COMMENTS TO: D. GAITSGORY, N. ROZENBLYUM [14]

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These are informal comments to [14] and Lurie's books [27, 28].

1. Preliminaries

1.0.1. For Lurie, Gaitsgory, Weil's conjecture for function fields. About Def 2.1.1 and Remark 2.1.2: the infinite complex

$$M = [\ldots \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \ldots]$$

is not K-injective. For example, consider $W' = [\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z}]$ placed in degrees $1, 0$. We have the evident inclusion $W' \subset W$, where $W = [\frac{1}{2}\mathbb{Z} \xrightarrow{1} \frac{1}{2}\mathbb{Z}/2\mathbb{Z}]$ placed in degrees $1, 0$. This is a quasi-isomorphism. Consider the map $W' \to M$ given by the diagram

$$\begin{array}{ccccc} \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} \ldots \\ & \uparrow 1 & & \uparrow 2 & \\ & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z}/2\mathbb{Z} & \to 0 \end{array}$$

This moprhism does not extend to a morphism of complexes $W \to M$.

Date: May 14, 2025.

1.0.2. I've read the note "An inroduction to simplicial sets" found in `preprinty chizhie/infty_categories` on my notebook. This is an MIT open course, it introduces the Kan extension condition for a simplicial set. There is a nice construction of a chain complex associated to a simplicial abelian group. What is the sense of its homology groups? Examples?

The simplicial sets $\Lambda^n_k \subset \triangle^n$ are used by Lurie in 'Higher alegbra'.

1.0.3. The definition of DG-category over a commutative ring k is found in ([28], Def 1.3.1.1), where Lurie uses chain convention (different from cochain conventions used by Keller in [39]). Things related to simplicial sets are discussed in ([27], A.2.7). For chain conventions, the DG-category structure on Chain(Λ), where Λ is a commutative ring, is as follows. For $X, Y \in$ Chain(Λ), so $X = [\ldots \to X_1 \to X_0 \to \ldots]$ the component $\mathrm{Map}(X,Y)_m$ is the Λ-module of a degree m graded morphism $f : X \to Y$. That is, a datum of a Λ-linear map $f : X_i \to Y_{i+m}$ for all i. The differential of f is given by

$$f \mapsto d_Y \circ f - (-1)^m f \circ d_X$$

1.0.4. Lurie uses the notation $[n] = \{0, \ldots, n\}$. Recall that $\triangle^n = \mathrm{Hom}(\cdot, \{0, \ldots, n\})$ is the simplicial set represented by $[n]$. Then $(\triangle^n)_n$ contains the identity $\mathrm{id} : [n] \to [n]$ denoted E_n, and $d_i E_n \in (\triangle^n)_{n-1}$ are the standard faces of $\triangle^n$ for $i = 0, \ldots n$. Then $\Lambda^n_k \subset \triangle^n$ is the smallest subsimplicial set that contains all the faces $d_i E_n$ for $i \neq k$. Lurie gives a different description in ([22], 2.1.7), where for a simplicial set X he defines $\Lambda^n_i(X)$ as a morphism of simplicial sets $\Lambda^n_i \to X$. It is given by a finite collection of data: something given for all the subsets $J \subset [n]$ such that $\{0, \ldots, i-1, i+1, \ldots, n\}$ is not contained in J.

Recall that for a simplicial set X one has $X_n = \mathrm{Hom}(\triangle^n, X)$, where Hom is calculated in the category of simplicial sets. So, the inclusion $\Lambda^n_i \hookrightarrow \triangle^n$ yields the restriction map $X_n \to \Lambda^n_i(X)$.

1.0.5. Write $\mathcal{S}\mathrm{et}_\triangle$ for the category of simplicial sets as in [28]. Given $X, Y \in \mathcal{S}\mathrm{et}_\triangle$, the simplicial set $\mathrm{Fun}(X,Y)$ from Notation 2.1.20 exists for the following reason. Recall that $\triangle^n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\triangle}(\cdot, [n])$. The functor $\mathcal{S}\mathrm{et}_\triangle \to$ Sets sending Z to $\mathrm{Hom}(Z \times X, Y)$ is representable by $\mathrm{Fun}(X,Y)$. Namely, $\mathrm{Fun}(X,Y)_n$ is defined as $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\triangle}(\triangle^n \times X, Y)$. For a non-decreasing map $f : [m] \to [n]$ we get $f_* : \triangle^m \to \triangle^n$ in $\mathcal{S}\mathrm{et}_\triangle$, hence a morphism

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\triangle}(\triangle^n \times X, Y) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\triangle}(\triangle^m \times X, Y)$$

which is the restriction under $f_* \times \mathrm{id} : \triangle^m \times X \to \triangle^n \times X$.

Recall that $X \times Y$ here is the categorical product in $\mathcal{S}\mathrm{et}_\triangle$. So, $(X \times Y)_n = X_n \times Y_n$, and for $f : [m] \to [n]$ the map $f^* : (X \times Y)_n \to (X \times Y)_m$ is the map

$$f^* \times f^* : X_n \times Y_n \to X_m \times Y_m$$

1.0.6. A simplicial category is a category enriched over the category $\mathcal{S}\mathrm{et}_\triangle$ of simplicial sets. Write $\mathcal{C}\mathrm{at}$ for the category of (small) categories. Then a simplicial category $\mathcal{C}$ gives a simplicial object $X_\bullet$ in $\mathcal{C}\mathrm{at}$. For $x, y \in \mathcal{C}$ we have the simplicial set $\mathcal{C}(x,y)$. Here X_n is the category, whose objects are $ob(\mathcal{C})$, and for $x, y \in X_n$ we let $X_n(x,y) = \mathcal{C}(x,y)_n$.

Given $x, y, z \in ob(\mathcal{C})$, the composition $X_n(y,z) \times X_n(x,y) \to X_n(x,z)$ comes from the morphism of simplicial sets

$$\mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(x,z)$$

by passing to n-simplexes:

$$\mathcal{C}(y,z)_n \times \mathcal{C}(x,y)_n \to \mathcal{C}(x,z)_n$$

Write $*$ for the constant simplicial set. Then for $x \in cC$ we have the identity map $\mathrm{id} : * \to \mathcal{C}(x,x)$, it gives for each $n \geq 0$ the element $\mathrm{id}(*)_n \in \mathcal{C}(x,x)$. The identity in $X_n(x,y)$ is $\mathrm{id}(*)_n$. For a non-decreasing map $f : [n] \to [m]$ the restriction functor $X_m \to X_n$ is the identity on objects $\mathrm{id} : ob(X_m) \to ob(X_n)$, and for $x, y \in X_m$ the corresponding map $X_m(x,y) \to X_n(x,y)$ is the strucure map

$$f^* : \mathcal{C}(x,y)_m \to \mathcal{C}(x,y)_n$$

of the simplicial set $\mathcal{C}(x,y)$ (cf. [27], Def 1.1.4.1).

Conversely, given a simplicial object $X_\bullet$ in $\mathcal{C}at$, assume that for each non-decreasing map $f : [n] \to [m]$ the corresponding map $f^* : ob(X_m) \to ob(X_n)$ is a bijection. Then we may identify all the sets $ob(X_n)$ with $ob(X_0)$ via the unique map $[n] \to [0]$. Then we get a simplicial category.

A functor $F : \mathcal{C} \to \mathcal{A}$ from a simplicial category $\mathcal{C}$ to a usual category $\mathcal{A}$ is a map $F : ob(\mathcal{C}) \to ob(\mathcal{A})$ and a map $F : \mathcal{C}(a_1, a_2)_0 \to \mathcal{A}(Fa_1, Fa_2)$ compatible with compositions and sending the identity $* \to \mathcal{C}(a,a)$ (here $*$ is the constant symplicial set $\mathrm{Hom}(\cdot, \{0\})$) to the identity via $\mathcal{C}(a,a)_0 \to \mathcal{A}(Fa, Fa)$.

1.0.7. The homotopy category of spaces $\mathcal{H}$ can be defined as formally inverting all the weak homotopy equivalences in $\mathcal{S}et_\Delta$. A map $f : S \to T$ of simplicial sets is a *weak homotopy equivalence* iff the induced map between the geometric realizations $\mid S \mid \to \mid T \mid$ is a weak homotopy equivalence ([27], 1.1.4.3).

1.0.8. In ([27], Remark 1.1.5.2) Lurie uses a functor $Sets \to \mathcal{S}et_\Delta$, which one? I think the only reasonable way is to say that a set A can be viewed as a category with the only morphisms the identity morphisms. Then we may take its nerve (which means disjoint union of constant simplicial sets corresponding to elements of A).

The notation $\mathcal{C}at_\Delta$ for the category of simplicial categories means the usual category $\mathcal{C}at_\Delta$, not a 2-category. Especially in ([27], Def 1.1.5.5), where one defines the simplicial nerve of $\mathcal{C} \in \mathcal{C}at_\Delta$.

1.0.9. Let A be a partially ordered set, X be a simlicial set. What does it mean to give a map of simplicial sets $f : N(A) \to X$? This may be described on "nondegenerate" simplxes of $N(A)$, I think. Namely, we are given a map $f : A \to X_0$, and for each collection of strictly increasing elements $a_\bullet = (a_0 < \ldots < a_n)$ in A an element $f(a_\bullet) \in X_n$ in a compatible way. This uniquely extends to f as above.

For example, if J is a linearly ordered set $[n] = \{0, \ldots, n\}$ and $A = \{I \subset J \mid 0, n \in I\}$ (so, A identifies with all the subsets in $\{1, n-1\}$) then a map $f : N(A) \to X$ means a collection of objects $x_0, \ldots, x_n \in X_0$, and for each strictly increasing collection of subsets $I_\bullet = (I_0 \subset \ldots \subset I_r)$ in A an element $f(I_\bullet) \in X_r$ in a compatible way. This is a finite datum.

Let $\boldsymbol{\Delta}$ as in [27] denote the category whose objects are $[n]$ for $n = 0, 1, \ldots$, and maps from $[n]$ to $[m]$ are non-decreasing maps $f : [n] \to [m]$. Recall that $[n]$ denotes the linearly ordered set $\{0, \ldots, n\}$. Then $\mathcal{S}\mathrm{et}_\triangle$ is the category of functors $\boldsymbol{\Delta}^{op} \to \mathrm{Sets}$.

For a simplicial set X write $\triangle{\downarrow}X$ for the category of elements of the presheaf $X : \boldsymbol{\Delta}^{op} \to \mathrm{Sets}$. Its objects are pairs (n, x), where $n \geq 0$ and an element $x \in X_n$. A map $(n, x) \to (m, y)$ is a morphism $f : [n] \to [m]$ in $\boldsymbol{\Delta}$ such that $f^*(y) = x$. Here $f^* : X_m \to X_n$ is the structure map of the simplicial set X.

We have the functor $\triangle{\downarrow}X \to \mathcal{S}\mathrm{et}_\triangle$ sending (n, x) to $\triangle^n$. The map $f : (n, x) \to (m, y)$ then goes to $f : \triangle^n \to \triangle^m$. By Proposition 3 here

`https://ncatlab.org/nlab/show/category+of+simplices`

we know that the colimit of this functor is X itself.

On ([27], p. 23, before 1.1.5.9) the functor $\mathfrak{C} : \boldsymbol{\Delta} \to \mathcal{C}\mathrm{at}_\triangle$ extends to a colimit-preserving functor $\mathcal{S}\mathrm{et}_\triangle \to \mathcal{C}\mathrm{at}_\triangle$ using the above colimit construction, I think. Namely, $X \in \mathcal{S}\mathrm{et}_\triangle$ gives rise via the composition to a functor $\triangle{\downarrow}X \to \mathcal{C}\mathrm{at}_\triangle$, and we take the colimit of the latter.

1.0.10. Recall that the homotopy category of a simplicial category $\mathcal{C}$ is the $\mathcal{H}$-enriched category $h\mathcal{C}$ with $ob(h\mathcal{C}) = ob(\mathcal{C})$, and maps from x to y in $h\mathcal{C}$ is the image of $\mathrm{Map}_{\mathcal{C}}(x, y)$ under $\mathcal{S}\mathrm{et}_\triangle \to \mathcal{H}$ ([27], 1.1.3).

Now for $X \in \mathcal{S}\mathrm{et}_\triangle$ the homotopy category hX is defined as the homotopy category $h\mathfrak{C}[X]$ of the simplical category $\mathfrak{C}[X]$ ([27], 1.1.5.14). Here $\mathfrak{C} : \mathcal{S}\mathrm{et}_\triangle \to \mathcal{C}\mathrm{at}_\triangle$ is the functor left adjoint to the simplicial nerve N.

1.0.11. Note that Λ^n_i is not an ∞-category for $0 < i < n$, because the identity: $\Lambda^n_i \to \Lambda^n_i$ does not lift to a map $\triangle^n \to \Lambda^n_i$.

Note that $\triangle^1$ is not a Kan complex. Since $\triangle^n$ is a nerve of the category $[n]$, $\triangle^n$ is a category (in particular, an ∞-category). According to the conventions of [27], we identify a category with its nerve.

1.0.12. Let S be an ∞-category. For ([27], 1.2.2.2) we have to explain the definition of the simplicial set $\mathrm{Hom}^R_S(x, y)$. Here for a finite linearly ordered set J one uses the notation $\triangle^J$ from ([27]), this is the simplicial set given by the functor $\boldsymbol{\Delta} \to \mathrm{Sets}$ represented by J. That is, $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\triangle}(\triangle^n, \triangle^J) = \mathrm{Hom}_{\mathcal{C}\mathrm{at}}([n], J)$ with evident structure maps.

If $0 \leq i \leq n$ we have to define the maps $d_i : \mathrm{Hom}^R_S(x, y)_n \to \mathrm{Hom}^R_S(x, y)_{n-1}$ and $s_i : \mathrm{Hom}^R_S(x, y)_n \to \mathrm{Hom}^R_S(x, y)_{n+1}$. In fact, for these i the diagrams commutes

$$\begin{array}{ccc} S_{n+1} & \stackrel{d_i}{\to} & S_n \\ \cup & & \cup \\ \mathrm{Hom}^R_S(x,y)_n & \to & \mathrm{Hom}^R_S(x,y)_{n-1} \end{array} \qquad \begin{array}{ccc} S_{n+1} & \stackrel{s_i}{\to} & S_{n+2} \\ \cup & & \cup \\ \mathrm{Hom}^R_S(x,y)_n & \to & \mathrm{Hom}^R_S(x,y)_{n+1} \end{array}$$

and define the corresponding maps. He used the inclusion $[n] \subset [n+1]$ giving $z \mid_{\triangle^n}$ for $z : \triangle^{n+1} \to S$.

About remark 1.2.2.5. I think for two partially ordered sets A, B the disjoint union $A \times B$ should be considered as partially ordered, namely $(a_1, b_1) \leq (a_2, b_2)$ iff $a_1 \leq a_2$

and $b_1 \leq b_2$. It is not clear where the inclusions of the last displayed formula before ([27], 1.2.3) come from. I think there are suitable morphisms $\vartriangle^n \times \vartriangle^1 \to \vartriangle^{n+1}$ that induce these inclusions. How to define them?

I suggest the map of sets $[n] \times [1] \to [n+1]$ sending (k, a) to

$$\begin{cases} k, & \text{for } k < n \\ n, & \text{for } k = n \text{ and } a = 0 \\ n+1, & \text{for } k = n \text{ and } a = 1 \end{cases}$$

This is a map of partially ordered sets. So, it induces by functoriality a morphism $\alpha : \vartriangle^n \times \vartriangle^1 \to \vartriangle^{n+1}$. Moreover, it induces the usual inclusion $\vartriangle^n \widetilde{\to} \vartriangle^n \times \vartriangle^{\{0\}} \to \vartriangle^{n+1}$.

1.0.13. The boundary $\partial \vartriangle^n \subset \vartriangle^n$ is the simplicial subset generated by all faces $d_i E_n \in (\vartriangle^n)_{n-1}$, here $E_n \in (\vartriangle^n)_n$ is the identity map $[n] \to [n]$. The simplicial set $\partial \vartriangle^n$ can also be described as the image of the map $\underset{0 \leq i \leq n}{\sqcup} \vartriangle^{n-1} \to \vartriangle^n$ given by all faces.

Recall that for a simplicial subset $Y \subset X$ the quotient $X/Y \in \mathcal{S}\mathrm{et}_\vartriangle$ is defined so that $(X/Y)_n$ is obtained from X_n by identifying the elements of Y_n to each other.

1.0.14. Let $f : X \to Y$ be a morphism in $\mathcal{S}\mathrm{et}_\vartriangle$. In ([27], 1.1.4.3) it is called a weak homotopy equivalence if the induced map of geometric realizations $\mid X \mid \to \mid Y \mid$ is a weak homotopy equivalence.

In `https://ncatlab.org/nlab/show/simplicial+homotopy+group` another definition is given. Namely, the above f is called a *weak homotopy equivalence* if it induces an isomorphism on all the symplicial homotopy groups. These homotopy groups are defined on the same web page, but usually these homotopy groups of $X \in \mathcal{S}\mathrm{et}_\vartriangle$ are only defined for X a Kan complex. To define homotopy groups of any symplectic set X, pick a weak homotopy equivalence $X \to X'$, where X' is a Kan complex. Then it induces an isomorphism of all the homotopy groups.

I hope the two definitions are equivalent.

Recall that for any $S \in \mathcal{S}\mathrm{et}_\vartriangle$ the canonical morphism $S \to Sing \mid S \mid$ is a weak homotopy equivalence ([27], 1.1.4.3). Here $Sing$ denotes the singular complex, which is a functor $\mathcal{CG} \to \mathcal{S}\mathrm{et}_\vartriangle$, and $Sing(Z)$ is a Kan complex for any $Z \in \mathcal{CG}$. Here $\mathcal{CG}$ is the category of compactly generated weakly Housedorf topological spaces. We see that any $S \in \mathcal{S}\mathrm{et}_\vartriangle$ admits a weak homotopy equivalence with a Kan complex.

Let J be a finite partially ordered set, $i \leq j \in J$. For the contant simplicial set $X = *$, the unique map $N(P_{ij}) \to X$ is a weak homotopy equivalence.

1.0.15. (Proposition 1.2.3.1, [27]) uses the functor $\mathcal{H} \to$ Sets associating to a homotopy type of a simplicial set some set. What is it? I think it sends X to $\pi_0(X)$. More precisely, we first need to replace X by a weakly homotopically equivalent Kan complex X'. Then $\pi_0(X)$ is the quotient of X'_0 by the equivalence relation: $x_0 \sim x_1$ iff there is $z \in X'_1$ with $d_1(z) = x_0$ and $d_0(z) = x_1$. Here $d_1 : [0] \to [1]$ has image $\{0\}$, and $d_0 : [0] \to [1]$ has image $\{1\}$.

Can one define π_0 of any simplicial set directly?

A good definition is given by Lurie: the functor $\pi_0 : \mathcal{S}\mathrm{et}_\vartriangle \to$ Sets is the left adjoint to the inclusion functor Sets $\to \mathcal{S}\mathrm{et}_\vartriangle$.

If $\mathcal{C}$ is an ∞-category then for $x, y \in \mathcal{C}_0$ the set $\mathrm{Map}_{\mathcal{C}}(x, y)$ is the set of equivalence classes of morphisms from x to y. A morphism z from x to y here is a map $z \in \mathcal{C}_1$ with $d_1(z) = x$ and $d_0(z) = y$. Two morphisms are equivalent if they are homotopic in the sense of the definition ([22], p. 31 after example 2.1.16). So, the set $\mathrm{Map}_{\mathcal{C}}(x, y)$ identifies with $\pi_0 \mathrm{Hom}^R_{\mathcal{C}}(x, y)$, the notation of ([27], Prop. 1.2.2.3).

The inclusion functor $i : \mathcal{C}at \to \mathcal{C}at_\Delta$ is right adjoint to the functor $\mathcal{C}at_\Delta \to \mathcal{C}at$, $\mathcal{C} \mapsto h\mathcal{C}$. Here $h\mathcal{C}$ is viewed as an ordinary category, that is, for $x, y \in \mathcal{C}$ we have $\mathrm{Hom}_{h\mathcal{C}}(x, y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(x, y)$. (see [27], the proof of 1.2.3.1).

For $n \geq 0$ and $X \in \mathrm{Sets}$ we get $\mathrm{Hom}_{\mathrm{Sets}}(\pi_0(\Delta^n), X) \widetilde{\to} \mathrm{Hom}_{\mathcal{S}et_\Delta}(\Delta^n, X_\bullet) \widetilde{\to} X$. Here $X_\bullet$ is the image of X under the inclusion functor $\mathrm{Sets} \to \mathcal{S}et_\Delta$. We conclude that $\pi_0(\Delta^n)$ is the point set.

1.0.16. Starting from Proposition 1.2.3.5 in [27] Lurie uses the following convention to denote a map $\Lambda^3_k \to \mathcal{C}$ of simplicial sets. The faces of Δ^3 are $(123), (023), (013), (012)$. They are ordered by the rule that the vertex $0, 1, 2, 3$ is missing in the corresponding face.

Here for example by (023) we mean a non-decreasing map $f : [2] \to [3]$ with image $\{0, 2, 3\}$. When he considers a map $\Lambda^3_1 \to \mathcal{C}$ it is denoted by a collection $(a, \bullet, b, c)$, which means the following. First, the face (023) is missing in Λ^3_1, which is denoted by the bullet $\bullet$ on the second place. The above notation means that $(123) \mapsto a, (013) \mapsto b, (012) \mapsto c$.

1.0.17. Last point of the proof of ([27], Prop. 1.2.3.9) not completely clear for me, where the equality $\mathrm{id}_y \circ \bar{\phi} = \bar{\phi}'$ comes from? I think a way to explain this would be as follows. $\sigma : \Delta^2 \to \mathcal{C}$ gives a functor $\mathfrak{C}[\Delta^2] \to \mathfrak{C}[\mathcal{C}]$, hence also a functor $h\,\Delta^2 \to h\mathcal{C}$. The desired equality holds already in $h\,\Delta^2$.

To be precise, for the definition of $\pi(\mathcal{C})$ for a ∞-category $\mathcal{C}$ in ([27], 1.2.3.4) we have the following. Given $f, g \in \mathcal{C}_1$ a homotopy from f to g is a datum of $\sigma \in \mathcal{C}_2$ whose border is given by the diagram

$$\begin{array}{ccc} & & 1 \\ & \nearrow f & \downarrow \mathrm{id} \\ 0 & \stackrel{g}{\to} & 2 \end{array}$$

1.0.18. We have the functor $\boldsymbol{\Delta} \to \mathcal{S}et_\Delta$ sending $[n]$ to the corresponding representable functor Δ^n, and a nondecreasing map $f : [m] \to [n]$ to the induced map $\Delta^m \to \Delta^n$. This is nothing but the Yoneda embedding. It induces a functor $\boldsymbol{\Delta}^{op} \to \mathcal{S}et_\Delta^{op}$. So, given any functor $F : \mathcal{S}et_\Delta^{op} \to \mathrm{Sets}$ we get a simplicial set X associated to F as the composition

$$\boldsymbol{\Delta}^{op} \to \mathcal{S}et_\Delta^{op} \xrightarrow{F} \mathrm{Sets}$$

Conversely, I think we can recover F out of X assuming that F commutes with small colimits. See 1.0.9 (and [23], Digression 1.8).

For example, for $X, Y \in \mathcal{S}et_\Delta$ the simplicial set $\mathrm{Fun}(X, Y)$ is obtained this way. Namely, we have the functor $F : \mathcal{S}et_\Delta^{op} \to \mathrm{Sets}$, $Z \mapsto \mathrm{Hom}_{\mathcal{S}et_\Delta}(Z \times X, Y)$. It is representable by $\mathrm{Fun}(X, Y)$.

1.0.19. Let X, Y be simplicial sets, $f : X \to Y$ be a morphism in $\mathcal{S}\mathrm{et}_\Delta$. According to Def. 1.2.10.1, we should say that f is an equivalence if f induces an equivalence of $\mathcal{H}$-enriched categories $hX \to hY$. This is the same that the notion of *categorical equivalence* introduced in Def 1.1.5.14. This corresponds to a map $f : X \to Y$ in $1-\mathcal{C}\mathrm{at}$ to be an equivalence.

1.0.20. I think a contractible Kan complex is the same as a Kan complex $S \in \mathcal{S}\mathrm{et}_\Delta$ such that all its homotopy groups are trivial.

1.0.21. Let $\mathcal{C}$ be an ∞-category, $p : K \to \mathcal{C}$ a morphism in $\mathcal{S}\mathrm{et}_\Delta$. Let f be an object of $\mathcal{C}/p$, that is, a diagram in $\mathcal{S}\mathrm{et}_\Delta$

$$\begin{array}{ccc} \Delta^0 \star K & \xrightarrow{f} & \mathcal{C} \\ \uparrow & \nearrow p & \\ K & & \end{array}$$

Here $\star$ is the join operation ([27], 1.2.8). Then f is final in $\mathcal{C}/p$ iff for any $n \geq 1$ and any diagram the dotted arrow h can be filled.

$$\begin{array}{ccc} \partial \Delta^n \star K & \hookrightarrow & \Delta^n \star K \\ \uparrow n & \searrow h_0 & \downarrow \\ \Delta^0 \star K & \xrightarrow{f} & \mathcal{C} \end{array}$$

Here n is the unique map sending the vertex 0 of Δ^0 to the vertex n of $\partial \Delta^n$.

A related description. Assume given another object g of $\mathcal{C}/p$ given by $g : \Delta^0 \star K \to \mathcal{C}$ extending p. What does it mean that $\mathrm{Hom}^R_{\mathcal{C}/p}(g, f)$ is a contractible Kan complex?

An n-simplex of $\mathrm{Hom}^R_{\mathcal{C}/p}(g, f)$ is a diagram

$$\begin{array}{ccc} \Delta^{\{n+1\}} \star K & & \\ \downarrow & \searrow f & \\ \Delta^{n+1} \star K & \xrightarrow{z} & \mathcal{C} \\ \uparrow & & \uparrow g \\ \Delta^n \star K & \xrightarrow{\epsilon \star \mathrm{id}} & \Delta^0 \star K, \end{array}$$

where $\epsilon : \Delta^n \to \Delta^0$ is the unique map is $\mathcal{S}\mathrm{et}_\Delta$. Here the top vertical arrow is obtained from $\Delta^{\{n+1\}} \hookrightarrow \Delta^{n+1}$, and $\Delta^n \to \Delta^{n+1}$ comes from the non-decreasing inclusion $[n] \to [n+1]$ whose image is $\{0, \ldots, n\}$.

1.0.22. If K is an ∞-category then $K^{\triangleleft} = \Delta^0 \star K$ is an ∞-category, so gives rise to the usual category $\pi(K^{\triangleleft})$. It has as objects the elements of K_0 and the distinguished object $\bullet \in K^{\triangleleft}$ belonging to Δ^0. For each $x \in K_0$ we have an arrow $f_x : \bullet \to x$ in $\pi(K^{\triangleleft})$, and for each $h \in K_1$ giving rise to a morphism $\bar{h} : x \to y$ in $\pi(K)$, we also get the morphism $\bar{h} \in \pi(K^{\triangleleft})$ and the diagram commutes in $\pi(K^{\triangleleft})$

$$\begin{array}{ccc} \bullet & \xrightarrow{f_x} & x \\ & \searrow f_y & \downarrow \bar{h} \\ & & y \end{array}$$

Therefore, given a map $p : K \to \mathcal{C}$ in $\mathcal{S}et_\triangle$, where $\mathcal{C}$ is an ∞-category, a datum of an object $z \in \mathcal{C}/p$ can be seen as a candidate for a limit of the functor p.

1.0.23. Let K be a Kan complex. The property of K to be contractible (that is, all the hotomopy groups are trivial), is not very visible for me. Formally, it means that K is connected and for $n > 0$ the group $\pi_n(K, x)$ vanishes (for $x \in K_0$). That is, for any diagram

$$\begin{array}{ccc} \partial \triangle^n & \to & \triangle^0 \\ \downarrow & & \downarrow x \\ \triangle^n & \stackrel{\alpha}{\to} & K \end{array}$$

the simplex α is equivalent to the constant simplex $\triangle^n \to \triangle^0 \stackrel{x}{\to} K$.

1.0.24. What is the limit of a functor $x : \triangle^0 \to \mathcal{C}$ for a given ∞-category $\mathcal{C}$? Note that for $n > 0$ we have $(\partial \triangle^n) \star \triangle^0 \widetilde{\to} \Lambda^{n+1}_{n+1}$. Therefore, the problem is to undestand the lifting for the diagram

$$\begin{array}{ccc} \Lambda^{n+1}_{n+1} & \hookrightarrow & \triangle^{n+1} \\ \uparrow & \searrow & \vdots \\ \triangle^1 & \to & \mathcal{C} \end{array}$$

Here the left vertical map sends the nondegenerate simplex of $\triangle^1$ to $\{n, n+1\}$. So, by Section 1.0.29, id : $x \to x$ veiwed as a map $\triangle^0 \star \triangle^0 \to \mathcal{C}$ is the limit of the functor $x : \triangle^0 \to \mathcal{C}$. Indeed, id is an equivalence in $\mathcal{C}$.

1.0.25. We have an equivalence $\boldsymbol{\Delta} \widetilde{\to} \boldsymbol{\Delta}$ sending a finite linearly ordered set J to the same set J with the opposite order. It is important for obtaining results about S^{op} for $S \in \mathcal{S}et_\triangle$.

We have the evident notion of an opposite to a simplicial category. Then I think for $S \in \mathcal{S}et_\triangle$ one has $\mathfrak{C}[S]^{op} \widetilde{\to} \mathfrak{C}[S^{op}]$.

Let us consider first a finite linearly ordered set J. For $i \le j$ in J we have the partially ordered set $P_{ij}(J)$ from ([27], 1.1.5.1). Now let $\bar{J}$ be the set J with the opposite order. Then $j \le_{\bar{J}} i$ in $\bar{J}$, and $P_{ji}(\bar{J}) = P_{ij}(J)$ as partially ordered sets. So, for $i, j \in J$ the set $\mathrm{Map}_{\mathfrak{C}[\triangle^{\bar{J}}]}(i, j)$ is nonempty for $i \le_{\bar{J}} j$, that is, $j \le i$, and in this case it equals $N(P_{ji}(J))$. So, $\mathfrak{C}[\triangle^{\bar{J}}] \widetilde{\to} \mathfrak{C}[\triangle^J]^{op}$ canonically.

