Note that for $ShvCat_{/Y}^{naive}$ we have similarly to (3)

$$ShvCat_{/Y}^{naive} \widetilde{\to} \lim_{(S\to S')\in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Fun}(Y(S'), ShvCat(S)) \tag{4}$$

taken over $Tw(\mathrm{Sch}^{aff})^{op}$. The category $Tw(\mathrm{Sch}^{aff})$ has no final object! The limit (4) understood as a limit in $1-\mathcal{C}\mathrm{at}$ gets the 1-category $ShvCat_{/Y}^{naive}$. It can also be understood as a limit in $2-\mathcal{C}\mathrm{at}$, and then gives an object of $2-\mathcal{C}\mathrm{at}$ mentioned in his Remark 4.19.2.

0.0.13. The functor $\mathrm{PreStk}^{op}\to 1-\mathcal{C}\mathrm{at}, Y\mapsto ShvCat_{/Y}^{naive}$ factors through the category of cocomplete categories and colimit-preserving functors. However, $ShvCat_{/Y}$ is not presentable already for $Y=\mathrm{Spec}\, k$. Given $Y\in \mathrm{PreStk}^{lax}$, we get

$$ShvCat_{/Y}^{naive} = \lim_{(c\to c')\in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Fun}(Y(c'), ShvCat(c))$$

If $f: Y_1 \rightleftarrows Y_2 : g$ are maps in $\mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, 1-\mathcal{C}\mathrm{at})$ with morphisms $fg\to \mathrm{id}$, $\mathrm{id}\to gf$ making them into an adjoint pair in the 2-category PreStk^{lax} then the functors

$$g^*: ShvCat_{/Y_1}^{naive} \rightleftarrows ShvCat_{/Y_2}^{naive} : f^*$$

also form an adjoint pair by ([2], Lm. 2.2.6). Indeed, apply the formula (2) and functoriality of **Map**.

0.0.14. For 4.19.2. Let Y be a lax prestack. Then QCoh_Y is the sheaf of categories on Y obtained as the pull-back along $Y\to *$ of Vect. If $\beta,\gamma: S\to Y$ are two S-points with $S\in \mathrm{Sch}^{aff}$ and $\theta:\beta\to\gamma$ is a map in $Y(S)$ then the corresponding map $\theta: \beta^*\mathrm{QCoh}_Y\to\gamma^*\mathrm{QCoh}_Y$ is the identity.

0.0.15. Let $Y\in\mathrm{PreStk}^{lax}$. Let us calculate $ShvCat_{/Y}^{naive}$ assuming in addition that $Y\in\mathrm{PreStk}$.

Lemma 0.0.16. *Let $C\in 1-\mathcal{C}\mathrm{at}$, $Y\in\mathrm{Fun}(C^{op}, 1-\mathcal{C}\mathrm{at})$, $S\in C$. We identify S with its image in $\mathrm{Fun}(C^{op}, 1-\mathcal{C}\mathrm{at})$ under the Yoneda. Then the category of natural transformations $S\to Y$ is canonically equivalent to $Y(S)$.*

More generally, if $Y_1\in\mathcal{P}(C)$ then the category of natural transformations $Y_1\to Y$ is equivalent to $\lim_{S\to Y_1} Y(S)$, the limit over $(C_{/Y_1})^{op}$ is taken in $1-\mathcal{C}\mathrm{at}$.

Proof. By ([4], Lm. 6.4), the category of natural transformations $Y_1 \to Y_2$ for $Y_i \in \mathrm{Fun}(C^{op}, 1-\mathcal{C}\mathrm{at})$ is the limit of the composition

$$Tw(C)^{op} \to C \times C^{op} \stackrel{Y_1^{op}\times Y_2}{\to} (1-\mathcal{C}\mathrm{at})^{op} \times 1-\mathcal{C}\mathrm{at} \stackrel{\mathrm{Fun}}{\to} 1-\mathcal{C}\mathrm{at}$$

So, $\mathbf{Map}_{\mathrm{Fun}(\mathcal{C}^{op},1-\mathcal{C}\mathrm{at})}(Y_1, Y_2) \widetilde{\to} \lim_{(c\to c')\in Tw(C)^{op}} \mathrm{Fun}(Y_1(c'), Y_2(c))$, here $c \to c'$ is an arrow in C.

By ([4], 6.9) we know that $\mathbf{Map}_{\mathrm{Fun}(\mathcal{C}^{op},1-\mathcal{C}\mathrm{at})}(Y_1, Y_2) \widetilde{\to} \mathrm{Fun}^{cart}_C(\mathcal{Y}_1, \mathcal{Y}_2)$, where $\mathcal{Y}_i \to C$ is the cartesian fibration corresponding to Y_i, and $\mathrm{Fun}^{cart}_C(\mathcal{Y}_1, \mathcal{Y}_2) \subset \mathrm{Fun}_C(\mathcal{Y}_1, \mathcal{Y}_2)$ is the full subcategory of functors sending cartesian arrows to cartesian arrows.

Assume $Y_1 = S$ for $S \in C$. The corresponding mapping category is $\mathrm{Fun}^{cart}_C(C_{/S}, \mathcal{Y}_2)$, as $C_{/S} \to C$ is a cartesian fibration in spaces. The functor of evaluation on the final object $S \stackrel{\mathrm{id}}{\to} S$ of $C_{/S}$ gives a map $\mathrm{Fun}^{cart}_C(C_{/S}, \mathcal{Y}_2) \to Y_2(S)$. To get a map in the other direction, we need to construct a functor $Y_2(S) \times C_{/S} \to \mathcal{Y}_2$ over C. Given $y \in Y_2(S)$ and $c \to S$ in $C_{/S}$, we get a morphism $c \to S$ in C. The desired functor sends this collection to $y' \in \mathcal{Y}_2(c)$, which is the source of a cartesian arrow $y' \to y$ over $c \to S$. I hope these two functors are inverse to each other.

More generally, if $Y_1 \in \mathcal{P}(C)$, then $Y_1 \widetilde{\to} \mathrm{colim}_{S\to Y_1} S$ in $\mathrm{Fun}(\mathcal{C}^{op}, 1-\mathcal{C}\mathrm{at})$ over $\mathcal{C}_{/Y_1}$. Then we should have

$$\mathbf{Map}_{\mathrm{Fun}(\mathcal{C}^{op},1-\mathcal{C}\mathrm{at})}(Y_1, Y_2) \widetilde{\to} \lim_{S\to Y_1} \mathbf{Map}_{\mathrm{Fun}(\mathcal{C}^{op},1-\mathcal{C}\mathrm{at})}(S, Y_2) \widetilde{\to} \lim_{S\to Y_1} Y_2(S)$$

To explain the first isomorphism, more generally, we claim that for any $Y_2 \in \mathrm{Fun}(\mathcal{C}^{op}, 1-\mathcal{C}\mathrm{at})$ the functor $(\mathrm{Fun}(\mathcal{C}^{op}, 1-\mathcal{C}\mathrm{at}))^{op} \to 1-\mathcal{C}\mathrm{at}, Y_1 \mapsto \mathbf{Map}_{\mathrm{Fun}(\mathcal{C}^{op},1-\mathcal{C}\mathrm{at})}(Y_1, Y_2)$ preserves limits. Indeed, if $Y_1 = \mathrm{colim}_{i\in I} Y_1^i$ in $\mathrm{Fun}(\mathcal{C}^{op}, 1-\mathcal{C}\mathrm{at})$ then

$$\mathbf{Map}_{\mathrm{Fun}(\mathcal{C}^{op},1-\mathcal{C}\mathrm{at})}(Y_1, Y_2) \widetilde{\to} \lim_{c\to c'} \mathrm{Fun}(Y_1(c'), Y_2(c)) \widetilde{\to} \lim_{c\to c'} \lim_{i\in I^{op}} \mathrm{Fun}(Y_1^i(c'), Y_2(c))$$

permuting the two latter limits, one gets the desired isomorphism. □

Is it true that the functor $\mathcal{D}^{op} \to 1-\mathcal{C}\mathrm{at}$, $x \mapsto \mathbf{Map}_{\mathcal{D}}(x, d)$ for a 2-category $\mathcal{D}$, preserves limits? No in general!!

We see that if $S \in \mathrm{Sch}^{aff}$ then $ShvCat^{naive}_{/S} \widetilde{\to} (\mathrm{QCoh}(S)-mod)$. More generally, if $Y \in \mathrm{PreStk}$ then $ShvCat^{naive}_{/Y} \widetilde{\to} ShvCat(Y)$ by the above lemma.

0.0.17. For 4.20.1. Let $Y \in \mathrm{PreStk}^{lax}$. We have

$$ShvCat_{/Y^{\mathrm{PreStk}}} \widetilde{\to} ShvCat(Y^{\mathrm{PreStk}}) \widetilde{\to} \lim_{S\to S'} \mathrm{Fun}(Y(S)^{\mathrm{Spc}}, ShvCat(S'))$$

in $1-\mathcal{C}\mathrm{at}$, the limit over $Tw((\mathrm{Sch}^{aff})^{op})$. To see that $ShvCat_{/Y} \to ShvCat_{/Y^{\mathrm{PreStk}}}$ is conservative, using (3) it would suffice to show that for any $S, S' \in \mathrm{Sch}^{aff}$ the map

$$(\mathrm{Fun}(Y(S), ShvCat(S'))_{llax})^{1-\mathcal{C}\mathrm{at}} \to \mathrm{Fun}(Y(S)^{\mathrm{Spc}}, ShvCat(S'))$$

is conservative. However, this is not the case (cf. remark below). So, I think $ShvCat_{/Y} \to ShvCat_{/Y^{\mathrm{PreStk}}}$ is not conservative.

Remark 0.0.18. *If $S \in 1-\mathcal{C}at, T \in 2-\mathcal{C}at$ then the natural map* $\mathrm{Fun}(S,T)^{1-\mathcal{C}at}_{rlax} \to \mathrm{Fun}(S^{\mathrm{Spc}},T)^{1-\mathcal{C}at} = \mathrm{Fun}(S^{\mathrm{Spc}},T^{1-\mathcal{C}at})$ *is not conservative in general. This happens already for $S=[1]$. Namely, if $x,y \in T$, $f,g : x \to y$ are 1-morphisms in T and $\alpha : f \to g$ is a 2-morphism in T, which is not an isomorphism, we get a 1-morphism in* $\mathrm{Fun}(S,T)^{1-\mathcal{C}at}_{rlax}$ *from f to g, which is not an isomorphism, but its restriction to S^{Spc} is an isomorphism. The notation* $\mathrm{Fun}(S,T)_{rlax} \in 2-\mathcal{C}at$ *is that of ([1], ch. A.1, 3.2.7).*

0.0.19. For 4.23. Let $\mathcal{X}$ be a prestack. My understanding is that for a unital sheaf of categories C on $\mathrm{Ran}_{\mathcal{X}}$, the fusion morphism is a 1-morphism in $ShvCat_{/\mathrm{Ran}^{\to}_{\mathcal{X},\emptyset}}$. Here $\mathrm{Ran}^{\to}_{\mathcal{X},\emptyset} \in \mathrm{PreStk}$. The property of being *adj-unital* here means that for any $S \in \mathrm{Sch}^{aff}$ with $f : S \to \mathrm{Ran}^{\to}_{\mathcal{X},\emptyset}$ the corresponding map in $ShvCat(S)$ admits a continuous right adjoint (it will be automatically a strict functor of $\mathrm{QCoh}(S)$-modules, as $\mathrm{QCoh}(S)$ is rigid).

The unit map from Section 4.24 is a 1-morphism in $ShvCat_{/\mathrm{Ran}_{\mathcal{X}}}$, no lax prestacks here!

0.0.20. For 4.25.1(5). It seems this comes from the following. Let $Y, C \in 1-\mathcal{C}at$. Assume given functors $f : Y \to C$ and $g : Y^{\mathrm{Spc}} \to C$. Write $i : Y^{\mathrm{Spc}} \to Y$ for the natural functor. Assume given a 1-morphism $\alpha : g \to f \circ i$ in $\mathrm{Fun}(Y^{\mathrm{Spc}}, C)$ such that for any $y \in Y$, $\alpha(y) : g(y) \to f(y)$ is a monomorphism in C. Assume also for any 1-morphism $h : y_1 \to y_2$ in Y the induced map $f(h) : f(y_1) \to f(y_2)$ in C is such that the composition

$$g(y_1) \stackrel{\alpha(y_1)}{\to} f(y_1) \stackrel{f(h)}{\to} f(y_2)$$

factors through $g(y_2) \stackrel{\alpha(y_2)}{\to} f(y_2)$. Then one gets a functor $g' : Y \to C$ together with a morphism $g' \to f$ in $\mathrm{Fun}(Y, C)$ such that the composition $Y^{\mathrm{Spc}} \to Y \stackrel{g'}{\to} C$ is g.

0.0.21. For 4.27. Given $\mathcal{X} \in \mathrm{PreStk}$, one first defines $[\mathrm{Ran}_{\mathcal{X},\emptyset} \times \mathrm{Ran}_{\mathcal{X},\emptyset}]_{disj} \in \mathrm{PreStk}$ so that its S-points is

$$\underset{I,J \in fSet^{op}_{\emptyset}}{\mathrm{colim}} [\mathcal{X}^I \times \mathcal{X}^J]_{disj}$$

Given prestack $\mathcal{X}$, the lax prestack $[\mathrm{Ran}^{un}_{\mathcal{X}} \times \mathrm{Ran}^{un}_{\mathcal{X}}]_{disj}$ is defined as follows. For each $S \in \mathrm{Sch}^{aff}$, the space underlying $(\mathrm{Ran}^{un}_{\mathcal{X}} \times \mathrm{Ran}^{un}_{\mathcal{X}}(S)$ is $(\mathrm{Ran}_{\mathcal{X},\emptyset} \times \mathrm{Ran}_{\mathcal{X},\emptyset})(S)$. We have to precise a full subspace in the latter space, which defines then the corresponding full subcategory. This full subspace is $[\mathrm{Ran}_{\mathcal{X},\emptyset} \times \mathrm{Ran}_{\mathcal{X},\emptyset}]_{disj}(S)$.

0.0.22. The proof of his Lemma 5.10.1. Let $F : \mathcal{J}^{op} \to \mathcal{P}\mathrm{r}^L$ be a functor, $\mathcal{P}\mathrm{r}^L$ is the notation from HTT, and $\mathcal{J}$ admits fibre products. Let $q : Groth(F) \to \mathcal{J}^{op}$ be the cocartesian fibration corresponding to F. Let $y_1 \leftarrow y \to y_2$ be a diagram in $Groth(F)$, let $i_1 \leftarrow i \to i_2$ be its image in $\mathcal{J}^{op}$, so $i_1 \to i \leftarrow i_2$ is the corresponding diagram in $\mathcal{J}$. Let $\bar{i} = i_1 \times_i i_2$ in $\mathcal{J}$. By (HTT, 4.3.1.11) there is a diagram

$$\begin{array}{ccc} y_1 & \to & \bar{y} \\ \uparrow & & \uparrow \\ y & \to & y_2 \end{array} \tag{5}$$

which is a q-colimit diagram, and lifts the diagram $\bar{i} = i_1 \sqcup_i i_2$ in $\mathcal{J}^{op}$. Now from (HTT, 4.3.1.5) we see that (5) is a colimit diagram in $Groth(F)$. So, $Groth(F)$ admits pushouts, and q preserves push-outs.

0.0.23. In Sect. 5.13 it is understood that in the last displayed formula the middle diagram is a push-out diagram.

0.0.24. For 5.15. A version with a module would be as follows. Let C be a symmetric monoidal $(\infty,1)$-category, $M \in 1-\mathcal{C}\mathrm{at}$ be a C-module. Let $\Phi_C : C \to 1-\mathcal{C}\mathrm{at}$ be right-lax symmetric monoidal, $\Phi_M : M \to 1-\mathcal{C}\mathrm{at}$ be a functor such that the map Φ from (C, M) to $(1-\mathcal{C}\mathrm{at}, 1-\mathcal{C}\mathrm{at})$ is right-lax (in the sense of [1], ch. I.1, 3.5.1), that is, Φ_M is right-lax compatible with actions. Let $f_C : \mathcal{X}_C \to C, f_M : \mathcal{X}_M \to M$ be the corresponding cocartesian fibrations. Then $\mathcal{X}_C$ is symmetric monoidal, f_C is symmetric monoidal, and $\mathcal{X}_C$ acts naturally on $\mathcal{X}_M$. Namely, $(c \in C, y \in \Phi_C(c))$ sends $(m \in M, z \in \Phi_M(m))$ to $(cm \in M, y \circ z \in \Phi_M(cm))$, where $y \circ z$ is the image of (y, z) under the natural map $\Phi_C(c) \times \Phi_M(m) \to \Phi_M(cm)$. So, the diagram commutes

$$\begin{array}{ccc} \mathcal{X}_C \times \mathcal{X}_M & \to & \mathcal{X}_M \\ \downarrow & & \downarrow \\ C \times M & \to & M \end{array}$$

0.0.25. For 5.15 more! Let $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad. Let $\mathcal{M} : \mathcal{O}^{\otimes} \to 1-\mathcal{C}\mathrm{at}$ be an $\mathcal{O}$-monoid in $1-\mathcal{C}\mathrm{at}$ in the sense of ([6], 2.4.2.1). Let $\mathcal{Y}^{\otimes} \xrightarrow{\pi} \mathcal{O}^{\otimes}$ be the cocartesian fibration attached to $\mathcal{M}$. Then $\mathcal{Y}^{\otimes}$ is an $\mathcal{O}$-monoidal category, in particular the composition $\mathcal{Y}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is an ∞-operad by ([6], 2.4.2.4).

We may always consider the category of cocartesian sections $\mathcal{O}^{\otimes} \to \mathcal{Y}^{\otimes}$ of π, they are automatically morphisms of ∞-operads. This is the category $\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathcal{O}, \mathcal{Y})$ of $\mathcal{O}$-monoidal functors from $\mathcal{O}$ to $\mathcal{Y}$.

0.0.26. Let Y be a lax prestack. We sometimes want to calculate the limit of the composition $Tw(\mathrm{Sch}^{aff})^{op} \to (Sch^{aff})^{op} \to 1-\mathcal{C}\mathrm{at}$, that is,

$$\lim_{(S \to S') \in (Tw(Sch^{aff}))^{op}} Y(S)$$

For any $S \in \mathrm{Sch}^{aff}$, the category $Tw(Sch^{aff}) \times_{\mathrm{Sch}^{aff}} \mathrm{Sch}^{aff}_{/S}$ has an initial object $S \leftarrow \emptyset \to *$, hence is contractible. This implies that the RKE of the above composition under $Tw(\mathrm{Sch}^{aff})^{op} \to (Sch^{aff})^{op}$ is the same functor $Y : (Sch^{aff})^{op} \to 1-\mathcal{C}\mathrm{at}$. So,

$$\lim_{(S \to S') \in (Tw(Sch^{aff}))^{op}} Y(S) \widetilde{\to} \lim_{S \in (Sch^{aff})^{op}} Y(S)$$

For example, let Y be the constant lax prestack with value $C \in 1-\mathcal{C}\mathrm{at}$. Then

$$ShvCat^{naive}_{/Y} \widetilde{\to} \lim_{S \in (\mathrm{Sch}^{aff})^{op}} \mathrm{Fun}(C, ShvCat(S)) \widetilde{\to} \mathrm{Fun}(C, \mathrm{DGCat}_{cont})$$

by (4). Indeed, Sch^{aff} has a final object $\mathrm{Spec}\, k$. Similarly,

$$ShvCat_{/Y} \widetilde{\to} \lim_{S \in (\mathrm{Sch}^{aff})^{op}} (\mathrm{Fun}(C, ShvCat(S))_{llax})^{1-\mathcal{C}\mathrm{at}} \widetilde{\to} (\mathrm{Fun}(C, \mathrm{DGCat}_{cont})_{llax})^{1-\mathcal{C}\mathrm{at}}$$

We get for the constant lax prestack Y with value $C \in 1-\mathcal{C}\mathrm{at}$, $\mathrm{QCoh}(Y) \widetilde{\to} \mathrm{Fun}(C, \mathrm{Vect})$.

Let Y be the constant prestack with value $C \in 1-\mathcal{C}\mathrm{at}$. Then $\mathrm{QCoh}_Y \in ShvCat^{naive}_{/Y}$ becomes an object of $\mathrm{Fun}(C, \mathrm{DGCat}_{cont})$, namely the constant functor $C \to * \to \mathrm{DGCat}_{cont}$ with value Vect.

If Y is a lax prestack let Y^{inv} be the prestack obtained by termwise inverting all arrows: for $S \in \mathrm{Sch}^{aff}$, $Y^{inv}(S)$ is obtained from $Y(S)$ by inverting all arrows. We have a natural functor $\mathrm{QCoh}(Y^{inv}) \to \mathrm{QCoh}(Y)$. It is not essentially surjective in general. Already for $Y = [1]$ it is not.

0.0.27. For 5.19. My understanding is that this construction is only used when working with lax prestacks. If we want only to consider $\mathrm{PreStk}^{ShvCat}_{corr}$ as a 2-category then already 5.14.1-5.14.2 are sufficient?

0.0.28. For 5.21.1. For each n we have a diagram $S^{\times n} \stackrel{m_1}{\leftarrow} mult_{n,S} \stackrel{m_2}{\to} S$, which realizes the n-ary multiplication in $\mathrm{PreStk}^{lax}_{corr}$ on S. Then for a multiplicative sheaf of categories Ψ on S we have $m_1^*(\Psi^{\boxtimes n}) \widetilde{\to} m_2^*(\Psi)$ over $mult_{n,S}$.

0.0.29. The notion of a weakly multiplicative sheaf of categories from 5.20.1 is very interesting in general. For example, let Y be a commutative algebra in PreStk. So, we have the product map $m : Y \times Y \to Y$ and the unit map $e : * \to Y$. Then this gives a sheaf of categories $C \in ShvCat(Y)$ together with maps $C \boxtimes C \to m^*C$, $\mathrm{Vect} \to e^*C$ satisfying the compatibilities. What are nontrivial examples in $MultCat^w(Y)$? What happens for Y an affine scheme?

Note that $MultCat^w(Y)$ is symmetric monoidal ∞-category. Example: let $Y = \mathrm{Spec}\, k$, so $C \in \mathrm{DGCat}_{cont}$ is equipped with $C \otimes C \to C$ and $\mathrm{Vect} \to C$ making it into a commutative algebra in DGCat_{cont}. My understanding is that

$$MultCat^w(\mathrm{Spec}\, k) \widetilde{\to} CAlg(\mathrm{DGCat}_{cont})$$

If $f : Y' \to Y$ is a map in $CAlg(\mathrm{PreStk})$ and C is a weakly multiplicative sheaf of categories on Y then f^*C is a weakly myltiplicative sheaf of categories on Y', and $f^* : MultCat^w(Y) \to MultCat^w(Y')$ is symmetric monoidal. So, for a commutative algebra C in DGCat_{cont} we get for $g : Y \to \mathrm{Spec}\, k$ the weakly multiplicative sheaf of categories g^*C.

Given a weakly multiplicative sheaf of categories D on $Y \in CAlg(\mathrm{PreStk})$, its restriction under $e : \mathrm{Spec}\, k \to Y$ is an object of $CAlg(\mathrm{DGCat}_{cont})$.

0.0.30. For 5.24. Assume $S \in CAlg(\mathrm{PreStk}^{lax}_{corr})$ and $\Psi, \Psi' \in MultCat^w(S)$. To understand the difference between $MultCat^w(S)$ and $MultCat^{w,lax}(S)$ note the following. If $\alpha : \Psi \to \Psi'$ is a morphism in $MultCat^w(S)$ then we get a commutative diagram in $ShvCat_{/mult_S}$ (that is, an explicit invertible 2-morphism in the 2-category $ShvCat_{/mult_S}$ is given making the square commute)

$$\begin{array}{ccc} m_1^*(\Psi \boxtimes \Psi) & \stackrel{\eta}{\to} & m_2^*\Psi \\ \downarrow {\scriptstyle m_1^*(\alpha\boxtimes\alpha)} & & \downarrow {\scriptstyle m_2^*(\alpha)} \\ m_1^*(\Psi' \boxtimes \Psi') & \stackrel{\eta'}{\to} & m_2^*\Psi' \end{array}$$

Here η, η' are the corresponding structure maps for the weakly multiplicative sheaves of categories Ψ, Ψ'. If α is only a morphism in $MultCat^{w,lax}(S)$ then for the above square we are only given a 2-morphism

$$\delta : \eta' \circ m_1^*(\alpha \boxtimes \alpha) \to m_2^*(\alpha) \circ \eta$$

in $ShvCat_{/mult_S}$. So, we here essentially use the 2-category structure of $ShvCat_{/mult_S}$.

For example, if $\Psi = \mathrm{QCoh}_S$ then the above map α is given by a section $s \in \Gamma(S, \Psi')$ together with a 1-morphism $\delta : \eta' m_1^*(s \boxtimes s) \to m_2^* s$ in $\Gamma(mult_S, m_2^*\Psi')$. If δ is an ismorphism then this is a map in $MultCat^w(S)$

So, we see that already for $S \in CAlg(\mathrm{PreStk}_{corr})$ there is a big difference between $MultCat^w(S)$ and $MultCat^{w,lax}(S)$, as the DG-category $\Gamma(mult_S, m_2^*\Psi')$ is not a groupoid.

My understanding is that Sam means by morphisms in $MultCat^{w,lax}(S)$ the nonunital right-lax maps, that is, the above morphism from Ψ to Ψ' in $MultCat^{w,lax}(S)$ contains as a part of data a 2-morphism

$$\eta'_e \to (e_2^*\alpha) \circ \eta_e$$

in $ShvCat_{/unit_S}$ for the diagram

$$\begin{array}{ccc} e_1^* \mathrm{Vect} & \stackrel{\eta_e}{\to} & e_2^*\Psi \\ & \searrow \eta'_e & \downarrow e_2^*\alpha \\ & & e_2^*\Psi' \end{array}$$

Here I used the notations from his Section 5.21, namely $* \stackrel{e_1}{\leftarrow} unit_S \stackrel{e_2}{\to} S$ is the unit map for the algebra S in $\mathrm{PreStk}^{lax}_{corr}$.

0.1. For 5.26. The functor $\Gamma(-,-) : Groth(ShvCat_{/-}) \to \mathrm{DGCat}_{cont}$ sends $(Y, C \in ShvCat_{/Y})$ to $\Gamma(Y, C)$. A map from (Y', C') to (Y, C) in $Groth(ShvCat_{/-})$ is a map of lax prestacks $f : Y \to Y'$ and $f^*C' \to C$ in $ShvCat_{/Y}$. The corresponding map $\Gamma(Y', C') \to \Gamma(Y, C)$ is the composition $\Gamma(Y', C') \to \Gamma(Y, f^*C') \to \Gamma(Y, C)$.

Similarly, consider the cocartesian fibration $\mathcal{X} \to Groth(ShvCat_{/-})$ corresponding to this functor. Then an object of $\mathcal{X}$ is $(Y \in \mathrm{PreStk}^{lax}, C \in ShvCat_{/Y}, \mathcal{F} \in \Gamma(Y, C))$. Consider a map $(Y_1, C_1) \to (Y_2, C_2)$ given by $f : Y_2 \to Y_1$, $f^*C_1 \to C_2$. A map from $(Y_1, C_1, \mathcal{F}_1)$ to $(Y_2, C_2, \mathcal{F}_2)$ in $\mathcal{X}$ over it is given in addition by a map $\mathcal{F} \to \mathcal{F}_2$ in $\Gamma(Y_2, C_2)$. Here $\mathcal{F}$ is the image of $f^*\mathcal{F}_1$ under the induced map $\Gamma(Y_2, f^*C_1) \to \Gamma(Y_2, C_2)$.

0.1.1. For 5.30. To be clear, $Mult^{op-w}(\Psi)$ is the category of maps $\mathrm{QCoh}_S \to \Psi$ in $MultCat^{op-w,lax}(S)$. What is the definition of $MultCat^{op-w,lax}(S)$? View S as a commutative coalgebra in $\mathrm{PreStk}^{lax}_{corr}$ by flipping correspondences. Then $MultCat^{op-w,lax}(S)$ is the category of commutative coalgebras in $\mathrm{PreStk}^{lax,ShvCat}_{corr}$ over S, where we allow lax morphisms of commutative coalgebras. However, should we allow "left-lax" or "right-lax" morphisms? According to the next paragraph, we have to allow the following.

Given $\alpha : \Psi \to \Psi'$ in $MultCat^{op-w,lax}(S)$, we will have the square

$$\begin{array}{ccc} m_2^*\Psi' & \stackrel{\tilde{\eta}'}{\to} & m_1^*(\Psi' \boxtimes \Psi') \\ \uparrow m_2^*\alpha & & \uparrow m_1^*(\alpha\boxtimes\alpha) \\ m_2^*\Psi & \stackrel{\tilde{\eta}}{\to} & m_1^*(\Psi \boxtimes \Psi) \end{array}$$

This diagram only "lax commutes" in the sense that we are given a 2-morphism $\tilde{\eta}' \circ m_2^*\alpha \to m_1^*(\alpha \boxtimes \alpha) \circ \tilde{\eta}$. And simiarly for the "counit" maps $\tilde{\eta}_e$.

Suppose the structure maps in $\Psi \in MultCat^{op-w}(S)$ are given by his (5.29.1), that is,

$$\begin{array}{l} \tilde{\eta}_m : m_2^*\Psi \to m_1^*(\Psi \boxtimes \Psi) \in ShvCat_{/mult_S} \text{ and} \\ \tilde{\eta}_e : e_2^*(\Psi) \to \mathrm{QCoh}_{unit_\Psi} \in ShvCat_{/unit_S} \end{array} \tag{6}$$

Then an op-weakly multiplicative object in Ψ is an element $s \in \Gamma(S, \Psi)$ together with maps

$$\tilde{\eta}_m m_2^*(s) \to m_1^*(s \boxtimes s) \ \in \Gamma(mult_S, m_1^*(\Psi \boxtimes \Psi)$$

and

$$\tilde{\eta}_e e_2^*(s) \to k \ \in \Gamma(unit_S, \mathrm{QCoh})$$

with the corresponding compatibilities.

0.1.2. In Lemma 5.30.1 there is a mistake: the category $MultCat^{op-w}(\mathcal{S})$ is not presentable in general, though it probably has colimits and limits. This already happens for $\mathcal{S} = *$, in which case this is the category of commutative coalgebras in DGCat_{cont}. The category $CCoalg(\mathrm{DGCat}_{cont})$ admits colimits, and the forgetful functor $CCoalg(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves colimits by (HA, 3.2.2.4).

0.1.3. For 5.35. Assume $\mathcal{T} : (\mathrm{PreStk}^{lax})^{op} \to 1-\mathcal{C}\mathrm{at}$ is a right-lax symmetric monoidal functor. Then Sam defined the notion of a multiplicative sheaf on $\mathcal{S} \in CAlg(\mathrm{PreStk}^{lax}_{corr})$ with values in $\mathcal{T}$. This is useful for considering other theories of sheaves as in ([2], Sect. 1.1).

0.1.4. For 5.27. For convenience, the definition of a weakly multiplicative object of a weakly multiplicafive sheaf of categories: let $S \in CAlg(\mathrm{PreStk}^{lax}_{corr})$, Ψ be a weakly multplicative sheaf of categories on S. So, we have the map

$$\eta_m : m_1^*(\Psi \boxtimes \Psi) \to m_2^*\Psi$$

in $ShvCat_{/mult_S}$ and the map

$$\eta_e : e_1^* \mathrm{Vect} \to e_2^*(\Psi)$$

in $ShvCat_{/unit_S}$. A weakly multiplicative object ψ of Ψ (defined in his Section 5.25.1) gives rise to morphisms

$$\eta_m m_1^*(\psi \boxtimes \psi) \to m_2^*\psi$$

in $\Gamma(mult_S, m_2^*\Psi)$, and

$$\eta_e(\mathcal{O}_{unit_S}) \to e_2^*\psi$$

in $\Gamma(unit_S, e_2^*\Psi)$ with the correspoding compatibilities.

0.1.5. For 6.3. For $\mathcal{X} \in \mathrm{PreStk}$, $[\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj}$ is a nonunital commutative algebra in PreStk_{corr} via the product

$$\begin{array}{ccc} [\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj} \times [\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj} & \leftarrow & [\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj} \\ & & \downarrow \\ & & [\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj} \end{array}$$

Here in the middle we have the prestack given by 4 subsets of $\mathcal{X}$, which are pairwise disjoint. The diagram

$$\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}} \stackrel{m_1}{\leftarrow} [\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj} \stackrel{add}{\to} \mathrm{Ran}_{\mathcal{X}}$$

is a diagram of maps of nonunital commutative algebras in PreStk_{corr}. So, if C is a chiral category for $\mathcal{X}$, both $\mathrm{add}^* C$ and $C \boxtimes C$ are multiplicative sheaves of categories on $[\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj}$. My understanding is that the isomorphism

$$m_1^*(C \boxtimes C) \widetilde{\to} add^* C$$

on $[\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj}$ is an isomorphism of multiplicative sheaves of categories. Here $m_1 : [\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{disj} \hookrightarrow \mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}$ is the inclusion. Simiarly also in the unital case for $\mathrm{Ran}_{\mathcal{X}}^{un}$. Besides, in the unital case the map $i : \mathrm{Spec}\, k \stackrel{\emptyset}{\to} \mathrm{Ran}_{\mathcal{X}}^{un}$ is a morphism of commutative algebras in $\mathrm{PreStk}_{corr}^{lax}$, so i^*C is a multiplicative sheaf of categories on $\mathrm{Spec}\, k$. I think for a unital factorization category C the isomorphism $i^*C \widetilde{\to} \mathrm{Vect}$ is an isomorphism of multiplicative sheaves of categories.

This looks like the claim that $CAlg(CAlg(C))$ is isomorphic to C for $C \in 1 - \mathcal{C}\mathrm{at}$. Sam says this is true for any $C \in 1 - \mathcal{C}\mathrm{at}$.

A version of this for unital categories exlplains the fact that $unit_C$ is a factorization algebra in C. We get isomorphisms $i^* unit_C \widetilde{\to} k$ and

$$\eta_m m_1^*(unit_C \boxtimes unit_C) \widetilde{\to} add^* unit_C$$

If $C, D \in ShvCat_{/ \mathrm{Ran}_{\mathcal{X}}^{un}}$ and $f : C \to D$ is a map in $ShvCat_{/ \mathrm{Ran}_{\mathcal{X}}^{un}}$ then we get units $unit_C \in \Gamma(\mathrm{Ran}_{\mathcal{X}}, C), unit_D \in \Gamma(\mathrm{Ran}_{\mathcal{X}}, D)$ as in his Section 4.24. Then f yields a natural map $unit_D \to f(unit_C)$ in $\Gamma(\mathrm{Ran}_{\mathcal{X}}, D)$. Indeed, for $S \in \mathrm{Sch}^{aff}$ and any $g \in \mathrm{Ran}_{\mathcal{X}}(S)$ consider the morphism $\emptyset \stackrel{\epsilon}{\to} g$ in $\mathrm{Ran}_{\mathcal{X}}^{un}(S)$. The description of f in his Section 4.17 gives a 2-morphism $unit_D \to f \circ unit_C$ for the square below

$$\begin{array}{ccc} \mathrm{QCoh}_S & \stackrel{unit_C}{\to} & g^*C \\ \downarrow \mathrm{id} & & \downarrow f \\ \mathrm{QCoh}_S & \stackrel{unit_D}{\to} & g^*D \end{array}$$

The corresponding universal 2-morphism becomes the desired map $unit_D \to f(unit_C)$ in $\Gamma(\mathrm{Ran}_{\mathcal{X}}, D)$.

Let now D be a unital chiral category for $\mathcal{X}$, and $s : \mathrm{QCoh} \to D$ be a morphism in $MultCat^w(\mathrm{Ran}_{\mathcal{X}}^{un})$ giving a unital factorization algebra s in D. View s just as a map in $ShvCat_{/ \mathrm{Ran}_{\mathcal{X}}^{un}}$. Note that $\mathcal{O}_{\mathrm{Ran}_{\mathcal{X}}} = unit_{QCoh} \in \Gamma(\mathrm{Ran}_{\mathcal{X}}, \mathrm{QCoh})$. The corresponding map $unit_D \to f(unit_{\mathrm{QCoh}}) = s$ is *the unit map* for the factorization algebra s as his map (6.3.2). He claims the latter is a map of unital factorization algebras in D. Ê

0.2. For Appendix A. If $Y \in$ PreStk then $ShvCat(Y)$ has a natural structure of an $(\infty, 2)$-category. For $Y \in \mathrm{Sch}^{aff}$ this is done in ([1], ch. I.1, 8.3). Namely, in this case the mapping category is the inner hom by ([1], ch. I.1, 8.3.3). Maybe for any prestack Y, the mapping category should be the inner hom for its symmetric monoidal structure? Not clear.

Given $f : S' \to S$ in Sch^{aff}, the functor $\mathrm{QCoh}(S) - mod \to \mathrm{QCoh}(S') - mod$ is a map in $2 - \mathcal{C}\mathrm{at}$, it is clear that we have the corresponding morphisms of the mapping categories (cf. [11], 6.0.6). Sam confirms this, as $2 - Cat$ can be thought of as the category of $(\infty, 1)$-categories enriched over $1-\mathcal{C}\mathrm{at}$. So, the limit $\lim_{S\to Y} \mathrm{QCoh}(S)-mod$ over $(\mathrm{Sch}^{aff}_{/Y})^{op}$ can be seen as a limit in $2 - \mathcal{C}\mathrm{at}$. Then according to ([11], 13.1.38), we get the mapping category for $C, D \in ShvCat(Y)$

$$\mathbf{Map}_{ShvCat(Y)}(C, D) \widetilde{\to} \lim_{S\to Y} \mathbf{Map}_{ShvCat(S)}(\Gamma(S, C), \Gamma(S, D)) \tag{7}$$

the limit over $(\mathrm{Sch}^{aff}_{/Y})^{op}$. Here for $S \in \mathrm{Sch}^{aff}$, $E_i \in \mathrm{QCoh}(S) - mod$,

$$\mathbf{Map}_{ShvCat(S)}(E_1, E_2) = \mathrm{Fun}_{\mathrm{QCoh}(S)}(E_1, E_2)$$

is the inner hom in $ShvCat(S)$. The limit (7) can be seen as a limit in DGCat_{cont}, as the projection $\mathrm{DGCat}_{cont} \to 1 - \mathcal{C}\mathrm{at}$ preserves limits. So, $ShvCat(Y)$ is enriched over DGCat_{cont}. My understanding is that the 2-category $ShvCat(Y)$ is actually enriched over $\mathrm{QCoh}(Y) - mod$.

For $C \in ShvCat(Y)$ we get $\mathbf{Map}_{ShvCat(Y)}(\mathrm{QCoh}_Y, C) \widetilde{\to} \Gamma(Y, C)$ canonically.

Actually, a stronger claim holds. Namely, for $C, D \in ShvCat_{/Y}$ there is an inner hom from C to D in the symmetric monoidal category $ShvCat_{/Y}$. Indeed, for any $S' \to S \to Y$ with $S, S' \in \mathrm{Sch}^{aff}$, there is a canonical isomorphism

$$\mathrm{Fun}_{\mathrm{QCoh}(S)}(\Gamma(S, C), \Gamma(S, D)) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S') \widetilde{\to} \mathrm{Fun}_{\mathrm{QCoh}(S')}(\Gamma(S', C), \Gamma(S', D))$$

It holds because $\mathrm{QCoh}(S), \mathrm{QCoh}(S')$ are rigid by ([11], 6.0.6). So, we get the inner hom $\mathcal{H}om(C, D) := \underline{\mathrm{Hom}}_{ShvCat_{/Y}}(C, D)$ whose sections over S are

$$\mathrm{Fun}_{\mathrm{QCoh}(S)}(\Gamma(S, C), \Gamma(S, D))$$

Now (7) is the category of global sections of $\underline{\mathrm{Hom}}_{ShvCat_{/Y}}(C, D)$. We may simply say that $ShvCat_{/Y}$ is a 2-category, because it admits inner homs, which are enriched over DGCat_{cont}.

More generally, we may view the functor $ShvCat_{/-} : (\mathrm{Sch}^{aff})^{op} \to 1 - \mathcal{C}\mathrm{at}$, $S \mapsto ShvCat(S)$ as a functor $ShvCat_{/-} : (\mathrm{Sch}^{aff})^{op} \to 2 - \mathcal{C}\mathrm{at}$, and consider its RKE as a functor $ShvCat_{/-} : \mathrm{PreStk}^{op} \to 2 - \mathcal{C}\mathrm{at}$. In particular, for any morphism $f : Y' \to Y$ in PreStk, we get a morphism $f^* : ShvCat_{/Y} \to ShvCat_{/Y'}$ of 2-categories.

Assume $f : X \to Y$ is 1-affine (cf. A.8). Then the functor $f_* : ShvCat_{/X} \to ShvCat_{/Y}$ from Prop. A.9.1(2) is a morphism in $2 - \mathcal{C}\mathrm{at}$. Indeed, assume first Y affine. Then f_* is the forgetful functor $\mathrm{QCoh}(X) - mod \to \mathrm{QCoh}(Y) - mod$. This is a map in $2 - \mathcal{C}\mathrm{at}$ by ([11], 6.0.6). Now let Y be arbitrary, $C, D \in ShvCat_{/X}$. We claim that there is a canonical map

$$f_*\mathcal{H}om(C, D) \to \mathcal{H}om(f_*C, f_*D)$$

in $ShvCat_{/Y}$. Indeed, from adjointness and the projection formula, we get a natural map

$$f_*C\otimes f_*\mathcal{H}om(C,D)\widetilde{\to} f_*(\mathcal{H}om(C,D)\otimes f^*f_*C)\to f_*(\mathcal{H}om(C,D)\otimes C)\to f_*D$$

Finally, passing to the global sections, we get the desired functor between the mapping categories.

0.2.1. For A.2.3. He uses ([1], ch. I.1, Lm 5.4.3) to see that $C\otimes_{A-mod} B-mod\to C$ is conservative.

0.2.2. For A.3. Let $f: X\to Y$ be a morphism in PreStk. Then I think in general $coind_f: ShvCat(X)\to ShvCat(Y)$ does not preserve all colimits. What happens for $f:\underset{n\geq 0}{\sqcup} *\to *$?

0.2.3. For A.4.3. If $F: D\to C$ is a map of A-module categories, this is a map in $\mathcal{A}-mod(\mathrm{DGCat}_{cont})$ with $\mathcal{A}=A-mod$. So, F is continuous. Let $\mathcal{X}\subset D\otimes_{A-mod}B-mod$ be the full subcategory of those x such that for any $y\in D\otimes_{A-mod}B-mod$ the map $\mathrm{Map}_{D\otimes_{A-mod}B-mod}(x,y)\to \mathrm{Map}_{C\otimes_{A-mod}B-mod}(F_B(x),F_B(y)$ is an isomorphism. Then it is easy to see that $\mathcal{X}$ is closed under colimits (since F_B preserves colimits). Since the image of the functor $D\to D\otimes_{A-mod}B-mod$ generates $D\otimes_{A-mod}B-mod$ under colimits, it suffices indeed to show that (A.4.1) is an isomorphism.

0.2.4. For A.4.5. The projection $\mathrm{DGCat}_{cont}\to 1-\mathcal{C}at$ preserves limits (cf.[11]). The limit of the constant family in $1-\mathcal{C}at$ over a contractible category is the same constant category. (Contractibility is needed: for example take $\mathcal{J}=*\sqcup *$).

0.2.5. For A.5. If $\mathrm{Spec}\, A\in \mathrm{Sch}^{aff}$, $F: C\to D$ is a morphism of A-linear categories then $F(C)\subset D$ is the smallest cocomplete presentable subcategory containing the objects $F(x), x\in C$. Then $F(C)\in \mathrm{DGCat}_{cont}$ and is preserved under $A-mod$-action on D, so $F(C)$ is an A-linear category.

If $F: C\to D$ is a morphism in $ShvCat(Y)$ for some $Y\in$ PreStk then $F(C)\in ShvCat(Y)$ is defined as follows. For $S\in \mathrm{Sch}^{aff}_{/Y}$ we let $\Gamma(S,F(C))=F(\Gamma(S,C))$. Here we also denoted by $F:\Gamma(S,C)\to\Gamma(S,D)$ the corresponding S-linear functor, and $F(\Gamma(S,C))\subset\Gamma(S,D)$ is the presentable stable subcategory generated under colimits by the image of $\Gamma(S,C)$. By A.5.1, this is indeed a sheaf of categories on Y, and $F(C)\subset D$ is locally fully faithful.

0.2.6. For A.6. Recall that for $Y\in$ PreStk, $ShvCat(Y)$ admits all colimits (computed locally). So, for a map $C\to D$ in $ShvCat(Y)$, we get $D/C\in ShvCat(Y)$. For $S\to Y$ with $S\in\mathrm{Sch}^{aff}$, $\Gamma(S,D/C)\widetilde{\to}\Gamma(S,D)/\Gamma(S,C)$, the latter colimit is taken in $\mathrm{QCoh}(S)-mod$.

0.2.7. Let $S = \operatorname{Spec} A \in \mathrm{Sch}^{aff}$, $h : C^0 \subset C$ a diagram in $\mathcal{A} - mod(\mathrm{DGCat}_{cont})$ with $\mathcal{A} = A - mod$. In particular, h is continuous. The quotient C/C_0 can also be calculated as a limit, namely let $h^R : C \to C^0$ be the right adjoint to h. It is a strict functor of $\mathcal{A}$-module categories, but h^R need not be continuous. Since colimits in DGCat_{cont} can be rewritten by passing to right adjoints as limits in DGCat, we get $C/C_0 \tilde{\to} 0 \times_{C^0} C = \mathrm{Ker}(h^R)$. So, the evaluation $e : C/C_0 \to C$ is fully faithful, hence conservative. It follows that its left adjoint $p : C \to C/C^0$ generates C/C^0 under colimits by ([1], ch. I.1, Lm.5.4.3), and p is a localization functor.

Lemma 0.2.8. *Under the above assumptions, the kernel of $p : C \to C/C^0$ is C^0.*

Proof. Set $D = \operatorname{Ker} p$, so $C^0 \subset D \subset C$ are full subcategories. By definition, $D = \{x \in C \mid \text{for any } y \in C/C^0, \mathrm{Map}_C(x, y) = *\}$. For $x \in D, y \in C/C^0$ we get $\mathcal{M}aps_{k,C}(x, y) = 0$, because p is Vect-linear. So, $D = \{x \in C \mid \text{for any } y \in C/C^0, \mathcal{M}aps_{k,C}(x, y) = *\}$.

Given $x \in C$, consider the fibre sequence $hh^R(x) \xrightarrow{\alpha} x \xrightarrow{\beta} y$ in C, where α is the natural map. Applying the exact functor h^R to this sequence, we get $h^R(x) \xrightarrow{\mathrm{id}} h^R(x) \to 0$, so $y \in \mathrm{Ker}(h^R) \tilde{\to} C/C^0$. We claim that $\beta : x \to y$ is isomorphic to the natural map $x \to ep(x)$ in C. Indeed, $p(\beta)$ is an isomorphism, and $p(y) \tilde{\to} y$. Assume in addition $x \in D$. Then $0 \tilde{\to} p(x)$, so the corresponding map $x \to ep(x)$ is the map $x \to 0$, hence α is an isomorphism. So, $x \in C^0$. (This also follows from [13], Lemma 1.8.15). $\square$

0.2.9. For Lm. A.7.4. The projection $\mathcal{A} - mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves colimits for $\mathcal{A} = A - mod$. Let $F : D \to C$ be a map in DGCat_{cont}, $C/D = \{0\} \sqcup_D C$ in DGCat_{cont}. The diagram $D \to F(D) \to C$ shows that we have a natural map $C/D \to C/\bar{D}$. To check this is an equivalence, for any $E \in \mathrm{DGCat}$ we check that $\mathrm{Map}(C/\bar{D}, E) \to \mathrm{Map}(C/D, E)$ is an isomorphism, where $\mathrm{Map} = \mathrm{Map}_{\mathrm{DGCat}_{cont}}$. This is the space $\mathrm{Map}(C, E) \times_{\mathrm{Map}(D,E)} \{0\}$, the full subspace in $\mathrm{Map}(C, E)$ of those functors whose restriction to $\bar{D}$ is isomorphic to zero. Indeed, $D \to F(D)$ generates $F(D)$ under colimits, and our functors are continuous.

0.2.10. For Pp. A.7.3. For each $i \in \mathcal{J}$ we have a full subcategory $\mathrm{Ker}(C_{i_0} \to C_i) \subset C_{i_0}$. Since $\mathcal{J}$ is filtered, it is contractible. Maybe it can be checked that $\mathrm{colim}_{i \in \mathcal{J}} \mathrm{Ker}(C_{i_0} \to C_i)$ is a full subcategory in C_{i_0}, but we don't need this because of Lemma A.7.4.

0.2.11. For Th. A.8.1. The category Sch_{aft} of schemes almost of finite type was defined in ([1], ch. I.3, 3.5.1). The claim is for $S \in \mathrm{Sch}_{aft}$, S_{dR} is 1-affine.

In A.8.2 he means schematic morphism.

0.2.12. For A.9. For a morphism of prestacks $f : X \to Y$ the functor $f_* : ShvCat(X) \to ShvCat(Y)$ does not need to preserve colimits. The reason is that if $S \to Y$ with $S \in \mathrm{Sch}^{aff}$, $S \times_Y X$ does not need to be affine. Already the functor $\Gamma^{enh}_X : ShvCat(X) \to \mathrm{QCoh}(X) - mod$ does not preserve colimits in general.

For A.9.1(3). Let $f : X \to Y$ be quasi-compact (quasi-separated) schematic morphism in PreStk, $C \in ShvCat(Y)$. Since f is 1-affine, by A.9.1(2) it satisfies the projection formula. That is, given $S \to Y$ with $S \in \mathrm{Sch}^{aff}$, $\Gamma(S, C) \to \Gamma(S \times_Y X, C)$ is the functor $\alpha : \Gamma(S, C) \to \Gamma(S, C) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S \times_Y X)$. This α admits a right adjoint, because the functor $f^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(S \times_Y X)$ admits a right adjoint

$f_* : \mathrm{QCoh}(S \times_Y X) \to \mathrm{QCoh}(S)$, the latter functor is continuous by ([1], ch. I.3, 2.2.2). So, α^R is a continuous fucntor of $\mathrm{QCoh}(S)$-module categories. As S varies, they form a map $f_* f^*(C) \to C$ in $ShvCat(Y)$.

How do we see $ShvCat(Y)$ as a 2-category? If $S \in \mathrm{Sch}^{aff}$ corresponding to an algebra $A \in VAlg(\mathrm{Vect}^{\le 0})$ then $\mathrm{QCoh}(S) - mod$ is naturally an $(\infty, 2)$-category, which I think is $(\mathrm{QCoh}(S) - mod^{St,cocmpl}_{cont})^{2-Cat}$ in the notations of ([1], ch. I.1, 8.3.1). Namely, for $C, D \in \mathrm{QCoh}(S)-mod$, the mapping category is $\mathrm{Fun}_A(C, D)$ with $A = \mathrm{QCoh}(S)-mod$, the category of continuous $\mathrm{QCoh}(S) - mod$-linear functors as in ([1], ch. I.1, 8.2.1).

So, existence of a right adjoint to $C \to f_* f^*(C)$ means just that for any $S \to Y$ with $S \in \mathrm{Sch}^{aff}$ the correspinding functor $\Gamma(S, f_* f^* C) \to \Gamma(S, C)$ is continuous (it is automatically $\mathrm{QCoh}(S)$-linear).

Now if $C, D \in ShvCat(Y)$, the mapping category from C to D in the corresponding 2-category is

$$\lim_{S \to Y} \mathrm{Fun}_{\mathrm{QCoh}(S)-mod}(\Gamma(S, C), \Gamma(S, D)),$$

the limit over $(\mathrm{Sch}^{aff}_{/Y})^{op}$.

0.2.13. For A.9.2. Let $f : Y \to Z$ be quasi-compact (quasi-separated) schematic morphism of prestacks, $C \in ShvCat(Z)$. Given a map $g : S' \to S$ in Sch^{aff}/Z, consider the diagram

$$\begin{array}{ccc} \Gamma(S, C) & \stackrel{\alpha_S}{\to} & \Gamma(S, C) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S \times_Z Y) \\ \downarrow g^* & & \downarrow g'^* \\ \Gamma(S', C) & \stackrel{\alpha_{S'}}{\to} & \Gamma(S', C) \otimes_{\mathrm{QCoh}(S')} \mathrm{QCoh}(S' \times_Z Y) \end{array}$$

the vertical maps being restrictions, and the horizontal ones are as in A.9.1(3). Let $\alpha^R_C, \alpha^R_{S'}$ denote the corresponding right adjoints, they are continuous, and by ([1], ch. I.3, 2.2.2) the base change gives $\alpha^R_{S'} g'^* \tilde{\to} g^* \alpha^R_S$. So, we may apply ([1], ch. I.1, Lm. 2.6.4) to see that the functor obtained by passing to the limit

$$f^*_C : \Gamma(Z, C) = \lim_{S \to Z} \Gamma(S, C) \to \lim_{S \to Z} \Gamma(S, C) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S \times_Z Y) = \Gamma(Z, f_* f^* C)$$

admits a right adjoint $(f_C)_*$, and for any $S \to Z$ with $S \in \mathrm{Sch}^{aff}$ the diagram commutes

$$\begin{array}{ccc} \Gamma(Z, C) & \stackrel{(f_C)_*}{\leftarrow} & \Gamma(Z, f_* f^* C) \\ \downarrow & & \downarrow \\ \Gamma(S, C) & \stackrel{\alpha^R_S}{\leftarrow} & \Gamma(S \times_Z Y, f^* C) \end{array}$$

Here the vertical arrows are evaluations (restrictions). Since each α^R_S is continuous, $(f_C)_*$ is also continuous, the transition maps in our projective system preserve limits, see ([11], 2.2.69).

0.2.14. For A.10.2. The proof is not clear. Let us show that for the map $p : Y \to *$ in PreStk^{lax} the functor $p^* : \mathrm{DGCat}_{cont} = ShvCat^{naive}_{/*} \to ShvCat^{naive}_{/Y}$ has a right adjoint. Clearly, p^* preserves colimits, however, $ShvCat^{naive}_{/Y}$ is not presentable usually, and we can not apply the adjoint functor theorem. The category $Tw(\mathrm{Sch}^{aff})$ is contractible, since it has an initial object. Write $ShvCat^{naive}_{/Y}$ as (4). For an object $(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})$ we have an ajoint pair

$$res : ShvCat(*) \leftrightarrows \mathrm{Fun}(Y(S'), ShvCat(S)) : \gamma_{S',S}$$

Here $\gamma_{S',S}$ is the composition $\mathrm{Fun}(Y(S'), ShvCat(S)) \stackrel{\lim}{\to} ShvCat(S) \stackrel{\Gamma(S,\cdot)}{\to} ShvCat(*)$. Passing to the limit over $Tw(\mathrm{Sch}^{aff})^{op}$ the functors res in the above yield our functor p^*. Given a map from $(S \stackrel{\alpha}{\to} S')$ to $(T \stackrel{\beta}{\to} T')$ in $Tw(\mathrm{Sch}^{aff})$, we have a commutative diagram

$$\begin{array}{ccc} ShvCat(*) & \stackrel{res}{\to} & \mathrm{Fun}(Y(S'), ShvCat(S)) \\ & \searrow_{res} & \uparrow t \\ & & \mathrm{Fun}(Y(T'), ShvCat(T)), \end{array}$$

the vertical arrows is the transition functor in our diagram indexed by $Tw(\mathrm{Sch}^{aff})^{op}$. The functor t admits a right adjoint t^R, which is not continuous in general.

Consider the functor $ShvCat^{naive}_{/Y} \to \mathrm{DGCat}_{cont}$ sending C to

$$\Gamma^{naive}(Y, C) = \lim_{(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} \Gamma(S, \lim_{Y(S')} C_\alpha), \tag{8}$$

where we denoted by C_α the image of C in $\mathrm{Fun}(Y(S'), ShvCat(S))$. In the above formula $\Gamma(S, \lim_{Y(S')} C_\alpha) \widetilde{\to} \lim_{Y(S')} \Gamma(S, C_\alpha)$. We claim that this is the right adjoint to p^*. Indeed, the mapping spaces in $ShvCat^{naive}_{/Y}$ are written as a similar limit over $Tw(\mathrm{Sch}^{aff})^{op}$, and we get

$$\mathrm{Map}_{ShvCat^{naive}_{/Y}}(p^*D, C) \widetilde{\to} \lim_{(S \stackrel{\alpha}{\to} S')} \mathrm{Map}_{\mathrm{Fun}(Y(S'), ShvCat(S))}((p^*D)_\alpha, C_\alpha) \widetilde{\to}$$
$$\lim_{(S \stackrel{\alpha}{\to} S')} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(D, \Gamma(S, \lim_{Y(S')} C_\alpha)) \widetilde{\to} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(D, \lim_{(S \stackrel{\alpha}{\to} S')} \Gamma(S, \lim_{Y(S')} C_\alpha))$$

Here $\lim\limits_{(S \stackrel{\alpha}{\to} S')}$ stands for the limit over $Tw(\mathrm{Sch}^{aff})^{op}$. We used the fact that there is a functor $Tw(\mathrm{Sch}^{aff})^{op} \to \mathrm{DGCat}_{cont}$ sending α as above to $\Gamma(S, \lim_{Y(S')} C_\alpha)$. Does $\Gamma^{naive}(Y, C)$ coincide with $\mathbf{Map}_{ShvCat^{naive}_{/Y}}(\mathrm{QCoh}_Y, C)$? Not clear.

Is $\Gamma^{naive}(Y, C)$ enriched over $\mathrm{QCoh}(Y)$ naturally? One gets

$$\Gamma^{naive}(Y, \mathrm{QCoh}_Y) \widetilde{\to} \lim_{(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} \lim_{Y(S')} \mathrm{QCoh}(S) \widetilde{\to}$$
$$\lim_{(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Fun}(Y^{inv}(S'), \mathrm{QCoh}(S)) \widetilde{\to} \mathrm{QCoh}(Y^{inv})$$

Here $Y^{inv} \in \mathrm{PreStk}$ is obtained by inverting all arrows in Y. We have the restriction functor $\mathrm{QCoh}(Y^{inv}) \to \mathrm{QCoh}(Y)$, which is not an equivalence in general.

We used the fact that for $C, D \in 1-\mathfrak{Cat}$, the limit of the constant functor $\lim_C D$ with value D is $\mathrm{Fun}(C^{inv}, D)$.

By functoriality, $\Gamma^{naive}(Y, C)$ is naturally a module over $\mathrm{QCoh}(Y^{inv}) = \Gamma^{naive}(Y, \mathrm{QCoh})$.

If now $f : Y \to T$, where $T \in \mathrm{Sch}^{aff}$ then we get the direct image functor $f_* : ShvCat^{naive}_{/Y} \to ShvCat_{/T}$ as the composition

$$ShvCat^{naive}_{/Y} \to \mathrm{QCoh}(Y^{inv}) - mod(\mathrm{DGCat}_{cont}) \to \mathrm{QCoh}(T) - mod(\mathrm{DGCat}_{cont})$$

Let now $f : Y \to Z$ be a map in PreStk^{lax}, $D \in ShvCat^{naive}_{/Z}, C \in ShvCat^{naive}_{/Y}$. We get

$$\mathrm{Map}_{ShvCat^{naive}_{/Y}}(f^*D, C) \widetilde{\to} \lim_{(S \overset{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Map}_{\mathrm{Fun}(Y(S'), ShvCat(S))}((f^*D)_\alpha, C_\alpha)$$

Given $S' \in \mathrm{Sch}^{aff}$, f yields the functor $Y(S') \to Z(S')$, so an adjoint pair

$$res_\alpha : \mathrm{Fun}(Z(S'), ShvCat(S)) \leftrightarrows \mathrm{Fun}(Y(S'), ShvCat(S)) : RKE_\alpha$$

and $res_\alpha(D_\alpha) = (f^*D)_\alpha$. Thus the above mapping space rewrites as

$$\lim_{(S \overset{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Map}_{\mathrm{Fun}(Z(S'), ShvCat(S))}(D_\alpha, RKE_\alpha C_\alpha)$$

Consider the functor $f^{naive}_* : ShvCat^{naive}_{/Y} \to ShvCat^{naive}_{/Z}$ sending C given by a projective system (C_α) as above to the functor $Tw(\mathrm{Sch}^{aff})^{op} \to 1 - \mathfrak{C}\mathrm{at}$ given by

$$\alpha \mapsto RKE_\alpha(C_\alpha) \in \mathrm{Fun}(Z(S'), ShvCat(S)) \tag{9}$$

for $(S \overset{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})$. Let finally

$$f^{naive}_* C = \lim_{(S \overset{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} RKE_\alpha(C_\alpha)$$

The above shows that

$$\mathrm{Map}_{ShvCat^{naive}_{/Y}}(f^*D, C) \widetilde{\to} \mathrm{Map}_{ShvCat^{naive}_{/Z}}(D, f^{naive}_* C)$$

If we are gven a map from $(S \overset{\alpha}{\to} S')$ to $(T \overset{\beta}{\to} T')$ in $Tw(\mathrm{Sch}^{aff})$ then write $\delta_{\alpha,\beta} : \mathrm{Fun}(Z(T'), ShvCat(T)) \to \mathrm{Fun}(Z(S'), ShvCat(S))$ for the transition functor in the projective system defining $ShvCat^{naive}_{/Z}$. Then we only have a natural map

$$\delta_{\alpha,\beta}(RKE_\beta C_\beta) \to RKE_\alpha C_\alpha \tag{10}$$

in $1 - \mathfrak{C}\mathrm{at}$. This is what (9) says. The existence of the functor (9) is not rigorously justified. I think if for the above map $\alpha \to \beta$ in $Tw(\mathrm{Sch}^{aff})$ the induced map $S' \to T'$ is an isomorphism then (10) is also an isomorphism.

Sam actually claims in A.10.4 that $\Gamma^{naive}(Y, C)$ is also given by a simplier formula:

$$\Gamma^{naive}(Y, C) = \lim_{S \in (\mathrm{Sch}^{aff})^{op}} \lim_{y \in Y(S)} \Gamma(S, y^*C) \tag{11}$$

To be more precise, we consider the cocartesian fibration $Groth(Y) \to (\mathrm{Sch}^{aff})^{op}$ corresponding to Y. Then the above limit is over $Groth(Y)$.

For $Y \in \mathrm{PreStk}^{lax}$ formulas (8) and (11) give the same result. This follows from the contractibility claim proved in the next subsection.

0.2.15. For A.10.3. For a map $f : A \to B$ in $1-\mathfrak{C}\mathrm{at}$ he says f is op-cofinal if $f^{op} : A^{op} \to B^{op}$ is cofinal.

Let Y be a lax prestack, $Groth(Y) \to (\mathrm{Sch}^{aff})^{op}$ be the cocartesian fibration corresponding to Y. Let $coGroth(Y) \to \mathrm{Sch}^{aff}$ be the cartesian fibration corresponding to Y. Consider the category $Tw(\mathrm{Sch}^{aff})^{op} \times_{\mathrm{Sch}^{aff}} coGroth(Y)$. It classifies $(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}$, and an object $y' \in Y(S')$. We have the map

$$Tw(\mathrm{Sch}^{aff})^{op} \times_{\mathrm{Sch}^{aff}} coGroth(Y) \to Groth(Y) \tag{12}$$

sending (α, y') as above to (S, y), where y is the composition $S \stackrel{\alpha}{\to} S' \stackrel{y'}{\to} Y$. To clarify things, if $(S \stackrel{\alpha}{\to} S', y' \in Y(S')$ and $(T \stackrel{\beta}{\to} T', y'' \in Y(T'))$ are two objects of $Tw(\mathrm{Sch}^{aff})^{op} \times_{\mathrm{Sch}^{aff}} coGroth(Y)$ then a map from the second to the first in this category is given by a map $\alpha \to \beta$ in $Tw(\mathrm{Sch}^{aff})$, and a map $(T', y'') \to (S', y')$ in $coGroth(Y)$. This means that for the corresponding map say $\mu : T' \to S'$ in Sch^{aff} we are given a map $y'' \to y'\mu$ in $Y(T')$. Even the existence of the functor (12) is already interesting.

Is the map (12) op-cofinal? In other words,

$$Tw(\mathrm{Sch}^{aff}) \times_{(\mathrm{Sch}^{aff})^{op}} Groth(Y^{op}) \to coGroth(Y^{op})$$

is cofinal? We used the fact that $(Groth(Y))^{op} \widetilde{\to} coGroth(Y^{op})$ canonically. Here Y^{op} denotes the lax prestack sending S to $Y(S)^{op}$.

Namely, given $(S, y) \in coGroth(Y^{op})$, consider the category classifying $(T \stackrel{\beta}{\to} T') \in Tw(\mathrm{Sch}^{aff}), y' \in Y(T')^{op}$ and a map $(S, y) \stackrel{\xi}{\to} (T, y'\beta)$ in $coGroth(Y^{op})$. We have to show it is contractible. The map ξ is simply a map $\tau : S \to T$ in Sch^{aff} and a morphism $y \to y'\beta\tau$ in $Y(S)^{op}$. The desired contractibility is not clear (though clear say in the case when Y is constant lax prestack).

Is (12) a cartesian fibration? There seems no reason for that. Let us show that

$$(Tw(\mathrm{Sch}^{aff})^{op} \times_{\mathrm{Sch}^{aff}} coGroth(Y)) \times_{Groth(Y)} Groth(Y)_{(S,y)/} = (Tw(\mathrm{Sch}^{aff})^{op} \times_{\mathrm{Sch}^{aff}} Groth(Y)_{(S,y)/}$$

is contractible, this is what we need. The projection $(Tw(\mathrm{Sch}^{aff})^{op} \to \mathrm{Sch}^{aff}$ is a cocartesian fibration, and $Groth(Y)_{(S,y)/}$ is contractible, as it has an initial object.

We have an adjoint pair

$$\mathrm{pr} : (Tw(\mathrm{Sch}^{aff})^{op} \leftrightarrows \mathrm{Sch}^{aff} : u,$$

where pr sends $(S \to S') \in Tw(\mathrm{Sch}^{aff})$ to S', and u sends S to $(\emptyset \to S)$. Moreover, the natural map $\mathrm{pr} \circ u \to \mathrm{id}$ is an equivalence, so u is fully faithful, pr is a localization. By ([11], 2.7.4), the projection $(Tw(\mathrm{Sch}^{aff})^{op} \times_{\mathrm{Sch}^{aff}} Groth(Y)_{(S,y)/} \to Groth(Y)_{(S,y)/}$ admits a right adjoint. Therefore, this projection induces an isomorphism of geometric realizations

$$\mid (Tw(\mathrm{Sch}^{aff})^{op} \times_{\mathrm{Sch}^{aff}} Groth(Y)_{(S,y)/} \mid \; \widetilde{\to} \; \mid Groth(Y)_{(S,y)/} \mid$$

(by [11], 2.2.106). The RHS is $*$, and we are done!

0.2.16. For Lemma A.11.2 this is ([1], Cor. 3.2.7).

0.2.17. For A.12.1. I think a stronger claim holds: there is a functor $ShvCat^{enh}_{/-}$: $\mathrm{PreStk}_{corr;all,1-aff} \to 2-\mathcal{C}\mathrm{at}$, whose composition with the forgetful functor $2-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$ is the one claimed by Sam. Here $2-\mathcal{C}\mathrm{at}$ is the $(\infty,1)$-category of $(\infty,2)$-categories. This is justified by my Section 0.2.

0.2.18. For A.13.3. here the colimit and limit are calculated in $ShvCat(Y)$, by tensor product $C \otimes_{\mathrm{QCoh}_Y} \bar{C}$ he means the tensor product in the symmetric monoidal $ShvCat(Y)$.

For A.13.4(3). This is an analog of ([1], ch. I.1, 6.4.3).

A.13.4(1) clearly has an analog for $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ instead of $ShvCat(Y)$.

For A.13.4(2). If the map (A.13.2) $D \otimes_{\mathrm{QCoh}_Y} \bar{C} \to \lim_{i\in I^{op}}(D \otimes_{\mathrm{QCoh}_Y} C_i^{\vee})$ is an isomorphism for any D then we get as in the proof that for any $D_1, D_2 \in ShvCat(Y)$ one has $\mathrm{Hom}(D_2, D_1 \otimes_{\mathrm{QCoh}_Y} \bar{C}) \widetilde{\to} \mathrm{Hom}(D_2 \otimes_{\mathrm{QCoh}_Y} C, D_1)$, where by Hom he means mapping spaces in $ShvCat(Y)$. This shows that C is dualizable.

0.3. For Sect. B.5. For any $[n] \in \boldsymbol{\Delta}$ we have $\mathrm{Grid}_{[n]} \in 1-\mathcal{C}\mathrm{at}$. This is a functor $\boldsymbol{\Delta} \to 1-\mathcal{C}\mathrm{at}$, $[n] \mapsto \mathrm{Grid}_{[n]}$. Namely, if $f : [m] \to [n]$ is a map in $\boldsymbol{\Delta}$, and $(i,j) \in \mathrm{Grid}_{[m]}$, that is, $0 \le i \le j \le m$ then $(f(i), f(j)) \in \mathrm{Grid}_{[n]}$. Moreover, it sends horizontal (resp., vertical) arrows to horizontal (resp., vertical).

This is why for a category with directions $(\mathcal{C}, hor, vert)$, $[n] \mapsto \mathrm{Grid}_{[n];hor,vert}(\mathcal{C})$ is indeed a functor $\boldsymbol{\Delta} \to \mathrm{Spc}$. Namely, given a map $f : [m] \to [n]$ in $\boldsymbol{\Delta}$, we precompose a n-grid $\mathrm{Grid}^{op}_{[n]} \to \mathcal{C}$ with $\mathrm{Grid}^{op}_{[m]} \to \mathrm{Grid}^{op}_{[n]}$.

0.3.1. For B.9. His $\mathcal{C}\mathrm{at}_{dir}$ is symmetric monoidal: Given $(C, hor_C, vert_C), (D, hor_D, vert_D)$, on $C \times D$ we get a structure of a category with directions: $hor_{C\times D} = hor_C \times hor_D$, and $vert_{C\times D} = vert_C \times vect_D$. Now $1-\mathcal{C}\mathrm{at}$ is equipped with the cartesian symmetric monoidal structure. Then $Tw : 1-\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_{dir}$ is symmetric monoidal. In particular, for $C, D \in 1-\mathcal{C}\mathrm{at}$, $Tw(C \times D) \widetilde{\to} Tw(C) \times Tw(D)$ canonically. The functor $\mathcal{C}\mathrm{at}_{dir} \to 1-\mathcal{C}\mathrm{at}$, $(C, hor, vert) \mapsto C_{corr,hor,vert}$ is also symmetric monoidal. In particular, we get an adjoint pair

$$Tw : Alg(1-\mathcal{C}\mathrm{at}) \leftrightarrows Alg(\mathcal{C}\mathrm{at}_{dir}) : corr$$

In fact, $1-\mathcal{C}\mathrm{at}$ and $\mathcal{C}\mathrm{at}_{dir}$ are naturally 2-categories, and so are $CAlg^{nu}(1-\mathcal{C}\mathrm{at})$, $CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$. For $D, D' \in \mathcal{C}\mathrm{at}_{dir}$ the category of maps between then in $\mathcal{C}\mathrm{at}_{dir}$ is the full subcategory $\mathrm{Fun}^{dir}(D, D') \subset \mathrm{Fun}(D, D')$ classifying functors preserving horizontal (resp., vertical) morphisms, and sending cartesian products of $x \xrightarrow{a} y \xleftarrow{b} z$ with a horizontal and b vertical to cartesian squares.

Given $E, E' \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$, the mapping category in $CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ from E to E' is $\mathrm{Fun}^{\otimes}(E, E')$. Namely, if $E^{\otimes} \to \mathrm{Surj}, E'^{\otimes} \to \mathrm{Surj}$ are the cocartesian fibrations corresponding to E, E' then $\mathrm{Fun}^{\otimes}(E, E') \subset \mathrm{Fun}_{\mathrm{Surj}}(E^{\otimes}, E'^{\otimes})$ is the full subcategory of functors sending Surj-cocartesian arrows to cocartesian arrows.

My understanding is that the adjoint pair $Tw : 1-\mathcal{C}\mathrm{at} \leftrightarrows \mathcal{C}\mathrm{at}_{dir} : corr$ is an adjoint pair of 2-categories, that is, we have

$$\mathrm{Fun}(E, C_{corr;hor,vert}) \widetilde{\to} \mathrm{Fun}^{dir}(Tw(E), C)$$

for $C \in \mathcal{C}\mathrm{at}_{dir}$, $E \in 1-\mathcal{C}\mathrm{at}$ naturally.

Moreover, the induced adjoint pair

$$Tw : CAlg^{nu}(1-\mathcal{C}\mathrm{at}) \leftrightarrows CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$$

is also an adjoint pair of 2-categories. For $D, D' \in CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$ the mapping category $\mathbf{Map}_{CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})}(D, D')$ is the category $\mathrm{Fun}^{\otimes,dir}(D, D')$ of those symmetric monoidal functors $f : D \to \mathrm{D}'$ whose image in $\mathrm{Fun}(D, D')$ lies in $\mathrm{Fun}^{dir}(D, D')$. I hope for $E \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ and $D \in CAlg^{nu}(\mathcal{C}\mathrm{at}_{dir})$ one has a naturall equivalence

$$\mathrm{Fun}^{\otimes}(E, C_{corr,hor,vert}) \widetilde{\to} \mathrm{Fun}^{\otimes,dir}(Tw(E), C)$$

in $1-\mathcal{C}\mathrm{at}$.

0.4. For 6.7. The natural functor $(fSet_{\emptyset})^{op} \hookrightarrow (Set_{<\infty})^{op}$ is not cofinal. Namely, given $J \in Set_{<\infty}$, the category $fSet_{\emptyset} \times_{Set_{<\infty}} (Set_{<\infty})_{/J}$ is not contractible in general. Namely, the geometric realization of the latter category is the disjoint union of points over all subsets $J_0 \subset J$.

For $\mathcal{G} \in \mathrm{Spc}$ we get a natural map

$$\underset{I\in(Set_{<\infty})^{op}}{\mathrm{colim}} \mathcal{G}^I \leftarrow \underset{I\in(fSet_{\emptyset})^{op}}{\mathrm{colim}} \mathcal{G}^I = \mathrm{Ran}_{\mathcal{G},\emptyset}$$

Is it an isomorphism? I hope yes according to his Section 4.8.

In the definition of $fSet_I$ it is assumed that J is finite. The symmetric monoidal category $fSet_{\emptyset}$ act on $fSet_I$. The action map $a : fSet_{\emptyset} \times fSet_I \to fSet_I$ sends $(J', I \overset{\alpha}{\to} J)$ to $I \to J \sqcup J'$. So, we get $\mathcal{G}^{J'} \times \mathcal{G}^J \widetilde{\to} \mathcal{G}^{J\sqcup J'}$. By adjointness, we get the canonical map

$$(\underset{J'\in fSet_{\emptyset}^{op}}{\mathrm{colim}} \mathcal{G}^{J'}) \times (\underset{(I\to J)\in fSet_I^{op}}{\mathrm{colim}} \mathcal{G}^J) \to \underset{(I\to \bar{J})\in fSet_I^{op}}{\mathrm{colim}} \mathcal{G}^{\bar{J}}$$

that is, $\mathrm{Ran}_{\mathcal{G},\emptyset} \times \mathrm{Ran}_{\mathcal{G},I} \to \mathrm{Ran}_{\mathcal{G},I}$. The canonical map $\mathrm{Ran}_{\mathcal{G},I} \to \mathcal{G}^I$ is invariant under the action of $\mathrm{Ran}^*_{\mathcal{X}}$.

The functor $fSet_I \to fSet_{\emptyset}$, $(I \to J) \mapsto J$ yields by functoriality a morphism $\mathrm{Ran}_{\mathcal{G},I} \to \mathrm{Ran}_{\mathcal{G}}$, which is $\mathrm{Ran}^*_{\mathcal{G}}$-equivariant.

The $\mathrm{Ran}^{ch}_{\mathcal{G}}$-module structure on $\mathrm{Ran}_{\mathcal{G},I}$ is given (via his example 6.6.3) by the correspondence

$$\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G},I} \leftarrow (\mathrm{Ran}_{\mathcal{G}} \times \mathrm{Ran}_{\mathcal{G},I})_{disj} \to \mathrm{Ran}_{\mathcal{G},I}$$

0.4.1. If $\mathcal{X} \in \mathrm{PreStk}$, C is a chiral category for $\mathcal{X}$, and $\mathcal{Z} \in \mathrm{PreStk}$ is a $\mathrm{Ran}^{ch}_{\mathcal{X}}$-module in PreStk_{corr}, let $M \in ShvCat_{/\mathcal{Z}}$. Then a structure of a chiral C-module category on M over $\mathcal{Z}$ means that $(\mathcal{Z}, M) \in \mathrm{PreStk}^{ShvCat}_{corr}$ is equipped with a structure of a $(\mathrm{Ran}^{ch}_{\mathcal{X}}, C)$-module in the symmetric monoidal category $\mathrm{PreStk}^{ShvCat}_{corr}$, and moreover the action map

$$\mathrm{act}_1^*(C \boxtimes M) \to \mathrm{act}_2^*(M)$$

is an isomorphism in $ShvCat_{/act_M}$. Here the diagram

$$\mathrm{Ran}_{\mathcal{X}} \times \mathcal{Z} \overset{act_1}{\leftarrow} act_M \overset{act_2}{\to} M$$

defined the action map as in his 5.6.

0.4.2. For 6.10.4. I think $\mathrm{Ran}_{\mathcal{X}}^{un,\to}, \mathrm{Ran}_{\mathcal{X},I}^{un}$ are not defined explicitely in the paper. For example, $\mathrm{Ran}_{\mathcal{X}}^{un,\to}$ should be defined as follows I think: for $\mathcal{X}$ a set consider the set $\mathrm{Ran}_{\mathcal{X}}^{\to}$ classifying pairs of finite subsets $S \subset T \subset \mathcal{X}$, here S, T could be empty. Now consider $\mathrm{Ran}_{\mathcal{X}}^{\to}$ as a partially ordered set, where $(S \subset T) \leq (S' \subset T')$ if $S \subset S'$ and $T \subset T'$. Then $\mathrm{Ran}_{\mathcal{X}}^{un,\to}$ is the category associated to this poset. Then extend this definition to any $\mathcal{X} \in \mathrm{Spc}$ by requiring that $\mathrm{Spc} \to 1-\mathcal{C}\mathrm{at}$, $\mathcal{X} \mapsto \mathrm{Ran}_{\mathcal{X}}^{un,\to}$ commutes with sifted colimits.

For a set $\mathcal{X}$, $\mathrm{Ran}_{\mathcal{X},I}$ should be the category associated to the following poset: its objects are (α, S), where $S \subset \mathcal{X}$ is a finite subset, and $\alpha : I \to S$ is any map. Then $(\alpha, S) \leq (\alpha', S')$ iff $S \subset S'$ and the composition $I \xrightarrow{\alpha} S \to S'$ equals α. Then probably we can extend this definition to any $\mathcal{X}$ requiring that $\mathcal{X} \mapsto \mathrm{Ran}_{\mathcal{X},I}$ commutes with sifted colimits.

We can also proceed as in 4.8. Namely, define the category $fSet_{I,[n]}^{\to}$ (generalizing $fSet_{\emptyset[n]}^{\to}$ from 4.8). Its objects are diagrams $I \xrightarrow{\alpha_I} I_0 \to I_1 \to \ldots \to I_n$ with I_i maybe empty finite set. The morphisms are diagrams

$$\begin{array}{ccccc} I_0 & \to & I_1 & \to \ldots \to & I_n \\ \downarrow & & \downarrow & & \downarrow \\ J_0 & \to & J_1 & \to \ldots \to & J_n, \end{array}$$

where the vertical maps are surjections. It is required that the above diagram is compatible with maps $I \xrightarrow{\alpha_I} I_0$ and $I \xrightarrow{\alpha_J} J_0$. Then proceed as in 4.8 to define $\mathrm{Ran}_{\mathcal{X},I}^{un} \in 1-\mathcal{C}\mathrm{at}$.

0.4.3. For 6.13.1, $\mathrm{Ran}_{\mathcal{X},I,\emptyset}$ is never defined? I think this is $\mathrm{Ran}_{\mathcal{X},I}$, where we added a disjoint point corresponding to empty set.

0.4.4. For 6.15. He starts with a commutative algebra in PreStk_{corr}, but an assumption is missing: he needs $m_2 : mult_S \to S$ and $e_2 : unit_S \to S$ to be 1-affine.

Under these assumptions, we get a symmetric monoidal convolution structure on $ShvCat_{/S}$ as he explains. Actually, $ShvCat_{/S}$ is a 2-category, and he claims implicitely that $ShvCat_{/S}$ is a symmetric monoidal 2-category. This means in particular, that the convolution $* : ShvCat_{/S} \times ShvCat_{/S} \to ShvCat_{/S}$ is a morphism in $2-\mathcal{C}\mathrm{at}$.

For 6.15.2. If $C \in MultCat^{op-w}(S)$ then we have maps $m_2^*C \to m_1^*(C \boxtimes C)$ in $ShvCat_{mult_S}$ and $e_2^*C \to \mathrm{QCoh}_{unit_S}$ in $ShvCat_{unit_S}$. A map from C to D in $MultCat^{op-w}(S)$ gives a morphism $\alpha : C \to D$ in $ShvCat_{/S}$ in particular such that the diagram commutes (an explicit invertible 2-morphism is given)

$$\begin{array}{ccc} m_2^*C & \to & m_1^*(C \boxtimes C) \\ \downarrow {\scriptstyle m_2^*\alpha} & & \downarrow \\ m_2^*D & \to & m_1^*(D \boxtimes D) \end{array}$$

Applying m_2^*, we get a diagram

$$\begin{array}{ccccc} C & \to & m_{2*}m_2^*C & \to & m_{2*}m_1^*(C \boxtimes C) \\ \downarrow {\scriptstyle \alpha} & & \downarrow {\scriptstyle m_{2*}m_2^*\alpha} & & \downarrow \\ D & \to & m_{2*}m_2^*D & \to & m_{2*}m_1^*(D \boxtimes D), \end{array} \tag{13}$$

Similarly for unit. This gives a coalgebra structure on C in the symmetric monoidal category $(ShvCat_{/S}, *)$.

Question. My understanding is that in (13) the outer square is commutative (that is, an explicit invertible 2-morphism in the 2-category $ShvCat_{/S}$ is given). Is this true? This would mean probably that the corresponding map from C to D in $ShvCat_{/S}$ is a true morphism of commutative algebras, and not a lax morphism. However, in the formulation of Prop. 6.15.2, $ComCoalg^{lax}$ appears. Maybe, $ComCoalg((ShvCat_{/S}, *))$ is meant in the formulation of Prop. 6.15.2? Or maybe you meant $MultCat^{op-w,lax}(S)$ instead?

0.4.5. For 6.16. Let $S \in CAlg(\mathrm{PreStk}_{corr})$ with e_2, m_2 1-affine. The unit in $(ShvCat_{/S}, *)$ is $e_{2*}\,\mathrm{QCoh}_{unit_S}$. If $C \in MultCat^{op-w,r.adj}(S)$ then the maps $\tilde{\eta}_m : m_2^*C \to m_1^*(C \boxtimes C)$ and $\tilde{\eta}_e : e_2^*C \to \mathrm{QCoh}_{unit_S}$ admit right adjoints in $ShvCat_{mult_S}$ and in $ShvCat_{unit_S}$ respectively. For $\tilde{\eta}_m$ this means that for any $T \to mult_S$ with $T \in \mathrm{Sch}^{aff}$ the corresponding right adjoint over T is continuous (similarly for $\tilde{\eta}_e$). So, the map

$$m_{2*}\tilde{\eta}_m : m_{2*}m_2^*C \to m_{2*}m_1^*(C \boxtimes C)$$

admits a right adjoint in $ShvCat_{/S}$. Assume m_2, e_2 are quasi-compact quasi-separated schematic morphisms. Then $C \to m_{2*}m_2^*C$ admits a right adjoint in $ShvCat_{/S}$ by his A.9.1(3). So, the composition $C \to m_{2*}m_1^*(C \boxtimes C) = C * C$ admits a right adjoint in $ShvCat_{/S}$. Similarly, $C \to e_{2*}\,\mathrm{QCoh}_{unit_S}$ has a right adjoint.

0.4.6. For 6.17. For clarity, if $\Psi \in MultCat^{op-w,r.adj}(S)$ then first we may see Ψ as an object of $ComCoalg^{r.adj}((ShvCat_{/S}, *))$ via his Section 6.16, hence as an object of $CAlg^{l.adj}((ShvCat_{/S}, *))$ via his (6.16.1). So, Ψ is a commutative algebra in $(ShvCat_{/S}, *)$, hence $\Gamma(S, \Psi)$ inherits a symmetric monoidal structure, which he still denotes by $*$.

Explicitely, view the maps (6) as

$$\tilde{\eta}_m : \Psi \to \Psi * \Psi \ \in ShvCat_{/S} \text{ and}$$
$$\tilde{\eta}_e : \Psi \to e_{2*}\,\mathrm{QCoh}_{unit_S} \ \in ShvCat_{/S}$$

They define the corresponding object of $ComCoalg^{r.adj}((ShvCat_{/S}, *))$. Let μ_m, μ_e be the functors right adjoint to $\tilde{\eta}_m$ and $\tilde{\eta}_e$ respectively. So,

$$(\Psi, \mu_m, \mu_e) \in CAlg^{l.adj}((ShvCat_{/S}, *))$$

The symmetric monoidal structure $(\Gamma(S, \Psi), *)$ is obtained by passing to global sections in the latter.

0.4.7. In his Prop. 6.17.1 in the RHS one has to replace $ComAlg(\Gamma(S, \Psi), *)$ by

$$ComCoalg(\Gamma(S, \Psi), *)$$

0.4.8. For 6.19. Explicitely, if $C \in Cat^{ch}(X_{dR})$ the non-unital commutative algebra structure on $\Gamma(\mathrm{Ran}_{X_{dR}}, C)$ in DGCat_{cont} is as follows. For the diagram

$$\mathrm{Ran}_{X_{dR}} \times \mathrm{Ran}_{X_{dR}} \stackrel{m_1}{\leftarrow} (\mathrm{Ran}_{X_{dR}} \times \mathrm{Ran}_{X_{dR}})_{disj} \stackrel{m_2}{\to} \mathrm{Ran}_{X_{dR}}$$

we have isomorphism $\eta_m : m_1^*(C \boxtimes C) \widetilde{\to} m_2^*C$. We consider $\eta_m^{-1} : m_2^*C \to m_1^*(C \boxtimes C)$, hence a morphism $\eta_m^{-1} : C \to m_{2*}m_1^*(C \boxtimes C) = C * C$, so C is a non-unital commutative coalgebra in $(ShvCat_{\mathrm{Ran}_{X_{dR}}}, *)$. Since m_2 is quasi-compact quasi-separated schematic morphism, the map $\eta_m^{-1}: C\to C{*}C$ admits a right adjoint $\mu: C{*}C\to C$ by his Section 6.16, which makes C a non-unital commutative algebra in $(ShvCat_{\mathrm{Ran}_{X_{dR}}}, *)$. In turn, $\Gamma(\mathrm{Ran}_{X_{dR}}, C)$ ihherits a non-unital symmetric monoidal structure with product $\overset{ch}{\otimes}$.

If $s\in\Gamma(\mathrm{Ran}_{X_{dR}}, C)$ then $s\overset{ch}{\otimes}s=\mu(m_1^*(s\boxtimes s))\in\Gamma(\mathrm{Ran}_{X_{dR}}, C)$. If $s\in ComCoalg^{ch}(C)$, we get a map

$$s\to\mu(m_1^*(s\boxtimes s))\in\Gamma(\mathrm{Ran}_{X_{dR}}, C)$$

For 6.20. Given a factorization category C on X_{dR}, I don't see what is the map

$$\Gamma(X_{dR}, C\mid_{X_{dR}})\to\Gamma(\mathrm{Ran}_{X_{dR}}, C)$$

This is like extenion by zero for $\mathcal{D}$-modules.

0.4.9. For 7.2.1. In the definition of a commutative weak chiral category he means it is a weakly multiplicative sheaf of categories on $\mathrm{Ran}^*_{\mathcal{X}}$. In fact, the map $\mathrm{id}: \mathrm{Ran}^{ch}_{\mathcal{X}}\to\mathrm{Ran}^*_{\mathcal{X}}$ is a morphism in $CAlg(\mathrm{PreStk}_{corr})$, so that his Section 5.34 applies, id^* is the desired functor.

0.4.10. For 7.7. Two lines before 7.8: replace Set_∞ by $Set_{<\infty}$.

If $S\in\mathrm{Sch}^{aff}$ then $\mathcal{Z}(S)$ is the cocartesian fibration coresponding to $\mathrm{Sets}_{<\infty}\to 1-\mathcal{C}\mathrm{at}$, $I\mapsto\mathrm{Ran}^{un,I}_{\mathcal{X},disj}(S)$.

For 7.8. There is a correction needed in the second displayed formmula, one has actually

$$ShvCat^{naive}_{/\,\mathrm{Sets}_{<\infty}}=\mathrm{Fun}(\mathrm{Sets}_{<\infty}, \mathrm{DGCat}_{cont})$$

as for any constant lax prestack, see my Section 0.0.26. The functor $CAlg(\mathrm{DGCat}_{cont})\to ShvCat^{naive}_{/\,\mathrm{Sets}_{<\infty}}$, $I\mapsto D^{\otimes I}$ is fine.

0.4.11. For 7.12. For a surjection $p: I\to J$ of finite sets, the subscheme $U(p)\subset X^I_{dR}$ seems not defined. I think this is teh scheme of collections $(x_i)_{i\in I}$ such that if $p(i)\neq p(i')$ then $(x_i, x_{i'})\in X^2_{disj}$.

0.4.12. The construction of $\mathrm{Loc}_{X_{dR}}(D)$ for $D\in CAlg(\mathrm{DGCat}_{cont})$ from 7.6-7.13 is based on the following: The category $\mathrm{Sets}_{<\infty}$ is symmetric monoidal under $(J_1, J_2)\mapsto J_1\sqcup J_2$. Consider the functor $\mathrm{Sets}_{<\infty}\to\mathrm{DGCat}_{cont}$, $I\mapsto D^{\otimes I}$. If $I\to J$ is any map then the corresponding map $D^{\otimes I}\to D^{\otimes J}$ is given by hte products along I_j for each $j\in J$. We view it as an object $\mathcal{F}$ of $ShvCat^{naive}_{\mathrm{Sets}_{<\infty}}$. Then for the above map $u: \mathrm{Sets}_{<\infty}\times\mathrm{Sets}_{<\infty}\to\mathrm{Sets}_{<\infty}$ we get $u^*\mathcal{F}\widetilde{\to}\mathcal{F}\boxtimes\mathcal{F}$, and for the empty set $e: *\to\mathrm{Sets}_{<\infty}$ we have $e^*\mathcal{F}\widetilde{\to}\mathrm{Vect}$. So, $\mathrm{Sets}_{<\infty}$ is naturally an object of $CAlg(1-\mathcal{C}\mathrm{at})$, and $\mathcal{F}$ is a multiplicative sheaf of categories on $\mathrm{Sets}_{<\infty}$.

It is this structure that yields a multiplicative structure on $\mathrm{Loc}_{X_{dR}}(D)$. So, let $\mathcal{C}\in CAlg(1-\mathcal{C}\mathrm{at})$ and $F\in\mathrm{Fun}(\mathcal{C}, \mathrm{DGCat}_{cont})$ giving an object $\mathcal{F}\in ShcCat^{naive}_{/\mathcal{C}}$, where $\mathcal{C}$ is viewed as a constant lax prestack. For which F the sheaf $\mathcal{F}$ is multiplicative? This is equivalent to $F:\mathcal{C}\to\mathrm{DGCat}_{cont}$ being a symmetric monoidal functor.

0.4.13. For 7.15.5. If $f : Y \to Y'$ is a proper morphism between schemes of finite type then we have an adjoint pair $f_! : \mathrm{D}(Y) \leftrightarrows \mathrm{D}(Y') : f^!$, where $f^!$ if the functor

$$f^* : \mathrm{QCoh}(Y'_{dR}) \to \mathrm{QCoh}(Y_{dR}) = D(Y),$$

and moreover $f_!$ is a strict morphism of $D(Y')$-modules ([2], 1.5.2). Recall that Y'_{dR} is 1-affine by ([3], Th. 2.6.3), so $ShvCat(Y'_{dR}) = \mathrm{D}(Y') - mod(\mathrm{DGCat}_{cont})$. Let $C \in ShvCat(Y'_{dR})$, which we view as a $D(Y')$-module. Let us explain that $f^* : \Gamma(Y'_{dR}, C) \to \Gamma(Y_{dR}, f^*C)$ has a left adjoint denoted $f_{*,dR,C}$.

This is a particular case of the following general observation. Let $h^! : B \to A$ be a map in $CAlg(\mathrm{DGCat}_{cont})$ admitting a left adjoint $h_! : A \to B$, so $h_!$ is a map in DGCat_{cont}. Assume the left-lax monoidal structure on the functor $h_!$ of B-modules is strict, so $h_!$ is a morphism in $B - mod(\mathrm{DGCat}_{cont})$, and the adjoint pair $h_! : A \leftrightarrows B : h^!$ takes place in $B - mod(\mathrm{DGCat}_{cont})$. Now if $C \in B - mod(\mathrm{DGCat}_{cont})$ then we get the adjoint pair

$$h_{C!} : C \otimes_B A \leftrightarrows C : h^!_C$$

in $B - mod(\mathrm{DGCat}_{cont})$.

In our case $f^!$ yields a morphism $f^* : C \to C \otimes_{D(Y')} D(Y)$ in $D(Y') - mod$. So, we get the left adjoint $f_{*,dR,C} : C \otimes_{D(Y')} D(Y) \to C$ to f^*.

For this reason if now Y is a pseudo-indscheme in the sense of Def. 7.15.1 then for $C \in ShvCat_{/Y}$ we get $\Gamma(Y_{dR}, C) \widetilde{\to} \mathrm{colim}_{j\in J}\, \Gamma(Y_{j,dR}, \psi_j^* C)$, the colimit taken in DGCat_{cont}. Here $Y = \mathrm{colim}_{j\in J}\, Y_j$, and $\psi_j : Y_j \to Y$ is the natural map.

The morphism $f_{*,dR,C}$ in his Pp-Construction 7.15.5 exists I think only if $f : Y \to Z$ is pseudo-indproper.

We underline that in the definition of a morphism of pseudo-ind-schemes in ([14], 7.15.2), we do not require some diagrams to be cartesian, there are no conditions. Here is a special case: assume $I \in 1 - \mathcal{C}\mathrm{at}$ and we are given a diagram $I \times [1] \to \mathrm{Sch}_{ft}$, where for $i \in I$ the corresponding morphism $f_i : Y_i \to Z_i$ is proper, and for $i \to j$ in I the transition maps $Y_i \to Y_j, Z_i \to Z_j$ are proper. For each i we have an adjoint pair $(f_i)_* : \Gamma(Y_i, f^*C) \leftrightarrows \Gamma(Z_i, C) : (f_i)^*$ defined as above (for any sheaf theory). Let $Z = \mathrm{colim}_{i\in I}\, Z_i, Y = \mathrm{colim}_{i\in I}\, Y_i$. Then $\Gamma(Z, C) \widetilde{\to} \lim_{i\in I^{op}} \Gamma(Z_i, C)$. The functor $f^*_C : \Gamma(Z, C) \to \Gamma(Y, f^*(C))$ is obtained by passing to lim in $f_i^* : \Gamma(Z_i, C) \to \Gamma(Y_i, C)$. Then the left adjoint $f_{*,C} : \Gamma(Y, f^*(C)) \to \Gamma(Z, C)$ is obtained by passing to colim_i in the diagram $(f_i)_* : \Gamma(Y_i, f^*C) \to \Gamma(Z_i, C)$ by ([11], 9.2.39).

0.5. More about lax prestacks. Using appendix, for $\mathcal{C} \in 1 - \mathcal{C}\mathrm{at}$ and a functor $G : \mathcal{C}^{op} \to 1 - \mathcal{C}\mathrm{at}$ one has

$$oplaxlim\, G = \lim_{(c'\to c)\in Tw(\mathcal{C})^{op}} \mathrm{Fun}(\mathcal{C}_{c/}, G(c')) \tag{14}$$

Here $c' \to c$ is a morphism in $\mathcal{C}$. We used the identification $Tw(\mathcal{C}) \widetilde{\to} Tw(\mathcal{C}^{op})$. We use this in the following result.

Let $\mathcal{C} \in 1 - \mathcal{C}\mathrm{at}$ and $Y : \mathcal{C} \to \mathrm{PreStk}^{lax}$ be a functor, which we also view as $Y : \mathcal{C} \times (\mathrm{Sch}^{aff})^{op} \to 1 - \mathcal{C}\mathrm{at}$. For $c \in \mathcal{C}$ let $Y_c \in \mathrm{PreStk}^{lax}$ be its value on c. For $S \in \mathrm{Sch}^{aff}$ let $Y_S : \mathcal{C} \to 1 - \mathcal{C}\mathrm{at}$ be the restriction of Y, where we fix the corresponding variable.

Define the *relative lax colimit* of Y over $(\mathrm{Sch}^{aff})^{op}$ as $\bar{Y}\in \mathrm{PreStk}^{lax}$ given by

$$\bar{Y}(S)=laxcolim\, Y_S=\underset{(c'\to c)\in Tw(\mathcal{C})}{\mathrm{colim}}\ \mathcal{C}_{c/}\times Y(S,c') \tag{15}$$

for $S\in \mathrm{Sch}^{aff}$, here the colimit is taken in $1-\mathcal{C}\mathrm{at}$.

Lemma 0.5.1. *Under the assumptions of Section 0.5 we have canonical equivalences*
1) $\mathrm{QCoh}(\bar{Y})\widetilde{\to} oplaxlim\, G$ *in* DGCat_{cont}*, where G is the composition*

$$\mathcal{C}^{op}\overset{Y^{op}}{\to}(\mathrm{PreStk}^{lax})^{op}\overset{\mathrm{QCoh}}{\to}\mathrm{DGCat}_{cont}$$

2) $ShvCat^{naive}_{/\bar{Y}}\widetilde{\to} oplaxlim\, G'$*, where G' is the composition*

$$\mathcal{C}^{op}\overset{Y^{op}}{\to}(\mathrm{PreStk}^{lax})^{op}\overset{ShvCat^{naive}_{/-}}{\to}\mathrm{DGCat}_{cont}$$

Proof. 1) For $S\in \mathrm{Sch}^{aff}$ using (15) and (1), we get

$$\begin{aligned}
&\mathrm{QCoh}(\bar{Y})\widetilde{\to}\lim_{(S'\to S)\in Tw(\mathrm{Sch}^{aff})^{op}}\mathrm{Fun}(\bar{Y}(S),\mathrm{QCoh}(S'))\widetilde{\to}\\
&\lim_{(S'\to S)\in Tw(\mathrm{Sch}^{aff})^{op}}\ \lim_{(c'\to c)\in Tw(\mathcal{C})^{op}}\mathrm{Fun}(\mathcal{C}_{c/},\mathrm{Fun}(Y(S,c'),\mathrm{QCoh}(S')))\widetilde{\to}\\
&\lim_{(c'\to c)\in Tw(\mathcal{C})^{op}}\mathrm{Fun}(\mathcal{C}_{c/},\lim_{(S'\to S)\in Tw(\mathrm{Sch}^{aff})^{op}}\mathrm{Fun}(Y(S,c'),\mathrm{QCoh}(S')))\widetilde{\to}\\
&\qquad\lim_{(c'\to c)\in Tw(\mathcal{C})^{op}}\mathrm{Fun}(\mathcal{C}_{c/},\mathrm{QCoh}(Y_{c'}))
\end{aligned}$$

By (14), we are done.
2) same argument. □

So, informally we may say that $\mathrm{QCoh}:(\mathrm{PreStk}^{lax})^{op}\to \mathrm{DGCat}_{cont}$ not only preserves limits. It actually trasforms relative lax colimits in PreStk^{lax} (over $(\mathrm{Sch}^{aff})^{op}$) into oplax limits.

The above does not seem to work for $ShvCat_{/-}$. Namely, For $T\in 2-\mathcal{C}\mathrm{at}$ the functor $(1-\mathcal{C}\mathrm{at})^{op}\to 1-\mathcal{C}\mathrm{at}$, $V\mapsto (\mathrm{Fun}(V,T)_{llax})^{1-\mathcal{C}\mathrm{at}}$ preserves limits by ([11], 13.1.45). However, for $A,B\in 1-\mathcal{C}\mathrm{at}$, $T\in 2-\mathcal{C}\mathrm{at}$, $\mathrm{Fun}(A\times B,T)^{1-\mathcal{C}\mathrm{at}}_{llax}$ does not seem to express as functors from A to $\mathrm{Fun}(B,T)^{1-\mathcal{C}\mathrm{at}}_{llax}$ or something like that. For that, given $X\in 1-\mathcal{C}\mathrm{at}$, we have to better understand $X\circledast(A\times B)\in 2-\mathcal{C}\mathrm{at}$.

0.5.2. As in [14], write $Y\mapsto Y^{\mathrm{PreStk}}$ for the functor $\mathrm{PreStk}^{lax}\to \mathrm{PreStk}$ right adjoint to the inclusion $\mathrm{PreStk}\hookrightarrow \mathrm{PreStk}^{lax}$. For $Y\in \mathrm{PreStk}^{lax}$, $S\in \mathrm{Sch}^{aff}$ we get $\mathrm{Map}_{\mathrm{PreStk}^{lax}}(S,Y)\widetilde{\to}\mathrm{Map}_{\mathrm{PreStk}}(S,Y^{\mathrm{PreStk}})=Y(S)^{\mathrm{Spc}}$.

1. More on commutative chiral categories

1.1. Recall that $fSet_{<\infty}$ denotes the category of finite sets. For $D\in CAlg(1-\mathcal{C}\mathrm{at})$ one has canonically $\mathrm{Fun}^{\otimes}(fSet_{<\infty},D)\widetilde{\to} CAlg(D)$, where $\mathrm{Fun}^{\otimes}(fSet_{<\infty},D)$ is the category of symmetric monoidal functors. Let $fSet$ be the category of non empty finite sets and surjections. Recall that $CAlg^{nu}(D)\widetilde{\to}\mathrm{Fun}^{\otimes}(fSet,D)$ canonically.

Fix a curve X for simplicity. We consider Ran spaces for X and remove X from the notation. Recall that Ran is a nonunital commutative algebra in PreStk_{lft}, view it now as a nonunital commutative algebra in $(\mathrm{PreStk}_{lft})_{corr}$ via the canonical inclusion $\mathrm{PreStk}_{lft} \hookrightarrow (\mathrm{PreStk}_{lft})_{corr}$. Assume given a non-unital commutative algebra Z in $(\mathrm{PreStk}_{lft})_{corr}$ together with a morphism $Z \to \mathrm{Ran}$ in $CAlg^{nu}((\mathrm{PreStk}_{lft})_{corr})$.

We think of Z as given by a symmetric monoidal functor $fSet \to (\mathrm{PreStk}_{lft})_{corr}$, $I \mapsto Z^I$. For a morphism $\phi : I \to J$ in $fSet$ we have the corresponding diagram $Z^I \stackrel{a_\phi}{\leftarrow} Z_{I,J} \stackrel{b_\phi}{\to} Z^J$ in PreStk_{lft}, where the first map is over Ran^I, and the second map covers the multiplication $\mathrm{Ran}^I \to \mathrm{Ran}^J$.

Write Ran^{ch} for the nonunital commutative algebra Ran in $(\mathrm{PreStk}_{lft})_{corr}$ with the chiral multiplication. So, for $I \to *$ in $fSet$ the multiplication diagram is $\mathrm{Ran}^I \leftarrow \mathrm{Ran}^I_d \stackrel{u}{\to} \mathrm{Ran}$, where u is the union. We have a canonical lax morphism $\mathrm{Ran}^{ch} \to \mathrm{Ran}$ in the 2-category $CAlg^{nu}((\mathrm{PreStk}_{lft})_{corr})$ coming from the inclusion $\mathrm{Ran}^I_d \subset \mathrm{Ran}^I$ for any finite nonempty set I.

For $I \to J$ in $fSet$ set

$$\mathcal{Z}_{I,J} = Z_{I,J} \times_{\mathrm{Ran}^I} \mathrm{Ran}^I_d$$

For any $I \to J$ in $fSet$ the restriction of the product diagram $Z^I \leftarrow \mathcal{Z}_{I,J} \to \mathcal{Z}^J$ defines a nonunital commutative algebra structure on the object $Z \in (\mathrm{PreStk}_{lft})_{corr}$. More precisely, the functor

$$fSet \to (\mathrm{PreStk}_{lft})_{corr},\ I \mapsto \mathcal{Z}_I; (I \to J) \mapsto (Z^I \leftarrow \mathcal{Z}_{I,J} \to Z^J)$$

becomes an object of $CAlg^{nu}((\mathrm{PreStk}_{lft})_{corr})$, we denote it by Z^{ch}. Assume in addition the latter functor is a factorization space over Ran. This means that for any $I \to J$ in $fSet$ the both squares in the daigram below are cartesian

$$\begin{array}{ccccc} Z^I & \leftarrow & \mathcal{Z}_{I,J} & \to & Z^J \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ran}^I & \leftarrow & \mathrm{Ran}^I_d & \to & \mathrm{Ran}^J \end{array}$$

Assume for simplicity we are in the constructible context, and for any $I \to J$ in $fSet$ the map $b_\phi : Z_{I,J} \to Z_J$ is proper (or pseudo-proper). Let

$$\mathrm{PreStk}_{lft,pseudo-proper} \subset \mathrm{PreStk}_{lft}$$

be the 1-full subcategory, where we keep all objects and morphisms which are pseudo-proper. Let $(\mathrm{PreStk}_{lft})_{corr,pseudo-proper} \subset (\mathrm{PreStk}_{lft})_{corr}$ be the 1-full subcategory, where we keep all objects and for $Y, Y' \in \mathrm{PreStk}_{lft}$ only 1-morphisms given by diagrams $Y \leftarrow V \stackrel{h}{\to} Y'$, where h is pseudo-proper. This makes sense, as the base change of a pseudo-proper morphism is pseudo-proper.

Since $Shv_{pseudo-proper} : (\mathrm{PreStk}_{lft})_{corr,pseudo-proper} \to \mathrm{DGCat}_{cont}$ is right-lax monoidal, it sends non-unital commutative algebras to non-unital commutative algebras. Here for a map $Z_1 \to Z_2$ in $(\mathrm{PreStk}_{lft})_{corr,pseud-proper}$ given by a diagram $Z_1 \stackrel{a}{\leftarrow} Z_{12} \stackrel{b}{\to} Z_2$ the functor $Shv(Z_1) \to Shv(Z_2)$ is $b_*a^! \widetilde{\to} b_!a^!$, because $b_* \widetilde{\to} b_!$. Thus, $Shv(Z)$ is an object of $CAlg^{nu}(\mathrm{DGCat}_{cont})$ given by applying $Shv_{pseudo-proper}$ to the above functor. We write $Shv(Z,*)$ for this non-unital symmetric monoidal structure.

In this setting we should be able to talk about commutative factorization categories on Z and commutative factorization algebras on Z in Shv, in particular.

I think a possible definition of a commutative factorization algebra on Z in Shv is a follows.

Definition 1.1.1. *A commutative factorization algebra on Z is an object*

$$F \in CAlg^{nu}(Shv(Z,*))$$

with the following property. For any $\phi : I \to J$ in $fSet$ consider the multiplication map $(b_\phi)_! a^!_\phi(F^{\boxtimes I}) \to F^{\boxtimes J}$ in $Shv(Z^J)$, and the map $a^!_\phi(F^{\boxtimes I}) \to b^!_\phi F^{\boxtimes J}$ over $Z_{I,J}$ obtained from it by adjointness. Then the restriction of the latter map to $\mathcal{Z}_{I,J}$ is an isomorphism.

1.1.2. For $I \in fSet$ consider the natural map $X \to \mathrm{Ran}^I$. Let $(Z^I)_X = Z^I \times_{\mathrm{Ran}^I} X$. For $\phi : I \to J$ in $fSet$ let $Z_{I,J,X} = Z_{I,J} \times_{\mathrm{Ran}^I} X$. Note that

$$Z_{I,J,X} \widetilde{\to} Z_{I,J} \times_{Z^J} (Z^J)_X$$

canonically. Here $(Z^I)_X$ is the product $\prod_{i\in I} Z_X$ taken in the category $\mathrm{PreStk}_{lft,/X}$. The functor $fSet \to (\mathrm{PreStk}_{lft,/X})_{corr}$, $I \mapsto (Z^I)_X$, $(\phi : I \to J) \mapsto ((Z^I)_X \leftarrow Z_{I,J,X} \to (Z^J)_X$ is symmetric monoidal, so $Z_X \in CAlg^{nu}((\mathrm{PreStk}_{lft,/X})_{corr})$ naturally.

For each $\phi : I \to J$ in $fSet$ the base change $Z_{I,J,X} \to (Z^J)_X$ of the map $b_\phi : Z_{I,J} \to Z^J$ under $X \to \mathrm{Ran}^J$ is still psedo-proper. As above this gives on $Shv(Z_X)$ a structure of a non-unital symmetric monoidal category, given by the convolution, which we denote by $Shv(Z_X,*)$. To be precise, for $\phi : I \to J$ the corresponding diagram

$$Z^I \xleftarrow{q} (Z^I)_X \xleftarrow{a_{\phi,X}} Z_{I,J,X} \xrightarrow{b_{\phi,X}} (Z^J)_X$$

gives the convolution sendng $\{K_i\}_{i\in I}$ with $K_i \in Shv(Z_X)$ to

$$(b_{\phi,X})_* a^!_{\phi,X} q^! (\boxtimes_{i\in I} K_i)$$

In fact we get $Shv(Z_X,*) \in CAlg^{nu}((Shv(X),\otimes^!) - mod)$, because the functor $\mathrm{PreStk}_{lft,/X} \to (Shv(X),\otimes^!) - mod$, $S \mapsto Shv(S)$ is right-lax symmetric monoidal.

1.1.3. Let $i : Z_X \to Z$ be the natural map. Since $X \to \mathrm{Ran}$ is pseudo-proper by ([2] Proposition 7.4.2), i is also pseudo-proper. We have an adjoint pair $i_! : Shv(Z_X) \leftrightarrows Shv(Z) : i^!$ in DGCat_{cont}. The functor $i^! : Shv(Z,*) \to Shv(Z_X,*)$ is non-unital symmetric monoidal. So, it induces a functor $CAlg^{nu}(Shv(Z,*)) \to CAlg^{nu}(Z_X,*)$. The latter functor preserves limits, so has a left adjoint.

This left adjoint produces the corresponding commutative factorization algebras as in [12], as far as I understand.

2. Appendix: Lax colimits

2.0.1. One has the notion of lax limit and colimit from [4]. For $\mathcal{C} \in 1-\mathcal{C}at$ they denote by $Tw(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ the cartesian fibration in spaces associated to the functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Spc}$, $(c', c) \mapsto \mathrm{Map}_{\mathcal{C}}(c', c)$. So, an object of $Tw(\mathcal{C})$ is an arrow $\alpha : c' \to c$ in

$\mathcal{C}$. A map from $\alpha : c' \to c$ to $\beta : d' \to d$ in $Tw(\mathcal{C})$ gives rise to maps $c' \to d'$ and $d \to c$ in $\mathcal{C}$ such that the diagram commutes

$$\begin{array}{ccc} c' & \stackrel{\alpha}{\to} & c \\ \downarrow & & \uparrow \\ d' & \stackrel{\beta}{\to} & d \end{array}$$

Another definition of $Tw(\mathcal{C})$ is given in ([14], Appendix B).

For $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ the projections $\mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$, $\mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$ on the source and target respectively are cartesian and cocartesian fibrations. The corresponding strengthenings are the functors $\mathcal{C}_{-/} : \mathcal{C}^{op} \to 1-\mathcal{C}\mathrm{at}$, $x \mapsto \mathcal{C}_{x/}$ and $\mathcal{C}_{/-} : \mathcal{C} \to 1-\mathcal{C}\mathrm{at}$, $x \mapsto \mathcal{C}_{/x}$.

Given a functor $F : \mathcal{C} \to 1-\mathcal{C}\mathrm{at}$, its lax colimit is

$$\underset{(c' \stackrel{\alpha}{\to} c) \in Tw(\mathcal{C})}{\mathrm{colim}} (\mathcal{C}_{c/})^{op} \times F(c')$$

and its oplax colimit is

$$\underset{(c' \stackrel{\alpha}{\to} c) \in Tw(\mathcal{C})}{\mathrm{colim}} (\mathcal{C}_{c/}) \times F(c')$$

The lax limit of F is

$$\lim_{(c' \stackrel{\alpha}{\to} c) \in Tw(\mathcal{C})^{op}} \mathrm{Fun}(\mathcal{C}_{/c'}, F(c))$$

The oplax limit of F is

$$\lim_{(c' \to c) \in Tw(\mathcal{C})^{op}} \mathrm{Fun}((\mathcal{C}_{/c'})^{op}, F(c))$$

There is a natural map $\xi : \lim F \to laxlim F$. Indeed, there is a natural morphism of functors from the composition $Tw(\mathcal{C})^{op} \to \mathcal{C} \stackrel{F}{\to} 1-\mathcal{C}\mathrm{at}$ to the functor $Tw(\mathcal{C})^{op} \to 1-\mathcal{C}\mathrm{at}$ sending $\alpha : c' \to c$ to $\mathrm{Fun}(\mathcal{C}_{/c'}, F(c))$. It comes from the fact that for $\alpha : c' \to c$ in $Tw(\mathcal{C})$, one has the natural map $const : F(c) \to \mathrm{Fun}(\mathcal{C}_{/c'}, F(c))$ functorial in $\alpha \in Tw(\mathcal{C})^{op}$. Here *const* is given by constant functors.

Note that for any $\alpha : c' \to c$ in $Tw(\mathcal{C})$ the above map $const : F(c) \to \mathrm{Fun}(\mathcal{C}_{/c'}, F(c))$ is fully faithful. Indeed, $L : \mathcal{C}_{/c'} \to *$ is left adjoint to $R : * \to \mathcal{C}_{/c'}$, here $R(*)$ is the final object of $\mathcal{C}_{/c'}$, and we apply ([11], 2.2.56). Now by ([11], Lemma 2.2.17), ξ is fully faithful.

2.0.2. *Example.* If $\mathcal{C} = [1]$, that is, the category with objects $0, 1$ and a unique nontrivial arrow $0 \to 1$ then given $F : [1] \to 1-\mathcal{C}\mathrm{at}$, the lax limit of F is as follows. Let $f : F(0) \to F(1)$ be the corresponding map in $1-\mathcal{C}\mathrm{at}$. Then $laxlim F = F(0) \times_{F(1)} \mathrm{Fun}([1], F(1))$, where the map $\mathrm{Fun}([1], F(1)) \to F(1)$ keeps the source of an arrow.

2.0.3. If $F : \mathcal{C}^{op} \to 1-\mathcal{C}\mathrm{at}$ is a map in $1-\mathcal{C}\mathrm{at}$, let $\mathcal{X} \to \mathcal{C}$ be the corresponding cartesian fibration. Then $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ is the oplax limit of F by ([4], 7.1). Recall that $\lim F$ is the category $\mathrm{Fun}^{cart}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$, the full subcategory of those functors in $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ that send cartesian arrows to cartesian arrows ([11], 2.2.67).

2.0.4. For $\mathcal{C} \in 1-\mathcal{C}at$ we have an isomorphism $Tw(\mathcal{C}) \widetilde{\to} Tw(\mathcal{C}^{op})$ compatible with the projections to $\mathcal{C} \times \mathcal{C}^{op}$. Indeed, for $c, c' \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(c', c) \widetilde{\to} \mathrm{Map}_{\mathcal{C}^{op}}(c, c')$.

For $\mathcal{A}, \mathcal{B} \in 1-\mathcal{C}at$ we get $Tw(\mathcal{A} \times \mathcal{B}) \widetilde{\to} Tw(\mathcal{A}) \times Tw(\mathcal{B})$ naturally.

2.0.5. For $F : [1] \to 1-\mathcal{C}at$ given by a functor $f : F(0) \to F(1)$ it oplax colimit is the coproduct $F(0) \times [1] \sqcup_{F(0)} F(1)$ in $1-\mathcal{C}at$, where the maps are $F(1) \stackrel{f}{\leftarrow} F(0) \stackrel{\mathrm{id} \times \{1\}}{\to} F(0) \times [1]$. This oplax colimit is the cocartesian fibration $X \to [1]$ corresponding to F by ([4], Th. 1.1).

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