1. Comments to Beilinson, Drinfeld, Chiral algebras

1.1. **For chapter 1.** The notion of a colored operad from ([14], 2.1.1.1) is the same as the pseudo-tensor category in [1]. The corresponding notion for ∞-catetories is that of ∞-operad. Their pseudo-tensor functor corresponds to a notion of a morphism of ∞-operads.

For Sect. 1.1.13. Let k be a commutative ring, R be an associative k-algebra, $\mathcal{B}$ be a k-operad. They write $\mathcal{B}_I$ for bove$k$-module of I-operations on the unique object $*$ of $\mathcal{B}$. Write also for a surjection $\pi : J \to I$ of finite sets, $\mathcal{B}_{J/I} = \otimes_{i\in I}\mathcal{B}_{J_i}$. The operad structure is given by k-linear maps $\mathcal{B}_I \otimes_k \mathcal{B}_{J/I} \to \mathcal{B}_J$ for any surjection of finite sets $\pi : J \to I$.

Then $\mathcal{B}_1$ for $I = \{1\}$ is a k-algebra, assume given an isomorphism of k-algebras $\mathcal{B}_1 \widetilde{\to} R$. Then $\mathcal{B}_I$ becomes a $(R - R^{\otimes I})$-bimodule, here $R^{\otimes I}$ is the tensor product over k. Indeed, $\underset{i\in I}{\otimes} r_i \in R^{\otimes I}$, $r \in R$ acts on $m \in \mathcal{B}_I$ as the composition $rm(r_i)$. The composition $\mathcal{B}_I \otimes_k \mathcal{B}_{J/I} \to \mathcal{B}_J$ factors evidently via $\mathcal{B}_I \otimes_{R^{\otimes I}} \mathcal{B}_{J/I} \to \mathcal{B}_J$. Note that $\mathcal{B}_{J/I}$ is naturally $R^{\otimes I} - R^{\otimes J}$-module, and the latter map is a morphism of $R - R^{\otimes J}$-bimodules.

The pseudo-tensor structure $P^{*\mathcal{B}}$ on $mod-R$ is as follows. For a finite nonempty set I,

$$P_I(\{L_i\}, M) = \mathrm{Hom}_{mod-R^{\otimes I}}(\otimes_{i\in I} L_i, M \otimes_R \mathcal{B}_I)$$

Now if $\pi : J \to I$ is a surjection of finite nonempty sets, assume given $M, L_i \in mod - R, K_j \in mod - R$, assume given morphism in $mod - R^{\otimes J_i}$

$$\alpha_i : \underset{j\in J_i}{\otimes} K_j \to L_i \otimes_R \mathcal{B}_{J_i}$$

and $\alpha : \underset{i\in I}{\otimes} L_i \to M \otimes_R \mathcal{B}_I$ in $mod - R^{\otimes I}$. Tensoring α_i over $i \in I$, we get a map

$$\underset{j\in J}{\otimes} K_j \to \underset{i\in I}{\otimes} (L_i \otimes_R \mathcal{B}_{J_i}) = (\otimes_{i\in I} L_i) \otimes_{R^{\otimes I}} \mathcal{B}_{J/I}$$

in $mod - R^{\otimes J}$. Applying $\otimes_{R^{\otimes I}}\mathcal{B}_{J/I}$ to α, we get a map

$$(\otimes_{i\in I} L_i) \otimes_{R^{\otimes I}} \mathcal{B}_{J/I} \to M \otimes_R \mathcal{B}_I \otimes_{R^{\otimes I}} \mathcal{B}_{J/I} \to M \otimes_R \mathcal{B}_J$$

Composing, we get the desired map $\underset{j\in J}{\otimes} K_j \to M \otimes_R \mathcal{B}_J$.

1.1.1. A way to think about the Lie operad Lie over a field k. They always assume $1/2 \in k$. The k-vector space Lie_n consists of Lie polynomials in variables $e_1, \dots, e_n$. For example, Lie_3 is generated by $[e_1, [e_2, e_3]], [e_3, [e_1, e_2]], [e_2, [e_3, e_1]]$ with the only (Jacobi) relation $[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] + [e_2, [e_3, e_1]] = 0$.

If I is a 2 elements set then Lie_I does not have a distinguished generator. Once you pick an isomorphism $I \widetilde{\to} \{1, 2\}$, you get the generator $[,]$, and if you change the order of I, you get $-[,]$.

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The k-operad Com from 1.1.7, 1.1.10 is given by $Com_n = k$ for any $n \geq 1$. We repeat that Com stands for usual operad, non the augmented one! So, Com-algebra in a tensor category is a commutative algebra without unit.

Let Σ be the operad of linear orders from 1.1.4, let $k[\Sigma]$ be its k-envelope. We have a pseudo-tensor functor Lie $\to k[\Sigma]$ given by the property that $[e_1, e_2] \mapsto e_1e_2 - e_2e_1$. In an expression like $e_2(e_1e_3)$ (parenthesis can be put in arbitrary way, as the product of e_i is associative) the corresponding order on $\{1, 2, 3\}$ is 213.

1.1.2. Proof of their Lemma 1.1.10 in details. Assume the operation $\cdot_M \in P_2^{\mathcal{M}}(\{M, M\}, M)$ commuttaive, set $[\cdot,\cdot]_M = \cdot_M \otimes [e_1, e_2] \in P_2^{\mathcal{M}\otimes\mathrm{Lie}}(\{M, M\}, M)$. If $\cdot_M$ is associative then for $m_i \in M$ we get, so to say,

$$[m_1, [m_2, m_3]_M]_M + [m_3, [m_1, m_2]_M]_M + [m_2, [m_3, m_1]_M]_M = \\ (m_1m_2m_3) \otimes ([e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] + [e_2, [e_3, e_1]]) = 0$$

by the Jacobi identity for the Lie polynomials.

Conversely, assume that $[\cdot,\cdot]_M$ satisfies the Jacobi identity. In $P_3^{\mathcal{M}\otimes\mathrm{Lie}}(\{M, M, M\}, M)$ we get so to say for $m_i \in M$

$$m_1(m_2m_3) \otimes [e_1, [e_2, e_3]] + m_3(m_1m_2) \otimes [e_3, [e_1, e_2]] + m_2(m_3m_1) \otimes [e_2, [e_3, e_1]] = 0$$

By the Jacobi identity $[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] + [e_2, [e_3, e_1]] = 0$ in Lie_3. So,

$$(m_1(m_2m_3) - m_2(m_3m_1)) \otimes [e_1, [e_2, e_3]] + (m_3(m_1m_2) - m_2(m_3m_1)) \otimes [e_3, [e_1, e_2]] = 0$$

Since $\{[e_1, [e_2, e_3]], [e_3, [e_1, e_2]]\}$ is a base of Lie_3, we have

$$P_3^{\mathcal{M}}(\{M, M, M\}, M) \otimes [e_1, [e_2, e_3]] \cap P_3^{\mathcal{M}}(\{M, M, M\}, M) \otimes [e_3, [e_1, e_2]] = 0$$

in $\mathrm{P}_3^{\mathcal{M}\otimes\mathrm{Lie}}(\{M, M, M\}, M)$. So, the above gives

$$m_1(m_2m_3) - m_2(m_3m_1) \quad \text{and} \quad m_3(m_1m_2) - m_2(m_3m_1)$$

so to say for any $m_i \in M$. This shows that $\cdot_M$ is associative.

1.1.3. Example 1.1.14 seems important. If R is a cocommutative Hopf algebra, set $\mathcal{B}_I = R^{\otimes I}$. The operad structure on $\mathcal{B}$ is given as follows. If $\pi : J \to I$ is a surjection of finite nonempty sets, the map $\mathcal{B}_I \otimes_{R^{\otimes I}} \mathcal{B}_{J/I} \to \mathcal{B}_J$ becomes $R^{\otimes I} \otimes_{R^{\otimes I}} \otimes_{i\in I}(R^{\otimes J_i}) \to R^{\otimes J}$, it is well defined in loc.cit.

In particular, the left R-module structure on $\mathcal{B}_I = R^{\otimes I}$ is given by the coproduct map $\triangle^{(I)}: R \to R^{\otimes I}$.

1.1.4. I think they call in 1.1.16 a category $\mathcal{C}$ k-linear if each $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is a k-vector space, and the composition is k-bilinear.

Now given k-linear catgeories $\mathcal{C}, \mathcal{C}'$ they call $\mathcal{C} \otimes \mathcal{C}'$ the category, whose objects are pairs (c, c') and morphisms $\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{C}'}((c, c'), (d, d')) = \mathrm{Hom}_{\mathcal{C}}(c, d) \otimes_k \mathrm{Hom}_{\mathcal{C}'}(c', d')$. You may ask Drinfeld.

1.1.5. For 1.2.11. If $\mathcal{A}$ is an operad in their sense and $\mathcal{C}$ is an $\mathcal{A}$-module operad, then this does not mean that $\mathcal{C}$ is an operad! The category $\mathcal{C}$ has two objects $., *$ and for I a nonempty finite set $P_I^{\mathcal{C}}(x_i, y)$ is nonempty in the following cases: either all x_i and y are ., or $y = *$ and exactly one of x_i equals $*$.

This is the case for example for their Σ-operad Σ^m defined in 1.2.12. This is the category having 2 objects $., *$. The set $P_I^{\Sigma^m}(\{x_i\}, y)$ is nonempty only in the above cases, and in those cases the latter set is the set of orders on I, not on $I \sqcup *$! This gives the gadget controlling the bimodules for associative algebras.

I think in 1.2.13, 1.2.16 they use the following notation without saying so explicitly: let $\mathcal{B}$ be an operad, $\mathcal{C}$ be a $\mathcal{B}$-module operad (so, $\mathcal{C}$ has objects $., *$). For nonempty finite set I they write $\mathcal{C}_I$ for the set of operations $P_I^{\mathcal{C}}(\{., *\}_{I \sqcup *}, *)$, here . are indexed by I.

Example 1.2.16 is great! It is easier to understand in a special case, say, when $\mathcal{B}$ is the operad of associative algebras, and $\mathcal{C}$ is the usual $\mathcal{B}$-module operad. Then, in their notations, L is a monoid in *Sets*, and $Sets(L, \mathcal{C})$ is the category $L - mod(Sets)$ of sets equipped with L-action. The question is then to understand the monoid $\mathcal{C}(L)$ of endomorphisms of the forgetful functor $L - mod(Sets) \to Sets$. We have a natural map from $L \to \mathcal{C}(L)$, which seems to be an isomorphism.

In general their description says the following: $\mathcal{C}(L)$ is generated by elements $((\ell_i, \phi)$, where I is a finite nonempty set, $\ell_i \in L$, and $\phi \in \mathcal{C}_I$ giving a map $\phi : (\prod_{i \in I} L) \times M \to M$ functorial in $M \in Sets(L, \mathcal{C})$. The relations in this $\mathcal{C}(L)$ are as follows. For a nonempty finite set I let $\tilde{I} = I \sqcup *$. Given nonempty finite sets I, J and a surjection $\pi : \tilde{J} \to \tilde{I}$ with $\pi(*) = *$, set $J. = \pi^{-1}(*) \cap J$. For $i \in I$ set $J_i = \pi^{-1}(i)$. Assume given elements $\phi_i \in \mathcal{B}_{J_i}$ for $i \in I$, $\phi_2 \in \mathcal{C}_{J.}$ and $\phi_1 \in \mathcal{C}_I$. So, ϕ_1 gives a map $(\prod_{i \in I} L) \times M \to M$ functorial in $M \in Sets(L, \mathcal{C})$, and ϕ_2 gives a map $(\prod_{j \in J.} L) \times M \to M$ functorial in $M \in Sets(L, \mathcal{C})$.

Then we want to compose the maps in the diagram

$$(\prod_{j \in J} L) \times M \stackrel{(\prod_{i \in I} \psi_i) \times \phi_2}{\to} (\prod_{i \in I} L) \times M \stackrel{\phi_1}{\to} M$$

This explains their formula (1.2.16.1).

I imagine, one may more generally define $\mathcal{C}(L)$ as the monad attached to the functor $\mathrm{oblv} : Sets(L, \mathcal{C}) \to Sets$, which should have a left adjoint.

To memorise: let say $E \in \mathrm{DGCat}_{cont}$ and $\mathcal{B}$ be a e-linear operad in the sense of [6] extended to a $\mathcal{B}$-module operad $\mathcal{C}$ as above. Fix a $\mathcal{B}$-module $L \in E$. Then we have the category $L - mod(E)$. In which generality $L - mod(E) \to E$ has a left adjoint ind? Then oblv ind will be a monad on E (it actually comes from a monad on Vect?). The corresponding monad on E is the universal enveloping algebra of L.

1.1.6. Example from Sect. 1.2.18 is great! Let k be a field (with $1/2 \in k$). If $\mathcal{M}$ is a pseudo-tensor k-category, L is a Lie algebra in $\mathcal{M}$, they denote by $\mathcal{M}(L)$ the category of L-modules in $\mathcal{M}$ (see 1.2.13). Then $\mathcal{M}(L)$ has a natural pseudo-tensor structure! It is defined in a way analogous to the definiton of the action of a Lie algebra on tensor product of representations.

A nice way to write down axioms (say, of a Lie algebra in a pseudo-tensor category) is to draw pictures: arrows going to the right; the objects $\{L_i\}$ in a column, M in the column to the right, so a map $\{L_i\} \to M$ is a collection of arrow going from each L_i to M. The compositions are compositions of arrows...

Let us explain a point from 1.2.18. Let $\mathcal{M}$ be a pseudo-tensor k-category, L a Lie algebra in $\mathcal{M}$, $\mathcal{M}(L)$ the pseudo-tensor category of L-modules in $\mathcal{M}$. Let $h : \mathcal{M} \to \mathrm{Vect}_k$ be an augmentation functor. If $M \in \mathcal{M}(L)$ then $h(M)$ is a $h(L)$-module. Let $h^L : \mathcal{M}(L) \to \mathrm{Vect}_k$ be the functor $M \mapsto h(M)^{h(L)}$ of $h(L)$-invariants in $h(M)$. Why this is an augmentation functor on $\mathcal{M}(L)$? Given a finite nonempty set I and $i_0 \in I$, let $M_i, N \in \mathcal{M}(L)$. Recall their notation $P_{LI}(\{M_i\}, N) \subset P_I(\{M_i\}, N)$ for the subspace of L-linear operations. The map

$$P_{LI}(\{M_i\}, N) \times h^L(M_{i_0}) \to P_{LI-\{i_0\}}(\{M_i\}, N)$$

is, I think, the restriction of the map

$$P_I(\{M_i\}, N) \times h(M_{i_0}) \to P_{I-\{i_0\}}(\{M_i\}, N)$$

Indeed, in the usual situation of L-modules if $\phi : \otimes_I M_i \to N$ is a morphism of L-representations over k then for $m \in M_{i_0}$ to define a morphism of representations $\otimes_{I-\{i_0\}} M_i \to N$ by restrictions, we need that $m \in M_{i_0}^L$, that is, $\ell m = 0$ for any $\ell \in L$.

If now $\mathcal{M}$ is a pseudo-tensor k-category, N is a Lie algebra in Vect_k then we have a notion of a N-module in $\mathcal{M}$. This is $M \in \mathcal{M}$ such that $N \otimes_k M$ exists, that is, the functor $\mathcal{M} \to \mathrm{Vect}_k, R \mapsto \mathrm{Hom}_{\mathrm{Vect}}(N, \mathrm{Hom}_{\mathcal{M}}(M, R))$ is corepresentable, and we are given a map $N \otimes M \to M$ satisfying the usual axioms of a Lie algebra action. In particular, I think, $N \otimes (N \otimes M)$ should exist. Probably, for $\dim N < \infty$ this is usually the case. Maybe this is true because they assume $\mathcal{M}$ to be abelian pseudo-tensor category. I think they say that if $N \otimes M$ exists for any $M \in \mathcal{M}$ then we may define the pseudo-tensor category $\{N\text{-modules in } \mathcal{M}\}$. Now they claim that if $L \in \mathcal{M}$ is a Lie algebra in $\mathcal{M}$ then the "identity" is the pseudo-tensor functor $\mathcal{M}(L) \to \{h(L)\text{-modules in } \mathcal{M}\}$. For example, L is an L-module, so $h(L)$ acts naturally on L.

1.1.7. Let $\mathcal{M}$ be an augmented pseudo-tensor category, $h : \mathcal{M} \to \mathrm{Sets}$ the corresponding augmentation functor. Let I be a finite set, $L_i \in \mathcal{M}, M \in \mathcal{M}$, assume the inner object $\mathcal{P}_I(\{L_i\}, M)$ exists. The natural map (1.2.7.2)

$$h(\mathcal{P}_I(\{L_i\}, M) \to P_I(\{L_i\}, M)$$

is the composition map with respect to the diagram $\emptyset \to I \to *$. Recall that for any $M \in \mathcal{M}$, $h(M) = P_\emptyset(M)$.

1.1.8. For Cor. 1.3.3.
i) If $S \leftarrow I \to T$ are surjections of finite nonempty sets, and S, T are transversal then $I \to S \times T$ is injective by Prop. 1.3.2 i).
ii) Let $S \leftarrow I \to T$ be complementary equivalence relations, $t \in T$. Then $I_t \to S$ is injective by the previous point. They claim it is actually bijective (on footnote 20 on p. 51, Sect.1.4.27), but this is wrong, say for $I = \{1, 2, 3\}$ et quotients $\{1, \{2, 3\}\}$, $\{\{1, 2\}, 3\}$.

iii) One may hope also the following holds. If $S, T \in Q(I)$ are transversal then for any $J \in Q(I)$, the equivalence relations $\inf\{S, J\}, \inf\{T, J\} \in Q(J)$ are transversal.

1.1.9. For Cor. 1.3.3. The following is actually used in 1.3.12.

Lemma 1.1.10. *Let $S \leftarrow I \rightarrow T$ be a complementary pair of equivalence relations. Let $H \in Q(I)$ with $I \geq S$, that is, $I \rightarrow H \rightarrow S$. Then $(inf(H, T), S)$ is a complementary pair in $Q(H)$.*

Proof. Let V be a finite-dimensional vector space. Then $V^T + V^S = V^I, V^T \cap V^S = V$. Set $T' = inf(H, T)$. Then $V^{T'} = V^H \cap V^T$ and $V^S + V^{T'} = V^H$. Indeed, if $v \in V^H$ and $v = v_1 + v_2$ with $v_1 \in V^S, v_2 \in V^T$ then $v_2 \in V^H \cap V^T = V^{T'}$. Finally, $V^{T'} \cap V^S = V^H \cap V^T \cap V^S = V$. The claim follows now from ([1], 1.3.2). □

1.1.11. For compound tensor structures, Section 1.3.5. I think they assume in 1.3.5 that the usual categories underlying the pseudotensor categories $(\mathcal{M}, P^*)$ and $(\mathcal{M}^0, P^!)$ are opposite to each other.

If $P^!$ is a pseudo-tensor structure on $\mathcal{M}^o$ (the opposite category), we write the composition as follows. For a surjection $\pi : J \rightarrow I$ of finite nonempty sets,

$$(\prod_{i \in I} P^!_{J_i}(L_i, \{K_j\}) \times P^!_I(M, \{L_i\}) \rightarrow P^!_J(M, \{K_j\})$$

sending (ψ_i, ϕ) to $(\psi_i)\phi$. Here $P^!_\cdot(M, L) = \mathrm{Hom}_{\mathcal{M}}(M, L)$. The order in compositions is very important!

An element of $\langle P^*_I(\{L_i\}, \cdot), P^!_J(\cdot, \{M_j\}\rangle$ is given by some $x \in \mathcal{M}$ and a diagram

$$\{L_i\} \stackrel{f}{\rightarrow} x \stackrel{g}{\rightarrow} \{M_j\}$$

If $\phi : x \rightarrow y$ is a map in $\mathcal{M}$ then given a diagram $\{L_i\} \stackrel{f}{\rightarrow} x \stackrel{\phi}{\rightarrow} y \stackrel{g}{\rightarrow} \{M_j\}$, we see that it is natural to impose the relation

$$(\phi f, g) \sim (f, {}^t\phi g),$$

here ${}^t\phi g = g\phi$. We may safely denote by gf the image of (f, g) in $P^{*!}_{I,J}(\{L_i\}, \{M_j\})$.

The composition map on the last line of p. 25 sends $\{M_j\} \stackrel{\tau_j}{\rightarrow} \{N_h\}$, $rf \in P^{*!}_{I,J}(\{L_i\}, \{M_j\})$, $\{K_g\} \stackrel{\eta_i}{\rightarrow} \{L_i\}$ given by the diagram

$$\{K_g\} \stackrel{\eta_i}{\rightarrow} \{L_i\} \stackrel{f}{\rightarrow} x \stackrel{r}{\rightarrow} \{M_j\} \stackrel{\tau_j}{\rightarrow} \{N_h\}$$

to their composition denoted $(\tau_j)(rf)(\eta_i)$. Here

$$(\eta_i) \in P^*_{G/I}(\{K_g\}, \{L_i\}) = \prod_{i \in I} P^*_{G_i}(\{K_g\}, L_i),$$

and $(\tau_j) \in P^!_{H/J}(\{M_j\}, \{N_h\}) = \prod_{j \in J} P^!_{H_j}(M_j, \{N_h\})$.

The maps given by (1.3.6.1) should be seen as "compositions". Namely, given a finite set I, and complementary pair of equivalence relations $S \leftarrow I \rightarrow T$, given a diagram

$$\{N_t\} \stackrel{(\psi_t)}{\rightarrow} \{L_i\} \stackrel{(\phi_s)}{\rightarrow} \{M_s\}$$

where (ϕ_s) is a map for P^*, (ψ_t) is a map for $P^!$, the operation $\langle\ \rangle^I_{S,T}$ gives their composition $\in P^{*!}_{T,S}(\{N_t\},\{M_s\})$. Memorize: $*$-operation decreases the number of variables, !-operation increases the number of variables.

When they say after (1.3.6.1) that $\langle\rangle^I_{S,T}$ must be functorial with respect to M_s and N_t-variables, they mean the following. Given $M'_s \in \mathcal{M}$ and maps $M_s \to M'_s$ in $\mathcal{M}$ for all $s \in S$, we may compose $P^*_{I/S}(\{L_i\},\{M_s\})$ with the latterr maps, so we get a map

$$P^*_{I/S}(\{L_i\},\{M_s\}) \to P^*_{I/S}(\{L_i\},\{M'_s\}).$$

Then the diagram should commute

$$\begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M'_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) & \to & P^{*!}_{T/S}(\{N_t\},\{M'_s\}) \\ \uparrow & & \uparrow \\ P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) & \to & P^{*!}_{T/S}(\{N_t\},\{M_s\}) \end{array}.$$

Similar situation with the N_t-variables.

Given in addition surjections $H \to S, G \to T$, the map $\langle\ \rangle^I_{S,T}$ extends to the following. Given an extension of the latter diagram to

$$\{A_g\} \stackrel{\eta_t}{\to} \{N_t\} \stackrel{(\psi_t)}{\to} \{L_i\} \stackrel{(\phi_s)}{\to} \{M_s\} \stackrel{(\kappa_s)}{\to} \{B_h\},$$

where $\eta,\phi \in P^*, \psi,\kappa \in P^!$, the composition $(\kappa_s)((\phi_s)(\psi_t))(\eta_t)$, where in the middle we have $\langle\ \rangle^I_{S,T}$-composition, depends only on the images of $(\kappa_s)(\phi_s)$ in

$$\prod_{s\in S} P^{*!}_{I_s,H_s}(\{L_i\},\{B_h\}),$$

and images of $(\psi_t)(\eta_t)$ in $\prod_{t\in T} P^{*!}_{G_t,I_t}(\{A_g\},\{L_i\})$.

For the axiom 1.3.7(i). They assume that $S,T \in Q(I)$ are complementary, let also $H \in Q(I)$ with $H \geq S$. Then let $V = inf\{H,T\}$. Then S,V are complementary on H, so we have the operation $\langle\rangle^H_{S,V}$. To make sense of 1.3.7(i) we first need to know that the operations $\langle\ \rangle^I_{S,T}$ defined by (1.3.6.1) indeed extend to more general operations attached to any diagram $G \to T \leftarrow I \to S \to H$ and still denoted by the same symbol $\langle\ \rangle^I_{S,T}$ (defined by the displayed formula just after (1.3.6.1)).

For 1.3.7(ii). They assume that $S,T \in Q(I)$ are complementary, let also $G \in Q(I)$ with $G \geq T$ and $U = inf\{S,G\}$. Then U,T are complementary on G first. Besides, for each $u \in U$, (S_u,G_u) are complementary on I_u.

Question: Why it suffices to define the operations $\langle\ \rangle^I_{S,T}$ for (S,T) *complementary* on I? If $S \leftarrow I \to T$ is a diagram of surjections of nonempty finite sets, let $V = inf\{S,T\}$. Then for each $v \in V$, (S_v,T_v) are complementary on I_v, so we have the operation $\langle\rangle^{I_v}_{S_v,T_v}$. Do they give rise to an operation

$$P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \to P^{*!}_{T,S}(\{N_t\},\{M_s\})$$

analogous to $\langle\rangle^I_{S,T}$ without the assumption that (S,T) are complementary? It suffices to define more generally the morphisms

$$\prod_{v\in V} P^{*!}_{T_v,S_v}(\{N_t\},\{M_s\}) \to P^{*!}_{T,S}(\{N_t\},\{M_s\})$$

For example, if $S \to V$ is an isomorphism then the result will be the map

$$P^*_{T/V}(\{N_t\},\{M_v\}) = \prod_{v\in V} P^*_{T_v}(\{N_t\}, M_v) \to P^{*!}_{T,V}(\{N_t\},\{M_v\})$$

Do we have such maps? Not clear.

1.1.12. For their 1.3.10. Let $\mathcal{M}$ be a compound pseudo-tensor category. Let $h : M \to Sets$ be an augmentation functor on $\mathcal{M}^*$. They use the fact that given a finite nonempty set T and $t_0 \in T$, the maps $h_{T,t_0} : P^*_T(\{N_t\}, M) \times h(N_{t_0}) \to P^*_{T\backslash t_0}(\{N_t\}, M)$ given by (1.2.5.2) extend as follows. Let S, T be nonempty finite sets, $t_0 \in T$. Then there are maps

$$P^{*!}_{T,S}(\{N_t\},\{M_s\}) \times h(N_{t_0}) \to P^{*!}_{T\backslash t_0,S}(\{N_t\},\{M_s\})$$

defined as follows. An element of the source given by a class of a diagram

$$\{N_t\} \overset{f}{\to} x \overset{g}{\to} \{M_s\}$$

for $x \in \mathcal{M}$, a $*$-opration f, !-operation g, is sent to the equivalence class of the composition

$$\{N_t\}_{t\neq t_0} \times h(N_{t_0}) \to \{N_t\} \overset{f}{\to} x \overset{g}{\to} \{M_s\}$$

It is useful to remember: if $S \overset{\pi_S}{\leftarrow} I \overset{\pi_T}{\to} T$ is a compemlentary pair on a finite nonempty set I let $i_0 \in I_0$, $s_0 = \pi_S(i_0), t_0 = \pi_T(i_0)$. Assume $\pi_T^{-1}(t_0) = \{i_0\}$ and $\mid \pi_S^{-1}(s_0) \mid \geq 2$. Set $T_0 = T\backslash t_0, I_0 = I\backslash i_0$. Then $(S \leftarrow I_0 \to T_0)$ is a complempentary pair!

1.1.13. **For Section 1.3.12.** In 1.3.12 line 1 there is a misprint. They say that if ! pseudo-tensor structure is representable then $P^{*!}_{I,J}(\{L_i\},\{M_j\}) \widetilde{\to} P^*_I(\{L_i\}, \otimes^!_J M_j)$ canonically. So, to define the compound pseudo-tensor structure, instead of $\langle\ \rangle^I_{S,T}$ of (1.3.6.1) it suffices to define these $\langle\ \rangle^I_{S,T}$ in the universal case. Namely, in the notations of 1.3.6, we have

$$P^!_{I/T}(\{N_t\},\{L_i\}) = \prod_{t\in T} \mathrm{Hom}(N_t, \otimes^!_{I_t} L_i)$$

So, it suffices to assume $N_t = \otimes^!_{I_t} L_i$ for any $t \in T$. Thus, we need to define the "compound tensor product maps"

$$\otimes^I_{S,T} : P^*_{I/S}(\{L_i\},\{M_s\}) \to P^*_T(\{\otimes^!_{I_t} L_i\}, \otimes^!_S M_s)$$

A way to memorize is to note that $\otimes^I_{S,T}$ converts a map $I \to S$ to a map $T \to pt$.

Def of compound pseudo-tensor structure assuming $P^!$ representable is comprehensible!!! Comments for the axioms from 1.3.12: assume $T \leftarrow I \to S$ is a complementary pair in $Q(I)$.

i) Take $H \in Q(I)$ with $H \geq S$, let $V = inf(H,T)$. Recall that (S, V) is complementary in $Q(H)$ by my Lemma 1.1.10. Besides, for any $v \in V$, (T_v, H_v) are complementary in $Q(I_v)$ by ([1], 1.3.3.i). The input data is a diagram of P^*-morphisms

$$\{L_i\} \overset{(\alpha_h)}{\to} \{K_h\} \overset{(\beta_s)}{\to} \{M_s\} \tag{1}$$

for $I \to H \to S$. The claim is that the composition of the above maps commutes with the compound tensor product. The diagram of sets behind is

$$\begin{array}{ccccc} I & \to & H & \to & S \\ \downarrow & & \downarrow & & \downarrow \\ T & \to & V & \to & pt \end{array}$$

The diagram (1) yields, taking the tensor products, the diagram

$$\{\otimes^!_{I_t} L_i\} \to \{\otimes^!_{H_v} K_h\} \to \otimes^!_S M_s$$

for the diagram of sets $T \to V \to pt$.

ii) We take $G \in Q(I)$ with $G \geq T$, so we get a diagram

$$\begin{array}{ccccc} I & \to & G & \to & T \\ \downarrow & & \downarrow & & \downarrow \\ S & \to & U & \to & pt, \end{array}$$

where $U = inf(S, G)$. Recall that (U, T) are complementary in $Q(G)$, and (S_u, G_u) are complementary in $Q(I_u)$ for any $u \in U$. This axiom says that the isomorphisms $\otimes^!_{I_t} L_i \widetilde{\to} \otimes^!_{G_t} (\otimes^!_{I_g} L_i)$ (for any $t \in T$) are compatible with P^*-operations.

These axioms are understandable!

1.1.14. For Remark 1.3.15. Let $\mathcal{M}^{*!}$ be a compound tensor category, so $\mathcal{M}^{o!}$ is a tensor category, hence $\mathcal{M}^!$ is also a tensor category (opposite one). Then the natural action of $\mathcal{M}^!$ on $\mathcal{M}$ extends to a $\mathcal{M}^!$-action on $\mathcal{M}^*$. Consider the diagram $I \stackrel{\mathrm{id,id}}{\leftarrow} I \sqcup I \stackrel{\mathrm{id},*}{\to} I \sqcup *$. These are complementary equivalence relations on $I \sqcup I$. Assume given $\phi \in P^*_I(\{M_i\}, N)$. Then consider the map $(\mathrm{id}, \ldots, \mathrm{id}, \phi) \in P^*_{I \sqcup I/I \sqcup *}(\{A_i, M_i\}, \{A_i, N\})$ and apply $\otimes^{I \sqcup I}_{I \sqcup *, I}$, we get a map

$$\{A_i \otimes^! M_i\} \to (\otimes^!_I A_i) \otimes^! N$$

1.1.15. **The Chevalley-Eilenberg complex** is defined in Sect. 1.4.5 in a very general case. The traditional definition is as follows. Given a Lie algebra L (over a field k) and a L-module M, set $C^p(L, M) = \mathrm{Hom}(\wedge^p L, M) \widetilde{\to} (\wedge^p L^*) \otimes M$, here $\wedge^p L$ is considered as a quotient of $L^{\otimes p}$, the skew-coinvariants of S^p. The differential $d : C^p(L, M) \to C^{p+1}(L, M)$ is given as follows:

- for $m \in M, l \in L$, $(dm)l = lm$.
- for $\alpha \in L^*$, $d\alpha : \wedge^2 L \to k$ is given by $(d\alpha)(x, y) = -\alpha([x, y])$ for $x, y \in L$. It extends to $\wedge^* L^*$ as an odd derivative, that is, by the rule $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$.
- it extends to $(\wedge^* L^*) \otimes M$ by $d(\omega \otimes m) = d\omega \otimes m + (-1)^{\deg \omega} \omega \wedge dm$ for $m \in M$, $\omega \in \wedge^* L^*$.

There is another formula for d. Given $\omega \in C^p(L,M)$, $x_i \in L$,

$$(d\omega)(x_1,\dots,x_{p+1}) = \sum_{1\le i<j\le p+1} (-1)^{i+j}\omega([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{p+1})+ \sum_{i=1}^{p+1}(-1)^{i+1}x_i\omega(x_1,\dots,\hat{x}_i,\dots,x_{p+1})$$

Then $\mathrm{H}^p(L,M)$ is the p-th cohomology group of this complex (p-th cohomology group of L with coefficients in M). The above is the cohomological Chevalley complex.

The homological Chevalley complex is $C_p(L,M) = M\otimes \wedge^p L$ with the differential $d: C_p(L,M)\to C_{p-1}(L,M)$ given by

$$d(m\otimes(x_1\wedge\ldots\wedge x_p)) = \sum_{i=1}^{p}(-1)^i x_i m\otimes(x_1\wedge\ldots\wedge\hat{x}_i\wedge\ldots\wedge x_p)+ \sum_{j<k}(-1)^{j+k}m\otimes([x_j,x_k]\wedge x_1\wedge\ldots\wedge\hat{x}_j\wedge\ldots\wedge\hat{x}_k\wedge\ldots\wedge x_p)$$

Its p-th homology group is the p-th Lie algebra homology group $\mathrm{H}_p(L,M)$ ([7], Sect. 9.2).

Remark 1.1.16. *i) In the above taking $M = k$ we get the complex $C := C^\bullet(L,k) = \wedge^* L^*$. It is a commutative DGA. The product map $C\otimes C\to C$ is $\alpha\otimes\beta\mapsto\alpha\wedge\beta$. Recall the standard conventions about cohmological complexes: if K, L are cohomological complexes, the differential in $K\otimes L$ is given by $d(k\otimes l) = dk\otimes l + (-1)^p k\otimes dl$, where $k\in K^p, l\in L^q$. The permutation isomorphism $K\otimes L\widetilde{\to} L\otimes K$ sends $k\otimes l$ as above to $(-1)^{pq} l\otimes k$.*
ii) The homological Chevalley complex of $C := C_\bullet(L,k)$ becomes a commutative DGA with the product $C\otimes C\to C$, $\alpha\otimes\beta\mapsto\alpha\wedge\beta\in\Lambda^{p+q}L$ for $\alpha\in\Lambda^pL, \beta\in\Lambda^qL$.

We will need in 3.4.11 to think about a homological Chevalley complex as a complex, where the differential augments the degrees. To this end, we may set $K^{-p} = C_p(L,k)$. Then $K = (\ldots\to K_{-2}\to K_{-1}\to K_0)$. In other words, $K = \mathrm{Sym}^*(L[1])$.

1.2. **chapter 2.** For Sect. 2.1.7. What they call the **de Rham complex** is also known as Spencer complex ([15], 1.4.2). Namely, if M is a right $\mathcal{D}_X$-module on X smooth, Θ_X is the vector bundle of vector fields then $\mathrm{DR}(M)^i = M\otimes_{\mathcal{O}_X}\wedge^{-i}\Theta_X$ with the differential

$$\delta(m\otimes\xi_1\wedge\ldots\wedge\xi_k) = \sum_{i=1}^{k}(-1)^{i-1}m\xi_i\otimes\xi_1\wedge\ldots\wedge\hat{\xi}_i\wedge\ldots\wedge\xi_k+ \sum_{i<j}(-1)^{i+j}m\otimes[\xi_i,\xi_j]\wedge\ldots\wedge\hat{\xi}_i\wedge\ldots\wedge\hat{\xi}_j\wedge\ldots\wedge\xi_k$$

By ([15], 1.4.4), $\mathrm{DR}(\mathcal{D}_X)$ is a locally free resolution of $\mathcal{O}_X$ by left $\mathcal{D}_X$-modules.

1.2.1. For Sect. 1.4.2: for the definition of the left non-degenerate pairing. If $\langle\rangle\in P_2^*(\{V',V\},1)$, $B\in\mathcal{M}$ then $\mathrm{id}_B\otimes\langle\rangle\in P_2^*(\{B\otimes V',V\},B)$ is defined by applying the binary map $\otimes_{1,2}^{\{1\},\{2,3\}}$ to $(\mathrm{id},\langle\rangle)\in \mathrm{P}_3^*(\{B,V',V\},\{B,1\})$.

1.2.2. For 1.4.3. If $f : I \to J$ is any map of finite sets, let $\pi : I \sqcup J \to J$ be the extension of f, which is id on J. Let $\tilde{I} = I \sqcup *$. The pair $(\tilde{I}, J) \in Q(I \sqcup J)$ is complementary, so we have the tensor product map $\otimes^{I\sqcup J}_{J,\tilde{I}}$ given by (1.4.3.1). This is also used in the next subsection.

1.2.3. For Sect. 2.3.1. I have not found a "classical" definition of a $\mathcal{D}_S$-scheme. Maybe a good definition is ([12], Definition 0.5), the notion of a crystal of schemes over a given smooth scheme S.

1.2.4. **passage to $\bar{\mathbb{Q}}_\ell$-sheaves from $\mathcal{D}$-modules** I think in their notations, the right $\mathcal{D}$-modules should correspond to perverse sheaves. The reasons: 1) if $f : X \to Y$ is affine then $f_* : \mathrm{D}\,\mathcal{M}(X) \to \mathrm{D}\,\mathcal{M}(Y)$ is right exact for $\mathcal{M}^r$ t-structure (sect. 2.1.2); 2) if X is not smooth, its only the notion of a right $\mathcal{D}$-module on X that makes sense, Remark 2.1.3.

Now for a map $f : X \to Y$ their functors $f_*, f^!$ given by (2.1.2.1) correspond to $f_* : \mathrm{D}(X, \bar{\mathbb{Q}}_\ell) \to \mathrm{D}(Y, \bar{\mathbb{Q}}_\ell)$, $f^! : \mathrm{D}(Y, \bar{\mathbb{Q}}_\ell) \to \mathrm{D}(X, \bar{\mathbb{Q}}_\ell)$. Besides, if both X and Y are smooth, their restriction functor $f^* : \mathcal{M}^\ell(Y) \to \mathcal{M}^\ell(X)$ is the functor $\mathrm{Perv}(Y) \to \mathrm{Perv}(X), M \mapsto \mathrm{H}^0(f^! M[\dim Y - \dim X])$, where H^0 denotes the 0-th perverse cohomology sheaf.

For example, if $f : X \hookrightarrow Y$ is a closed immersion and both X, Y are smooth, E is a local system on Y then $\mathrm{IC}(E) = E[\dim Y]$ and $f^! \,\mathrm{IC}(E) \,\tilde{\to}\, \mathrm{IC}(f^* E)[\dim X - \dim Y]$.

In their notations, $\mathcal{M}(X)$ is the category of right $\mathcal{D}$-modules on a smooth scheme. Under the above correspondence, $\omega_X \in \mathcal{M}(X)$ corresponds to $\mathrm{IC}_X \in \mathrm{Perv}(X)$.

1.2.5. For Sect. 2.2. The !-tensor structure on $\mathrm{D}(X, \bar{\mathbb{Q}}_\ell)$, where X is a smooth variety of dimension $\dim X$ is given as in Sect. 2.2.2. That is, for $M_i \in \mathrm{D}(X, \bar{\mathbb{Q}}_\ell)$,

$$\text{(2)} \qquad \overset{L}{\otimes}{}^!_I M_i = \triangle^{(I)!}\,(\boxtimes M_i[\dim X])[-\dim X] \,\tilde{\to}\, \mathbb{D}((\underset{I}{\otimes}\, \mathbb{D} M_i[-\dim X])[\dim X])$$

If M_i are perverse then the latter complex is placed in perverse degrees ≤ 0 (cf. Asterisque 100, 4.1.10ii)). Indeed, $X \to X \times X$ is a regular embedding, so locally in Zarizki topology on X is given by $\dim X$ equations.

They suggest a notation $\otimes^!_I M_i$ for the 0-th perverse cohomology of the above complex. It is indeed essential not to get the shifts outside the brackets! For various sign problems!!! For example, $\mathrm{Aut}\, I$ acts on X^I hence on $\overset{L}{\otimes}{}^!_I M_i$ and we want this isomorphism to be canonical, hence compatible with all natural things. With this definition, IC_X is the unit in $\mathrm{Perv}(X)$ equipped with the "shift definition of $\otimes^!$", that is, $\otimes_{i\in I} M_i$ is the ${}^pH^0$ of the complex (2).

If $\pi : J \to I$ is a surjection of finite nonempty sets, $K_j \in \mathrm{D}(X, \bar{\mathbb{Q}}_\ell)$ for $j \in J$ then the maps

$$\epsilon_\pi : \overset{L}{\otimes}{}^!_J K_j \to \overset{L}{\otimes}{}^!_I (\overset{L}{\otimes}{}^!_{J_i} K_j)$$

given by ([1], (1.1.3.1)) are isomorphisms in $\mathrm{D}(X, \bar{\mathbb{Q}}_\ell)$. The same also for $\mathcal{D}$-modules.

Let us underline that $\overset{L}{\otimes}{}^!$ equips $\mathrm{D}(X, \bar{\mathbb{Q}}_\ell)$ with a symmetric monoidal structure.

1.2.6. Let X be a smooth variety. For the category of perverse sheaves $\mathrm{Perv}(X)$ on X, one can define the compound tensor structure on it exactly as in Sect. 2.2. I think an analog of the middle de Rham cohomology functor will be the functor $h : \mathrm{Perv}(X) \to \mathrm{Vect}_{\bar{\mathbb{Q}}_\ell}$, $h(F) = H^0(X, F)$. With this definition I think it is an augmentation functor on $\mathrm{Perv}(X)^*$ with the operation defined as in (2.2.7.1). Is it true that h is reliable (in the sence of Sect. 1.4.7)?

Drinfeld: for $F \in \mathrm{Perv}(X)$ one may also try to consider the sheaf on etale or Zarisky topology of X associated to the presheaf $U \mapsto \mathrm{H}^0(U, F)$. For $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaves, this is pretty tautological, but not tautological for perverse sheaves over $\mathbb{Z}_\ell$ or $\bar{\mathbb{Q}}_\ell$.

The analogy between $\mathcal{D}$-modules and perverse sheaves is not perfect. In particular, in the constructible world we have no analogs of non-holonomic $\mathcal{D}$-modules.

If $\mathcal{B}$ is a k-operad then we get, as in Sect. 1.4.1, the category $\mathcal{B}(\mathrm{Perv}(X)^*)$ of $\mathcal{B}$-algebras in $\mathrm{Perv}(X)^*$. In particular, the categorie of Lie*-algebras and of unital associative algebras. A Lie*-algebra in $\mathrm{Perv}(X)^*$ is a perverse sheaf F on X together with a bracket map $[.,.] : F \boxtimes F \to \vartriangle_* F$, which is anti-commutative and satisfies the Jacobi identity. The Jacobi identity means that the corresponding map $F^{\boxtimes 3} \to \vartriangle_* F$ on X^3 vanishes.

For such a Lie*-algebra L the category $\mathcal{M}(L)$ of L-modules in $\mathrm{Perv}(X)^*$ is defined as in Sect. 1.2.13. This is $F \in \mathrm{Perv}(X)$ together with an action map $L \boxtimes F \to \vartriangle_* F$ on X^2 that satisfies the usual axiom: for $l_i \in L$, $f \in F$, $l_1(l_2 f) - l_2 l_1(f) = [l_1, l_2] f$. By Lemma 1.2.14, $\mathcal{M}(L)$ is an abelian category. By 1.2.18, $\mathcal{M}(L)$ is an abelian pseudo-tensor category. In fact, it is an abelian compound tensor category by Lemma 1.4.4.

The Lie algebra cohomology of $F \in \mathcal{M}(L)$ is defined according to Sect. 1.4.5.

Question. What is the structure of the tensor category $\mathrm{Perv}(X)^!$? In particular, it contains inside the tensor subcategory of local systems on X, I think.

By Remark 1.3.15, the tensor category $\mathrm{Perv}(X)^!$ acts on $\mathrm{Perv}(X)^*$. The action map $\mathrm{Perv}(X)^! \times \mathrm{Perv}(X)^* \to \mathrm{Perv}(X)^*$ sends (F, N) to $F \otimes^! N$. So, for example a commutative algebra $F \in \mathrm{Perv}(X)^*$ yields a pseudo-tensor functor $\mathrm{Perv}(X)^! \to \mathrm{Perv}(X)^*$, $M \mapsto M \otimes^! F$.

1.2.7. For 2.6.1. Let L be a Lie algebra in $\mathrm{Perv}(X)^*$, recall that $\mathcal{M}(X, L)$ denotes the category of L-modules in $\mathrm{Perv}(X)^*$. The tensor structure on this category is defined as in 1.4.4: if $M, N \in \mathcal{M}(X, L)$ then $M \otimes^! N \in \mathrm{Perv}(X)$ is a L-module, where L acts by Leibnitz rule. Namely, for surjections $\tau : \{1,2,3\} \to \{1,2\}$, $\pi : \{1,2,3\} \to \{1,2\}$, $\xi : \{1,2,3\} \to \{1,2\}$ with $\tau^{-1}(2) = \{2,3\}$, $\pi^{-1}(1) = \{1,2\}$, $\xi^{-1}(2) = \{1,3\}$ we have a cartesian squares

$$\begin{array}{ccc} X^2 & \overset{\vartriangle^{(\tau)}}{\to} & X^3 \\ \uparrow & & \uparrow \vartriangle^{(\pi)} \\ X & \overset{\vartriangle}{\to} & X^2 \end{array} \qquad \begin{array}{ccc} X^2 & \overset{\vartriangle^{(\tau)}}{\to} & X^3 \\ \uparrow & & \uparrow \vartriangle^{(\xi)} \\ X & \overset{\vartriangle}{\to} & X^2 \end{array}$$

Write a_M, a_N for the L-actions on M, N. We get maps

$$L \boxtimes (M \otimes^! N) \widetilde{\to} \vartriangle^{(\tau)!} (L \boxtimes M \boxtimes N)[1] \overset{a_M}{\to} \vartriangle^{(\tau)!} \vartriangle_*^{(\pi)} (M \boxtimes N)[1] \widetilde{\to} \vartriangle_* (M \otimes^! N)$$

and

$$L\boxtimes(M\otimes^! N)\widetilde{\to}\ \triangle^{(\tau)!}\,(L\boxtimes M\boxtimes N)[1]\stackrel{a_N}{\to}\triangle^{(\tau)!}\triangle_*^{(\xi)}\,(M\boxtimes N)[1]\widetilde{\to}\ \triangle_*\,(M\otimes^! N),$$

where all the functors are understood in the nonderived perverse sense. Their sum is the desired action map of L on $M\otimes^! N$. Maybe there is a better way to write taking into account signs.

Definition 1.2.8. *A coisson algebra in* $\mathrm{Perv}(X)$ *is a unital commutative algebra A in* $\mathrm{Perv}(X)^!$ *together with a structure of Lie algebra* $\{.,.\}$ *on* $A\in\mathrm{Perv}(X)^*$ *such that the maps* $A\otimes^! A\to A$ *and* $1_A:\mathrm{IC}\to A$ *are morphisms in* $\mathcal{M}(X,A)$*, that is, these morphisms commute with actions of the Lie algebra A. Of course, the action* $A\boxtimes\mathrm{IC}\to\triangle_* A$ *is the zero map. So, for* $a,b,c\in A$ *we have* $\{a,bc\}=\{a,b\}c+a\{b,c\}$*.*

1.3. **For chapter 3.** Work over an algebraically closed field k. We will ignore the Tate twists everywhere.

1.3.1. For **Chiral operations** (Sect. 3.1.1). If X is our curve, $j:U^{(I)}\hookrightarrow X^I$ is an affine open embedding, so j_* is exact for the $\mathcal{M}^r$ t-structure on $\mathcal{D}$-modules. For this reason the functor $P_I^{ch}(\{L_i\},M)$ defined in (3.1.1.1) is left exact in each variable $\in\mathcal{M}(X)$.

If $\pi:J\to I$ is a surjection of finite sets, they denote by $\triangle^{(\pi)}\colon X^I\to X^J$ the diagonal, $j^{(J)}:U^{(J)}\hookrightarrow X^J$ the complement to all the diagonals, and

$$U^{(\pi)}=\{(x_j)\in X^J\mid x_{j_1}\neq x_{j_2}\text{ if }\pi(j_1)\neq\pi(j_2)\},$$

$j^{(\pi)}:U^{(\pi)}\to X^J$. So, $U^{(I)}\subset U^{(\pi)}\subset X^J$, and the square is cartesian

$$\begin{array}{ccc} X^I & \stackrel{\triangle^{(\pi)}}{\hookrightarrow} & X^J\\ \uparrow j^{(I)} & & \uparrow j^{(\pi)}\\ U^{(I)} & \hookrightarrow & U^{(\pi)}\end{array}$$

One defines the pseudo-tensor category $\mathrm{Perv}(X)^{ch}$ as in 3.1.2. The augmentation functor $\mathrm{Perv}(X)^{ch}\to\mathrm{Vect}_k$ sends F to $\mathrm{H}^0(X,F)$. The structure map

$$h_{I,i_0}:P_I^{ch}(\{L_i\},M)\otimes h(L_{i_0})\to P_{I-i_0}^{ch}(\{L_i\},M)$$

sends $\phi:j_*^{(I)}j^{(I)*}(\boxtimes_I L_i)\to\triangle_*^{(I)}\,M$ to the composition

$$\mathrm{H}^0(X,L_{i_0})\otimes(j_*^{(I-i_0)}j^{(I-i_0)*}\boxtimes_{I-i_0}L_i)\to\mathrm{pr}_*\,j_*^{(I)}j^{(I)*}(\boxtimes_I L_i)\stackrel{\mathrm{pr}_*\phi}{\to}\triangle_*^{(I)}\,M,$$

where $\mathrm{pr}:X^I\to X^{I-i_0}$ is the projection.

Remark: If $J\to I$ is an isomorphism of finite nonempty sets, then the composition map $P_I^{ch}(\{L_i\},M)\otimes(\otimes_I P_{J_i}^{ch}(K_j,L_i))\to P_J^{ch}(\{K_j\},M)$ is the evident one. Namely, we are given maps $\phi_i:K_i\to L_i$ and $\psi:j_*^{(I)}j^{(I)*}(\boxtimes L_i)\to M$. The composition is

$$j_*^{(I)}j^{(I)*}(\boxtimes K_i)\stackrel{\boxtimes\phi_i}{\to}j_*^{(I)}j^{(I)*}(\boxtimes L_i)\stackrel{\psi}{\to}M$$

1.3.2. The projection formula for the derived !-tensor product in $\mathrm{D}(X,\bar{\mathbb{Q}}_\ell)$. Let $f : X \to Y$ be a morphism, $F \in \mathrm{D}(X,\bar{\mathbb{Q}}_\ell), G \in \mathrm{D}(Y,\bar{\mathbb{Q}}_\ell)$. Then, understanding the functors in the derived sense, we have $(f_*F) \overset{L}{\otimes}{}^! G \widetilde{\to} f_*(F \overset{L}{\otimes}{}^! f^!G)$. More precisely, this would be true if we understand $\overset{L}{\otimes}{}^!_I L_i$ as $\mathbb{D}(\otimes_I \mathbb{D}L_i)$, without any shifts!

For the "shifts definition in the derived sense" of $\otimes^!$, we get the isomorphism

$$(f_*F) \overset{L}{\otimes}{}^! G \widetilde{\to} f_*(F \overset{L}{\otimes}{}^! f^!G)[\dim Y - \dim X] \tag{3}$$

This is canonical because the symmetry between the two factors in the tensor product is broken!

Now assume f is an affine open immersion and F, G are perverse sheaves. Then passing to the 0-th perverse cohomology sheaf, we get an isomorphism of perverse sheaves $(f_*F) \otimes^! G \widetilde{\to} f_*(F \otimes^! f^!G)$, where now f_* is understood in the non derived sense (here we may use either the definition with shifts or without them).

This is used in Sect 3.1.3 (In Sect. 3.1.3 and around by the tensor product in $\mathcal{M}(X)$ they understand the !-tensor product of $\mathcal{D}$-modules). For any $N, A_i \in \mathrm{Perv}(X)$, one has an isomorphism

$$(\triangle_*^{(I)} N) \otimes^! (\boxtimes_I A_i) \widetilde{\to} \triangle_*^{(I)} (N \otimes^! (\otimes^!_I A_i)) \otimes \lambda_I \tag{4}$$

in $\mathrm{Perv}(X)$. Recall that $\otimes^!_I A_i$ is defined as the 0-th perverse cohomology sheaf of $\triangle^{(I)!} (\boxtimes_I A_i[1])[-1]$ by (2.2.2.1), here H^d denotes the d-th perverse cohomology sheaf.

For $M_i, A_i \in \mathrm{D}(X,\bar{\mathbb{Q}}_\ell)$ one has canonically

$$(\boxtimes_I M_i) \overset{L}{\otimes}{}^! (\boxtimes_I A_i) \widetilde{\to} (\boxtimes_I (M_i \overset{L}{\otimes}{}^! A_i)) \otimes \lambda_I$$

The action of $\mathrm{Perv}(X)^!$ on $\mathrm{Perv}(X)^{ch}$ is defined as in Sect. 3.1.3, namely as follows. Given $\phi : j_*^{(I)} j^{(I)*}(\boxtimes_I M_i) \to \triangle_* N$ and $A_i \in \mathrm{Perv}(X)$, applying $\triangle^!$, where $\triangle$: $X^I \to X^I \times X^I$ is the diagonal map, we get from

$$j_*^{(I)} j^{(I)*}(\boxtimes_I M_i) \boxtimes (\boxtimes_I A_i) \overset{\phi\boxtimes\mathrm{id}}{\to} (\triangle_* N) \boxtimes (\boxtimes_I A_i)$$

the desired map

$$j_*^{(I)} j^{(I)*}(\boxtimes_I (M_i \otimes^! A_i)) \to \triangle_*^{(I)} N \otimes^! (\otimes^!_I A_i)$$

1.3.3. About the **Cousin complex**, Sect. 3.1.5. One of its definition is found in [8].

Consider in the case of $\bar{\mathbb{Q}}_\ell$-sheaves the following situation. Let Y be a stack with a stratification by $Y = \sqcup_{k\ge 0} Y_k$ such that the closure $\bar{Y}_m = \cup_{k\ge m} Y_k$. So, Y_0 is open and dense in Y. Assume the inclusion $j_m : Y_m \to Y$ is affine. Let F be a perverse sheaf on Y. Assume for any $m \ge 0, j_m^* F$ is placed in perverse degree $-m$. Let $F_m = (j_m)_! j_m^* F[-m]$, so $Y \in \mathrm{Perv}(Y)$. Then the following complex

$$\ldots \to F_2 \to F_1 \to F_0 \to F \to 0$$

is an exact sequence of perverse sheaves on Y. The transition maps in the complex are as follows. Let $Y_{[k,m]} = \cup_{k\le i\le m} Y_i$. Then Y_m is closed in $Y_{[m-1,m]}$, and its complement Y_{m-1} is open in $Y_{[m-1,m]}$. This gives an exact triangle on $Y_{[m-1,m]}$

$$(j_{m-1})_! j_{m-1}^* F \to F \mid_{Y_{[m-1,m]}} \to (j_m)_! j_m^* F,$$

where we used the $*$-restriction to $Y_{[m-1,m]}$. We extend the latter by zero to Y, this gives the transition maps.

We can also dualize the above exact sequence, then we get the following claim. Let $\mathcal{F}$ be a perverse sheaf on Y such that $j_m^!\mathcal{F}[m]$ is perverse for any $m \geq 0$. Set $\mathcal{F}_m = (j_m)_*j_m^!\mathcal{F}[m]$, this is a perverse sheaf on Y. The exact triangle

$$(j_m)_*j_m^!\mathcal{F} \to \mathcal{F} \mid^!_{Y_{[m-1,m]}} \to (j_{m-1})_*j_{m-1}^!\mathcal{F},$$

which we extend to Y by $*$-extension, gives the transition maps, and we get the exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \dots$$

of perverse sheaves on Y.

Example appearing in 3.1.5. Let I be a finite set, assume X is a curve. Consider $Y = X^I$ with the stratification given by all the diagonals. Take now $\bar{Y}_m$ be the closed subscheme, the union of all the diagonals of dimension $\leq\mid I \mid -m$, and $Y_m = \bar{Y}_m - \bar{Y}_{m+1}$. The open immersion $j_m : Y_m \to \bar{Y}_m$ is affine. Take $\mathcal{F} = \mathrm{IC}$ on $Y = X^I$. Then $j_m^! \mathrm{IC}_{X^I} \widetilde{\to} \mathrm{IC}[-m]$, and we get the exact sequence

$$0 \to \mathrm{IC}_{X^I} \to (j_0)_* \mathrm{IC} \to (j_1)_* \mathrm{IC} \to \dots \to (j_{|I|-1})_* \mathrm{IC}$$

If $\dim X > 1$ then the above does not work! That seems to be a first place where the assumption $\dim X = 1$ made on p. 157, Sect. 3.1, is used!

1.3.4. As in 3.1.4, if $\lambda_I = (\bar{\mathbb{Q}}_\ell[1])^{\otimes I}[- \mid I \mid]$, $\mathrm{Aut}\, I$ acts on this line by the sign character. One has canonically $\mathrm{IC}_X^{\boxtimes I} \otimes \lambda_I \widetilde{\to} \mathrm{IC}_{X^I}$. We have an exact sequence of perverse sheaves

$$0 \to \mathrm{IC} \to j_*j^* \mathrm{IC} \to \triangle_* \mathrm{IC} \to 0$$

on X^2 (we ignore the Tate twists). This gives $\lambda_I \widetilde{\to} \mathrm{Hom}_{X^2}(j_*j^*(\mathrm{IC}_X^{\boxtimes I}), \triangle_* \mathrm{IC})$ for a set I of 2 elements. Let

$$[,] \in \mathrm{Hom}_{X^2}(j_*j^*(\mathrm{IC}_X \boxtimes \mathrm{IC}_X), \triangle_* \mathrm{IC})$$

be a generator given by an order $I \widetilde{\to} \{1,2\}$ on the set I. It is skew-symmetric.

For the proof of 3.1.5: the beginning of the Cousin complex on X^3 is

$$(j_0)_*j_0^* \mathrm{IC}^{\boxtimes 3} \otimes \lambda_3 \to (j_1)_* \mathrm{IC} \to (j_2)_* \mathrm{IC}$$

Write the equivalence relations $(1 = 2)$ on $I = \{1,2,3\}$ as $\{12,3\}$. Then the construction is that on any quotient $\pi : I \to J$ of order 2, the element $j \in J$ such that $\mid \pi^{-1}(j) \mid= 2$ should be smaller then the second element of J. Then further pick an order on the fibre $\pi^{-1}(j)$. This yields an order on I, the lexicographical one. This order on I can be seen as a permutation σ, what matters is $\mathrm{sign}(\sigma)$. The corresponding summand is $(-1)^{\mathrm{sign}(\sigma)}[[,]_{\pi^{-1}(j)}, i]$, where $J = \{j, i\}$ and $\pi^{-1}(j)$ has 2 elements. We get the Jacobi identity!

The Cousin complex on X^I is easier to write for the dualizing copmplex $\omega_{X^I} \in \mathrm{D}(X^I, \bar{\mathbb{Q}}_\ell)$. Namely, if $\mid I \mid= n$ then it is the complex

$$j_*^{(I)}(j^{(I)})^*\omega_{X^I} \to \bigoplus_{T\in Q(I)_{n-1}} \triangle_*^{(I/T)}\, j_*^{(T)}(j^{(T)})^*\omega_{X^T} \to \bigoplus_{T\in Q(I)_{n-2}} \triangle_*^{(I/T)}\, j_*^{(T)}(j^{(T)})^*\omega_{X^T} \to \dots,$$

where $Q(I)_m$ is the set of equivalence relations on I viewed as $I \to J$ with J with m elements. This is a resolution of ω_{X^I} by their Section 3.1.5.

1.3.5. For 3.1.6. See my note on the category of special perverse sheaves (it is an abelian category for any scheme with a finite stratification).

If P is any k-operad, that is, for a finite nonempty set I, P_I is a k-vector space, and the compositions are k-linear, we can define the following category. Pick a finite nonempty set I. Let $Q(I)$ be the category, whose objects are quotients $I \to S$, and a morphism from S to T exists if T is a quotient of S, in which case $\mathrm{Hom}_{Q(I)}(S,T) = \otimes_{t\in T} P_{S_t}$. Then $Q(I)$ is a k-linear category. Given $S \stackrel{\pi}{\to} T \stackrel{\kappa}{\to} J$, the composition $\mathrm{Hom}(T,J)\otimes \mathrm{Hom}(S,T) \to \mathrm{Hom}(S,J)$ is the composition map $P_{T/J}\otimes P_{S/T} \to P_{S/J}$.

1.3.6. For 3.1.12. If $M \in \mathrm{Perv}(X)$ then for $\vartriangle: X \hookrightarrow X\times X$ one has canonically $\vartriangle^! (\mathrm{IC}\boxtimes M)[1] \widetilde{\to} M$ (even in the derived category, not only on the level of abelian categories of perverse sheaves). For any $M \in \mathrm{Perv}(X)$ we have an exact sequence of perverse sheaves $0 \to \mathrm{IC}\boxtimes M \to j_*j^*(\mathrm{IC}\boxtimes M) \to \vartriangle_* M \to 0$ on $X\times X$, where $j : U^{(2)} \hookrightarrow X\times X$. This gives the *unit operation* $\varepsilon_M \in P_2^{ch}(\{\mathrm{IC}, M\}, M)$.

Their 3.1.13 is a great phenomenon. It says that for a finite nonempty set I and $\phi \in P_I^{ch}(\{M_i\}, N)$ we should think about the composition $\epsilon_N\phi \in P_{\tilde I}^{ch}(\{\mathrm{IC}_X, M_i\}, N)$ as the derivative of ϕ. Here $\tilde I = I \sqcup *$.

The exact sequence (3.1.13.1) is the short exact sequence of perverse sheaves on $X^{\tilde I}$

$$0 \to \mathrm{IC}_X \boxtimes j_*^{(I)} j^{(I)*}(\boxtimes_i M_i) \to j_*^{(\tilde I)} j^{(\tilde I)*}(\mathrm{IC}_X \boxtimes(\boxtimes_i M_i)) \to \bigoplus_{m\in I} \vartriangle_*^m j_*^{(I)} j^{(I)*}(\boxtimes_i M_i)$$

Here $\vartriangle^i: X^I \to X^I \times I$ is the graph of i-th projection, $j^{(I)} : U^{(I)} \hookrightarrow X^I$ is the complement to all the diagonals, and similarly for $j^{(\tilde I)} : U^{(\tilde I)} \hookrightarrow X^{\tilde I}$. Here $M_i \in \mathrm{Perv}(X)$ for $i \in I$.

1.3.7. For 3.2.2. It seems in many places in Sect. 3 they abbreviate $\otimes^!$ to $\otimes$, for example, in 3.1.3 and 3.2.2. For perverse sheaves, the map $\mathrm{Perv}(X)^! \otimes \mathrm{Lie} \to \mathrm{Perv}(X)^{ch}$ is defined as the following extension of the identity functor. For a finite nonempty set I, $L_i, M \in \mathrm{Perv}(X)$ the map

$$\alpha_I : \mathrm{Hom}(\otimes^! L_i, M)\otimes \mathrm{Lie}_I \to \mathrm{Hom}(j_*^{(I)} j^{(I)*}\boxtimes L_i, \vartriangle_*^{(I)} M) = P_I^{ch}(\{L_i\}, M)$$

is defined as follows. Let $\phi \in \mathrm{Hom}(\otimes^! L_i, M), \nu \in \mathrm{Lie}_I = P_I^{ch}(\mathrm{IC})$. Recall that we have $\mathrm{IC}\otimes^! j^{(I)}F \widetilde{\to} j^{(I)}F$ for any $F \in \mathrm{Perv}(X^I)$, where $\otimes^!$ is understood in the "shifts definition" sense. So,

$$\begin{aligned} j_*^{(I)} j^{(I)*}(\boxtimes_I L_i) &\widetilde{\to} j_*^{(I)}(\mathrm{IC}\otimes^! j^{(I)*}(\boxtimes_I L_i)) \widetilde{\to} (j_*^{(I)}\mathrm{IC})\otimes^! (\boxtimes_I L_i) \widetilde{\to} \\ & j_*^{(I)}(\boxtimes_I \mathrm{IC})\otimes^! (\boxtimes_I L_i)\otimes \lambda_I \stackrel{\nu\otimes \mathrm{id}}{\to} (\vartriangle_*^{(I)}\mathrm{IC})\otimes^! (\boxtimes_I L_i)\otimes\lambda_I \widetilde{\to} \\ & \vartriangle_*^{(I)}(\mathrm{IC}\otimes^!(\otimes^! L_i)) \widetilde{\to} \vartriangle_*^{(I)}(\otimes^! L_i) \stackrel{\phi}{\to} \vartriangle_*^{(I)} M \end{aligned}$$

Here the last isomorphism in the 2nd line is (4).

It is clear that the above map α_I is injective. Indeed,

$$j_*^{(I)} j^{(I)*}(\boxtimes_I \mathrm{IC}) \to (\vartriangle_*^{(I)}\mathrm{IC})\otimes \mathrm{Lie}_I^*$$

is surjective, here Lie_I^* is the vector space dual to Lie_I. So, the corresponding chiral operation lies in the image of α_I iff the map $j_*^{(I)} j^{(I)*}(\boxtimes L_i) \to \triangle_*^{(I)} M$ factors through the quotient

$$j_*^{(I)} j^{(I)*}(\boxtimes L_i) \to (\triangle_*^{(I)} (\otimes^! L_i)) \otimes \mathrm{Lie}_I^*$$

(here $\otimes^!$ is understood in the "shifts definition sense").

1.3.8. For 3.2.3. On our curve X the functor

$$\mathrm{Perv}(X) \times \mathrm{Perv}(X) \to \mathrm{D}(X),\ L_1, L_2 \mapsto \triangle^! (L_1 \boxtimes L_2)[1] = L_1 \overset{L}{\otimes}{}^! L_2$$

is not exact for the perverse t-structure. For example, if $i_x : \{x\} \hookrightarrow X$ then

$$i_{x!}\bar{\mathbb{Q}}_\ell \overset{L}{\otimes}{}^! i_{x!}\bar{\mathbb{Q}}_\ell \widetilde{\to} i_{x!}\bar{\mathbb{Q}}_\ell[1]$$

is placed in perverse degree -1. For this reason it seemed surprising for me that $0 \to P_2^{!\,\mathrm{Lie}} \to P_2^{ch} \to P_2^*$ is exact. But this is easy:

For $L_i \in \mathrm{Perv}(X)$ the triangle is distinguished $L_1 \boxtimes L_2 \to j_*j^*(L_1 \boxtimes L_2) \to \triangle_*\triangle^! (L_1 \boxtimes L_2)[1]$, where ! is understood in the derived sense. So, using the "shifts definition of $\otimes^!$", one gets an exact sequence on X^2

$$0 \to \mathrm{H}^{-1}(\triangle_*\triangle^! (L_1 \boxtimes L_2)[1]) \to L_1 \boxtimes L_2 \to j_*j^*(L_1 \boxtimes L_2) \to \triangle_* (L_1 \otimes^! L_2) \to 0,$$

here H^i denotes the perverse cohomology sheaf. So, for $M \in \mathrm{Perv}(X)$ the sequence is exact $0 \to \mathrm{Hom}_{X^2}(\triangle_* (L_1 \otimes^! L_2), \triangle_* M) \to P^{ch}(\{L_1, L_2\}, M) \to P^*(\{L_1, L_2\}, M)$.

1.3.9. For 3.2.4. Let $L_1 \subset \ldots \subset L_n = L$ be a filtration on $L \in \mathrm{Perv}(Y)$, Y smooth of some pure dimension, $M \in \mathrm{Perv}(Y)$. Set $V_i = L_i/L_{i-1}$.

Lemma 1.3.10. *There is a filtration $\{W_i\}$ on $L \otimes^! M$ with natural surjections $V_i \otimes^! M \to (W_i/W_{i-1}) \otimes^! M$, here we use the "shifts definition of $\otimes^!$", so that $\otimes^!$ is right exact.*

Proof. Abbreviate $\otimes^! = \otimes$. This is a general phenomenon, any abelian category $\mathcal{A}$ and the functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ right exact in each variable. We define by induction quotients $V_i \otimes M \to \overline{V_i \otimes M}$ and the corresponding filtration.

First, $0 \to V_{n-1} \to L_n/L_{n-2} \to V_n \to 0$ yields an exact sequence $V_{n-1} \otimes M \to (L_n/L_{n-2}) \otimes M \to V_n \otimes M \to 0$. Hence an exact sequence $0 \to \overline{V_{n-1} \otimes M} \to (L_n/L_{n-1}) \otimes M \to \overline{V_n \otimes M} \to 0$, where $\overline{V_{n-1} \otimes M}$ is the image of the first map, and $V_n \otimes M = \overline{V_n \otimes M}$.

Now $0 \to V_{n-2} \to L_n/L_{n-3} \to L_n/L_{n-2} \to 0$ gives $V_{n-2} \otimes M \to (L_n/L_{n-3}) \otimes M \to (L_n/L_{n-2}) \otimes M \to 0$. Denote by $\overline{V_{n-2} \otimes M}$ the image of the first map. We get a filtration on $(L_n/L_{n-3}) \otimes M$ with succesive quotients $\overline{V_i \otimes M}$ for $i = n-2, n-1, n$.

Now continue with $0 \to V_{n-3} \to L_n/L_{n-4} \to L_n/L_{n-3} \to 0$ and so on. □

1.3.11. For Sect. 3.3.1. If X is a smooth scheme, we have the tensor category $\mathrm{Perv}(X)^!$, and $Com^!(X) := Com(\mathrm{Perv}(X)^!)$ is the category of nonunital commutative algebras in $\mathrm{Perv}(X)^!$. Here for $L_i \in \mathrm{Perv}(X)$, $L_1 \otimes^! L_2 = \mathrm{H}^0(\triangle^! ((L_1[\dim X]) \boxtimes (L_2[\dim X]))[-\dim X])$, where H^0 denotes the 0-th perverse cohomology.

Difficult question. Given A a commutative algebra with unit in $\mathrm{Perv}(X)^!$, can we define $\mathrm{Spec}\, A$, to which extend we can localize? I mean for $\mathcal{D}$-modules there is a notion of a $\mathcal{D}_X$-scheme, what is the analog of this notion for perverse sheaves? More precise question: if say $A = E[\dim X]$, where E is a local system, then E becomes a sheaf of commutative algebras with unit on X, so for each $x \in X$ we get a $\bar{\mathbb{Q}}_\ell$-algebra E_x, and we have the set $\operatorname{Spec} E_x$. What is the analog of this set over those points where A is no longer smooth?

1.3.12. For 3.3.2. Let A be a Lie algebra in $\mathrm{Perv}(X)^{ch}$, we write simply Lie^{ch} algebra. Then $\mathrm{H}^0(X, A)$ is a Lie algebra over $\bar{\mathbb{Q}}_\ell$, and it acts on A by the adjoint action. Namely, if $\mu : j_*j^*(A \boxtimes A) \to \triangle_* A$ is the chiral product then taking the direct image under $X^2 \stackrel{\mathrm{pr}_2}{\to} X$, we get the action map $\mathrm{H}^0(X, A) \otimes A \to A$ in $\mathrm{Perv}(X)$. The latter map depends only on the Lie*-bracket $\beta(\mu)$.

For 3.3.3. If A is a Lie^{ch}-algebra (in $\mathrm{Perv}(X)$) then a unit of A is a morphism $1_A : \mathrm{IC}_X \to A$ in $\mathrm{Perv}(X)$ such that the composition $\{\mathrm{IC}_X, A\} \stackrel{1_A, \mathrm{id}}{\to} \{A, A\} \stackrel{\mu}{\to} A$ in $\mathrm{Perv}(X)^{ch}$ is the unit operation ε_A from 3.1.12.

Notation: $\mathrm{Lie}^{ch}(X)$ the k-category of Lie algebras in $\mathrm{Perv}(X)^{ch}$, and $\mathcal{C}\mathrm{A}(X) \subset \mathrm{Lie}^{ch}(X)$ the subcategory of chiral algebras (with unit) and morphisms preserving units. This is not a full subcategory, I think.

If $A \in \mathrm{Perv}(X)$ then a morphism $\mathrm{IC}_X \to A$ is completely determined by its restriction to the generic point. Indeed, if $i : Z \hookrightarrow X$ is closed (with $\dim Z = 0$) and $j : U \hookrightarrow X$ is the complement then j is affine, and $0 \to \mathrm{H}^0(i^*A) \to A \to j_*j^*A$ is an exact sequence of of perverse sheaves with $\mathrm{Hom}(\mathrm{IC}_X, \mathrm{H}^0(i^*A)) = 0$.

Note that if A is a Lie algebra in $\mathrm{Perv}(X)^{ch}$, which vanishes at the generic point of X, then $A = \oplus_{i=1}^n A_{x_i}$ for some distinct points $x_1, \ldots, x_n$. Here we identify the $*$-fibre A_{x_i} with $i_! A_{x_i}$ for $i : \{x_i\} \hookrightarrow X$. Then $j^*(A \boxtimes A) = \oplus_{i \neq j} A_{x_i} \otimes A_{x_j}$. So, any map $j_*j^*(A \boxtimes A) \to \triangle_* A$ vanishes. So, A can not have a unit, because the composition

$$j_*j^*(\mathrm{IC} \boxtimes A) \stackrel{1_A \boxtimes \mathrm{id}}{\to} j_*j^*(A \boxtimes A) \to \triangle_* A \tag{5}$$

must be surjective. So, A is not a chiral algebra. Thus, a chiral algebra does not vanish at the generic point of X. If A is a chiral algebra (and $A \neq 0$) then the unit $1_A : \mathrm{IC} \to A$ does not vanish at the generic point. Indeed, otherwise, $1_A = 0$ and the composition (5) would vanish.

If A is a commutative chiral algebra (with unit) on X then the unit axiom says: $1_A \to A$ is such that the composition $A = \mathrm{IC} \otimes^! A \stackrel{1_A \otimes \mathrm{id}}{\to} A \otimes^! A \to A$ is the identity. The proof of the uniqueness of unit in A in this case is the same as in any commutative algebra: if $1_A, 1'_A$ are units then $1_A = m(1_A \otimes^! 1'_A) = 1'_A$, where $m : A \otimes^! A \to A$ is the product map.

If A is a Lie algebra in $\mathrm{Perv}(X)^{ch}$ and $1_A : \mathrm{IC} \to A$ any section then the composition $\mathrm{IC} \boxtimes A \to j_*j^*(\mathrm{IC} \boxtimes A) \stackrel{1_A \boxtimes \mathrm{id}}{\to} j_*j^*(A \boxtimes A) \to \triangle_* A$ vanishes, because $\mathrm{Hom}_{X^2}(\mathrm{IC} \boxtimes A, \triangle_* A) = 0$ for any perverse sheaf $A \in \mathrm{Perv}(X)$. So, we get the map, say

$$\bar{1}_A : j_*j^*(\mathrm{IC} \boxtimes A)/(\mathrm{IC} \boxtimes A) = \triangle_* A \to \triangle_* A$$

Then 1_A is a unit section iff $\bar{1}_A = \mathrm{id} : A \to A$ on X.

Lemma 1.3.13. *Let A be a Lie algebra in* $\mathrm{Perv}(X)^{ch}$*. If a unit section* $1_A : \mathrm{IC} \to A$ *exists, it is unique.*

Proof. We have an exact sequence $0 \to \mathrm{IC} \boxtimes \mathrm{IC} \to j_*j^*(\mathrm{IC} \boxtimes \mathrm{IC}) \stackrel{[,]}{\to} \triangle_* \mathrm{IC} \to 0$, where $[,]$ is a canonical generator of Lie_2 (see Sect. 3.1.5). Let $1, 1' : \mathrm{IC} \to A$ be unit sections. The composition $j_*j^*(\mathrm{IC} \boxtimes \mathrm{IC}) \stackrel{1\boxtimes 1'}{\to} j_*j^*(A \boxtimes A) \to \triangle_* A$ factors as $j_*j^*(\mathrm{IC} \boxtimes \mathrm{IC}) \stackrel{[,]}{\to} \triangle_* \mathrm{IC} \stackrel{1'}{\to} \triangle_* A$. On the other hand, applying the same for $1' \boxtimes 1$ instead of $1 \boxtimes 1'$, we get that the compositions $j_*j^*(\mathrm{IC} \boxtimes \mathrm{IC}) \stackrel{[,]}{\to} \triangle_* \mathrm{IC} \stackrel{1'}{\to} \triangle_* A$ and $j_*j^*(\mathrm{IC} \boxtimes \mathrm{IC}) \stackrel{[,]}{\to} \triangle_* \mathrm{IC} \stackrel{1}{\to} \triangle_* A$ are the same. So, $1 = 1'$. □

They write $\mathrm{Comu}^!(X)$ for the category of commutative unital algebras in $\mathrm{Perv}(X)^!$.

1.3.14. If I is a 2 elements set and A is a Lie^{ch}-algebra on X then on X^I we get a canonical morphism $j_*j^*(A[1]^{\boxtimes I}) \to \triangle_* A[2]$. Indeed, $A[1]^{\boxtimes I} \tilde{\to} (A^{\boxtimes I}) \otimes \lambda_I[2]$, and $\mathrm{Lie}_I \tilde{\to} \lambda_I$ canonically, because Aut_I acts on Lie_I by the sign character. So, the map $\mathrm{Lie}_I \to P_I^{ch}(\{A, A\}, A)$ is a distinguished element in $\mathrm{Hom}(\lambda_I \otimes j_*j^*(A^{\boxtimes I}), \triangle_* A)$. This is used in 3.4.11.

1.3.15. Let $A \in \mathcal{CA}(X)$ be a chiral algebra. An *A*-**module** is $M \in \mathrm{Perv}(X)$ together with an action $\mu_M : j_*j^*(A \boxtimes M) \to \triangle_* M$ such that this is a Lie^{ch}-action of a Lie^{ch}-algebra, and the composition $j_*j^*(\mathrm{IC} \boxtimes M) \stackrel{1\boxtimes \mathrm{id}}{\to} j_*j^*(A \boxtimes M) \to \triangle_* M$ equals the unit operation ε_M from Sect. 3.1.12.

A morphism of A-modules is just a morphism $f : M \to N$ in $\mathrm{Perv}(X)$ such that the diagram commutes

$$\begin{array}{ccc} j_*j^*(A \boxtimes M) & \stackrel{\mu_M}{\to} & \triangle_* M \\ \downarrow \mathrm{id} \boxtimes f & & \downarrow f \\ j_*j^*(A \boxtimes N) & \stackrel{\mu_N}{\to} & \triangle_* N \end{array}$$

If A is a chiral algebra as in ([1], 3.3.4) write $\mathcal{M}(X, A)$ for the category of chiral A-modules. This is an abelian augmented pseudo-tensor category.

1.3.16. If E is a local system on our curve X, and $A = E[1]$ is a chiral algebra then A is commutative. Indeed, $\mathrm{Hom}_{X^2}(A \boxtimes A, \triangle_* A) = 0$.

1.3.17. For Remarks at the end of Sect. 3.3.4. Recall that $\mathrm{Perv}(X)^!$ acts on $\mathrm{Perv}(X)^{ch}$, see my Sect. 1.3.2. Now if A is a chiral algebra, $M \in \mathcal{M}(X, A)$, $N \in \mathrm{Perv}(X)$ then $N \otimes^! M$ is naturally an object of $\mathcal{M}(X, A)$, here we are using the "shifts definition of $\otimes^!$". Indeed, the pair $\{\mathrm{IC}, N\}$ for $\{A, N\}$ via the above action applied to $\mu : j_*j^*(A \boxtimes M) \to \triangle_* M$ yields a map (taking into account that $A \otimes^! \mathrm{IC} \tilde{\to} A$ canonically)

$$j_*j^*(A \boxtimes (N \otimes^! M)) \to \triangle_* (N \otimes^! M)$$

which defines a structure of a Lie^{ch}-module on $N \otimes^! M$. They claim that automatically $N \otimes^! M \in \mathcal{M}(X, A)$.

1.3.18. For 3.3.7. Let A be a chiral algebra and $M \in \mathcal{M}(X, A)$. The augmentation structure on $\mathrm{Perv}(X)^{ch}$ yields from $\mu : j_*j^*(A \boxtimes M) \to \triangle_* M$ a morphism $\mathrm{H}^0(X, M) \otimes A \to M$. We also have by restriction of μ a morphism $\mathrm{Hom}_X(\mathrm{IC}, M) \otimes A \to M$. More precisely, given $m \in \mathrm{Hom}(\mathrm{IC}, M)$ the composition $A \boxtimes \mathrm{IC} \stackrel{\mathrm{id}\boxtimes m}{\to} j_*j^*(A \boxtimes M) \to \triangle_* M$ vanishes, so induces a map $A \otimes^! \mathrm{IC} = A \to M$. The kernel of this map is called $\mathrm{Cent}(m)$ the centralizer of m. This is a chiral subalgebra.

To give $m : \mathrm{IC} \to M$ on X means to give a "horizontal section" of M.

3.3.7, Exercice ii) means that the map $\mathrm{Hom}_{\mathcal{M}(X,A)}(A, M) \to \mathrm{Hom}_X(\mathrm{IC}, M)$ sending f to the composition $\mathrm{IC} \stackrel{1_A}{\to} A \stackrel{f}{\to} M$ is an isomorphism of vector spaces.

They say that $\mathrm{Cent}(m)$ for a given local section $m \in \mathrm{Hom}_U(\mathrm{IC}, M)$ (for some open $U \subset X$) have etale local nature, so we can define the perverse subsheaf of those $a \in A$ that commute with any local section $m \in \mathrm{Hom}_U(\mathrm{IC}, M)$. It is denoted $\mathrm{Cent}(M)$, the centralizer of M, this is a chiral subalgebra of A. The centralizer $\mathrm{Cent}(A)$ is called the center $Z(A)$ of A. This should be a commutative chiral subalgebra of A I think.

1.3.19. If $A \in \mathrm{Perv}(X)$ is smooth then any chiral algebra structure on A is commutative, because any Lie*-algebra structure in trivial.

1.3.20. For 3.3.12. This seems a useful thing! Let A be a Lie algebra in $\mathrm{Perv}(X)^{ch}$. A *filtration* on A is an increasing filtration $A_0 \subset A_1 \subset \ldots \subset A$ such that $\cup_i A_i = A$ and $\mu(j_*j^*(A_i \boxtimes A_j)) \subset \triangle_* A_{i+j}$. Then grA is a Lie algebra in $\mathrm{Perv}(X)^{ch}$. If A is unital, that is, a chiral algebra, unless is stated otherwise, they assume $1 \in A_0$. Then grA is a chiral algebra. This is further used in 3.7.13.

If the chiral algebra grA is moreover commutative then grA has a structure of a coisson algebra: the coisson bracket $\{.,.\} \in \mathrm{P}_2^*(\{A_i/A_{i-1}, A_j/A_{j-1}\}, A_{i+j-1}$. Namely, the Lie bracket $[.,.] : A_i \boxtimes A_j \to \triangle_* A_{i+j-1}$ of A^{Lie} induces the desired map $\{.,.\}$.

1.3.21. **Factorization.** Let X be a smooth projective geometrically connected curve. Recall the definition of the Ran space $\mathcal{R}(X)$. One considers the category $\mathcal{S}$, whose objects are finite non empty sets I, and morphisms are surjections $\pi : J \to I$. Let PreStk be the category of prestacks over k. We have a functor $\mathcal{S} \to \mathrm{PreStk}$ sending $I \to X^I$, and a surjection π to the diagonal map $\triangle^{(\pi)} = \triangle^{(J/I)}: X^I \to X^J$. Then $\mathcal{R}(X) = \mathrm{colim}_{I \in \mathcal{S}^o} X^I$.

1.3.22. The first definition of perverse $\bar{\mathbb{Q}}_\ell$-sheaves on $\mathcal{R}(X)$ is from ([1], 3.4.2). We need first to associate to each finite nonempty set I a $\bar{\mathbb{Q}}_\ell$-line η_I placed in cohomological degree $-\mid I \mid$ with the following properties. We want that for any surjection of finite nonempty sets $\pi : J \to I$, $\eta_J \widetilde{\to} \underset{I}{\otimes} \eta_{J_i}$ canonically. In particular $\mathrm{Aut}(I)$ should acts on η_I. For a finite nonempty set I set $\eta_I = (\bar{\mathbb{Q}}_\ell[1])^{\otimes I}$. Recall their line λ_I defined in Sect. 3.1.4, we have $\lambda_I = \eta_I[- \mid I \mid]$.

A **perverse $\bar{\mathbb{Q}}_\ell$-sheaf** on $\mathcal{R}(X)$ is a data for any $I \in \mathcal{S}$ of a perverse $\bar{\mathbb{Q}}_\ell$-sheaf F_{X^I} on X^I and for any surjection $\pi : J \to I$ of finite nonempty sets (so, $\triangle^{(\pi)}: X^I \to X^J$) of an isomorphism

$$\nu^{(\pi)} : \triangle^{(\pi)!} F_{X^J} \otimes \eta_J \widetilde{\to} F_{X^I} \otimes \eta_I \qquad (6)$$

compatible with compositions of π's. Here the inverse image is understood in the "non-derived perverse sense". That is, for any perverse sheaf M on X^J the complex $\vartriangle^{(\pi)!} M[\dim X^J - \dim X^I]$ is placed in perverse degrees ≤ 0 and the isomorphism (6) says $\vartriangle^{(\pi)!} F_{X^J} \otimes \eta_J \otimes \eta_I^{-1} \tilde{\to} F_{X^I}$. Here the LHS is the 0-th perverse cohomology of the derived pullback with the corresponding shift. If we forget the actions of $\mathrm{Aut}_J, \mathrm{Aut}_I$ then the latter is $\vartriangle^{(\pi)!} F_{X^J}[\dim X^J - \dim X^I] \tilde{\to} F_{X^I}$.

It is required that for the divisor of all the diagonals $i : D \subset X^I$ the complex $i^! F_{X^I}$ is placed in perverse degrees > 0. (apriori, it is placed in degrees ≥ 0). Write $\mathcal{M}(\mathcal{R}(X))$ for the category of perverse sheaves on $\mathcal{R}(X)$. This is an exact tensor $\bar{\mathbb{Q}}_\ell$-linear category. Compare with the definition of a perverse $\bar{\mathbb{Q}}_\ell$-sheaf on any prestack! If $F \in \mathcal{M}(\mathcal{R}(X))$ then for $\pi : J \to I$ in $\mathcal{S}$ the group $\mathrm{Aut}(J/I)$ acts trivially on $\vartriangle^{(\pi)!} F_{X^J} \otimes \eta_J$.

For example, take $F_{X^I} = \boxtimes_I \mathrm{IC}$. This gives the perverse sheaf $\mathrm{IC}_{\mathcal{R}(X)}$. Indeed, to construct ν, we are reduced to the case of $I = \{1, 2\}$ and the diagonal $\vartriangle: X \hookrightarrow X^2$. In this case $\vartriangle^! (\mathrm{IC}[1] \boxtimes \mathrm{IC}[1])[-1] \tilde{\to} \mathrm{IC} \otimes^! \mathrm{IC} \tilde{\to} \mathrm{IC}$ canonically!

Actually, if F is a perverse sheaf on $\mathcal{R}(X)$ then for any $\pi : J \to I$ as above $\vartriangle^{(\pi)!} F_{X^J}[\dim X^J - \dim X^I]$ is perverse, and $\vartriangle^{(\pi)!}$ can be understood in the derived sense. This is proved by induction passing to a diagonal of codimension 1 several times.

For convenience, given $F \in \mathcal{M}(\mathcal{R}(X))$, for $I \in \mathcal{S}$ set $\bar{F}_{X^I} = F_{X^I} \otimes \lambda_I$. Then for $\pi : J \to I$ in $\mathcal{S}$ we get $\vartriangle^{(\pi)!} \bar{F}_{X^J}[\dim X^J] \tilde{\to} \bar{F}_{X^I}[\dim X^I]$. This is an alternative way to write down a perverse sheaf on $\mathcal{R}(X)$.

Remark 1.3.23. *Let $f : Y \to Z$ be a morphism with Y, Z smooth of pure dimensions. If $F_i \in \mathrm{D}(Z, \bar{\mathbb{Q}}_\ell), G_i \in \mathrm{D}(Y, \bar{\mathbb{Q}}_\ell)$ are equipped with $f^! F_i[\dim Z] \tilde{\to} G_i[\dim Y]$ then we have canonically $f^!(F_1 \overset{L}{\otimes} {}^! F_2)[\dim Z] \tilde{\to} (G_1 \overset{L}{\otimes} {}^! G_2)[\dim Y]$, where we used the "shifts definition" of $\overset{L}{\otimes}$.*

So, the tensor product on $\mathcal{M}(\mathcal{R}(X))$ is as follows. Given $F, G \in \mathcal{M}(\mathcal{R}(X))$, $F \otimes^! G \in \mathcal{M}(\mathcal{R}(X))$ is such that for $I \in \mathcal{S}$,

$$\overline{F \otimes^! G}_{X^I} = \bar{F}_{X^I} \otimes^! \bar{G}_{X^I}, \tag{7}$$

where the tensor product is understood in the nonderived sense.

An analog of Lemma 3.4.3 holds:

Lemma 1.3.24. *Let F be a perverse sheaf on $\mathcal{R}(X)$. Let $I \in \mathcal{S}$. Let $\ell : V \subset X^I$ be the complement to all the diagonal strata of codimension ≥ 2. The natural map $F_{X^I} \to \ell_* \ell^* F_{X^I}$ is an isomorphism, where ℓ_* is understood in the nonderived perverse sence.*

Proof. By definition, $F_{X^I} \to \ell_* \ell^* F_{X^I}$ is injective. Let C be the cokernel of this map. Let $\pi : I \to J$ in $\mathcal{S}$ be such that C does not vanish at the generic point of the diagonal $X^J \subset X^I$, but vanishes on the complement to all the diagonals of dimension $\le \mid I \mid$. Then the 0-th perverse cohomology sheaf of $\vartriangle^{(\pi)!} F_{X^I}[1] \tilde{\to} F_{X^J} \otimes \eta_J \otimes \eta_I^{-1}[1]$ is nonzero (at the generic point of X^J). So, $\mid J \mid = \mid I \mid -1$, a contradiction. □

1.3.25. The definition of factorization algebras from ([1], 3.4.4) makes sense also for perverse $\bar{\mathbb{Q}}_\ell$-sheaves. For a finite nonempty set J they denote by $j^{(J)} : U^{[J]} \hookrightarrow X^J$ the complement to all the diagonals, that is, $U^{[J/J]}$ for the identity map $J \to J$. For a surjection $\pi : J \to I$ of finite nonempty sets they write $U^{[J/I]} = U^{(\pi)} = \{(x_j) \in X^J \mid x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\}$, and $j^{[J/I]} = j^{(\pi)} : U^{[J/I]} \hookrightarrow X^J$ for the open immersion.

Let B be a perverse sheaf on $\mathcal{R}(X)$. A *factorization structure on* B is a rule that assigns to every such $\pi : J \to I$ an isomorphism

$$c_{[J/I]} : j^{[J/I]*}(\boxtimes_I B_{X^{J_i}}) \tilde{\to} j^{[J/I]*} B_{X^J} \tag{8}$$

They demand the c's are mutually compatible: for a surjection $K \to J$ the isomorphism $c_{[K/J]}$ equals the composition $c_{[K/I]}(\boxtimes c_{[K_i/J_i]})$, that is, the diagram commutes

$$\begin{array}{ccc} \boxtimes_J B_{X^{K_j}} & \tilde{\to} & B_{X^K} \\ & \nwarrow & \uparrow \\ & & \boxtimes_I B_{X^{K_i}}, \end{array} \tag{9}$$

where we omited the restrictions to $U^{[K/J]}$. Here $U^{[K/J]} \subset U^{[K/I]} \subset X^K$. Besides, it is required that c is compatible with ν. For every $J \to J' \to I$ one has the cartesian square

$$\begin{array}{ccc} U^{[J/I]} & \hookrightarrow & X^J \\ \uparrow & & \uparrow \\ U^{[J'/I]} & \hookrightarrow & X^{J'} \end{array} \tag{10}$$

It is required that $\nu^{(J/J')} \vartriangle^{(J/J')!} c_{[J/I]} = c_{[J'/I]}(\boxtimes \nu^{(J_i/J'_i)})$ in the diagram

$$\begin{array}{ccccccc} \vartriangle^{(J/J')!}(\boxtimes_I B_{X^{J_i}} \otimes \eta_{J_i}) & \tilde{\to} & \vartriangle^{(J/J')!}(\boxtimes_I B_{X^{J_i}}) \otimes \eta_J & \overset{c_{[J/I]}}{\to} & \vartriangle^{(J/J')!} B_{X^J} \otimes \eta_J \\ \downarrow & & & & \downarrow \nu \\ (\boxtimes_I B_{X^{J'_i}} \otimes \eta_{J'_i}) & \tilde{\to} & (\boxtimes_I B_{X^{J'_i}}) \otimes \eta_{J'} & \overset{c_{[J'/I]}}{\to} & B_{X^{J'}} \otimes \eta_{J'} \end{array} \tag{11}$$

The left vertical arrow is the map $\boxtimes_I \nu^{(J_i/J'_i)}$. We have used here for $\vartriangle^{(J_i/J'_i)} : X^{J'_i} \to X^{J_i}$ the isomorphisms

$$\nu^{(J_i/J'_i)} : \vartriangle^{(J_i/J'_i)!} F_{X^{J_i}} \otimes \eta_{J_i} \tilde{\to} F_{X^{J'_i}} \otimes \eta_{J'_i}$$

We also used the *factorization property* of the line η_J, see the previous subsection.

If B is a perverse $\bar{\mathbb{Q}}_\ell$-sheaf and factorization algebra on $\mathcal{R}(X)$ in this sense, for any finite nonempty set I we get $B_{X^I} \subset j^{(I)}_* j^{(I)*}(B_X)^{\boxtimes I}$, because it has no local subsections supported on diagonals. The factorization algebras form a tensor category $\mathcal{F}\mathrm{A}(X)$. But we insist that the tensor product of two factoirzation algebras is given by (7).

The evident tensor structure on $\mathrm{Comu}^!(X)$ coincides, when we transalte them into factorization algebras, with the above tensor structure on commutative factorization algebras.

1.3.26. In the definition of a factorization algebra on $\mathcal{R}(X)$ (p. 175) the requirement for the factorization structure to be unital is local. Namely, it is required that there is a global section $1 = 1_B : \mathrm{IC} \to B_X$ such that the map $\mathrm{IC} \boxtimes B_X \to j_* j^*(B_X \boxtimes B_X)$ factors through B_{X^2}, and applying $\vartriangle^!$ to the map $\mathrm{IC} \boxtimes B_X \to B_{X^2}$ and using $\nu^{(2)}$, we get the identity map $\mathrm{id} : B_X \to B_X$. We used here that $\mathrm{IC} \otimes^! B_X \tilde{\to} B_X$ canonically.

This is their requirement $\nu^{(2)}(\vartriangle^! (1 \boxtimes f)[1]) = f$ in B_X. Here we destroyed the symmetry in the set $\{1, 2\}$, because $1 \boxtimes f$ is not symmetric.

1.3.27. For 3.4.5. The **second description** of factorization algebras for perverse $\bar{\mathbb{Q}}_\ell$-sheaves. Assume we are given a rule that assigns to each $I \in \hat{\mathcal{S}}$ (a possible empty finite set) a perverse sheaf $B_{X^I} \in \mathrm{Perv}(X^I)$, and to every map $\pi : J \to I$ in $\hat{\mathcal{S}}$ a morphism of perverse sheaves

$$\nu^{(\pi)} : \mathrm{H}^{top}(\vartriangle^{(\pi)!} B_{X^J}) \otimes \eta_J \to B_{X^I} \otimes \eta_I, \tag{12}$$

here H^{top} denotes the top perverse cohomology sheaf. Suppose also we are given isomorphism of perverse sheaves (8). The properties are required:

- $\nu^{(\pi)}$ are compatible with compositions of π;
- for π surjective $\nu^{(\pi)}$ is an isomorphism;
- for $K \to J \to I$ in $\hat{\mathcal{S}}$ the diagram (9) commutes over $U^{[K/J]}$.
- for $J \to J' \to I$ in $\hat{\mathcal{S}}$ the diagram (11) commutes, where now ν are not necessarily isomorphisms (the diagram (10) is still cartesian);
- the B_{X^I} have no nonzero perverse subsheaves supported at the diagonal divisor;
- $B_{X^\emptyset} \neq 0$.

The latter property yields a canonical isomorphism $B_{X^\emptyset} \widetilde{\to} \bar{\mathbb{Q}}_\ell$. For $\pi : \emptyset \to \{1\} = I$ we get $\vartriangle^{(\pi)}: X \to \operatorname{Spec} k$, and $\nu^{(\pi)} : \mathrm{IC} \to B_X$ is the section 1_B. This is the same as a factorization algebra.

Note that $\vartriangle^{(\pi)!} B_{X^J} \otimes \eta_J$ is placed in perverse degrees $\leq -\dim(X^I)$, so the datum of (12) is the same as a map

$$\nu^{(\pi)} : \vartriangle^{(\pi)!} B_{X^J} \otimes \eta_J \to B_{X^I} \otimes \eta_I$$

in the derived category, where $\vartriangle^{(\pi)!}$ is understood in the derived sense.

Note also that if $\pi : J \to I$ is injective then $\vartriangle^{(\pi)}: X^I \to X^J$ is smooth, so that $\vartriangle^{(\pi)!} B_{X^J} \otimes \eta_J$ is placed in perverse degree $- \mid I \mid$, and in this case no need to pass to the top perverse cohomology sheaves.

1.3.28. For Sect. 3.4.6. The **third description of factorization algebras**. Let Aff be the category of affine schemes over k. Let QCoh be the category whose objects are an affine scheme $Z \in \mathrm{Aff}$ and a quasi-coherent sheaf F on Z. The map QCoh $\to$ Aff is a cartesian fibration. So, QCoh $\to$ Aff is a fibred category.

Let $\mathcal{C}(X)$ be the category whose objects are pairs $Z \in \mathrm{Aff}$ and an element of $\mathcal{C}(X)_Z$, an equivalence class of an effective Cartier divisor on $X \times Z/Z$ proper over Z. A morphism from $(Z', D' \hookrightarrow Z \times X')$ to $(Z, D \hookrightarrow Z \times X)$ is a morphism $f : Z' \to Z$ in Aff such that the composition $D'_{red} \to Z' \times X \overset{f \times \mathrm{id}}{\to} Z \times X$ factors through $D_{red} \hookrightarrow Z \times X$. Then $\mathcal{C}(X) \to \mathrm{Aff}$ is a cartesian fibration. The corresponding inverse image functor between the category fibres sends $D \hookrightarrow Z \times X$ to its pull-back under $Z' \times X \to Z \times X$.

We need to consider perverse sheaves instead of quasi-coherent sheaves. Let now Aff_{sm} denote the category of affine k-schemes of finite type, which are smooth of pure dimension. Let Perv denote the category, whose objects are pairs $Z \in \mathrm{Aff}_{sm}$ and $F \in \mathrm{Perv}(Z)$. A morphism in Perv from (Z', F') to (Z, F) is a morphism $f : Z' \to Z$ in Aff and a morphism $F' \to \mathrm{H}^{top} f^! F[\dim Z - \dim Z']$, here H^{top} is the 0-th perverse cohomology sheaf. (Indeed, write f as the composition $Z' \stackrel{\Gamma_f}{\to} Z' \times Z \stackrel{\mathrm{pr}}{\to} Z$, where Γ_f is the graph of f. Since $\mathrm{pr}^!$ has perverse amplitude $-\dim Z'$, and $\Gamma_f^!$ has perverse amplitude $\leq \dim Z$, our claim follows). They compose naturally, and $\mathrm{Perv} \to \mathrm{Aff}_{sm}$ is a cartesian fibration. Denote by $\mathcal{C}(X)_{sm}$ the restriction of $\mathcal{C}(X)$ to Aff_{sm}.

Consider a pair (B, c), where:

i) $B : \mathcal{C}(X) \to \mathrm{Perv}$ is a functor over Aff_{sm}, which sends a cartesian arrow to a cartesian one. So, B assigns to each $(Z, D) \in \mathcal{C}(X)_{sm}$, where $D \hookrightarrow Z \times X$ is an effective Cartier divisor proper over Z, a perverse sheaf B_D on Z. If $D' \leq D$ in $\mathcal{C}(X)_Z$ then we have a morphism $B_{D'} \to B_D$, and everything is compatible with base changes. In particular, we have for each $n \geq 0$ the perverse sheaf $B_{X^{(n)}}$ on the symmetric power $X^{(n)}$.

ii) c is a rule that assigns to every pair of mutually disjoint divisors $D_1, D_2 \in \mathcal{C}(X)_Z$ with $Z \in \mathrm{Aff}_{sm}$, an isomorphism $c_{D_1,D_2} : B_{D_1} \otimes^! B_{D_2} \widetilde{\to} B_{D_1+D_2}$, where $\otimes^!$ is understood in the nonderived sense. It is required that c are commutative and associative in the obvious manner and compatible with the morphisms from i).

Remark 1.3.29. *If $Z \stackrel{f}{\to} Z' \stackrel{g}{\to} Z''$ are morphisms of smooth schemes of finite type and pure dimension then for the functor $\bar{f}^! := \mathrm{H}^{top}(f^![\dim Z' - \dim Z)) : \mathrm{Perv}(Z') \to \mathrm{Perv}(Z)$ we have $\bar{f}^!\bar{g}^! \widetilde{\to} \overline{gf}^!$. This is always true for trangular functors between triangular categories, which are right exact for fixed t-structures.*

They further assume that $B_{X^{(n)}}$ has no perverse subsheaves supported on the diagonal divisor, and $B_{X^{(0)}} \neq 0$. The latter property gives $B_{X^{(0)}} \widetilde{\to} \bar{\mathbb{Q}}_\ell$. The equivalence of tensor categories (3.4.6.2) still holds.

The relation with second construction: assume we are given a functor $F : \mathcal{C}(X) \to \mathrm{Perv}$ satisfying the above properties. For a finite nonempty set I we have the universal divisor $D_I \subset X^I \times X$, it gives the perverse sheaf F_{X^I} on X^I. Define B_{X^I} by $B_{X^I} \otimes \eta_I = F_{X^I}[\dim X^I]$. If $\pi : J \to I$ is a surjection of finite sets, $\vartriangle^{(\pi)}: X^I \to X^J$ is the diagonal. Let $\bar{D}_J$ be the preimage of D_J under $X^I \times X \to X^J \times X$. Then $D_I \sim \bar{D}_J$, hence we get an isomorphism

$$F_{X^I}[\dim X^I] \widetilde{\to} \vartriangle^{(\pi)!} F_{X^J}[\dim X^J]$$

It yields the desired isomorphism $B_{X^I} \otimes \eta_I \widetilde{\to} \vartriangle^{(\pi)!} B_{X^J} \otimes \eta_J$.

If $\pi : J \to I$ is not necessarily surjective morphism then in the above we get $\bar{D}_J \leq D_I$, hence a morphism $\vartriangle^{(\pi)!} F_{X^J}[\dim X^J] \to F_{X^I}[\dim X^I]$. It yields the desired morphism $\vartriangle^{(\pi)!} B_{X^J} \otimes \eta_J \to B_{X^I} \otimes \eta_I$.

Now consider for a surjection $\pi : J \to I$ the open part $U^{(\pi)} \hookrightarrow X^J$, let $p_i : U^{(\pi)} \to X^{J_i}$ be the projection for $i \in I$. Let $D_{J,\pi}$ denote the restriction of D_J under $U^{(\pi)} \times X \to X^J \times X$. For each $i \in I$ we have the divisor $D_i := D_{J_i} \hookrightarrow X^{J_i} \times X$, let $\bar{D}_i$ be its restriction to $U^{(\pi)}$. Then $D_{J,\pi} = \sum_I \bar{D}_i$. Now $F_{\bar{D}_i}[\dim U^{(\pi)}] = p_i^! B_{D_i} \otimes \eta_{J_i}$ on $U^{(\pi)}$. We have the isomorphism

$$c : F_{D_{J,\pi}} \widetilde{\to} \otimes^!_I F_{\bar{D}_i} \widetilde{\to} \vartriangle^! (\boxtimes_I(F_{\bar{D}_i}[\dim U^{(\pi)}]))[-\dim U^{(\pi)}]$$

for the diagonal $\vartriangle: U^{(\pi)} \hookrightarrow \prod_I U^{(\pi)}$. So,

$$\vartriangle^! (\boxtimes_I(p_i^! B_{D_i} \otimes \eta_{J_i}) \widetilde{\to} B_{D_J} \otimes \eta_J \mid_{U^{(\pi)}}$$

over $U^{(\pi)}$. Since $\boxtimes p_i^! = (\prod p_i)^!$ and the composition $U^{(\pi)} \overset{\triangle}{\hookrightarrow} \prod_I U^{(\pi)} \overset{\prod p_i}{\to} X^J$ is the canonical inclusion, it yields the desired isomorphism (8) over $U^{(\pi)}$.

1.3.30. For the canonical connection in 3.4.7. The section $1 \in B_{X^\emptyset}$ is horizontal. Indeed, we have as in (3.4.5.2)(f), $B_{X^\emptyset} \otimes B_{X^\emptyset} \widetilde{\to} B_{X^\emptyset}$, this is an isomorphism of $\mathcal{D}$-modules over $\operatorname{Spec} k$, and $1 \in B_{X^\emptyset}$, but $\Omega_{k/k} = 0$, so 1 is a horizontal section in $B_{X^\emptyset}$. Now for a finite set I, $\emptyset \to I$ yields $\triangle$: $X^I \to \operatorname{Spec} k$, and $\mathcal{O} =\triangle^* B_{X^\emptyset} \to B_{X^I}$ is a morphism of $\mathcal{D}$-modules, so $d(1) = 0$ in B_{X^I}.

1.3.31. For 3.4.8. Let B be a factorization algebra on X. Then for a finite nonempty set J consider $\triangle^{(J)}$: $X \to X^J$. Recall the line $\lambda_J[\dim X^J] = \eta_J$. By the projection formula (3), we have

$$(\triangle_*^{(J)} \operatorname{IC}) \otimes^! B_{X^J} \widetilde{\to} \triangle_*^{(J)} (\operatorname{IC} \otimes^! \triangle^{(J)!} B_{X^J})[\dim X^J - \dim X],$$

here $\dim X = 1$ and the $\otimes^!$ product is understood in the "shift sense" (derived or nonderived, this is the same). The above gives

$$(\triangle_*^{(J)} \operatorname{IC}) \otimes^! B_{X^J} \otimes \lambda_J \widetilde{\to} \triangle_*^{(J)} (\operatorname{IC} \otimes^! \triangle^{(J)!} B_{X^J} \otimes \lambda_J)[\dim X^J - \dim X] \widetilde{\to} \triangle_*^{(J)} B_X$$

Note that $\lambda_{\{1\}} = \bar{\mathbb{Q}}_\ell$ canonically. We have

$$j_*^{(J)} j^{(J)*}(B_X^{\boxtimes J}) \widetilde{\to} j_*^{(J)} j^{(J)*}(\operatorname{IC}_X^{\boxtimes J}) \otimes^! B_{X^J} \otimes \lambda_J,$$

because of the factorization isomorphism c. So, we get maps

$$\operatorname{Lie}_J = P_J^{ch}(\operatorname{IC}_X) \to P_J^{ch}(\{B_X, \dots, B_X\}, B_X)$$

compatible with compositions. So, B_X acquires a Lie^{ch}-algebra structure. The unit section $1_B : \operatorname{IC} \to B_X$ is the unit of B_X, so B_X is a chiral algebra.

Their equivalence $\mathcal{FA}(X) \widetilde{\to} \mathcal{CA}(X)$ from 3.4.9 should hold also in the $\bar{\mathbb{Q}}_\ell$-setting.

1.3.32. For 3.4.10. The category $\mathcal{M}(X^{\mathcal{S}})$ is defined as the category, whose objects are collections (M, θ), where for each $I \in \mathcal{S}$, $M_{X^I} \in \operatorname{Perv}(X^I)$ and for each map $\pi : J \to I$ in $\mathcal{S}$, $\theta^{(\pi)} :\triangle_*^{(\pi)} M_{X^I} \to M_{X^J}$ is a morphism in $\operatorname{Perv}(X^J)$ such that $\theta^{(\pi)}$ are compatible with the compositions of π's. That is, if $J_2 \overset{\pi_2}{\to} J_1 \overset{\pi_1}{\to} I$ then for $X^I \overset{\triangle^{(\pi_1)}}{\to} X^{J_1} \overset{\triangle^{(\pi_2)}}{\to} X^{J_2}$ the diagram commutes

$$\begin{array}{ccc} \triangle_*^{(\pi_2)} \triangle_*^{(\pi_1)} M_{X^I} & \overset{\triangle_*^{(\pi_2)}\theta}{\to} & \triangle_*^{(\pi_2)} M_{X^{J_1}} \\ & \searrow_\theta & \downarrow \theta \\ & & M_{X^{J_2}} \end{array}$$

In Sect. 3.4.10 we have a beautiful example of a colimit: the functor $a_n : \mathcal{M}(X^{\mathcal{S}_n}) \to \mathcal{M}(X^{\mathcal{S}})$ left adjoint to the projection $\mathcal{M}(X^{\mathcal{S}}) \to \mathcal{M}(X^{\mathcal{S}_n})$. Let $I \in \mathcal{S}$ and $n <\mid I \mid$. Then we take the inductive limit over the ordered set $Q(I, \leq n)$. We have the set $Q_n(I)$ of quotients of I of order n, so $Q_n(I) \subset Q(I, \leq n)$ is a subcategory, and $Q_n(I)$ is a set, there are no nontrivial morphisms. So, for an object $(N, \theta) \in \mathcal{M}(X^{\mathcal{S}_n})$ we have $\oplus_{J \in Q_n(I)} N_{X^J} = \operatorname{colim}_{J \in Q_n(I)} N_{X^J} \to \operatorname{colim}_{J \in Q(I, \leq n)} N_{X^J}$ naturally, this is a surjection. We have to quotient by some relations.

If $M \in \mathcal{M}(X^{\mathcal{S}})$ then for any isomorphism $\pi : J \to I$ in $\mathcal{S}$ the map $\theta^{(\pi)} : \vartriangle_*^{(\pi)} M_{X^I} \to M_{X^J}$ is an isomorphism.

The $\otimes^*$-tensor product on $\mathcal{M}(X^{\mathcal{S}})$. Given a collection M_i, $i \in I$ with I a finite nonempty set, we let $\otimes_I^* M_i \in \mathcal{M}(X^{\mathcal{S}})$ be given by $(\otimes_I^* M_i)_{X^J} = \underset{J \to I \in \mathcal{S}}{\oplus} \underset{i \in I}{\boxtimes} (M_i)_{X^{J_i}}$. If now $\pi : J \to J'$ a surjection in $\mathcal{S}$ (recall that $\mathcal{S}$ is the category of nonempty finite sets with morphisms=surjections) then we get $Q(J') \subset Q(J)$, and for any surjection $\phi : J' \to I$ we get a morphism

$$\vartriangle_!^{(\pi)} (\underset{i \in I}{\boxtimes} (M_i)_{X^{J'_i}}) \to \underset{i \in I}{\boxtimes} (M_i)_{X^{J_i}} \tag{13}$$

as the product of map $\vartriangle_!^{(\pi_i)} (M_i)_{X^{J'_i}} \to (M_i)_{X^{J_i}}$ for $\pi_i : J_i \to J'_i$. Taking the sum over the set of surjections $\phi : J' \to I$, we get the desired morphism

$$\vartriangle_!^{(\pi)} (\otimes_I^* M_i)_{X^{J'}} \to (\otimes_I^* M_i)_{X^J}$$

This defines the object $\otimes_I^* M_i \in \mathcal{M}(X^{\mathcal{S}})$.

The $\otimes^{ch}$-tensor product on $\mathcal{M}(X^{\mathcal{S}})$. Given $I \in \mathcal{S}$, $M_i \in \mathcal{M}(X^{\mathcal{S}})$ for $i \in I$ and $J \in \mathcal{S}$, the object $(\otimes_I^{ch} M_i)_{X^J}$ is defined as

$$(\otimes_I^{ch} M_i)_{X^J} = \underset{J \to I \in \mathcal{S}}{\oplus} j_*^{[J/I]} j^{[J/I]*} (\underset{i \in I}{\boxtimes} M_{X^{J_i}})$$

Now given $\pi : J \to J'$ in $\mathcal{S}$, we use $Q(J') \subset Q(J)$. For any surjection $\phi : J' \to I$ we get a morphism

$$\vartriangle_!^{(\pi)} j_*^{[J'/I]} j^{[J'/I]*} (\underset{i \in I}{\boxtimes} (M_i)_{X^{J'_i}}) \to j_*^{[J/I]} j^{[J/I]*} (\underset{i \in I}{\boxtimes} (M_i)_{X^{J_i}}) \tag{14}$$

It is obtained by applying the functor $j_*^{[J/I]} j^{[J/I]*}$ to the morphism (13) and using base change. Summing up the maps (14) over all the surjections $J' \to I$, we get the desired morphism

$$\vartriangle_!^{(\pi)} (\otimes_I^{ch} M_i)_{X^{J'}} \to (\otimes_I^{ch} M_i)_{X^J}$$

This defines the object $\otimes_I^{ch} M_i \in \mathcal{M}(X^{\mathcal{S}})$.

Note that $\otimes_I^* M_i$, $\otimes_I^{ch} M_i \in \mathcal{M}(X^{\geq n})$, where $n = \mid I \mid$. For the definition of free commutative non-unital algebras in a given symmetric monoidal ∞-category see ([10], 3.0.40).

1.3.33. For Sect. 3.4.10. For the definition of the $\otimes^{ch}$-tensor structure on $\mathcal{M}(X^{\mathcal{S}})$, we use the fact that (10) is cartesian. So, for any object $M \in \mathcal{M}(X^{\mathcal{S}})$ and any diagram $J \stackrel{\pi}{\to} J' \to I$ in $\mathcal{S}$, there is a natural map

$$\vartriangle_!^{(\pi)} j_*^{[J'/I]} j^{[J'/I]^*} M_{X^{J'}} \to j_*^{[J/I]} j^{[J'/I]*} M_{X^J}$$

Note that for $M_i \in \mathcal{M}(X^{\mathcal{S}}), i \in I$ the group $\operatorname{Aut} I$ acts on $\otimes_I^* M_i$ and on $\otimes_I^{ch} M_i$, as in any tensor category!

The identity functor $\beta^{\mathcal{S}} : \mathcal{M}(X^{\mathcal{S}})^{ch} \to \mathcal{M}(X^{\mathcal{S}})^*$ is right-lax non-unital symmetric monoidal.

A subtelty in the Examples on p. 181: for Sym_*^m we may take invariants or coinvariants for the S_m-action. I think they will be isomorphic in any characteristic, because our object is a direct sum. However, the isomorphism is maybe not "the obvious one, which always holds in characteristic zero". As for free commutative non-unital algebras in any symmetric monoidal ∞-category, we shoiuld take coinvariants, that is, colimit over $B(S_m)$ to get Sym^m.

For $N \in \mathrm{Perv}(X)$, $I, J \in \mathcal{S}$ we get

$$(\otimes_I^{ch} \vartriangle_*^{(\mathcal{S})} N)_{X^J} = \underset{J\to I}{\oplus} j_*^{[J/I]} j^{[J/I]*} \vartriangle_*^{(J/I)} (N^{\boxtimes I}),$$

the sum being taken over all surjections $\pi : J \to I$. Note that $j^{(I)} : U^{(I)} \hookrightarrow X^I$ is obtained from $U^{(\pi)} \hookrightarrow X^J$ by the base change $\vartriangle^{(J/I)}: X^I \hookrightarrow X^J$, so we may rewrite

$$(\otimes_I^{ch} \vartriangle_*^{(\mathcal{S})} N)_{X^J} = \underset{J\to I}{\oplus} \vartriangle_*^{(J/I)} j_*^{(I)} j^{(I)*}(N^{\boxtimes I})$$

This gives for $m > 0$

$$(\mathrm{Sym}_{ch}^m \vartriangle_*^{(\mathcal{S})} N)_{X^J} = \underset{T\in Q(J,m)}{\oplus} \vartriangle_*^{(J/T)} j_*^{(T)} j^{(T)*}(N^{\boxtimes T}),$$

here $Q(J, m)$ is formed by quotients sets T of order m of J.

Lemma 1.3.34. *The fully faithful functor $\vartriangle_*^{(\mathcal{S})}: \mathcal{M}(X) \to \mathcal{M}(X^{\mathcal{S}})$ extends canonically to fully faithful pseudo-tensor embeddings*

$$\mathcal{M}(X)^* \hookrightarrow \mathcal{M}(X^{\mathcal{S}})^*, \quad \mathcal{M}(X)^{ch} \hookrightarrow \mathcal{M}(X^{\mathcal{S}})^{ch}.$$

Proof. We prove this for $*$-version. The argument for ch-version is similar. Let $I \in \mathcal{S}$. If $N, M_i \in \mathcal{M}(X), i \in I$ are given then $\phi \in \mathrm{Hom}_{\mathcal{M}(X^{\mathcal{S}})}(\otimes_I^* \vartriangle_*^{(\mathcal{S})} M_i, \vartriangle_*^{(\mathcal{S})} N)$ is completely determined by the restriction of ϕ to the direct summand

$$\underset{i\in I}{\boxtimes} M_i \subset (\otimes_I^* \vartriangle_*^{(\mathcal{S})} M_i)_{X^I} = \underset{\eta\in \mathrm{Aut}\, I}{\oplus} \eta^*(\underset{i\in I}{\boxtimes} M_i)$$

Indeed, for any $\mu \in \mathrm{Aut}\, I$, the composition

$$(\otimes_I^* \vartriangle_*^{(\mathcal{S})} M_i)_{X^I} \stackrel{\theta(\mu)}{\to} (\otimes_I^* \vartriangle_*^{(\mathcal{S})} M_i)_{X^I} \stackrel{\phi}{\to} (\vartriangle_*^{(\mathcal{S})} N)_{X^I} = \vartriangle_* N$$

equals ϕ. Now for any J with $\mid J \mid \geq \mid I \mid$ and any surjection $\pi : J \to I$ we may take $J' = I$ and present the latter as a composition $J \to J' \tilde{\to} I$. We see that the corresponding X^J-part $(\otimes_I^* \vartriangle_*^{(\mathcal{S})} M_i)_{X^J} \to (\vartriangle_*^{(\mathcal{S})} N)_{X^J}$ of the map ϕ is determined by the X^I-part of the map ϕ. This shows that we have canonically

$$\mathrm{Hom}_{\mathcal{M}(X^{\mathcal{S}})}(\otimes_I^* \vartriangle_*^{(\mathcal{S})} M_i, \vartriangle_*^{(\mathcal{S})} N) \tilde{\to} \mathrm{Hom}_{\mathrm{Perv}(X^I)}(\boxtimes_I M_i, \vartriangle_* N)$$

Our claim follows. □

1.3.35. As in 3.4.10, we have an action of the tensor category $\mathcal{M}(\mathcal{R}(X))$ on $\mathcal{M}(X^{\mathcal{S}})$. For $F \in \mathcal{M}(\mathcal{R}(X)), M \in \mathcal{M}(X^{\mathcal{S}})$ the tensor product, which should be rather denoted $M \otimes^! F \in \mathcal{M}(X^{\mathcal{S}})$ is given as follows. Recall our notation $\bar{F}_{X^I} = F_{X^I} \otimes \lambda_I$. For $I \in \mathcal{S}$ set

$$(M \otimes^! F)_{X^I} = M_{X^I} \otimes^! \bar{F}_{X^I}$$

By the projection formula (3), for $\pi : J \to I$ in $\mathcal{S}$ we have

$$\vartriangle_*^{(\pi)} (M_{X^I} \otimes^! \bar{F}_{X^I}) \tilde{\to} \vartriangle_*^{(\pi)} (M_{X^I} \otimes^! \vartriangle^{(\pi)!} \bar{F}_{X^J})[\dim X^J - \dim X^I] \tilde{\to} (\vartriangle_*^{(\pi)} M_{X^I}) \otimes^! \bar{F}_{X^J},$$

where $\otimes^!$ is understood in "the nonderived shifted sense". The corresponding morphism $\theta^{(\pi)} : \triangle_*^{(\pi)} (M \otimes^! F)_{X^I} \to (M \otimes^! F)_{X^J}$ is defined as the composition

$$\triangle_*^{(\pi)} (M_{X^I} \otimes^! \bar{F}_{X^I}) \tilde{\to} (\triangle_*^{(\pi)} M_{X^I}) \otimes^! \bar{F}_{X^J} \stackrel{\theta^{(\pi)}\otimes \mathrm{id}}{\to} M_{X^J} \otimes^! \bar{F}_{X^J}$$

1.3.36. For 3.4.11. If A is a Chiral algebra on X, we may consider the following version of the homological Chevalley complex for A, denoted $C(A)^{\cdot}_{X^I}$ in 3.4.11, this is a complex of perverse sheaves on X^I for some $I \in \mathcal{S}$.

We write it using the notations of 3.4.11 as a cohomological complex (though according to my Section 1.1.15 it is rather a homological complex). As a plain $\mathbb{Z}$-graded perverse sheaf, this is

$$C(A)^{\cdot}_{X^I} = \underset{T\in Q(I)}{\oplus} \triangle_*^{(I/T)} j_*^{(T)} j^{(T)*} (A[1])^{\boxtimes T}$$

The differential is described in 3.4.11. Namely, the component $d_{T,T'}$ of the differential is nonzero only for $T' \in Q(T, \mid T \mid -1)$. Then $T = T'' \sqcup \{\alpha', \alpha''\} \to T' = T'' \sqcup \{\alpha\}$, and $d_{T,T'}$ is the exteriour product of the chiral product map $j_* j^*((A[1])^I) \to \triangle_* A[2]$ and the identity map for $A^{\boxtimes T''}$. Here $I = \{\alpha', \alpha''\}$, and the order on I is not needed, see my Section 1.3.14.

Note that the usual chiral product is a map $j_* j^*((A[1])^I) \to \triangle_* A[2]$, so the differential is well-defined as a map $C(A)^{\cdot}_{X^I} \to C(A)^{\cdot}_{X^I}[1]$ in the derived category on X^I.

In fact, $C(A)^{\cdot}$ is

$$\mathrm{Sym}^*_{ch}(\triangle_*^{(\mathcal{S})} A[1]) = \underset{m>0}{\oplus} \mathrm{Sym}^m_{ch}(\triangle_*^{(\mathcal{S})} A[1]),$$

where we are using the tensor structure $\mathcal{M}(X^{\mathcal{S}})^{ch}$ on $\mathcal{M}(X^{\mathcal{S}})$. For $m \geq 1$ we have

$$(\mathrm{Sym}^m_{ch} \triangle_*^{(\mathcal{S})} A[1])_{X^I} = \underset{T\in Q(I,m)}{\oplus} \triangle_*^{(J/T)} j_*^{(T)} j^{(T)*} (A[1])^{\boxtimes T}$$

by Examples after (3.4.10.4). Here Sym^m_{ch} means that we are taking the symmetric power with respect to the ch-tensor structure.

Our $C(A) \in \mathcal{M}(X^{\mathcal{S}})$. So, for $\pi : J \to I$ in $\mathcal{S}$ for $\triangle^{(\pi)}$: $X^I \to X^J$ we are given structure maps $\triangle_*^{(\pi)} C(A)_{X^I} \to C(A)_{X^J}$. This map is

$$\triangle_*^{(\pi)} \underset{T\in Q(I)}{\oplus} \triangle_*^{(I/T)} j_*^{(T)} j^{(T)*} (A[1]^{\boxtimes T}) \to \underset{T\in Q(J)}{\oplus} \triangle_*^{(J/T)} j_*^{(T)} j^{(T)*} (A[1]^{\boxtimes T})$$

This is just the inclusion of those direct summands that correspond to the subset $Q(I) \subset Q(J)$.

If now $\pi : J \to I$ is a surjection why do we have (3.4.11.3)? If $T \in Q(J)$ and say $S = \inf(T, I) \in Q(I)$ then the square is cartesian

$$\begin{array}{ccc} X^I & \stackrel{\triangle^{(\pi)}}{\to} & X^J \\ \uparrow & & \uparrow \\ X^S & \to & X^T \end{array}$$

If $S < T$ then $\triangle^{(T/S)!}\, j_*^{(T)} K = 0$ for any $K \in \mathrm{D}(U^{(T)})$. So, $\triangle^{(\pi)!}\, C(A)_{X^J}$ will be a sum over those $T \in Q(J)$ such that $T \in Q(I)$, that is, by $Q(I)$, and we get

$$\triangle^{(\pi)!}\, (\triangle_*^{(J/T)}\, j_*^{(T)} j^{(T)*}(A[1])^{\boxtimes T}) = \triangle_*^{(I/T)}\, j_*^{(T)} j^{(T)*}(A[1])^{\boxtimes T}$$

for $T \in Q(I)$, because $X^T \hookrightarrow X^I \hookrightarrow X^J$. By adjointness, this gives a map $\triangle_!^{(\pi)}$ $C(A)_{X^I} \to C(A)_{X^J}$. They are compatible with compositions of π's, so we constructed a complex in the abelian category $\mathcal{M}(X^{\mathcal{S}})$ (placed in degrees ≤ -1).

Why do we have the factorization property (3.4.11.5) of $C(A)$? We need the following.

Claim: Let $T \overset{s}{\leftarrow} J \overset{\pi}{\to} I$ be a diagram in $\mathcal{S}$. Then $U^{[J/I]} \times_{X^J} X^T$ is empty unless $I \in Q(T)$, that is, $J \to T \to I$. In the latter case the square is cartesian

$$\begin{array}{ccc} U^{[J/I]} & \overset{j^{[J/I]}}{\hookrightarrow} & X^J \\ \uparrow \triangle^{(J/T)} & & \uparrow \triangle^{(J/T)} \\ U^{[T/I]} & \hookrightarrow & X^T \end{array}$$

Proof. Let $\bar{x} : T \to X$ be a map, $x : J \to X$ its restriction to J. Assume $x \in U^{[J/I]} \times_{X^J} X^T$. If $j_1, j_2 \in J$ such that $s(j_1) = s(j_2)$ and $\pi(j_1) \neq \pi(j_2)$ then we have $x_{j_1} \neq x_{j_2}$, and also $x_{j_1} = x_{j_2}$. Contradiction. So, such pair (j_1, j_2) does not exist. The first part is proved. The second claim is the diagram (10). □

The above claim shows that for $K \in \mathrm{D}(X^T)$ we have $j^{[J/I]^*}\ \triangle_*^{(J/T)}\ K = 0$ unless $I \in Q(T)$. So,

$$j^{[J/I]^*} C(A)_{X^J} \widetilde{\to} \bigoplus_{T \in Q(J), T \geq I} j^{[J/I]^*}\ \triangle_*^{(J/T)}\ j_*^{(T)} j^{(T)*}(A[1]^{\boxtimes T}) \widetilde{\to}$$
$$\bigoplus_{T \in Q(J), T \geq I} \triangle_*^{(J/T)}\ (j_I^T)_* j^{(T)*}(A[1]^{\boxtimes T})$$

For each $T \in Q(J), T \geq I$ we get the equivalence relation $T_i \in Q(J_i)$ for each $i \in I$, and $\{T \in Q(J) \mid T \geq I\} = \prod_I Q(J_i)$. We have denoted

$$U^{(T)} \overset{j_I^T}{\to} U^{[T/I]} \overset{j^{[T/I]}}{\to} X^T$$

Note that

$$C(A)_{X^{J_i}} = \bigoplus_{T_i \in Q(J_i)} \triangle_*^{(J_i/T_i)}\ j_*^{(T_i)} j^{(T_i)*}(A[1]^{\boxtimes T_i})$$

and

$$(j_I^T)_* j^{(T)*}(A[1]^{\boxtimes T}) \widetilde{\to} j^{[T/I]*}(\boxtimes_I (j_*^{(T_i)} j^{(T_i)*}(A[1]^{\boxtimes T_i}))$$

Finally,

$$j^{[J/I]*}(\boxtimes_I C(A)_{X^{J_i}}) \widetilde{\to} \bigoplus_{\{T_i\} \in \prod_I Q(J_i)} \triangle_*^{(J/T)}\ j^{[T/I]*}(\boxtimes_I (j_*^{(T_i)} j^{(T_i)*}(A[1]^{\boxtimes T_i}))$$

So, (3.4.11.5) is proved, this is only the geometry.

Lemma from their Section 3.4.12 allows to construct the factorization algebra out of a chiral algebra A as follows. Define B_{X^I} by

$$B_{X^I}\otimes\lambda_I := \mathrm{H}^{-|I|}C(A)_{X^I} = \mathrm{Ker}(d : C(A)^{-|I|}_{X^I}\to C(A)^{1-|I|}_{X^I})$$

Then $\{B_{X^I}\}$ are equipped with isomorphisms (8) coming from (3.4.11.5), at this place it is crucial that $\otimes_I\eta_{J_i}\widetilde{\to}\eta_J$ canonically. The isomorphisms (3.4.11.3) now become

$$\vartriangle^{(\pi)!} B_{X^J}\otimes\eta_J\widetilde{\to} B_{X^I}\otimes\eta_I$$

for any $\pi : J\to I$ in $\mathcal{S}$. Thus, B is a factorization algebra in the first sense. This is the functor $\mathcal{C}\mathrm{A}(X)\to\mathcal{F}\mathrm{A}(X)$ inverse to (Theorem 3.4.9).

1.3.37. *Example.* Consider a chiral algebra A on X in $\mathrm{Perv}(X)$. In the previous section we constructed explicitly the corresponding factorization algebra B on $\mathcal{R}(X)$. In particular, B_{X^2} is the kernel of $\mu : j_*j^*(A\boxtimes A)\to\vartriangle_* A$ on X^2, and μ is surjective. Note that $\mathrm{IC}\boxtimes A, A\boxtimes\mathrm{IC}\subset B_{X^2}$. We implicitly used the fact that the unit $1:\mathrm{IC}\to A$ is always injective.

If A is commutative then $0\to B\boxtimes B\to B_{X^2}\to\vartriangle_*\mathcal{F}\to 0$ is an exact sequence on X^2, where $\mathcal{F}$ is the kernel of the product map $A\otimes^! A\to A$. Moreover, $B\overset{L}{\otimes}{}^! B$ is placed in perverse degree zero! Indeed, the latter exact sequence after applying $\vartriangle^!$ yields an exact triangle $\mathcal{F}\otimes\lambda_2\to B\overset{L}{\otimes}{}^! B\overset{m}{\to} B$ on X. Since $\mathcal{F}, B$ are perverse, the middle term is also perverse. Here m is the product map (see Prop. 3.4.20).

1.3.38. For 3.4.14. One can easily define analog of the categories $\mathcal{Q}\mathrm{FA}(X)$, $\mathcal{P}\mathrm{FA}(X)$ of quasi-factorization algebras and pre-factorization algebras for perverse sheaves. We have obvious tensor functors $\mathcal{F}\mathrm{A}(X)\to\mathcal{Q}\mathrm{FA}(X)\to\mathcal{P}\mathrm{FA}(X)$. However, to affirm the existence of a left adjoint $\mathcal{P}\mathrm{FA}(X)\to\mathcal{Q}\mathrm{FA}(X)$ one needs to replace $\mathrm{Perv}(X)$ in all the related definitions by its ind-completion.

There is the following difference: the category of $\mathcal{D}$-modules on a smooth scheme admits, say, filtered colimits. A $\mathcal{D}$-module on X is not necessarily of finite length, but an object of $\mathrm{Perv}(X)$ is of finite length. The category $\mathrm{Perv}(X)$ does not admit filtered colimits. In their proof of Lemma on p. 185 they need to take a colimit over the category $(\mathcal{S}/I)^o$ (for any $I\in\mathcal{S}$) in $\mathcal{D}$-modules.

The category $(\mathcal{S}/I)^o$ is not filtered. However, the category $\mathrm{Perv}(X)$ and $\mathcal{D}$-modules on X admit finite colimits, hence by ([13], 5.5.1.1), $\mathbb{P}\mathrm{erv}(X) := \mathrm{Ind}(\mathrm{Perv}(X))$ is presentable, so admits all small colimits.

Convention: assume that we replace $\mathrm{Perv}(X)$ by $\mathbb{P}\mathrm{erv}(X)$ everywhere in all the definitions of chiral algebras and factorization algebras. Their Theorem 3.4.9 still holds.

It is understood that in the definition of a pre-factorization algebra the following condition $(*)_n$ is required: let $\pi : J\to I$ be a surjection with $|J| = |I| + 1$. So, there is a unique nontrivial automorphism σ of J over I. It is required that σ acts trivially on $\vartriangle^{(\pi)!} T_{X_J}\otimes\eta_J$.

Then the analog of lemma in 3.4.14 holds. In the proof of Lemma in 3.4.14 the quotient qB_{X^I} is the quotient of B_{X^I} by the maximal perverse subsheaf, which is the extension by zero from the diagonal divisor in X^I. So, we have the left adjoint functors

$$\mathcal{P}\mathrm{FA}(X)\to\mathcal{Q}\mathrm{FA}(X)\to\mathcal{F}\mathrm{A}(X) \tag{15}$$

Example: let $N \in \mathrm{Perv}(X)$, set $T(N)_{X^I} = j_*^{(I)} j^{(I)*}(N \oplus \mathrm{IC})^{\boxtimes I}$ on X^I. This is a pre-factoirzation algebra $T(N) \in \mathcal{P}\mathrm{FA}(X)$ naturally. For $I = \{1, 2\}$ and the diagonal $\triangle\colon X \to X^2$ we get $\triangle^! \, T(N)_{X^2} = 0$, so this is not a quasi-factorization algebra. The associated quasi-factorization algebra is zero!

The formulation of their Theorem in 3.4.14 for perverse shaves should be as follows, I think.

Theorem 1.3.39. *Let $N \in \mathrm{Perv}(X), T_{X^2} \subset j_*j^*((N \oplus \mathrm{IC})^{\boxtimes 2}) = T(N)_{X^2}$ a perverse subsheaf such that $T_{X_2} \mid_U = (N \oplus \mathrm{IC})^{\boxtimes 2} \mid_U$ for the complement of the diagonal $U \subset X^2$. Assume S_2 acts trivially on $\triangle^! \, T_{X^2} \otimes \eta_2$, the latter is understood in "perverse non-derived sense". Assume that $(N \oplus \mathrm{IC}) \boxtimes \mathrm{IC}$ and $\mathrm{IC} \boxtimes (N \oplus \mathrm{IC})$ are contained in $T(N)_{X^2}$. Consider the functor $\mathcal{F}\mathrm{A}(X) \to Sets$ sending B to the set of $\phi \in \mathrm{Hom}_X(N, B)$ such that the diagram commutes*

$$\begin{array}{ccc} j_*j^*(N \oplus \mathrm{IC})^{\boxtimes 2} & \stackrel{\bar\phi \boxtimes \bar\phi}{\to} & j_*j^*(B \boxtimes B) \\ \uparrow & & \uparrow \\ T_{X^2} & \to & B_{X^2}, \end{array}$$

where $\bar\phi = \phi \oplus 1_B : N \oplus \mathrm{IC} \to B$, $1_B : \mathrm{IC} \to B$ is the unit section. Then this functor is corepresentable (by a factorizable algebra maybe lying in $\mathbb{P}\mathrm{erv}(X)$).

Proof. Define a pre-factorization subalgebra T of $T(N)$ as follows. We set $T_{X^\emptyset} = \bar{\mathbb{Q}}_\ell$, $T_X = T(N)_X$, we have already T_{X^2}.

If $I \in \mathcal{S}$ with $\mid I \mid \geq 3$, define T_{X^I} as follows. For any surjection $\pi : I \to I'$ with $\mid I \mid = \mid I' \mid +1$, consider $j^{[I/I']} : U^{[I/I']} \subset X^I$. Let U be the union of $U^{[I/I']}$ for all such surjections $\pi : I \to I'$. Let $j : U \hookrightarrow X^I$ be the open immersion. We will have $T_{X^I} = j_*j^*T_{X^I}$. Now $j^*T_{X^I}$ is defined as a gluing of perverse sheaves $j^{[I/I']*}T_{X^I}$ over $U^{[I/I']}$ for each $\pi : I \to I'$ as above. Namely, if $I' = \bar I \sqcup i'$ and $I = \bar I \sqcup \{i_1, i_2\}$ then we set

$$j^{[I/I']*}T_{X^I} = j^{[I/I']*}((T_{X^2}) \mid_{X^{\{i_1,i_2\}}} \boxtimes (T_X^{\boxtimes \bar I}) \mid_{X^{\bar I}})$$

We have the natural gluing data giving so a perverse sheaf $j^*T_{X^I}$ over U, hence also T_{X^I}.

For any $\pi : I \to I'$ in $\mathcal{S}$ with $\mid I \mid = \mid I' \mid +1$, the perverse sheaf $\triangle^{(I/I')!} \, T_{X^I} \otimes \eta_I$ has no perverse subsheaves, which are extensions by zero from the diagonal divisor of $X^{I'}$, because $T_{X^I} = j_*j^*T_{X^I}$. This implies the condition $(*)_n$ in the definition of a pre-factorization algebra. The desired factorization algebra is the image of T under (15). □

Example: let $N \in \mathrm{Perv}(X)$ be such that $N \overset{L}{\otimes}{}^! N$ is placed in degree zero. Then $(N \oplus \mathrm{IC}) \overset{L}{\otimes}{}^! (N \oplus \mathrm{IC})$ is also placed in perverse degree zero. Let T_{X^2} be the kernel of $j_*j^*(N \oplus \mathrm{IC})^{\boxtimes 2} \to \triangle_* \, (N \oplus \mathrm{IC}) \otimes^! (N \oplus \mathrm{IC}) \to \triangle_* \, (\mathrm{Sym}^2 N)$, where Sym^2 is understood in the sense of the tensor structure $\mathrm{Perv}(X)^!$. Then T_{X^2} contains $N \boxtimes N$ on X^2, and S_2 acts trivially on $\triangle^! \, (T_{X^2} \otimes \eta_2)$. The corresponding factorizable algebra will be $\mathrm{Sym}(N)$, where Sym is understood in the sense of the tensor structure $\mathrm{Perv}(X)^!$.

1.3.40. A notion of a **module over a factorization algebra** from Sect. 3.4.18. For $I \in \hat{\mathcal{S}}$ write $\tilde{I} = I \sqcup \cdot$. Let B be a factorization algebra on X. A factorization B-module is a triple $(B(M), \tilde{\nu}, \tilde{c})$, where

- $B(M)$ is a rule that assigns to $I \in \hat{\mathcal{S}}$ a perverse sheaf $B(M)_{X^{\tilde{I}}} \in \mathbb{P}\mathrm{erv}(X^{\tilde{I}})$, which has no perverse subsheaves supported on the diagonal divisors. Set $M = B(M)_X$.
- $\tilde{\nu}$ assigns to every surjection $\pi : \tilde{J} \to \tilde{I}$ preserving the $\cdot$'s an isomorphism

$$\tilde{\nu}^{(\pi)} : \triangle^{(\pi)*} B(M)_{X^{\tilde{J}}} \otimes \eta_{\tilde{J}} \widetilde{\to} B(M)_{X^{\tilde{I}}} \otimes \eta_{\tilde{I}}$$

  The $\tilde{\nu}^{(\pi)}$'s must be compatible with compositions of π's.
- Consider a surjection $\tilde{J} \to \tilde{I}$ preserving $\cdot$. Then $\tilde{J}$ is the union of $\tilde{J}_\cdot$ and $J_i, i \in I$. Our $\tilde{c}$ assigns to this surjection an isomorphism

$$\tilde{c}_{[\tilde{J}/\tilde{I}]} : j^{[\tilde{J}/\tilde{I}]*}((\boxtimes_I B_{X^{J_i}}) \boxtimes B(M)_{X^{\tilde{J}_\cdot}}) \widetilde{\to} j^{[\tilde{J}/\tilde{I}]*} B(M)_{X^{\tilde{J}}}$$

  It is required that c's are mutually compatible: for $\tilde{K} \to \tilde{J} \to \tilde{I}$, $\tilde{c}_{[\tilde{K}/\tilde{J}]}$ equals the composition $\tilde{c}_{[\tilde{K}/\tilde{I}]}((\boxtimes c_{[K_i/J_i]}) \boxtimes \tilde{c}_{[\tilde{K}_\cdot/\tilde{J}_\cdot]})$. Besides, $\tilde{c}$ must be compatible with the isomorphisms $\tilde{\nu}$ (and ν for B) in the same way as for factorization algebra. Finally, the composition $\mathrm{IC} \boxtimes M \stackrel{1 \boxtimes \mathrm{id}}{\to} j_* j^*(B_X \boxtimes M)$ on X^2 must factor through $B(M)_{X^2}$, and applying $\triangle^!$ to the obtained map $\mathrm{IC} \boxtimes M \to B(M)_{X^2}$, we should get the identity $\mathrm{id} : M \to M$ for

$$\triangle^! (\mathrm{IC} \boxtimes M)[1] \widetilde{\to} M \to \triangle^! B(M)_{X^2}[1] \widetilde{\to} M$$

Write $\mathcal{M}^\ell(X, B)$ for the category of factorization B-modules. After (3.4.18.2) there is a minor mistake: the functor $\mathcal{M}^\ell(X, B) \to \mathrm{Perv}(X)$, $(B(M), \tilde{\nu}, \tilde{c}) \to M$ is faithful (not fully faithful).

1.3.41. For Sect. 3.4.19. Let B be a factorization algebra. If $(B(M), \tilde{\nu}, \tilde{c})$ is a factorization B-module, the corresponding chiral B-module is $M := B(M)_X$. The chiral action is the map

$$j_* j^*(B \boxtimes M) \stackrel{\tilde{c}}{\to} j_* j^* B(M)_{X^2} \to j_* j^* B(M)_{X^2}/B(M)_{X^2} \widetilde{\to} \triangle_* i^! B(M)_{X^2}[1] \stackrel{\tilde{\nu}}{\to} \triangle_* M$$

We have the las isomorphism, as we broked the symmetry of $\{1, 2\}$.

The description of chiral B-operations (after Remark on p. 190): there is a MISPRINT in the definition of $B(\{M_i\})_{X^{J \sqcup I}}$. The correct definition is as follows. Let M_i, N be chiral B-modules for $I \in \mathcal{S}$, let $J \in \mathcal{S}$. Let G be the factorization algebra corresponding to the chiral algebra $B \oplus (\oplus_I M_i)$. Then $B(\{M_i\})_{X^{J \sqcup I}} \subset G_{X^{J \sqcup I}}$ is the submodule of those sections whose restriction to $U^{(J \sqcup I)}$ lies in $B^{\boxtimes J} \boxtimes (\boxtimes_I M_i)$.

Lemma 1.3.42. *A map $\phi : j_*^{(I)} j^{(I)*}(\boxtimes M_i) \to \triangle_*^{(I)} N$ is a chiral B-operation, that is, lying in $P^{ch}_{BI}(\{M_i\}, N)$ iff for $J \in \mathcal{S}$ the map $\mathrm{id}_{B_{X^J}} \boxtimes \phi : B_{X^J} \boxtimes j_*^{(I)} j^{(I)*}(\boxtimes M_i) \to B_{X^J} \boxtimes \triangle_*^{(I)} N$ sends $B(\{M_i\})_{X^{J \sqcup I}}$ to $B(N)_{X^{\tilde{J}}}$, here $\tilde{J} = J \sqcup \cdot$.*

Proof. If $\phi \in P^{ch}_{BI}(\{M_i\}, N)$ then the desired property of ϕ is evident. Assume now our property of ϕ for the set $J = \{\cdot\}$. We get a diagram

$$\begin{array}{ccccc} B(\{M_i\})_{X^{J\sqcup I}} & \to & j_*^{(J\sqcup I)} j^{(J\sqcup I)*}(B^{\boxtimes J} \boxtimes (\boxtimes_I M_i)) & \to & \underset{i_0\in I}{\oplus} j_*^{(I)} j^{(I)*}(\boxtimes M_i) \\ \downarrow & & \downarrow \mathrm{id}\boxtimes\phi & & \downarrow \\ B(N)_{X^2} & \to & j_*j^*(B_X \boxtimes N) & \to & \vartriangle_* N, \end{array}$$

where the right arrow in the top row is the sum of maps μ_{i_0} coming from action of B on M_{i_0}. Moreover, the right vertical map will be the sum over i_0 of the maps ϕ. Our claim follows. □

1.3.43. For 3.4.20. If B is a chiral algebra, and $\{B_{X^I}, c, \nu\}$ is the corresponding factorization algebra then $0 \to B_{X^2} \to j_*j^*(B \boxtimes B) \to \vartriangle_* B \to 0$ is exact. So, $[.,.] : B \boxtimes B \to \vartriangle_* B$ vanishes iff c extends to a morphism $B \boxtimes B \to B_{X^2}$.

Their claim 3.4.20 Prop b) follows from my Lemma 1.3.24.

1.3.44. *Example.* See 3.4.23. Let E be a local system on X, $A = E[1]$ and $B = \oplus_{n\geq 0} \mathrm{Sym}^n(A)$, where Sym^n is taken in $\mathrm{Perv}(X)^!$. Then $\mathrm{Sym}^n(A) = (\mathrm{Sym}^n E)[1]$ naturally. Our B is a commutative algebra in $\mathbb{P}\mathrm{erv}(X)^!$. Describe explicitly the corresponding factorization algebra $\{B_{X^I}\}$.

For $I \in \mathcal{S}$ let $Z_I \subset X \times X^I$ be the union of the subschemes $x = x_i$, $i \in I$. Let $\pi_I : Z_I \to X, p_I : Z_I \to X^I$ be the projections. Then π_I is a locally complete intersection of pure relative dimension $n-1$, so $\pi_I^! M[1-n]$ is perverse for any $M \in \mathrm{Perv}(X)$. Set $M_{X^I} = (p_I)_* \pi_I^! A[1-n] \in \mathrm{Perv}(X^I)$. Over $U^{(I)}$ we get $M_{X^I} \widetilde{\to} \sum_I \mathrm{pr}_i^* A[n-1]$ for the projections $\mathrm{pr}_i : U^{(I)} \to X$. That is, $M_{X^I} \mid_{U^{(I)}}$ is a smooth perverse sheaf corresponding to the local system with fibre $\sum_I E_{x_i}$ at $(x_i) \in U^{(I)}$. I think their Prop. (3.4.23.3) becomes: there is a canonical isomorphism

$$B_{X^I} \widetilde{\to} \lambda_I \otimes \mathrm{Sym}(M_{X^I}),$$

where Sym is understood in the sense of the tensor structure $\mathrm{Perv}(X^I)^!$.

For example, take $E = 0$ then $B = \mathrm{IC}_X$, and $B_{X^I} = \boxtimes_I \mathrm{IC}$, $M_{X^I} = 0$.

1.3.45. Let $L \in \mathrm{Perv}(X)$, write $(L \boxtimes L)^-$ for the quotient of $L \boxtimes L$ by ${}^p\mathrm{H}^0(\vartriangle^! (L \boxtimes L))$, so the sequence on X^2 is exact $0 \to (L \boxtimes L)^- \to j_*j^*(L \boxtimes L) \to \vartriangle_* (L \otimes^! L) \otimes \lambda_2 \to 0$ and S_2-equivariant.

The sheaf $(L \boxtimes L)^-$ may have nontrivial quotients supported on X. Namely, let $S = \mathrm{H}^0(L)$, where H^0 is for the usual t-structure. We have a surjection $f : L \to S$, and $\mathrm{H}^0(\mathrm{Ker}\, f) = 0$, so $\mathrm{Ker}\, f$ has no quotients supported on points of X. For example, if $j_x : X - x \to X$ then $0 \to \mathrm{IC} \to j_{x*}\mathrm{IC} \to i_{x*}\bar{\mathbb{Q}}_\ell \to 0$ is exact, and taking $L = j_{x*}\mathrm{IC}$ we get a surjection $L \boxtimes L \to (i_{x,x})_* \bar{\mathbb{Q}}_\ell$.

If $P \subset j_*j^*(L\boxtimes L)$ is a perverse subsheaf such that $P \mid_U = L\boxtimes L$ then $j_{!*}j^*(L\boxtimes L) \subset P$.

Lemma 1.3.46. *Let $j_{!*}j^*(L \boxtimes L) \subset P \subset j_*j^*(L \boxtimes L)$ be a S_2-invariant subsheaf. Then there is the smallest perverse subsheaf $P \subset \bar{P} \subset j_*j^*(L\boxtimes L)$ such that $\bar{P}$ is S_2-invariant and S_2 acts trivially on ${}^p\mathrm{H}^{-1}(\vartriangle^! \bar{P} \otimes \eta_2)$.*

Proof. Define $F \in \mathrm{Perv}(X)$ as the quotient $j_*j^*(L \boxtimes L)/P$. We have canonically $F \otimes \lambda_2 \widetilde{\to} \vartriangle^! P \otimes \lambda_2[1]$. Any S_2-invariant subsheaf $P \subset \bar{P} \subset j_*j^*(L \boxtimes L)$ is the preimage of a S_2-invariant subsheaf $\bar{F} \subset F$ in $j_*j^*(L \boxtimes L)$. For any such $\bar{P}$ we have canonically $(F/\bar{F}) \otimes \lambda_2 \widetilde{\to} \vartriangle^! \bar{P} \otimes \lambda_2[1]$, and here $\vartriangle^!$ can be understood both in derived ot nonderived perverse sense. The desired $\bar{P}$ correspond to the smallest S_2-invariant subsheaf $\bar{F} \subset F$ such that S_2 acts trivially on $(F/\bar{F}) \otimes \lambda_2$, that is, we are taking coinvariants. □

With the above lemma we can perform the construction of the *chiral enveloping algebra* from Sect. 3.7:

Theorem 1.3.47. *The forgetful functor* $\mathcal{C}\mathrm{A}(X) \to \mathit{Lie}^*(X), A \mapsto A^{\mathrm{Lie}}$ *admits a left adjoint* $L \mapsto U(L)$.

Proof. Let L be a Lie algebra in $\mathrm{Perv}(X)^*$. Let P be the kernel of $[.,.] : L \boxtimes L \to \vartriangle_* L$ on X^2. Define T_{X^2} as the smallest S_2-invariant perverse subsheaf of $j_*j^*((L \oplus \mathrm{IC})^{\boxtimes 2})$ on X^2 such that $T_{X^2} \mid_U = (L \oplus \mathrm{IC})^{\boxtimes 2} \mid_U$, $\mathrm{IC} \boxtimes (L \oplus \mathrm{IC})$, $(L \oplus \mathrm{IC}) \boxtimes \mathrm{IC}$ and P are contained in T_{X^2}, S_2 acts trivially on $\vartriangle^! T_{X^2} \otimes \eta_2$. By the above lemma it exists. By Theorem 1.3.39, we get a factorization algebra B corepresenting the corresponding functor. View it as a chiral algebra. Let $f : L \to B$ be the natural map. Finally, define $U(L)$ as the biggest quotient of B by the ideal that coequalizes the diagram on X^2

$$\begin{array}{ccc} L \boxtimes L & \stackrel{f \boxtimes f}{\to} & B \boxtimes B \\ \downarrow [.,.] & & \downarrow [.,.] \\ \vartriangle_* L & \stackrel{f}{\to} & \vartriangle_* B \end{array}$$

That is, we take the image of the map $f \circ [.,.] - [.,.] \circ (f \boxtimes f) : L \boxtimes L \to \vartriangle_* B$ and take the ideal generated by this image, then quotient by this ideal. □

Of course, the corresponding $U(L)$ is in $\mathbb{P}\mathrm{erv}(X)$ in general.

1.3.48. For 3.7.13. Let $A \in \mathcal{C}\mathrm{A}(X)$ equipped with a filtration $A_0 \subset A_1 \subset \dots$ in the sense of their Section 3.3.12. We have their notion of 1-generated filtration. Now if A is a chiral algebra, $R \subset A$ is a chiral subalgebra and $R \subset M \subset A$ is a R-submodule of A that generates A as a chiral algebra then there is a unique 1-generated filtration $A.$ of A with $A_0 = R, A_1 = M$. If M is a Lie*-subalgebra of A^{Lie}, R^{Lie} is commutative and M normalizes R (that is, $[M, R] \subset R$) then grA is a commutative chiral algebra.

1.3.49. For 3.7.15. Their Proposition 3.7.15 remains valid for $\bar{\mathbb{Q}}_\ell$-perverse sheaves. Namely, let L be a Lie algebra in $\mathrm{Perv}(X)^*$. The forgetful functor $\mathcal{M}(X, U(L)) \to \mathcal{M}(X, L)$ admits a left adjoint $\mathrm{Ind} : \mathcal{M}(X, L) \to \mathcal{M}(X, U(L))$. Their notion of a 1-generated filtration makes sense in our setting of perverse sheaves.

1.3.50. Let $A \in \mathrm{Perv}(X)$, a precision about chiral algebra structure on A. Define $F \in \mathrm{Perv}(X)$ by the exact sequence $0 \to j_{!*}(A \boxtimes A) \to j_*j^*(A \boxtimes A) \to F \to 0$. So, we have a surjection $F \to A \otimes^! A$ on X, the tensor product is in the nonderived sense. Then $F = {}^p\mathrm{H}^0(\vartriangle^* j_*j^*(A \boxtimes A))$ and ${}^p\mathrm{H}^{-1}(\vartriangle^* j_{!*}j^*(A \boxtimes A)$. The chiral product is given by a map $\mu : F \to A$ in $\mathrm{Perv}(X)$.

1.3.51. Let B be a quasi-factorization algebra (as in Sect. 3.4.14, p. 185). Then the construction from their Section 3.4.8 (as in my Sect. 1.3.31) works and provides a structure of Lie^{ch}-algebra on $B_X \in \mathrm{Perv}(X)$. We can also take as input a version of a quasi-factorization algebra B, where B_{X^J} is a complex of perverse sheaves, not necessarily placed in one perverse degree.

1.3.52. Let A be a constant complex of $\bar{\mathbb{Q}}_\ell$-vector spaces on $\mathrm{Spec}\, k$. (We may assume if needed that the differential is zero). Let us also write A for its $*$-restriction to X. Then $\phi \in P^*(\{A, A\}, A)$, that is, $\phi : A \boxtimes A \to \triangle_* A$ is given by a map $f : A \otimes A = \triangle^* (A \boxtimes A) \to A$ on X. Since X is proper, f is costant map, that is, we are given a map $f : A \otimes A \to A$ on $\mathrm{Spec}\, k$. Then (A, ϕ) is a Lie^*-algebra on X iff f is skew-symmetric and satisfies the Jacobi identity. That is, (A, f) is a Lie algebra in the tensor category of complexes on $\mathrm{Spec}\, k$.

1.3.53. Let L be a Lie algebra in $\mathrm{Perv}(X)^*$. Then $\mathrm{R}\Gamma(X, L)$ has a structure of a Lie algebra in the category of complexes on $\mathrm{Spec}\, k$. Namely, the bracket $[,] : L \boxtimes L \to \triangle_* L$ gives a map $[,] : \mathrm{R}\Gamma(X, L) \otimes \mathrm{R}\Gamma(X, L) \to \mathrm{R}\Gamma(X, L)$ by passing to cohomologies, and it satisfies the axioms of a Lie algebra.

1.3.54. Let $E^{\cdot}$ be a commutative DGA in the tensor category of local systems on X. So, $E^{\cdot} \otimes E^{\cdot} \to E^{\cdot}$ is (super)-commutative and associative, $E^{\cdot} = \ldots \to E^i \to E^{i+1} \to \ldots$, where each E^i is a local system on X. Then $A := E[1]$ is a complex of smooth perverse sheaves on X, and A is a commutative DGA in $\mathrm{Perv}(X)^!$ with the product $A \otimes^! A = (E \otimes E)[1] \to E[1] = A$. Applying the construction of the corresponding factorization algebra, we get for each $I \in \mathcal{S}$ a complex of perverse sheaves B_{X^I} on X^I, which is a total complex of the double Chevalley complex $C(A)^{\cdot}_{X^I}$. Maybe, according to (stack project, Lemma 12.22.7), we may also see B_{X^I} as the subcomplex of $j^{(I)}_* j^{(I)*}(A^{\boxtimes I})$, namely the kernel of the differential in the Chevalley complex $C(A)^{\cdot}_{X^I}$. We have $B_X = A$. For $\triangle \colon X \to X^I$ we have $\triangle^! \, B_{X^I} \otimes \eta_I \widetilde{\to} B_X[1]$.

In particular, we know all the !-fibres of B_{X^I}. Namely, for $mx = (x, \ldots, x) \in X^I$ let $i_{mx} : \mathrm{Spec}\, k \to X^I$ be this point. Then $i^!_{mx} B_{X^I} \otimes \eta_I \widetilde{\to} E^{\cdot}_x$.

Note that $E^{\cdot *}$ is a cocommutative and coassociative coalgebra in local systems on X. Similarly, $\mathbb{D}A$ is a complex of smooth perverse sheaves on X, this is a cocommutative and coassociative DG-coalgebra. The corresponding factorizable object is a collection of complexes: for $I \in \mathcal{S}$ the complex $\mathbb{D}(B_{X^I} \otimes \eta_I)$ on X^I. The $*$-fibre of $\mathbb{D}(B_{X^I} \otimes \eta_I)$ at $mx = (x, \ldots, x) \in X^I$ is the complex $E^{\cdot *}_x$. So, the $*$-fibre of $\mathbb{D}(B_{X^I} \otimes \eta_I)$ at $(x_i)_{i \in I}$ is the tensor product $\otimes_j E^{\cdot *}_{y_j}$, where $\{y_i\} = \{x_i\}$ as a set, and y_j are pairwise distinct. This is the construction used by Dennis in ([2], Section 3.1, p. 1805). In particular, if $E^{\cdot}$ is placed in usual degree zero then $\mathbb{D}(B_{X^I} \otimes \eta_I)$ is a constructible sheaf on X^I, because each of its $*$-fibres is placed in usual degree zero.

If in addition $E^{\cdot}$ is Λ^{pos}-graded, where Λ^{pos} is a free abelian semigroup of finite type, then assume the component of $E^{\cdot}$ corresponding to $0 \in \Lambda^{pos}$ is $\bar{\mathbb{Q}}_\ell$. Then $A = E^{\cdot}[1]$ gets a structure of a Λ^{pos}-graded commutative chiral DG-algebra. As in my Section 2.2.7, we attach to it a DG-object (B, c, ν) in $\mathcal{F}\mathrm{A}(X)_{\Lambda^{pos}}$, namely we take the total complex of the corresponding part of the Chevalley complex. Then for any $(I, \lambda) \in \mathcal{S}_{\Lambda^{pos}}$ we get the corresponding complex B_{X^I} on X^I, which yields one on $X^{\mathfrak{B}(\mu)}$, where $\mathfrak{B}(\mu)$ is the decomposition $\sum_s n_s\mu_s$ of $\mu=\sum_{i\in I}\lambda_i$. Here $\lambda: I\to\Lambda^{pos}$ takes distinct values μ_s with multiplicities n_s. If $f: X^I\to X^{\mathfrak{B}(\mu)}$ is the natural map then take $\operatorname{Aut}(I,\lambda)$-invariants in $f_*(B_{X^I}\otimes\eta_I)$, the result is a complex of perverse sheaves denoted $B_{X^{\mathfrak{B}(\mu)}}$ on $X^{\mathfrak{B}(\mu)}$.

For $\mu\in\Lambda^{pos}$ write $E^{\cdot}_\mu$ for the μ-component of $E^{\cdot}$. Consider a k-point $(x_i)\in X^I$. Assume $\sum_{i\in I}\lambda_i x_i=\sum_{j\in J}\mu_j y_j$, where y_j are pairwise distinct, so we get a surjection $\pi: I\to J$ and a map $\mu: J\to\Lambda^{pos}_*$ given by $\mu_j=\sum_{i\in\pi^{-1}(j)}\lambda_i$. The !-fibre of $B_{X^I}\otimes\eta_I$ at $(x_i)\in X^I$ is $\underset{j\in J}{\otimes}(E^{\cdot}_{\mu_j})_{y_j}$. So, the $*$-fibre of $\mathbb{D}(B_{X^I}\otimes\eta_I)$ at $(x_i)\in X^I$ is

$$\underset{j\in J}{\otimes}(E^*_{\mu_j})_{y_j}$$

Here $E^*=\mathcal{H}om(E^{\cdot},\bar{\mathbb{Q}}_\ell)$. The $*$-fibre of $B_{X^{\mathfrak{B}(\mu)}}$ at $\sum_j\mu_j y_j$ is not clear.

We especially apply the above in the following case. Let $\lambda\in\Lambda^{pos}$. Write α_j for the set of generators of Λ^{pos}. Write $\lambda=\sum_{j\in J}n_j\alpha_j$ with $n_j\geq 0$. Then take I a set and $\lambda_I: I\to\Lambda^{pos}_*$ a map taking value α_j precisely n_j times. We get a map $f: X^I\to X^\lambda$, and $X^{\mathfrak{B}(\lambda)}\stackrel{\sim}{\to}X^\lambda$. For this $(I,\lambda_I)\in\mathcal{S}_{\Lambda^{pos}}$ we get a complex $B_{X^\lambda}:=B_{X^{\mathfrak{B}(\lambda)}}$ on X^λ. They form a Λ^{pos}-factorization algebra in the first 'nonrigorous' sense of Section 2.2. The $*$-fibre of $\mathbb{D}B_{X^\lambda}$ at $\lambda x\in X^\lambda$ is $(E^*_\lambda)_x$, because $\operatorname{Aut}(I,\lambda_I)$ acts trivially on the $*$-fibre of $f_*\mathbb{D}(B_{X^I}\otimes\eta_I)$ at x.

So, the $*$-fibre of $\mathbb{D}B_{X^\lambda}$ at $\sum\mu_j y_i\in X^\lambda$ with y_i parwise distinct is $\otimes_j(E^*_{\mu_j})_{y_j}$.

By construction, for $\lambda_i\in\Lambda^{pos}$ there is a natural map

$$B_{X^{\lambda_1}}\star B_{X^{\lambda_2}}\to B_{X^{\lambda_1+\lambda_2}} \tag{16}$$

on $X^{\lambda_1+\lambda_2}$. Here for $S_i\in D(X^{\lambda_i})$, $S_1\star S_2=\operatorname{add}_!(S_1\boxtimes S_2)$ for $\operatorname{add}: X^{\lambda_1}\times X^{\lambda_2}\to X^{\lambda_1+\lambda_2}$.

1.3.55. if $E^{\cdot},E'^{\cdot}$ are commutative DGA in local systems on X then $E^{\cdot}\otimes E'^{\cdot}$ is also a commutative DGA. Let B,B' be the DG-objects in factorization algebras corresponding to $E^{\cdot},E'^{\cdot}$ respectively. Then the factorization algebra corresponding to $E^{\cdot}\otimes E'^{\cdot}$ is the tensor product of B and B'. So, for $I\in\mathcal{S}$, $(B\otimes^! B')_{X^I}=B_{X^I}\otimes^! B'_{X^I}\otimes\lambda_I$.

1.3.56. Let $E^{\cdot},E'^{\cdot}$ be Λ^{pos}-graded commutative DGA in local systems on X. Then $E^{\cdot}\otimes E'^{\cdot}$ is also a Λ^{pos}-graded commutative DGA in local systems on X. In Section 1.3.54 we associated to $E^{\cdot},E'^{\cdot}$ the DG-objects $B,B'\in\mathcal{FA}(X)_{\Lambda^{pos}}$, and in turn for any $\lambda\in\Lambda^{pos}_*$ complexes $B_{X^\lambda},B'_{X^\lambda}$ on X^λ. Let B'' be the DG-object of $\mathcal{FA}(X)_{\Lambda^{pos}}$ similarly associated to $E^{\cdot}\otimes E'^{\cdot}$. Then for any $\lambda\in\Lambda^{pos}_*$ the complex B''_{X^λ} is given by

$$B''_{X^\lambda}=\underset{\lambda_1+\lambda_2=\lambda}{\oplus}B_{X^{\lambda_1}}\star B'_{X^{\lambda_2}} \tag{17}$$

This is done also in ([5], Sections 2.1-2.3).

Indeed, write $A=E[1],A'=E'[1],A''=(E\otimes E')[1]$. Let $\lambda=\sum_{j\in\mathcal{J}}n_j\alpha_j\in\Lambda^{pos}_*$. Here $\{\alpha_j\}_{j\in\mathcal{J}}$ is the set of generators of Λ^{pos}. Pick a set I and a map $\lambda_I: I\to\Lambda^{pos}_*$ taking values α_j with multiplicities n_j. By definition, to this (I,λ_I) is attached B''_{X^I}.

For any $i\in I$ we have $\lambda_i=\alpha_j$ for some j. So, $(E\otimes E')_{\lambda_i}=E_{\lambda_i}\oplus E'_{\lambda_i}$. For any decomposition $I=I_0\sqcup I'_0$ denote by $\lambda_0: I_0\to\Lambda^{pos}_*,\lambda'_0: I'_0\to\Lambda^{pos}_*$ the restrictions of

λ_I. We get

$$B''_{X^I} = \underset{(I,\lambda_I)=(I_0,\lambda_0)\sqcup(I'_0,\lambda'_0)}{\oplus} B_{X^{I_0}} \boxtimes B'_{X^{I'_0}}$$

Taking the direct image of $B''_{X^I} \otimes \eta_I$ under $X^I \to X^\lambda$ and $\mathrm{Aut}(I, \lambda_I)$-invariants we get B''_{X^λ}. Using the latter formula, this should yield (17).

1.3.57. **Example**. Let V be a local system on X. Take $E = \wedge^{\cdot} V = \mathrm{Sym}(V[-1])$. This is a commutative DGA in the category of local systems on X. The construction of Section 1.3.54 yields for $I \in \mathcal{S}$ a complex of perverse sheaves B_{X^I} on X^I, here B is a 'complex of factorization algebras', $B_X = E[1]$.

1.3.58. **Example**. Let $E^{\cdot}$ be a Λ^{neg}-graded commutative DG-algebra in local systems on X. Let $\lambda \in \Lambda^{neg}$. Consider a DG-object B in $\mathcal{F}\mathrm{A}(X)_{\Lambda^{neg}}$ attached to it in Section 1.3.54. Let $\lambda = \sum_i n_i\alpha_i \in \Lambda^{neg}$, where α_i is the set of generators of Λ^{neg}. Let $(I, \lambda_I) \in \mathcal{S}_{\Lambda^{neg}}$ be such that λ_I takes values α_i with multipilicity n_i. Consider the corresponding complexes $B_{X^I} \in D(X^I), B_{X^\lambda} \in \mathrm{D}(X^\lambda)$. We should not think that $B_{X^I} \subset j^{(I)}_* j^{(I)*}(\boxtimes_i A_{\lambda_i})$, where $A = E^{\cdot}[1]$.

This is illustrated in the following example. Let $E^{\cdot} = \mathrm{Sym}(\mathfrak{n}^*[-1])$ be the costandard complex for $\mathfrak{n}$, the unipotent radical of the standard Borel of GL_3. Let α_i, $i = 1, 2$ be the two negative simple roots of GL_3, so $\beta = \alpha_1 + \alpha_2$ is also a negative root. The Λ^{neg}-grading is the one given by the action of T on $E^{\cdot}$. Take $\lambda = \alpha_1 + \alpha_2$, $I = \{1, 2\}, \lambda_I : I \to \Lambda^{neg}$ taking values α_1, α_2 respectively. Then the (I, λ_I)-graded component of the Chevalley double complex on X^I is given by the diagram

$$\begin{array}{ccc} j_*j^*(\mathfrak{n}^*_{\alpha_1} \boxtimes \mathfrak{n}^*_{\alpha_2}) & \to & \triangle_* \,(\mathfrak{n}^*_{\alpha_1} \otimes \mathfrak{n}^*_{\alpha_2}) \\ & & \uparrow d \\ & & \triangle_* \,\mathfrak{n}^*_\beta \end{array}$$

(In fact, the vertical arrow d is an isomorphism). We see that the corresponding total complex is $\mathfrak{n}^*_\beta \oplus j_*j^*(\mathfrak{n}^*_{\alpha_1} \boxtimes \mathfrak{n}^*_{\alpha_2}) \to \triangle_* \,(\mathfrak{n}^*_{\alpha_1} \otimes \mathfrak{n}^*_{\alpha_2})$. So, $B_{X^I} \widetilde{\to} j_*j^*(\mathfrak{n}^*_{\alpha_1} \boxtimes \mathfrak{n}^*_{\alpha_2})[2]$ on X^2. $\mathbb{D}B_{X^\lambda}$ is the extension by zero from $j : U \hookrightarrow X^\lambda = X^2$. It has a filtration given by the subsheaf $\triangle_* \,\mathfrak{n}_{\alpha_1} \otimes \mathfrak{n}_{\alpha_2}[1] \subset \mathbb{D}B_{X^\lambda}$. This filtration is generalized for any reductive group in ([2], Sect. 3.3).

1.3.59. **Example**. Let E be a Λ^{pos}-graded commutative bialgebra in the tensor category of local systems on X (it could be DG-bialgebra also). So, both product and coproduct are compatible with gradings, but the coproduct is not necessarily commutative. We assume the 0-th component is $E_0 = \bar{\mathbb{Q}}_\ell$. In Section 1.3.54 we associated to it a DG-object of $\mathcal{F}\mathrm{A}(X)_{\Lambda^{pos}}$ and a collection B_{X^λ} of complexes on X^λ for $\lambda \in \Lambda^{pos}$. They are equipped with maps (16). Now $E \otimes E$ is also a Λ^{pos}-graded commutative algebra in the tensor category of local systems on X, and the corresponding Λ^{pos}-factorization algebra was described in Section 1.3.56.

The coproduct map $E \to E \otimes E$ is a morphism of Λ^{pos}-graded commutative algebras. So, it gives a morphism of the corresponding DG-objects in $\mathcal{F}\mathrm{A}(X)_{\Lambda^{pos}}$. Thus, we get for any $\lambda \in \Lambda^{pos}$ a morphism

$$B_{X^\lambda} \to \underset{\lambda_1+\lambda_2=\lambda,\ \lambda_i\in\Lambda^{pos}}{\oplus} B_{X^{\lambda_1}} \star B_{X^{\lambda_2}}$$

Example: let $\mathfrak{n}^-$ be the unipotent radical of the negative Borel subalgebra of a reductive Lie algebra. Then $U(\mathfrak{n}^-)$ is Λ^{neg}-graded cocommutative bialgebra. Take E to be its graded dual, it is a Λ^{pos}-graded commutative bialgebra.

2. Constructions via twisted arrow categories

2.1. If $\mathcal{C}$ is a category, let $TA(\mathcal{C})$ be the category, whose objects are arrow $f : a \to b$ in $\mathcal{C}$. A morphism from $f : a \to b$ to $g : c \to d$ is a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} a & \stackrel{p}{\leftarrow} & c \\ \downarrow f & & \downarrow g \\ b & \stackrel{q}{\to} & d \end{array}$$

We have the projection $TA(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$ sending $f : a \to b$ to (a, b), this is a cocartesian fibration corresponding to the functor $\mathcal{C}^{op} \times \mathcal{C} \to Sets$, $(a, b) \mapsto \operatorname{Hom}_{\mathcal{C}}(a, b)$.

We have a functor $\operatorname{Fun}(\mathcal{C}^o \times \mathcal{C}, \mathcal{D}) \to \operatorname{Fun}(TA(\mathcal{C}), \mathcal{D})$ sending F to the functor $\bar{F} : TA(\mathcal{C}) \to \mathcal{D}$, $f \mapsto F(src(f), trg(f)$ (see [9], Def. 1.12).

For a functor $F : \mathcal{C}^o \times \mathcal{C} \to \mathcal{C}$ the limit $\lim_{TA(\mathcal{C})} \bar{F}$ is called the end of F and is denoted $\int_c F = \lim_{TA(\mathcal{C})} \bar{F}$. The colimit $\operatorname{colim}_{TA(\mathcal{C})} \bar{F}$ is called the coend of F. The notation is $\int^c F = \operatorname{colim}_{TA(\mathcal{C})} \bar{F}$, see [9].

2.1.1. Recall that $\mathcal{S}$ denotes the category, whose objects are finite nonempty sets, and morphisms are surjections. For $I \in \mathcal{S}$ let $a_I : X^I \to \mathcal{R}(X)$ be the natural map. Let $A \in \operatorname{Perv}(X)$ be a commutative chiral algebra on X. Then for each $\pi : J \to I$ in $\mathcal{S}$ we get the product map $m : \otimes^!_J A \to \otimes^!_I A$ compatible with compositions of π's. In other words, we get a functor $\mathcal{S} \to \operatorname{Perv}(X)$, $I \mapsto \otimes^!_I A$.

We want to consider for π as above $(\underset{I}{\boxtimes}(\overset{L}{\underset{J_i}{\otimes}}{}^! A)) \otimes \eta_I$ on X^I as a functor of π. More precisely, for a morphism $\pi \to \pi'$ in $TW(\mathcal{S})$ given by the diagram

$$\begin{array}{ccc} J & \leftarrow & J' \\ \downarrow \pi & & \downarrow \pi' \\ I & \to & I' \end{array}$$

we get a natural map on X^I

$$(\underset{I}{\boxtimes}(\overset{L}{\underset{J_i}{\otimes}}{}^! A)) \otimes \eta_I \leftarrow \vartriangle^{(I/I')}_* (\underset{I'}{\boxtimes}(\overset{L}{\underset{J'_i}{\otimes}}{}^! A)) \otimes \eta_{I'}, \qquad (18)$$

here $\vartriangle^{(I/I')}$: $X^{I'} \to X^I$. Note that $X^I = \prod_{I'} X^{I_{i'}}$, and the latter map is the exteriour product of the following maps over I'. The factors correspond to the squares obtained by passing to the fibre over $i' \in I'$ in the above square, that is, to

$$\begin{array}{ccc} J_{i'} & \leftarrow & J'_{i'} \\ \downarrow \pi & & \downarrow \pi' \\ I_{i'} & \to & i' \end{array}$$

Thus, it suffices to define (18) in the case when $I' = \cdot$. In this case our map corresponds by adjointness to the map

$$(\overset{L}{\underset{J'}{\otimes}}{}^! A)[1] \overset{product}{\to} (\overset{L}{\underset{I}{\otimes}}{}^! \overset{L}{\underset{J_i}{\otimes}}{}^! A)[1] = \vartriangle^{(I/I')!} (\boxtimes_I (\overset{L}{\underset{J_i}{\otimes}}{}^! A)) \otimes \eta_I$$

Now imagine that $(a_I)_*$ makes some sense then (18) gives a map over $\mathcal{R}(X)$

$$(a_I)_*(\underset{I}{\boxtimes}(\overset{L}{\underset{J_i}{\otimes}}{}^! A)) \otimes \eta_I \leftarrow (a_{I'})_*(\underset{I'}{\boxtimes}(\overset{L}{\underset{J'_i}{\otimes}}{}^! A)) \otimes \eta_{I'}$$

Dennis: there is a functor $TA(\mathcal{S})^o \to \mathrm{D}(\mathcal{R}(X), \bar{\mathbb{Q}}_\ell)$ sending $\pi : J \to I$ to

$$(a_I)_*(\underset{I}{\boxtimes}(\otimes^!_{J_i} A)[1]) = (a_I)_*(\underset{I}{\boxtimes}(\otimes^!_{J_i} A)) \otimes \eta_I$$

Dennis claims in ([4], Sect. 2.3): the colimit of the latter functor is the desired "factorization algebra" on $\mathcal{R}(X)$. Maybe what we actually get is rather a pre-factorization algebra? Maybe the colimit has to be calculated in the corresponding stable infinity category.

Let B be the factorization algebra corresponding to A. In the above paper by Dennis he propose an explicit formula for the complex B_{X^I} for any $I \in \mathcal{S}$. In addition, if A is placed in one cohomological degree, B_{X^I} is also placed in one degree.

2.2. Λ^{pos}-graded version. Let Λ^{pos} be a free abelian semigroup isomorphic to $(\mathbb{Z}^+)^r$ for some $r > 0$. Let $\{\alpha_i\}$, $i \in \mathcal{I}$ be the set of generators of order r, it is unique. For $\lambda \in \Lambda^{pos}$ write X^λ for the moduli scheme of Λ^{pos}-valued divisors of degree λ. Let add : $X^\lambda \times X^\mu \to X^{\lambda+\mu}$ be the sum map. Its restriction to $(X^\lambda \times X^\mu)_{disj}$ is etale, the latter is the open subscheme of divisors with disjoint supports. Let add_{disj} : $(X^\lambda \times X^\mu)_{disj} \to X^{\lambda+\mu}$ be the restriction of add.

Consider the following notion of a Λ^{pos}-*quasi-factorization algebra* over $\{X^\lambda\}_{\lambda\in\Lambda^{pos}}$. This is a collection $B^\lambda \in \mathrm{Perv}(X^\lambda)$ together with isomorphisms

$$c : \mathrm{add}^*_{disj} B^{\lambda+\mu} \tilde{\to} B^\lambda \boxtimes B^\mu$$

over $(X^\lambda \times X^\mu)_{disj}$, which are "commutative and associative in a natural sense". In my opinion, this is a bad formulation, because it hiddens the symmetries! We will give a more precise definition below. We will also compare it with the notions from ([1], 3.4.14).

Let now $A = \oplus_{\lambda\in\Lambda^{pos}} A^\lambda$ be a Λ^{pos}-graded unital commutative algebra in $\mathrm{Perv}(X)^!$. We want to associate to it a Λ^{pos}-quasi-factorization algebra.

Set $\Lambda^{pos}_* = \Lambda^{pos} - 0$. We may consider the category $\mathcal{S}_{\Lambda^{pos}}$ whose objects are diagrams $\Lambda^{pos}_* \overset{\lambda}{\leftarrow} J$, and a morphism from $\Lambda^{pos}_* \overset{\lambda'}{\leftarrow} J'$ to $\Lambda^{pos}_* \overset{\lambda}{\leftarrow} J$ is a morphism $\phi_J : J' \to J$ in $\mathcal{S}$ such that $\lambda_j = \sum_{J'_j} \lambda'_{j'}$ for $j \in J$. Set

$$\mathcal{R}(X)_{\Lambda^{pos}} = (\mathcal{R}(X) \otimes \Lambda^{pos}) - \{0\} = (\prod_{i\in\mathcal{I}} (\mathcal{R}(X) \sqcup *)) - \{0\}$$

where the operation on $\mathcal{R}(X)$ is the union.

For an object $\Lambda^{pos}_* \stackrel{\lambda}{\leftarrow} I$ of $\mathcal{S}_{\Lambda^{pos}}$ consider the map $a_I : X^I \to \sqcup_{\lambda\in\Lambda^{pos}_*} X^\lambda$ sending $(x_i)_{i\in I}$ to $\sum_{i\in I}\lambda_i x_i$. For a map $\Lambda^{pos}_* \stackrel{\lambda}{\leftarrow} I \stackrel{\phi_I}{\to} I'$ in $\mathcal{S}_{\Lambda^{pos}}$ the diagram commutes

$$\begin{array}{ccc} X^{I'} & \stackrel{\vartriangle^{(I/I')}}{\to} & X^I \\ & \searrow a_{I'} & \downarrow a_I \\ & & \underset{\lambda\in\Lambda^{pos}_*}{\sqcup} X^\lambda \end{array}$$

Consider the category $TA(\mathcal{S}_{\Lambda^{pos}})$. Its object is a diagram $\Lambda^{pos}_* \stackrel{\lambda}{\leftarrow} J \stackrel{\pi}{\to} I$, where $\pi : J \to I$ is a map in $\mathcal{S}$. The corresponding map $I \to \Lambda^{pos}_*$ is recovered as an integral. A morphism in $TA(\mathcal{S}_{\Lambda^{pos}})$ is a diagram

$$\begin{array}{ccc} \Lambda^{pos}_* & & \Lambda^{pos}_* \\ \uparrow \lambda & & \uparrow \lambda' \\ J & \stackrel{\phi_J}{\leftarrow} & J' \\ \downarrow \pi & & \downarrow \pi' \\ I & \stackrel{\phi_I}{\to} & I' \end{array} \tag{19}$$

where the bottom part is a map $\pi \to \pi'$ in $\mathcal{S}$, and $\sum_{J'_j} \lambda'_{j'} = \lambda_j$ for $j \in J$.

A map in $\mathcal{S}_{\Lambda^{pos}}$ yields the product morphism

$$m : \otimes^!_J A^{\lambda_j} \to \otimes^!_{J'} A^{\lambda'_j}$$

on X compatible with compositions. So, we get a functor $\mathcal{S}_{\Lambda^{pos}} \to \mathrm{Perv}(X)$ sending the above object to $\otimes^!_J A^{\lambda_j}$.

As in the previous section, we want to consider for an object $\Lambda^{pos}_* \stackrel{\lambda}{\leftarrow} J \stackrel{\pi}{\to} I$ of $TA(\mathcal{S})_{\Lambda^{pos}}$ the perverse sheaf

$$(\underset{I}{\boxtimes}(\overset{L}{\underset{J_i}{\otimes}}{}^! A^{\lambda_j})) \otimes \eta_I$$

as a functor on $TA(\mathcal{S}_{\Lambda^{pos}})$. More precisely, exactly as in the previous section a map (19) yields a morphism on X^I

$$(\underset{I}{\boxtimes}(\overset{L}{\underset{J_i}{\otimes}}{}^! A^{\lambda_j})) \otimes \eta_I \leftarrow \vartriangle^{(I/I')}_* (\underset{I'}{\boxtimes}(\overset{L}{\underset{J'_i}{\otimes}}{}^! A^{\lambda'_{j'}})) \otimes \eta_{I'}, \tag{20}$$

here $\vartriangle^{(I/I')}$: $X^{I'} \to X^I$. Now if $(a_I)_*$ makes sense then (20) yields a map over $\underset{\lambda\in\Lambda^{pos}_*}{\sqcup} X^\lambda$

$$(a_I)_*(\underset{I}{\boxtimes}(\overset{L}{\underset{J_i}{\otimes}}{}^! A^{\lambda_j})) \otimes \eta_I \leftarrow (a_{I'})_*(\underset{I'}{\boxtimes}(\overset{L}{\underset{J'_i}{\otimes}}{}^! A^{\lambda'_{j'}})) \otimes \eta_{I'}$$

We get a functor $TA(\mathcal{S}_{\Lambda^{pos}})^o \to \mathrm{D}(\underset{\lambda\in\Lambda^{pos}_*}{\sqcup} X^\lambda, \bar{\mathbb{Q}}_\ell) = \prod_{\lambda\in\Lambda^{pos}_*} \mathrm{D}(X^\lambda, \bar{\mathbb{Q}}_\ell)$ sending $\Lambda^{pos}_* \stackrel{\lambda}{\leftarrow} J \stackrel{\pi}{\to} I$ to

$$(a_I)_*(\underset{I}{\boxtimes}(\overset{L}{\underset{J_i}{\otimes}}{}^! A^{\lambda_j})) \otimes \eta_I$$

Then the colimit of the latter functor should be the desired Λ^{pos}-quasi-factorization algebra. Maybe the colimit has to be calculated in the corresponding stable infinity category.

2.2.1. A definition in the style of [1]. Let $\alpha_i, i \in \mathcal{J}$ be the generators of Λ^{pos}. Let $\lambda = \sum_{i\in\mathcal{J}} n_i\alpha_i$. Recall that a Z-point of X^λ is a collection $D = (D_i)_{i\in\mathcal{J}}$ of effective relative Cartier divisors $D_i \hookrightarrow Z \times X$ over Z of degre n_i for $i \in \mathcal{J}$. We consider the equivalence relation on Λ^{pos}-valued relative Cartier divisors on $Z \times X/Z$. Say that $D \sim D'$ if $D_{i,red} = D'_{i,red}$ for all $i \in \mathcal{J}$. The equivalence classes of Λ^{pos}-valued relative Cartier divisors on $Z \times X/Z$ form an ordered set denoted $\mathcal{C}(X)_{Z,\Lambda^{pos}}$. Namely, $D \le D'$ iff $D_{i,red} \subset D'_{i,red}$ for any $i \in \mathcal{J}$. We consider $\mathcal{C}(X)_{Z,\Lambda^{pos}}$ as a category.

Recall the category Aff_{sm} of smooth affine k-schemes of finite type, which are of pure dimension. We get a category $\mathcal{C}(X)_{\Lambda^{pos}}$ fibred over Aff_{sm}. Its object is a pair $Z \in \mathrm{Aff}_{sm}$ and an element of $\mathcal{C}(X)_{Z,\Lambda^{pos}}$. A morphism from (Z', D') to (Z, D) is a morphism $f : Z' \to Z$ in Aff_{sm} such that the composition $D'_{i,red} \hookrightarrow Z' \times X \stackrel{f\times\mathrm{id}}{\to} Z \times X$ factors through $D_{i,red} \stackrel{Z}{\hookrightarrow} \times X$. The projection $\mathcal{C}(X)_{\Lambda^{pos}} \to \mathrm{Aff}_{sm}$ is a cartesian fibration.

Consider a pair (B, c), where

i) $B : \mathcal{C}(X)_{\Lambda^{pos}} \to \mathrm{Perv}$ is a functor over Aff_{sm}, which sends a cartesian arrow to a cartesian arrow. So, B assigns to every Λ^{pos}-valued relative Cartier disivor D on $Z \times X/Z$ a perverse sheaf B_D on Z. If $D' \le D$ in $\mathcal{C}(X)_{Z,\Lambda^{pos}}$ then we get a morphism $B_{D'} \to B_D$, and everything is compatibe with base changes. In particular, for $\lambda \in \Lambda^{pos}$ we have the universal Cartier divisor $D = (D_i)_{i\in\mathcal{J}}$ on $X \times X^\lambda/X^\lambda$, hence a perverse sheaf B_{X^λ} on X^λ.

ii) c is a rule that assigns to every pair of mutually disjoint divisors $D_1, D_2 \in \mathcal{C}(X)_{Z,\Lambda^{pos}}$ an isomorphism $c_{D_1,D_2} : B_{D_1} \otimes^! B_{D_2} \widetilde{\to} B_{D_1+D_2}$, here $\otimes^!$ is understood in the "nonderived shifts sense". It is required that c are commutative and associative in the obvious sense and compatible with morphisms from i).

We require in addition $B_{X^0} \ne 0$. Then $B_{X^0} \widetilde{\to} \bar{\mathbb{Q}}_\ell$. Denote by $\mathcal{Q}\mathrm{FA}(X)_{\mathcal{R}\Lambda^{pos}}$ the category of these objects, call them $\mathcal{R}\Lambda^{pos}$-quasi-factorization algebras, the letter $\mathcal{R}$ refers to the Ran space.

If we require in addition that B_{X^λ} have no perverse subsheaves supported on the diagonal divisor, we get the notion of a $\mathcal{R}\Lambda^{pos}$-factorization algebra, the corresponding category is denoted $\mathcal{F}\mathrm{A}(X)_{\mathcal{R}\Lambda^{pos}}$.

2.2.2. Say that a Λ^{pos}-*quasi-perverse sheaf* is a data for any $(I, \lambda) \in \mathcal{S}_{\Lambda^{pos}}$ of a perverse sheaf B_{X^I} on X^I and for any $\pi : (J, \lambda_J) \to (I, \lambda_I)$ in $\mathcal{S}_{\Lambda^{pos}}$ of an isomorphism $\nu^{(\pi)} : \vartriangle^{(\pi)!} B_{X^J} \otimes \eta_J \widetilde{\to} B_{X^I} \otimes \eta_I$ compatible with compositions of π's. The inverse image is understood in the "nonderived perverse sense". Write $\mathrm{QPerv}_{\Lambda^{pos}}$ for the category of Λ^{pos}-quasi-perverse sheaves.

If we require in addition that B_{X^I} has no perverse subsheaves supported on the diagonal divisor, we get the full subcategory of Λ^{pos}-*perverse sheaves* inside denoted $\mathrm{Perv}_{\Lambda^{pos}}$. As in Section 1.3.22, the above inverse image can also be understood in the derived sense. This is a tensor $\bar{\mathbb{Q}}_\ell$-category, the tensor product is defined as in my Section 1.3.22.

2.2.3. Here is a rigorous definition of a Λ^{pos}-*quasi-factorization algebra* in the style of ([1], 3.4.4). This is a Λ^{pos}-quasi-perverse sheaf $B \in \mathrm{Perv}_{\Lambda^{pos}}$ with the following data and properties. If π is a map in $\mathcal{S}_{\Lambda^{pos}}$ as above, for $i \in I$ the set J_i is equipped with the restriction $\lambda_{J_i} : J_i \to \Lambda^{pos}$ of λ_J. A factorization structure on B is a rule that assigns to any map π in $\mathcal{S}_{\Lambda^{pos}}$ as above an isomorphism $c_{[J/I]}$ given by (8), where J_i is equipped with the restriction of λ_J. We require c to be mutually compatible as in Section 1.3.25. Besides, we require that c is compatible with ν. Namely, for every diagram $J \to J' \to I$ in $\mathcal{S}_{\Lambda^{pos}}$ the diagram (11) commutes.

This is a tensor category naturally, we write $\mathcal{Q}\mathrm{FA}(X)_{\Lambda^{pos}}$. If we impose in addition the property that $B \in \mathrm{Perv}_{\Lambda^{pos}}$, we get the full subcategory $\mathcal{F}\mathrm{A}(X)_{\Lambda^{pos}}$ of Λ^{pos}-factorization algebras. This is also a tensor category. Our Λ^{pos}-quasi-factorization algebras and Λ^{pos}-factorization algebras are nonunital.

2.2.4. For $\lambda \in \Lambda^{pos}_*$ let $\mathcal{S}_\lambda \subset \mathcal{S}_{\Lambda^{pos}}$ be the full subcategory of pairs (I, λ_I) such that $\sum_{i\in I} \lambda_I(i) = \lambda$. Then for $\lambda \neq \mu$ and $I \in \mathcal{S}_\lambda, J \in \mathcal{S}_\mu$ there are no maps between J and I in $\mathcal{S}_{\Lambda^{pos}}$. So, $\mathcal{S}_{\Lambda^{pos}}$ is the disjoint union of the categories $\mathcal{S}_\mu$ for all $\mu \in \Lambda$. Therefore, a Λ^{pos}-quasi-perverse sheaf is a data of a perverse sheaf B_{X^I} for any $(I, \lambda) \in \mathcal{S}_{\Lambda^{pos}}$ such that λ takes values in the generators of Λ^{pos} together with isomorphisms ν_π for any isomoprhism $\pi : J \widetilde{\to} I$ preserving the functions λ_I, λ_J. To such a collection we associate perverse sheaves $B^\lambda \in \mathrm{Perv}(X^\lambda)$ for all $\lambda \in \Lambda^{pos}_*$ as follows. Take the direct image of B_{X^I} under $sum : X^I \to X^\lambda$ and then take $\mathrm{Aut}(I, \lambda_I)$-invariants. If $B \in \mathrm{Perv}_{\Lambda^{pos}}$ then we will have $sum^!(B^\lambda) \widetilde{\to} B_{X^I}$. This should be proved as in ([1], Lemma in 3.4.6). Thus, $\mathrm{Perv}_{\Lambda^{pos}}$ is the category of collections $(B_{X^\lambda}) \in \prod_{\lambda\in\Lambda^{pos}_*} \mathrm{Perv}(X^\lambda)$ such that each B_{X^λ} has no perverse subsheaves supported on the diagonal divisor.

2.2.5. For $\lambda \in \Lambda$ we have the universal Λ^{pos}-valued relative Cartier divisor D on $X \times X^\lambda$. So, a datum of $F \in \mathcal{Q}\mathrm{FA}(X)_{\mathcal{R}\Lambda^{pos}}$ gives $F^\lambda \in \mathrm{Perv}(X^\lambda)$ corresponding to D. They factorize as follows. Given $(J, \lambda : J \to \Lambda^{pos}_*) \in \mathcal{S}_{\Lambda^{pos}}$ set $\lambda = \sum_{j\in J} \lambda_j$. We have the étale sum map

$$\mathrm{add}_{disj} : (\prod_{j\in J} X^{\lambda_j})_{disj} \to X^\lambda$$

Let D^j be the universal Λ^{pos}-valued divisor on $X \times X^{\lambda_j}$, let $\bar{D}^j$ be its restriction to $(\prod_{j\in J} X^{\lambda_j})_{disj}$. Let D be the universal Λ^{pos}-valued divisor on $X \times X^\lambda$, $\bar{D}$ be its restriction to $(\prod_{j\in J} X^{\lambda_j})_{disj}$. So, $\bar{D} = \sum_{j\in J} \bar{D}_j$. We get from F an isomorphism $F_{\bar{D}} \widetilde{\to} \otimes^!_{j\in J} F_{\bar{D}_j}$. Set for brevity $Y = \prod_{j\in J} X^{\lambda_j}$. Let $p_j : Y_{disj} \to X^{\lambda_j}$ be the j-th projection. We get

$$p_j^! F^{\lambda_j}[\dim X^{\lambda_j}] \widetilde{\to} F_{\bar{D}^j}[\dim Y]$$

let $i : Y \to Y^J$ be the diagonal. We get an isomorphism

$$i^!(\boxtimes_J p_j^! F^{\lambda_j}[\dim X^{\lambda_j}])[-\dim Y] \widetilde{\to} \mathrm{add}^*_{disj} F^\lambda \qquad (21)$$

The left hand side identifies with $\boxtimes_J F^{\lambda_j}$, but the latter identification breaks the symmetry!! For example, if all λ_j are equal then the actions of Aut_J are different. I think we need a twist to get a good factorization algebra as we did in the case $\Lambda^{pos} = \mathbb{Z}_+$.

Define for $\lambda=\sum_{i\in\mathcal{J}} n_i\alpha_i$ a line

$$\eta_\lambda=\underset{i\in\mathcal{J}}{\otimes}\eta_{n_i},$$

where $\eta_{n_i}=(\bar{\mathbb{Q}}_\ell[1])^{\otimes n_i}$. This is maybe not the best way to formulate things, but we get $\eta_\lambda\widetilde{\to}\otimes_{j\in J}\eta_{\lambda_j}$. Now define $B^\lambda\in \mathrm{Perv}(X^\lambda)$ by $B^\lambda\otimes\eta_\lambda=F^\lambda[\dim X^\lambda]$. Then (21) yields a factorization isomorphism

$$\boxtimes_J B^{\lambda_j}\widetilde{\to} B^\lambda$$

on Y_{disj}. So, $\{B_{X^\lambda}\}$ form a Λ^{pos}-quasi-factorization algebra.

A better explanation is to use te definition from my Section 2.2.3. Then for any $I\in\mathcal{S}_{\Lambda^{pos}}$ we have the universal Λ^{pos}-valued relative Cartier divisor D on $X\times X^I$. So, $F\in\mathcal{Q}\mathrm{FA}(X)_{\mathcal{R}\Lambda^{pos}}$ gives a perverse sheaf F_{X^I} (depending on λ_I of course, but this is not reflected in the notation). For $I\in\mathcal{S}_{\Lambda^{pos}}$ define $B_{X^I}\in\mathrm{Perv}(X^I)$ by $B_{X^I}\otimes\eta_I=F_{X^I}[\dim X^I]$. If $\pi: J\to I$ is a map in $\mathcal{S}_{\Lambda^{pos}}$, we have the open immersion $U^{[J/I]}\hookrightarrow X^J$. For each $i\in I$ we equip J_i with the restriction of the map $\lambda_J: J\to\Lambda^{pos}_*$.

Let D^i, $i\in I$ be the universal Λ^{pos}-valued relative Cartier divisor on $X\times X^{J_i}$. Let $\bar{D}^i$ be its restriction to $U^{[J/I]}$. Let D be the universal Λ^{pos}-valued relative Cartier divisor on $X\times X^J$, let $\bar{D}$ be its restriction to $U^{[J/I]}$. Then $\bar{D}=\sum_{i\in I}\bar{D}^i$. So, $F_{\bar{D}}\widetilde{\to}\otimes^!_{i\in I}F_{\bar{D}^i}$ on $U^{[J/I]}$. As above, we get canonically the isomorphism $c_{[J/I]}$ now for our B_{X^J} and $B_{X^{J_i}}$ given by (8). The advantage is that we dot need lines η_λ with this rigorous definition.

2.2.6. Let now $B\in\mathrm{Perv}_{\Lambda^{pos}}$. For $J=*$ and $\lambda: *\to\Lambda^{pos}_*$ given by λ, we get the perverse sheaf $B_\lambda:=B_X\in\mathrm{Perv}(X)$. This way we attach to B a Λ^{pos}_*-graded perverse sheaf $B.:=\oplus_{\lambda\in\Lambda^{pos}}B_\lambda$ on X.

2.2.7. Let $B\in\mathcal{F}\mathrm{A}(X)_{\Lambda^{pos}}$. Consider the Λ^{pos}_*-graded perverse sheaf $B.$ on X attached to B in the previous section. It gets a structure of a Λ^{pos}_*-graded nonunital chiral algebra as follows.

Let $(J,\lambda_J)\in\mathcal{S}_{\Lambda^{pos}}$, set $\lambda=\sum_{j\in J}\lambda_j$. As in my Section 1.3.31, we get a map

$$P^{ch}_J(\mathrm{IC}_X)\to P^{ch}_J(\{B_{\lambda_j}\}, B_\lambda)$$

In other words, for $B.$ we get a map

$$P^{ch}_J(\mathrm{IC}_X)\to P^{ch}_J(\{B.\}, B.),$$

where operations on the right hand side preserve the grading. They are compatible with compositions, so $B.$ is a Lie-algebra in $\mathrm{Perv}(X)^{ch}$. Then $\mathrm{IC}\oplus B.$ is a Λ^{pos}-graded chiral algebra on X. We constructed a functor

$$\mathcal{F}\mathrm{A}(X)_{\Lambda^{pos}}\to\{\Lambda^{pos}\text{-graded chiral algebras } B. \text{ with } B_0=\mathrm{IC}_X\}\tag{22}$$

We can go in the opposite direction as follows. Let $A.$ be a Λ^{pos}-graded chiral algebra with $A_0=\mathrm{IC}_X$. For $(I,\lambda)\in\mathcal{S}_{\Lambda^{pos}}$ consider the Chevalley-Cousin complex $C(A.)_{X^I}$ defined as in my Section 1.3.36, but with the correction that in the product $\boxtimes_I A[1]$ on X^I the factor $A[1]$ is replaced by $A_{\lambda_i}[1]$, and similarly for every $T\in Q(I)$. Then for $(I,\lambda)\in\mathcal{S}_{\Lambda^{pos}_*}$ define the complex B_{X^I} on X^I by

$$B_{X^I}\otimes\eta_I=C(A.)_{X^I}$$

So, B_{X^I} depends on λ, though this is not reflected in the notation. In fact, B_{X^I} is not always placed in perverse degree $- \mid I \mid$, it may have perverse cohomology in several degrees! [1] An analog of ([1], Lemma 3.4.12) does not hold in the graded setting.

As above (as in [1] in fact) one then shows that the collection (B_{X^I}, c, ν) is a DG-object of $\mathcal{F}\mathrm{A}(X)_{\Lambda^{pos}}$, it can be placed in many perverse degrees. The functor (22) does not seem to be an equivalence, but maybe it is an equivalence between the corresponding DG-objects. A version of this claim is found in [5]. For $I = *$ and $\lambda : * \to \Lambda^{pos}$ given by λ we get $B_X = A_\lambda$.

If $(x_i) \in X^I$ is a k-point and $\lambda : I \to \Lambda^{pos}_*$ is a map, assume that $\sum_{i\in I} \lambda_i x_i = \sum_{j\in J} \mu_j y_j$, where y_j are pairwise distinct for $j \in J$. We get a surjection $\pi : I \to J$, hence a function $\mu_j = \sum_{i\in\pi^{-1}(j)} \lambda_i$ on J. Consider the perverse sheaf B_{X^I} on X^I attached to $(I, \lambda) \in \mathcal{S}_{\Lambda^{pos}}$. The !-fibre of $B_{X^I} \otimes \eta_I$ at $(x_i) \in X^I$ is $\underset{j\in J}{\otimes} (i^!_{y_j} A_{\mu_j}[1])$.

For $(I, \lambda) \in \mathcal{S}_{\Lambda^{pos}}$ we get the complex $\mathfrak{B}(\mu)$ on on $X^{\mathfrak{B}(\mu)}$, where $\mathfrak{B}(\mu)$ is the decomposition $\sum_s n_s \mu_s$ of $\mu = \sum_{i\in I} \lambda_i$. Here $\lambda : I \to \Lambda^{pos}$ takes distinct values μ_s with multiplicities n_s. If $f : X^I \to X^{\mathfrak{B}(\mu)}$ is the natural map then take $\mathrm{Aut}(I, \lambda)$-invariants in $f_*(B_{X^I} \otimes \eta_I)$, the result is a complex denoted $B_{X^{\mathfrak{B}(\mu)}}$ on $X^{\mathfrak{B}(\mu)}$.

We should especially apply the above in the following case. Let $\lambda \in \Lambda^{pos}$. Write α_j for the set of generators of Λ^{pos}. Write $\lambda = \sum_{j\in J} n_j \alpha_j$ with $n_j \geq 0$. Then take I a set and $\lambda_I : I \to \Lambda^{pos}_*$ a map taking value α_j precisely n_j times. We get a map $f : X^I \to X^\lambda$, and $X^{\mathfrak{B}(\lambda)} \widetilde{\to} X^\lambda$. For this $(I, \lambda_I) \in \mathcal{S}_{\Lambda^{pos}}$ we get a complex $B_{X^\lambda} := B_{X^{\mathfrak{B}(\lambda)}}$ on X^λ. They form a Λ^{pos}-factorization algebra in the first 'nonrigorous' sense of Section 2.2.

We convent that for $\lambda = 0$ we have $B_{X^\lambda} = \bar{\mathbb{Q}}_\ell$ on $X^\lambda = \mathrm{Spec}\, k$.

Appendix A. Later ideas

A.0.1. Consider any of our 4 sheaf theories given by a functor $(\mathrm{Sch}_{ft})^{op} \to \mathrm{DGCat}_{cont}$, $S \mapsto Shv(S)$. Let $q : \mathcal{X} \to \mathrm{Sch}_{ft}$ be the cartesian fibration attached to this functor. One considers the category $\mathcal{S}$, whose objects are finite non empty sets I, and morphisms are surjections $\pi : J \to I$. We have a functor $\xi : \mathcal{S}^{op} \to \mathrm{Sch}_{ft}$ sending I to X^I, and π to the diagonal embedding $\triangle^{(\pi)}: X^I \to X^J$.

Set $Shv(X^{\mathcal{S}}) = \mathrm{Fun}(\mathcal{S}^{op}, \mathcal{X}) \times_{\mathrm{Fun}(\mathcal{S}^{op}, \mathrm{Sch}_{ft})} \{\xi\}$. So, an object of this category is a lifting $\mathcal{S}^{op} \to \mathcal{X}$ of the above functor $\mathcal{S}^{op} \to \mathrm{Sch}_{ft}$. Informally, such lifting is given for $I \in \mathcal{S}$ by $M_{X^I} \in Shv(X^I)$, and for $\pi : J \to I$ by a map $\triangle^{(\pi)}_! M_{X^I} \to M_{X^J}$ in $Shv(X^J)$. So, $Shv(X^{\mathcal{S}})$ is an analog in the DG-world of the category denoted by $\mathcal{M}(X^{\mathcal{S}})$ in ([1], Section 3.4.10). By ([3], 7.1), $Shv(X^{\mathcal{S}})$ is the oplax limit of the composition $\mathcal{S} \to \mathrm{Sch}^{op}_{ft} \to \mathrm{DGCat}_{cont}$. In other words,

$$Shv(X^{\mathcal{S}}) \widetilde{\to} \lim_{(I'\to I)\in Tw(\mathcal{S})^{op}} \mathrm{Fun}((\mathcal{S}_{/I'})^{op}, Shv(X^I)) \tag{23}$$

Since the projection $\mathrm{DGCat}_{cont} \to 1-\mathrm{Cat}$ preserves limits, we may view the latter limit as one in DGCat_{cont}, thus $Shv(X^{\mathcal{S}}) \in \mathrm{DGCat}_{cont}$ naturally.

[1] An example is given by $A = E[1]$, where E is the Λ^{pos}-graded commutative algebra dual to the cocommutative coalgebra $U(\mathfrak{n})$, here $\mathfrak{n}$ is the unipotent radical of a Borel subalgebra of a reductive Lie algebra.

Let us precise that given a morphism $\beta \to \alpha$ in $\mathrm{Tw}(\mathcal{S})^{op}$ given by a diagram

$$\begin{array}{ccc} I' & \stackrel{\alpha}{\to} & I \\ \downarrow & & \uparrow \\ J' & \stackrel{\beta}{\to} & J \end{array}$$

the transition functor in the above limit is the composition

$$\mathrm{Fun}((\mathcal{S}_{/J'})^{op}, Shv(X^J)) \stackrel{a}{\to} \mathrm{Fun}((\mathcal{S}_{/I'})^{op}, Shv(X^J)) \stackrel{b}{\to} \mathrm{Fun}((\mathcal{S}_{/I'})^{op}, Shv(X^I)), \tag{24}$$

where the first functor is the composition with $(\mathcal{S}_{/I'})^{op} \to (\mathcal{S}_{/J'})^{op}$, and the second one is the composition with $\vartriangle^{(J/I)!}\colon Shv(X^J) \to Shv(X^I)$ for $\vartriangle^{(J/I)}\colon X^I \to X^J$.

Both functors in (24) admit left adjoints. Indeed, a^L is the left Kan extension along $(\mathcal{S}_{/I'})^{op} \to (\mathcal{S}_{/J'})^{op}$, and b^L is the compostion with $\vartriangle_!^{(J/I)}\colon Shv(X^I) \to Shv(X^J)$.

Thus, we may pass to left adjoints in the above diagram $Tw(\mathcal{S})^{op} \to \mathrm{DGCat}_{cont}$, and get a functor $\mathrm{Tw}(\mathcal{S}) \to \mathrm{DGCat}_{cont}$, moreover we get

$$Shv(X^{\mathcal{S}}) \widetilde{\to} \underset{(I' \to I) \in Tw(\mathcal{S})}{\mathrm{colim}} \mathrm{Fun}((\mathcal{S}_{/I'})^{op}, Shv(X^I)), \tag{25}$$

colimit taken in DGCat_{cont}.

For $(I' \to I) \in \mathrm{Tw}(\mathcal{S})$ the projection $Shv(X^{\mathcal{S}}) \to \mathrm{Fun}((\mathcal{S}_{/I'})^{op}, Shv(X^I))$ sends $f : \mathcal{S}^{op} \to \mathcal{X}$ to the following functor. First, ξ gives a map $(\mathcal{S}_{/I'})^{op} \to (\mathrm{Sch}_{ft})_{X^{I'}/}$, so the compostion $(\mathcal{S}_{/I'})^{op} \to \mathcal{S}^{op} \to \mathcal{X}$ extends to a functor $(\mathcal{S}_{/I'})^{op} \to (\mathrm{Sch}_{ft})_{X^{I'}/} \times_{\mathrm{Sch}_{ft}} \mathcal{X}$. The latter is composed with the map $(\mathrm{Sch}_{ft})_{X^{I'}/} \times_{\mathrm{Sch}_{ft}} \mathcal{X} \to Shv(X^{I'})$ given by the !-pullback, and in turn with the !-pullback for $\vartriangle\colon X^I \to X^{I'}$.

A.0.2. We have the embedding $\{*\} \to \mathcal{S}^{op}$ given by the 1-element set, this is an initial object of $\mathcal{S}^{op}$. Restriction to $\{*\}$ yields a functor $Shv(X^{\mathcal{S}}) \to Shv(X)$. This functor preserves limits, and the two categories are presentable, so it admits a left adjoint. This left adjoint sends M to the functor $\mathcal{S}^{op} \to \mathcal{S}$, $I \mapsto \vartriangle_* M$ for $\vartriangle\colon X \to X^I$ (defined naturally on morphisms). Probably, this is a relative left Kan extension in the sense of ([13], Def. 4.3.2.2).

Let $\mathcal{S}_n \subset \mathcal{S}$ be the full subcategory of sets of order $\leq n$. We have the truncated version $Shv(X^{\mathcal{S}_n}) = \mathrm{Fun}(\mathcal{S}_n^{op}, \mathcal{X}) \times_{\mathrm{Fun}(\mathcal{S}_n^{op}, \mathrm{Sch}_{ft})} \{\xi_n\}$, where ξ_n is the composition $\mathcal{S}_n^{op} \to \mathcal{S}^{op} \stackrel{\xi}{\to} \mathrm{Sch}_{ft}$. Let

$$\pi_n : Shv(X^{\mathcal{S}}) \to Shv(X^{\mathcal{S}_n})$$

be the restriction functor. It also admits a left adjoint L^{π_n}, because it preserves limits. This left adjoint should be given by the relative left Kan extension in the sense of ([13], Def. 4.3.2.2). According to ([1], 3.4.10), it should be given by the formula: for $F \in \mathrm{Fun}(\mathcal{S}_n^{op}, \mathcal{X})$ lying in $Shv(X^{\mathcal{S}_n})$, $L^{\pi_n}(F)$ sends $I \in \mathcal{S}$ to the pair $(X^I, M_{X^I} \in Shv(X^I))$, where

$$M_{X^I} = \underset{(I \to J) \in (\mathcal{S}^{op})_{/I} \times_{\mathcal{S}^{op}} \mathcal{S}_n^{op}}{\mathrm{colim}} \vartriangle_*^{(I/J)} F_{X^J},$$

the colimit taken in $Shv(X^I)$. The index category is precisely the one usually used for the left Kan extension along $\mathcal{S}_n^{op} \hookrightarrow \mathcal{S}^{op}$. In fact, $L^{\pi_n}(F)$ should be the q-left Kan extension of $F : \mathcal{S}_n^{op} \to \mathcal{X}$ along $\mathcal{S}_n^{op} \to \mathcal{S}^{op}$.

For this we need to check that for any $I \in \mathcal{S}$, the diagram

$$\begin{array}{ccc} (\mathcal{S}_n^{op})_{/I} & \stackrel{F_I}{\to} & \mathcal{X} \\ \downarrow & \nearrow & \downarrow \\ (\mathcal{S}^{op})_{/I} & \to & \mathrm{Sch}_{ft} \end{array}$$

is a q-colimit of F_I. Here $(\mathcal{S}_n^{op})_{/I} := (\mathcal{S}^{op})_{/I} \times_{\mathcal{S}^{op}} \mathcal{S}_n^{op}$.

As in the previous section, we may rewrite

$$Shv(X^{\mathcal{S}_n}) = \lim_{(I' \to I) \in \mathrm{Tw}(\mathcal{S}_n)^{op}} \mathrm{Fun}(((\mathcal{S}_n)_{/I'})^{op}, Shv(X^I))$$

as a colimit

$$Shv(X^{\mathcal{S}_n}) = \operatorname*{colim}_{(I' \to I) \in \mathrm{Tw}(\mathcal{S}_n)} \mathrm{Fun}(((\mathcal{S}_n)_{/I'})^{op}, Shv(X^I))$$

obtained by passing to left adjoints. By ([10], Section 9.2.40), the functor L^{π_n} comes from a compatible system of functors: for each $(I' \to I) \in \mathrm{Tw}(\mathcal{S}_n)$ the functor

$$ins : \mathrm{Fun}(((\mathcal{S}_n)_{/I'})^{op}, Shv(X^I)) \to Shv(X^{\mathcal{S}}),$$

which is the left adjoint to the projection $ev : Shv(X^{\mathcal{S}}) \to \mathrm{Fun}(((\mathcal{S}_n)_{/I'})^{op}, Shv(X^I))$.

We should have $\mathrm{id} \widetilde{\to} \pi_n L^{\pi_n}$, so that L^{π_n} is fully fauthful. In turn, by ([11], Lemma 1.8.15), $Shv(X^{\mathcal{S}})/Shv(X^{\mathcal{S}_n})$ identifies with the right orthogonal of $Shv(X^{\mathcal{S}_n})$ in $Shv(X^{\mathcal{S}})$. This right orthogonal is the fibre of π_n.

From this we may derive that $Shv(X^{\mathcal{S}_n})/Shv(X^{\mathcal{S}_{n-1}})$ is the category S_n-equivariant sheaves on X^n.

A.0.3. The appearance of $Shv(X^{\mathcal{S}})$ is related to the definition of $Shv(\mathrm{Ran}(X)) = \lim_{I \in \mathcal{S}} Shv(X^I)$. This limit identifies with a full subcategory of $Shv(X^{\mathcal{S}})$ consisting of those functors $\mathcal{S}^{op} \to \mathcal{S}^{op} \times_{Sch_{ft}} \mathcal{X}$ over $\mathcal{S}^{op}$ sending any arrow to a cartesian one, see ([10], 2.2.67).

Recall that the disjoint union of finite sets equips $\mathcal{S}$ with a symmetric monoidal structure, and ξ is symmetric monoidal, where Sch_{ft} is equipped with the cartesian sylmmetric monoidal structure. Now as in ([16], 5.15), one equips $\mathcal{X}$ with a symmetric monoidal structure such that the projection $\mathcal{X} \to \mathrm{Sch}_{ft}$ is symmetric monoidal. Namely, for $(S_1, M_1), (S_2, M_2) \in \mathcal{S}$, where $S_i \in \mathrm{Sch}_{ft}, M_i \in Shv(S_i)$ we have $(S_1, M_1) \otimes (S_2, M_2) = (S_1 \times S_2, M_1 \boxtimes M_2)$.

So, we may also consider the category $\mathrm{Fun}^{\otimes}(\mathcal{S}^{op}, \mathcal{X})$ of symmetric monoidal functors from $\mathcal{S}^{op}$ to $\mathcal{X}$, and similarly

$$\mathrm{Fun}^{\otimes}(\mathcal{S}^{op}, \mathcal{X}) \times_{\mathrm{Fun}^{\otimes}(\mathcal{S}^{op}, \mathrm{Sch}_{ft})} \{\xi\}$$

A.0.4. If $C \in \mathrm{CAlg}(\mathrm{DGCat}_{cont})$, on $\mathrm{Fun}(\mathcal{S}^{op}, C)$ we may define a symmetric monoidal structure similar to the $*$-tensor structure on $\mathcal{M}(X^{\mathcal{S}})$. Namely, for $I \in \mathcal{S}$, $M_i \in \mathrm{Fun}(\mathcal{S}^{op}, C)$ we let $\underset{i \in I}{\otimes} M_i \in \mathrm{Fun}(\mathcal{S}^{op}, C)$ be the functor sending J to

$$\bigoplus_{J \twoheadrightarrow I} (\otimes (M_i)_{J_i})$$

If $J \xrightarrow{\pi} J' \to I$ is a diagram in $\mathcal{S}$, for $i \in I$ we get $\pi_i : J_i \to J'_i$, hence maps $(M_i)_{J'_i} \to (M_i)_{J_i}$. Taking tensor product over I, we get $\otimes_I (M_i)_{J'_i} \to \otimes_I (M_i)_{J_i}$. This gives a natural map $(\underset{I}{\otimes} M_i)_{J'} \to (\underset{I}{\otimes} M_i)_J$. Is it of any use???

A.0.5. The category $Shv(\mathrm{Ran}_X)$ acts on $Shv(X^{\mathcal{S}})$ as in ([1], 3.4.10.6). Namely, we may say for any $(I' \to I) \in \mathrm{Tw}(\mathcal{S})$, $\mathrm{Fun}((\mathcal{S}_{/I'})^{op}, Shv(X^I))$ is a category acted on by $Shv(\mathrm{Ran}_X)$, namely, $M \in Shv(\mathrm{Ran}_X)$ sends $f \in \mathrm{Fun}((\mathcal{S}_{/I'})^{op}, Shv(X^I))$ to the functor $(J \to I') \mapsto f(J) \otimes^! (i^!_{X^I} M)$, here $i_{X^I} : X^I \to \mathrm{Ran}_X$ is the natural map. The transition maps in the limit diagram (23) are maps of $Shv(\mathrm{Ran}_X)$-modules. Since $Shv(\mathrm{Ran}_X) - mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves limits, the limit (23) may be understood in $Shv(\mathrm{Ran}_X) - mod(\mathrm{DGCat}_{cont})$.

In the colimit system (25) the transition functors are also maps of $Shv(\mathrm{Ran}_X)$-modules, so this colimit can also be understood in $Shv(\mathrm{Ran}_X) - mod(\mathrm{DGCat}_{cont})$.

Is $Shv(X^{\mathcal{S}})$ a factorization category over Ran_X?

A.1. Relation to the notions from [14].

A.1.1. A pseudo-tensor category in the sense of [1] is precisely an ∞-operad $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ such that $\mathcal{O}$ is a usual category.

Given pseudo-tensor categories $\mathcal{M}_i$ for $i \in I$, by $\prod_i \mathcal{M}_i$ in ([1], 1.1.6 iv) they mean the following. Let $\mathcal{M}_i^{\otimes} \to \mathrm{Fin}_*$ be ∞-operads. Then $\prod_{i \in I, / \mathrm{Fin}_*} \mathcal{M}_i^{\otimes}$ is an ∞-operad also ([10], 3.0.9). Here the product is taken over Fin_*.

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