Remark 1.0.26. *If $\mathcal{I}$ is a usual category and $F : \mathcal{I} \to \mathcal{C}at_\triangle$ is a functor then it induces a functor $\bar{F} : \mathcal{I} \to \mathcal{C}at_\triangle$ sending $i \in \mathcal{I}$ to $F(i)^{op}$ and $\tau : i \to j$ to the induced functor $F(i)^{op} \to F(j)^{op}$. Then $\mathcal{X} \in \mathcal{C}at_\triangle$ is the colimit of F iff $\mathcal{X}^{op} \in \mathcal{C}at_\triangle$ is the colimit of $\bar{F}$.*

Recall that each $S \in \mathcal{S}et_\triangle$ is written as the colimit of the functor $\triangle{\downarrow}S \to \mathcal{S}et_\triangle$. So, S^{op} is the colimit of the corresponding functor $\triangle{\downarrow}(S^{op}) \to \mathcal{S}et_\triangle$. An object of $\triangle{\downarrow}(S^{op})$ is a pair: a finite linearly ordered set I and a map $x : \bar{I} \to S$. So, $\mathfrak{C}[S^{op}]$ is the colimit of the functor $\triangle{\downarrow}(S^{op}) \to \mathcal{C}at_\triangle$ sending (I, x) to $\mathfrak{C}[\triangle^J]^{op}$. Its colimit is $\mathfrak{C}[S]^{op} \widetilde{\to} \mathfrak{C}[S^{op}]$.

We conclude that the homotopy category of $S \in \mathcal{S}et_\triangle$, as a $\mathcal{H}$-enriched category, is the opposite to $h(S^{op})$.

1.0.27. It is easy to see that for $A, B \in \mathcal{S}et_\Delta$ one has $\mathrm{Fun}(A,B)^{op} \widetilde{\to} \mathrm{Fun}(A^{op}, B^{op})$ in $\mathcal{S}et_\Delta$ canonically.

For $A, B \in \mathcal{S}et_\Delta$ one has $(A \star B)^{op} \widetilde{\to} B^{op} \star A^{op}$ naturally.

If $p : K \to S$ is a morphism in $\mathcal{S}et_\Delta$ one has the map $p^{op} : K^{op} \to S^{op}$ and we have canonically

$$(S_{/p})^{op} \widetilde{\to} S_{p^{op}/}$$

Let now S be an ∞-category, and $p : K \to S$ is a morphism in $\mathcal{S}et_\Delta$. We see that an intial object of $S_{p^{op}/}$ is the same as a final object of $S_{/p}$. That is, a colimit of p^{op} is the same as a limit of p.

1.0.28. A vertex $x \in \mathcal{C}$ for $\mathcal{C} \in \mathcal{S}et_\Delta$ is strongly initial if any map $f_0 : \partial \Delta^n \to \mathcal{C}$ with $f_0(0) = X$ can be extended to a map $f : \Delta^n \to \mathcal{C}$.

1.0.29. Let $\phi : x \to y$ be a morphism in a ∞-category $\mathcal{C}$. Then ϕ is an equivalence iff $\phi : y \to x$ is an equivalence in $\mathcal{C}^{op}$. Therefore, ([27], 1.2.4.3) rewrites as follows: let $\mathcal{C}$ be an ∞-category, $\phi : \Delta^1 \to \mathcal{C}$ a morphism in $\mathcal{C}$. Then f is an equivalence iff for any $n \geq 2$ and any $f_0 : \Lambda^n_n \to \mathcal{C}$ with $f_0 \mid_{\Delta^{\{n-1,n\}}} = \phi$ there is an extension of f_0 to Δ^n.

1.0.30. Let $X, Y, Z \in \mathcal{S}et_\Delta$. The map $\mathrm{Fun}(Y,Z) \times \mathrm{Fun}(X,Y) \to \mathrm{Fun}(X,Z)$ is as follows. Given $A \in \mathcal{S}et_\Delta$, a A-point of $\mathrm{Fun}(Y,Z)$ is a map $A \times Y \to A \times Z$, whose first component is the projection $A \times Y \to A$. Given A-points $f : A \times Y \to A \times Z$ and $g : A \times X \to A \times Y$, the composition fg is the corresponding A-point of $\mathrm{Fun}(X,Z)$.

According to ([27], 1.2.16.2), if $A, B \in \mathcal{S}et_\Delta$ are Kan complexes then $\mathrm{Fun}(A,B)$ is a Kan complex (no proof was given).

1.0.31. The product of two categories $\mathcal{A}, \mathcal{B} \in \mathcal{C}at$ in $\mathcal{C}at$ is the category whose objects are pairs (a,b), $a \in ob(\mathcal{A}), b \in ob(\mathcal{B})$. A morphism from (a_1,b_1) to (a_2,b_2) is a pair (f,g), where $f \in \mathrm{Hom}_{\mathcal{A}}(a_1,a_2)$, $g \in \mathrm{Hom}_{\mathcal{B}}(b_1,b_2)$. Note that $N(\mathcal{A} \times \mathcal{B}) \widetilde{\to} N(\mathcal{A}) \times N(\mathcal{B})$ canonically in $\mathcal{S}et_\Delta$. The following is proved in ([23], Lemma 2.2). Write $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ for the category of functors from $\mathcal{A}$ to $\mathcal{B}$. Then

$$N(\mathrm{Fun}(\mathcal{A},\mathcal{B})) \widetilde{\to} \mathrm{Fun}(N\mathcal{A}, N\mathcal{B})$$

naturally.

1.0.32. Given an ∞-category $\mathcal{C}$, for $x, y \in \mathcal{C}$ the role of n-morphisms from x to y is played by $\mathrm{Hom}^R_{\mathcal{C}}(x,y)_n$, see ([23], Remark 1.16).

1.0.33. Fact: a small cocomplete category is a partially ordered set (cf. MacLane...)

Let $F : C \times D \to$ Sets be a functor, where C, D are usual categories. Then there is a canonical map in Sets

$$\lambda : \varinjlim_C \varprojlim_D F \to \varprojlim_D \varinjlim_C F$$

Definition. One says that $\varprojlim_D F$ commutes with $\varinjlim_C F$ iff the latter map is an isomorphism.

Fact: In Sets the filtered colimits commute with finite limits.

1.0.34. If A is a small category, $\mathcal{C}$ is a cocomplete category then the category $\mathrm{Fun}(A, \mathcal{C})$ of functors is cocomplete (and colimits are computed pointwise) ([23], before Th. 3.2). Actually, it is also complete.

1.0.35. Let Grpd $\subset$ $\mathcal{C}$at be the full subcategory of groupoids in the category $\mathcal{C}$at of categories. This inclusion admits a left adjoint $L : \mathcal{C}\mathrm{at} \to \mathrm{Grpd}$ called groupoidification in ([23], p. 8).

The composition $\pi_1 : \boldsymbol{\Delta} \to \mathcal{C}\mathrm{at} \stackrel{L}{\to} \mathrm{Grpd}$ sends Δ^n to the 'free groupoid on $[n]$', it has objects $\{0, \dots, n\}$. For $0 \le i < n$ it has an arrow $a_i : i \to i{+}1$ and $a_i^{-1} : i{+}1 \to i$, which is its inverse. It also has the identity maps for each vertices and various compositions of a_i and a_j^{-1}. The automoprhism group of an object in this grupoid is trivial. The procedure of ([23], Digression 1.8) gives an adjunction $(\pi_1, N) : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathrm{Grpd}$. So, $\pi_1(X)$ for $X \in \mathcal{S}\mathrm{et}_\Delta$ can be calculated as a colimit. Can similarly $\pi_i(X)$ be calculated as similar colimits?

For a simplicial set X the notation $h(X)$ of [27] is the same as $\tau_1(X) \in \mathcal{C}\mathrm{at}$ from ([23], p. 8). So, by ([23], Digression 1.8), $h : \mathcal{S}\mathrm{et}_\Delta \to \mathcal{C}\mathrm{at}$ preserves the small colimits.

1.0.36. I don't understand why in ([23], Th. 1.18) in the base change diagram after this theorem, F_λ is contractible. If i^* is an acyclic Kan fibration then $F_\lambda \to \Delta^0$ is a Kan complex (because in any model category fibrations are stable by base change). But why all the fundamental groups of F_λ are trivial?

Theorem 1.0.37 ([21], Th.11.2, p. 65)**.** *Let $g : X \to Y$ be a morphism in $\mathcal{S}\mathrm{et}_\Delta$. Then g is a Kan fibration and a weak homotopical equivalence iff g has the right lifting property with respect to all inclusions $\partial\, \Delta^n \hookrightarrow \Delta^n$, $n \ge 0$.*

This implies immediately that the property ($g : X \to Y$ is a Kan fibration and a weak homotopical equivalence) is stable under base change. For this reason $F_\lambda \to \Delta^0$ is a weak homotopy equivalence in ([23], Th. 1.18)!

1.0.38. The category $\mathcal{S}\mathrm{et}_\Delta$ is monoidal with respect to the operation of product of simplicial sets. I think the fact from [27] that each simplicial category gives rise to a $\mathcal{H}$-enriched category uses in a hidden way the fact that the category of spaces $\mathcal{H}$ is monoidal, where the monoidal structure is induced by the above monoidal structure on $\mathcal{S}\mathrm{et}_\Delta$. Then the projection $\mathcal{S}\mathrm{et}_\Delta \to \mathcal{H}$ should be a monoidal functor.

1.0.39. The category $\mathcal{C}$at has a 'natural' model structure, where the weak equivalences are the equivalences in the 2-category of small categories ([23], Perspective 1.33).

1.0.40. Important: a zig-zag of categorical equivalences $A \leftarrow B \to C$ in $\mathcal{S}\mathrm{et}_\Delta$ can always be replaced by a single categorical equivalence $f : A \to C$ in $\mathcal{S}\mathrm{et}_\Delta$ ([23], Remark 2.8).

1.0.41. If $C \in \mathcal{S}\mathrm{et}_\Delta$ and $x, y \in C_0$ then we have a canonical cartesian diagram in $\mathcal{S}\mathrm{et}_\Delta$

$$\begin{array}{ccc} \mathrm{Hom}^R_C(x,y) & \to & C/y \\ \downarrow & & \downarrow \\ \Delta^0 & \stackrel{x}{\to} & C \end{array}$$

That is, the corresponding product identifies with $\mathrm{Hom}^R_C(x,y)$ canonically.

1.0.42. If $u : A \to B$ is a functor between small categories and $\mathcal{C}$ is a cocomplete (usual) category then the left Kan extension $Lan_u : \mathcal{C}^A \to \mathcal{C}^B$ always exists ([23], Perspective 2.31).

1.0.43. For any $K \in \mathcal{S}\mathrm{et}_\triangle$ one has canonically $\mathrm{Fun}(\triangle^0, K) \widetilde{\to} K$.

1.0.44. View $\mathcal{S}\mathrm{et}_\triangle$ as a simplicial category with the usual enrichment, that is, for $A, B \in \mathcal{S}\mathrm{et}_\triangle$ we use $\mathrm{Fun}(A, B)$, the simplicial set of maps. If $\mathcal{B}$ is a simplicial category then we get the simplicial functor $\mathcal{B}^{op} \times \mathcal{B} \to \mathcal{S}\mathrm{et}_\triangle$ sending (a, b) to $\mathrm{Map}_{\mathcal{B}}(a, b)$. On the level of morphisms, the morphism space from (a, b) to (a', b') is $\mathrm{Map}_{\mathcal{B}}(a', a) \times \mathrm{Map}_{\mathcal{B}}(b, b')$. The corresponding morphism in $\mathcal{S}\mathrm{et}_\triangle$

$$\mathrm{Map}_{\mathcal{B}}(a', a) \times \mathrm{Map}_{\mathcal{B}}(b, b') \to \mathrm{Fun}(\mathrm{Map}_{\mathcal{B}}(a, b), \mathrm{Map}_{\mathcal{B}}(a', b'))$$

comes from the composition map

$$\mathrm{Map}_{\mathcal{B}}(a', a) \times \mathrm{Map}_{\mathcal{B}}(a, b) \times \mathrm{Map}_{\mathcal{B}}(b, b') \to \mathrm{Map}_{\mathcal{B}}(a', b')$$

Recall that if $A, B \in \mathcal{K}\mathrm{an}$ then $\mathrm{Fun}(A, B)$ is a Kan complex ([27]). Then we may view $\mathcal{K}\mathrm{an}$ as a symplicial category, where $\mathrm{Map}_{\mathcal{K}\mathrm{an}}(A, B) = \mathrm{Fun}(A, B)$. The simplicial nerve of this simplicial category is the ∞-category $\mathcal{S}$ of spaces.

Notation: for a simplicial set K, $\mathcal{P}(K) = \mathrm{Fun}(K^{op}, \mathcal{S})$ is the infinity category of simplicial presheaves on K. For example, $\mathcal{P}(\triangle^0) \widetilde{\to} \mathcal{S}$ canonically, so $\mathcal{S}$ is the "cocompletion of the point".

1.0.45. Lurie starts by choosing a regular cardinal κ. This means that for any maps of sets $f : X \to Y$ with $\mid Y \mid < \kappa$, here $<$ means strictly less, and assuming for any $y \in Y$, $\mid X_y \mid < \kappa$ then $\mid X \mid < \kappa$. This garantees the following: the category $\mathrm{Sets}_{<\kappa}$ of sets of size $< \kappa$ has all colimits of size $< \kappa$.

He later assumes κ is strongly inaccessible. Then something is small iff it is κ-small. For example, $X \in \mathcal{S}\mathrm{et}_\triangle$ is small iff for any n, X_n is small (that is, X_n is κ-small).

1.0.46. a "canonical definition" of a right-lax monoidal functor between monoidal categories in given in ([27], A.1.3.5).

If $\mathcal{C}$ is a right-closed monoidal category then $\mathcal{C}$ is enriched over itself as in ([27], Ex. A.1.4.1). Namely, for $a, b, c \in \mathcal{C}$ the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(a \otimes b, c) \widetilde{\to} \mathrm{Hom}_{\mathcal{C}}(a, \mathrm{Map}(b, c))$$

defines the object $\mathrm{Map}(b, c) \in \mathcal{C}$. It is equipped with a canonical map $ev : \mathrm{Map}(b, c) \otimes b \to c$. The composition $\gamma : \mathrm{Map}(y, z) \otimes \mathrm{Map}(x, y) \to \mathrm{Map}(x, z)$ is then defined as follows. The composition

$$\mathrm{Map}(y, z) \otimes \mathrm{Map}(x, y) \otimes x \stackrel{\mathrm{id} \otimes ev}{\to} \mathrm{Map}(y, z) \otimes y \stackrel{ev}{\to} z$$

yields by adjontness the desired map γ. For $b, c \in C$ Lurie uses rather the notation c^b for the above $\mathrm{Map}_{\mathcal{C}}(b, c)$.

1.0.47. Let $\mathcal{C}$ be a right-closed monoidal category, $\mathcal{D}$ a $\mathcal{C}$-enriched category. Let $x \in \mathcal{D}, c \in \mathcal{C}$. The functor $\mathcal{D} \to \mathcal{C}$, $y \mapsto \mathrm{Map}_{\mathcal{D}}(x, y)$ may be viewed as a $\mathcal{C}$-enriched functor between $\mathcal{C}$-enriched categories. Further, the functor $\mathcal{C} \to \mathcal{C}$, $y \mapsto y^c = \mathrm{Map}_{\mathcal{C}}(c, y)$ is also naturally $\mathcal{C}$-enriched.

There is a minor mistake in ([27], p. 792, A.1.4.4). If under the above assumptions the functor $\mathcal{D} \to \mathcal{C}$, $y \mapsto \mathrm{Map}_{\mathcal{C}}(c, \mathrm{Map}_{\mathcal{D}}(x, y))$ is corepresentable then the object that corepresents it should be denoted $c \otimes x \in \mathcal{D}$. It is defined by the isomorphism $\mathrm{Map}_{\mathcal{C}}(c, \mathrm{Map}_{\mathcal{D}}(x, y)) \widetilde{\to} \mathrm{Map}_{\mathcal{D}}(c \otimes x, y)$ functorial in $y \in \mathcal{D}$. If the object $c \otimes x$ exists for any $c \in \mathcal{C}, x \in \mathcal{D}$ then we say that $\mathcal{D}$ is tensored over $\mathcal{C}$. In this case we get a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$, $(c, x) \mapsto c \otimes x$.

1.0.48. About Grothendieck opfibrations ([23], Def. 4.5). Let $p : \mathcal{C} \to \mathcal{D}$ be a functor. Any isomorphism $f : c_1 \widetilde{\to} c_2$ in $\mathcal{C}$ is p-cocartesian. The composition of two p-cocartesian maps $c_1 \xrightarrow{f} c_2 \xrightarrow{g} c_3$ is also p-cocartesian.

1.0.49. Recall that $\mathcal{C}at$ and $\mathcal{S}et_\Delta$ are locally presentable ([23], Example 3.4). Since $(\tau_1, N) : \mathcal{S}et_\Delta \rightleftarrows \mathcal{C}at$ is an adjunction, N preserves small limits. In particular, one has $N(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \widetilde{\to} N(\mathcal{A}) \times_{N(\mathcal{B})} N(\mathcal{C})$ for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{C}at$.

1.0.50. Let S be an ∞-category, $a, b \in S$, $f : a \to b$ a morphism in S.

Question 1. is it true that f is an equivalence iff for any $c \in S$ the natural map $\mathrm{Map}_S(c, a) \to \mathrm{Map}_S(c, b)$ is an isomorphism in $\mathcal{H}$? Here $\mathcal{H}$ is the notation from [27], the category of spaces. Yes, as the infinity analog of Yoneda lemma is a full embedding.

1.0.51. Given a simplicial category $\mathcal{C}$, a morphism $\mathfrak{C}[\Delta^0] \to \mathcal{C}$ in $\mathcal{C}at_\Delta$ is just an object $c \in \mathcal{C}$. A morphism $\mathfrak{C}[\Delta^1] \to \mathcal{C}$ in $\mathcal{C}at_\Delta$ is the same as an element $f \in \mathrm{Map}_{\mathcal{C}}(x, y)$, $x, y \in \mathcal{C}$.

1.0.52. Given a category $\mathcal{C}$, a map $\Lambda^3_0 \to N(\mathcal{C})$ does not always extend to $\Delta^3 \to N(\mathcal{C})$. For example, if $\mathcal{C}$ is the category of abelian groups, A_i is an abelian group for $i = 0, \dots, 3$ with $A_0 = 0$, and we are given any maps $f : A_1 \to A_2$, $g : A_2 \to A_3$, $h : A_1 \to A_3$ in $\mathcal{C}$, then this gives a map $\Lambda^3_0 \to N(\mathcal{C})$. It extends to $\Delta^3 \to N(\mathcal{C})$ iff $h = gf$.

On the other hand, if $\mathcal{C}$ is a category and $d > 3$ then any map $\Lambda^d_0 \to N(\mathcal{C})$ extends uniquely to $\Delta^d \to N(\mathcal{C})$ ([27], proof of 2.1.1.3).

1.0.53. Let $K \in \mathcal{S}et_\Delta$. There is a canonical map $K \times \Delta^1 \to K \star K$. Indeed, given a finite ordered set J, a J-point of $K \times \Delta^1$ gives $J_0 = J \times_{\Delta^1} \{0\}$, $J_1 = J \times_{\Delta^1} \{1\}$ and $J \to K$. Restricting the latter map to J_0 and J_1, we get an element of $K(J_0) \times K(J_1) \subset K \star K$.

Composing with $\mathrm{id} \star \mathrm{pr} : K \star K \to K \star \Delta^0$, we get a map $K \times \Delta^1 \to K \star \Delta^0$. It is used for colimits in ([27], 4.2.4.3). Namely, if $\mathcal{C}$ is an ∞-category, $\bar{p} : K \star \Delta^0 \to \mathcal{C}$ extending $p : K \to \mathcal{C}$ with cone point $x : \Delta^0 \to \mathcal{C}$ yields a morphism $\alpha : K \times \Delta^1 \to \mathcal{C}$ with $\alpha \mid_{K \times \{0\}} = p$ and $\alpha \mid_{K \times \{1\}} = \delta(x)$. Here $\delta : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ is the diagonal map.

1.0.54. If $\mathcal{A} \to \mathcal{B}$ is an inner fibration, $\mathcal{B}$ is an ∞-category then $\mathcal{A}$ is also an ∞-category.

1.0.55. Given a diagram of simplicial sets

$$\begin{array}{ccc} B & \xrightarrow{p} & X \\ \uparrow & & \downarrow q \\ A & \to & S \end{array}$$

with $r = qp$, $p_0 = p\mid_A$, $r_0 = r\mid_A$, we get morphisms $X_{p/} \to X_{p_0/} \times_{S_{r_0}/} S_{r/}$ and $X_{/p} \to X_{/p_0} \times_{S_{/r_0}} S_{/r}$.

1.0.56. If $n \geq 2$ then Λ^n_0 is the coproduct in the diagram

$$\begin{array}{ccc} \{0\}\star \triangle^{n-2} & & \\ \uparrow & & \\ \{0\}\star\partial\triangle^{n-2} & \hookrightarrow & \triangle^1 \star\partial\triangle^{n-2} \end{array}$$

This is used in ([27], 1.2.4.3), proof after 2.1.2.2. Dualzing, we see that Λ^n_n for $n \geq 2$ is the coproduct

$$\begin{array}{ccc} \triangle^{n-2}\star\{1\} & & \\ \uparrow & & \\ \partial\triangle^{n-2}\star\{1\} & \hookrightarrow & \partial\triangle^{n-2}\star\triangle^1 \end{array}$$

This is used in Remark 2.4.1.4 for description of p-cartesian morphisms.

1.0.57. If $i : A \hookrightarrow B$ is a left anodyne inclusion of simplicial sets, X is a Kan complex then $X^B \to X^A$ is a trivial fibration ([27], 2.1.2.9). For example, $\{0\} \hookrightarrow \triangle^1$ is left anodyne. In fact, for $X \in \mathcal{S}et_\triangle$, X is a Kan complex iff $X^{\triangle^1} \to X^{\{0\}}$ is a trivial fibration ([27], 2.1.2.10).

1.0.58. Consider a diagram of simplicial sets $X \xrightarrow{f} Y \xrightarrow{g} Z$. If g and gf are trivial fibrations then f is a homotopy equivalence.

1.0.59. Recall that the class of left anodyne maps is weekly saturated ([27], A.1.2.2), the same for anodyne (right anodine, inner anodyne) maps. Let X be any symplicial set. Then $X \times \{0\} \to X \times \triangle^1$ is left anodyne. Indeed, any weekly saturated class is closed under push-outs. Pick a point $x \in X$ then $X \to X \sqcup_{\{0\}} \triangle^1$ is the push-out of $\{0\} \to \triangle^1$, so is left anodyne. By ([27], 2.1.2.7) the map $X \sqcup_{\{0\}} \triangle^1 \to X \times \triangle^1$ is left anodyne, so their composition is also left anodyne ([27], A.1.2.3). This was used after ([27], 2.1.1.3).

1.0.60. Question: does a trivial fibration of simplicial sets always admit a section?

A trivial fibration is the same as a Kan fibration and homotopy equivalence. In ([27], 2.0.0.2) the definition of trivial fibration is different, via the right lifting property with respect to every map $\partial\triangle^n \to \triangle^n$.

To memorize: let $p : S \to T$ be a Kan fibration of simplicial sets. Then p is a trivial fibration iff each fibre of p is a trivial fibration ([27], 2.1.3.4).

1.0.61. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between usual categories. If $N(F) : N(\mathcal{A}) \to N(\mathcal{B})$ is a trivial fibration then F is an equivalence of categories, and F is surjective on objects. As far as I undestand, F still may be non injective on objects (so, F is not always an isomorphism of categories).

1.0.62. Let X, Y be contractible Kan complexes, $f : X \to Y$ a right fibration. Then f is a trivial fibration. Proof: f is a homotopy equivalence. By ([27], 2.1.3.3), f is a Kan fibration. So, f is a trivial fibration.

1.0.63. Let $X \stackrel{\alpha}{\to} Y \stackrel{\beta}{\to} Z$ be a diagram of simplicial sets, α be a trivial fibration, β a right fibration such that $\beta\alpha$ is a trivial fibration. Then β is a trivial fibration. Proof: for $z \in Z$ consider the trivial fibration of fibres $X_z \stackrel{\alpha}{\to} Y_z$. We know that Y_z is a Kan complex, and $Y_z \to \Delta^0$ is a homotopy equivalence. So, Y_z is a contractible Kan complex.

1.0.64. In ([27], 2.4.1.7) it is used also that $\Delta^{\{0,1\}} \to \Lambda^2_1$ is left anodyne as a push-out of $\{1\} \to \Delta^{\{12\}}$. It was used that for a diagram of simplicial subsets $A' \leftarrow A \to B$ in Δ^2 with B' coproduct, the square is cartesian

$$\begin{array}{ccc} C/B \times_{D/B} D/\ \Delta^2 & \to & C/A \times_{D/A} D/\ \Delta^2 \\ \uparrow & & \uparrow \\ C/B' \times_{D/B'} D/\ \Delta^2 & \to & C/A' \times_{D/A'} D/\ \Delta^2 \end{array}$$

Here $p : C \to D$ is an inner fibration of simplicial sets.

1.0.65. Let $p : X \to Y$ be an inner fibration, f an edge in X. From ([27], 2.4.1.3, point (2)) it follows that if f is cartesian then it is locally cartesian.

1.0.66. In ([27], 2.4.3.2) it is used that $(\Lambda^n_n \star \Delta^m) \sqcup_{\Lambda^n_n \star \partial \Delta^m} (\Delta^n \star \partial \Delta^m) \hookrightarrow \Delta^n \star \Delta^m$ is equivalent to $\Lambda^{n+m+1}_{n+1} \hookrightarrow \Delta^{n+m+1}$.

1.0.67. Let C be an ∞-category, $x, y \in \mathcal{C}$. Recall that $\mathrm{Hom}^R_C(x,y) = C/y \times_C \{x\}$. For the discussion just before ([27], 2.4.4.2). First, $\sigma = s_0(\bar{e})$. The diagram consists of cartesian squares

$$\begin{array}{ccccc} C/e & \to & C/y \times_{D/p(y)} D/\bar{e} & \to & C \times_D D/\bar{e} \\ \uparrow & & \uparrow & & \uparrow (x,\sigma) \\ F & \to & \phi^{-1}(\bar{e}) & \to & \Delta^0 \end{array}$$

Here $C_{p(x)} = C \times_D \{p(x)\}$. For $x, x' \in C_{p(x)}$ the space $\mathrm{Hom}^R_{C_{p(x)}}(x,x')$ is the space of those elements of $\mathrm{Hom}^R_C(x,x')$ that "induce the identity map" $p(x) \to p(x)$ in D.

1.0.68. comment for ([27], 2.4.4.3). Recall that $\{0\} \hookrightarrow \Delta^1$ is left anodyne. If $\mathcal{C}$ is an ∞-category, $f : y \to z$ is an edge in $\mathcal{C}$ then $\mathcal{C}/f \to \mathcal{C}/y$ is a trivial fibration by ([27], 2.1.2.5). So, $\mathcal{C}/f \times_{\mathcal{C}} \{x\}$ represents $\mathrm{Map}_{\mathcal{C}}(x,y)$ in the homotopy category of spaces $\mathcal{H}$. Indeed,

$$\mathcal{C}/f \times_{\mathcal{C}} \{x\} \to \mathcal{C}/y \times_{\mathcal{C}} \{x\}$$

is a trivial fibration, and $\mathcal{C}/y \times_{\mathcal{C}} \{x\} = \mathrm{Hom}^R_{\mathcal{C}}(x,y)$. Therefore, the diagram in the proof really represents the diagram in the claim of ([27], 2.4.4.3).

Lemma 1.0.69. *Let $p : \mathcal{C} \to \mathcal{D}$ be fully faithful map between ∞-categories. Then any edge in $\mathcal{C}$ is p-cartesian.*

Proof. We want to deduced this from ([27], 2.4.4.3(2)). For any $y \in \mathcal{C}$, $\mathcal{C}/y \to \mathcal{D}/p(y) \times_{\mathcal{D}} \mathcal{C}$ is a trivial fibration. Indeed, it is a right fibration, and after any base change $\{x\} \to \mathcal{C}$ by a vertex, this becomes a trivial fibration (see [27], 2.4.4.1).

Let $f : y \to z$ be any edge in $\mathcal{C}$. We want to prove that f is p-Cartesian. Let $x \in \mathcal{C}$. Consider the diagram from ([27], proof of 2.4.4.3)

$$\begin{array}{ccc} \mathcal{C}/f \times_{\mathcal{C}} \{x\} & \to & \mathcal{C}/z \times_{\mathcal{C}} \{x\} \\ \downarrow & & \downarrow \\ \mathcal{D}/p(f) \times_{\mathcal{D}} \{p(x)\} & \to & \mathcal{D}/p(z) \times_{\mathcal{D}} \{p(x)\} \end{array}$$

All the vertices of this diagram are Kan complexes (see 1.0.68). The right vertical arrow is a trivial fibration. The left one is a homotopy equivalence and a right fibration (2.1.2.1). By 2.1.3.3, the left vertical arrow is a Kan fibration, so it is a trivial fibration.

As in ([27], 2.4.4.3), the induced map $\phi_X : \mathcal{C}/f \times_{\mathcal{C}} \{x\} \to \mathcal{C}/z \times_{\mathcal{D}/p(z)} \mathcal{D}/p(f) \times_{\mathcal{C}} \{x\}$ is a Kan fibration. We see also it is a homotopy equivalence from the digram

$$\begin{array}{ccc} \mathcal{C}/f \times_{\mathcal{C}} \{x\} & \stackrel{\phi_X}{\to} & \mathcal{C}/z \times_{\mathcal{D}/p(z)} \mathcal{D}/p(f) \times_{\mathcal{C}} \{x\} \\ \downarrow & \swarrow & \\ \mathcal{D}/p(f) \times_{\mathcal{D}} \{p(x)\}, & & \end{array}$$

where the unnamed arrows are trivial fibrations. So, ϕ_X is a trivial fibration. □

In fact, a categorical equivalence is not necessarily a cartesian fibration. Indeed, by ([27], 2.4.4.6) it would be a trivial fibration. But if $\mathcal{A} \to \mathcal{B}$ is an equivalence of usual categories, which is not surjective on objects, then $N(\mathcal{A}) \to N(\mathcal{B})$ is a categorical equivalence, but it is not a trivial fibration.

1.0.70. Explanation of the proof of 2.4.4.4. One does not have to assume p inner. Take $x, y \in \mathcal{C}$. We have the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x, y) & \stackrel{b}{\to} & \mathrm{Map}_{\mathcal{D}}(p(x), p(y)) \\ \downarrow & \swarrow & \\ \mathrm{Map}_{\mathcal{E}}(qp(x), qp(y)) & & \end{array}$$

Take any $\bar{e} : qp(x) \to qp(y)$. Pick a locally qp-cartesian section $\alpha : x' \to y$ over $\bar{e}$. By the assumptions, $p(\alpha) : p(x') \to p(y)$ is locally q-cartesian. Let $z = qp(x)$. By the assumptions, $\mathrm{Map}_{\mathcal{C}_z}(x, x') \to \mathrm{Map}_{\mathcal{D}_z}(p(x); p(x'))$ is an isomorphism in $\mathcal{H}$. The map b is represented by a Kan fibration $\mathrm{Hom}^R_{\mathcal{C}}(x, y) \to \mathrm{Hom}^R_{\mathcal{D}}(p(x), p(y))$, whose each fibre is a homotopy equivalence. So, each fibre is a trivial fibration, so b itself is an isomorphism in $\mathcal{H}$.

1.0.71. For a diagram of ∞-categories $\mathcal{C} \to \mathcal{D} \leftarrow \mathcal{D}'$ set $\mathcal{C}' = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$. Let $u', v' \in \mathcal{C}'$ objects with images $u, v \in \mathcal{C}$, $x', y' \in \mathcal{D}'$, $x, y \in \mathcal{D}$. Then the square is cartesian

$$\begin{array}{ccc} \mathcal{C}'/v' \times_{\mathcal{C}'} \{u'\} & \to & \mathcal{D}'/y' \times_{\mathcal{D}'} \{x'\} \\ \downarrow & & \downarrow \\ \mathcal{C}/v \times_{\mathcal{C}} \{v\} & \to & \mathcal{D}_y \times_{\mathcal{D}} \{x\} \end{array}$$

and consists of Kan complexes. The vertical arrows are Kan fibrations. If the right vertical arrow is a homotopy equivalence, it is a trivial fibration, so the left one is also a trivial fibration. So, if $\mathcal{D}' \to \mathcal{D}$ is fully-faithful then $\mathcal{C}' \to \mathcal{C}$ is also fully faithful. This is used in ([27], 2.4.4.5).

1.0.72. In [27], Proposition 2.1.2.5 is stated incorrectly; in place of "Suppose either that i is right anodyne or that π is a left fibration", it should read "Suppose either that i is right anodyne or that i is anodyne and π is a left fibration." Also, the parenthetical "(right fibration)" in the proof should be replaced by "(left fibration)". Jacob told me this in his email.

1.0.73. Let K be a Kan complex. Then for $x, y \in K$ the mapping space $\mathrm{Map}_K(x, y)$ is usually not contractible.

Lemma 1.0.74. *Let K be a nonempty Kan complex. Assume for any $x, y \in K$, $Map_K(x, y)$ is contractible. Then K is contractible.*

Proof. For any $y \in K$, $K_y \to K$ is a trivial fibration (its a right fibration, and its fibres are contractible). So, any map $f_0 : \partial \vartriangle^n \to K$ with $f_0(n) = y$ may b extended to $f : \vartriangle^n \to K$. Since y was arbitrary, K is contractible. □

Lemma 1.0.75. *Let $p : \mathcal{C} \to \mathcal{D}$ be a fully faithful map between ∞-categories. Assume p is a cartesian fibration. Then p is a right fibration.*

Proof. Apply [27], 2.4.2.4. □

1.0.76. Consider a cartesian square of simplicial spaces

$$\begin{array}{ccc} X' & \overset{f}{\to} & X \\ \downarrow p' & & \downarrow p \\ S' & \to & S \end{array}$$

Assume p (so p' also) is a cartesian fibration. Then an edge e in X' is p'-cartesian iff $f(e)$ is p-cartesian. Indeed, any locally p-cartesian edge is p-cartesian by ([27], 2.4.2.8). The claim follows now from ([27], 2.4.1.12).

1.0.77. If $p : K \to C$ is a map of simplicial sets, $\bar{p} : \vartriangle^0 * K \to C$ extends p, view $\bar{p} : \vartriangle^0 \to C/p$. Then $(C/p)/\bar{p} = C/\bar{p}$ naturally. So, $\bar{p}$ is a limit of p iff $C/\bar{p} \to C/p$ is a trivial fibration. See ([27], 4.3.1).

1.0.78. If $p : X \to S$ is a trivial fibration of simplicial sets then p is a categorical equivalence. Indeed, p is a cartesian fibration. Now apply ([27], 2.4.4.6).

1.0.79. If $f : C \to D$ is an inner fibration, $c \in C$, the property that c is f-initial object of C means by definition that $C_{c/} \to C \times_D D_{f(c)/}$ is a trivial fibration. This implies that for any $x \in C$, $\mathrm{Map}_C(c, x) \to \mathrm{Map}_D(f(c), f(x))$ is an isomorphism.

1.0.80. Let $\mathcal{C}$ be an infinity category, $\mathcal{C}^0 \subset \mathcal{C}$ is a full subcategory, $c \in \mathcal{C}^0$, $\mathcal{C}^0/c := \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}/c$. Then $\mathrm{id} : c \to c$ is the final object of $\mathcal{C}^0/c$. Indeed, $(\mathcal{C}^0/c)/\mathrm{id} \to \mathcal{C}^0/c$ is an isomorphism of simplicial sets.

Similarly for any simplicial set C, $(C^{\triangleright})/\mathit{cone\,point} \to C^{\triangleright}$ is an isomorphism of simplicial sets, so the cone point is the final object of $C^{\triangleright}$.

1.0.81. Explanation for ([27], 4.3.2.2), left Kan extension. We have $F : \mathcal{C} \to \mathcal{D}$ extending $F_0 : \mathcal{C}^0 \to \mathcal{D}$. For $c \in C$ the functor F yields a composition $\mathcal{C}^0/c \hookrightarrow \mathcal{C}/c \to \mathcal{D}/f(c)$, which can be seen as a map $(\mathcal{C}^0/c)\star \vartriangle^0 \to \mathcal{D}$ extending $F_c : \mathcal{C}^0/c \to \mathcal{D}$.

The sense of the definition ([27], 4.3.2.2) of left Kan extension. Consider a diagram $\mathcal{C}^0 \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$, F_0 is the composition. Usually, one takes $\mathcal{D}' = *$. The idea, I think, is that $F : \mathcal{C} \to \mathcal{D}$ behaves as if the objects of $\mathcal{C}$ were colimits of $\mathcal{C}^0/c = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}/c$, and F "preserves this colimits".

By the way, taking $F = \mathrm{id}, \mathcal{D} = \mathcal{C}$, the property that id is the LKE of the inclusion $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ becomes a property of this subcategory. Is there a name for such subcategories?

1.0.82. About ([27], 4.3.2.8). If $\mathcal{D}' = *$ then the assumption that $\mathcal{D} \to \mathcal{D}'$ is a categorical fibration is automatically satisfied, which establishes that the LKE is transitive.

2. Justification of Gaitsgory-Rozenblyum's vocabulary from [14]

2.1. For a usual category C one has $N(C)^{ordn} = C$ canonically. A morphism $f : x \to y$ in $N(C)$ is an isomorphism iff it is an isomorphism in C. So, $N(C)^{Spc}$ is $N(C^{Spc})$.

Here C^{Spc} is the groupoid, whose objects are the same as objects of C. A morphism in C^{Spc} is a morphism in C, which is moreover an isomorphism.

2.2. The ∞-category of ∞-categories denoted $\mathcal{C}at_\infty$ by Lurie is defined in ([27], 3.0.0.1). Since $\mathcal{K}an \subset \mathcal{C}at_\infty^\vartriangle$ is a full simplicial subcategory, $\mathcal{S} \subset \mathcal{C}at_\infty$ is a full subcategory. Here $\mathcal{S}$ is the ∞-category of spaces.

If $C \in \mathcal{S}et_\vartriangle$ then $\pi_0(C^{Spc})$ is the set of isomorphism classes of objects of C. The natural map $\pi_0(C^{Spc}) \to \pi_0(C)$ is surjective, but not injective in general.

$(1-\mathcal{C}at)^{ordn}$ is the category whose objects are ∞-categories, morphisms from $\mathcal{C}$ to $\mathcal{D}$ are isomorphism classes of functors $f : \mathcal{C} \to \mathcal{D}$. In particular, $f : \mathcal{C} \to \mathcal{D}$ is an equivalence iff there is a functor $g : \mathcal{D} \to \mathcal{C}$ such that fg and gf are isomorphic to the identity functors. Is it the same notion as 'categorical equivalence' from Lurie [27]? Yes.

If A, B are usual categories then $\mathrm{Funct}(N(A), N(B)) \widetilde{\to} N(\mathrm{Fun}(A, B))$ canonically. A functor $f : N(A) \to N(B)$ is the same as a functor $f : A \to B$. A morphism of functors $e : \vartriangle^1 \to \mathrm{Funct}(N(A), N(B))$ is the same as a natural transformation $e : f \to g$, that is, a morphism in $\mathrm{Fun}(A, B)$. Further, $e : \vartriangle^1 \to \mathrm{Funct}(N(A), N(B))$ is an isomorphism iff $e : f \to g$ is an isomorphism in $\mathrm{Fun}(A, B)$. Thus, the isomorphism classes of functors $N(A) \to N(B)$ are precisely the isomorphism classes of functors $A \to B$ in the usual category $\mathrm{Fun}(A, B)$.

So, a functor $f : N(A) \to N(B)$ is an equivalence (in the sense of [14], 1.1.7) iff $f : A \to B$ is an equivalence of usual categories in the usual sense.

2.2.1. For $A, B \in 1-\mathcal{C}at$, $\mathrm{Funct}(A, B)^{op} \widetilde{\to} \mathrm{Funct}(A^{op}, B^{op})$ canonically.

2.2.2. I think one may add ([27], 2.3.2.2) to the vocabulary of Dennis. Namely, for $\mathcal{C} \in 1-\mathcal{C}at$ the natural map $\mathrm{Funct}(\vartriangle^2, \mathcal{C}) \to \mathrm{Funct}(\Lambda^2_1, \mathcal{C})$ is an isomorphism in $1-\mathcal{C}at$.

Besides, for any space $X \in \mathrm{Spc}$ the map $\mathrm{Funct}(\vartriangle^1, X) \to \mathrm{Funct}(\{0\}, X) = X$ is an isomoprhism in Spc ([27], 2.1.2.10). One may probably add that for $X \in 1-\mathcal{C}at$ we have $X \in \mathrm{Spc}$ iff the natural map $\mathrm{Funct}(\Delta^1, X) \to \mathrm{Funct}(\{0\}, X)$ is an isomorphism in $1-\mathcal{C}\mathrm{at}$.

This implies the following. If $X \in \mathrm{Spc}$, $x, y \in X$ then $\mathrm{Map}_X(x,y) \widetilde{\to} X \times_{X\times X} \{x,y\}$. Indeed, $\{x\} \times_X \mathrm{Funct}([1], X) \times_X \{y\} \widetilde{\to} \mathrm{Map}_X(x,y)$.

2.2.3. The ordinary category Spc^{ordn} is as follows. Its objects are Kan complexes. For two Kan complexes K, K', $\mathrm{Hom}_{\mathrm{Spc}^{ordn}}(K,K') = \pi_0(\mathrm{Funct}(K,K'))$. Recall that $\mathrm{Funct}(K,K')$ is a Kan complex, so $\pi_0(\mathrm{Funct}(K,K'))$ is the set of isomorphism classes of maps $K \to K'$. This is true because $\mathcal{K}\mathrm{an}$ is a fibrant simplicial category.

2.2.4. If A, B are usual categories we have the usual groupoid $\mathrm{Fun}(A,B)^{Spc}$. Then $\mathrm{Map}_{1-Cat}(N(A), N(B)) = N(\mathrm{Fun}(A,B)^{Spc})$ is a Kan complex.

2.2.5. For $\mathcal{C} \in \mathcal{S}\mathrm{et}_\Delta$, $\pi_0(C)$ glues all the arrows (into identities). If X is a set, $b : a \to a'$ is an arrow in $\mathcal{C}$, $f : \mathcal{C} \to X$ is a map in $\mathcal{S}\mathrm{et}_\Delta$ then $f(b) : f(a) \to f(a') = f(a)$ is the identity.

2.2.6. If $\mathcal{A}$ is a usual category, $x, y \in \mathcal{A}$ then $\mathrm{Map}_{N(\mathcal{A})}(x,y)$ is not just a space, but it is actually a set $\mathrm{Hom}_{\mathcal{A}}(x,y)$. Indeed, $\mathcal{A}/y \times_{\mathcal{A}} [0]$ is a set, not just a usual category, and $N(\mathcal{A})/y \times_{N(\mathcal{A})} \{x\}$ identifies with its nerve.

2.2.7. My understanding is that ([27], 1.2.3.3) means $\mathcal{H}$-enriched categories. This would mean the following. Given a fibrant simplicial category $\mathcal{C}$, if $x, y \in N(\mathcal{C})$ then $\mathrm{Map}_{N(\mathcal{C})}(x,y)$ is represented in Spc by $\mathrm{Map}_{\mathcal{C}}(x,y)$. Is it true?

At least, ([14], 1.1.7) gives the following. Given $\mathcal{C}, \mathcal{D} \in \mathrm{Spc}$, one has $\mathrm{Map}_{\mathrm{Spc}}(\mathcal{C}, \mathcal{D}) = \mathrm{Funct}(\mathcal{C}, \mathcal{D})$.

2.2.8. If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ and $f : a \to b$ is an isomorphism in $\mathcal{C}$ then one should add to Dennis' axioms that for any $z \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(z,a) \to \mathrm{Map}_{\mathcal{C}}(z,b)$ is an isomorphism in Spc. Any isomorphism has to be cartesian with respect to $\mathcal{C} \to *$.

2.2.9. If $h, h' : \mathcal{C} \to \mathcal{D}$ are two functors between $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}$, $u : \mathcal{A} \to \mathcal{C}$ is a morphism in $1-\mathcal{C}\mathrm{at}$, and $\alpha : h \to h'$ is an isomorphism in $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ then composing with u one gets an isomorphism $hu \to h'u$ in $\mathrm{Funct}(\mathcal{A}, \mathcal{D})$. Similarly for a composition with functors $\mathcal{D} \to \mathcal{B}$.

More generally, if $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, $a, b, c \in \mathcal{C}$, $h, h' \in \mathrm{Map}_{\mathcal{C}}(b,c)$, $\alpha : h \to h'$ is an isomorphism then for any $v \in \mathrm{Map}_{\mathcal{C}}(a,b)$ the induced map $hv \to h'v$ is an isomorphism in $\mathrm{Map}_{\mathcal{C}}(a,c)$. Probably, if we denote $\bar{h}, \bar{h}' : \mathrm{Map}_{\mathcal{C}}(a,b) \to \mathrm{Map}_{\mathcal{C}}(a,c)$ the two induced maps then α yields an isomorphism $\bar{h} \to \bar{h}'$ in

$$\mathrm{Map}_{\mathcal{S}}(\mathrm{Map}_{\mathcal{C}}(a,b), \mathrm{Map}_{\mathcal{C}}(a,c))$$

2.2.10. If $h, h' : \mathcal{C} \to \mathcal{D}$ are two maps in $1-\mathcal{C}\mathrm{at}$ and $\alpha : h \to h'$ is a morphism in $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ then for any $x, y \in \mathcal{C}$ we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x,y) & \to & \mathrm{Map}_{\mathcal{D}}(hx,hy) \\ \downarrow & & \downarrow \alpha_2 \\ \mathrm{Map}_{\mathcal{D}}(h'x,h'y) & \stackrel{\alpha_1}{\to} & \mathrm{Map}_{\mathcal{D}}(hx,h'y) \end{array}$$

This resembles of course the definition of an enriched natural transformation from the enriched category theory. If, moreover, α is an isomorphism then α_1, α_2 are isomorphisms.

Lemma 2.2.11. *Let $f : \mathcal{C} \to \mathcal{D}$ be an isomorphism in $1-\mathcal{C}\mathrm{at}$. Then f is fully faithful and essentially surjective.*

Proof. Let $g : \mathcal{D} \to \mathcal{C}$ be a functor such that fg and gf are isomorphic to the identity functors. Pick isomorphisms $\alpha : fg \widetilde{\to} \mathrm{id}$, $\alpha' : \mathrm{id} \widetilde{\to} fg$ such that $\alpha\alpha'$ and $\alpha'\alpha$ are isomorphic to the identities. Pick isomorphism $\beta : gf \widetilde{\to} \mathrm{id}$, $\beta' : \mathrm{id} \widetilde{\to} gf$ such that $\beta\beta'$ and $\beta'\beta$ are isomorphic to the identities.

Take $x, y \in \mathcal{C}$. We must show that the natural map $q : \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(fx, fy)$ is an isomorphism. The above choices yield a map

$$q' : \mathrm{Map}_{\mathcal{D}}(fx, fy) \to \mathrm{Map}_{\mathcal{C}}(gf(x), gf(y)) \widetilde{\to} \mathrm{Map}_{\mathcal{C}}(x, y)$$

It remains to show that $q'q$ and qq' are in the connected component of the identity. This follows now from Section 2.2.8.

If $z \in \mathcal{D}$ then $fg(z)$ is isomorphic to z according to Sect. 2.2.9. □

Note that if $\mathcal{C}_0 \subset \mathcal{C}$ is a full subcategory, $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, then $(\mathcal{C}_0)^{op} \subset \mathcal{C}^{op}$ is a full subcategory.

2.2.12. For a pointed space $\mathcal{S} \in \mathrm{Spc}$ with $x : \triangle^0 \to S$, its 0-th loop space is the pointed space $\Omega^0(S, x) = (S, x)$, its 1st loop space is the pointed space $\Omega^1(S, x) = * \times_S *$ with the diagonal point. Then $\Omega^n(S, x) = \Omega^1(\Omega^{n-1}(S, x))$ for $n \geq 1$. Finally, $\pi_n(S, x) = \pi_0(\Omega^n(S, x))$.

2.2.13. Let $f : \mathcal{C} \to \mathcal{D}$ be a map in $1-\mathcal{C}\mathrm{at}$. Assume f 1-replete. If $h : c_1 \to c_2$ is a morphism in $\mathcal{C}$ such that $fh : f(c_1) \to f(c_2)$ is an isomorphism in $\mathcal{D}$ then h is an isomorphism. So, f is conservative.

2.2.14. The functor $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}^{ordn}$, $C \mapsto C^{ordn}$ preserves finite products, because so does $\pi_0 : \mathrm{Spc} \to \mathrm{Sets}$.

If $\mathcal{A}, \mathcal{B}$ are usual categories then $\mathrm{Funct}(\mathcal{A}, \mathcal{B})$ is also a usual category, not just an object of $1-\mathcal{C}\mathrm{at}$. Now given $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}$, it is not clear how Dennis' axiomatics provides a morphism $\mathrm{Funct}(\mathcal{D}, \mathcal{C}) \to \mathrm{Funct}(\mathcal{D}^{ordn}, \mathcal{C}^{ordn})$. The latter should come from a morphism $\mathrm{Funct}(\mathcal{D}, \mathcal{C})^{ordn} \to \mathrm{Funct}(\mathcal{D}^{ordn}, \mathcal{C}^{ordn})$. Should not this be added to the axiomatics?

Let $I \in 1-\mathcal{C}\mathrm{at}, D \in 1-\mathcal{C}\mathrm{at}^{ordn}$. The natural map $\mathrm{Fun}(I^{ordn}, D) \to \mathrm{Fun}(I, D)$ is an equivalence. Indeed, it suffices to show that for $n \geq 0$ the natural map

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \mathrm{Fun}(I^{ordn}, D)) \to \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \mathrm{Fun}(I, D))$$

is an isomorphism in Spc. This is true, because $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}^{ordn}$, $C \mapsto C^{ordn}$ preserves finite products. Thus, when calculating limits or colimits in $\mathcal{D}$ indexed by I, we may replace I by I^{ordn}.

2.2.15. Assume given a map $f: D^0 \to D$ in $1-\mathcal{C}\mathrm{at}$. Damien Calaque claims that f is fully faithfull iff the induced functor $\mathrm{Funct}([1], D^0) \to \mathrm{Funct}([1], D) \times_{D\times D} D^0 \times D^0$ is an equivalence ([14], Ch. A.1, 1.4.3).

If this is true then one may derive the following.

Lemma 2.2.16. *Let $D^0 \subset D$ be a full subcategory, $C, D \in 1-\mathcal{C}\mathrm{at}$. Then $\mathrm{Funct}(C, D^0) \to \mathrm{Funct}(C, D)$ is fully faithful.*

Proof. The functor $\mathrm{Funct}(C, \cdot)$ commutes with limits according to Lemma 2.4.18. So, the equivalence $\mathrm{Funct}([1], D^0) \to \mathrm{Funct}([1], D) \times_{D\times D} D^0 \times D^0$ yields

$$\begin{aligned}&\mathrm{Funct}(C, \mathrm{Funct}([1], D^0)) \widetilde{\to}\\ &\qquad \mathrm{Funct}(C, \mathrm{Funct}([1], D)) \times_{\mathrm{Funct}(C,D)\times \mathrm{Funct}(C,D)} (\mathrm{Funct}(C, D^0) \times \mathrm{Funct}(C, D^0))\end{aligned}$$

This reads

$$\begin{aligned}&\mathrm{Funct}([1], \mathrm{Funct}(C, D^0)) \widetilde{\to}\\ &\qquad \mathrm{Funct}([1], \mathrm{Funct}(C, D)) \times_{\mathrm{Funct}(C,D)\times \mathrm{Funct}(C,D)} (\mathrm{Funct}(C, D^0) \times \mathrm{Funct}(C, D^0))\end{aligned}$$

Our claim follows. □

Lemma 2.2.17. *Let $I \times [1] \to 1-\mathcal{C}\mathrm{at}$ be a functor $i \mapsto (\mathcal{A}_i \stackrel{f_i}{\to} \mathcal{B}_i)$. Assume f_i is fully faithful for all i. Let $f: \mathcal{A} \to \mathcal{B}$ be obtained by passing to the limit over I. Then f is fully faithful.*

Proof. We will check that the natural functor $\mathrm{Funct}([1], \mathcal{A}) \to \mathrm{Funct}([1], \mathcal{B}) \times_{\mathcal{B}\times\mathcal{B}} \mathcal{A} \times \mathcal{A}$ is an equivalence. This functor is obtained by passing to the limit over $i \in I$ in the diagram in $1-\mathcal{C}\mathrm{at}$

$$\mathrm{Funct}([1], \mathcal{A}_i) \to \mathrm{Funct}([1], \mathcal{B}_i) \times_{\mathcal{B}_i\times\mathcal{B}_i} \mathcal{A}_i \times \mathcal{A}_i$$

We used also the fact that the functor $\lim: \mathrm{Funct}(I, 1-\mathcal{C}\mathrm{at}) \to 1-\mathcal{C}\mathrm{at}$ preserves small limits. □

Example 0: let $\mathrm{Idem} \times [1] \to 1-\mathcal{C}\mathrm{at}$ be a functor given on the unique object of Idem by a full subcategory $D^0 \subset D$. Then passing to the limit over Idem, we get a full subcategory (this is passing to retracts).

Example 1: consider a diagram $C \stackrel{\gamma}{\to} D \stackrel{\alpha}{\to} C$ in $1-\mathcal{C}\mathrm{at}$ with $\alpha\gamma \widetilde{\to} \mathrm{id}$, and $D \stackrel{\beta}{\to} E$ in $1-\mathcal{C}\mathrm{at}$ with $\beta \widetilde{\to} \beta\gamma\alpha$. Then $\beta\gamma$ is a retract of β. So, if β is fully faithful then $\beta\gamma$ is also fully faithful.

Example 2: let $Y \stackrel{i}{\to} X \stackrel{p}{\to} Y$ be a diagram in $1-\mathcal{C}\mathrm{at}$ with $pi \widetilde{\to} \mathrm{id}$. Let $\beta: Z \to Y$ be a functor and $\alpha = i\beta$. If α is fully faithful then β is also fully faithful as a retract of α. Indeed, the diagram $\beta: Z \to Y$ is obtained from $\alpha: Z \to X$ by passing to the limit over Idem. The corresponding idempotent acts on X as ip and on Z as id.

Remark 2.2.18. *If $I \times [1] \to 1-\mathcal{C}\mathrm{at}$ is a functor $i \mapsto (C_i \stackrel{f_i}{\to} D_i)$ and each f_i is 1-fully faithful, let $C = \lim_i C_i$, $D = \lim_i D_i$. Then $C \to D$ is 1-fully faithful.*

2.2.19. If $f : X \to S$ in $1-\mathcal{C}at$ is a bicartesian fibration in spaces then for any $a : s \to s'$ in S the induced functor $a_! : X_s \to X_{s'}$ is an equivalence ([27], 2.1.3.1). This is also seen in the following. If $X, Y \in$ Spc, $f : X \to Y$ is left adjoint to $g : Y \to X$ then f and g are isomorphisms in Spc. Indeed, any map $\mathrm{id} \to gf$ in $\mathrm{Funct}(X, X)$ is an isomorphism, similarly for $fg \to \mathrm{id}$.

Note also that ([27], 2.1.3.4) says: if $f : X \to Y$ is a cocartesian fibration in spaces, and for any $y \in Y$, $X_y \widetilde{\to} *$ in Spc then f is an equivalence in Spc.

2.2.20. For the terminology of [27]: a Kan fibration is a bi-cartesian fibration in spaces. Left fibration (resp., right fibration) is a cocartesian fibration in spaces (resp., cartesian fibration in spaces). The notion of a locally cartesian fibration ([27], 2.4.2.6) makes sense in the model-independent setting. If $f : X \to S$, $h : [1] \to S$ is a map in $1-\mathcal{C}at$, $h : s_1 \to s_2$ and $x_2 \in X$ with $f(x_2) \widetilde{\to} s_2$ then an arrow $\bar{h} : x_1 \to x_2$ in X over h is locally f-cartesian if $\bar{h}$ is a cartesian arrow for the projection $X \times_S [1] \to [1]$.

2.2.21. For 1.3.1. Let $F : D \to C$ be a map in $1-\mathcal{C}at$. The definition of a F-cartesian arrow in 1.3.1 is different from that of [27]. The equivalence of the two definitions is ([27], 2.4.4.3). Recall that for any $\alpha : d_0 \to d_1$ in D the induced functor $\xi : D_{/\alpha} \to D_{/d_1} \times_{C_{/F(d_1)}} C_{/F(\alpha)}$ is a cartesian fibration in spaces ([27], 2.1.2.1). According to [27], α is F-cartesian iff ξ is an equivalence.

Note that the diagram

$$\begin{array}{ccc} D_{/\alpha} & \to & D_{/d_1} \times_{C_{/F(d_1)}} C_{/F(\alpha)} \\ & \searrow & \downarrow \\ & & D \end{array}$$

is a morphism of cartesian fibrations in spaces over D. According to my Section 2.2.99, ξ is an equivalence iff its any base change by a point $d' : * \to D$ is an equivalence. This base change is essentially calculated in ([27], 2.4.1.10), it becomes the map

$$\mathrm{Map}_D(d', d_0) \to \mathrm{Map}_D(d', d_1) \times_{\mathrm{Map}_C(F(d'), F(d_1))} \mathrm{Map}_C(F(d'), F(d_0))$$

Namely, Lurie claims there that the diagram

$$\begin{array}{ccc} \{d'\} \times_D (D_{/d_1} \times_{C_{/F(d_1)}} C_{/F(\alpha)}) & \to & \{F(d')\} \times_C C_{/F(d_0)} \\ \downarrow & & \downarrow \\ \{d'\} \times_D D_{/d_1} & \to & \{F(d')\} \times_C C_{/F(d_1)} \end{array}$$

is cartesian. Note also that $D_{/\alpha} \widetilde{\to} D_{/d_0}$ naturally in $1-\mathcal{C}at$.

2.2.22. The fact that for $\mathcal{C} \in 1-\mathcal{C}at$, $F : I \to \mathcal{C}$ a morphism in $1-\mathcal{C}at$, $\mathcal{C}_{/F} \to \mathcal{C}$ is a cartesian fibration in spaces is ([27], 2.1.2.2).

If $p : \mathcal{X} \to \mathcal{S}$ is a cartesian fibration, let $\mathcal{X}' \subset \mathcal{X}$ be the 1-full subcategory, where we keep only those (connected components of) edges $f : x \to x'$, which are p-cartesian. Then the restriction $p' : \mathcal{X}' \to \mathcal{S}$ is a cartesian fibration in spaces ([27], 2.4.2.5).

2.2.23. If $p : \mathcal{X} \to \mathcal{S}$ is a cartesian fibration in spaces then any edge $f : x \to x'$ in $\mathcal{X}$ is p-cartesian ([27], 2.4.2.4). This is used by Dennis in 1.4.1. Namely, $(coCart/C)_{strict} \cap 0 - coCart/C$ simply coincides with $0 - coCart/C$.

question for 1.4.1: if $f : D_1 \to D_2$ is a morphism in $coCart/\mathcal{C}$, why the condition that f sends $\mathcal{C}$-cocartesian arrows to $\mathcal{C}$-cocartesian arrows depends only on the connected component of f in $\pi_0 \operatorname{Map}_{coCart/\mathcal{C}}(D_1, D_2)$?

2.2.24. If $\mathcal{C} \in 1 - \mathcal{C}\text{at}$ then $\mathcal{C} \to *$ is a cartesian fibration (cf. [27], 2.4.1.5), hence also cocartesian fibration.

If $\mathcal{C}$ is a simplicial set then $\mathcal{C}^{\Delta^1} \to \mathcal{C}^{\{0\}}$ is an inner fibration by ([27], 2.3.2.5). It is actually, a cartesian fibration by ([27], 2.4.7.12). Moreover, given $f : x \to y, f' : x' \to y'$ objects of $\mathcal{C}^{\Delta^1}$, assume given a morphism $\alpha : \Delta^1 \to \operatorname{Funct}(\Delta^1, \mathcal{C})$ from f to f' such that evaluating at 1 it gives a map $h : y \to y'$. Then α is cartesian over $\mathcal{C}^{\{0\}}$ iff h is an equivalence in $\mathcal{C}$. So, h is cartesian with respect to the projection $\mathcal{C} \to *$.

Therefore, for $\mathcal{C} \in 1-\mathcal{C}\text{at}$ the functor $\operatorname{Funct}([1], \mathcal{C}) \to \mathcal{C}^{\{0\}} \times \mathcal{C}^{\{1\}}$ takes $\mathcal{C}^{\{0\}}$-cartesian edges to $\mathcal{C}^{\{0\}}$-cartesian edges.

2.2.25. For Yoneda and 1.5.4. For $\mathcal{C} \in 1 - \mathcal{C}\text{at}$ the diagram

$$\begin{array}{ccc} \operatorname{Funct}([1], \mathcal{C}) & \to & \mathcal{C}^{\{0\}} \times \mathcal{C}^{\{1\}} \\ & \searrow & \downarrow \\ & & \mathcal{C}^{\{0\}} \end{array}$$

is a morphism in $(Cart/\mathcal{C})_{strict}$, it gives rise to the diagram

$$\begin{array}{ccc} \mathcal{C}^{op} & \to & (1 - \mathcal{C}\text{at})/\mathcal{C} \\ & \searrow & \downarrow \\ & & 1 - \mathcal{C}\text{at} \end{array}$$

where the horizontal arrow sends c to the object $(\mathcal{C}_{c/} \to \mathcal{C})$ of $(1 - \mathcal{C}\text{at})/\mathcal{C}$. Since $\mathcal{C}_{c/} \to \mathcal{C}$ is a cocartesian fibration in spaces, we got the functor $\mathcal{C}^{op} \to 0 - coCart/\mathcal{C}$. That is, a functor $\operatorname{Yon}_{\mathcal{C}} : \mathcal{C}^{op} \to \operatorname{Funct}(\mathcal{C}, \operatorname{Spc})$. The functor $\operatorname{Yon}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \operatorname{Spc}$ sends (c, x) to $\operatorname{Map}_{\mathcal{C}}(c, x)$.

2.2.26. About 1.6.1. An arrow in $\operatorname{Funct}^{coCart}([1], 1 - \mathcal{C}\text{at})$ gives rise a diagram

$$\begin{array}{ccc} D_1 & \stackrel{G_D}{\to} & D_2 \\ \downarrow F_1 & & \downarrow F_2 \\ C_1 & \to & C_2, \end{array}$$

the adjective commutative' diagram here does not seem to mean something, just the fact that this comes from $[1] \times [1] \to 1 - \mathcal{C}\text{at}$.

The fact that (1.5) is a cartesian fibration is obtained as follows. Given a map $C_1 \to C_2$ in $1 - \mathcal{C}\text{at}$ and an object $D_2 \stackrel{F_2}{\to} C_2$ of $\operatorname{Funct}^{coCart}([1], 1 - \mathcal{C}\text{at})_{strict}$ over C_2, set $D_1 = C_1 \times_{C_2} D_2$. The corresponding map $F_1 : D_1 \to C_1$ is a cocartesian fibration. Besides, the obtained map $D_1 \to D_2$ sends C_1-cocartesian arrows to C_2-cocartesian arrows (this follows from my Section 1.0.76).

For any $\mathcal{C} \in 1 - \mathcal{C}\text{at}$, the evaluation $\operatorname{Funct}([1], \mathcal{C}) \to \mathcal{C}$ at 1 is a cocartesian fibration, so the above is strange. Does this mean that we actually get a Kan fibration here?

2.2.27. For section 1.7.2, adjoint functors. Let $\alpha : E \to [1]$ be a bi-cartesian fibration. Set $C_0 = \alpha^{-1}(0), C_1 = \alpha^{-1}(1)$. Let $G : C_0 \to C_1$ and $F : C_1 \to C_0$ be the functors corresponding to E via unstrenthening. Let $c_0 \in C_0, c_1 \in C_1$. Pick α-cartesian edge $\alpha_1 : \tilde{c}_0 \to c_1$, and α-cocartesian edge $\alpha_0 : c_0 \to \tilde{c}_1$. Then, roughly, we should take $F(c_1) = \tilde{c}_0$, $G(c_0) = \tilde{c}_1$. From definition, the canonical maps

$$\mathrm{Map}_E(c_0, \tilde{c}_0) \to \mathrm{Map}_E(c_0, c_1) \text{ and } \mathrm{Map}_E(\tilde{c}_1, c_1) \to \mathrm{Map}_E(c_0, c_1)$$

are isomorphisms. Now ([27], 2.4.4.2) gives isomorphisms $\mathrm{Map}_{C_0}(c_0, \tilde{c}_0) \widetilde{\to} \mathrm{Map}_E(c_0, \tilde{c}_0)$ and $\mathrm{Map}_{C_1}(\tilde{c}_1, c_1) \widetilde{\to} \mathrm{Map}_E(\tilde{c}_1, c_1)$. This gives an isomorphism in Spc

$$\mathrm{Map}_{C_0}(c_0, F(c_1)) \widetilde{\to} \mathrm{Map}_{C_1}(G(c_0), c_1)$$

2.2.28. For convenience, in 1.7.5 we formulate the notion of partially defined left adjoint. Let $F : C_1 \to C_0$ be a map in $1-\mathcal{C}\mathrm{at}$. View it as a map $[1]^{op} \to 1-\mathcal{C}\mathrm{at}$, let $\alpha : E \to [1]$ be the corresponding cartesian fibration. Let $C'_0 \subset C_0$ be the full subcategory of those $c_0 \in C_0$ for which there is a α-cocartesian edge $c_0 \to c_1$ over $0 \to 1$. Let $\tilde{E}$ be the corresponding full subcategory of E, so its fibres over $0, 1$ are respectively C'_0, C_1. Then $\tilde{E} \to [1]$ is a cocartesian fibration, let $G : C'_0 \to C_1$ be the corresponding functor. This is the partially defined left adjoint to F.

For $c_0 \in C'_0$, $c_1 \in C_1$ we have canonically $\mathrm{Map}_{C_1}(G(c_0), c_1) \widetilde{\to} \mathrm{Map}_{C_0}(c_0, F(c_1))$.

2.2.29. If $C_2 \xrightarrow{F} C_1 \xrightarrow{F'} C_0$ is a diagram in $1-\mathcal{C}\mathrm{at}$, $G' : C'_0 \to C_1$ is the partially defined left adjoint to F', $G : C'_1 \to C_2$ is the partially defined left adjoint to F. If $c \in C_0$ such that $c \in C'_0$ and $G'(c) \in C'_1$ then the partially defined left adjoint $\mathcal{G}$ to $F'F$ is defined at c, and $\mathcal{G}(c) = GG'(c)$. I don't see how to prove this, but no doute this should be true. See also ([27], 5.2.2.6).

Remark 2.2.30. *If $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}$ then by ([27], 5.2.6.2), $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \widetilde{\to} \mathrm{Fun}^R(\mathcal{D}, \mathcal{C})^{op}$ canonically. Here $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the full subcategory spanned by left adjoint functors and $\mathrm{Fun}^R(\mathcal{D}, \mathcal{C}) \subset \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is the full subcategory spanned by right adjoint functors.*

2.2.31. For 2.1.1, left and right Kan extensions. Let $F : D \to C$ be a map in $1-\mathcal{C}\mathrm{at}$, $E \in 1-\mathcal{C}\mathrm{at}$. Let $a : \mathrm{Funct}(C, E) \to \mathrm{Funct}(D, E)$ be the composition with F.

Then LKE_F is as follows. It defines a full subcategory $\mathrm{Funct}(D, E)' \subset \mathrm{Funct}(D, E)$ and a functor $b : \mathrm{Funct}(D, E)' \to \mathrm{Funct}(C, E)$ with the following property. Given $f \in \mathrm{Funct}(C, E), g \in \mathrm{Funct}(D, E)'$ we have canonically

$$\mathrm{Map}_{\mathrm{Funct}(C,E)}(b(g), f) \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(D,E)}(g, a(f))$$

Let $\Phi : D \to E$ be a functor such that $LKE_F(\Phi) : C \to E$ exists. Then the functor $LKE_F(\Phi) : C^{op} \to E^{op}$ is the right Kan extension of $\Phi : D^{op} \to E^{op}$ with respect to $F : D^{op} \to C^{op}$.

2.2.32. For 2.1.2. Let $D, E \in 1-\mathcal{C}\mathrm{at}$, consider the functor $a : E \to \mathrm{Funct}(D, E)$ of "constant functors". Its left Kan extension is the colimit functor $\mathrm{colim}_D : \mathrm{Funct}(D, E) \to E$. It is not everywhere defined in general.

Given $f \in E$, $g : D \to E$ a functor, we have canonically

$$\mathrm{Map}_E(\mathrm{colim}_D(g), f) \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(D,E)}(g, a(f))$$

Here $a(f)$ is the constant functor $D \to * \xrightarrow{f} E$. The compatibility with Lurie is given in ([27], 4.2.4.3).

The right Kan extension of a is the limit functor $\lim_D : \mathrm{Funct}(D, E) \to E$. Fiven f, g as above, we have canonically

$$\mathrm{Map}_E(f, \lim_D(g)) \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(D,E)}(a(f), g)$$

2.2.33. Let $g : D \to E$ be a map in $1-\mathcal{C}\mathrm{at}$. If $e = \mathrm{colim}_D\, g$ then for $g^{op} : D^{op} \to E^{op}$ the element e is the limit $\lim_{D^{op}} g^{op}$. Assume $K \in 1-\mathcal{C}\mathrm{at}$, and D admits K-indexed limits. Then D^{op} admits K^{op}-indexed colimits. If in addition g preserves K-indexed limits then g^{op} preserves K^{op}-indexed colimits.

2.2.34. If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}^{ordn}$, that is, $\mathcal{C}$ is an ordinary category then the notion of an $(\infty, 1)$-limit/colimit in $\mathcal{C}$ coincides with the usual notion of limit/colimit.

2.2.35. Let $p : K \to \mathcal{C}$ be a diagram in $1-\mathcal{C}\mathrm{at}$. Any $z \in \mathcal{C}_{p/}$ is the same as a map $\bar{p} : K^{\triangleright} \to \mathcal{C}$ extending p. Then $(\mathcal{C}_{p/})_{z/} \widetilde{\to} \mathcal{C}_{\bar{p}/}$. So, $\bar{p}$ is a colimit diagram iff the natural map $\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/}$ is an equivalence.

2.2.36. If $C_2 \xrightarrow{F} C_1 \xrightarrow{F'} C_0$ is a diagram in $1-\mathcal{C}\mathrm{at}$, $E \in 1-\mathcal{C}\mathrm{at}$, consider the composition

$$\mathrm{Funct}(C_0, E) \xrightarrow{a'} \mathrm{Funct}(C_1, E) \xrightarrow{a} \mathrm{Funct}(C_2, E)$$

Let $\Phi \in \mathrm{Funct}(C_2, E)$. Assuming that $\Phi' := LKE_F(\Phi)$ exists, and $LKE_{F'}(\Phi')$ exists, it follows that $LKE_{F'F}(\Phi)$ exists, and

$$LKE_{F'F}(\Phi) = LKE_{F'} LKE_F(\Phi)$$

2.2.37. A nice application of the transitivity of right Kan extensions. For a map $f : I \to J$ in $1-\mathcal{C}\mathrm{at}$ and $C \in 1-\mathcal{C}\mathrm{at}$, write $f^* : \mathrm{Funct}(J, C) \to \mathrm{Funct}(I, C)$ for the composition. Write $f_* : \mathrm{Funct}(I, C) \to \mathrm{Funct}(J, C)$ for its right adjoint when it exists. Then for a diagram $I \xrightarrow{f} J \xrightarrow{g} K$ in $1-\mathcal{C}\mathrm{at}$ we get $g_* f_* \widetilde{\to} (gf)_*$ when both f_*, g_* exist.

Now assume we are given a map $F : I \times J \to C$ in $1-\mathcal{C}\mathrm{at}$. Consider the commutative diagram

$$\begin{array}{ccc} I \times J & \xrightarrow{p} & J \\ \downarrow q & & \downarrow \\ I & \to & * \end{array}$$

The functor p_* sends F to the functor $j \mapsto \lim_{i \in I} F(i, j)$. The functor q_* sends F to the functor $i \mapsto \lim_{j \in J} F(i, j)$. So, the transitivity of right Kan extensions gives in this case

$$\lim_{j \in J}(\lim_{i \in I} F(i, j)) \widetilde{\to} \lim_{I \times J} F \widetilde{\to} \lim_{i \in I}(\lim_{j \in J} F(i, j))$$

For colimits it is similar. For $f : I \to J$ call $f_! : \mathrm{Funct}(I, C) \to \mathrm{Funct}(J, C)$ its left adjoint when it exists and repeat.

For example, if $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admits limits, and for $i \in I$ we are given a diagram $a_i \to b_i \leftarrow c_i$, let $d_i = a_i \times_{b_i} c_i$. Then $\prod_{i \in I} d_i \widetilde{\to} (\prod a_i) \times_{\prod b_i} \prod c_i$.

Example: if $K, I, D \in 1-\mathcal{C}at$, K, I are small, D admits I-indexed limits then let $\mathcal{E} \subset \mathrm{Funct}({}^{\triangleleft}K, D)$ be the full subcategory spanned by the limit diagrams. Then $\mathcal{E}$ is stable under I-indexed limits.

2.2.38. For 2.1.3. We are in the situation of 2.1.1, He says if for each $c \in C$ the colimit of the composition $D \times_C C/c \to D \stackrel{\Phi}{\to} E$ exists then $LKE_F(\Phi)$ exists, it is a functor $C \to E$ sending c to $\mathrm{colim}_{D\times_C C/c}\, \Phi$.

Proposition 2.2.39 ([27], 4.3.2.15)**.** *Let $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}at$, $\mathcal{C}^0 \subset \mathcal{C}$ be a full subcategory. Let $\mathcal{K} \subset \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by those functors which are LKE of their restriction to $\mathcal{C}^0$. Let $\mathcal{K}' \subset \mathrm{Funct}(\mathcal{C}^0, \mathcal{D})$ be the full subcategory of those functors F such that for any $c \in \mathcal{C}$, the diagram $\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/c} \to \mathcal{C}^0 \stackrel{F}{\to} \mathcal{D}$ has a colimit. The restriction map $\mathcal{K} \to \mathcal{K}'$ is an isomorphism in $1-\mathcal{C}at$.*

Example: let $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}at$, $c \in \mathcal{C}$ be an initial object, $d \in \mathcal{D}$ giving $\Phi : * \stackrel{d}{\to} \mathcal{D}$. Then the $LKE(\Phi)$ along $* \stackrel{c}{\to} \mathcal{C}$ is the constant functor $\mathcal{C} \to \mathcal{D}$ with value d.

Another important case is ([27], 5.2.6.6): Let $f : \mathcal{C} \to \mathcal{D}$ be a left adjoint to $g : \mathcal{D} \to \mathcal{C}$, these are maps in $1-\mathcal{C}at$. Let $T : \mathcal{C} \to \mathcal{X}$ be any functor. Then $Tg : \mathcal{D} \to \mathcal{X}$ is a LKE of T along f.

If $\mathcal{A} \in 1-\mathcal{C}at$ is small then $\mathrm{id} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ is the LKE of the Yoneda embedding $j : \mathcal{A} \to \mathcal{P}(\mathcal{A})$ along itself ([27], 5.1.5.3). So, for $\mathcal{G} \in \mathcal{P}(\mathcal{A})$ we get

$$\mathcal{G} \,\widetilde{\to}\, \underset{\mathcal{A}\times_{\mathcal{P}(\mathcal{A})}\mathcal{P}(\mathcal{A})/\mathcal{G}}{\mathrm{colim}} jp,$$

where $p : \mathcal{A} \times_{\mathcal{P}(\mathcal{A})} \mathcal{P}(\mathcal{A})/\mathcal{G} \to \mathcal{A}$ is the projection.

Lemma 2.2.40. *Let $\Phi : \mathcal{A} \to \mathcal{D}$ be a functor, assume $\mathcal{D}$ cocomplete. Let $\bar{\Phi} : \mathcal{P}(\mathcal{A}) \to \mathcal{D}$ be the LKE of Φ along $j : \mathcal{A} \to \mathcal{P}(\mathcal{A})$. Let $R : \mathcal{D} \to \mathcal{P}(\mathcal{A})$ be the functor correspnding to $\mathcal{A}^{op} \times \mathcal{D} \to \mathrm{Spc}$, $(a, d) \mapsto \mathrm{Map}_{\mathcal{D}}(\Phi(a), d)$. Then R is right adjoint to $\bar{\Phi}$.*

Proof. Recall that $\bar{\Phi}$ is comilit preserving. For $\mathcal{G} \in \mathcal{P}(\mathcal{A})$ we have $\mathcal{G} \,\widetilde{\to} \underset{\mathcal{A}\times_{\mathcal{P}(\mathcal{A})}\mathcal{P}(\mathcal{A})/\mathcal{G}}{\mathrm{colim}} jp$, where $p : \mathcal{A} \times_{\mathcal{P}(\mathcal{A})} \mathcal{P}(\mathcal{A})/\mathcal{G} \to \mathcal{A}$ is the projection. For $d \in \mathcal{D}, \mathcal{G} \in \mathcal{P}(\mathcal{A})$ we get

$$\begin{aligned}\mathrm{Map}_{\mathcal{D}}(\bar{\Phi}(\mathcal{G}), d) \,\widetilde{\to}\, \mathrm{Map}_{\mathcal{D}}(\mathrm{colim}_{\mathcal{A}\times_{\mathcal{P}(\mathcal{A})}\mathcal{P}(\mathcal{A})/\mathcal{G}}\, \Phi p, d) \,\widetilde{\to} \underset{(\mathcal{A}\times_{\mathcal{P}(\mathcal{A})}\mathcal{P}(\mathcal{A})/\mathcal{G})^{op}}{\lim} \mathrm{Map}_{\mathcal{D}}(\Phi(a), d) \\ \widetilde{\to} \underset{(\mathcal{A}\times_{\mathcal{P}(\mathcal{A})}\mathcal{P}(\mathcal{A})/\mathcal{G})^{op}}{\lim} \mathrm{Map}_{\mathcal{P}(\mathcal{A})}(j(a), R(d)) \,\widetilde{\to}\, \mathrm{Map}_{\mathcal{P}(\mathcal{A})}(\underset{\mathcal{A}\times_{\mathcal{P}(\mathcal{A})}\mathcal{P}(\mathcal{A})/\mathcal{G}}{\mathrm{colim}} jp, R(d)) \\ \widetilde{\to}\, \mathrm{Map}_{\mathcal{P}(\mathcal{A})}(\mathcal{G}, R(d))\end{aligned}$$

□

For more details on this see ([32], Remark 4.4.4) and ([27], 5.2.6.5). Closely related claims: ([27], 5.2.6.3, 5.3.5.13).

Example: let $\mathcal{C} \in 1-\mathcal{C}at$ admit finite colimits. Recall that $\mathrm{Ind}(\mathcal{C})$ is presentable by ([27], 5.5.1.1). The inclusion $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ has a LKE $\bar{\Phi} : \mathcal{P}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ along $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$. Let $R : \mathrm{Ind}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$ be the right adjoint to $\bar{\Phi}$ then R is the natural inclusion, so $\bar{\Phi}$ is a localization functor.

Lemma 2.2.41. *Let $\mathcal{C}$ be a small category, $\mathcal{D}$ be cocomplete, $Y \in \mathcal{P}(\mathcal{C})$, $f : \mathcal{C}_{/Y} \to \mathcal{D}$ be a functor. Here $\mathcal{C}_{/Y} = \mathcal{C} \times_{\mathcal{P}(Y)} \mathcal{P}(Y)_{/Y}$. Let $\bar{f} : \mathcal{P}(\mathcal{C})_{/Y} \to \mathcal{D}$ be the LKE of f under $\mathcal{C}_{/Y} \hookrightarrow \mathcal{P}(\mathcal{C})_{/Y}$. Then f preserves colimits.*

Proof. Let $F : \mathcal{P}(\mathcal{C}_{/Y}) \to \mathcal{D}$ be the LKE of f under the Yoneda embedding $\mathcal{C}_{/Y} \hookrightarrow \mathcal{P}(\mathcal{C}_{/Y})$. Recall that F preserves colimits. Let $g : \mathcal{P}(\mathcal{C})_{/Y} \to \mathcal{P}(\mathcal{C}_{/Y})$ be the functor sending $Z \to Y$ to the presheaf $(c \overset{\alpha}{\to} Y) \mapsto Z(c) \times_{Y(c)} \{\alpha\}$, here $c \overset{\alpha}{\to} Y$ is an object of $\mathcal{C}_{/Y}$. Then the composition Fg is isomorphic to $\bar{f}$. Indeed, for $(Z \to Y) \in \mathcal{P}(\mathcal{C})_{/Y}$, we get from definitions $Fg(Z) = \underset{(c \to Z) \in \mathcal{C}_{/Z}}{\operatorname{colim}} F(c) \widetilde{\to} \bar{f}(Z)$. Recall that the projection $\mathcal{C}_{/Y} \to \mathcal{C}$ preserves colimits. Now g preserves colimits, because for any $c \in \mathcal{C}_{/Y}$ the composition $\mathcal{P}(\mathcal{C})_{/Y} \overset{g}{\to} \mathcal{P}(\mathcal{C}_{/Y}) \overset{c}{\to} \mathrm{Spc}$ preserves colimits. The latter is true, because the colimits in Spc are universal. Since F preserves colimits, we are done. □

The above claimed is strengthened in the next subsection.

2.2.42. Let $\mathcal{C}$ be a small category, $Y \in \mathcal{P}(\mathcal{C})$. Consider the functor $a : \mathcal{P}(\mathcal{C})_{/Y} \to \mathcal{P}(\mathcal{C}_{/Y})$ sending Z to the presheaf $(c \overset{\alpha}{\to} Y) \mapsto Z(c) \times_{Y(c)} \{\alpha\}$. Consider also the functor $b : \mathcal{P}(\mathcal{C}_{/Y}) \to \mathcal{P}(\mathcal{C})_{/Y}$ sending $Z' : (\mathcal{C}_{/Y})^{op} \to \mathrm{Spc}$ to the presheaf given informally by $S \mapsto \{\alpha \in \mathcal{Y}(S), z \in Z'(S, \alpha)\}$. The formal definition: let $\bar{Z}' \to (\mathcal{C}_{/Y})^{op}$ be the cocartesian fibration corresponding to Z'. Then $b(Z')$ is the functor $\mathcal{C}^{op} \to \mathrm{Spc}$ such that the corresponding cocartesian fibration in spaces over $\mathcal{C}^{op}$ is the composition $\bar{Z}' \to (\mathcal{C}_{/Y})^{op} \to \mathcal{C}^{op}$. Then a and b are inverses of each other.

Another definition of a: the projection $\mathcal{C}_{/Y} \to \mathcal{C}$ is a cartesian fibration in spaces corresponding to $Y : \mathcal{C}^{op} \to \mathrm{Spc}$. Given $Z \in \mathcal{P}(\mathcal{C})_{/Y}$, let $f : \mathcal{X} \to \mathcal{C}_{/Y}$ be the corresponding morphism of cartesian fibrations in spaces over $\mathcal{C}$ via strengthening for cartesian fibrations. Then f itself is a cartesian fibration in spaces by remark below. Then a sends the above point to the corresponding functor $(\mathcal{C}_{/Y})^{op} \to \mathrm{Spc}$.

Remark 2.2.43. *If $\mathcal{X}_i \to \mathcal{C}$ are cartesian fibrations in spaces, and $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a map over $\mathcal{C}$ then f itself is a cartesian fibration in spaces. (reason: any map in $\mathcal{X}_i$ is cartesian over $\mathcal{C}$).*

Remark 2.2.44. *An accessible category may be non complete. If κ is regular cardinal, $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ does not admit κ-small colimits then $\mathrm{Ind}_\kappa(\mathcal{C})$ may be non complete. Here is an example. Let R be a commutative ring, $\mathcal{C}$ the category of finitely generated free R-modules, $\kappa = \omega$. Then $\mathcal{C}$ does not admit finite colimits, for example, $R \overset{a}{\to} R$ does not always have a cokernel. By ([27], 5.4.2.3), $\mathrm{Ind}\,\mathcal{C}$ is the category of flat R-modules. Then $\mathrm{Ind}\,\mathcal{C}$ does not admit finite colimits in general. For example, $R \overset{a}{\to} R$ does not always have a cokernel.*

([28], 5.3.6.8) seems important! If κ is a regular cardinal, $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ be small and admitting κ-small colimits then for any cocomplete category $\mathcal{D}$ we have an equivalence $\mathrm{Funct}_{\mathcal{K}'}(\mathrm{Ind}_\kappa \mathcal{C}, \mathcal{D}) \widetilde{\to} \mathrm{Funct}_{\mathcal{K}}(\mathcal{C}, \mathcal{D})$. Here the subscript $\mathcal{K}'$ means that we take all colimit preserving functors, and $\mathcal{K}$ that we take all functors preserving κ-small colimits.

2.2.45. *Relative colimits.* They are defined in ([27], 4.3.1). Let $K^{\triangleright} \stackrel{\bar{p}}{\to} \mathcal{C} \stackrel{f}{\to} \mathcal{D}$ be a diagram in $1-\mathcal{C}$at, let $p : K \to \mathcal{C}$ be the restriction of $\bar{p}$. Then $\bar{p}$ is a f-colimit diagram iff $\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/} \times_{\mathcal{D}_{fp/}} \mathcal{D}_{f\bar{p}/}$ is an equivalence in $1-\mathcal{C}$at.

Important example ([27], 4.3.1.4): for $K = *$ a map $\bar{p} : [1] = K^{\triangleright} \to \mathcal{C}$ is a f-colimit iff the corresponding arrow in $\mathcal{C}$ is f-cocartesian.

([27], 4.3.1.11) is a nice result providing existence of some relative colimits (its claim is model-independent).

By definition, $c \in \mathcal{C}$ is f-initial in $\mathcal{C}$ if $(c, \mathrm{id}_{f(c)})$ is an initial object of $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{f(c)/}$. In other words, the partially defined left adjoint L to f is defined at $f(c)$, and the natural map $Lf(c) \to c$ is an isomorphism.

Question: how to reformulate the notion of a relative colimit in terms of adjoint functors? One may consider the following notion. Let $\mathcal{D} \in 1-\mathcal{C}$at, $h : \mathcal{C}_1 \to \mathcal{C}_2$ be a morphism in $1-\mathcal{C}\mathrm{at}_{/\mathcal{D}}$. For $\mathcal{E} \in 1-\mathcal{C}\mathrm{at}_{/\mathcal{D}}$ we have the restriction functor $\mathrm{Funct}_{\mathcal{D}}(\mathcal{C}_2, \mathcal{E}) \to \mathrm{Funct}_{\mathcal{D}}(\mathcal{C}_1, \mathcal{E})$. What about its left/right adjoints for the role of relative Kan extension? This is a wrong answer already in the case $\mathcal{C}_1 = \emptyset, \mathcal{C}_2 = *$.

It seems completely about the functor $\mathcal{C}_{p/} \to \mathcal{D}_{fp/}$. Let $q : K^{\triangleright} \to D$ be an extension of $fp : K \to D$. It is given by an object, say $h \in D_{fp/}$, and $(D_{fp/})_{h/} \widetilde{\to} D_{q/}$. Now giving $\bar{p} : K^{\triangleright} \to \mathcal{C}$ extending p means giving an object $r \in \mathcal{C}_{p/}$, so that $\mathcal{C}_{\bar{p}/} \widetilde{\to} (\mathcal{C}_{p/})_{r/}$. Let now r be such that its image in $D_{fp/}$ is denoted by h. We get a morphism

$$(\mathcal{C}_{p/})_{r/} \to \mathcal{C}_{p/} \times_{\mathcal{D}_{fp/}} (\mathcal{D}_{fp/})_{h/}$$

This is an isomorphism in $1-\mathcal{C}$at iff the partially defined left adjoint L to the natural functor $\bar{f} : \mathcal{C}_{p/} \to \mathcal{D}_{fp/}$ is defined at $h = \bar{f}(r)$, and the natural map $L\bar{f}(r) \to r$ is an isomorphism. In other words, this means that

$$\mathrm{Map}_{\mathcal{C}_{p/}}(r, z) \widetilde{\to} \mathrm{Map}_{\mathcal{D}_{fp/}}(\bar{f}(r), \bar{f}(z))$$

functorially on $z \in \mathcal{C}_{p/}$. In other words, r is $\bar{f}$-initial object of $\mathcal{C}_{p/}$.

Cofinality is applicable to relative colimits as for usual colimits. Namely, we have ([27], 4.3.1.7), which follows immediately from ([27], 4.1.1.8), and the proof is model-independent.

2.2.46. For the convenience of the reader, here is the notion of a relative p-limit. Let ${}^{\triangleleft}K \stackrel{\bar{p}}{\to} \mathcal{C} \stackrel{f}{\to} \mathcal{D}$ be a diagram in $1-\mathcal{C}$at, $p : K \to \mathcal{C}$ be the restriction of p. Then $\bar{p}$ is a f-limit diagram iff $\mathcal{C}_{/\bar{p}} \to \mathcal{C}_{/p} \times_{\mathcal{D}_{/fp}} \mathcal{D}_{/f\bar{p}}$ is an equivalence.

Let $\bar{f} : \mathcal{C}_{/p} \to \mathcal{D}_{/fp}$ be the induced map. As above, the notion of a f-limit of p is the same as a $\bar{f}$-final object of $\mathcal{C}_{/p}$.

2.2.47. Lurie defines also *relative Kan extensions* in ([27], 4.3.2.2). The definition is model independent. Namely, given a commutative diagram in $1-\mathcal{C}$at

$$\begin{array}{ccc} \mathcal{C}^0 & \stackrel{F_0}{\to} & \mathcal{D} \\ \downarrow & \nearrow F & \downarrow p \\ \mathcal{C} & \to & \mathcal{D}', \end{array} \tag{1}$$

where $\mathcal{C}^0 \to \mathcal{C}$ is a full subcategory, this is a p-left Kan extension of F_0 at $c \in \mathcal{C}$ iff the induced diagram

$$\begin{array}{ccc} \mathcal{C}^0_{/c} & \stackrel{F_c}{\to} & \mathcal{D} \\ \downarrow & \nearrow & \downarrow p \\ (\mathcal{C}^0_{/c})^{\triangleright} & \to & \mathcal{D}', \end{array}$$

exhibits $F(c)$ as a p-colimit of F_c. Here $\mathcal{C}^0_{/c} := \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/c}$, F_c is the composition $\mathcal{C}^0_{/c} \to \mathcal{C}^0 \stackrel{F_0}{\to} \mathcal{D}$. Here the functor $(\mathcal{C}^0_{/c})^{\triangleright} \to \mathcal{D}$ is obtained from $\mathcal{C}_{/c} \stackrel{\mathrm{pr}}{\to} \mathcal{C} \stackrel{F}{\to} \mathcal{D}$ via the composition with $(\mathcal{C}^0_{/c})^{\triangleright} \to \mathcal{C}/c$, the cone point is sent to $c \stackrel{\mathrm{id}}{\to} c \in \mathcal{C}_{/c}$.

F is a p-left Kan extension of F_0 if it is a p-left Kan extension of F_0 at each $c \in \mathcal{C}$.

If F is a p-left Kan extension of F_0 then $\mathcal{D}_{F/} \to \mathcal{D}'_{pF/} \times_{\mathcal{D}'_{pF_0/}} \mathcal{D}_{F_0/}$ is an equivalence ([27], 4.3.2.7), the proof seems depending on a model of quasicategories. The criterium for existence of the p-left Kan extension of F_0 is the same as in the absolute case ([27], 4.3.2.13). We also have a description of the full subcategory of

2.2.48. For the convenience of the reader, the notion of a *relative right Kan extension*. Given a diagram (1), where $\mathcal{C}^0 \subset \mathcal{C}$ is a full subcategory, F is a p-right Kan extension of F_0 at $c \in \mathcal{C}$ iff the induced diagram

$$\begin{array}{ccc} \mathcal{C}^0_{c/} & \stackrel{F_c}{\to} & \mathcal{D} \\ \downarrow & \nearrow & \downarrow p \\ {}^{\triangleleft}(\mathcal{C}^0_{c/}) & \to & \mathcal{D}', \end{array}$$

exhibits $F(c)$ as a p-limit of F_c.

2.2.49. Let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory, $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$. Let $F : I \to \mathcal{C}'$ be a functor, I small. Assume that c is the colimit of the composition $I \stackrel{F}{\to} \mathcal{C}' \to \mathcal{C}$ and $c \in \mathcal{C}'$. Then c is the colimit of F. Indeed, $\mathrm{Funct}(I, \mathcal{C}') \subset \mathrm{Funct}(I, \mathcal{C})$ is a full subcategory and for $c' \in \mathcal{C}'$, $\mathrm{Map}_{\mathcal{C}'}(c, c') \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(I,\mathcal{C})}(F, \delta(c')) \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(I,\mathcal{C}')}(F, \delta(c'))$.

2.2.50. ([27], 5.2.7.11) is a model-independent proof, ([27], 5.2.7.12) is also model-independent modulo ([27], 4.3.2.15).

2.2.51. Let $f : C \to D$ be a map in $1-\mathcal{C}\mathrm{at}$. You may define the notion of f-initial object of C. This is an object $c \in C$ such that for any $x \in C$, $\mathrm{Map}_C(c, x) \to \mathrm{Map}_D(f(c), f(x))$ is an isomorphism. See ([27], 4.3.1.1). Then ([27], 4.3.1.13) claims that c is a f-initial object of C iff (c, id) is an initial object of $C \times_D D_{f(c)/}$.

2.2.52. Important: Lurie means by a finite colimit a colimit in an infinity category $\mathcal{C}$ with respect to a functor $K \to \mathcal{C}$, where K is a simplicial set, which has only finitely many nondegenerate simplices!

For example, if $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admits finite colimits, it may be idempotent non complete (see example HTT, 4.4.5.1). That is, a colimit over $Idem$ is not a finite colimit.

In a model independent setting we may define the property "$\mathcal{C}$ has finite colimits" by requiring that $\mathcal{C}$ admits finite direct sums and push-out squares. I don't know an "official definition".

2.2.53. It seems in 2.1.3 one may add "if and only if", that is, the converse is also true? At least, if $\mathcal{C}^0 \subset \mathcal{C}$ is a full subcategory, and we consider the LKE of the functor $\mathrm{Funct}(\mathcal{C}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{C}^0, \mathcal{D})$ then we have the corresponding pointwise property in ([27], 4.3.2.13).

2.2.54. For a full subcategory $\mathcal{C}^0 \subset \mathcal{C}$ there is a property that the identity $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is the left Kan extension of the inclusion $\mathcal{C}^0 \to \mathcal{C}$. There should be a name for such full subcategories? There is one, see [32].

If $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is the LKE of the full embedding $i : \mathcal{C}^0 \hookrightarrow \mathcal{C}$ under i then the following holds. For any full subcategory $\mathcal{C}^0 \subset \mathcal{C}' \subset \mathcal{C}$, $\mathrm{id} : \mathcal{C}' \to \mathcal{C}'$ is the LKE of the full embedding $\mathcal{C}^0 \to \mathcal{C}'$ under itself.

If $f : \mathcal{C} \to \mathcal{D}$ is map in $1 - \mathcal{C}\mathrm{at}$, f *strongly generates* $\mathcal{D}$ if $\mathrm{id} : \mathcal{D} \to \mathcal{D}$ is the LKE of f along f ([32], 4.4.2).

Lemma 2.2.55. *Let $L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors, $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}$, L is left adjoint to R.*
i) R is fully faithfull iff $LR \to \mathrm{id}$ is an equivalence.
ii) L is fully faitfull iff $\mathrm{id} \to RL$ is an equivalence.

Proof. i) Assume $co : LR \to \mathrm{id}$ is an equivalence. Let $u : \mathrm{id} \to RL$ be the unit map. Then $L \stackrel{\mathrm{id}\circ u}{\to} LRL \stackrel{co\circ \mathrm{id}}{\to} L$ and $R \stackrel{u\circ \mathrm{id}}{\to} RLR \stackrel{\mathrm{id}\circ co}{\to} R$ are isomorphisms. So, $\mathrm{id} \circ u : L \to LRL$ and $u \circ \mathrm{id} : R \to RLR$ are isomorphism. The assumptions of ([27], 5.2.7.4(3)) are verified, so ([27], 5.2.7.4) shows that R is fully faithful. □

Note that L (or R) is an equivalence iff both $\mathrm{id} \to RL$ and $LR \to \mathrm{id}$ are isomorphisms.

Lemma 2.2.56. *Let $L : \mathcal{B} \to \mathcal{B}'$ be a left adjoint to $R : \mathcal{B}' \to \mathcal{B}$. Let $\mathcal{A} \in 1 - \mathcal{C}\mathrm{at}$.*
1) Let $\bar{L} : \mathrm{Funct}(\mathcal{A}, \mathcal{B}) \to \mathrm{Funct}(\mathcal{A}, \mathcal{B}')$ be the composition with L, $\bar{R} : \mathrm{Funct}(\mathcal{A}, \mathcal{B}') \to \mathrm{Funct}(\mathcal{A}, \mathcal{B})$ be the composition with R. Then $\bar{L}$ is left adjoint to $\bar{R}$.
2) Let $\tilde{L} : \mathrm{Funct}(\mathcal{B}', \mathcal{A}) \to \mathrm{Funct}(\mathcal{B}, \mathcal{A})$ be the composition with L, $\tilde{R} : \mathrm{Funct}(\mathcal{B}, \mathcal{A}) \to \mathrm{Funct}(\mathcal{B}', \mathcal{A})$ be the composition with R. Then $\tilde{L}$ is right adjoint to $\tilde{R}$.

Proof. 1) We want to use ([27], 5.2.2.8). We have unit and counit transformations $u : \mathrm{id} \to RL$ in $\mathrm{Funct}(\mathcal{B}, \mathcal{B})$, $c : LR \to \mathrm{id}$ in $\mathrm{Funct}(\mathcal{B}', \mathcal{B}')$. The unit transformation is constructed in ([27], 5.2.2.8). The counit transformation can be obtained from the unit transformation for the pair of adjoint functors $R^{op} : \mathcal{B}'^{op} \to \mathcal{B}^{op}$, $L^{op} : \mathcal{B}^{op} \to \mathcal{B}'^{op}$. For $b \in \mathcal{B}, b' \in \mathcal{B}'$ the composition

$$\mathrm{Map}_{\mathcal{B}'}(Lb, b') \to \mathrm{Map}_{\mathcal{B}}(RLb, Rb') \stackrel{u(c)}{\to} \mathrm{Map}_{\mathcal{B}}(b, Rb')$$

is an isomorphism in Spc. Similarly for the counit transformation.

For $F \in \mathrm{Funct}(\mathcal{A}, \mathcal{B}), G \in \mathrm{Funct}(\mathcal{A}, \mathcal{B}')$ we get maps

$$\mathrm{Map}_{\mathrm{Funct}(\mathcal{A},\mathcal{B})}(F, RG) \to \mathrm{Map}_{\mathrm{Funct}(\mathcal{A},\mathcal{B}')}(LF, LRG) \stackrel{c}{\to} \mathrm{Map}_{\mathrm{Funct}(\mathcal{A},\mathcal{B}')}(LF, G)$$

and

$$\mathrm{Map}_{\mathrm{Funct}(\mathcal{A},\mathcal{B}')}(LF, G) \to \mathrm{Map}_{\mathrm{Funct}(\mathcal{A},\mathcal{B})}(RLF, RG) \stackrel{u}{\to} \mathrm{Map}_{\mathrm{Funct}(\mathcal{A},\mathcal{B})}(F, RG)$$

It suffices to show they are inverse to each other. This should be a consequence of the following. The compositions $L \stackrel{L\circ u}{\to} LRL \stackrel{co L}{\to} L$ and $R \stackrel{u\circ R}{\to} RLR \stackrel{R\circ c}{\to} R$ are isomorphisms.

Justin Campbell suggest to apply a general thing: if $\mathcal{F}:\mathcal{C}\to\mathcal{D}$ is a functor between $(\infty,2)$-categories and $l:b\to b'$, $r:b'\to b$ are 1-morphisms in $\mathcal{C}$ which are adjoint (in the sence of [14], ch. 12, 1.1.4) then $\mathcal{F}(l):\mathcal{F}(b)\to\mathcal{F}(b')$ and $\mathcal{F}(r):\mathcal{F}(b')\to\mathcal{F}(b)$ are also adjoint. □

Corollary 2.2.57. *Let $L:\mathcal{B}\leftrightarrows\mathcal{B}':R$ be an adjoint pair of functors, $\mathcal{A},\mathcal{B},\mathcal{B}'\in 1-\mathcal{C}\mathrm{at}$. If R is fully faithful then $\tilde{L}:\mathrm{Fun}(\mathcal{B}',\mathcal{A})\to\mathrm{Fun}(\mathcal{B},\mathcal{A})$ given by composing with L, is fully faithful.*

Proof. Combine Lemma 2.2.56 and Lemma 2.2.55. □

Remark 2.2.58. *For $A,B\in 1-\mathcal{C}\mathrm{at}$ let $\mathrm{Fun}^R(A,B)\subset\mathrm{Fun}(A,B)$ be the full subcategory of functors which are right adjoint, $\mathrm{Fun}^L(A,B)\subset\mathrm{Fun}(A,B)$ the left adjoints. One has canoncally $\mathrm{Fun}^R(A,B)\,\widetilde{\to}\,(\mathrm{Fun}^L(A^{op},B^{op}))^{op}$ sending $f:A\to B$ to $f^{op}:A^{op}\to B^{op}$.*

2.2.59. Dennis' claims 2.1.5-2.1.6 are proved in Nick's email of 3.09.2016. Namely, if $F:\mathcal{C}\to\mathrm{Spc}$ is a functor, $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}, X\in\mathrm{Spc}$ let $\tilde{\mathcal{C}}\to\mathcal{C}$ be the cocartesian fibration in spaces associated to F. Then

$$\mathrm{Map}_{\mathrm{Spc}}(\mathrm{colim}\,F,X)\,\widetilde{\to}\,\mathrm{Map}_{\mathrm{Funct}(\mathcal{C},\mathrm{Spc})}(F,const(X))\,\widetilde{\to}\,\mathrm{Map}_{0-cocart/\mathcal{C}}(\tilde{\mathcal{C}},\mathcal{C}\times X)\,\widetilde{\to}\,\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(\tilde{C},X)$$

Now first if F is the constant functor with value $*$ the above shows that $\mathcal{C}\mapsto\mathrm{colim}_{\mathcal{C}}\,*$ is the left adjoint to the inclusion $\mathrm{Spc}\to 1-\mathcal{C}\mathrm{at}$. Besides, 2.1.6 also follows from the above.

2.2.60. Dennis' claim 2.1.8 follows from ([27], 5.1.5.6). The important observation here is as follows. If $X\in\mathrm{Spc}$ then $*\times_{\mathrm{Spc}}\mathrm{Spc}/X\,\widetilde{\to}\,X$. The projection on Spc/X yields a morphism $X\to\mathrm{Spc}/X$ whose composition with $\mathrm{Spc}/X\to\mathrm{Spc}$ factors through $*\to\mathrm{Spc}$. If $i:*\to\mathrm{Spc}$ is the inclusion, let $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$ and $c\in\mathcal{C}$. $F:\mathrm{Spc}\to\mathcal{C}$ is the left Kan extension of $c:*\to\mathcal{C}$ via i then for any $X\in\mathrm{Spc}$, the colimit of the composition $X=*\times_{\mathrm{Spc}}\mathrm{Spc}/X\to *\stackrel{c}{\to}C$ is $F(X)$ by ([27], 4.3.2.2).

2.2.61. *Enhaced version of strenthening for spaces.* Consider the full subcategory

$$\mathrm{Funct}^{0-cocart}([1],1-\mathcal{C}\mathrm{at})\subset\mathrm{Funct}([1],1-\mathcal{C}\mathrm{at})$$

whose objects are functors $F:D\to C$ which are cocartesian fibrations in spaces. Evaluation at $1\in[1]$ defines a functor

$$\mathrm{Funct}^{0-cocart}([1],1-\mathcal{C}\mathrm{at})\to 1-\mathcal{C}\mathrm{at},$$

which is a cartesian fibration (for this see my Section 2.2.26). The functor $1-\mathcal{C}\mathrm{at}^{op}\to 1-\mathcal{C}\mathrm{at}$ corresponding to this cartesian fibration is canonically isomorphic to the functor $C\mapsto\mathrm{Funct}(C,\mathrm{Spc})$.

2.2.62. Let $p:C\to E$ be a map in $1-\mathcal{C}\mathrm{at}$. Then $E_{p/}\,\widetilde{\to}\,(\mathrm{Funct}(C,E)_{p/})\times_{\mathrm{Funct}(C,E)}E$ in $1-\mathcal{C}\mathrm{at}$ naturally according to Dennis' definition from 1.3.6. Let $\bar{p}:[1]\times C\to E$ lie in $E_{p/}$. Why the propety that $\bar{p}$ is an initial object of $E_{p/}$ is equivalent to requiring that $\bar{p}$ is a colimit of p in the sense of 2.1.2? This can be deduced from my Corollary 2.2.116.

2.2.63. Normalization of strengthening for cartesian fibrations. Let $C \in 1-\mathcal{C}\mathrm{at}$. The functor $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$, $C \mapsto C^{op}$ is an equivalence. Composing with $\mathrm{Funct}(C, .)$, it gives an equivalence $a : \mathrm{Funct}(C, 1-\mathcal{C}\mathrm{at}) \to \mathrm{Funct}(C, 1-\mathcal{C}\mathrm{at})$. The composition

$$\mathrm{Funct}(C, 1-\mathcal{C}\mathrm{at}) \stackrel{SCC}{\to} (cocart/C)_{strict} \widetilde{\to} (cart/C^{op})_{strict} \stackrel{SC}{\to} \mathrm{Funct}(C, 1-\mathcal{C}\mathrm{at})$$

is a. Here SCC (resp., SC) is the strengthening for cocartesian (resp., cartesian fibrations), and the arrow in the middle $\widetilde{\to}$ sends a cocartesian fibration $X \to C$ to the cartesian one $X^{op} \to C^{op}$.

2.2.64. $1-\mathcal{C}\mathrm{at}$ admits all small limits and colimits ([27], 4.2.4.8 and 3.3.3).

([27], 3.1.2.1) seems important and maybe should be added to Dennis' book? It says in Dennis' framework that given a cartesian fibration $p : X \to S$ and $K \in 1-\mathcal{C}\mathrm{at}$, the induced map $p^K : X^K \to S^K$ is a cartesian fibration. An edge $f : [1] \to X^K$ is p^K-cartesian iff for any $k \in K$ the composition $[1] \stackrel{k}{\to} [1] \times K \stackrel{f}{\to} X$ is p-cartesian.

Dually, if $p : X \to S$ is a cocartesian fibration then $p^K : X^K \to S^K$ is a cocartesian fibration.

If $p : X \to S$ is a cocartesian fibration in spaces then $X^K \to S^K$ is also a cocartesian fibration in spaces ([27], 2.1.2.9).

2.2.65. ([27], 3.3.3.4) can be formulated in Dennis' framework as follows. Let $p : K \to \mathrm{Spc}$ be a morphism in $1-\mathcal{C}\mathrm{at}$, here Spc is the ∞-category of spaces. Let $X \to K$ be the cocartesian fibration in spaces associated to p. Then there is a canonical isomorphism in Spc

$$\lim p \widetilde{\to} \mathrm{Funct}_K(K, X)$$

(confirmed by Nick). Here $\mathrm{Funct}_K(K, X)$ is the space $X^K \times_{K^K} \{\mathrm{id}\}$ of sections of $X \to K$. Actually, $\mathrm{Funct}_K(K, X)$ is a space, because $X^K \to K^K$ is a a cocartesian fibration in spaces.

For the convenience of the reader, a version for cartesian fibrations: Let $p : K^{op} \to \mathrm{Spc}$ be a functor, $X \to K$ the corresponding cartesian fibration in spaces. Then $\lim p \widetilde{\to} \mathrm{Funct}_K(K, X)$.

Important question: how to reformulate ([27], 3.3.3.2) in Dennis' framework?

Notation. Given a diagram $X \to S \stackrel{q}{\leftarrow} K$ in $1-\mathcal{C}\mathrm{at}$, write $\mathrm{Funct}_S(K, X)$ for the object $X^K \times_{S^K} \{q\}$ of $1-\mathcal{C}\mathrm{at}$. One is tempted to give the following definition (according to
`https://en.wikipedia.org/wiki/Fibred_category`).

Let $f : X \to \mathcal{C}$ be a cocartesian fibration, $p : \mathcal{C} \to 1-\mathcal{C}\mathrm{at}$ be the corresponding functor via unstrenthening. Then we may define the full subcategory $\mathrm{Funct}^{cocart}_{\mathcal{C}}(\mathcal{C}, X) \subset \mathrm{Funct}_{\mathcal{C}}(\mathcal{C}, X)$ whose objects are functors $f : \mathcal{C} \to X$ over $\mathcal{C}$ such that f is a morphism in $(coCart/\mathcal{C})_{strict}$. In other words, f sends any edge of $\mathcal{C}$ to a f-cocartesian edge of X.

Given a cocartesian fibration $f : X \to \mathcal{C}$, one may define a 1-full subcategory $X^\natural \subset X$, where we keep only those edges of X, which are f-cocartesian. We have a natural map $\mathrm{Funct}_{\mathcal{C}}(\mathcal{C}, X^\natural) \to \mathrm{Funct}^{cocart}_{\mathcal{C}}(\mathcal{C}, X)$, which is not an equivalence (the 2-morphisms are not the same). A way to formulate ([27], 3.3.3.2) in Dennis' framework would be as follows.

Proposition 2.2.66 (Nick). *Let* $f : X \to \mathcal{C}$ *be a cocartesian fibration in* $1-\mathcal{C}\mathrm{at}$*,* $p : \mathcal{C} \to (1-\mathcal{C}\mathrm{at})$ *be the corresponding functor via unstrenthening. Then there is an isomorphism* $\lim p \widetilde{\to} \mathrm{Funct}_{\mathcal{C}}^{cocart}(\mathcal{C}, X)$.

Proof. It suffices to establish an isomorphism

$$\mathrm{Funct}(Z, \lim p)^{\mathrm{Spc}} \widetilde{\to} \mathrm{Funct}(Z, \mathrm{Funct}_{\mathcal{C}}^{cocart}(\mathcal{C}, X))^{\mathrm{Spc}}$$

in Spc functorial in $Z \in 1-\mathcal{C}\mathrm{at}$. Write $co(Z) : \mathcal{C} \to 1-\mathcal{C}\mathrm{at}$ for the constant functor with value Z. We have

$$\mathrm{Funct}(Z, \lim p)^{\mathrm{Spc}} \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(\mathcal{C}, 1-\mathcal{C}\mathrm{at})}(co(Z), p) \widetilde{\to} \mathrm{Map}_{(coCart_{/\mathcal{C}})_{strict}}(\mathcal{C} \times Z, X)$$

For an arrow $\alpha : c_1 \to c_2$ in $\mathcal{C}$, $z \in Z$ the map $(\mathrm{id}, \alpha) : (z, c_1) \to (z, c_2)$ in $\mathcal{C} \times Z$ is cocartesian over $\mathcal{C}$. So, a functor $\mathcal{C} \times Z \to X$ over $\mathcal{C}$ is a morphism in $(coCart_{/\mathcal{C}})_{strict}$ iff the corresponding functor $Z \to \mathrm{Funct}_{\mathcal{C}}(\mathcal{C}, X)$ factors through $\mathrm{Funct}_{\mathcal{C}}^{cocart}(\mathcal{C}, X)$. We are done. □

For the convenience of the reader, the version of the above claim for cartesian fibrations. It comes from the fact that the functor $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$, $Z \mapsto Z^{op}$, being an equivalence, preserves limits and colimits.

Proposition 2.2.67. *Let* $f : X \to \mathcal{C}$ *be a cartesian fibration in* $1-\mathcal{C}\mathrm{at}$*,* $p : C^{op} \to 1-\mathcal{C}\mathrm{at}$ *the corresponding functor (strengthening for cartesian fibrations). One has canonically* $\lim p \widetilde{\to} \mathrm{Funct}_{\mathcal{C}}^{cart}(\mathcal{C}, X)$.

We have denoted by $\mathrm{Funct}_{\mathcal{C}}^{cart}(\mathcal{C}, X) \subset \mathrm{Funct}_{\mathcal{C}}(\mathcal{C}, X)$ the full subcategory of those functors that send any arrow in $\mathcal{C}$ to a cartesian arrow in X.

For a cartesian fibration $f : X \to \mathcal{C}$, write $X^{\natural}$ for the 1-full subcategory of X, where we keep only f-cartesian edges.

Lemma 2.2.68. *1) Let* $p : I \to 1-\mathcal{C}\mathrm{at}$ *be a functor,* $i \mapsto C_i$*, let* $C = \lim p$*. For* $i \in I$ *let* $ev_i : C \to C_i$ *be the canonical projection. Let* $K \in 1-\mathcal{C}\mathrm{at}$*,* $p : K^{\triangleright} \to C$ *be a diagram such that for any* $i \in I$ *the composition* $K^{\triangleright} \to C \to C_i$ *is a colimit diagram. Then* $K^{\triangleright} \to C$ *is a colimit diagram.*
2) Let $p : K^{\triangleleft} \to C$ *be a diagram such that for any* $i \in I$ *the composition* $K^{\triangleleft} \to C \to C_i$ *is a limit diagram. Then* $K^{\triangleleft} \to C$ *is a limit diagram.*

Proof. 1) Let $X \to I^{op}$ be the cartesian fibration corresponding to p. By Proposition 2.2.67, $C \widetilde{\to} \mathrm{Funct}_{I^{op}}^{cart}(I^{op}, X)$. The functor $C \to C_i$ is given by the evaluation at $i \in I^{op}$. Let $p : K^{\triangleright} \to \mathrm{Funct}_{I^{op}}^{cart}(I^{op}, X)$ be a diagram such that for any i the induced functor $K^{\triangleright} \to X_i$ is a colimit diagram. Then, by ([27], 5.1.2.2), the composition

$$K^{\triangleright} \xrightarrow{p} \mathrm{Funct}_{I^{op}}^{cart}(I^{op}, X) \hookrightarrow \mathrm{Funct}_{I^{op}}(I^{op}, X)$$

is a colimit diagram. So, p is also a colimit diagram (we are passing to a full subcategory).
2) The functor $p^{op} : (K^{op})^{\triangleright} \to C^{op} \widetilde{\to} \lim_{i \in I} C_i^{op}$ satisfies the assumptions of 1). So, $p^{op} : (K^{op})^{\triangleright} \to C^{op}$ is a colimit diagram. □

The previous lemma may be strengthened in the case any limits replaced by a cartesian square, namely one has GREAT CLAIMS ([27], 5.4.5.4 and 5.4.5.5). The analog of ([27], 5.4.5.5) with K-colimits replaced by K-limits is also true and is obtained by passing to opposite categories.

It can also be strengthened in the case when any limit is replaced by a product indexed by a set, see Corollary 2.5.5 below.

Lemma 2.2.69. *1) Let $p : I \to 1-\mathcal{C}\mathrm{at}$ be a functor $i \mapsto C_i$, let $C = \lim p$. Let $q : K \to C$ be a map in $1-\mathcal{C}\mathrm{at}$, denote the composition $K \to C \to C_i$ by q_i for $i \in I$. Then $C_{q/} \widetilde{\to} \lim\limits_{i\in I}(C_i)_{q_i/}$ in $1-\mathcal{C}\mathrm{at}$.*
2) Assume that each q_i admits a colimit e_i in C_i. For any map $i \to j$ in I let $F_{ij} : C_i \to C_j$ be the corresponding transition functor. Assume that for any $i \to j$ in I the induced map $e_j \to F_{ij}(e_i)$ is an isomorphism. Then the colimit e of q exists, and the image of e in C_i identifies with the colimit of q_i.
3) Assume each q_i admits a limit e_i in C_i. For any map $i \to j$ in I let $F_{ij} : C_i \to C_j$ be the transition functor. Assume for any $i \to j$ in I the induced map $F_{ij}(e_i) \to e_j$ is an isomorphism in C_j. Then the limit e of q exists, and the image of e in each C_i identifies with e_i.

Proof. 1) Transitivity of Kan extensions. We have

$$\begin{aligned} C_{q/} \widetilde{\to} \{q\} \times_{\mathrm{Funct}(K,C)} \mathrm{Funct}(K \times [1], C) \times_{\mathrm{Funct}(K,C)} C \widetilde{\to} \\ \{q\} \times_{\lim_{i\in I}\mathrm{Funct}(K,C_i)} \lim_{i\in I} \mathrm{Funct}(K \times [1], C_i) \times_{\lim_{i\in I}\mathrm{Funct}(K,C_i)} \lim C_i \widetilde{\to} \\ \lim_{i\in I}(\{q_i\} \times_{\mathrm{Funct}(K,C_i)} \mathrm{Funct}(K \times [1], C_i) \times_{\mathrm{Funct}(K,C)} C) \end{aligned}$$

2) Each $(C_i)_{q_i/}$ admits an initial object and the transition functors $(C_i)_{q_i/} \to (C_j)_{q_j/}$ preserve the initial objects. Now apply Lemma 2.2.70 below.
3) Is obtained from 2) by applying $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$, $D \mapsto D^{op}$. □

Lemma 2.2.70. *Let $I \to 1-\mathcal{C}\mathrm{at}$ be a functor $i \mapsto C_i$, let $C = \lim C_i$ in $1-\mathcal{C}\mathrm{at}$. Assume that each C_i admits an initial object, and for any $i \to j$ in I the transition functor $C_i \to C_j$ preserves initial objects. Then C admits an initial object c, and the image of c in any C_i is an initial object of C_i.*

Proof. Let $q : X \to I^{op}$ be a cartesian fibration corresponding to $I \to 1-\mathcal{C}\mathrm{at}$. By ([27], 2.4.4.9), there is a section $p : I^{op} \to X$ in $\mathrm{Funct}_{I^{op}}(I^{op}, X)$ such that for any $i \in I$, $p(i)$ is intial object of $X_i \widetilde{\to} C_i$. We have to show that $p \in \mathrm{Funct}^{cart}_{I^{op}}(I^{op}, X)$ that is, for any $a : i \to j$ in I^{op} the map $p(a) : p(i) \to p(j)$ is q-cartesian. Indeed, let $x \to p(j)$ be a q-cartesian arrow over a. Then $x, p(i)$ are an initial objects of $X_i \widetilde{\to} C_i$. By the universal properties of cartesian arrows, there is a map $p(i) \to x$ in X_i such that the composition $p(i) \to x \to p(j)$ is $p(a)$. Since $p(i) \to x$ is an isomorphism, $p(a)$ is a q-cartesian arrow. □

The proof of ([27], 2.4.4.9) given by Lurie depends on a model, here is a model-independent proof of this result:

Lemma 2.2.71. *Let $f : X \to S$ be a cartesian fibration. Assume for any $s \in S$, X_s admits an initial object. Let $X' \subset X$ be the full subcategory spanned by those objects x*

such that x is initial in $X_{f(x)}$. The composition $X' \to X \xrightarrow{f} S$ is an isomorphism in $1-\mathcal{C}\mathrm{at}$.

Proof. Since X_s has an initial object for any s, f is essentially surjective. Let $x_1, x_2 \in X'$. It remains to show that $\mathrm{Map}_X(x_1, x_2) \to \mathrm{Map}_S(f(x_1), f(x_2))$ is an equivalence. It is surjective. Indeed, any $g : f(x_1) \to f(x_2)$ admits a cartesian lifting to an arrow $x_0 \to x_2$ with $x_0 \in X_{f(x_1)}$. Since we have a map $x_1 \to x_0$, we get a morphism $x_1 \to x_2$ over g.

Pick now $g : f(x_1) \to f(x_2)$ in S. It remains to show that $\mathrm{Map}_X(x_1, x_2)_g \widetilde{\to} *$ in spaces. Pick a cartesian arrow $\bar{g} : x_0 \to x_2$ over g. By definitions,

$$\mathrm{Map}_{X_{f(x_1)}}(x_1, x_0) \widetilde{\to} \mathrm{Map}_X(x_1, x_2)_g$$

Since $x_1 \in X_{f(x_1)}$ is initial, the latter space is $*$. □

Lemma 2.2.72. *Assume given two functors $p, p' : I \to 1-\mathcal{C}\mathrm{at}$ and a map $\alpha : p \to p'$ in $\mathrm{Funct}(I, 1-\mathcal{C}\mathrm{at})$. Let p be given by $i \mapsto C_i$, p' by $i \mapsto C'_i$. Let $C = \lim p, C' = \lim p'$. Assume given a map $K \to C$ in $1-\mathcal{C}\mathrm{at}$. Assume each C_i, C'_i admits K-indexed limits, and each functor $\alpha_i : C_i \to C'_i$ preserves K-indexed limits. Assume for any $i \to j$ in I the transition functors $C_i \to C_j$ and $C'_i \to C'_j$ preserve K-indexed limits. Then C, C' admits K-indexed limits, and the induced functor $C \to C'$ preserves K-indexed limits.*

Proof. This follows from Lemma 2.2.69 and 2.2.68. □

Lemma 2.2.73. *Let $I \to 1-\mathcal{C}\mathrm{at}$, $i \mapsto C_i$ be a functor, $\mathcal{D} \in 1-\mathcal{C}\mathrm{at}$. Then $(\mathrm{colim}_{i\in I} C_i) \times \mathcal{D} \widetilde{\to} \mathrm{colim}_{i\in I} C_i \times \mathcal{D}$ canonically.*

Proof. For any $\mathcal{E} \in 1-\mathcal{C}\mathrm{at}$ there is a canonical equivalence

$$\mathrm{Funct}((\mathrm{colim}\, C_i) \times \mathcal{D}, \mathcal{E})^{\mathrm{Spc}} \widetilde{\to} \mathrm{Funct}(\mathrm{colim}\, C_i, \mathrm{Funct}(\mathcal{D}, \mathcal{E}))^{\mathrm{Spc}} \widetilde{\to}$$

$$\lim_{i\in I^{op}} \mathrm{Funct}(C_i, \mathrm{Funct}(\mathcal{D}, \mathcal{E}))^{\mathrm{Spc}} \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Funct}(C_i \times \mathcal{D}, \mathcal{E})^{\mathrm{Spc}} \widetilde{\to} \mathrm{Funct}(\mathrm{colim}(C_i \times \mathcal{D}), \mathcal{E})^{\mathrm{Spc}}$$

□

2.2.74. Important: the strengthening for cartesian fibrations is normalized as follows. If $X \to \mathcal{C}$ is a cartesian fibration then the corresponding functor $F : \mathcal{C}^{op} \to 1-\mathcal{C}\mathrm{at}$ sends $c \in \mathcal{C}$ to X_c. For a map $\alpha : c_2 \to c_1$ in $\mathcal{C}^{op}$, $F(c)$ is the pull-back functor $X_{c_2} \to X_{c_1}$ with respect to the corresponding map $c_1 \to c_2$ in $\mathcal{C}$.

2.2.75. The analog of ([27], 3.3.4.3) in Dennis' should be as follows. Let $p : K^{op} \to 1-\mathcal{C}\mathrm{at}$ be a map corresponding to a cartesian fibration $X \to K$. Then $\mathrm{colim}\, p$ is characterised by the property: for $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$,

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(\mathrm{colim}\, p, \mathcal{C}) \subset \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(X, \mathcal{C})$$

is the full subspace consisting of those functors $X \to \mathcal{C}$ that send every cartesian edge to an equivalence.

For example, if for $K \in 1-\mathcal{C}\mathrm{at}$ we consider the constant functor $p : K^{op} \to 1-\mathcal{C}\mathrm{at}$ with value $*$ then $\mathrm{colim}\, p \widetilde{\to} \mid K \mid$. Since $\mathrm{Spc} \hookrightarrow 1-\mathcal{C}\mathrm{at}$ admits a right adjoint, it preserves colimits by the adjoint functors theorem ([27], cor. 5.5.2.9). So, the colimit of the constant functor $p : K^{op} \to \mathrm{Spc}$ with value $*$ is also $\mid K \mid$.

Actually, $\mathrm{Spc} \hookrightarrow 1-\mathcal{C}\mathrm{at}$ preserves both limits and colimits.

2.2.76. One is also tempted to define a 'cartesian equivalence' ([27], 3.1.3.3) in Dennis' model independent framework. Let $S \in 1-\mathcal{C}\mathrm{at}$ and $p: X \to Y$ be a morphism in $(1-\mathcal{C}\mathrm{at})/S$. One may try the following definition: p is a cartesian equivalence iff for any cartesian fibration $Z \to S$, that is, an object of $Cart/S$, the induced map $\mathrm{Funct}_S(Y, Z^\natural) \to \mathrm{Funct}_S(X, Z^\natural)$ is an isomorphism in $1-\mathcal{C}\mathrm{at}$.

2.2.77. ([27], 4.1.1.5) in a model independent way says the following. Let $p: K \to \mathcal{C}, q: K' \to \mathcal{C}$ be maps in $1-\mathcal{C}\mathrm{at}$. There is a canonical isomorphism in Spc

$$\mathrm{Funct}(K, \mathcal{C}_{/q}) \times_{\mathrm{Funct}(K,\mathcal{C})} \{p\} \widetilde{\to} \mathrm{Funct}(K', \mathcal{C}_{p/}) \times_{\mathrm{Funct}(K',\mathcal{C})} \{q\}$$

Recall that $\mathcal{C}/q \to \mathcal{C}$ is a cartesian fibration in spaces, so $(\mathcal{C}/q)^K \to \mathcal{C}^K$ is also a cartesian fibration in spaces, so the left hand side is a space. Similarly for the right hand side. The above isomorphism rewrites

$$\mathrm{Funct}_{\mathcal{C}}(K, \mathcal{C}_{/q}) \widetilde{\to} \mathrm{Funct}_{\mathcal{C}}(K', \mathcal{C}_{p/})$$

2.2.78. Let $p: K \to \mathcal{C}$ be a map in $1-\mathcal{C}\mathrm{at}$, $x \in \mathcal{C}$. Then we have canonically

$$\mathcal{C}_{p/} \times_{\mathcal{C}} \{x\} \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(K,\mathcal{C})}(p, \delta(x))$$

Here $\delta: \mathcal{C} \to \mathrm{Funct}(K, \mathcal{C})$ is the precomposition with $K \to *$. Indeed, use Dennis' description of under category. Taking in Section 2.2.77, $K' = *$ and $x = q: * \to \mathcal{C}$, we get canonically

$$\mathcal{C}_{p/} \times_{\mathcal{C}} \{x\} \widetilde{\to} \mathrm{Funct}_{\mathcal{C}}(K, \mathcal{C}_{/x})$$

The map $\mathcal{C}_{/x} \to \mathcal{C}$ is a cartesian fibration in spaces corresponding to the functor $\mathcal{C}^{op} \to \mathrm{Spc}$, $c \mapsto \mathrm{Map}_{\mathcal{C}}(c, x)$. So, $\mathcal{C}_{/x} \times_{\mathcal{C}} K \to K$ is a cartesian fibration on spaces corresponding to the composition $\tilde{p}: K^{op} \xrightarrow{p} \mathcal{C}^{op} \to \mathrm{Spc}$. From Section 2.2.65 we get

$$\lim_{k \in K^{op}} \mathrm{Map}_{\mathcal{C}}(p(k), x) := \lim \tilde{p} \widetilde{\to} \mathrm{Funct}_{\mathcal{C}}(K, \mathcal{C}_{/x}) \widetilde{\to} \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}\, p, x)$$

Example: if x is a final object of $\mathcal{C}$ then $\delta(c)$ is a final object of $\mathrm{Funct}(K, \mathcal{C})$.

2.2.79. The previous subsection allows to prove a part of the adjoint functor theorem. Let $L: \mathcal{C} \to \mathcal{D}$ be left adjoint to $R: \mathcal{D} \to \mathcal{C}$ in $1-\mathcal{C}\mathrm{at}$. Let $p: I \to \mathcal{C}$ be a functor, $i \mapsto c_i$. Assume $c = \mathrm{colim}\, p$ exists. Let us show that $L(c)$ is the colimit $\mathrm{colim}_{i \in I} L(c_i)$. For $y \in \mathcal{D}$ this follows from

$$\mathrm{Map}_{\mathcal{D}}(L(c), y) \widetilde{\to} \mathrm{Map}_{\mathcal{C}}(c, R(y)) \widetilde{\to} \lim_{i \in I^{op}} \mathrm{Map}_{\mathcal{C}}(c_i, R(y)) \widetilde{\to} \lim_{i \in I^{op}} \mathrm{Map}_{\mathcal{D}}(L(c_i), y)$$
$$\widetilde{\to} \mathrm{Map}_{\mathcal{D}}(\mathrm{colim}_{i \in I} L(c_i), y)$$

2.2.80. A dual version of Section 2.2.78 with colim replaced by lim. Let $p: K \to C$ be a functor, $x \in \mathcal{C}$. Then

$$\mathrm{Map}_{\mathcal{C}}(x, \lim p) \widetilde{\to} \lim_{k \in K} \mathrm{Map}_{\mathcal{C}}(x, p(k))$$

It is obtained by applying the functor *op* to the last isomorphism in Section 2.2.78.

We use the observation that $(\mathrm{Map}_{\mathcal{C}}(x, y))^{op}$ identifies canonically with $\mathrm{Map}_{\mathcal{C}^{op}}(y, x)$. Besides, $\mathrm{Spc} \to \mathrm{Spc}, S \mapsto S^{op}$ is an equivalence, so preserves limits.

2.2.81. Apply Section 2.2.78 for $\mathcal{C} = 1-\mathcal{C}\mathrm{at}$. We get that for $p : K \to 1-\mathcal{C}\mathrm{at}$, $k \mapsto C_k$, and $Y \in 1-\mathcal{C}\mathrm{at}$ we get

$$\lim_{k\in K^{op}}(\mathrm{Funct}(C_k, Y)^{\mathrm{Spc}}) \widetilde{\to} \mathrm{Funct}(C, Y)^{\mathrm{Spc}},$$

here $C = \mathrm{colim}\, p$. On the LHS, the limit is taken in Spc.

See Lemma 2.4.17 for a generalization.

2.2.82. If $\mathcal{C}, K \in 1-\mathcal{C}\mathrm{at}$, $x \in \mathcal{C}$, let $p : K \to * \stackrel{x}{\to} \mathcal{C}$ be the composition. If $y \in \mathcal{C}$ then $\mathrm{Funct}(K, \mathrm{Map}_{\mathcal{C}}(x, y)) \widetilde{\to} \mathcal{C}_{p/} \times_{\mathcal{C}} \{y\}$. A colimit of p is described in ([27], 4.4.4.9). Namely, assume $K \in \mathrm{Spc}$ in addition. Then $y \in \mathcal{C}$ is the colimit of p iff the corresponding object of $\mathcal{C}_{p/} \times_{\mathcal{C}} \{y\} \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(K, \mathrm{Map}_{\mathcal{C}}(x, y))$ induces an equivalence for any $z \in \mathcal{C}$

$$\mathrm{Map}_{\mathcal{C}}(y, z) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(K, \mathrm{Map}_{\mathcal{C}}(x, z))$$

The notation to be used by Lurie for this colimit is $K \otimes x \in \mathcal{C}$. If $\mathcal{C}$ admits all small colimits then we see that $\mathcal{C}$ is tensored over Spc.

2.2.83. For 2.1.5. Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$. The space $\mid \mathcal{C} \mid \in \mathrm{Spc}$ is characterized by the property that for any $Y \in \mathrm{Spc}$ one has a natural isomorphism

$$\mathrm{Map}_{\mathrm{Spc}}(\mid \mathcal{C} \mid, Y) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(\mathcal{C}, Y) = \mathrm{Funct}(\mathcal{C}, Y)^{Spc}$$

On the other hand, the colimit $\mathrm{colim}_{\mathcal{C}} *$ of the constant functor $\mathcal{C} \to \mathrm{Spc}$ with value $*$ is characterized by the following property. For any $Y \in \mathrm{Spc}$,

$$\mathrm{Map}_{\mathrm{Spc}}(\mathrm{colim}_{\mathcal{C}} *, Y) \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(\mathcal{C},\mathrm{Spc})}(\delta(*), \delta(Y)) \widetilde{\to} \mathrm{Spc}_{\delta(*)/} \times_{\mathrm{Spc}} \{Y\} \widetilde{\to} \mathrm{Funct}_{\mathrm{Spc}}(\mathcal{C}, \mathrm{Spc}_{/Y})$$

Here $\delta(Y) : \mathcal{C} \to \mathrm{Spc}$ is the constant functor with value Y.

Proof of 2.1.6. Let $\Phi : \mathcal{C} \to \mathrm{Spc}$ be a functor, $\tilde{\mathcal{C}} \to \mathcal{C}$ be the corresponding cocartesian fibration in spaces, $X \in \mathrm{Spc}$. We have

$$\mathrm{Map}_{\mathrm{Spc}}(\mathrm{colim}\,\Phi, X) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}^{\mathcal{C}}}(\Phi, \delta(X)) \widetilde{\to} \mathrm{Map}_{0-cocart/\mathcal{C}}(\tilde{\mathcal{C}}, \mathcal{C} \times X)$$

The latter identifies with $\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(\tilde{\mathcal{C}}, X) = \mathrm{Map}_{\mathrm{Spc}}(\mid \tilde{\mathcal{C}} \mid, X)$, see my Section 2.2.88. So, $\mathrm{colim}\,\Phi \widetilde{\to} \mid \tilde{\mathcal{C}} \mid$.

For $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ we have $\mid \mathcal{C}^{op} \mid \widetilde{\to} \mid \mathcal{C} \mid$.

2.2.84. For $\mathcal{C}_i \in 1-\mathcal{C}\mathrm{at}$ we have naturally $\mid \mathcal{C}_1 \times \mathcal{C}_2 \mid \widetilde{\to} \mid \mathcal{C}_1 \mid \times \mid \mathcal{C}_2 \mid$. Indeed, for $A \in \mathrm{Spc}$,

$$\mathrm{Fun}(\mathcal{C}_1 \times \mathcal{C}_2, A) \widetilde{\to} \mathrm{Fun}(\mathcal{C}_1, \mathrm{Fun}(\mathcal{C}_2, A)) \widetilde{\to} \mathrm{Fun}(\mathcal{C}_1, \mathrm{Fun}(\mid \mathcal{C}_2 \mid, A)) \widetilde{\to} \mathrm{Fun}(\mid \mathcal{C}_1 \mid, \mathrm{Fun}(\mid \mathcal{C}_2 \mid, A)) \widetilde{\to} \mathrm{Fun}(\mid \mathcal{C}_1 \mid \times \mid \mathcal{C}_2 \mid, A)$$

2.2.85. As far as I understand, the inclusion $\mathcal{G}\mathrm{rp}(\mathrm{Spc}) \hookrightarrow Mon(\mathrm{Spc})$ admits a left adjoint L, which we think of as inverting all morphisms of a monoid.

Denote by $\triangle_{set}$ the following non-full subcategory in $1-\mathcal{C}\mathrm{at}^{ordn}$, whose objects are indexed by $n \geq 0$. The object $\bar{n}$ corresponding to n is a set $\{0, \dots, n\}$ considered as a category, so there are no nontrivial maps between $i \neq j$ in this category. A morphism $\bar{n} \to \bar{m}$ is an order-preserving map $\{0, \dots, n\} \to \{0, \dots, m\}$. So, as abstract categories, we have an equivalence $\triangle_{set} \widetilde{\to} \triangle$. However, as subcategories in $1-\mathcal{C}\mathrm{at}^{ordn} \subset 1-\mathcal{C}\mathrm{at}$ they are different. The inclusion into $1-\mathcal{C}\mathrm{at}^{ordn}$ gives a natural functor $\triangle_{set} \to \triangle$ sending

$\{0,\dots,n\}$ to $[n]$. So, given a functor $1-\mathcal{C}at\to\mathcal{D}$, we can restrict to to $\triangle$ and further to $\triangle_{set}$.

Lemma 2.2.86. *Let $\mathcal{C}\in 1-\mathcal{C}at$, $c\in\mathcal{C}$. Then* $\mathrm{Map}_C(c,c)=\mathrm{Funct}([1],\mathcal{C})\times_{C\times C}\{c,c\}$ *has a natural structure of a monoid in* Spc.

Proof. Recall that we have a functor $1-\mathcal{C}at^{op}\to 1-\mathcal{C}at$, $D\mapsto\mathrm{Funct}(D,\mathcal{C})$. We want to consider the natural functor $\triangle\to 1-\mathcal{C}at$, $[n]\mapsto[n]\sqcup_{\{0,\dots,n\}}*$. We have $\mathrm{Funct}([n]\sqcup_{\{0,\dots,n\}}*,C)\widetilde{\to}\mathrm{Funct}([n],C)\times_{C^{n+1}}C$. We set $F([n])=\mathrm{Funct}([n],C)\times_{C^{n+1}}C$. The functor $F:\triangle^{op}\to 1-\mathcal{C}at$ is almost the desired one.

To get the correct one, we use a characterization of $1-\mathcal{C}at$ via the complete Segal spaces ([14], ch. 10, 1.2). Consider the category $CSeg(\mathrm{Spc})$ of complete Segal spaces in Spc from *loc.cit.*, it is a subcategory in $\mathrm{Funct}(\triangle^{op},\mathrm{Spc})$. Given $\mathcal{C}\in 1-\mathcal{C}at$, let $\bar{\mathcal{C}}\in\mathrm{Funct}(\triangle^{op},\mathrm{Spc})$ be the corresponding complete Segal space, set $X=\bar{\mathcal{C}}_0=\mathcal{C}^{\mathrm{Spc}}$. Consider now $Y\in\mathrm{Funct}(\triangle^{op},\mathrm{Spc})$ the Cech nerve of $X\to *$. So, we get $Y_1=X\times X$, $Y_n\widetilde{\to}X^{n+1}$. The inclusion $\{0,\dots,n\}\hookrightarrow[n]$ yields a morphism $\bar{\mathcal{C}}_n\to X^{n+1}$. This is in fact a morphism of functors $\bar{\mathcal{C}}\to Y$ in $\mathrm{Funct}(\triangle^{op},\mathrm{Spc})$. Consider $co(*)\in\mathrm{Funct}(\triangle^{op},\mathrm{Spc})$ the constant functor with value $*$. Now $c\in\mathcal{C}$ gives a map $co(*)\to Y$ in $\mathrm{Funct}(\triangle^{op},\mathrm{Spc})$ such that the corresponding map $*\to Y_n=X^{n+1}$ is c^{n+1}. The product $\bar{\mathcal{C}}\times_Y co(*)$ in $\mathrm{Funct}(\triangle^{op},\mathrm{Spc})$ is the desired monoid in Spc. □

Assume $\mathcal{C}\in 1-\mathcal{C}at$ has just one object c. Then we can consider $M=\mathrm{Map}_C(c,c)$ and the corresponding group in spaces $L(M)$. I wonder if $\mid\mathcal{C}\mid\in\mathrm{Spc}$ identifies with $B(L(M))$? Check with the construction of $1-\mathcal{C}at$ via Spc-valued Segal spaces.

Let $1-\mathcal{C}at(*)$ be the ∞-category of pairs $\mathcal{C}\in 1-\mathcal{C}at$, and an isomorphism $\mathcal{C}^{\mathrm{Spc}}\widetilde{\to}*$ in Spc. From ([14], ch. 10, 1.3.4) we get an equivalence $1-\mathcal{C}at(*)\widetilde{\to}Mon(\mathrm{Spc})$.

2.2.87. If I is a set, $\mathcal{C}\in 1-\mathcal{C}at$, $f:I\to\mathcal{C}$ is a functor then the colimit $c=\sqcup_{i\in I}f(i)$ of f is caracterized by $\mathrm{Map}_{\mathcal{C}}(c,x)\widetilde{\to}\prod_{i\in I}\mathrm{Map}_{\mathcal{C}}(f(i),x)$, the product being taken in Spc.

2.2.88. Let $\mathcal{C}\in 1-\mathcal{C}at$, $\alpha:c\to a$ be a map in $\mathcal{C}$. Assume $\mathcal{C}$ admits fibred products. Then we get a functor $\mathcal{C}_{/a}\to\mathcal{C}_{/c}$, $(x\to a)\mapsto x\times_a c\to c$. This is the "composition" $\mathcal{C}/a\to\mathrm{Funct}(\Lambda^2_2,\mathcal{C})\overset{\lim}{\to}\mathcal{C}/c$. How to define it rigourously? Consider the natural map $\mathcal{C}_{/c}\to\mathcal{C}_{/a}$ given by composing with α. Its right adjoint should be the desired functor.

For $\mathcal{C}\in 1-\mathcal{C}at$ the projection $\mathcal{C}_{/c}\to\mathcal{C}$ preserves fibred products. Indeed, $*\overset{c}{\to}\mathcal{C}$ preserves fibred products, because the diagram $*\leftarrow *\to *$ in $1-\mathcal{C}at$ is contractible, now apply my Lemma 2.2.69.

Remark 2.2.89. *i) Let $\mathcal{X}\in 1-\mathcal{C}at$ be presentable, $q:I\to\mathcal{X}$ be a small diagram, $x=\operatorname{colim}q$. Then viewing q as a map $\tilde{q}:I\to\mathcal{X}/x$ we have $\operatorname{colim}\tilde{q}=x$, that is, the object* $\mathrm{id}:x\to x$ *of $\mathcal{X}/x$.*
ii) Assume in addition that colimits in $\mathcal{X}$ are universal. Let $y\to x$ be a map in $\mathcal{X}$. Let $\tilde{\tilde{q}}$ be the composition $I\overset{\tilde{q}}{\to}\mathcal{X}/x\to\mathcal{X}/y$. Then $\operatorname{colim}\tilde{\tilde{q}}=y$ in $\mathcal{X}/y$, so $\underset{i\in I}{\operatorname{colim}}(q(i)\times_x y)\widetilde{\to}y$ in $\mathcal{X}$ also.

Proof. i) The functor $\mathcal{X}/x\to\mathcal{X}$, $(y\to x)\mapsto y$ admits a right adjoint, so preserves colimits. The category $\mathcal{X}/x$ is also presentable, so admits colimits. Let $\tilde{x}\to x$ be the

colimit of $\tilde{q}$. Then $\tilde{x}$ is the colimit of q.
ii) the projection $\mathcal{X}/y \to \mathcal{X}$ preserves colimits. $\square$

Note also that if $\gamma : a' \to a, \beta : a'' \to a$ are maps in $\mathcal{C}$ then we should have canonically

$$\mathrm{Map}_{\mathcal{C}/a}(\gamma, \beta) \widetilde{\to} \mathrm{Funct}(\Delta^2, \mathcal{C}) \times_{\mathrm{Funct}(\Lambda^2_0, \mathcal{C})} \{\gamma, \beta\}$$

For $y \in \mathcal{C}$ and a cartesian square in $\mathcal{C}$

$$\begin{array}{ccc} x' & \to & a' \\ \downarrow & & \downarrow \\ x & \to & a \end{array}$$

the square has to be cartesian in Spc

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(y, x') & \to & \mathrm{Map}_{\mathcal{C}}(y, a') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}}(y, x) & \to & \mathrm{Map}_{\mathcal{C}}(y, a) \end{array}$$

2.2.90. If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is presentable, $x \in \mathcal{C}$ then let $f : I \to \mathcal{C}/x$ be a map in $1-\mathcal{C}\mathrm{at}$, $y \to x$ be a colimit of f. The functor $\mathcal{C}/x \to \mathcal{C}$ preserves colimits, so y is the colimit of the composition $f_0 : I \to \mathcal{C}/x \to \mathcal{C}$. Now if $F : \mathcal{C} \to \mathcal{C}'$ is a colimit preserving functor between presentable categories and $x \in \mathcal{C}$ then the corresponding functor $\mathcal{C}/x \to \mathcal{C}'/F(x)$ is colimit preserving. Indeed, if $f : I \to \mathcal{C}/x$ is a diagram, $y \to x$ is a colimit of f then let $\bar{f} : I^{\triangleright} \to \mathcal{C}$ be the corresponding colimit diagram extending $f_0 : I \to \mathcal{C}$. Then $F\bar{f} : I^{\triangleright} \to \mathcal{C}'$ is a colimit diagram extending Ff_0. Since the functor $\mathcal{C}'/F(y) \to \mathcal{C}/F(x)$ is colimit preserving, our claim follows from Remark 2.2.89. More generally, colimits in the slice diagrams are described in ([27], 1.2.13.8).

If in addition $F : \mathcal{C} \to \mathcal{C}'$ is left exact then for any $x \in \mathcal{C}$ the functor $\mathcal{C}/x \to \mathcal{C}'/F(x)$ is also left exact by ([27], 5.3.2.8).

2.2.91. For the slice categories. The mapping space is described in ([27], 5.5.5.12): Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$. If $f : c \to d, g : c \to e$ are maps in $\mathcal{C}$ then $\mathrm{Map}_{\mathcal{C}_{c/}}(f, g)$ is the fibre of $\mathrm{Map}_{\mathcal{C}}(d, e) \to \mathrm{Map}_{\mathcal{C}}(c, e)$ over g.

Lemma 2.2.92. *Let $X : D \to C$ be a map in $1-\mathcal{C}\mathrm{at}$, $K \in 1-\mathcal{C}\mathrm{at}$. One has canonically*

$$\mathrm{Funct}(K \diamond D, C) \times_{\mathrm{Funct}(D,C)} \{X\} \widetilde{\to} \mathrm{Fun}(K, C_{/X})$$

and $C_{/X} \widetilde{\to} \mathrm{Funct}({}^{\triangleleft}D, C) \times_{\mathrm{Funct}(D,C)} \{X\}$.

Proof. We have in the model independent setting

$$C_{/X} = C \times_{\mathrm{Funct}(D,C)} \mathrm{Funct}([1] \times D, C) \times_{\mathrm{Funct}(D,C)} \{X\} \widetilde{\to} \mathrm{Funct}({}^{\triangleleft}D, C) \times_{\mathrm{Funct}(D,C)} \{X\}$$

This gives

$$\mathrm{Funct}(K, C/X) \widetilde{\to} \mathrm{Funct}(K, C) \times_{\mathrm{Funct}(K \times D,C)} \mathrm{Funct}([1] \times K \times D, C) \times_{\mathrm{Funct}(K \times D,C)} \{X\}$$

Recall that $K \diamond D = K \sqcup_{K \times D \times \{0\}} K \times D \times [1] \sqcup_{K \times D \times \{1\}} D$. This gives

$$\mathrm{Funct}(K \diamond D, C) \widetilde{\to} \mathrm{Funct}(K, C) \times_{\mathrm{Funct}(K \times D,C)} \mathrm{Funct}([1] \times K \times D, C) \times_{\mathrm{Funct}(K \times D,C)} \mathrm{Funct}(D, C)$$

Our claim follows. $\square$

For $Y \in 1-\mathcal{C}\mathrm{at}$ the functor $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}_{Y/}$ given by $X \mapsto X \diamond Y$ commutes with colimits. This is the transitivity of left Kan extensions combined with Lemma 2.2.73. Similarly for the functor $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}_{Y/}$ given by $X \mapsto Y \diamond X$.

By ([27], 1.2.8.1) for $n, m \geq 1$ we have $[n-1] \diamond [m-1] \widetilde{\to} [n+m-1]$. This by the above implies the associativity of the join construction in the model-independent setting: for $X, Y, Z \in 1-\mathcal{C}\mathrm{at}$, $X \diamond (Y \diamond Z) \widetilde{\to} (X \diamond Y) \diamond Z$. In particular, ${}^{\triangleleft}(X \diamond Y) \widetilde{\to} ({}^{\triangleleft}X) \diamond Y$.

The limits in the slice categories are calculated as follows:

Lemma 2.2.93. *Let $X : D \to \mathcal{C}$ be a map in $1-\mathcal{C}\mathrm{at}$, $F : K \to \mathcal{C}_{/X}$ be a diagram. Let $\bar{F} : K \diamond D \to \mathcal{C}$ be the functor given by F in the sense of Lemma 2.2.92. Then $(\mathcal{C}_{/X})_{/F} \widetilde{\to} \mathcal{C}_{/\bar{F}}$. So, $\lim F$ can be identified with $\lim \bar{F}$. More precisely, if ${}^{\triangleleft}(K \diamond D) \to \mathcal{C}$ is a limiting cone for $\bar{F}$ then $({}^{\triangleleft}K) \diamond D \widetilde{\to} {}^{\triangleleft}(K \diamond D) \to \mathcal{C}$ defines a map ${}^{\triangleleft}K \to \mathcal{C}/X$, which is a limiting cone for F.*

Proof. 1) Model-dependent proof. Use the interpetation of join via quasi-categories then it is clear that this is a monoidal operation on simplicial sets. The claim follows easily from the description of the overcategory $\mathcal{C}/X$ via the simplicial set representing the functor $\mathcal{S}\mathrm{et}_{\Delta} \to Sets$, $I \mapsto \mathrm{Hom}_X(I \star D, \mathcal{C})$.

2) Model-independent proof. First, $\bar{F}$ corresponds to F in the sense of Lemma 2.2.92. We get

$$(\mathcal{C}/X)/F \widetilde{\to} \mathrm{Funct}({}^{\triangleleft}K, \mathcal{C}/X) \times_{\mathrm{Funct}(K, \mathcal{C}/X)} \{F\} \widetilde{\to}$$
$$\mathrm{Funct}({}^{\triangleleft}K \diamond D, \mathcal{C}) \times_{\mathrm{Funct}(K \diamond D, \mathcal{C})} \{F\} \widetilde{\to} \mathcal{C}_{/\bar{F}}$$

□

Remark 2.2.94. *Given a diagram $\mathcal{C} \to \mathcal{D} \leftarrow \mathcal{D}'$ in $1-\mathcal{C}\mathrm{at}$ let $\mathcal{C}' = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$, let $q' : K \to \mathcal{C}'$ be a map in $1-\mathcal{C}\mathrm{at}$. Denote $q : K \to \mathcal{C}$, $p' : K \to \mathcal{D}'$, $p : K \to \mathcal{C}$ the composition maps. Then $\mathcal{C}'_{q'/} \widetilde{\to} \mathcal{C}_{q/} \times_{\mathcal{D}_{p/}} \mathcal{D}'_{p'/}$ naturally.*

2.2.95. The parametrized join construction: given $X \to S \leftarrow Y$ in $1-\mathcal{C}\mathrm{at}$, set

$$X \diamond_S Y = X \bigsqcup_{X \times_S Y \times \{0\}} (X \times_S Y \times [1]) \bigsqcup_{X \times_S Y \times \{1\}} Y$$

Given $S \in 1-\mathcal{C}\mathrm{at}$ and a map $p : K \to Y$ in $1-\mathcal{C}\mathrm{at}/S$, the relative undercategory $Y_{p_S/} \in 1-\mathcal{C}\mathrm{at}/S$ could possibly be defined by the property that we have an equivalence functorial in $X \in 1-\mathcal{C}\mathrm{at}/S$

$$\mathrm{Fun}_S(X, Y_{p_S/}) \widetilde{\to} \{p_S\} \times_{\mathrm{Funct}_S(K,Y)} \mathrm{Funct}_S(K \diamond_S X, Y) \tag{2}$$

This is inspired by ([27], 4.2.2.1). However, its existence is not clear! It is a question of presentability. Namely, we ask if the functor

$$(1-\mathcal{C}\mathrm{at}/S)^{op} \to \mathrm{Spc},\ X \to \{p_S\} \times_{\mathrm{Map}_{1-\mathcal{C}\mathrm{at}/S}(K,Y)} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}/S}(K \diamond_S X, Y) \tag{3}$$

is representable. Since $1-\mathcal{C}\mathrm{at}/S$ is presentable, this is equivalent to the fact that this functor preserves limits ([27], 5.5.2.2). The analog of Lemma 2.2.92 here is what?

The parametrized join construction does not seem to preserve colimits in X and Y respectively. The situation becomes better for the following its version. For $S \in 1-\mathcal{C}\mathrm{at}$

consider the functor $1-\mathcal{C}\mathrm{at}\times(1-\mathcal{C}\mathrm{at}/S)\to 1-\mathcal{C}\mathrm{at}$ given by

$$(L, X\to S)\mapsto (L\times S)\diamond_S X\widetilde{\to}(L\times S)\underset{L\times X}{\sqcup} L^{\triangleright}\times X$$

This functor preserves colimits separately in each variable.

Now for a map $p: K=L\times S\to Y$ in $1-\mathcal{C}\mathrm{at}/S$ the functor (3) preserves limits, so is representable (see also Remark 2.2.96 below), the corresponding object of $1-\mathcal{C}\mathrm{at}/S$ is denoted $Y_{pS/}$. The isomorphism (2) then holds for this $Y_{pS/}$, and moreover $Y_{pS/}$ is given by the formula from my Section 3.0.21.

Remark 2.2.96. *Let $Y\to S$ be a map in $1-\mathcal{C}\mathrm{at}$. The functor $(1-\mathcal{C}\mathrm{at}/S)^{op}\to 1-\mathcal{C}\mathrm{at}$, $Z\mapsto \mathrm{Fun}_S(Z,Y)$ preserves limits.*

2.2.97. Let $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$ be presentable, $c\in C$. Recall that $\mathcal{C}/c$ is presentable. We equip it with the cartesian monoidal structure. Then (for any $c\in\mathcal{C}$, $\mathcal{C}/c$ admits inner homs) iff the colimits in $\mathcal{C}$ are universal.

Indeed, given $b,d\in\mathcal{C}/c$ the representability of the functor $(\mathcal{C}/c)^{op}\to\mathrm{Spc}$, $a\mapsto \mathrm{Map}_{\mathcal{C}/c}(a\times_c b, d)$ is equivalent (by [27], 5.5.2.2) to the fact that it preserves limits. However, given a functor $I\to\mathcal{C}/c$, $i\mapsto a_i$, we get

$$\lim_{i\in I^{op}}\mathrm{Map}_{\mathcal{C}/c}(a_i\times_c b, d)\widetilde{\to}\mathrm{Map}_{\mathcal{C}/c}(\mathrm{colim}_{i\in I}(a_i\times_c b), d)$$

The latter identifies with $\mathrm{Map}_{\mathcal{C}/c}((\mathrm{colim}\, a_i)\times_c b, d)$ iff $(\mathrm{colim}\, a_i)\times_c b$ and $\mathrm{colim}_{i\in I}(a_i\times_c b)$ corepresent the same functor. We are done.

For example, the category $1-\mathcal{C}\mathrm{at}/S$ with the cartesian monoidal structure does not admit inner homs.

2.2.98. Let $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$, assume for $i\in\mathbb{N}$ we are given a full subcategory $\mathcal{C}_i\subset\mathcal{C}$ such that if $i\le j$ then $\mathcal{C}_i\subset\mathcal{C}_j$. Assume $\mathcal{C}=\cup_{i\in\mathbb{N}}\mathcal{C}_i$. Then $\mathcal{C}=\mathrm{colim}_{i\in\mathbb{N}}\,\mathcal{C}_i$, the colimit taken in $1-\mathcal{C}\mathrm{at}$. This follows from the description of the maps spaces in $\mathrm{colim}_{i\in\mathbb{N}}\,\mathcal{C}_i$ given in [46].

2.2.99. Nick: if $F,G:\mathcal{C}_1\to\mathcal{C}_2$ are two maps in $1-\mathcal{C}\mathrm{at}$, $f:F\to G$ is a map in $\mathrm{Funct}(\mathcal{C}_1,\mathcal{C}_2)$ then f is invertible iff for any $c\in C_1$, $f(c):F(c)\to G(c)$ is invertible in $\mathcal{C}_2$. (Proof in his email 6.09.2016).

In particular, if X is a space, $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$ then $\mathrm{Funct}(\mathcal{C},X)$ is a space.

Lemma 2.2.100. *Let $Y\in 1-\mathcal{C}\mathrm{at}$, $X\to Y$, $Z\to Y$ be cocartesian fibrarions and $f:X\to Z$ a morphism in $(cocart/Y)_{strict}$. Then f is an equivalence iff for any $y\in Y$ the base change $X_y\to Z_y$ is an equivalence.*

Proof. Assume $f_y:X_y\to Z_y$ is an equivalence for any $y\in Y$. We must show f is an equivalence.

1) Assume this claim is established under additional assumption: $X,Y,Z\in\mathrm{Spc}$. Clearly, f is essentially surjective. Let us show that f is fully faithful. Let $x_1,x_2\in X$ over $y_i=f(x_i)$. Let $\alpha:y_1\to y_2$ be a map in Y. It suffices to show that $\mathrm{Map}_X(x_1,x_2)_\alpha\to\mathrm{Map}_Y(f(x_1),f(x_2))_\alpha$ is an isomorphism in Spc. Here $\mathrm{Map}_X(x_1,x_2)_\alpha$ is the fibre of $\mathrm{Map}_X(x_1,x_2)$ over α.

Pick a cocartesian arrow $\tilde{\alpha}: x_1 \to x_2'$ in X over α. By definition,

$$\mathrm{Map}_X(x_1, x_2)_\alpha \widetilde{\to} \mathrm{Map}_X(x_2', x_2) \widetilde{\to} \mathrm{Map}_{X_{c_2}}(x_2', x_2) \widetilde{\to} \mathrm{Map}_{Z_{c_2}}(f(x_2'), f(x_2))$$

Since $f(\tilde{\alpha}): f(x_1) \to f(x_2')$ is cocartesian over Y, the latter space also identifies with

$$\mathrm{Map}_Z(f(x_2'), f(x_2)) \widetilde{\to} \mathrm{Map}_Z(f(x_1), f(x_2))_\alpha$$

We are reduced to the special case when $X, Y, Z \in \mathrm{Spc}$.

2) Now we assume $X, Y, Z \in \mathrm{Spc}$, so $f: X \to Z$ is a cocartesian fibration in spaces. By Lemma 2.2.101 below, it suffices to prove that for any $z \in Z$, $X_z \widetilde{\to} *$ in Spc. Let $y \in Y$ be the image of z. Then $X_z \widetilde{\to} X_y \times_{Z_y} *$. Since $X_y \widetilde{\to} Z_y$ by assumption, we are done. □

Lemma 2.2.101. *If $f: X \to Z$ is a cocartesian fibration in spaces, for any $z \in Z$, $X_z \widetilde{\to} *$ in* Spc *then f is an equivalence.*

Proof. According to the strengthening, f corresponds to the functor $F: Z \to \mathrm{Spc}$ with $F(z) \widetilde{\to} *$ for any $z \in Z$. □

Example of an application: let $\mathcal{C} \in 1-\mathcal{C}at$ and $a = b \times_d c$ in $\mathcal{C}$. Then the square is cartesian in $1-\mathcal{C}at$

$$\begin{array}{ccc} \mathcal{C}/b & \to & \mathcal{C}/d \\ \uparrow & & \uparrow \\ \mathcal{C}/a & \to & \mathcal{C}/c, \end{array}$$

where each map is given by the composition. For example, the top horizontal map is given by the composition with $b \to d$. Indeed, this is a diagram in $0-Cart/\mathcal{C}$, and for each $r \in \mathcal{C}$ after base change $\{r\} \to \mathcal{C}$ it becomes an equivalence.

2.2.102. If X is a space, consider the usual category $\mathcal{A}$ consisting of objects $0, 1$ and two morphisms $0 \to 1$ and $1 \to 0$ (and the identity morphisms). View $[1]^{op}$ as the category with objects $0, 1$ and one morphism $1 \to 0$. We get a diagram

$$\{1\} \to [1]^{op} \to \mathcal{A} \leftarrow [1] \leftarrow \{0\}$$

Applying $\mathrm{Funct}(., X)$, we get a diagram

$$X \leftarrow \mathrm{Funct}([1], X) \leftarrow \mathrm{Funct}(\mathcal{A}, X) \to \mathrm{Funct}([1]^{op}, X) \to X$$

Since $\mathcal{A} \leftarrow \{0\}$ is an equivalence of usual categories, it is an isomorphism in $1-\mathcal{C}at$, so the induced map $X \leftarrow \mathrm{Funct}(\mathcal{A}, X)$ is an equivalence. From ([27], 2.1.2.10) we see now that all the maps in this diagram are isomorphisms in $1-\mathcal{C}at$. Pick inverse equivalences, then the obtained map $X \to X$ sends x_0 to the end x_1 of "an arrow" $x_0 \to x_1$.

2.2.103. (Nick) Let C be a space. One may construct an equivalence $C \to C^{op}$ as follows. The diagonal morphism $C \to C \times C$ is a cocartesian fibration, so gives a functor $C \to \mathrm{Funct}(C, 1-\mathcal{C}at)$. By construction, we may assume it sends c to the corepresentable functor $c' \mapsto \mathrm{Map}_C(c, c')$. By Yoneda, this defines a functor $C \to C^{op}$, which is an equivalence.

2.2.104. *Cofinality.* Why the definition of cofinality in Dennis and ([27], 4.1.1.1) are the same? A category $\mathcal{C} \in 1-\mathcal{C}at$ is contractible iff there is an isomorphism $\mathrm{Funct}(\mathcal{C}, X) \widetilde{\to} X$ in Spc functorial in $X \in \mathrm{Spc}$. If $\mathcal{C}$ was a simplicial set, this would mean that the map $\mathcal{C} \to *$ in Sets_Δ gives an isomorphism in the homotopy category of spaces $\mathcal{H}$ (notation from [27]), that is, $\mathcal{C}$ is contractible.

Let $f : \mathcal{C} \to \mathcal{D}$ be a map in $1-\mathcal{C}at$. By ([27], 4.1.3.1), f is cofinal iff for any $d \in \mathcal{D}$, $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is contractible. Dennis takes this as a definition of cofinal maps.

Let $F : \mathcal{D} \to \mathcal{C}$ be a map in $1-\mathcal{C}at$, $c \in \mathcal{C}$. Let $h : \mathcal{C} \to \mathrm{Spc}$ be the map corresponding to the cocartesian fibration in spaces $\mathcal{C}_{c/} \to \mathcal{C}$. We get a natural map $\mid \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{c/} \mid \widetilde{\to} \operatorname{colim}(hF) \to \operatorname{colim} h \widetilde{\to} \mid \mathcal{C}_{c/} \mid$.

Lemma 2.2.105. *Let $\mathcal{C} \in 1-\mathcal{C}at$ with an initial object $e \in \mathcal{C}$. Then $\mathcal{C}$ is contractible.*

Proof. Since $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is cofinal, from ([27], 4.1.3.1) we conclude that for any $c \in \mathcal{C}$, $\mathcal{C}_{c/}$ is contractible. If $e \in \mathcal{C}$ is an initial object that $\mathcal{C}_{e/} \to \mathcal{C}$ is an isomorphism in $1-\mathcal{C}at$, so $\mathcal{C}$ is also contractible. □

For example, if $\mathcal{C} \in 1-\mathcal{C}at$ has a final object $c \in \mathcal{C}$ then the map $* \overset{c}{\to} \mathcal{C}$ is cofinal. So, a cofinal map is a generalization of the inclusion of a final object.

I think the **key property** here is ([27], 4.1.1.8): let $v : K' \to K$ be a map in $1-\mathcal{C}at$. Then v is cofinal iff for any $\mathcal{C} \in 1-\mathcal{C}at$ and any map $p : K \to \mathcal{C}$ the induced map $\mathcal{C}_{p/} \to \mathcal{C}_{pv/}$ is an equivalence.

In particular, if $X \in \mathrm{Spc}, x \in X$ then $X_{/x} \widetilde{\to} X_{x/} \widetilde{\to} *$ in Spc. A generalization: if $f : \mathcal{C} \to \mathcal{D}$ is a cartesian fibration in spaces and $c \in \mathcal{C}$ then $\mathcal{C}_{/c} \to \mathcal{D}_{/f(c)}$ is an equivalence in $1-\mathcal{C}at$.

If a map $f : K \to K'$ in $1-\mathcal{C}at$ is cofinal then $\mid K \mid \to \mid K' \mid$ is an isomorphism in Spc (HTT, 4.1.1.3(3)).

Remark. Let $\mathcal{A} \subset \mathcal{B}$ be a full subcategory, $\mathcal{B} \in 1-\mathcal{C}at$. Let $Y' \in \mathcal{P}(\mathcal{A})$, and Y be the LKE of Y' along $\mathcal{A}^{op} \subset \mathcal{B}^{op}$. Then the inclusion $\mathcal{A}_{/Y} \hookrightarrow \mathcal{B}_{/Y}$ is cofinal. Here $\mathcal{B}_{/Y} = \mathcal{B} \times_{\mathcal{P}(\mathcal{B})} \mathcal{P}(\mathcal{B})_{/Y}$.
Proof. For $\alpha \in Y(b)$ with $b \in B$ we check that $\mathcal{A}_{/Y} \times_{\mathcal{B}_{/Y}} (\mathcal{B}_{/Y})_{b/}$ is contractible. Recall that

$$Y(b) \widetilde{\to} \operatorname*{colim}_{a \in (\mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{b/})^{op}} Y'(a),$$

the colimit taken in Spc. Since the colimits in Spc are universal,

$$* \widetilde{\to} Y(b) \times_{Y(b)} \{\alpha\} \widetilde{\to} \operatorname*{colim}_{a \in (\mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{b/})^{op}} Y'(a) \times_{Y(b)} \{\alpha\}$$

The latter colimit is the space obtained from $\mathcal{A}_{/Y} \times_{\mathcal{B}_{/Y}} (\mathcal{B}_{/Y})_{b/}$ by inverting all morphisms by ([14], ch. I.1, 2.1.6). □

2.2.106. Let $L : \mathcal{B} \to \mathcal{B}'$ be left adjoint to $R : \mathcal{B}' \to \mathcal{B}$, maps in $1-\mathcal{C}at$. Then the induces maps $\mid L \mid : \mid \mathcal{B} \mid \to \mid \mathcal{B}' \mid$ and $\mid R \mid : \mid \mathcal{B} \mid \to \mid \mathcal{B}' \mid$ are equivalences in Spc. Indeed, for any space A the induced map $\tilde{R} : \mathrm{Fun}(\mathcal{B}, A) \to \mathrm{Fun}(\mathcal{B}', A)$ is left adjoint to $\tilde{L} : \mathrm{Fun}(\mathcal{B}', A) \to \mathrm{Fun}(\mathcal{B}, A)$, hence $\tilde{L}$ and $\tilde{R}$ are equivalences. Note that $\mathrm{Fun}(\mathcal{B}, A) \widetilde{\to} \mathrm{Fun}(\mid \mathcal{B} \mid, A)$ is a space. The map $\mid L \mid$ is an isomorphism, because for any space A it induces an equivalence $\mathrm{Fun}(\mid \mathcal{B}' \mid, A) \to \mathrm{Fun}(\mid \mathcal{B} \mid, A)$ in Spc.

According to Lurie's terminology, a functor $F : \mathcal{A} \to \mathcal{B}$ in $1-\mathcal{C}at$ should be called a weak homotopy equivalence iff $\mid F \mid : \mid \mathcal{A} \mid \to \mid \mathcal{B} \mid$ is an isomorphism in Spc.

2.2.107. Why the property 2.2.2(iv) is equivalent to $F : \mathcal{D} \to \mathcal{C}$ being cofinal? Here is the proof that 2.2.2(iv) implies that F is cofinal. Take in 2.2.2(iv) $\Phi' : \mathcal{C} \to E$ to be the constant functor $\delta(e)$ with value e, it sends all morphisms to isomorphisms. Let $\Phi : \mathcal{C} \to E$ be any functor. Then $\mathrm{Map}_{\mathrm{Funct}(\mathcal{C},E)}(\Phi, \delta(e)) \widetilde{\to} \mathrm{Funct}_E(\mathcal{C}, E_{/e})$ according to my Section 2.2.78. So, by our assumption the map

$$\mathrm{Funct}_E(\mathcal{C}, E_{/e}) \to \mathrm{Funct}_E(\mathcal{D}, E_{/e})$$

is an isomorphism in Spc. Note that $E_{/e} \to E$ is a right fibration in spaces. Let $\mathcal{X} \to \mathcal{C}$ be any cartesian fibration in spaces. It corresponds to a map $\mathcal{C} \to \mathrm{Spc}^{op}$ such that $(\mathrm{Spc}^{op})_{/*} \times_{\mathrm{Spc}^{op}} \mathcal{C} \to \mathcal{C}$ is isomorphic to $\mathcal{X} \to \mathcal{C}$ in $1-\mathcal{C}at_{/\mathcal{C}}$. The above isomorphism implies that $\mathrm{Funct}_{\mathcal{C}}(\mathcal{C}, \mathcal{X}) \to \mathrm{Funct}_{\mathcal{C}}(\mathcal{D}, \mathcal{X})$ is an isomorphism in Spc. So, F is cofinal according to ([27], def. 4.1.1.1).

Lemma 2.2.108. *(Nick) Let $F : \mathcal{D} \to \mathcal{C}$ be a map in $1-\mathcal{C}at$, $q : \mathcal{C} \to \mid \mathcal{C} \mid$ be the natural map. Then F is cofinal iff for any map $\Phi : \mathcal{C} \to E$ in $1-\mathcal{C}at$ the natural map $LKE_{qF}(\Phi F) \to LKE_q(\Phi)$ is an isomorphism.*

Proof. We use the diagram

$$\begin{array}{ccc} \mathrm{Funct}(\mid \mathcal{C} \mid, E) & \stackrel{q}{\to} & \mathrm{Funct}(\mathcal{C}, E) \\ & \searrow qF & \downarrow F \\ & & \mathrm{Funct}(\mathcal{D}, E) \end{array}$$

The natural map $LKE_F(\Phi F) \to \Phi$ yields the natural map in the lemma. By Yoneda, we may assume $E = \mathrm{Spc}$.

Note that $F : \mathcal{D} \to \mathcal{C}$ is cofinal iff for any map $\tau : \mathcal{C} \to \mathrm{Spc}$ the natural map $\mathrm{colim}\, \tau F \to \mathrm{colim}\, \tau$ is an isomorphism. In other words, for any cocartesian fibration in spaces $\mathcal{C}' \to \mathcal{C}$ the natural map $\mid \mathcal{C}' \times_{\mathcal{C}} \mathcal{D} \mid \to \mid \mathcal{C}' \mid$ is an equivalence in Spc. Our claim is reduced to Lemma 2.2.109 below. Indeed, let Φ correspond to a cocartesian fibration in spaces $\mathcal{C}' \to \mathcal{C}$, let $\mathcal{D}' = \mathcal{C}' \times_{\mathcal{C}} \mathcal{D}$. Then $LKE_q(\Phi)$ is the cocartesian fibration in spaces $\mid \mathcal{C}' \mid \to \mid \mathcal{C} \mid$, and $LKE_{qF}(\Phi F)$ is the cocartesian fibration in spaces $\mid \mathcal{D}' \mid \to \mid \mathcal{C} \mid$. □

Lemma 2.2.109. *Let $\Phi : \mathcal{D} \to \mathrm{Spc}$ be a functor given by a cocartesian fibration in spaces $X \to \mathcal{D}$. Let $F : \mathcal{D} \to C$ be a functor, where C is a space. Then the cocartesian fibration corresponding to $LKE_F(\Phi) : C \to \mathrm{Spc}$ is given by $\mid X \mid \widetilde{\to} \mathrm{colim}_{\mathcal{D}} \Phi \to \mathrm{colim}_{\mathcal{D}} * = \mid \mathcal{D} \mid \to C$.*

Proof. Let $Y \to C$ be a cocartesian fibration in spaces. Its image under $\mathrm{Funct}(C, \mathrm{Spc}) \to \mathrm{Funct}(\mathcal{D}, \mathrm{Spc})$ is $Y \times_C \mathcal{D} \to \mathcal{D}$. Note that Y is a space, because composition of left fibrations is a left fibration. Now

$$\mathrm{Map}_{0-cocart/\mathcal{D}}(X, Y \times_C \mathcal{D}) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}at_{/C}}(X, Y) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}_{/C}}(\mid X \mid, Y)$$

Since $0-cocart_{/C} = \mathrm{Spc}_{/C}$, we are done. □

Here is the proof that that $F : \mathcal{D} \to \mathcal{C}$ cofinal implies 2.2.2(iv). First, we may assume E has all colimits (by embedding it into a cocomplete category). We may also assume $\Phi' = \Psi q$, where $\Psi : \mid \mathcal{C} \mid \to E$ is some functor, $q : \mathcal{C} \to \mid \mathcal{C} \mid$ is the natural map.

By adjointness for $\mathrm{Funct}(\mid \mathcal{C} \mid, E) \xrightarrow{q} \mathrm{Funct}(\mathcal{C}, E)$, we get

$$\mathrm{Map}_{\mathrm{Funct}(\mathcal{C},E)}(\Phi, \Psi q) \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(|\mathcal{C}|,E)}(LKE_q(\Phi), \Psi)$$

By adjointness for $\mathrm{Funct}(\mid \mathcal{C} \mid, E) \xrightarrow{qF} \mathrm{Funct}(\mathcal{D}, E)$ we get

$$\mathrm{Map}_{\mathrm{Funct}(\mathcal{D},E)}(\Phi F, \Psi qF) \widetilde{\to} \mathrm{Map}_{\mathrm{Funct}(|\mathcal{C}|,E)}(LKE_{qF}(\Phi F), \Psi)$$

By Lemma 2.2.108, $LKE_{qF}(\Phi F) \to LKE_q(\Phi)$ is an isomorphism, and our claim follows.

2.2.110. If X is a space then $0 - cocart_{/X} = \mathrm{Spc}_{/X}$. In other words, if $f : Y \to X$ is a morphism in Spc then f is cocartesian fibration.

2.2.111. If $\mathcal{C} \in 1 - \mathcal{C}at$ then the natural functor $\mathcal{C} \to \mid \mathcal{C} \mid$ is cofinal. Indeed, for any cartesian fibration in spaces $X \to \mid \mathcal{C} \mid$, we have $X \in \mathrm{Spc}$. By ([27], 4.1.1.1) it suffices to show that the natural map $\mathrm{Map}_{|\mathcal{C}|}(\mid \mathcal{C} \mid, X) \to \mathrm{Map}_{|\mathcal{C}|}(\mathcal{C}, X)$ is an isomorphism in Spc. This is nothing but the map

$$\mathrm{Funct}(\mid \mathcal{C} \mid, X) \times_{\mathrm{Funct}(|\mathcal{C}|,|\mathcal{C}|)} \{\mathrm{id}\} \to \mathrm{Funct}(\mathcal{C}, X) \times_{\mathrm{Funct}(\mathcal{C},|\mathcal{C}|)} \{\alpha\}$$

Here $\alpha : \mathcal{C} \to \mid \mathcal{C} \mid$ is the canonical map. Since X is a space, $\mathrm{Funct}(\mathcal{C}, X) \widetilde{\to} \mathrm{Funct}(\mid \mathcal{C} \mid, X)$ canonically, and both are spaces. We are done.

2.2.112. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor, assume it admits a left adjoint F^L. Pick $c \in C$, let $\alpha : c \to FF^L(c)$ be the canonical map. We get a diagram (of cocartesian fibration in spaces over $\mathcal{D}$)

$$\begin{array}{ccc} \mathcal{D}_{F^L(c)/} & \to & \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{c/} \\ & \searrow & \downarrow \\ & & \mathcal{D} \end{array}$$

The horizontal map sends $F^L(c) \to d$ to the composition $c \xrightarrow{\alpha} FF^L(c) \to F(d)$. The horizontal maps exists even on the level of simplicial sets, it comes as the composition $\mathcal{D}_{F^L(c)/} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{FF^L(c)/} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{c/}$.

The fact that F^L is left adjoint to F implies that for each $d \in \mathcal{D}$, the fibre of the above horizontal map over d is an isomorphism

$$\mathrm{Map}_{\mathcal{D}}(F^L(c), d) \widetilde{\to} \mathrm{Map}_{\mathcal{C}}(c, F(d))$$

From my Section 2.2.99 we conclude that $\mathcal{D}_{F^L(c)/} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{c/}$ is an equivalence, so $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{c/}$ has an initial object.

2.2.113. Let $F : \mathcal{D} \to \mathcal{C}$ be a cocartesian fibration. Then for any $c \in \mathcal{C}$, $\alpha \in \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/c}$ given by $d \in \mathcal{D}, \alpha : F(d) \to c$, the category $\mathcal{D}_{d/} \times_{\mathcal{C}_{F(d)/}} \{\alpha\}$ admits an initial object. Such initial objects are precisely a F-cocartesian morphisms $d \to d'$ over $\alpha : F(d) \to c$.

We have an equivalence

$$\mathcal{D}_c \times_{(\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/c})} (\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/c})_{\alpha/} \widetilde{\to} \mathcal{D}_{d/} \times_{\mathcal{C}_{F(d)/}} \{\alpha\}$$

So, if F is cocartesian then $\mathcal{D}_c \to \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/c}$ is cofinal, as is claimed in 2.2.4.

If $F : \mathcal{D} \to \mathcal{C}$ is cocartesian then any $d \in \mathcal{D}$ gives rise to a functor $\mathcal{L} : \mathcal{C}_{F(d)/} \to \mathcal{D}_{d/}$ sending $\alpha : F(d) \to c$ to a cocartesian arrow $d \to d'$ over α. It has the property that it sends any arrow $\alpha : F(d) \to c$ to an initial object of $\mathcal{D}_{d/} \times_{\mathcal{C}_{F(d)/}} \{\alpha\}$. The existence of $\mathcal{L}$ is explained below.

Let $\bar{\beta} : d \to d_1$ be a map in $\mathcal{D}$. Recall that $\bar{\beta}$ is cocartesian iff

$$\xi : D_{\bar{\beta}/} \to D_{d/} \times_{C_{F(d)/}} C_{F(\bar{\beta})/}$$

is an equivalence. Given an object $\alpha : d \to d'$ of $D_{d/}$ the fibre of ξ over α becomes a morphism in Spc

$$\mathrm{Map}_{\mathcal{D}_{d/}}(\bar{\beta}, \alpha) \to \mathrm{Map}_{\mathcal{C}_{F(d)/}}(F(\bar{\beta}), F(\alpha))$$

So, $\bar{\beta}$ is cocartesian iff the latter map is an isomorphism for any α.

We have the functor $\mathcal{F} : \mathcal{D}_{d/} \to \mathcal{C}_{F(d)/}$ sending $\alpha : d \to d'$ to $F(\alpha) : F(d) \to F(d')$. Note that $\mathcal{F}$ is a cocartesian fibration by ([27], 2.4.3.1). The desired functor $\mathcal{L}$ is its left adjoint. Indeed, for any $\beta \in \mathcal{C}_{F(d)/}$ the category

$$D_{d/} \times_{C_{F(d)/}} C_{\beta/} \widetilde{\to} D_{d/} \times_{C_{F(d)/}} (C_{F(d)/})_{\beta/}$$

has an initial object. So, by Corollary 2.2.116 below, $\mathcal{F}$ has a left adjoint $\mathcal{L}$. If $\bar{\beta} : d \to d_1$ is cocartesian over $\mathcal{C}$ then $\mathcal{L}$ sends $F(\bar{\beta})$ to $\bar{\beta}$. The natural map $\mathrm{id} \to \mathcal{F}\mathcal{L}$ an isomorphism of functors, so $\mathcal{L}$ is fully faithful.

If, in addition, $F : \mathcal{D} \to \mathcal{C}$ is a cocartesian fibration in spaces then $\mathcal{F} : \mathcal{D}_{d/} \to \mathcal{C}_{F(d)/}$ is an equivalence. Indeed, in this case any arrow in $\mathcal{D}$ is F-cocartesian, so $\mathcal{L}\mathcal{F} \widetilde{\to} \mathrm{id}$.

Lemma 2.2.114. *(Nick) Let $C \in 1-\mathcal{C}\mathrm{at}$, let $F : X \to C$ be a coCartesian fibration in spaces. The corresponding functor $C \to \mathrm{Spc}$ is corepresentable iff X has an initial object.*

Proof. If X is corepresented by $c \in C$, we have that $X \widetilde{\to} C_{c/}$ over C, and it has an initial object. Now assume X has an initial object $x \in X$. Then we get a functor $\mathcal{L} : C_{F(x)/} \to X$ sending $\alpha : F(x) \to c'$ to the end x' of a cocartesian arrow $x \to x'$ over α, see my Sect 2.2.113. We will show this is an isomorphism. Our $\mathcal{L}$ is left adjoint to $\mathcal{R} : X \to C_{F(x)/}$, $y \mapsto ((F(x) \to y)$. For $y \in X$ let $\beta : x \to y$ be a map in X. By ([27], 2.4.2.4), β is F-cocartesian. So, the natural arrow $\mathcal{L}\mathcal{R} \to \mathrm{id}$ is an isomorphism. We know already that $\mathrm{id} \to \mathcal{R}\mathcal{L}$ is an equivalence. We are done. □

The dual claim is ([27], 4.4.4.5):

Lemma 2.2.115. *If $f : \tilde{\mathcal{C}} \to \mathcal{C}$ is a cartesian fibration in spaces, let $\tilde{c} \in \tilde{\mathcal{C}}$ and $c = f(\tilde{c})$. Let $F : \mathcal{C}^{op} \to \mathrm{Spc}$ be the functor corresponding to f. Note that $\tilde{C} \widetilde{\to} \mathcal{C} \times_{\mathcal{P}(\mathcal{C})} \mathcal{P}(\mathcal{C})_{/F}$. Then $(C, \tilde{c} \in F(c))$ represents F iff $\tilde{c}$ is a final object of $\tilde{C}$.*

Corollary 2.2.116. *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor, $c \in \mathcal{C}$.*
i) The partially defined left adjoint F^L to F is defined at c iff $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{c/}$ has an initial object.
ii) The partially defined right adjoint F^R to F is defined at c iff $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/c}$ has a final object.

Proof. i) It is defined at c iff the functor $\mathcal{D} \to \mathrm{Spc}$, $d \mapsto \mathrm{Map}_{\mathcal{C}}(c, F(d))$ is corepresentable. This functor is given by the cocartesian fibration in spaces $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{c/} \to \mathcal{D}$. □

2.2.117. As in ([27], 3.3.2) let $q^0 : \mathcal{Z}^0 \to \mathrm{Spc}^{op}$ be the universal cartesian fibration in spaces, it corresponds to $\mathrm{id} : \mathrm{Spc} \to \mathrm{Spc}$. Let $*_{\mathcal{Z}} \in \mathcal{Z}^0$ be the unique object of $\mathrm{id}(*) = *$. Recall that $*_{\mathcal{Z}}$ is final in $\mathcal{Z}^0$ ([27], 3.3.2.6). The functor $\mathrm{id} : (\mathrm{Spc}^{op})^{op} \to \mathrm{Spc}$ is represented by $* \in \mathrm{Spc}^{op}$. Indeed, for $X \in \mathrm{Spc}$, $\mathrm{Map}_{\mathrm{Spc}^{op}}(X, *) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(*, X)^{op} \widetilde{\to} X^{op} \widetilde{\to} X$ functorially.

2.2.118. For 2.4.1. Let $C_I : I \to 1-\mathcal{C}\mathrm{at}$ be a functor, $\tilde{C} \to I$ the corresponding cocartesian fibration. Assume that for any $\alpha : i_0 \to i_1$ in I, the corresponding functor $F_\alpha : C_{i_0} \to C_{i_1}$ admits a right adjoint. Why $\tilde{C} \to I$ is a cartesian fibration? By assumption, it is a locally cartesian fibration (after the base change $\alpha : [1] \to I$, it becomes bi-cartesian, now apply [27], 2.4.1.12). Now from ([27], 5.2.2.6) we see that a composition of two locally cartesian arrows in $\tilde{C}$ is a locally cartesian arrow. The claim follows now from ([27], 2.4.2.8).

The following is useful for future applications.

Remark 2.2.119. *Let $f : X \to S$ be a cartesian fibration in $1-\mathcal{C}\mathrm{at}$, $f' : X' \to S'$ be obtained by base change $S' \to S$ in $1-\mathcal{C}\mathrm{at}$. Then an arrow h in X' is f'-cartesian iff its image in X is f-cartesian (combine* [27]*, 2.4.1.3, 2.4.1.12, 2.4.2.8).*

([27], 2.4.2.13) implies: if $f : X \to S$ is a cartesian fibration then an arrow h in X is f-cartesian iff it is locally f-cartesian.

2.2.120. Example of passing to right adjoints: The map $\xi : \mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}^{\{1\}}$ is a cocartesian fibration always. Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit fibred products. Consider the functor $\mathcal{C} \to 1-\mathcal{C}\mathrm{at}$ sending $c \in \mathcal{C}$ to $\mathcal{C}_{/c}$ and $\alpha : c_1 \to c_2$ to the functor $\alpha_! : \mathcal{C}_{/c_1} \to \mathcal{C}_{/c_2}$ given by the composition with α. It exists according to my Section 2.2.24. The functor $\alpha_!$ has a right adjoint $\alpha^! : \mathcal{C}_{/c_2} \to \mathcal{C}_{/c_1}$, $x \mapsto x \times_{c_2} c_1$. By ([14], 2.4.1), we may pass to right adjoints and get a functor $\mathcal{C}^{op} \to 1-\mathcal{C}\mathrm{at}$, $c \mapsto \mathcal{C}_{/c}$. It sends $\alpha : c_1 \to c_2$ to the pull-back functor $\alpha^! : \mathcal{C}_{/c_2} \to \mathcal{C}_{/c_1}$. We have just proved that under our assumption ξ is bicartesian fibration.

2.2.121. 2.5.7 can be explained as follows. Write $\mathcal{P}\mathrm{r}^L$, $\mathcal{P}\mathrm{r}^R$ for the categories defined in ([27], 5.5.3.1). Consider the canonical inclusion $\mathcal{F} : \mathcal{P}\mathrm{r}^L \subset 1-\mathcal{C}\mathrm{at}$. Applying ([14], 2.4), we may pass to the write adjoints $\mathcal{F}^R : (\mathcal{P}\mathrm{r}^L)^{op} \to 1-\mathcal{C}\mathrm{at}$. Then ([28], 5.5.3.3) means that the functor $\mathcal{F}^R$ factors uniquely through the 1-full subcategory $\mathcal{P}\mathrm{r}^R \subset 1-\mathcal{C}\mathrm{at}$, and the resulting functor $(\mathcal{P}\mathrm{r}^L)^{op} \to \mathcal{P}\mathrm{r}^R$ is an equivalence. So, if $I^{\triangleright} \to \mathcal{P}\mathrm{r}^L$ is a colimit diagram then $(I^{\triangleright})^{op} \to (\mathcal{P}\mathrm{r}^L)^{op} \widetilde{\to} \mathcal{P}\mathrm{r}^R$ is a limit diagram. Besides, $\mathcal{P}\mathrm{r}^R \subset 1-\mathcal{C}\mathrm{at}$ preserves small limits.

Sam Raskin claims $\mathcal{P}\mathrm{r}^L$ is not presentable, there is a mathoverflow discussion of this [42].

Let $I \to \mathcal{P}\mathrm{r}^L$ be a diagram, $i \mapsto C_i$, let $C = \mathrm{colim}_{i \in I} C_i$ in $\mathcal{P}\mathrm{r}^L$. For $i \in I$ let $ins_i : C_i \to C$ be the natural functor, $ev_i : C \to C_i$ its right adjoint, which is the projection $ev_i : \lim_{j \in I^{op}} C_j \to C_i$. For $c \in C$ the natural map $\mathrm{colim}_{i \in I}\, ins_i ev_i(c)$ is an isomorphism (same proof as in my Section 9.2.6).

2.2.122. About the last claim before 2.5.8. I think in general, $\lim_{I^{op}} C_i$ can not be calculated in $1-\mathcal{C}\mathrm{at}_{Prs}$, because $C^R_{I^{op}} : I^{op} \to 1-\mathcal{C}\mathrm{at}$ does not factor through $1-\mathcal{C}\mathrm{at}_{Prs}$. So, in (2.6) the limit in the RHS is calculated in $1-\mathcal{C}\mathrm{at}$.

About 2.6.1, there $I \to 1-\mathcal{C}\mathrm{at}$ should be actually a functor $I \to 1-\mathcal{C}\mathrm{at}_{Prs}$, otherwise $ev_i : C \to C_i$ for $i \in I$ are not defined.

2.3. For 2.6.2. I think there the assumption for Lemma 2.6.2 is the following. For a map $\alpha : i \to j$ in I, we have the right adjoint $\alpha^! : C_j \to C_i$ to the functor $\alpha_! : C_i \to C_j$ given by $C_I : I \to 1-\mathcal{C}\mathrm{at}$. Then they assume that $\alpha^!(\lim_{a\in A} c^a_j) \to \lim_{a\in A} c^a_i$ is an isomorphism. Then they claim their lemma 2.6.2.

2.4. For 2.6.3. For $i \to j$ in I we have an isomorphism $\Phi_j F^D_{ij} \to F^C_{ij}\Phi_i$, it gives $F^D_{ij} \to \Phi^R_j F^C_{ij}\Phi_i$. Composing with Φ^R_i, we get

$$F^D_{ij}\Phi^R_i \to \Phi^R_j F^C_{ij}\Phi_i\Phi^R_i \to \Phi^R_j F^C_{ij}$$

We used in the latter map the natural transformation $\Phi_i\Phi^R_i \to \mathrm{id}$.

Lemma 2.4.1. *Let $I \in 1-\mathcal{C}\mathrm{at}$, $I \times [1] \to 1-\mathcal{C}\mathrm{at}$ be the diagram sending i to $f_i : C_i \to D_i$. For $i \to j$ in I let $F^C_{ij} : C_i \to C_j$, $F^D_{ij} : D_i \to D_j$ be the corresponding transition functors. Assume each f_i has a left adjoint $g_i : D_i \to C_i$, and the natural transformation $g_j F^D_{ij} \to F^C_{ij} g_i$ is an isomorphism. Then $f := \lim f_i : C = \lim C_i \to D = \lim D_i$ admits a left adjoint $g : D \to C$, and for any $i \in I$ the natural transformation $g_i ev^D_i \to ev^C_i g$ is an isomorphism.*

Proof. Consider the composition $I \to 1-\mathcal{C}\mathrm{at} \stackrel{op}{\to} 1-\mathcal{C}\mathrm{at}$. This is the diagram sending i to $f^{op}_i : C^{op}_i \to D^{op}_i$. Then each f^{op}_i has a right adjoint g^{op}_i, and the natural transformation $F^{C^{op}}_{ij} g^{op}_i \to g^{op}_j F^{D^{op}}_{ij}$ is an isomorphism. Our claim follows so from (ch. 1, 2.6.4). □

2.4.2. The description of $1-\mathcal{C}\mathrm{at}$ via complete Segal spaces gives the following. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor in $1-\mathcal{C}\mathrm{at}$. To show that it is an equivalence, it suffices to show that for any $n \geq 0$ it induces an isomorphism of spaces

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \mathcal{C}) \to \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \mathcal{D})$$

2.4.3. The alternative join construction from ([27], 4.2.1.1) makes sense in model independent framework. So,

$$X \diamond Y = X \underset{X\times Y\times\{0\}}{\sqcup} (X \times Y \times [1]) \underset{X\times Y\times\{1\}}{\sqcup} Y$$

It is equivalent to the usual join in $1-\mathcal{C}\mathrm{at}$, see ([27], 4.2.1.2). In particular, $X^{\triangleright} = (X \times [1]) \underset{X\times\{1\}}{\sqcup} *$.

The *relative undercategory construction* from ([27], 4.2.2.1) could maybe be defined as follows (I am not sure, a good definition is given in my Section 2.2.95). Let $S \in 1-\mathcal{C}\mathrm{at}$, $I, C \in 1-\mathcal{C}\mathrm{at}/S$, let $p_S : I \to C$ be a morphism of $1-\mathcal{C}\mathrm{at}/S$. Then $C_{p_S/}$ should be defined as

$$\{p_S\} \times_{\mathrm{Funct}_S(I,C)} \mathrm{Funct}([1], \mathrm{Funct}_S(I,C)) \times_{\mathrm{Funct}_S(I,C)} \mathrm{Funct}_S(S,C)$$

2.4.4. Let κ be a regular cardinal, ω the first infinite cardinal. As in ([27], 5.3.1.7), $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is κ-filtered iff for any κ-small $K \in 1-\mathcal{C}\mathrm{at}$, every functor $K \to \mathcal{C}$ can be extended to a functor $K^{\triangleright} \to \mathcal{C}$. Say that $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is filtered iff it is ω-filtered.

For example, $\boldsymbol{\Delta}^{op}$ is not filtered.

If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ has a final object then $\mathcal{C}$ is κ-filtered for every regular cardinal κ ([27], 5.3.1.15).

Definition 2.4.5. *Let $f : \mathcal{A} \to \mathcal{B}$ be a map in $1-\mathcal{C}\mathrm{at}$. As in ([27], 5.3.2.1), say that f is right exact iff for any cartesian fibration in spaces $\mathcal{B}' \to \mathcal{B}$, where $\mathcal{B}'$ is filtered, $\mathcal{A} \times_{\mathcal{B}} \mathcal{B}'$ is also filtered. If in addition $\mathcal{A}$ admits all finite colimits then this property is equivalent to requiring that F preserves finite colimits ([27], 5.3.2.3).*

Claim ([27], 5.3.3.3): Let $I \in 1-\mathcal{C}\mathrm{at}$, κ be a regular cardinal. Then I is κ-filtered iff the functor $colim : \mathrm{Funct}(I, \mathrm{Spc}) \to \mathrm{Spc}$ preserves κ-small limits.

It does not seem true in general that filtered colimits are left exact. In ([27], 7.3.4.2) Lurie considers a class of presentable categories with this property. For example, in an ∞-topos this holds ([27], 7.3.4.7).

Definition 2.4.6. *([27], 5.3.4.5). 1) Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit filtered colimits. A functor $f : \mathcal{C} \to \mathcal{D}$ in $1-\mathcal{C}\mathrm{at}$ is continuous iff it preserves filtered colimits. Let $c \in \mathcal{C}$. Then c is called compact iff the functor $\mathcal{C} \to \mathrm{Spc}$, $y \mapsto \mathrm{Map}_{\mathcal{C}}(c, y)$ is continuous.*
2) Let κ be a regular cardinal, let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit small κ-filtered colimits. A functor $f : \mathcal{C} \to \mathcal{D}$ is κ-continuous iff it preserves κ-filtered colimits. In addition, $c \in \mathcal{C}$ is κ-compact if the functor $\mathcal{C} \to \hat{\mathrm{Spc}}$, $z \mapsto \mathrm{Map}_{\mathcal{C}}(c, z)$ is κ-continuous. Here $\hat{\mathrm{Spc}}$ is the ∞-category of not necessarily small spaces.

([27], 5.3.4.12) has a model independent meaning I think: let $\dots C_2 \overset{f_2}{\to} C_1 \overset{f_1}{\to} C_0$ be a tower of ∞-categories. Assume each C_i admits small κ-filtered colimits, and each f_i is κ-continuous. Let $C = \lim_i C_i$ in $1-\mathcal{C}\mathrm{at}$. Then C admits small κ-filtered colimits, and each projection $C \to C_i$ is κ-continuous. (Lurie's assumption that each f_i is a categorical fibration is not needed). Assume κ uncountable. Then if $c \in C$ has κ-compact image in each C_n for $n \geq 0$ then c is κ-compact. (All this follows from my Lemma 2.2.69).

Remark 2.4.7. *The following is also obtained from my Lemma 2.2.69 and (HTT, 5.3.4.7). Let κ be a regular cardinal, I a κ-small ∞-category. Let $f : I \to 1-\mathcal{C}\mathrm{at}$ be a diagram such that for any $i \in I$, $\mathcal{C}_i = f(i)$ admits κ-filtered colimits, and for $i \to j$ in I, $\mathcal{C}_i \to \mathcal{C}_j$ is κ-continuous. Let $\mathcal{C} = \lim_I f$. Then $\mathcal{C}$ admits κ-filtered colimits. If $c \in \mathcal{C}$ is such that for any $i \in I$ its projection $c_i \in (\mathcal{C}_i)^{\kappa}$ then $c \in \mathcal{C}^{\kappa}$.*

Definition 2.4.8. *([27], 5.1.5.7). Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit all small colimits. Let A be a collection of objects of $\mathcal{C}$. Then A generates $\mathcal{C}$ under colimits if the following holds: for any full subcategory $\mathcal{C}' \subset \mathcal{C}$ such that $A \subset \mathcal{C}'$, if $\mathcal{C}'$ is stable under colimits then $\mathcal{C} = \mathcal{C}'$. A functor $f : S \to \mathcal{C}$ generates $\mathcal{C}$ under colimits if the image $f(S)$ generates $\mathcal{C}$ under colimits.*

2.4.9. Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit small colimits. In ([27], 5.1.6.2) Lurie defines a notion of a completely compact object of $\mathcal{C}$. Let us show that the only completely compact object of Spc is $*$. If $X \in \mathrm{Spc}$ is completely compact then for any $Y \in \mathrm{Spc}$ we have the following. Let $F : Y \to \mathrm{Spc}$ be the constant functor with value $*$ then $Y \widetilde{\to} \mathrm{colim}_Y F$, so $\mathrm{Funct}(X, Y) \widetilde{\to} \mathrm{colim}_{y \in Y} \mathrm{Funct}(x, F(y)) \widetilde{\to} Y$. So, $X \widetilde{\to} \mid X \mid \widetilde{\to} *$.

2.4.10. Let κ be a regular cardinal. For $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ the category $\mathrm{Ind}_\kappa(\mathcal{C})$ of ind-objects of $\mathcal{C}$ is defined in ([27], 5.3.5.1). This is the full subcategory of those $F \in \mathcal{P}(\mathcal{C}) = \mathrm{Funct}(\mathcal{C}^{op}, \mathrm{Spc})$ such that for the corresponding cartesian fibration in spaces $\tilde{\mathcal{C}} \to \mathcal{C}$, $\tilde{\mathcal{C}}$ is κ-filtered.

The key thing about ind-objects seems to be ([27], 5.3.5.10).

Important special case: Let κ be a regular cardinal, let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit κ-small colimits. Then the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by functors $F : \mathcal{C}^{op} \to \mathrm{Spc}$ preserving κ-small limits coincides with $\mathrm{Ind}_\kappa(\mathcal{C})$ ([27], 5.3.5.4).

A related claim in ([27], 5.3.5.14), its proof is badly explained in my opinion. It can be reformulated as the following improvement of ([27], 5.3.5.4):

Lemma 2.4.11. *Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ be small, $F : \mathcal{C}^{op} \to \mathrm{Spc}$ be an object of $\mathrm{Ind}_\kappa(\mathcal{C})$. Then*
1) F preserves all κ-small limits that exist in $\mathcal{C}^{op}$.
2) $j : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ preserves all κ-small colimits which exist in $\mathcal{C}$.

Proof. 1) Let $K \in 1-\mathcal{C}\mathrm{at}$ be κ-small, $K \to \mathcal{C}$ be a diagram $k \mapsto c_k$ having a colimit c in $\mathcal{C}$. So, $c = \lim_{k \in K^{op}} \; c_k$ in $\mathcal{C}^{op}$. Using ([27], 5.3.5.4), pick a small κ-filtered $\mathcal{J} \in 1-\mathcal{C}\mathrm{at}$ and a diagram $p : \mathcal{J} \to \mathcal{C}$ such that F is the colimit of $\mathcal{J} \xrightarrow{p} \mathcal{C} \to \mathcal{P}(\mathcal{C})$, $j \mapsto F_j \in \mathcal{P}(\mathcal{C})$. Each F_j preserves all κ-small limits that exist in $\mathcal{C}^{op}$. We have to establish an isomorphism $F(c) \widetilde{\to} \lim_{k \in K^{op}} F(c_k)$, here $F(c) \widetilde{\to} \mathrm{colim}_{j \in J} F_j(c)$ and $F(c_k) \widetilde{\to} \mathrm{colim}_{j \in J} F_j(c_k)$. So, we are looking for an isomorphism

$$\operatorname*{colim}_{j \in \mathcal{J}} \lim_{j \in K^{op}} F_j(c_k) \widetilde{\to} \lim_{j \in K^{op}} \operatorname*{colim}_{j \in \mathcal{J}} F_j(c_k)$$

It follows from ([27], 5.3.3.3).
2) Let $\bar{p} : K^{\triangleright} \to \mathcal{C}$ be a colimit diagram, where $K \in 1-\mathcal{C}\mathrm{at}$ be κ-small. It suffices to show that for any $C \in \mathrm{Ind}_\kappa(\mathcal{C})$, the composition ${}^{\triangleleft}(K^{op}) \xrightarrow{\bar{p}} \mathcal{C}^{op} \xrightarrow{j} (\mathrm{Ind}_\kappa(\mathcal{C}))^{op} \xrightarrow{F_C} \mathrm{Spc}$ is a limit diagram, where F_C is the functor represented by C. But $F_C \circ j$ identifies with C, and C preserves all κ-small limits which exist in $\mathcal{C}^{op}$. $\square$

2.4.12. If $\kappa \leq \tau$ are regular cardinals, $\mathcal{A}$ admits small κ-filtered colimits, $f : \mathcal{A} \to \mathcal{B}$ is κ-continuous then f is also τ-continuous. Note that each τ-filtered category is also κ-filtered. Roughly, we should think that all reasonable functors are τ-continuous for τ large enough I think ([27], 5.4.2.5).

Strange question: if $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, is there always a regular cardinal κ such that $\mathcal{C}$ admits κ-filtered colimits? It is adressed in ([27], 5.4.3): if $\mathcal{C}$ is small and idempotent complete then $\mathcal{C}$ is accessible.

If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admits κ-filtered colimits then it admits τ-filtered colimits for any $\tau \geq \kappa$, and $\mathcal{C}^\kappa \subset \mathcal{C}^\tau$.

I think the definition of Ind_κ in families from ([27], 5.4.2.18) extends that of ([27], 5.3.5.1).

(HTT, 5.4.7.7): Let $G : C \to C'$ be a functor between accessible ∞-categories. If G admits a right or a left adjoint, then G is accessible.

2.4.13. Any presentable category $\mathcal{C}$ is cotensored over Spc ([27], 5.5.2.6). For any $K \in \mathrm{Spc}, X \in \mathcal{C}$ there is $X^K \in \mathcal{C}$ and a collection of natural isomorphisms for $Y \in \mathcal{C}$

$$\mathrm{Map}_{\mathcal{C}}(Y, X^K) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(K, \mathrm{Map}_{\mathcal{C}}(Y, X))$$

A presentable $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admits all (small) limits and colimits ([27], 5.5.2.4).

2.4.14. *Description of accessible localizations.* If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, S is a collection of morphisms in $\mathcal{C}$, Lurie says that $z \in \mathcal{C}$ is S-local if for any $s : x \to y$ in S, $\mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Map}_{\mathcal{C}}(x, z)$ is an isomorphism in Spc ([27], 5.5.4.1).

If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is presentable, S is a small set of morphisms in $\mathcal{C}$ the full subcategory $S^{-1}\mathcal{C} \subset \mathcal{C}$ is defined in ([27], 5.5.4.15) as the full subcategory consisting of S-local objects. This is an accessible localization of $\mathcal{C}$, and $S^{-1}\mathcal{C}$ is presentable ([27], 5.5.4.15). See also the notion of strongly reflective subcategory in a presentable category (HTT, 5.5.4.16). In [28] Lurie uses the notation $S^{-1}\mathcal{C}$ in a more general case, for example, when discussing inverting the quasi-isomorphisms in the derived infinity-categories ([28], 1.3.4.1).

Lemma 2.4.15. *If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, S is a collection of morphisms of $\mathcal{C}$, for $X \in 1-\mathcal{C}\mathrm{at}$ write $\mathrm{Fun}_S(\mathcal{C}, X) \subset \mathrm{Fun}(\mathcal{C}, X)$ for the full subcategory of functors sending elements of S to isomorphisms. Consider the functor $h : 1-\mathcal{C}\mathrm{at} \to \mathrm{Spc}$, $X \mapsto \mathrm{Fun}_S(\mathcal{C}, X)^{\mathrm{Spc}}$. This functor is corepresentable by a category that should be denoted $S^{-1}\mathcal{C}$.*

Proof. $1-\mathcal{C}\mathrm{at}$ is presentable. So, to see that h is corepresentable, it suffices to show by (HTT, 5.5.2.7) that h preserves limits and is accessible. If $X = \lim_{i\in I} X_i$ in $1-\mathcal{C}\mathrm{at}$ then $\mathrm{Fun}(\mathcal{C}, X)^{\mathrm{Spc}} \widetilde{\to} \lim_{i\in I} \mathrm{Fun}(\mathcal{C}, X_i)^{\mathrm{Spc}}$ as always. Restricting to full subcategories we get a map $\alpha : \mathrm{Fun}_S(\mathcal{C}, X)^{\mathrm{Spc}} \to \lim_{i\in I} \mathrm{Fun}_S(\mathcal{C}, X_i)^{\mathrm{Spc}}$, which is fully faithful. Let now $f : \mathcal{C} \to X$ be a functor such that for any $i \in I$ the composition $\mathcal{C} \xrightarrow{f} X \to X_i$ sends elements of S to isomorphisms. Then $f \in \mathrm{Fun}_S(\mathcal{C}, X)^{\mathrm{Spc}}$. Thus, α is essentially surjective. □

(If β is a map in $X = \lim_{i\in I} X_i$ such that for any i its image is an isomorphism in X_i then β is an isomorphism. Indeed, the functor $\lim : \mathrm{Fun}(I, 1-\mathcal{C}\mathrm{at}) \to 1-\mathcal{C}\mathrm{at}$ sends an isomorphism to an isomorphism). If in the situation of Lemma 2.4.15, $\mathcal{C}$ is presentable, and S is of small generation then $S^{-1}\mathcal{C}$ is an accessible localization of $\mathcal{C}$ by (HTT, 5.5.4.20).

Remark In the situation of Lemma 2.4.15 the canonical functor $h : C \to S^{-1}C$ is cofinal. Proof by Nick: our h gives the full embedding $\mathrm{Fun}(S^{-1}C, \mathrm{Spc}) \hookrightarrow \mathrm{Fun}(C, \mathrm{Spc})$. Let $a : S^{-1}C \to \mathrm{Spc}$ be a functor. Then the LKE of $a \circ h$ along h identifies with a. So, $\mathrm{colim}\, a \widetilde{\to} \mathrm{colim}\, a \circ h$. This implies that h is cofinal.

2.4.16. Consider the diagram $[n] \xleftarrow{n} * \xrightarrow{0} [m]$. The colimit in $1-\mathcal{C}\mathrm{at}$ identifies with $[n+m]$. This has to be taken as a definition (expressing the fact that the compositions are unique). In the framework of [27], let $S(n, m)$ be the colimit in the category of simplicial sets. The natural map $S(n,m) \to \Delta^{n+m}$ is an acyclic cofibration in the Joyal model structure (combinatorial exercice).

This implies that the square is cartesian

$$\begin{array}{ccc} \mathrm{Funct}([m],C) & \to & \mathrm{Funct}(*,C) \\ \uparrow & & \uparrow \\ \mathrm{Funct}([n+m],C) & \to & \mathrm{Funct}([n],C) \end{array}$$

Lemma 2.4.17. *If $h: I \to 1-\mathcal{C}\mathrm{at}$ is a functor, $i \mapsto C_i$, let $C = \mathrm{colim}_{i\in I} C_i$. If $D \in 1-\mathcal{C}\mathrm{at}$ then*

$$\mathrm{Funct}(C,D) \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Funct}(C_i,D)$$

Proof. I think we accept without a proof the fact that Funct : $1-\mathcal{C}\mathrm{at}^{op} \times 1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}, (C,D) \mapsto \mathrm{Funct}(C,D)$ is a functor. So, h yields $h: I^{op} \to 1-\mathcal{C}\mathrm{at}^{op}$, the latter yields the functor $I^{op} \to 1-\mathcal{C}\mathrm{at}$, $i \mapsto \mathrm{Funct}(C_i,D)$.

Let $\bar{h}: I^{\triangleright} \to 1-\mathcal{C}\mathrm{at}$ be the map realizing C as the colimit of h. Composing $\bar{h}: {}^{\triangleleft}(I^{op}) \to 1-\mathcal{C}\mathrm{at}^{op}$ with the functor $\mathrm{Funct}(.,D)$, we get a cone ${}^{\triangleleft}(I^{op}) \to 1-\mathcal{C}\mathrm{at}$ for the natural functor $I^{op} \to 1-\mathcal{C}\mathrm{at}$, $i \mapsto \mathrm{Funct}(C_i,D)$. So, we get a natural map $\mathrm{Funct}(C,D) \to \lim_{i\in I^{op}} \mathrm{Funct}(C_i,D)$.

Let $n \geq 0$. Let us check that the induced map

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \mathrm{Funct}(C,D)) \to \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \lim_{i\in I^{op}} \mathrm{Funct}(C_i,D))$$

is an isomorphism in Spc. We have

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \mathrm{Funct}(C,D)) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n]\times C, D) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(\mathrm{colim}_{i\in I}([n]\times C_i), D)$$
$$\widetilde{\to} \lim_{i\in I^{op}} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n]\times C_i, D) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \lim_{i\in I^{op}} \mathrm{Funct}(C_i,D))$$

□

Reversing the arrows in the above proof, one similarly gets the following.

Lemma 2.4.18. *Let $D \in 1-\mathcal{C}\mathrm{at}$, $h: I \to 1-\mathcal{C}\mathrm{at}$ be a functor $i \mapsto C_i$, let $C = \lim_{i\in I} C_i$. Then the canonical map*

$$\mathrm{Funct}(D,C) \to \lim_{i\in I} \mathrm{Funct}(D,C_i)$$

is an isomorphism in $1-\mathcal{C}\mathrm{at}$.

2.4.19. For ([14], Chapter I.2, Sect. 2.2.1). Let $C \in 1-\mathcal{C}\mathrm{at}$, $c \in C$. The functor $\mathcal{F}\mathrm{in}^{op} \to C$, $I \mapsto c^I$ is rigorously defined as follows.

In general, given $\mathcal{A}_1, \mathcal{A}_2 \in 1-\mathcal{C}\mathrm{at}$ and two functors $F_i: \mathcal{A}_i \to \mathrm{Spc}$, one gets the functor $\mathcal{A}_1^{op} \times \mathcal{A}_2 \to \mathrm{Spc}$, $(a_1,a_2) \mapsto \mathrm{Map}_{\mathrm{Spc}}(F_1(a_1), F_2(a_2)) = \mathrm{Funct}(F_1(a_1), F_2(a_2))$, because there is a functor $1-\mathcal{C}\mathrm{at}^{op} \times 1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$ given by $\mathcal{A}, \mathcal{B} \mapsto \mathrm{Funct}(\mathcal{A},\mathcal{B})$.

It suffices to construct the corresponding functor $\mathcal{F}\mathrm{in}^{op} \times C^{op} \to \mathrm{Spc}$, $(I,c') \mapsto \mathrm{Map}_{\mathcal{C}}(c', c^I) = \mathrm{Map}_{\mathrm{Spc}}(I, \mathrm{Map}_{\mathcal{C}}(c',c))$. This can be done as above, namely we have two functors $\mathcal{F}\mathrm{in} \to \mathrm{Spc}$, $I \mapsto I$, and $\mathcal{C}^{op} \to \mathrm{Spc}$, $c' \mapsto \mathrm{Map}_{\mathcal{C}}(c',c)$. As above this yields the desired functor $\mathcal{F}\mathrm{in}^{op} \times C^{op} \to \mathrm{Spc}$.

2.4.20. Let $\mathcal{C} \in 1-\mathcal{C}at$ be nonempty and admit push-outs. Assume also for any pair $x, y \in \mathcal{C}$ there is a diagram $x \to z \leftarrow y$ in $\mathcal{C}$. