

COMMENTS TO: RASKIN, CHIRAL CATEGORIES

for version Sept. 4, 2019

0.0.1. For 4.3.3. $\mathbf{Sets} = \text{Ind}(\mathbf{Set}_{<\infty})$, and \mathbf{Spc} is generated by \mathbf{Sets} under geometric realizations (HTT, 5.5.8.14).

Every object $Y \in \mathbf{Sets}_{<\infty}$ is projective in \mathbf{Spc} in the sense of (HTT, 5.5.8.18), because $\mathbf{\Delta}^{op}$ is sifted. Now the canonical inclusion $\mathbf{Sets}_{<\infty} \subset \mathbf{Spc}$ extends uniquely by (HTT, 5.5.8.15) to a functor $f : \mathcal{P}_{\Sigma}(\mathbf{Sets}_{<\infty}) \rightarrow \mathbf{Spc}$ preserving sifted colimits. We see that the conditions (i),(ii), (iii) of (HTT, 5.5.8.22) are verified, so $f : \mathcal{P}_{\Sigma}(\mathbf{Sets}_{<\infty}) \xrightarrow{\sim} \mathbf{Spc}$ is an equivalence.

The functors $F, F' : \mathbf{Spc} \rightarrow \mathbf{Spc}$, $F(Z) = \text{Ran}_Z$ and $F'(Z) = \text{Ran}_{Z, \emptyset}$ clearly commute with sifted colimits indeed. Thus, in the notations of (HTT, 5.5.8.15), these are functors in $\text{Fun}_{\Sigma}(\mathcal{P}_{\Sigma}(\mathbf{Sets}_{<\infty}), \mathbf{Spc})$. To see that F, F' are LKE of their restriction to $\mathbf{Sets}_{<\infty}$, decompose F (resp., F') as

$$\mathcal{P}_{\Sigma}(\mathbf{Sets}_{<\infty}) \xrightarrow{i} \mathcal{P}(\mathbf{Sets}_{<\infty}) \xrightarrow{b} \mathbf{Spc},$$

where the first map is the canonical inclusion, and the second is the colimi preserving functor extending $F : \mathbf{Sets}_{<\infty} \rightarrow \mathbf{Spc}$ (resp., $F' : \mathbf{Sets}_{<\infty} \rightarrow \mathbf{Spc}$). Now b is the LKE from $\mathbf{Sets}_{<\infty}$, and its composition with i is also a LKE from $\mathbf{Sets}_{<\infty}$, because i is a right adjoint!

0.0.2. For 2.8. The fact that his \mathbf{Vect} coincides with the definition from Dennis and Nick's book follows from (HA, 7.1.1.15-16).

0.0.3. For 4.3.2. Let A be a set. Then $\text{colim}_{I \in f\mathbf{Set}^{op}} A^I \xrightarrow{\sim} F(A)$, where $F(A)$ is the set of finite subsets in A . Indeed, for $B \in F(A)$ let $A_B^I = \{f : I \rightarrow A \mid \text{Im}(f) = B\}$. Then $A^I = \sqcup_{B \in F(A)} A_B^I$. Now

$$\text{colim}_{I \in f\mathbf{Set}^{op}} \sqcup_{B \in F(A)} A_B^I \xrightarrow{\sim} \sqcup_{B \in F(A)} \text{colim}_{I \in f\mathbf{Set}^{op}} A_B^I,$$

and $\text{colim}_{I \in f\mathbf{Set}^{op}} A_B^I \xrightarrow{\sim} \text{colim}_{I \in I \in f\mathbf{Set}^{op}, I \twoheadrightarrow B} * \xrightarrow{\sim} *$. We used the fact that the category $(I \in I \in f\mathbf{Set}^{op}, I \twoheadrightarrow B)$ has an initial object, so is contractible and the last colimit is $*$.

0.0.4. For 4.5. Once we defined $\text{Ran}_{\mathcal{G}}^{un}$ for $\mathcal{G} \in \mathbf{Set}_{<\infty}$, we may extend this definition to $\mathcal{P}_{\Sigma}(\mathbf{Sets}_{<\infty}) \xrightarrow{\sim} \mathbf{Spc}$ by requiring that the functor $\mathbf{Spc} \rightarrow 1 - \mathbf{Cat}$, $\mathcal{G} \mapsto \text{Ran}_{\mathcal{G}}^{un}$ commutes with sifted colimits using (HTT, 5.5.8.15).

For 4.6.1(1) see below. The forgetful functor $0 - \mathbf{Cart}/I \rightarrow 1 - \mathbf{Cat}$ preserves colimits (see below). For a cartesian fibration in spaces $p : \mathcal{X} \rightarrow I$ any arrow in \mathcal{X} is p -cartesian. So, $'\text{Ran}_{\mathcal{G}}^{un}$ can be defined as the category obtained from $\text{coGroth}(\Psi_{\mathcal{G}})$ inverting all arrows lying over surjective morphisms in $\mathbf{Sets}_{<\infty}$.

Date: March 18, 2026.

For 4.9: For $\mathcal{G} \in \mathbf{Spc}$, $\mathrm{Ran}_{\mathcal{G},\emptyset,[0]}^{\rightarrow} = \mathrm{Ran}_{\mathcal{G},\emptyset}$. Thus, the space underlying the category " $\mathrm{Ran}_{\mathcal{G}}^{\mathrm{un}} \xrightarrow{\sim} \mathrm{Ran}_{\mathcal{G}}$ " is $\mathrm{Ran}_{\mathcal{G},\emptyset}$.

For any $n \geq 0$ the category $f\mathrm{Set}_{\emptyset,[n]}^{\rightarrow}$ has a symmetric monoidal structure given by the disjoint union. This provides for any $\mathcal{G} \in \mathbf{Spc}$ a structure of a commutative monoid on $\mathrm{Ran}_{\mathcal{G},\emptyset,[n]}^{\rightarrow}$ as in ([11], 3.0.75). For example, if $\mathcal{G} \in \mathbf{Sets}$ then recall that $\mathrm{Ran}_{\mathcal{G},\emptyset,[n]}^{\rightarrow}$ is the set of collections $\{S_0 \subset \dots \subset S_n \subset \mathcal{G}\}$ with S_i finite. Then the operation sends a pair $(S_0 \subset \dots \subset S_n), (S'_0 \subset \dots \subset S'_n)$ to $(S_0 \cup S'_0 \subset \dots \subset S_n \cup S'_n)$.

0.0.5. For 4.13. Since $\pi_0 : \mathbf{Spc} \rightarrow \mathbf{Sets}$ is a left adjoint, for $\mathcal{G} \in \mathbf{Spc}$ we get isomorphisms $\pi_0 \mathrm{Ran}_{\mathcal{G}} \xrightarrow{\sim} \mathrm{Ran}_{\pi_0(\mathcal{G})}$ and $\pi_0(\mathrm{Ran}_{\mathcal{G},\emptyset}) \xrightarrow{\sim} \mathrm{Ran}_{\pi_0(\mathcal{G}),\emptyset}$.

For 4.13.2: misprint I think: the space underlying $\mathrm{Ran}_{\mathcal{G}}^{\mathrm{un}}$ is $\mathrm{Ran}_{\mathcal{G},\emptyset}$. However, it is clear how to define $[\mathrm{Ran}_{\mathcal{G},\emptyset} \times \mathrm{Ran}_{\mathcal{G},\emptyset}]_{\mathrm{disj}}$, and hence we also get $[\mathrm{Ran}_{\mathcal{G}}^{\mathrm{un}} \times \mathrm{Ran}_{\mathcal{G}}^{\mathrm{un}}]_{\mathrm{disj}}$ as a full subcategory restricting objects in the same way.

0.0.6. Let $I \in 1 - \mathbf{Cat}$. Sam claims $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}$ sending $\mathcal{X} \rightarrow I$ to \mathcal{X} preserves colimits. I prove this using ([4], Th. 1.1). Given $F : \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Spc}$, let $\mathcal{X} \rightarrow \mathcal{C}$ be the cartesian fibration in spaces classifying F . By ([4], Th. 1.1), \mathcal{X} is the lax colimit of F . That is, $\mathcal{X} \xrightarrow{\sim} \mathrm{colim}_{(c' \xrightarrow{\alpha} c) \in \mathrm{Tw}(\mathcal{C}^{\mathrm{op}})} \mathcal{C}_{/c} \times F(c')$ in $1 - \mathbf{Cat}$. If now $F \xrightarrow{\sim} \mathrm{colim}_{j \in J} F^j$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Spc})$ then we get

$$\mathcal{X} \xrightarrow{\sim} \mathrm{colim}_{(c' \xrightarrow{\alpha} c) \in \mathrm{Tw}(\mathcal{C}^{\mathrm{op}}), j \in J} \mathcal{C}_{/c} \times F^j(c') \xrightarrow{\sim} \mathrm{colim}_{j \in J} \mathrm{colim}_{(c' \xrightarrow{\alpha} c) \in \mathrm{Tw}(\mathcal{C}^{\mathrm{op}})} \mathcal{C}_{/c} \times F^j(c') \xrightarrow{\sim} \mathrm{colim}_{j \in J} \mathcal{X}^j,$$

where $\mathcal{X}^j \rightarrow \mathcal{C}$ is the cartesian fibration in spaces classifying F^j . Here the colimit $\mathrm{colim}_{j \in J} \mathcal{X}^j$ is calculated in $1 - \mathbf{Cat}$.

The same argument actually shows that $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}_{/I}$ preserves colimits, hence admits a right adjoint, because these categories are presentable. Recall that the functor $1 - \mathbf{Cat}_{/I} \rightarrow 1 - \mathbf{Cat}$, $(X \rightarrow I) \mapsto X$ preserves colimits, as this is a left adjoint.

Lemma 0.0.7. *Let $I, J \in 1 - \mathbf{Cat}$, $J \subset I$ be a 1-full subcategory with the same class of objects as I . The functor $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}$, $(X \rightarrow I) \mapsto S^{-1}X$ preserves colimits. Here S is the class of those morphisms of X which lie over a morphism of J .*

Proof. For $(X \rightarrow I) \in 1 - \mathbf{Cat}_{/I}$ write $X_J = X \times_I J$. Given $Y \in 1 - \mathbf{Cat}$, one gets

$$\mathrm{Map}_{1 - \mathbf{Cat}}(S^{-1}X, Y) \xrightarrow{\sim} \mathrm{Map}_{1 - \mathbf{Cat}}(X, Y) \times_{\mathrm{Map}_{1 - \mathbf{Cat}}(X_J, Y)} \mathrm{Map}_{1 - \mathbf{Cat}}(X_J, Y^{\mathrm{Spc}})$$

Assume $X \xrightarrow{\sim} \mathrm{colim}_{k \in K} X_k$ in $0 - \mathbf{Cart}_{/I}$. The functor $0 - \mathbf{Cart}_{/I} \rightarrow 1 - \mathbf{Cat}_{/I}$ preserves colimits by the above, so this is also a colimit diagram in $1 - \mathbf{Cat}$, hence $\mathrm{Map}_{1 - \mathbf{Cat}}(X, Y) \xrightarrow{\sim} \mathrm{lim}_{k \in K^{\mathrm{op}}} \mathrm{Map}_{1 - \mathbf{Cat}}(X_k, Y)$. The functor $0 - \mathbf{Cart}_{/I} \rightarrow 0 - \mathbf{Cart}_{/J}$, $X' \mapsto X'_J$ preserves colimits, as we may interpret this as $\mathrm{Fun}(I^{\mathrm{op}}, \mathbf{Spc}) \rightarrow \mathrm{Fun}(J^{\mathrm{op}}, \mathbf{Spc})$. So, $\mathrm{Map}_{1 - \mathbf{Cat}}(X_J, Y) \xrightarrow{\sim} \mathrm{lim}_{k \in K^{\mathrm{op}}} \mathrm{Map}_{1 - \mathbf{Cat}}((X_k)_J, Y)$ and

$$\mathrm{Map}_{1 - \mathbf{Cat}}(X_J, Y^{\mathrm{Spc}}) \xrightarrow{\sim} \mathrm{lim}_{k \in K^{\mathrm{op}}} \mathrm{Map}_{1 - \mathbf{Cat}}((X_k)_J, Y^{\mathrm{Spc}})$$

Limits commute with limits, so the natural map

$$\mathrm{Map}_{1 - \mathbf{Cat}}(S^{-1}X, Y) \rightarrow \mathrm{Map}_{1 - \mathbf{Cat}}(\mathrm{colim}_{k \in K} S^{-1}(X_k), Y)$$

is an isomorphism. \square

The above lemma implies Pp4.6.1(1) of Sam.

Question. What is the right adjoint to the inclusion $0 - \mathcal{C}art_{/I} \rightarrow 1 - \mathcal{C}at_{/I}$?

0.0.8. For Sect. 4.14. Since $\mathbf{1-Cat}$ is a 2-category, $\text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat})$ is also a 2-category. This is a general construction associating two a pair of 2-categories, the 2-category of functors between them. I think the corresponding mapping category is given by ([4], 6.9).

Given $Y \in \text{PreStk}^{lax}$, the object $\text{QCoh}(Y) \in 1 - \mathcal{C}at$ is $\mathbf{Map}_{\text{PreStk}^{lax}}(Y, \text{QCoh})$ in the sense of Dennis and Nick book.

Let $i : \text{PreStk} \rightarrow \text{PreStk}^{lax}$ be the inclusion, it has a right adjoint $R : \text{PreStk}^{lax} \rightarrow \text{PreStk}$ sending Y to the prestack $(S \mapsto Y(S)^{\text{SpC}})$ for $S \in \text{Sch}^{aff}$. The functor i also has a left adjoint $\text{PreStk}^{lax} \rightarrow \text{PreStk}$ sending Y to the prestack $(S \mapsto |Y(S)|)$ for $S \in \text{Sch}^{aff}$, here $|Z|$ denotes the inverting of all arrows in the category Z . This is an application of ([11], 2.2.56).

The *RKE* of $\text{QCoh}_{\text{PreStk}} : \text{PreStk}^{op} \rightarrow \text{DGCat}_{cont}$ under $\text{PreStk}^{op} \hookrightarrow (\text{PreStk}^{lax})^{op}$ is the composition $(\text{PreStk}^{lax})^{op} \xrightarrow{R^{op}} \text{PreStk}^{op} \rightarrow \text{DGCat}_{cont}$, where the second functor is $\text{QCoh}_{\text{PreStk}}$. It sends Y to $\text{QCoh}(R(Y))$, so this is not the same thing as $\text{QCoh}(Y)$ that Sam means in this paper.

According to ([2], 2.2.1), for $Y \in \text{PreStk}^{lax}$, let $\mathcal{X}_Y \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to $Y : (\text{Sch}^{aff})^{op} \rightarrow 1 - \mathcal{C}at$. Let also $\text{Shv}_{\text{Sch}^{aff}} \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to the composition $(\text{Sch}^{aff})^{op} \rightarrow \text{DGCat}_{cont} \rightarrow 1 - \mathcal{C}at$, where the first functor is $\text{QCoh}_{\text{Sch}^{aff}}^*$. Then $\text{QCoh}(Y)$ can be defined as the full subcategory of $\text{Fun}_{\text{Sch}^{aff}}(\mathcal{X}_Y, \text{Shv}_{\text{Sch}^{aff}})$ consisting of functors sending cartesian arrows to cartesian arrows.

There is one more way to understand $\text{QCoh}(Y)$ for a lax prestack Y as follows. By ([4], Lemma 6.4), the category $\text{QCoh}(Y)$ is the limit of the composition

$$Tw(\text{Sch}^{aff})^{op} \rightarrow \text{Sch}^{aff} \times (\text{Sch}^{aff})^{op} \xrightarrow{Y^{op} \times \text{QCoh}} (1 - \mathcal{C}at)^{op} \times 1 - \mathcal{C}at \xrightarrow{\text{Fun}} 1 - \mathcal{C}at$$

This allows to view $\text{QCoh}(Y)$ as an object of DGCat_{cont} as follows. Consider the composition

$$Tw(\text{Sch}^{aff})^{op} \rightarrow \text{Sch}^{aff} \times (\text{Sch}^{aff})^{op} \xrightarrow{Y^{op} \times \text{QCoh}} (1 - \mathcal{C}at)^{op} \times \text{DGCat}_{cont} \xrightarrow{\text{Fun}} \text{DGCat}_{cont}$$

and take the limit. Since the projection $\text{DGCat}_{cont} \rightarrow 1 - \mathcal{C}at$ preserves limits, we are done.

We used the fact that given any $C \in 1 - \mathcal{C}at, D \in \text{DGCat}_{cont}, \text{Fun}(C, D) \in \text{DGCat}_{cont}$ naturally (see [11], 9.2.13). Once again, for $Y \in \text{PreStk}^{lax}$,

$$(1) \quad \text{QCoh}(Y) = \lim_{(\alpha: S' \rightarrow S) \in Tw(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S), \text{QCoh}(S'))$$

This implies the following.

Remark 0.0.9. Write $\text{QCoh}_{\text{PreStk}^{lax}}^* : (\text{PreStk}^{lax})^{op} \rightarrow \text{DGCat}_{cont}$ for the functor sending $Y \mapsto \text{QCoh}(Y)$ and $f : Y \rightarrow Y'$ to $f^* : \text{QCoh}(Y') \rightarrow \text{QCoh}(Y)$. This functor preserves limits. The functor f^* is a map in DGCat_{cont} by construction.

The fact that for $Y \in \text{PreStk}^{lax}$, $\text{QCoh}(Y)$ coincides with the usual notion, provided that $Y \in \text{PreStk}$ in addition, follows from Lemma 0.0.17 below.

Given a map $\alpha \rightarrow \beta$ in $\text{Tw}(\text{Sch}^{aff})$ with $\alpha : S' \rightarrow S, \beta : T' \rightarrow T$, the transition functor $\text{Fun}(Y(T), \text{QCoh}(T')) \rightarrow \text{Fun}(Y(S), \text{QCoh}(S'))$ in (1) admits a right adjoint, however this right adjoint is not continuous in general.

0.0.10. For 4.18. In ([1], ch. A.1, 3.2.7) they defined for a pair $\mathbb{S}, \mathbb{T} \in 2 - \text{Cat}$ the object $\text{Fun}(\mathbb{S}, \mathbb{T})_{rlax} \in 2 - \text{Cat}$. Sam means by $\text{Hom}^{lax}(\mathcal{C}, \mathcal{D})$ the underlying 1-category of left-lax functors $(\text{Fun}(\mathbb{S}, \mathbb{T})_{lax})^{1-\text{Cat}}$ (see my file [11], 12.1.44).

Sam claims in 4.18.3 that there is an object, say $2 - \text{CAT}_{lax}$ of $2 - \text{Cat}$, which extends $2 - \text{Cat}$ as follows. The objects of $2 - \text{CAT}_{lax}$ are those of $2 - \text{Cat}$, and given $C, D \in 2 - \text{Cat}$, the mapping category from C to D is $(\text{Fun}(C, D)_{lax})^{1-\text{Cat}}$. Let's accept this.

0.0.11. For 4.19. Recall Sam's notation $\text{ShvCat}_{/-}$ for the functor

$$\text{ShvCat}_{/-} : (\text{Sch}^{aff})^{op} \rightarrow 1 - \text{Cat}, S \mapsto \text{ShvCat}(S) = \text{QCoh}(S) - \text{mod}$$

Let $\mathcal{X}_{Shv} \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to this functor. Given a lax prestack Y , the definition of $\text{ShvCat}_{/Y}^{naive}$ is as follows. Let $\mathcal{X}_Y \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to $Y : (\text{Sch}^{aff})^{op} \rightarrow 1 - \text{Cat}$. Then $\text{ShvCat}_{/Y}^{naive}$ is the full subcategory in $\text{Fun}_{\text{Sch}^{aff}}(\mathcal{X}_Y, \mathcal{X}_{Shv})$ consisting of functors sending cartesian arrows to cartesian arrows. In other words,

$$(2) \quad \text{ShvCat}_{/Y}^{naive} = \mathbf{Map}_{\text{PreStk}^{lax}}(Y, \text{ShvCat}_{/-}),$$

where we view PreStk^{lax} naturally as a $(\infty, 2)$ -category.

So, for a lax prestack Y ,

$$\text{ShvCat}_{/Y}^{naive} \xrightarrow{\simeq} \lim_{(S' \rightarrow S) \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S), \text{QCoh}(S') - \text{mod})$$

taken in $1 - \text{Cat}$. In the latter limit system each category admits colimits, and the transition functors preserve colimits. For this reason $\text{ShvCat}_{/Y}^{naive}$ has colimits, they are computed termwise in $\text{Fun}(Y(S), \text{QCoh}(S') - \text{mod})$ in the above projective system.

0.0.12. My understanding is that $\text{ShvCat}_{/Y}$ has to be a 2-category, and this is precisely how it differs from the $(\infty, 1)$ -category $\text{ShvCat}_{/Y}^{naive}$.

A candidate for his definition of $\text{ShvCat}_{/Y}$ for a lax prestack Y is as follows. First, we consider $1 - \text{Cat}$ as a 2-category (as in [1], ch. A.1, Sect. 2.4) denoted $\mathbf{1-Cat}$ in *loc.cit.* So, we have the 2-categories

$$\text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat}), \text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat})_{rlax}, \text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat})_{lax}$$

defined in ([1], ch. A.1, 2.5.4 and 3.2.7).

Question: Consider the corresponding mapping category

$$\mathbf{Map}_{\text{Fun}((\text{Sch}^{aff})^{op}, \mathbf{1-Cat})_{lax}}(Y, \text{ShvCat}_{/-})$$

does it coincide with $\text{ShvCat}_{/Y}$ by definition? Sam says yes, but I think this is wrong!

He proposes the following in email of 14 Oct 2019.

Definition 0.0.13. We have the natural functor $\mathbf{2-Cat} \rightarrow 2 - \mathcal{CAT}_{lax}$, here $\mathbf{2-Cat}$ is the 2-category defined in ([1], ch. A.1, Sect. 6). View $ShvCat_{/_Y}$ as a functor $(\text{Sch}^{aff})^{op} \rightarrow \mathbf{2-Cat}$. It is essential here to use the 2-category structure of $ShvCat(S)$ for $S \in \text{Sch}^{aff}$, see Section 0.2 here. Given $Y \in \text{PreStk}^{lax}$, we may also view Y as the composition $(\text{Sch}^{aff})^{op} \rightarrow 1 - \mathcal{Cat} \rightarrow \mathbf{2-Cat} \rightarrow 2 - \mathcal{CAT}_{lax}$. Now $ShvCat_{/_Y}$ is the category of natural transformations $Y \rightarrow ShvCat_{/_Y}$ of functors $(\text{Sch}^{aff})^{op} \rightarrow 2 - \mathcal{CAT}_{lax}$.

My understanding is that

$$ShvCat_{/_Y} = \mathbf{Map}_{\text{Fun}((\text{Sch}^{aff})^{op}, 2 - \mathcal{CAT}_{lax})}(Y, ShvCat_{/_Y})$$

Using [4], the latter can be rewritten as the limit in $1 - \mathcal{Cat}$

$$(3) \quad ShvCat_{/_Y} \xrightarrow{\sim} \lim_{(S \rightarrow S') \in Tw(\text{Sch}^{aff})^{op}} (\text{Fun}(Y(S'), ShvCat(S))_{lax})^{1 - \mathcal{Cat}}$$

taken over $Tw(\text{Sch}^{aff})^{op}$.

Sam claims $ShvCat_{/_Y}$ is not presentable, already $DGCat_{cont}$ is not presentable.

It is not clear which 2-categorical enhancement of $ShvCat_{/_Y}$ for a lax prestack Y he meant, but there is the following one: write $ShvCat_{/_Y}^{2 - \mathcal{Cat}}$ for the limit in $2 - \mathcal{Cat}$

$$\lim_{S \rightarrow S'} (\text{Fun}(Y(S'), ShvCat(S))_{lax})$$

taken over $Tw(\text{Sch}^{aff})^{op}$. Here we view $ShvCat(S)$ as an object of $2 - \mathcal{Cat}$. This is a good idea, because the functor $2 - \mathcal{Cat} \rightarrow 1 - \mathcal{Cat}$, $D \mapsto D^{1 - \mathcal{Cat}}$ preserves limits.

Note that for $ShvCat_{/_Y}^{naive}$ we have similarly to (3)

$$(4) \quad ShvCat_{/_Y}^{naive} \xrightarrow{\sim} \lim_{(S \rightarrow S') \in Tw(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S'), ShvCat(S))$$

taken over $Tw(\text{Sch}^{aff})^{op}$. The category $Tw(\text{Sch}^{aff})$ has no final object! The limit (4) understood as a limit in $1 - \mathcal{Cat}$ gets the 1-category $ShvCat_{/_Y}^{naive}$. It can also be understood as a limit in $2 - \mathcal{Cat}$, and then gives an object of $2 - \mathcal{Cat}$ mentioned in his Remark 4.19.2.

0.0.14. The functor $\text{PreStk}^{op} \rightarrow 1 - \mathcal{Cat}$, $Y \mapsto ShvCat_{/_Y}^{naive}$ factors through the category of cocomplete categories and colimit-preserving functors. However, $ShvCat_{/_Y}$ is not presentable already for $Y = \text{Spec } k$. Given $Y \in \text{PreStk}^{lax}$, we get

$$ShvCat_{/_Y}^{naive} = \lim_{(c \rightarrow c') \in Tw(\text{Sch}^{aff})^{op}} \text{Fun}(Y(c'), ShvCat(c))$$

If $f : Y_1 \rightrightarrows Y_2 : g$ are maps in $\text{Fun}((\text{Sch}^{aff})^{op}, 1 - \mathcal{Cat})$ with morphisms $fg \rightarrow \text{id}$, $\text{id} \rightarrow gf$ making them into an adjoint pair in the 2-category PreStk^{lax} then the functors

$$g^* : ShvCat_{/_Y_1}^{naive} \rightrightarrows ShvCat_{/_Y_2}^{naive} : f^*$$

also form an adjoint pair by ([2], Lm. 2.2.6). Indeed, apply the formula (2) and functoriality of \mathbf{Map} .

0.0.15. For 4.19.2. Let Y be a lax prestack. Then \mathbf{QCoh}_Y is the sheaf of categories on Y obtained as the pull-back along $Y \rightarrow *$ of \mathbf{Vect} . If $\beta, \gamma : S \rightarrow Y$ are two S -points with $S \in \mathbf{Sch}^{aff}$ and $\theta : \beta \rightarrow \gamma$ is a map in $Y(S)$ then the corresponding map $\theta : \beta^* \mathbf{QCoh}_Y \rightarrow \gamma^* \mathbf{QCoh}_Y$ is the identity.

0.0.16. Let $Y \in \mathbf{PreStk}^{lax}$. Let us calculate $\mathbf{ShvCat}/_Y^{naive}$ assuming in addition that $Y \in \mathbf{PreStk}$.

Lemma 0.0.17. *Let $C \in 1 - \mathbf{Cat}$, $Y \in \mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})$, $S \in C$. We identify S with its image in $\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})$ under the Yoneda. Then the category of natural transformations $S \rightarrow Y$ is canonically equivalent to $Y(S)$.*

More generally, if $Y_1 \in \mathcal{P}(C)$ then the category of natural transformations $Y_1 \rightarrow Y$ is equivalent to $\lim_{S \rightarrow Y_1} Y(S)$, the limit over $(C/_Y)^{op}$ is taken in $1 - \mathbf{Cat}$.

Proof. By ([4], Lm. 6.4), the category of natural transformations $Y_1 \rightarrow Y_2$ for $Y_i \in \mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})$ is the limit of the composition

$$Tw(C)^{op} \rightarrow C \times C^{op} \xrightarrow{Y_1^{op} \times Y_2} (1 - \mathbf{Cat})^{op} \times 1 - \mathbf{Cat} \xrightarrow{\mathbf{Fun}} 1 - \mathbf{Cat}$$

So, $\mathbf{Map}_{\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})}(Y_1, Y_2) \xrightarrow{\sim} \lim_{(c \rightarrow c') \in Tw(C)^{op}} \mathbf{Fun}(Y_1(c'), Y_2(c))$, here $c \rightarrow c'$ is an arrow in C .

By ([4], 6.9) we know that $\mathbf{Map}_{\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})}(Y_1, Y_2) \xrightarrow{\sim} \mathbf{Fun}_C^{cart}(\mathcal{Y}_1, \mathcal{Y}_2)$, where $\mathcal{Y}_i \rightarrow C$ is the cartesian fibration corresponding to Y_i , and $\mathbf{Fun}_C^{cart}(\mathcal{Y}_1, \mathcal{Y}_2) \subset \mathbf{Fun}_C(\mathcal{Y}_1, \mathcal{Y}_2)$ is the full subcategory of functors sending cartesian arrows to cartesian arrows.

Assume $Y_1 = S$ for $S \in C$. The corresponding mapping category is $\mathbf{Fun}_C^{cart}(C/_S, \mathcal{Y}_2)$, as $C/_S \rightarrow C$ is a cartesian fibration in spaces. The functor of evaluation on the final object $S \xrightarrow{\text{id}} S$ of $C/_S$ gives a map $\mathbf{Fun}_C^{cart}(C/_S, \mathcal{Y}_2) \rightarrow Y_2(S)$. To get a map in the other direction, we need to construct a functor $Y_2(S) \times C/_S \rightarrow \mathcal{Y}_2$ over C . Given $y \in Y_2(S)$ and $c \rightarrow S$ in $C/_S$, we get a morphism $c \rightarrow S$ in C . The desired functor sends this collection to $y' \in \mathcal{Y}_2(c)$, which is the source of a cartesian arrow $y' \rightarrow y$ over $c \rightarrow S$. I hope these two functors are inverse to each other.

More generally, if $Y_1 \in \mathcal{P}(C)$, then $Y_1 \xrightarrow{\sim} \text{colim}_{S \rightarrow Y_1} S$ in $\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})$ over $C/_Y$. Then we should have

$$\mathbf{Map}_{\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})}(Y_1, Y_2) \xrightarrow{\sim} \lim_{S \rightarrow Y_1} \mathbf{Map}_{\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})}(S, Y_2) \xrightarrow{\sim} \lim_{S \rightarrow Y_1} Y_2(S)$$

To explain the first isomorphism, more generally, we claim that for any $Y_2 \in \mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})$ the functor $(\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat}))^{op} \rightarrow 1 - \mathbf{Cat}$, $Y_1 \mapsto \mathbf{Map}_{\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})}(Y_1, Y_2)$ preserves limits. Indeed, if $Y_1 = \text{colim}_{i \in I} Y_1^i$ in $\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})$ then

$$\mathbf{Map}_{\mathbf{Fun}(C^{op}, 1 - \mathbf{Cat})}(Y_1, Y_2) \xrightarrow{\sim} \lim_{c \rightarrow c'} \mathbf{Fun}(Y_1(c'), Y_2(c)) \xrightarrow{\sim} \lim_{c \rightarrow c'} \lim_{i \in I^{op}} \mathbf{Fun}(Y_1^i(c'), Y_2(c))$$

permuting the two latter limits, one gets the desired isomorphism. \square

Is it true that the functor $\mathcal{D}^{op} \rightarrow 1 - \mathbf{Cat}$, $x \mapsto \mathbf{Map}_{\mathcal{D}}(x, d)$ for a 2-category \mathcal{D} , preserves limits? No in general!!

We see that if $S \in \mathbf{Sch}^{aff}$ then $\mathbf{ShvCat}/_S^{naive} \xrightarrow{\sim} (\mathbf{QCoh}(S) - \text{mod})$. More generally, if $Y \in \mathbf{PreStk}$ then $\mathbf{ShvCat}/_Y^{naive} \xrightarrow{\sim} \mathbf{ShvCat}(Y)$ by the above lemma.

0.0.18. For 4.20.1. Let $Y \in \text{PreStk}^{lax}$. We have

$$\text{ShvCat}/_{Y\text{PreStk}} \xrightarrow{\sim} \text{ShvCat}(Y^{\text{PreStk}}) \xrightarrow{\sim} \lim_{S \rightarrow S'} \text{Fun}(Y(S)^{\text{Spc}}, \text{ShvCat}(S'))$$

in $1 - \text{Cat}$, the limit over $\text{Tw}((\text{Sch}^{aff})^{op})$. To see that $\text{ShvCat}/_Y \rightarrow \text{ShvCat}/_{Y\text{PreStk}}$ is conservative, using (3) it would suffice to show that for any $S, S' \in \text{Sch}^{aff}$ the map

$$(\text{Fun}(Y(S), \text{ShvCat}(S'))_{llax})^{1-\text{Cat}} \rightarrow \text{Fun}(Y(S)^{\text{Spc}}, \text{ShvCat}(S'))$$

is conservative. However, this is not the case (cf. remark below). So, I think $\text{ShvCat}/_Y \rightarrow \text{ShvCat}/_{Y\text{PreStk}}$ is not conservative.

Remark 0.0.19. *If $S \in 1 - \text{Cat}, T \in 2 - \text{Cat}$ then the natural map $\text{Fun}(S, T)_{rlax}^{1-\text{Cat}} \rightarrow \text{Fun}(S^{\text{Spc}}, T)^{1-\text{Cat}} = \text{Fun}(S^{\text{Spc}}, T^{1-\text{Cat}})$ is not conservative in general. This happens already for $S = [1]$. Namely, if $x, y \in T$, $f, g : x \rightarrow y$ are 1-morphisms in T and $\alpha : f \rightarrow g$ is a 2-morphism in T , which is not an isomorphism, we get a 1-morphism in $\text{Fun}(S, T)_{rlax}^{1-\text{Cat}}$ from f to g , which is not an isomorphism, but its restriction to S^{Spc} is an isomorphism. The notation $\text{Fun}(S, T)_{rlax} \in 2 - \text{Cat}$ is that of ([1], ch. A.1, 3.2.7).*

0.0.20. For 4.23. Let \mathcal{X} be a prestack. My understanding is that for a unital sheaf of categories C on $\text{Ran}_{\mathcal{X}}$, the fusion morphism is a 1-morphism in $\text{ShvCat}/_{\text{Ran}_{\mathcal{X}, \emptyset}}$. Here $\text{Ran}_{\mathcal{X}, \emptyset}^{\rightarrow} \in \text{PreStk}$. The property of being *adj-unital* here means that for any $S \in \text{Sch}^{aff}$ with $f : S \rightarrow \text{Ran}_{\mathcal{X}, \emptyset}^{\rightarrow}$ the corresponding map in $\text{ShvCat}(S)$ admits a continuous right adjoint (it will be automatically a strict functor of $\text{QCoh}(S)$ -modules, as $\text{QCoh}(S)$ is rigid).

The unit map from Section 4.24 is a 1-morphism in $\text{ShvCat}/_{\text{Ran}_{\mathcal{X}}}$, no lax prestacks here!

0.0.21. For 4.25.1(5). It seems this comes from the following. Let $Y, C \in 1 - \text{Cat}$. Assume given functors $f : Y \rightarrow C$ and $g : Y^{\text{Spc}} \rightarrow C$. Write $i : Y^{\text{Spc}} \rightarrow Y$ for the natural functor. Assume given a 1-morphism $\alpha : g \rightarrow f \circ i$ in $\text{Fun}(Y^{\text{Spc}}, C)$ such that for any $y \in Y$, $\alpha(y) : g(y) \rightarrow f(y)$ is a monomorphism in C . Assume also for any 1-morphism $h : y_1 \rightarrow y_2$ in Y the induced map $f(h) : f(y_1) \rightarrow f(y_2)$ in C is such that the composition

$$g(y_1) \xrightarrow{\alpha(y_1)} f(y_1) \xrightarrow{f(h)} f(y_2)$$

factors through $g(y_2) \xrightarrow{\alpha(y_2)} f(y_2)$. Then one gets a functor $g' : Y \rightarrow C$ together with a morphism $g' \rightarrow f$ in $\text{Fun}(Y, C)$ such that the composition $Y^{\text{Spc}} \rightarrow Y \xrightarrow{g'} C$ is g .

0.0.22. For 4.27. Given $\mathcal{X} \in \text{PreStk}$, one first defines $[\text{Ran}_{\mathcal{X}, \emptyset} \times \text{Ran}_{\mathcal{X}, \emptyset}]_{disj} \in \text{PreStk}$ so that its S -points is

$$\text{colim}_{I, J \in f\text{Set}_0^{op}} [\mathcal{X}^I \times \mathcal{X}^J]_{disj}$$

Given prestack \mathcal{X} , the lax prestack $[\text{Ran}_{\mathcal{X}}^{un} \times \text{Ran}_{\mathcal{X}}^{un}]_{disj}$ is defined as follows. For each $S \in \text{Sch}^{aff}$, the space underlying $(\text{Ran}_{\mathcal{X}}^{un} \times \text{Ran}_{\mathcal{X}}^{un})(S)$ is $(\text{Ran}_{\mathcal{X}, \emptyset} \times \text{Ran}_{\mathcal{X}, \emptyset})(S)$. We have to precise a full subspace in the latter space, which defines then the corresponding full subcategory. This full subspace is $[\text{Ran}_{\mathcal{X}, \emptyset} \times \text{Ran}_{\mathcal{X}, \emptyset}]_{disj}(S)$.

0.0.23. The proof of his Lemma 5.10.1. Let $F : \mathcal{J}^{op} \rightarrow \mathcal{P}r^L$ be a functor, $\mathcal{P}r^L$ is the notation from HTT, and \mathcal{J} admits fibre products. Let $q : Groth(F) \rightarrow \mathcal{J}^{op}$ be the cocartesian fibration corresponding to F . Let $y_1 \leftarrow y \rightarrow y_2$ be a diagram in $Groth(F)$, let $i_1 \leftarrow i \rightarrow i_2$ be its image in \mathcal{J}^{op} , so $i_1 \rightarrow i \leftarrow i_2$ is the corresponding diagram in \mathcal{J} . Let $\bar{i} = i_1 \times_i i_2$ in \mathcal{J} . By (HTT, 4.3.1.11) there is a diagram

$$(5) \quad \begin{array}{ccc} y_1 & \rightarrow & \bar{y} \\ \uparrow & & \uparrow \\ y & \rightarrow & y_2 \end{array}$$

which is a q -colimit diagram, and lifts the diagram $\bar{i} = i_1 \sqcup_i i_2$ in \mathcal{J}^{op} . Now from (HTT, 4.3.1.5) we see that (5) is a colimit diagram in $Groth(F)$. So, $Groth(F)$ admits pushouts, and q preserves push-outs.

0.0.24. In Sect. 5.13 it is understood that in the last displayed formula the middle diagram is a push-out diagram.

0.0.25. For 5.15. A version with a module would be as follows. Let C be a symmetric monoidal $(\infty, 1)$ -category, $M \in 1 - \mathcal{C}at$ be a C -module. Let $\Phi_C : C \rightarrow 1 - \mathcal{C}at$ be right-lax symmetric monoidal, $\Phi_M : M \rightarrow 1 - \mathcal{C}at$ be a functor such that the map Φ from (C, M) to $(1 - \mathcal{C}at, 1 - \mathcal{C}at)$ is right-lax (in the sense of [1], ch. I.1, 3.5.1), that is, Φ_M is right-lax compatible with actions. Let $f_C : \mathcal{X}_C \rightarrow C, f_M : \mathcal{X}_M \rightarrow M$ be the corresponding cocartesian fibrations. Then \mathcal{X}_C is symmetric monoidal, f_C is symmetric monoidal, and \mathcal{X}_C acts naturally on \mathcal{X}_M . Namely, $(c \in C, y \in \Phi_C(c))$ sends $(m \in M, z \in \Phi_M(m))$ to $(cm \in M, y \circ z \in \Phi_M(cm))$, where $y \circ z$ is the image of (y, z) under the natural map $\Phi_C(c) \times \Phi_M(m) \rightarrow \Phi_M(cm)$. So, the diagram commutes

$$\begin{array}{ccc} \mathcal{X}_C \times \mathcal{X}_M & \rightarrow & \mathcal{X}_M \\ \downarrow & & \downarrow \\ C \times M & \rightarrow & M \end{array}$$

0.0.26. For 5.15 more! Let $\mathcal{O}^\otimes \rightarrow \mathcal{F}in_*$ be an ∞ -operad. Let $\mathcal{M} : \mathcal{O}^\otimes \rightarrow 1 - \mathcal{C}at$ be an \mathcal{O} -monoid in $1 - \mathcal{C}at$ in the sense of ([6], 2.4.2.1). Let $\mathcal{Y}^\otimes \xrightarrow{\pi} \mathcal{O}^\otimes$ be the cocartesian fibration attached to \mathcal{M} . Then \mathcal{Y}^\otimes is an \mathcal{O} -monoidal category, in particular the composition $\mathcal{Y}^\otimes \rightarrow \mathcal{O}^\otimes \rightarrow \mathcal{F}in_*$ is an ∞ -operad by ([6], 2.4.2.4).

We may always consider the category of cocartesian sections $\mathcal{O}^\otimes \rightarrow \mathcal{Y}^\otimes$ of π , they are automatically morphisms of ∞ -operads. This is the category $\mathcal{F}un_{\mathcal{O}}^\otimes(\mathcal{O}, \mathcal{Y})$ of \mathcal{O} -monoidal functors from \mathcal{O} to \mathcal{Y} .

For 5.16. The assumption here should be as follows: $\mathcal{J} \in 1 - \mathcal{C}at$ admits fibre products, and is viewed as a category with directions, where the horizontal and vertical arrows are arbitrary. We then assume that $\mathcal{J} \in \mathcal{C}Alg(\mathcal{C}at_{dir})$, a commutative algebra object of the category of categories with directions, so $\mathcal{J}_{corr} \in \mathcal{C}Alg(1 - \mathcal{C}at)$.

0.0.27. Let Y be a lax prestack. We sometimes want to calculate the limit of the composition $Tw(\mathcal{S}ch^{aff})^{op} \rightarrow (\mathcal{S}ch^{aff})^{op} \rightarrow 1 - \mathcal{C}at$, that is,

$$\lim_{(S \rightarrow S') \in (Tw(\mathcal{S}ch^{aff}))^{op}} Y(S)$$

For any $S \in \text{Sch}^{aff}$, the category $Tw(\text{Sch}^{aff}) \times_{\text{Sch}^{aff}} \text{Sch}_{/S}^{aff}$ has an initial object $S \leftarrow \emptyset \rightarrow *$, hence is contractible. This implies that the RKE of the above composition under $Tw(\text{Sch}^{aff})^{op} \rightarrow (\text{Sch}^{aff})^{op}$ is the same functor $Y : (\text{Sch}^{aff})^{op} \rightarrow 1 - \text{Cat}$. So,

$$\lim_{(S \rightarrow S') \in (Tw(\text{Sch}^{aff}))^{op}} Y(S) \xrightarrow{\sim} \lim_{S \in (\text{Sch}^{aff})^{op}} Y(S)$$

For example, let Y be the constant lax prestack with value $C \in 1 - \text{Cat}$. Then

$$\text{ShvCat}_{/Y}^{naive} \xrightarrow{\sim} \lim_{S \in (\text{Sch}^{aff})^{op}} \text{Fun}(C, \text{ShvCat}(S)) \xrightarrow{\sim} \text{Fun}(C, \text{DGCat}_{cont})$$

by (4). Indeed, Sch^{aff} has a final object $\text{Spec } k$. Similarly,

$$\text{ShvCat}_{/Y} \xrightarrow{\sim} \lim_{S \in (\text{Sch}^{aff})^{op}} (\text{Fun}(C, \text{ShvCat}(S))_{llax})^{1-\text{Cat}} \xrightarrow{\sim} (\text{Fun}(C, \text{DGCat}_{cont})_{llax})^{1-\text{Cat}}$$

We get for the constant lax prestack Y with value $C \in 1 - \text{Cat}$, $\text{QCoh}(Y) \xrightarrow{\sim} \text{Fun}(C, \text{Vect})$.

Let Y be the constant prestack with value $C \in 1 - \text{Cat}$. Then $\text{QCoh}_Y \in \text{ShvCat}_{/Y}^{naive}$ becomes an object of $\text{Fun}(C, \text{DGCat}_{cont})$, namely the constant functor $C \rightarrow * \rightarrow \text{DGCat}_{cont}$ with value Vect .

If Y is a lax prestack let Y^{inv} be the prestack obtained by termwise inverting all arrows: for $S \in \text{Sch}^{aff}$, $Y^{inv}(S)$ is obtained from $Y(S)$ by inverting all arrows. We have a natural functor $\text{QCoh}(Y^{inv}) \rightarrow \text{QCoh}(Y)$. It is not essentially surjective in general. Already for $Y = [1]$ it is not.

0.0.28. For 5.19. My understanding is that this construction is only used when working with lax prestacks. If we want only to consider $\text{PreStk}_{corr}^{ShvCat}$ as a 2-category then already 5.14.1-5.14.2 are sufficient?

0.0.29. For 5.21.1. For each n we have a diagram $S^{\times n} \xleftarrow{m_1} \text{mult}_{n,S} \xrightarrow{m_2} S$, which realizes the n -ary multiplication in $\text{PreStk}_{corr}^{lax}$ on S . Then for a multiplicative sheaf of categories Ψ on S we have $m_1^*(\Psi^{\boxtimes n}) \xrightarrow{\sim} m_2^*(\Psi)$ over $\text{mult}_{n,S}$.

0.0.30. The notion of a weakly multiplicative sheaf of categories from 5.20.1 is very interesting in general. For example, let Y be a commutative algebra in PreStk . So, we have the product map $m : Y \times Y \rightarrow Y$ and the unit map $e : * \rightarrow Y$. Then this gives a sheaf of categories $C \in \text{ShvCat}(Y)$ together with maps $C \boxtimes C \rightarrow m^*C$, $\text{Vect} \rightarrow e^*C$ satisfying the compatibilities. What are nontrivial examples in $\text{MultCat}^w(Y)$? What happens for Y an affine scheme?

Note that $\text{MultCat}^w(Y)$ is symmetric monoidal ∞ -category. Example: let $Y = \text{Spec } k$, so $C \in \text{DGCat}_{cont}$ is equipped with $C \otimes C \rightarrow C$ and $\text{Vect} \rightarrow C$ making it into a commutative algebra in DGCat_{cont} . My understanding is that

$$\text{MultCat}^w(\text{Spec } k) \xrightarrow{\sim} \text{CAlg}(\text{DGCat}_{cont})$$

If $f : Y' \rightarrow Y$ is a map in $\text{CAlg}(\text{PreStk})$ and C is a weakly multiplicative sheaf of categories on Y then f^*C is a weakly multiplicative sheaf of categories on Y' , and $f^* : \text{MultCat}^w(Y) \rightarrow \text{MultCat}^w(Y')$ is symmetric monoidal. So, for a commutative algebra C in DGCat_{cont} we get for $g : Y \rightarrow \text{Spec } k$ the weakly multiplicative sheaf of categories g^*C .

Given a weakly multiplicative sheaf of categories D on $Y \in \mathcal{CAlg}(\text{PreStk})$, its restriction under $e : \text{Spec } k \rightarrow Y$ is an object of $\mathcal{CAlg}(\text{DGCat}_{\text{cont}})$.

0.0.31. For 5.24. Assume $S \in \mathcal{CAlg}(\text{PreStk}_{\text{corr}}^{\text{lax}})$ and $\Psi, \Psi' \in \text{MultCat}^w(S)$. To understand the difference between $\text{MultCat}^w(S)$ and $\text{MultCat}^{w,\text{lax}}(S)$ note the following. If $\alpha : \Psi \rightarrow \Psi'$ is a morphism in $\text{MultCat}^w(S)$ then we get a commutative diagram in $\text{ShvCat}_{/\text{mult}_S}$ (that is, an explicit invertible 2-morphism in the 2-category $\text{ShvCat}_{/\text{mult}_S}$ is given making the square commute)

$$\begin{array}{ccc} m_1^*(\Psi \boxtimes \Psi) & \xrightarrow{\eta} & m_2^*\Psi \\ \downarrow m_1^*(\alpha \boxtimes \alpha) & & \downarrow m_2^*(\alpha) \\ m_1^*(\Psi' \boxtimes \Psi') & \xrightarrow{\eta'} & m_2^*\Psi' \end{array}$$

Here η, η' are the corresponding structure maps for the weakly multiplicative sheaves of categories Ψ, Ψ' . If α is only a morphism in $\text{MultCat}^{w,\text{lax}}(S)$ then for the above square we are only given a 2-morphism

$$\delta : \eta' \circ m_1^*(\alpha \boxtimes \alpha) \rightarrow m_2^*(\alpha) \circ \eta$$

in $\text{ShvCat}_{/\text{mult}_S}$. So, we here essentially use the 2-category structure of $\text{ShvCat}_{/\text{mult}_S}$.

For example, if $\Psi = \text{QCoh}_S$ then the above map α is given by a section $s \in \Gamma(S, \Psi')$ together with a 1-morphism $\delta : \eta' m_1^*(s \boxtimes s) \rightarrow m_2^*s$ in $\Gamma(\text{mult}_S, m_2^*\Psi')$. If δ is an isomorphism then this is a map in $\text{MultCat}^w(S)$.

So, we see that already for $S \in \mathcal{CAlg}(\text{PreStk}_{\text{corr}})$ there is a big difference between $\text{MultCat}^w(S)$ and $\text{MultCat}^{w,\text{lax}}(S)$, as the DG -category $\Gamma(\text{mult}_S, m_2^*\Psi')$ is not a groupoid.

My understanding is that Sam means by morphisms in $\text{MultCat}^{w,\text{lax}}(S)$ the nonunital right-lax maps, that is, the above morphism from Ψ to Ψ' in $\text{MultCat}^{w,\text{lax}}(S)$ contains as a part of data a 2-morphism

$$\eta'_e \rightarrow (e_2^*\alpha) \circ \eta_e$$

in $\text{ShvCat}_{/\text{unit}_S}$ for the diagram

$$\begin{array}{ccc} e_1^* \text{Vect} & \xrightarrow{\eta_e} & e_2^*\Psi \\ & \searrow \eta'_e & \downarrow e_2^*\alpha \\ & & e_2^*\Psi' \end{array}$$

Here I used the notations from his Section 5.21, namely $* \xleftarrow{e_1} \text{unit}_S \xrightarrow{e_2} S$ is the unit map for the algebra S in $\text{PreStk}_{\text{corr}}^{\text{lax}}$.

0.1. For 5.26. The functor $\Gamma(-, -) : \text{Groth}(\text{ShvCat}_{/-}) \rightarrow \text{DGCat}_{\text{cont}}$ sends $(Y, C \in \text{ShvCat}_{/Y})$ to $\Gamma(Y, C)$. A map from (Y', C') to (Y, C) in $\text{Groth}(\text{ShvCat}_{/-})$ is a map of lax prestacks $f : Y \rightarrow Y'$ and $f^*C' \rightarrow C$ in $\text{ShvCat}_{/Y}$. The corresponding map $\Gamma(Y', C') \rightarrow \Gamma(Y, C)$ is the composition $\Gamma(Y', C') \rightarrow \Gamma(Y, f^*C') \rightarrow \Gamma(Y, C)$.

Similarly, consider the cocartesian fibration $\mathcal{X} \rightarrow \text{Groth}(\text{ShvCat}_{/-})$ corresponding to this functor. Then an object of \mathcal{X} is $(Y \in \text{PreStk}^{\text{lax}}, C \in \text{ShvCat}_{/Y}, \mathcal{F} \in \Gamma(Y, C))$. Consider a map $(Y_1, C_1) \rightarrow (Y_2, C_2)$ given by $f : Y_2 \rightarrow Y_1, f^*C_1 \rightarrow C_2$. A map from

$(Y_1, C_1, \mathcal{F}_1)$ to $(Y_2, C_2, \mathcal{F}_2)$ in \mathcal{X} over it is given in addition by a map $\mathcal{F} \rightarrow \mathcal{F}_2$ in $\Gamma(Y_2, C_2)$. Here \mathcal{F} is the image of $f^*\mathcal{F}_1$ under the induced map $\Gamma(Y_2, f^*C_1) \rightarrow \Gamma(Y_2, C_2)$.

0.1.1. For 5.30. To be clear, $Mult^{op-w}(\Psi)$ is the category of maps $QCoh_S \rightarrow \Psi$ in $MultCat^{op-w, lax}(S)$. What is the definition of $MultCat^{op-w, lax}(S)$? View S as a commutative coalgebra in $PreStk_{corr}^{lax}$ by flipping correspondences. Then $MultCat^{op-w, lax}(S)$ is the category of commutative coalgebras in $PreStk_{corr}^{lax, ShvCat}$ over S , where we allow lax morphisms of commutative coalgebras. However, should we allow "left-lax" or "right-lax" morphisms? According to the next paragraph, we have to allow the following. Given $\alpha : \Psi \rightarrow \Psi'$ in $MultCat^{op-w, lax}(S)$, we will have the square

$$\begin{array}{ccc} m_2^*\Psi' & \xrightarrow{\tilde{\eta}'} & m_1^*(\Psi' \boxtimes \Psi') \\ \uparrow m_2^*\alpha & & \uparrow m_1^*(\alpha \boxtimes \alpha) \\ m_2^*\Psi & \xrightarrow{\tilde{\eta}} & m_1^*(\Psi \boxtimes \Psi) \end{array}$$

This diagram only "lax commutes" in the sense that we are given a 2-morphism $\tilde{\eta}' \circ m_2^*\alpha \rightarrow m_1^*(\alpha \boxtimes \alpha) \circ \tilde{\eta}$. And similarly for the "counit" maps $\tilde{\eta}_e$.

Suppose the structure maps in $\Psi \in MultCat^{op-w}(S)$ are given by his (5.29.1), that is,

$$(6) \quad \begin{array}{l} \tilde{\eta}_m : m_2^*\Psi \rightarrow m_1^*(\Psi \boxtimes \Psi) \in ShvCat_{/mult_S} \text{ and} \\ \tilde{\eta}_e : e_2^*(\Psi) \rightarrow QCoh_{unit_\Psi} \in ShvCat_{/unit_S} \end{array}$$

Then an op-weakly multiplicative object in Ψ is an element $s \in \Gamma(S, \Psi)$ together with maps

$$\tilde{\eta}_m m_2^*(s) \rightarrow m_1^*(s \boxtimes s) \in \Gamma(mult_S, m_1^*(\Psi \boxtimes \Psi))$$

and

$$\tilde{\eta}_e e_2^*(s) \rightarrow k \in \Gamma(unit_S, QCoh)$$

with the corresponding compatibilities.

0.1.2. In Lemma 5.30.1 there is a mistake: the category $MultCat^{op-w}(S)$ is not presentable in general, though it probably has colimits and limits. This already happens for $S = *$, in which case this is the category of commutative coalgebras in $DGCat_{cont}$. The category $CCoalg(DGCat_{cont})$ admits colimits, and the forgetful functor $CCoalg(DGCat_{cont}) \rightarrow DGCat_{cont}$ preserves colimits by (HA, 3.2.2.4).

0.1.3. For 5.35. Assume $\mathcal{T} : (PreStk^{lax})^{op} \rightarrow 1 - Cat$ is a right-lax symmetric monoidal functor. Then Sam defined the notion of a multiplicative sheaf on $S \in CAlg(PreStk_{corr}^{lax})$ with values in \mathcal{T} . This is useful for considering other theories of sheaves as in ([2], Sect. 1.1).

0.1.4. For 5.27. For convenience, the definition of a weakly multiplicative object of a weakly multiplicative sheaf of categories: let $S \in CAlg(PreStk_{corr}^{lax})$, Ψ be a weakly multiplicative sheaf of categories on S . So, we have the map

$$\eta_m : m_1^*(\Psi \boxtimes \Psi) \rightarrow m_2^*\Psi$$

in $ShvCat_{/mult_S}$ and the map

$$\eta_e : e_1^* Vect \rightarrow e_2^*(\Psi)$$

in $ShvCat/unit_S$. A weakly multiplicative object ψ of Ψ (defined in his Section 5.25.1) gives rise to morphisms

$$\eta_m m_1^*(\psi \boxtimes \psi) \rightarrow m_2^* \psi$$

in $\Gamma(mult_S, m_2^* \Psi)$, and

$$\eta_e(\mathcal{O}_{unit_S}) \rightarrow e_2^* \psi$$

in $\Gamma(unit_S, e_2^* \Psi)$ with the corresponding compatibilities.

0.1.5. For 6.3. For $\mathcal{X} \in \text{PreStk}$, $[\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj}$ is a nonunital commutative algebra in PreStk_{corr} via the product

$$\begin{array}{ccc} [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \times [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} & \leftarrow & [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \\ & & \downarrow \\ & & [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \end{array}$$

Here in the middle we have the prestack given by 4 subsets of \mathcal{X} , which are pairwise disjoint. The diagram

$$\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}} \xleftarrow{m_1} [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \xrightarrow{add} \text{Ran}_{\mathcal{X}}$$

is a diagram of maps of nonunital commutative algebras in PreStk_{corr} . So, if C is a chiral category for \mathcal{X} , both $add^* C$ and $C \boxtimes C$ are multiplicative sheaves of categories on $[\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj}$. My understanding is that the isomorphism

$$m_1^*(C \boxtimes C) \xrightarrow{\sim} add^* C$$

on $[\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj}$ is an isomorphism of multiplicative sheaves of categories. Here $m_1 : [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \hookrightarrow \text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}$ is the inclusion. Similarly also in the unital case for $\text{Ran}_{\mathcal{X}}^{un}$. Besides, in the unital case the map $i : \text{Spec } k \xrightarrow{\emptyset} \text{Ran}_{\mathcal{X}}^{un}$ is a morphism of commutative algebras in $\text{PreStk}_{corr}^{lax}$, so $i^* C$ is a multiplicative sheaf of categories on $\text{Spec } k$. I think for a unital factorization category C the isomorphism $i^* C \xrightarrow{\sim} \text{Vect}$ is an isomorphism of multiplicative sheaves of categories.

This looks like the claim that $\mathcal{C}Alg(\mathcal{C}Alg(C))$ is isomorphic to C for $C \in 1 - \text{Cat}$. Sam says this is true for any $C \in 1 - \text{Cat}$.

A version of this for unital categories explains the fact that $unit_C$ is a factorization algebra in C . We get isomorphisms $i^* unit_C \xrightarrow{\sim} k$ and

$$\eta_m m_1^*(unit_C \boxtimes unit_C) \xrightarrow{\sim} add^* unit_C$$

If $C, D \in ShvCat/\text{Ran}_{\mathcal{X}}^{un}$ and $f : C \rightarrow D$ is a map in $ShvCat/\text{Ran}_{\mathcal{X}}^{un}$ then we get units $unit_C \in \Gamma(\text{Ran}_{\mathcal{X}}, C)$, $unit_D \in \Gamma(\text{Ran}_{\mathcal{X}}, D)$ as in his Section 4.24. Then f yields a natural map $unit_D \rightarrow f(unit_C)$ in $\Gamma(\text{Ran}_{\mathcal{X}}, D)$. Indeed, for $S \in \text{Sch}^{aff}$ and any $g \in \text{Ran}_{\mathcal{X}}(S)$ consider the morphism $\emptyset \xrightarrow{c} g$ in $\text{Ran}_{\mathcal{X}}^{un}(S)$. The description of f in his Section 4.17 gives a 2-morphism $unit_D \rightarrow f \circ unit_C$ for the square below

$$\begin{array}{ccc} \text{QCoh}_S & \xrightarrow{unit_C} & g^* C \\ \downarrow \text{id} & & \downarrow f \\ \text{QCoh}_S & \xrightarrow{unit_D} & g^* D \end{array}$$

The corresponding universal 2-morphism becomes the desired map $unit_D \rightarrow f(unit_C)$ in $\Gamma(\text{Ran}_{\mathcal{X}}, D)$.

Let now D be a unital chiral category for \mathcal{X} , and $s : \mathrm{QCoh} \rightarrow D$ be a morphism in $\mathrm{MultCat}^w(\mathrm{Ran}_{\mathcal{X}}^{un})$ giving a unital factorization algebra s in D . View s just as a map in $\mathrm{ShvCat}/\mathrm{Ran}_{\mathcal{X}}^{un}$. Note that $\mathcal{O}_{\mathrm{Ran}_{\mathcal{X}}} = \mathrm{unit}_{\mathrm{QCoh}} \in \Gamma(\mathrm{Ran}_{\mathcal{X}}, \mathrm{QCoh})$. The corresponding map $\mathrm{unit}_D \rightarrow f(\mathrm{unit}_{\mathrm{QCoh}}) = s$ is *the unit map* for the factorization algebra s as his map (6.3.2). He claims the latter is a map of unital factorization algebras in D . \hat{E}

0.2. For Appendix A. If $Y \in \mathrm{PreStk}$ then $\mathrm{ShvCat}(Y)$ has a natural structure of an $(\infty, 2)$ -category. For $Y \in \mathrm{Sch}^{aff}$ this is done in ([1], ch. I.1, 8.3). Namely, in this case the mapping category is the inner hom by ([1], ch. I.1, 8.3.3). Maybe for any prestack Y , the mapping category should be the inner hom for its symmetric monoidal structure? Not clear.

Given $f : S' \rightarrow S$ in Sch^{aff} , the functor $\mathrm{QCoh}(S) - \mathrm{mod} \rightarrow \mathrm{QCoh}(S') - \mathrm{mod}$ is a map in $2 - \mathrm{Cat}$, it is clear that we have the corresponding morphisms of the mapping categories (cf. [11], 6.0.6). Sam confirms this, as $2 - \mathrm{Cat}$ can be thought of as the category of $(\infty, 1)$ -categories enriched over $1 - \mathrm{Cat}$. So, the limit $\lim_{S \rightarrow Y} \mathrm{QCoh}(S) - \mathrm{mod}$ over $(\mathrm{Sch}_{/Y}^{aff})^{op}$ can be seen as a limit in $2 - \mathrm{Cat}$. Then according to ([11], 13.1.38), we get the mapping category for $C, D \in \mathrm{ShvCat}(Y)$

$$(7) \quad \mathbf{Map}_{\mathrm{ShvCat}(Y)}(C, D) \xrightarrow{\sim} \lim_{S \rightarrow Y} \mathbf{Map}_{\mathrm{ShvCat}(S)}(\Gamma(S, C), \Gamma(S, D))$$

the limit over $(\mathrm{Sch}_{/Y}^{aff})^{op}$. Here for $S \in \mathrm{Sch}^{aff}$, $E_i \in \mathrm{QCoh}(S) - \mathrm{mod}$,

$$\mathbf{Map}_{\mathrm{ShvCat}(S)}(E_1, E_2) = \mathrm{Fun}_{\mathrm{QCoh}(S)}(E_1, E_2)$$

is the inner hom in $\mathrm{ShvCat}(S)$. The limit (7) can be seen as a limit in DGCat_{cont} , as the projection $\mathrm{DGCat}_{cont} \rightarrow 1 - \mathrm{Cat}$ preserves limits. So, $\mathrm{ShvCat}(Y)$ is enriched over DGCat_{cont} . My understanding is that the 2-category $\mathrm{ShvCat}(Y)$ is actually enriched over $\mathrm{QCoh}(Y) - \mathrm{mod}$.

For $C \in \mathrm{ShvCat}(Y)$ we get $\mathbf{Map}_{\mathrm{ShvCat}(Y)}(\mathrm{QCoh}_Y, C) \xrightarrow{\sim} \Gamma(Y, C)$ canonically.

Actually, a stronger claim holds. Namely, for $C, D \in \mathrm{ShvCat}/_Y$ there is an inner hom from C to D in the symmetric monoidal category $\mathrm{ShvCat}/_Y$. Indeed, for any $S' \rightarrow S \rightarrow Y$ with $S, S' \in \mathrm{Sch}^{aff}$, there is a canonical isomorphism

$$\mathrm{Fun}_{\mathrm{QCoh}(S)}(\Gamma(S, C), \Gamma(S, D)) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S') \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{QCoh}(S')}(\Gamma(S', C), \Gamma(S', D))$$

It holds because $\mathrm{QCoh}(S), \mathrm{QCoh}(S')$ are rigid by ([11], 6.0.6). So, we get the inner hom $\mathcal{H}om(C, D) := \underline{\mathrm{Hom}}_{\mathrm{ShvCat}/_Y}(C, D)$ whose sections over S are

$$\mathrm{Fun}_{\mathrm{QCoh}(S)}(\Gamma(S, C), \Gamma(S, D))$$

Now (7) is the category of global sections of $\underline{\mathrm{Hom}}_{\mathrm{ShvCat}/_Y}(C, D)$. We may simply say that $\mathrm{ShvCat}/_Y$ is a 2-category, because it admits inner homs, which are enriched over DGCat_{cont} .

More generally, we may view the functor $\mathrm{ShvCat}/_Y : (\mathrm{Sch}^{aff})^{op} \rightarrow 1 - \mathrm{Cat}$, $S \mapsto \mathrm{ShvCat}(S)$ as a functor $\mathrm{ShvCat}/_Y : (\mathrm{Sch}^{aff})^{op} \rightarrow 2 - \mathrm{Cat}$, and consider its RKE as a functor $\mathrm{ShvCat}/_Y : \mathrm{PreStk}^{op} \rightarrow 2 - \mathrm{Cat}$. In particular, for any morphism $f : Y' \rightarrow Y$ in PreStk , we get a morphism $f^* : \mathrm{ShvCat}/_Y \rightarrow \mathrm{ShvCat}/_{Y'}$ of 2-categories.

Assume $f : X \rightarrow Y$ is 1-affine (cf. A.8). Then the functor $f_* : ShvCat_{/X} \rightarrow ShvCat_{/Y}$ from Prop. A.9.1(2) is a morphism in $2 - \mathcal{C}at$. Indeed, assume first Y affine. Then f_* is the forgetful functor $QCoh(X) - mod \rightarrow QCoh(Y) - mod$. This is a map in $2 - \mathcal{C}at$ by ([11], 6.0.6). Now let Y be arbitrary, $C, D \in ShvCat_{/X}$. We claim that there is a canonical map

$$f_* \mathcal{H}om(C, D) \rightarrow \mathcal{H}om(f_* C, f_* D)$$

in $ShvCat_{/Y}$. Indeed, from adjointness and the projection formula, we get a natural map

$$f_* C \otimes f_* \mathcal{H}om(C, D) \xrightarrow{\sim} f_*(\mathcal{H}om(C, D) \otimes f^* f_* C) \rightarrow f_*(\mathcal{H}om(C, D) \otimes C) \rightarrow f_* D$$

Finally, passing to the global sections, we get the desired functor between the mapping categories.

0.2.1. For A.2.3. He uses ([1], ch. I.1, Lm 5.4.3) to see that $C \otimes_{A-mod} B - mod \rightarrow C$ is conservative.

0.2.2. For A.3. Let $f : X \rightarrow Y$ be a morphism in $PreStk$. Then I think in general $coind_f : ShvCat(X) \rightarrow ShvCat(Y)$ does not preserve all colimits. What happens for $f : \bigsqcup_{n \geq 0} * \rightarrow *$?

0.2.3. For A.4.3. If $F : D \rightarrow C$ is a map of A -module categories, this is a map in $\mathcal{A} - mod(DGCat_{cont})$ with $\mathcal{A} = A - mod$. So, F is continuous. Let $\mathcal{X} \subset D \otimes_{A-mod} B - mod$ be the full subcategory of those x such that for any $y \in D \otimes_{A-mod} B - mod$ the map $Map_{D \otimes_{A-mod} B - mod}(x, y) \rightarrow Map_{C \otimes_{A-mod} B - mod}(F_B(x), F_B(y))$ is an isomorphism. Then it is easy to see that \mathcal{X} is closed under colimits (since F_B preserves colimits). Since the image of the functor $D \rightarrow D \otimes_{A-mod} B - mod$ generates $D \otimes_{A-mod} B - mod$ under colimits, it suffices indeed to show that (A.4.1) is an isomorphism.

0.2.4. For A.4.5. The projection $DGCat_{cont} \rightarrow 1 - \mathcal{C}at$ preserves limits (cf.[11]). The limit of the constant family in $1 - \mathcal{C}at$ over a contractible category is the same constant category. (Contractibility is needed: for example take $\mathcal{J} = * \sqcup *$).

0.2.5. For A.5. If $Spec A \in Sch^{aff}$, $F : C \rightarrow D$ is a morphism of A -linear categories then $F(C) \subset D$ is the smallest cocomplete presentable subcategory containing the objects $F(x), x \in C$. Then $F(C) \in DGCat_{cont}$ and is preserved under $A - mod$ -action on D , so $F(C)$ is an A -linear category.

If $F : C \rightarrow D$ is a morphism in $ShvCat(Y)$ for some $Y \in PreStk$ then $F(C) \in ShvCat(Y)$ is defined as follows. For $S \in Sch_{/Y}^{aff}$ we let $\Gamma(S, F(C)) = F(\Gamma(S, C))$. Here we also denoted by $F : \Gamma(S, C) \rightarrow \Gamma(S, D)$ the corresponding S -linear functor, and $F(\Gamma(S, C)) \subset \Gamma(S, D)$ is the presentable stable subcategory generated under colimits by the image of $\Gamma(S, C)$. By A.5.1, this is indeed a sheaf of categories on Y , and $F(C) \subset D$ is locally fully faithful.

0.2.6. For A.6. Recall that for $Y \in \text{PreStk}$, $\text{ShvCat}(Y)$ admits all colimits (computed locally). So, for a map $C \rightarrow D$ in $\text{ShvCat}(Y)$, we get $D/C \in \text{ShvCat}(Y)$. For $S \rightarrow Y$ with $S \in \text{Sch}^{aff}$, $\Gamma(S, D/C) \xrightarrow{\sim} \Gamma(S, D)/\Gamma(S, C)$, the latter colimit is taken in $\text{QCoh}(S) - \text{mod}$.

0.2.7. Let $S = \text{Spec } A \in \text{Sch}^{aff}$, $h : C^0 \subset C$ a diagram in $\mathcal{A} - \text{mod}(\text{DGCat}_{cont})$ with $\mathcal{A} = A - \text{mod}$. In particular, h is continuous. The quotient C/C_0 can also be calculated as a limit, namely let $h^R : C \rightarrow C^0$ be the right adjoint to h . It is a strict functor of \mathcal{A} -module categories, but h^R need not be continuous. Since colimits in DGCat_{cont} can be rewritten by passing to right adjoints as limits in DGCat , we get $C/C_0 \xrightarrow{\sim} 0 \times_{C^0} C = \text{Ker}(h^R)$. So, the evaluation $e : C/C_0 \rightarrow C$ is fully faithful, hence conservative. It follows that its left adjoint $p : C \rightarrow C/C_0$ generates C/C_0 under colimits by ([1], ch. I.1, Lm.5.4.3), and p is a localization functor.

Lemma 0.2.8. *Under the above assumptions, the kernel of $p : C \rightarrow C/C_0$ is C^0 .*

Proof. Set $D = \text{Ker } p$, so $C^0 \subset D \subset C$ are full subcategories. By definition, $D = \{x \in C \mid \text{for any } y \in C/C_0, \text{Map}_C(x, y) = *\}$. For $x \in D, y \in C/C_0$ we get $\text{Maps}_{k,C}(x, y) = 0$, because p is Vect-linear. So, $D = \{x \in C \mid \text{for any } y \in C/C_0, \text{Maps}_{k,C}(x, y) = *\}$.

Given $x \in C$, consider the fibre sequence $hh^R(x) \xrightarrow{\alpha} x \xrightarrow{\beta} y$ in C , where α is the natural map. Applying the exact functor h^R to this sequence, we get $h^R(x) \xrightarrow{\text{id}} h^R(x) \rightarrow 0$, so $y \in \text{Ker}(h^R) \xrightarrow{\sim} C/C_0$. We claim that $\beta : x \rightarrow y$ is isomorphic to the natural map $x \rightarrow ep(x)$ in C . Indeed, $p(\beta)$ is an isomorphism, and $p(y) \xrightarrow{\sim} y$. Assume in addition $x \in D$. Then $0 \xrightarrow{\sim} p(x)$, so the corresponding map $x \rightarrow ep(x)$ is the map $x \rightarrow 0$, hence α is an isomorphism. So, $x \in C^0$. (This also follows from [13], Lemma 1.8.15). \square

0.2.9. For Lm. A.7.4. The projection $\mathcal{A} - \text{mod}(\text{DGCat}_{cont}) \rightarrow \text{DGCat}_{cont}$ preserves colimits for $\mathcal{A} = A - \text{mod}$. Let $F : D \rightarrow C$ be a map in DGCat_{cont} , $C/D = \{0\} \sqcup_D C$ in DGCat_{cont} . The diagram $D \rightarrow F(D) \rightarrow C$ shows that we have a natural map $C/D \rightarrow C/\bar{D}$. To check this is an equivalence, for any $E \in \text{DGCat}$ we check that $\text{Map}(C/\bar{D}, E) \rightarrow \text{Map}(C/D, E)$ is an isomorphism, where $\text{Map} = \text{Map}_{\text{DGCat}_{cont}}$. This is the space $\text{Map}(C, E) \times_{\text{Map}(D, E)} \{0\}$, the full subspace in $\text{Map}(C, E)$ of those functors whose restriction to \bar{D} is isomorphic to zero. Indeed, $D \rightarrow F(D)$ generates $F(D)$ under colimits, and our functors are continuous.

0.2.10. For Pp. A.7.3. For each $i \in \mathcal{J}$ we have a full subcategory $\text{Ker}(C_{i_0} \rightarrow C_i) \subset C_{i_0}$. Since \mathcal{J} is filtered, it is contractible. Maybe it can be checked that $\text{colim}_{i \in \mathcal{J}} \text{Ker}(C_{i_0} \rightarrow C_i)$ is a full subcategory in C_{i_0} , but we don't need this because of Lemma A.7.4.

0.2.11. For Th. A.8.1. The category Sch_{aft} of schemes almost of finite type was defined in ([1], ch. I.3, 3.5.1). The claim is for $S \in \text{Sch}_{aft}$, S_{dR} is 1-affine.

In A.8.2 he means schematic morphism.

0.2.12. For A.9. For a morphism of prestacks $f : X \rightarrow Y$ the functor $f_* : \text{ShvCat}(X) \rightarrow \text{ShvCat}(Y)$ does not need to preserve colimits. The reason is that if $S \rightarrow Y$ with $S \in \text{Sch}^{aff}$, $S \times_Y X$ does not need to be affine. Already the functor $\Gamma_X^{enh} : \text{ShvCat}(X) \rightarrow \text{QCoh}(X) - \text{mod}$ does not preserve colimits in general.

For A.9.1(3). Let $f : X \rightarrow Y$ be quasi-compact (quasi-separated) schematic morphism in PreStk , $C \in \text{ShvCat}(Y)$. Since f is 1-affine, by A.9.1(2) it satisfies the projection formula. That is, given $S \rightarrow Y$ with $S \in \text{Sch}^{aff}$, $\Gamma(S, C) \rightarrow \Gamma(S \times_Y X, C)$ is the functor $\alpha : \Gamma(S, C) \rightarrow \Gamma(S, C) \otimes_{\text{QCoh}(S)} \text{QCoh}(S \times_Y X)$. This α admits a right adjoint, because the functor $f^* : \text{QCoh}(S) \rightarrow \text{QCoh}(S \times_Y X)$ admits a right adjoint $f_* : \text{QCoh}(S \times_Y X) \rightarrow \text{QCoh}(S)$, the latter functor is continuous by ([1], ch. I.3, 2.2.2). So, α^R is a continuous functor of $\text{QCoh}(S)$ -module categories. As S varies, they form a map $f_* f^*(C) \rightarrow C$ in $\text{ShvCat}(Y)$.

How do we see $\text{ShvCat}(Y)$ as a 2-category? If $S \in \text{Sch}^{aff}$ corresponding to an algebra $A \in \text{VAlg}(\text{Vect}^{\leq 0})$ then $\text{QCoh}(S) - \text{mod}$ is naturally an $(\infty, 2)$ -category, which I think is $(\text{QCoh}(S) - \text{mod}_{\text{cont}}^{\text{St,coimpl}})^{2\text{-Cat}}$ in the notations of ([1], ch. I.1, 8.3.1). Namely, for $C, D \in \text{QCoh}(S) - \text{mod}$, the mapping category is $\text{Fun}_A(C, D)$ with $A = \text{QCoh}(S) - \text{mod}$, the category of continuous $\text{QCoh}(S) - \text{mod}$ -linear functors as in ([1], ch. I.1, 8.2.1).

So, existence of a right adjoint to $C \rightarrow f_* f^*(C)$ means just that for any $S \rightarrow Y$ with $S \in \text{Sch}^{aff}$ the corresponding functor $\Gamma(S, f_* f^*(C)) \rightarrow \Gamma(S, C)$ is continuous (it is automatically $\text{QCoh}(S)$ -linear).

Now if $C, D \in \text{ShvCat}(Y)$, the mapping category from C to D in the corresponding 2-category is

$$\lim_{S \rightarrow Y} \text{Fun}_{\text{QCoh}(S) - \text{mod}}(\Gamma(S, C), \Gamma(S, D)),$$

the limit over $(\text{Sch}_Y^{aff})^{op}$.

0.2.13. For A.9.2. Let $f : Y \rightarrow Z$ be quasi-compact (quasi-separated) schematic morphism of prestacks, $C \in \text{ShvCat}(Z)$. Given a map $g : S' \rightarrow S$ in Sch^{aff}/Z , consider the diagram

$$\begin{array}{ccc} \Gamma(S, C) & \xrightarrow{\alpha_S} & \Gamma(S, C) \otimes_{\text{QCoh}(S)} \text{QCoh}(S \times_Z Y) \\ \downarrow g^* & & \downarrow g'^* \\ \Gamma(S', C) & \xrightarrow{\alpha_{S'}} & \Gamma(S', C) \otimes_{\text{QCoh}(S')} \text{QCoh}(S' \times_Z Y) \end{array}$$

the vertical maps being restrictions, and the horizontal ones are as in A.9.1(3). Let $\alpha_C^R, \alpha_{S'}^R$ denote the corresponding right adjoints, they are continuous, and by ([1], ch. I.3, 2.2.2) the base change gives $\alpha_{S'}^R g'^* \xrightarrow{\sim} g^* \alpha_S^R$. So, we may apply ([1], ch. I.1, Lm. 2.6.4) to see that the functor obtained by passing to the limit

$$f_C^* : \Gamma(Z, C) = \lim_{S \rightarrow Z} \Gamma(S, C) \rightarrow \lim_{S \rightarrow Z} \Gamma(S, C) \otimes_{\text{QCoh}(S)} \text{QCoh}(S \times_Z Y) = \Gamma(Z, f_* f^*(C))$$

admits a right adjoint $(f_C)_*$, and for any $S \rightarrow Z$ with $S \in \text{Sch}^{aff}$ the diagram commutes

$$\begin{array}{ccc} \Gamma(Z, C) & \xleftarrow{(f_C)_*} & \Gamma(Z, f_* f^*(C)) \\ \downarrow & & \downarrow \\ \Gamma(S, C) & \xleftarrow{\alpha_S^R} & \Gamma(S \times_Z Y, f^*(C)) \end{array}$$

Here the vertical arrows are evaluations (restrictions). Since each α_S^R is continuous, $(f_C)_*$ is also continuous, the transition maps in our projective system preserve limits, see ([11], 2.2.69).

0.2.14. For A.10.2. The proof is not clear. Let us show that for the map $p : Y \rightarrow *$ in PreStk^{lax} the functor $p^* : \text{DGCat}_{cont} = \text{ShvCat}_{/*}^{naive} \rightarrow \text{ShvCat}_Y^{naive}$ has a right adjoint. Clearly, p^* preserves colimits, however, ShvCat_Y^{naive} is not presentable usually, and we can not apply the adjoint functor theorem. The category $\text{Tw}(\text{Sch}^{aff})$ is contractible, since it has an initial object. Write ShvCat_Y^{naive} as (4). For an object $(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})$ we have an adjoint pair

$$res : \text{ShvCat}(*) \rightleftarrows \text{Fun}(Y(S'), \text{ShvCat}(S)) : \gamma_{S', S}$$

Here $\gamma_{S', S}$ is the composition $\text{Fun}(Y(S'), \text{ShvCat}(S)) \xrightarrow{\lim} \text{ShvCat}(S) \xrightarrow{\Gamma(S, \cdot)} \text{ShvCat}(*)$. Passing to the limit over $\text{Tw}(\text{Sch}^{aff})^{op}$ the functors res in the above yield our functor p^* . Given a map from $(S \xrightarrow{\alpha} S')$ to $(T \xrightarrow{\beta} T')$ in $\text{Tw}(\text{Sch}^{aff})$, we have a commutative diagram

$$\begin{array}{ccc} \text{ShvCat}(*) & \xrightarrow{res} & \text{Fun}(Y(S'), \text{ShvCat}(S)) \\ & \searrow_{res} & \uparrow_t \\ & & \text{Fun}(Y(T'), \text{ShvCat}(T)), \end{array}$$

the vertical arrows is the transition functor in our diagram indexed by $\text{Tw}(\text{Sch}^{aff})^{op}$. The functor t admits a right adjoint t^R , which is not continuous in general.

Consider the functor $\text{ShvCat}_Y^{naive} \rightarrow \text{DGCat}_{cont}$ sending C to

$$(8) \quad \Gamma^{naive}(Y, C) = \lim_{(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \Gamma(S, \lim_{Y(S')} C_\alpha),$$

where we denoted by C_α the image of C in $\text{Fun}(Y(S'), \text{ShvCat}(S))$. In the above formula $\Gamma(S, \lim_{Y(S')} C_\alpha) \xrightarrow{\sim} \lim_{Y(S')} \Gamma(S, C_\alpha)$. We claim that this is the right adjoint to p^* . Indeed, the mapping spaces in ShvCat_Y^{naive} are written as a similar limit over $\text{Tw}(\text{Sch}^{aff})^{op}$, and we get

$$\begin{aligned} \text{Map}_{\text{ShvCat}_Y^{naive}}(p^* D, C) &\xrightarrow{\sim} \lim_{(S \xrightarrow{\alpha} S')} \text{Map}_{\text{Fun}(Y(S'), \text{ShvCat}(S))}((p^* D)_\alpha, C_\alpha) \xrightarrow{\sim} \\ &\lim_{(S \xrightarrow{\alpha} S')} \text{Map}_{\text{DGCat}_{cont}}(D, \Gamma(S, \lim_{Y(S')} C_\alpha)) \xrightarrow{\sim} \text{Map}_{\text{DGCat}_{cont}}(D, \lim_{(S \xrightarrow{\alpha} S')} \Gamma(S, \lim_{Y(S')} C_\alpha)) \end{aligned}$$

Here $\lim_{(S \xrightarrow{\alpha} S')}$ stands for the limit over $\text{Tw}(\text{Sch}^{aff})^{op}$. We used the fact that there is a functor $\text{Tw}(\text{Sch}^{aff})^{op} \rightarrow \text{DGCat}_{cont}$ sending α as above to $\Gamma(S, \lim_{Y(S')} C_\alpha)$. Does $\Gamma^{naive}(Y, C)$ coincide with $\mathbf{Map}_{\text{ShvCat}_Y^{naive}}(\text{QCoh}_Y, C)$? Not clear.

Is $\Gamma^{naive}(Y, C)$ enriched over $\text{QCoh}(Y)$ naturally? One gets

$$\begin{aligned} \Gamma^{naive}(Y, \text{QCoh}_Y) &\xrightarrow{\sim} \lim_{(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \lim_{Y(S')} \text{QCoh}(S) \xrightarrow{\sim} \\ &\lim_{(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Fun}(Y^{inv}(S'), \text{QCoh}(S)) \xrightarrow{\sim} \text{QCoh}(Y^{inv}) \end{aligned}$$

Here $Y^{inv} \in \text{PreStk}$ is obtained by inverting all arrows in Y . We have the restriction functor $\text{QCoh}(Y^{inv}) \rightarrow \text{QCoh}(Y)$, which is not an equivalence in general.

We used the fact that for $C, D \in 1 - \mathcal{C}at$, the limit of the constant functor $\lim_C D$ with value D is $\text{Fun}(C^{inv}, D)$.

By functoriality, $\Gamma^{naive}(Y, C)$ is naturally a module over $\text{QCoh}(Y^{inv}) = \Gamma^{naive}(Y, \text{QCoh})$.

If now $f : Y \rightarrow T$, where $T \in \text{Sch}^{aff}$ then we get the direct image functor $f_* : \text{ShvCat}_{/Y}^{naive} \rightarrow \text{ShvCat}_{/T}$ as the composition

$$\text{ShvCat}_{/Y}^{naive} \rightarrow \text{QCoh}(Y^{inv}) - \text{mod}(\text{DGCat}_{cont}) \rightarrow \text{QCoh}(T) - \text{mod}(\text{DGCat}_{cont})$$

Let now $f : Y \rightarrow Z$ be a map in PreStk^{lax} , $D \in \text{ShvCat}_{/Z}^{naive}$, $C \in \text{ShvCat}_{/Y}^{naive}$. We get

$$\text{Map}_{\text{ShvCat}_{/Y}^{naive}}(f^*D, C) \xrightarrow{\sim} \lim_{(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})^{op}} \text{Map}_{\text{Fun}(Y(S'), \text{ShvCat}(S))}((f^*D)_\alpha, C_\alpha)$$

Given $S' \in \text{Sch}^{aff}$, f yields the functor $Y(S') \rightarrow Z(S')$, so an adjoint pair

$$\text{res}_\alpha : \text{Fun}(Z(S'), \text{ShvCat}(S)) \rightleftarrows \text{Fun}(Y(S'), \text{ShvCat}(S)) : \text{RKE}_\alpha$$

and $\text{res}_\alpha(D_\alpha) = (f^*D)_\alpha$. Thus the above mapping space rewrites as

$$\lim_{(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})^{op}} \text{Map}_{\text{Fun}(Z(S'), \text{ShvCat}(S))}(D_\alpha, \text{RKE}_\alpha C_\alpha)$$

Consider the functor $f_*^{naive} : \text{ShvCat}_{/Y}^{naive} \rightarrow \text{ShvCat}_{/Z}^{naive}$ sending C given by a projective system (C_α) as above to the functor $Tw(\text{Sch}^{aff})^{op} \rightarrow 1 - \mathcal{C}at$ given by

$$(9) \quad \alpha \mapsto \text{RKE}_\alpha(C_\alpha) \in \text{Fun}(Z(S'), \text{ShvCat}(S))$$

for $(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})$. Let finally

$$f_*^{naive} C = \lim_{(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})^{op}} \text{RKE}_\alpha(C_\alpha)$$

The above shows that

$$\text{Map}_{\text{ShvCat}_{/Y}^{naive}}(f^*D, C) \xrightarrow{\sim} \text{Map}_{\text{ShvCat}_{/Z}^{naive}}(D, f_*^{naive} C)$$

If we are given a map from $(S \xrightarrow{\alpha} S')$ to $(T \xrightarrow{\beta} T')$ in $Tw(\text{Sch}^{aff})$ then write $\delta_{\alpha, \beta} : \text{Fun}(Z(T'), \text{ShvCat}(T)) \rightarrow \text{Fun}(Z(S'), \text{ShvCat}(S))$ for the transition functor in the projective system defining $\text{ShvCat}_{/Z}^{naive}$. Then we only have a natural map

$$(10) \quad \delta_{\alpha, \beta}(\text{RKE}_\beta C_\beta) \rightarrow \text{RKE}_\alpha C_\alpha$$

in $1 - \mathcal{C}at$. This is what (9) says. The existence of the functor (9) is not rigorously justified. I think if for the above map $\alpha \rightarrow \beta$ in $Tw(\text{Sch}^{aff})$ the induced map $S' \rightarrow T'$ is an isomorphism then (10) is also an isomorphism.

Sam actually claims in A.10.4 that $\Gamma^{naive}(Y, C)$ is also given by a simpler formula:

$$(11) \quad \Gamma^{naive}(Y, C) = \lim_{S \in (\text{Sch}^{aff})^{op}} \lim_{y \in Y(S)} \Gamma(S, y^* C)$$

To be more precise, we consider the cocartesian fibration $\text{Groth}(Y) \rightarrow (\text{Sch}^{aff})^{op}$ corresponding to Y . Then the above limit is over $\text{Groth}(Y)$.

For $Y \in \text{PreStk}^{lax}$ formulas (8) and (11) give the same result. This follows from the contractibility claim proved in the next subsection.

0.2.15. For A.10.3. For a map $f : A \rightarrow B$ in $1 - \mathcal{C}at$ he says f is op-cofinal if $f^{op} : A^{op} \rightarrow B^{op}$ is cofinal.

Let Y be a lax prestack, $Groth(Y) \rightarrow (\text{Sch}^{aff})^{op}$ be the cocartesian fibration corresponding to Y . Let $coGroth(Y) \rightarrow \text{Sch}^{aff}$ be the cartesian fibration corresponding to Y . Consider the category $Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} coGroth(Y)$. It classifies $(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})^{op}$, and an object $y' \in Y(S')$. We have the map

$$(12) \quad Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} coGroth(Y) \rightarrow Groth(Y)$$

sending (α, y') as above to (S, y) , where y is the composition $S \xrightarrow{\alpha} S' \xrightarrow{y'} Y$. To clarify things, if $(S \xrightarrow{\alpha} S', y' \in Y(S'))$ and $(T \xrightarrow{\beta} T', y'' \in Y(T'))$ are two objects of $Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} coGroth(Y)$ then a map from the second to the first in this category is given by a map $\alpha \rightarrow \beta$ in $Tw(\text{Sch}^{aff})$, and a map $(T', y'') \rightarrow (S', y')$ in $coGroth(Y)$. This means that for the corresponding map say $\mu : T' \rightarrow S'$ in Sch^{aff} we are given a map $y'' \rightarrow y'\mu$ in $Y(T')$. Even the existence of the functor (12) is already interesting.

Is the map (12) op-cofinal? In other words,

$$Tw(\text{Sch}^{aff}) \times_{(\text{Sch}^{aff})^{op}} Groth(Y^{op}) \rightarrow coGroth(Y^{op})$$

is cofinal? We used the fact that $(Groth(Y))^{op} \xrightarrow{\sim} coGroth(Y^{op})$ canonically. Here Y^{op} denotes the lax prestack sending S to $Y(S)^{op}$.

Namely, given $(S, y) \in coGroth(Y^{op})$, consider the category classifying $(T \xrightarrow{\beta} T') \in Tw(\text{Sch}^{aff}), y' \in Y(T')^{op}$ and a map $(S, y) \xrightarrow{\xi} (T, y'\beta)$ in $coGroth(Y^{op})$. We have to show it is contractible. The map ξ is simply a map $\tau : S \rightarrow T$ in Sch^{aff} and a morphism $y \rightarrow y'\beta\tau$ in $Y(S)^{op}$. The desired contractibility is not clear (though clear say in the case when Y is constant lax prestack).

Is (12) a cartesian fibration? There seems no reason for that. Let us show that

$$(Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} coGroth(Y)) \times_{Groth(Y)} Groth(Y)_{(S,y)/} = \\ (Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} Groth(Y)_{(S,y)/})$$

is contractible, this is what we need. The projection $(Tw(\text{Sch}^{aff})^{op} \rightarrow \text{Sch}^{aff})$ is a cocartesian fibration, and $Groth(Y)_{(S,y)/}$ is contractible, as it has an initial object.

We have an adjoint pair

$$\text{pr} : (Tw(\text{Sch}^{aff})^{op} \rightleftarrows \text{Sch}^{aff}) : u,$$

where pr sends $(S \rightarrow S') \in Tw(\text{Sch}^{aff})$ to S' , and u sends S to $(\emptyset \rightarrow S)$. Moreover, the natural map $\text{pr} \circ u \rightarrow \text{id}$ is an equivalence, so u is fully faithful, pr is a localization. By ([11], 2.7.4), the projection $(Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} Groth(Y)_{(S,y)/} \rightarrow Groth(Y)_{(S,y)/})$ admits a right adjoint. Therefore, this projection induces an isomorphism of geometric realizations

$$|(Tw(\text{Sch}^{aff})^{op} \times_{\text{Sch}^{aff}} Groth(Y)_{(S,y)/})| \xrightarrow{\sim} |Groth(Y)_{(S,y)/}|$$

(by [11], 2.2.106). The RHS is $*$, and we are done!

0.2.16. For Lemma A.11.2 this is ([1], Cor. 3.2.7).

0.2.17. For A.12.1. I think a stronger claim holds: there is a functor $ShvCat_{/-}^{enh} : \text{PreStk}_{corr;all,1-aff} \rightarrow 2 - \text{Cat}$, whose composition with the forgetful functor $2 - \text{Cat} \rightarrow 1 - \text{Cat}$ is the one claimed by Sam. Here $2 - \text{Cat}$ is the $(\infty, 1)$ -category of $(\infty, 2)$ -categories. This is justified by my Section 0.2.

0.2.18. For A.13.3. here the colimit and limit are calculated in $ShvCat(Y)$, by tensor product $C \otimes_{\text{QCoh}_Y} \bar{C}$ he means the tensor product in the symmetric monoidal $ShvCat(Y)$.

For A.13.4(3). This is an analog of ([1], ch. I.1, 6.4.3).

A.13.4(1) clearly has an analog for $1 - \text{Cat}_{cont}^{St,cocmpl}$ instead of $ShvCat(Y)$.

For A.13.4(2). If the map (A.13.2) $D \otimes_{\text{QCoh}_Y} \bar{C} \rightarrow \lim_{i \in I^{op}} (D \otimes_{\text{QCoh}_Y} C_i^\vee)$ is an isomorphism for any D then we get as in the proof that for any $D_1, D_2 \in ShvCat(Y)$ one has $\text{Hom}(D_2, D_1 \otimes_{\text{QCoh}_Y} \bar{C}) \xrightarrow{\sim} \text{Hom}(D_2 \otimes_{\text{QCoh}_Y} C, D_1)$, where by Hom he means mapping spaces in $ShvCat(Y)$. This shows that C is dualizable.

0.3. For Sect. B.5. For any $[n] \in \mathbf{\Delta}$ we have $\text{Grid}_{[n]} \in 1 - \text{Cat}$. This is a functor $\mathbf{\Delta} \rightarrow 1 - \text{Cat}$, $[n] \mapsto \text{Grid}_{[n]}$. Namely, if $f : [m] \rightarrow [n]$ is a map in $\mathbf{\Delta}$, and $(i, j) \in \text{Grid}_{[m]}$, that is, $0 \leq i \leq j \leq m$ then $(f(i), f(j)) \in \text{Grid}_{[n]}$. Moreover, it sends horizontal (resp., vertical) arrows to horizontal (resp., vertical).

This is why for a category with directions $(\mathcal{C}, hor, vert)$, $[n] \mapsto \text{Grid}_{[n];hor,vert}(\mathcal{C})$ is indeed a functor $\mathbf{\Delta} \rightarrow \text{Spc}$. Namely, given a map $f : [m] \rightarrow [n]$ in $\mathbf{\Delta}$, we precompose a n -grid $\text{Grid}_{[n]}^{op} \rightarrow \mathcal{C}$ with $\text{Grid}_{[m]}^{op} \rightarrow \text{Grid}_{[n]}^{op}$.

0.3.1. For B.9. His Cat_{dir} is symmetric monoidal: Given $(C, hor_C, vert_C), (D, hor_D, vert_D)$, on $C \times D$ we get a structure of a category with directions: $hor_{C \times D} = hor_C \times hor_D$, and $vert_{C \times D} = vert_C \times vert_D$. Now $1 - \text{Cat}$ is equipped with the cartesian symmetric monoidal structure. Then $Tw : 1 - \text{Cat} \rightarrow \text{Cat}_{dir}$ is symmetric monoidal. In particular, for $C, D \in 1 - \text{Cat}$, $Tw(C \times D) \xrightarrow{\sim} Tw(C) \times Tw(D)$ canonically. The functor $\text{Cat}_{dir} \rightarrow 1 - \text{Cat}$, $(C, hor, vert) \mapsto C_{corr,hor,vert}$ is also symmetric monoidal. In particular, we get an adjoint pair

$$Tw : \text{Alg}(1 - \text{Cat}) \rightleftarrows \text{Alg}(\text{Cat}_{dir}) : corr$$

In fact, $1 - \text{Cat}$ and Cat_{dir} are naturally 2-categories, and so are $CAlg^{nu}(1 - \text{Cat})$, $CAlg^{nu}(\text{Cat}_{dir})$. For $D, D' \in \text{Cat}_{dir}$ the category of maps between them in Cat_{dir} is the full subcategory $\text{Fun}^{dir}(D, D') \subset \text{Fun}(D, D')$ classifying functors preserving horizontal (resp., vertical) morphisms, and sending cartesian products of $x \xrightarrow{a} y \xleftarrow{b} z$ with a horizontal and b vertical to cartesian squares.

Given $E, E' \in CAlg^{nu}(1 - \text{Cat})$, the mapping category in $CAlg^{nu}(1 - \text{Cat})$ from E to E' is $\text{Fun}^\otimes(E, E')$. Namely, if $E^\otimes \rightarrow \text{Surj}, E'^\otimes \rightarrow \text{Surj}$ are the cocartesian fibrations corresponding to E, E' then $\text{Fun}^\otimes(E, E') \subset \text{Fun}_{\text{Surj}}(E^\otimes, E'^\otimes)$ is the full subcategory of functors sending Surj -cocartesian arrows to cocartesian arrows.

My understanding is that the adjoint pair $Tw : 1 - \text{Cat} \rightleftarrows \text{Cat}_{dir} : corr$ is an adjoint pair of 2-categories, that is, we have

$$\text{Fun}(E, C_{corr;hor,vert}) \xrightarrow{\sim} \text{Fun}^{dir}(Tw(E), C)$$

for $C \in \mathcal{C}at_{dir}$, $E \in 1 - \mathcal{C}at$ naturally.

Moreover, the induced adjoint pair

$$Tw : CAlg^{nu}(1 - \mathcal{C}at) \rightleftarrows CAlg^{nu}(\mathcal{C}at_{dir})$$

is also an adjoint pair of 2-categories. For $D, D' \in CAlg^{nu}(\mathcal{C}at_{dir})$ the mapping category $\mathbf{Map}_{CAlg^{nu}(\mathcal{C}at_{dir})}(D, D')$ is the category $\text{Fun}^{\otimes, dir}(D, D')$ of those symmetric monoidal functors $f : D \rightarrow D'$ whose image in $\text{Fun}(D, D')$ lies in $\text{Fun}^{dir}(D, D')$. I hope for $E \in CAlg^{nu}(1 - \mathcal{C}at)$ and $D \in CAlg^{nu}(\mathcal{C}at_{dir})$ one has a natural equivalence

$$\text{Fun}^{\otimes}(E, C_{corr, hor, vert}) \xrightarrow{\sim} \text{Fun}^{\otimes, dir}(Tw(E), C)$$

in $1 - \mathcal{C}at$.

0.4. For 6.7. The natural functor $(fSet_{\emptyset})^{op} \hookrightarrow (Set_{<\infty})^{op}$ is not cofinal. Namely, given $J \in Set_{<\infty}$, the category $fSet_{\emptyset} \times_{Set_{<\infty}} (Set_{<\infty})_{/J}$ is not contractible in general. Namely, the geometric realization of the latter category is the disjoint union of points over all subsets $J_0 \subset J$.

For $\mathcal{G} \in \text{Spc}$ we get a natural map

$$\text{colim}_{I \in (Set_{<\infty})^{op}} \mathcal{G}^I \leftarrow \text{colim}_{I \in (fSet_{\emptyset})^{op}} \mathcal{G}^I = \text{Ran}_{\mathcal{G}, \emptyset}$$

Is it an isomorphism? I hope yes according to his Section 4.8.

In the definition of $fSet_I$ it is assumed that J is finite. The symmetric monoidal category $fSet_{\emptyset}$ act on $fSet_I$. The action map $a : fSet_{\emptyset} \times fSet_I \rightarrow fSet_I$ sends $(J', I \xrightarrow{\alpha} J)$ to $I \rightarrow J \sqcup J'$. So, we get $\mathcal{G}^{J'} \times \mathcal{G}^J \xrightarrow{\sim} \mathcal{G}^{J \sqcup J'}$. By adjointness, we get the canonical map

$$\left(\text{colim}_{J' \in fSet_{\emptyset}^{op}} \mathcal{G}^{J'} \right) \times \left(\text{colim}_{(I \rightarrow J) \in fSet_I^{op}} \mathcal{G}^J \right) \rightarrow \text{colim}_{(I \rightarrow J) \in fSet_I^{op}} \mathcal{G}^J$$

that is, $\text{Ran}_{\mathcal{G}, \emptyset} \times \text{Ran}_{\mathcal{G}, I} \rightarrow \text{Ran}_{\mathcal{G}, I}$. The canonical map $\text{Ran}_{\mathcal{G}, I} \rightarrow \mathcal{G}^I$ is invariant under the action of $\text{Ran}_{\mathcal{X}}^*$.

The functor $fSet_I \rightarrow fSet_{\emptyset}$, $(I \rightarrow J) \mapsto J$ yields by functoriality a morphism $\text{Ran}_{\mathcal{G}, I} \rightarrow \text{Ran}_{\mathcal{G}}$, which is $\text{Ran}_{\mathcal{G}}^*$ -equivariant.

The $\text{Ran}_{\mathcal{G}}^{ch}$ -module structure on $\text{Ran}_{\mathcal{G}, I}$ is given (via his example 6.6.3) by the correspondence

$$\text{Ran}_{\mathcal{G}} \times \text{Ran}_{\mathcal{G}, I} \leftarrow (\text{Ran}_{\mathcal{G}} \times \text{Ran}_{\mathcal{G}, I})_{disj} \rightarrow \text{Ran}_{\mathcal{G}, I}$$

0.4.1. If $\mathcal{X} \in \text{PreStk}$, C is a chiral category for \mathcal{X} , and $\mathcal{Z} \in \text{PreStk}$ is a $\text{Ran}_{\mathcal{X}}^{ch}$ -module in PreStk_{corr} , let $M \in \text{ShvCat}_{/\mathcal{Z}}$. Then a structure of a chiral C -module category on M over \mathcal{Z} means that $(\mathcal{Z}, M) \in \text{PreStk}_{corr}^{ShvCat}$ is equipped with a structure of a $(\text{Ran}_{\mathcal{X}}^{ch}, C)$ -module in the symmetric monoidal category $\text{PreStk}_{corr}^{ShvCat}$, and moreover the action map

$$\text{act}_1^*(C \boxtimes M) \rightarrow \text{act}_2^*(M)$$

is an isomorphism in $\text{ShvCat}_{/act_M}$. Here the diagram

$$\text{Ran}_{\mathcal{X}} \times \mathcal{Z} \xleftarrow{act_1} act_M \xrightarrow{act_2} M$$

defined the action map as in his 5.6.

0.4.2. For 6.10.4. I think $\text{Ran}_{\mathcal{X}}^{un, \rightarrow}, \text{Ran}_{\mathcal{X}, I}^{un}$ are not defined explicitly in the paper. For example, $\text{Ran}_{\mathcal{X}}^{un, \rightarrow}$ should be defined as follows I think: for \mathcal{X} a set consider the set $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ classifying pairs of finite subsets $S \subset T \subset \mathcal{X}$, here S, T could be empty. Now consider $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ as a partially ordered set, where $(S \subset T) \leq (S' \subset T')$ if $S \subset S'$ and $T \subset T'$. Then $\text{Ran}_{\mathcal{X}}^{un, \rightarrow}$ is the category associated to this poset. Then extend this definition to any $\mathcal{X} \in \text{Spc}$ by requiring that $\text{Spc} \rightarrow 1 - \text{Cat}$, $\mathcal{X} \mapsto \text{Ran}_{\mathcal{X}}^{un, \rightarrow}$ commutes with sifted colimits.

For a set \mathcal{X} , $\text{Ran}_{\mathcal{X}, I}$ should be the category associated to the following poset: its objects are (α, S) , where $S \subset \mathcal{X}$ is a finite subset, and $\alpha : I \rightarrow S$ is any map. Then $(\alpha, S) \leq (\alpha', S')$ iff $S \subset S'$ and the composition $I \xrightarrow{\alpha'} S' \rightarrow S$ equals α . Then probably we can extend this definition to any \mathcal{X} requiring that $\mathcal{X} \mapsto \text{Ran}_{\mathcal{X}, I}$ commutes with sifted colimits.

We can also proceed as in 4.8. Namely, define the category $f\text{Set}_{I, [n]}^{\rightarrow}$ (generalizing $f\text{Set}_{\emptyset, [n]}^{\rightarrow}$ from 4.8). Its objects are diagrams $I \xrightarrow{\alpha_I} I_0 \rightarrow I_1 \rightarrow \dots \rightarrow I_n$ with I_i maybe empty finite set. The morphisms are diagrams

$$\begin{array}{ccccccc} I_0 & \rightarrow & I_1 & \rightarrow & \dots & \rightarrow & I_n \\ \downarrow & & \downarrow & & & & \downarrow \\ J_0 & \rightarrow & J_1 & \rightarrow & \dots & \rightarrow & J_n, \end{array}$$

where the vertical maps are surjections. It is required that the above diagram is compatible with maps $I \xrightarrow{\alpha_I} I_0$ and $I \xrightarrow{\alpha_I} J_0$. Then proceed as in 4.8 to define $\text{Ran}_{\mathcal{X}, I}^{un} \in 1 - \text{Cat}$.

0.4.3. For 6.13.1, $\text{Ran}_{\mathcal{X}, I, \emptyset}$ is never defined? I think this is $\text{Ran}_{\mathcal{X}, I}$, where we added a disjoint point corresponding to empty set.

0.4.4. For 6.15. He starts with a commutative algebra in $\text{PreStk}_{\text{corr}}$, but an assumption is missing: he needs $m_2 : \text{mult}_S \rightarrow S$ and $e_2 : \text{unit}_S \rightarrow S$ to be 1-affine.

Under these assumptions, we get a symmetric monoidal convolution structure on $\text{ShvCat}_{/S}$ as he explains. Actually, $\text{ShvCat}_{/S}$ is a 2-category, and he claims implicitly that $\text{ShvCat}_{/S}$ is a symmetric monoidal 2-category. This means in particular, that the convolution $* : \text{ShvCat}_{/S} \times \text{ShvCat}_{/S} \rightarrow \text{ShvCat}_{/S}$ is a morphism in $2 - \text{Cat}$.

For 6.15.2. If $C \in \text{MultCat}^{op-w}(S)$ then we have maps $m_2^* C \rightarrow m_1^*(C \boxtimes C)$ in $\text{ShvCat}_{\text{mult}_S}$ and $e_2^* C \rightarrow \text{QCoh}_{\text{unit}_S}$ in $\text{ShvCat}_{\text{unit}_S}$. A map from C to D in $\text{MultCat}^{op-w}(S)$ gives a morphism $\alpha : C \rightarrow D$ in $\text{ShvCat}_{/S}$ in particular such that the diagram commutes (an explicit invertible 2-morphism is given)

$$\begin{array}{ccc} m_2^* C & \rightarrow & m_1^*(C \boxtimes C) \\ \downarrow m_2^* \alpha & & \downarrow \\ m_2^* D & \rightarrow & m_1^*(D \boxtimes D) \end{array}$$

Applying m_2^* , we get a diagram

$$(13) \quad \begin{array}{ccccc} C & \rightarrow & m_{2*} m_2^* C & \rightarrow & m_{2*} m_1^*(C \boxtimes C) \\ \downarrow \alpha & & \downarrow m_{2*} m_2^* \alpha & & \downarrow \\ D & \rightarrow & m_{2*} m_2^* D & \rightarrow & m_{2*} m_1^*(D \boxtimes D), \end{array}$$

Similarly for unit. This gives a coalgebra structure on C in the symmetric monoidal category $(\text{ShvCat}_{/S}, *)$.

Question. My understanding is that in (13) the outer square is commutative (that is, an explicit invertible 2-morphism in the 2-category $ShvCat/S$ is given). Is this true? This would mean probably that the corresponding map from C to D in $ShvCat/S$ is a true morphism of commutative algebras, and not a lax morphism. However, in the formulation of Prop. 6.15.2, $ComCoalg^{lax}$ appears. Maybe, $ComCoalg((ShvCat/S, *))$ is meant in the formulation of Prop. 6.15.2? Or maybe you meant $MultCat^{op-w,lax}(S)$ instead?

0.4.5. For 6.16. Let $S \in CAlg(\text{PreStk}_{corr})$ with e_2, m_2 1-affine. The unit in $(ShvCat/S, *)$ is $e_{2*} \text{QCoh}_{unit_S}$. If $C \in MultCat^{op-w,r.adj}(S)$ then the maps $\tilde{\eta}_m : m_2^* C \rightarrow m_1^*(C \boxtimes C)$ and $\tilde{\eta}_e : e_2^* C \rightarrow \text{QCoh}_{unit_S}$ admit right adjoints in $ShvCat_{mult_S}$ and in $ShvCat_{unit_S}$ respectively. For $\tilde{\eta}_m$ this means that for any $T \rightarrow mult_S$ with $T \in \text{Sch}^{aff}$ the corresponding right adjoint over T is continuous (similarly for $\tilde{\eta}_e$). So, the map

$$m_{2*} \tilde{\eta}_m : m_{2*} m_2^* C \rightarrow m_{2*} m_1^*(C \boxtimes C)$$

admits a right adjoint in $ShvCat/S$. Assume m_2, e_2 are quasi-compact quasi-separated schematic morphisms. Then $C \rightarrow m_{2*} m_2^* C$ admits a right adjoint in $ShvCat/S$ by his A.9.1(3). So, the composition $C \rightarrow m_{2*} m_1^*(C \boxtimes C) = C * C$ admits a right adjoint in $ShvCat/S$. Similarly, $C \rightarrow e_{2*} \text{QCoh}_{unit_S}$ has a right adjoint.

0.4.6. For 6.17. For clarity, if $\Psi \in MultCat^{op-w,r.adj}(S)$ then first we may see Ψ as an object of $ComCoalg^{r.adj}((ShvCat/S, *))$ via his Section 6.16, hence as an object of $CAlg^{l.adj}((ShvCat/S, *))$ via his (6.16.1). So, Ψ is a commutative algebra in $(ShvCat/S, *)$, hence $\Gamma(S, \Psi)$ inherits a symmetric monoidal structure, which he still denotes by $*$.

Explicitly, view the maps (6) as

$$\begin{aligned} \tilde{\eta}_m &: \Psi \rightarrow \Psi * \Psi \in ShvCat/S \text{ and} \\ \tilde{\eta}_e &: \Psi \rightarrow e_{2*} \text{QCoh}_{unit_S} \in ShvCat/S \end{aligned}$$

They define the corresponding object of $ComCoalg^{r.adj}((ShvCat/S, *))$. Let μ_m, μ_e be the functors right adjoint to $\tilde{\eta}_m$ and $\tilde{\eta}_e$ respectively. So,

$$(\Psi, \mu_m, \mu_e) \in CAlg^{l.adj}((ShvCat/S, *))$$

The symmetric monoidal structure $(\Gamma(S, \Psi), *)$ is obtained by passing to global sections in the latter.

0.4.7. In his Prop. 6.17.1 in the RHS one has to replace $ComAlg(\Gamma(S, \Psi), *)$ by

$$ComCoalg(\Gamma(S, \Psi), *)$$

0.4.8. For 6.19. Explicitly, if $C \in Cat^{ch}(X_{dR})$ the non-unital commutative algebra structure on $\Gamma(\text{Ran}_{X_{dR}}, C)$ in $DGCat_{cont}$ is as follows. For the diagram

$$\text{Ran}_{X_{dR}} \times \text{Ran}_{X_{dR}} \xleftarrow{m_1} (\text{Ran}_{X_{dR}} \times \text{Ran}_{X_{dR}})_{disj} \xrightarrow{m_2} \text{Ran}_{X_{dR}}$$

we have isomorphism $\eta_m : m_1^*(C \boxtimes C) \xrightarrow{\sim} m_2^* C$. We consider $\eta_m^{-1} : m_2^* C \rightarrow m_1^*(C \boxtimes C)$, hence a morphism $\eta_m^{-1} : C \rightarrow m_{2*} m_1^*(C \boxtimes C) = C * C$, so C is a non-unital commutative coalgebra in $(ShvCat_{\text{Ran}_{X_{dR}}}, *)$. Since m_2 is quasi-compact quasi-separated schematic

morphism, the map $\eta_m^{-1} : C \rightarrow C * C$ admits a right adjoint $\mu : C * C \rightarrow C$ by his Section 6.16, which makes C a non-unital commutative algebra in $(ShvCat_{\text{Ran}_{X_{dR}}}, *)$. In turn, $\Gamma(\text{Ran}_{X_{dR}}, C)$ inherits a non-unital symmetric monoidal structure with product $\overset{ch}{\otimes}$.

If $s \in \Gamma(\text{Ran}_{X_{dR}}, C)$ then $s \overset{ch}{\otimes} s = \mu(m_1^*(s \boxtimes s)) \in \Gamma(\text{Ran}_{X_{dR}}, C)$. If $s \in ComCoalg^{ch}(C)$, we get a map

$$s \rightarrow \mu(m_1^*(s \boxtimes s)) \in \Gamma(\text{Ran}_{X_{dR}}, C)$$

For 6.20. Given a factorization category C on X_{dR} , I don't see what is the map

$$\Gamma(X_{dR}, C |_{X_{dR}}) \rightarrow \Gamma(\text{Ran}_{X_{dR}}, C)$$

This is like extension by zero for \mathcal{D} -modules.

0.4.9. For 7.2.1. In the definition of a commutative weak chiral category he means it is a weakly multiplicative sheaf of categories on Ran_X^* . In fact, the map $\text{id} : \text{Ran}_X^{ch} \rightarrow \text{Ran}_X^*$ is a morphism in $CAlg(\text{PreStk}_{corr})$, so that his Section 5.34 applies, id^* is the desired functor.

0.4.10. For 7.7. Two lines before 7.8: replace Set_∞ by $Set_{<\infty}$.

If $S \in \text{Sch}^{aff}$ then $\mathcal{Z}(S)$ is the cocartesian fibration corresponding to $\text{Sets}_{<\infty} \rightarrow 1 - \text{Cat}$, $I \mapsto \text{Ran}_{X, disj}^{un, I}(S)$.

For 7.8. There is a correction needed in the second displayed formula, one has actually

$$ShvCat_{\text{Sets}_{<\infty}}^{naive} = \text{Fun}(\text{Sets}_{<\infty}, \text{DGCat}_{cont})$$

as for any constant lax prestack, see my Section 0.0.27. The functor $CAlg(\text{DGCat}_{cont}) \rightarrow ShvCat_{\text{Sets}_{<\infty}}^{naive}$, $I \mapsto D^{\otimes I}$ is fine.

0.4.11. For 7.12. For a surjection $p : I \rightarrow J$ of finite sets, the subscheme $U(p) \subset X_{dR}^I$ seems not defined. I think this is the scheme of collections $(x_i)_{i \in I}$ such that if $p(i) \neq p(i')$ then $(x_i, x_{i'}) \in X_{disj}^2$.

0.4.12. The construction of $\text{Loc}_{X_{dR}}(D)$ for $D \in CAlg(\text{DGCat}_{cont})$ from 7.6-7.13 is based on the following: The category $\text{Sets}_{<\infty}$ is symmetric monoidal under $(J_1, J_2) \mapsto J_1 \sqcup J_2$. Consider the functor $\text{Sets}_{<\infty} \rightarrow \text{DGCat}_{cont}$, $I \mapsto D^{\otimes I}$. If $I \rightarrow J$ is any map then the corresponding map $D^{\otimes I} \rightarrow D^{\otimes J}$ is given by the products along I_j for each $j \in J$. We view it as an object \mathcal{F} of $ShvCat_{\text{Sets}_{<\infty}}^{naive}$. Then for the above map $u : \text{Sets}_{<\infty} \times \text{Sets}_{<\infty} \rightarrow \text{Sets}_{<\infty}$ we get $u^* \mathcal{F} \xrightarrow{\sim} \mathcal{F} \boxtimes \mathcal{F}$, and for the empty set $e : * \rightarrow \text{Sets}_{<\infty}$ we have $e^* \mathcal{F} \xrightarrow{\sim} \text{Vect}$. So, $\text{Sets}_{<\infty}$ is naturally an object of $CAlg(1 - \text{Cat})$, and \mathcal{F} is a multiplicative sheaf of categories on $\text{Sets}_{<\infty}$.

It is this structure that yields a multiplicative structure on $\text{Loc}_{X_{dR}}(D)$. So, let $\mathcal{C} \in CAlg(1 - \text{Cat})$ and $F \in \text{Fun}(\mathcal{C}, \text{DGCat}_{cont})$ giving an object $\mathcal{F} \in ShvCat_{\mathcal{C}}^{naive}$, where \mathcal{C} is viewed as a constant lax prestack. For which F the sheaf \mathcal{F} is multiplicative? This is equivalent to $F : \mathcal{C} \rightarrow \text{DGCat}_{cont}$ being a symmetric monoidal functor.

0.4.13. For 7.15.5. If $f : Y \rightarrow Y'$ is a proper morphism between schemes of finite type then we have an adjoint pair $f_! : D(Y) \rightleftharpoons D(Y') : f^!$, where $f^!$ if the functor

$$f^* : \mathrm{QCoh}(Y'_{dR}) \rightarrow \mathrm{QCoh}(Y_{dR}) = D(Y),$$

and moreover $f_!$ is a strict morphism of $D(Y')$ -modules ([2], 1.5.2). Recall that Y'_{dR} is 1-affine by ([3], Th. 2.6.3), so $\mathrm{ShvCat}(Y'_{dR}) = D(Y') - \mathrm{mod}(\mathrm{DGCat}_{cont})$. Let $C \in \mathrm{ShvCat}(Y'_{dR})$, which we view as a $D(Y')$ -module. Let us explain that $f^* : \Gamma(Y'_{dR}, C) \rightarrow \Gamma(Y_{dR}, f^*C)$ has a left adjoint denoted $f_{*,dR,C}$.

This is a particular case of the following general observation. Let $h^! : B \rightarrow A$ be a map in $\mathrm{CAlg}(\mathrm{DGCat}_{cont})$ admitting a left adjoint $h_! : A \rightarrow B$, so $h_!$ is a map in DGCat_{cont} . Assume the left-lax monoidal structure on the functor $h_!$ of B -modules is strict, so $h_!$ is a morphism in $B - \mathrm{mod}(\mathrm{DGCat}_{cont})$, and the adjoint pair $h_! : A \rightleftharpoons B : h^!$ takes place in $B - \mathrm{mod}(\mathrm{DGCat}_{cont})$. Now if $C \in B - \mathrm{mod}(\mathrm{DGCat}_{cont})$ then we get the adjoint pair

$$h_C! : C \otimes_B A \rightleftharpoons C : h_C^!$$

in $B - \mathrm{mod}(\mathrm{DGCat}_{cont})$.

In our case $f^!$ yields a morphism $f^* : C \rightarrow C \otimes_{D(Y')} D(Y)$ in $D(Y') - \mathrm{mod}$. So, we get the left adjoint $f_{*,dR,C} : C \otimes_{D(Y')} D(Y) \rightarrow C$ to f^* .

For this reason if now Y is a pseudo-indscheme in the sense of Def. 7.15.1 then for $C \in \mathrm{ShvCat}/_Y$ we get $\Gamma(Y_{dR}, C) \xrightarrow{\sim} \mathrm{colim}_{j \in J} \Gamma(Y_{j,dR}, \psi_j^* C)$, the colimit taken in DGCat_{cont} . Here $Y = \mathrm{colim}_{j \in J} Y_j$, and $\psi_j : Y_j \rightarrow Y$ is the natural map.

The morphism $f_{*,dR,C}$ in his Pp-Construction 7.15.5 exists I think only if $f : Y \rightarrow Z$ is pseudo-indproper.

We underline that in the definition of a morphism of pseudo-ind-schemes in ([14], 7.15.2), we do not require some diagrams to be cartesian, there are no conditions. Here is a special case: assume $I \in 1 - \mathrm{Cat}$ and we are given a diagram $I \times [1] \rightarrow \mathrm{Sch}_{ft}$, where for $i \in I$ the corresponding morphism $f_i : Y_i \rightarrow Z_i$ is proper, and for $i \rightarrow j$ in I the transition maps $Y_i \rightarrow Y_j, Z_i \rightarrow Z_j$ are proper. For each i we have an adjoint pair $(f_i)_* : \Gamma(Y_i, f^*C) \rightleftharpoons \Gamma(Z_i, C) : (f_i)^*$ defined as above (for any sheaf theory). Let $Z = \mathrm{colim}_{i \in I} Z_i, Y = \mathrm{colim}_{i \in I} Y_i$. Then $\Gamma(Z, C) \xrightarrow{\sim} \lim_{i \in I} \Gamma(Z_i, C)$. The functor $f_C^* : \Gamma(Z, C) \rightarrow \Gamma(Y, f^*(C))$ is obtained by passing to \lim in $f_i^* : \Gamma(Z_i, C) \rightarrow \Gamma(Y_i, C)$. Then the left adjoint $f_{*,C} : \Gamma(Y, f^*(C)) \rightarrow \Gamma(Z, C)$ is obtained by passing to colim_i in the diagram $(f_i)_* : \Gamma(Y_i, f^*C) \rightarrow \Gamma(Z_i, C)$ by ([11], 9.2.39).

0.5. More about lax prestacks. Using appendix, for $\mathcal{C} \in 1 - \mathrm{Cat}$ and a functor $G : \mathcal{C}^{op} \rightarrow 1 - \mathrm{Cat}$ one has

$$(14) \quad \mathrm{oplaxlim} G = \lim_{(c' \rightarrow c) \in \mathrm{Tw}(\mathcal{C})^{op}} \mathrm{Fun}(\mathcal{C}_{c'}, G(c'))$$

Here $c' \rightarrow c$ is a morphism in \mathcal{C} . We used the identification $\mathrm{Tw}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Tw}(\mathcal{C}^{op})$. We use this in the following result.

Let $\mathcal{C} \in 1 - \mathrm{Cat}$ and $Y : \mathcal{C} \rightarrow \mathrm{PreStk}^{lax}$ be a functor, which we also view as $Y : \mathcal{C} \times (\mathrm{Sch}^{aff})^{op} \rightarrow 1 - \mathrm{Cat}$. For $c \in \mathcal{C}$ let $Y_c \in \mathrm{PreStk}^{lax}$ be its value on c . For $S \in \mathrm{Sch}^{aff}$ let $Y_S : \mathcal{C} \rightarrow 1 - \mathrm{Cat}$ be the restriction of Y , where we fix the corresponding variable.

Define the *relative lax colimit* of Y over $(\text{Sch}^{aff})^{op}$ as $\bar{Y} \in \text{PreStk}^{lax}$ given by

$$(15) \quad \bar{Y}(S) = \text{laxcolim } Y_S = \text{colim}_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})} \mathcal{C}_{c'} \times Y(S, c')$$

for $S \in \text{Sch}^{aff}$, here the colimit is taken in $1 - \text{Cat}$.

Lemma 0.5.1. *Under the assumptions of Section 0.5 we have canonical equivalences*
 1) $\text{QCoh}(\bar{Y}) \xrightarrow{\sim} \text{oplaxlim } G$ in DGCat_{cont} , where G is the composition

$$\mathcal{C}^{op} \xrightarrow{Y^{op}} (\text{PreStk}^{lax})^{op} \xrightarrow{\text{QCoh}} \text{DGCat}_{cont}$$

2) $\text{ShvCat}_{\bar{Y}}^{naive} \xrightarrow{\sim} \text{oplaxlim } G'$, where G' is the composition

$$\mathcal{C}^{op} \xrightarrow{Y^{op}} (\text{PreStk}^{lax})^{op} \xrightarrow{\text{ShvCat}_{\bar{Y}}^{naive}} \text{DGCat}_{cont}$$

Proof. 1) For $S \in \text{Sch}^{aff}$ using (15) and (1), we get

$$\begin{aligned} \text{QCoh}(\bar{Y}) &\xrightarrow{\sim} \lim_{(S' \rightarrow S) \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Fun}(\bar{Y}(S), \text{QCoh}(S')) \xrightarrow{\sim} \\ &\lim_{(S' \rightarrow S) \in \text{Tw}(\text{Sch}^{aff})^{op}} \lim_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})^{op}} \text{Fun}(\mathcal{C}_{c'}, \text{Fun}(Y(S, c'), \text{QCoh}(S'))) \xrightarrow{\sim} \\ &\lim_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})^{op}} \text{Fun}(\mathcal{C}_{c'}, \lim_{(S' \rightarrow S) \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S, c'), \text{QCoh}(S'))) \xrightarrow{\sim} \\ &\lim_{(c' \rightarrow c) \in \text{Tw}(\mathcal{C})^{op}} \text{Fun}(\mathcal{C}_{c'}, \text{QCoh}(Y_{c'})) \end{aligned}$$

By (14), we are done.

2) same argument. □

So, informally we may say that $\text{QCoh} : (\text{PreStk}^{lax})^{op} \rightarrow \text{DGCat}_{cont}$ not only preserves limits. It actually transforms relative lax colimits in PreStk^{lax} (over $(\text{Sch}^{aff})^{op}$) into oplax limits.

The above does not seem to work for $\text{ShvCat}_{\bar{Y}}$. Namely, For $T \in 2 - \text{Cat}$ the functor $(1 - \text{Cat})^{op} \rightarrow 1 - \text{Cat}$, $V \mapsto (\text{Fun}(V, T)_{llax})^{1 - \text{Cat}}$ preserves limits by ([11], 13.1.45). However, for $A, B \in 1 - \text{Cat}$, $T \in 2 - \text{Cat}$, $\text{Fun}(A \times B, T)_{llax}^{1 - \text{Cat}}$ does not seem to express as functors from A to $\text{Fun}(B, T)_{llax}^{1 - \text{Cat}}$ or something like that. For that, given $X \in 1 - \text{Cat}$, we have to better understand $X \otimes (A \times B) \in 2 - \text{Cat}$.

0.5.2. As in [14], write $Y \mapsto Y^{\text{PreStk}}$ for the functor $\text{PreStk}^{lax} \rightarrow \text{PreStk}$ right adjoint to the inclusion $\text{PreStk} \hookrightarrow \text{PreStk}^{lax}$. For $Y \in \text{PreStk}^{lax}$, $S \in \text{Sch}^{aff}$ we get $\text{Map}_{\text{PreStk}^{lax}}(S, Y) \xrightarrow{\sim} \text{Map}_{\text{PreStk}}(S, Y^{\text{PreStk}}) = Y(S)^{\text{SpC}}$.

1. MORE ON COMMUTATIVE CHIRAL CATEGORIES

1.1. Recall that $f\text{Set}_{<\infty}$ denotes the category of finite sets. For $D \in \text{CAlg}(1 - \text{Cat})$ one has canonically $\text{Fun}^{\otimes}(f\text{Set}_{<\infty}, D) \xrightarrow{\sim} \text{CAlg}(D)$, where $\text{Fun}^{\otimes}(f\text{Set}_{<\infty}, D)$ is the category of symmetric monoidal functors. Let $f\text{Set}$ be the category of non empty finite sets and surjections. Recall that $\text{CAlg}^{nu}(D) \xrightarrow{\sim} \text{Fun}^{\otimes}(f\text{Set}, D)$ canonically.

Fix a curve X for simplicity. We consider Ran spaces for X and remove X from the notation. Recall that Ran is a nonunital commutative algebra in PreStk_{lft} , view it

now as a nonunital commutative algebra in $(\text{PreStk}_{lft})_{corr}$ via the canonical inclusion $\text{PreStk}_{lft} \hookrightarrow (\text{PreStk}_{lft})_{corr}$. Assume given a non-unital commutative algebra Z in $(\text{PreStk}_{lft})_{corr}$ together with a morphism $Z \rightarrow \text{Ran}$ in $\mathcal{CAlg}^{nu}((\text{PreStk}_{lft})_{corr})$.

We think of Z as given by a symmetric monoidal functor $fSet \rightarrow (\text{PreStk}_{lft})_{corr}$, $I \mapsto Z^I$. For a morphism $\phi : I \rightarrow J$ in $fSet$ we have the corresponding diagram $Z^I \xleftarrow{a_\phi} Z_{I,J} \xrightarrow{b_\phi} Z^J$ in PreStk_{lft} , where the first map is over Ran^I , and the second map covers the multiplication $\text{Ran}^I \rightarrow \text{Ran}^J$.

Write Ran^{ch} for the nonunital commutative algebra Ran in $(\text{PreStk}_{lft})_{corr}$ with the chiral multiplication. So, for $I \rightarrow *$ in $fSet$ the multiplication diagram is $\text{Ran}^I \leftarrow \text{Ran}_d^I \xrightarrow{u} \text{Ran}$, where u is the union. We have a canonical lax morphism $\text{Ran}^{ch} \rightarrow \text{Ran}$ in the 2-category $\mathcal{CAlg}^{nu}((\text{PreStk}_{lft})_{corr})$ coming from the inclusion $\text{Ran}_d^I \subset \text{Ran}^I$ for any finite nonempty set I .

For $I \rightarrow J$ in $fSet$ set

$$\mathcal{Z}_{I,J} = Z_{I,J} \times_{\text{Ran}^I} \text{Ran}_d^I$$

For any $I \rightarrow J$ in $fSet$ the restriction of the product diagram $Z^I \leftarrow \mathcal{Z}_{I,J} \rightarrow Z^J$ defines a nonunital commutative algebra structure on the object $Z \in (\text{PreStk}_{lft})_{corr}$. More precisely, the functor

$$fSet \rightarrow (\text{PreStk}_{lft})_{corr}, I \mapsto \mathcal{Z}_I; (I \rightarrow J) \mapsto (Z^I \leftarrow \mathcal{Z}_{I,J} \rightarrow Z^J)$$

becomes an object of $\mathcal{CAlg}^{nu}((\text{PreStk}_{lft})_{corr})$, we denote it by Z^{ch} . Assume in addition the latter functor is a factorization space over Ran . This means that for any $I \rightarrow J$ in $fSet$ the both squares in the daigram below are cartesian

$$\begin{array}{ccccc} Z^I & \leftarrow & \mathcal{Z}_{I,J} & \rightarrow & Z^J \\ \downarrow & & \downarrow & & \downarrow \\ \text{Ran}^I & \leftarrow & \text{Ran}_d^I & \rightarrow & \text{Ran}^J \end{array}$$

Assume for simplicity we are in the constructible context, and for any $I \rightarrow J$ in $fSet$ the map $b_\phi : Z_{I,J} \rightarrow Z^J$ is proper (or pseudo-proper). Let

$$\text{PreStk}_{lft,pseudo-proper} \subset \text{PreStk}_{lft}$$

be the 1-full subcategory, where we keep all objects and morphisms which are pseudo-proper. Let $(\text{PreStk}_{lft})_{corr,pseudo-proper} \subset (\text{PreStk}_{lft})_{corr}$ be the 1-full subcategory, where we keep all objects and for $Y, Y' \in \text{PreStk}_{lft}$ only 1-morphisms given by diagrams $Y \leftarrow V \xrightarrow{h} Y'$, where h is pseudo-proper. This makes sense, as the base change of a pseudo-proper morphism is pseudo-proper.

Since $Shv_{pseudo-proper} : (\text{PreStk}_{lft})_{corr,pseudo-proper} \rightarrow \text{DGCat}_{cont}$ is right-lax monoidal, it sends non-unital commutative algebras to non-unital commutative algebras. Here for a map $Z_1 \rightarrow Z_2$ in $(\text{PreStk}_{lft})_{corr,pseudo-proper}$ given by a diagram $Z_1 \xleftarrow{a} Z_{12} \xrightarrow{b} Z_2$ the functor $Shv(Z_1) \rightarrow Shv(Z_2)$ is $b_* a^! \xrightarrow{\sim} b_! a^!$, because $b_* \xrightarrow{\sim} b_!$. Thus, $Shv(Z)$ is an object of $\mathcal{CAlg}^{nu}(\text{DGCat}_{cont})$ given by applying $Shv_{pseudo-proper}$ to the above functor. We write $Shv(Z, *)$ for this non-unital symmetric monoidal structure.

In this setting we should be able to talk about commutative factorization categories on Z and commutative factorization algebras on Z in Shv , in particular.

I think a possible definition of a commutative factorization algebra on Z in Shv is a follows.

Definition 1.1.1. *A commutative factorization algebra on Z is an object*

$$F \in CAlg^{nu}(Shv(Z, *))$$

with the following property. For any $\phi : I \rightarrow J$ in $fSet$ consider the multiplication map $(b_\phi)_! a_\phi^!(F^{\boxtimes I}) \rightarrow F^{\boxtimes J}$ in $Shv(Z^J)$, and the map $a_\phi^!(F^{\boxtimes I}) \rightarrow b_\phi^! F^{\boxtimes J}$ over $Z_{I,J}$ obtained from it by adjointness. Then the restriction of the latter map to $Z_{I,J}$ is an isomorphism.

1.1.2. For $I \in fSet$ consider the natural map $X \rightarrow \text{Ran}^I$. Let $(Z^I)_X = Z^I \times_{\text{Ran}^I} X$. For $\phi : I \rightarrow J$ in $fSet$ let $Z_{I,J,X} = Z_{I,J} \times_{\text{Ran}^I} X$. Note that

$$Z_{I,J,X} \xrightarrow{\sim} Z_{I,J} \times_{Z^J} (Z^J)_X$$

canonically. Here $(Z^I)_X$ is the product $\prod_{i \in I} Z_X$ taken in the category $\text{PreStk}_{lft./X}$. The functor $fSet \rightarrow (\text{PreStk}_{lft./X})_{corr}$, $I \mapsto (Z^I)_X$, $(\phi : I \rightarrow J) \mapsto ((Z^I)_X \leftarrow Z_{I,J,X} \rightarrow (Z^J)_X)$ is symmetric monoidal, so $Z_X \in CAlg^{nu}((\text{PreStk}_{lft./X})_{corr})$ naturally.

For each $\phi : I \rightarrow J$ in $fSet$ the base change $Z_{I,J,X} \rightarrow (Z^J)_X$ of the map $b_\phi : Z_{I,J} \rightarrow Z^J$ under $X \rightarrow \text{Ran}^J$ is still pseudo-proper. As above this gives on $Shv(Z_X)$ a structure of a non-unital symmetric monoidal category, given by the convolution, which we denote by $Shv(Z_X, *)$. To be precise, for $\phi : I \rightarrow J$ the corresponding diagram

$$Z^I \xleftarrow{q} (Z^I)_X \xleftarrow{a_{\phi,X}} Z_{I,J,X} \xrightarrow{b_{\phi,X}} (Z^J)_X$$

gives the convolution sending $\{K_i\}_{i \in I}$ with $K_i \in Shv(Z_X)$ to

$$(b_{\phi,X})_* a_{\phi,X}^! (q^! (\boxtimes_{i \in I} K_i))$$

In fact we get $Shv(Z_X, *) \in CAlg^{nu}((Shv(X), \otimes^!) - mod)$, because the functor $\text{PreStk}_{lft./X} \rightarrow (Shv(X), \otimes^!) - mod$, $S \mapsto Shv(S)$ is right-lax symmetric monoidal.

1.1.3. Let $i : Z_X \rightarrow Z$ be the natural map. Since $X \rightarrow \text{Ran}$ is pseudo-proper by ([2] Proposition 7.4.2), i is also pseudo-proper. We have an adjoint pair $i_! : Shv(Z_X) \rightleftarrows Shv(Z) : i^!$ in DGCat_{cont} . The functor $i^! : Shv(Z, *) \rightarrow Shv(Z_X, *)$ is non-unital symmetric monoidal. So, it induces a functor $CAlg^{nu}(Shv(Z, *)) \rightarrow CAlg^{nu}(Z_X, *)$. The latter functor preserves limits, so has a left adjoint.

This left adjoint produces the corresponding commutative factorization algebras as in [12], as far as I understand.

2. APPENDIX: LAX COLIMITS

2.0.1. One has the notion of lax limit and colimit from [4]. For $\mathcal{C} \in 1 - \text{Cat}$ they denote by $Tw(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{op}$ the cartesian fibration in spaces associated to the functor $\mathcal{C}^{op} \times \mathcal{C} \rightarrow \text{Spc}$, $(c', c) \mapsto \text{Map}_{\mathcal{C}}(c', c)$. So, an object of $Tw(\mathcal{C})$ is an arrow $\alpha : c' \rightarrow c$ in

\mathcal{C} . A map from $\alpha : c' \rightarrow c$ to $\beta : d' \rightarrow d$ in $Tw(\mathcal{C})$ gives rise to maps $c' \rightarrow d'$ and $d \rightarrow c$ in \mathcal{C} such that the diagram commutes

$$\begin{array}{ccc} c' & \xrightarrow{\alpha} & c \\ \downarrow & & \uparrow \\ d' & \xrightarrow{\beta} & d \end{array}$$

Another definition of $Tw(\mathcal{C})$ is given in ([14], Appendix B).

For $\mathcal{C} \in 1 - \mathcal{Cat}$ the projections $\text{Fun}([1], \mathcal{C}) \rightarrow \mathcal{C}$, $\text{Fun}([1], \mathcal{C}) \rightarrow \mathcal{C}$ on the source and target respectively are cartesian and cocartesian fibrations. The corresponding strengthenings are the functors $\mathcal{C}_{-/} : \mathcal{C}^{op} \rightarrow 1 - \mathcal{Cat}$, $x \mapsto \mathcal{C}_{x/}$ and $\mathcal{C}_{/_} : \mathcal{C} \rightarrow 1 - \mathcal{Cat}$, $x \mapsto \mathcal{C}_{/x}$.

2.0.2. (*Co*)ends. Given a functor $F : C \times C^{op} \rightarrow D$ in $1 - \mathcal{Cat}$, its coend is the colimit of the composition

$$Tw(C) \rightarrow C \times C^{op} \xrightarrow{F} D.$$

Given a functor $F : C^{op} \times C \rightarrow D$, its end is the limit of the composition

$$Tw(C)^{op} \rightarrow C^{op} \times C \xrightarrow{F} D.$$

2.0.3. Given a functor $F : \mathcal{C} \rightarrow 1 - \mathcal{Cat}$, its lax colimit is

$$\text{colim}_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C})} (\mathcal{C}_{c'})^{op} \times F(c')$$

and its oplax colimit is

$$\text{colim}_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C})} (\mathcal{C}_{c'}) \times F(c')$$

The lax limit of F is

$$\lim_{(c' \xrightarrow{\alpha} c) \in Tw(\mathcal{C})^{op}} \text{Fun}(\mathcal{C}_{c'}, F(c))$$

The oplax limit of F is

$$\lim_{(c' \rightarrow c) \in Tw(\mathcal{C})^{op}} \text{Fun}((\mathcal{C}_{c'})^{op}, F(c))$$

There is a natural map $\xi : \lim F \rightarrow \text{laxlim} F$. Indeed, there is a natural morphism of functors from the composition $Tw(\mathcal{C})^{op} \xrightarrow{\text{pr}} \mathcal{C} \xrightarrow{F} 1 - \mathcal{Cat}$ to the functor $Tw(\mathcal{C})^{op} \rightarrow 1 - \mathcal{Cat}$ sending $\alpha : c' \rightarrow c$ to $\text{Fun}(\mathcal{C}_{c'}, F(c))$. It comes from the fact that for $\alpha : c' \rightarrow c$ in $Tw(\mathcal{C})$, one has the natural map $\text{const} : F(c) \rightarrow \text{Fun}(\mathcal{C}_{c'}, F(c))$ functorial in $\alpha \in Tw(\mathcal{C})^{op}$. Here const is given by constant functors.

Note that for any $\alpha : c' \rightarrow c$ in $Tw(\mathcal{C})$ the above map $\text{const} : F(c) \rightarrow \text{Fun}(\mathcal{C}_{c'}, F(c))$ is fully faithful. Indeed, $L : \mathcal{C}_{c'} \rightarrow *$ is left adjoint to $R : * \rightarrow \mathcal{C}_{c'}$, here $R(*)$ is the final object of $\mathcal{C}_{c'}$, and we apply ([11], 2.2.56). Now by ([11], Lemma 2.2.17), the map $\lim_{Tw(\mathcal{C})^{op}} F \circ \text{pr} \rightarrow \text{laxlim} F$ is fully faithful. Is the map $\lim F \rightarrow \lim_{Tw(\mathcal{C})^{op}} F \circ \text{pr}$ fully faithful or maybe equivalence?

The projection $Tw(\mathcal{C}) \rightarrow \mathcal{C}^{op}$ is a cartesian fibration by ([5], 2.4.2.3). The corresponding functor $\mathcal{C} \rightarrow 1 - \mathcal{Cat}$ sends c to $\mathcal{C}_{/c}$, it sends an arrow $\alpha : c_1 \rightarrow c_2$ to the functor $\mathcal{C}_{/c_1} \rightarrow \mathcal{C}_{/c_2}$ given by composing with α .

The projection $Tw(\mathcal{C})^{op} \rightarrow \mathcal{C}^{op}$, $(d \rightarrow c) \mapsto d$ is a cocartesian fibration for the same reason. The corresponding functor $\mathcal{C}^{op} \rightarrow 1 - \text{Cat}$ sends d to $\mathcal{C}_{d/}$, and it sends a map $\alpha : d \rightarrow d'$ to the functor $\mathcal{C}_{d'/} \rightarrow \mathcal{C}_{d/}$ given by composing with α .

We have a natural functor $oplax \text{ colim } F \rightarrow \text{colim}_C F$. This map is always cofinal by ([11], 2.4.15).

Proposition 2.0.4. *Let $F : \mathcal{C} \rightarrow 1 - \text{Cat}$ be a functor. Let $X \rightarrow \mathcal{C}$ (resp., $Y \rightarrow \mathcal{C}^{op}$) be the cocartesian (resp., cartesian) fibration attached to F . Then*

i) $laxlim F \xrightarrow{\sim} \text{Fun}_{\mathcal{C}}(\mathcal{C}, X)$. In particular, $\text{Fun}_{\mathcal{C}}^{cocart}(\mathcal{C}, X) \xrightarrow{\sim} \lim F \hookrightarrow laxlim F$ is fully faithful.

ii) $oplaxlim F \xrightarrow{\sim} \text{Fun}_{\mathcal{C}^{op}}(\mathcal{C}^{op}, Y)$. In particular, $\text{Fun}_{\mathcal{C}^{op}}^{cart}(\mathcal{C}^{op}, Y) \xrightarrow{\sim} \lim F \hookrightarrow oplaxlim F$ is fully faithful.

iii) $laxcolim F \xrightarrow{\sim} Y$

iv) $oplaxcolim F \xrightarrow{\sim} X$.

Proof. i) is given by ([4], Corollary 7.7), ii) is ([4], Proposition 7.1), and iii), iv) are given by Theorem 1.1 in [4]. \square

Given $(c' \xrightarrow{a} c) \in Tw(\mathcal{C})$, the composition $C_{c/} \times F(c') \rightarrow oplaxcolim F \rightarrow X$ in the above proposition sends $(c \xrightarrow{b} u, z \in F(c'))$ to $(u, (F(ba))(z) \in F(u))$.

2.0.5. Example. If $\mathcal{C} = [1]$, that is, the category with objects $0, 1$ and a unique nontrivial arrow $0 \rightarrow 1$ then given $F : [1] \rightarrow 1 - \text{Cat}$, the lax limit of F is as follows. Let $f : F(0) \rightarrow F(1)$ be the corresponding map in $1 - \text{Cat}$. Then $laxlim F = F(0) \times_{F(1)} \text{Fun}([1], F(1))$, where the map $\text{Fun}([1], F(1)) \rightarrow F(1)$ keeps the source of an arrow.

2.0.6. If $F : \mathcal{C}^{op} \rightarrow 1 - \text{Cat}$ is a map in $1 - \text{Cat}$, let $\mathcal{X} \rightarrow \mathcal{C}$ be the corresponding cartesian fibration. Then $\text{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ is the oplax limit of F by ([4], 7.1). Recall that $\lim F$ is the category $\text{Fun}_{\mathcal{C}}^{cart}(\mathcal{C}, \mathcal{X})$, the full subcategory of those functors in $\text{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ that send cartesian arrows to cartesian arrows ([11], 2.2.67).

2.0.7. For $\mathcal{C} \in 1 - \text{Cat}$ we have an isomorphism $Tw(\mathcal{C}) \xrightarrow{\sim} Tw(\mathcal{C}^{op})$ compatible with the projections to $\mathcal{C} \times \mathcal{C}^{op}$. Indeed, for $c, c' \in \mathcal{C}$, $\text{Map}_{\mathcal{C}}(c', c) \xrightarrow{\sim} \text{Map}_{\mathcal{C}^{op}}(c, c')$.

For $\mathcal{A}, \mathcal{B} \in 1 - \text{Cat}$ we get $Tw(\mathcal{A} \times \mathcal{B}) \xrightarrow{\sim} Tw(\mathcal{A}) \times Tw(\mathcal{B})$ naturally.

2.0.8. For $F : [1] \rightarrow 1 - \text{Cat}$ given by a functor $f : F(0) \rightarrow F(1)$ its oplax colimit is the coproduct $F(0) \times [1] \sqcup_{F(0)} F(1)$ in $1 - \text{Cat}$, where the maps are $F(1) \xleftarrow{f} F(0) \xrightarrow{\text{id} \times \{1\}} F(0) \times [1]$. This oplax colimit is the cocartesian fibration $X \rightarrow [1]$ corresponding to F by ([4], Th. 1.1).

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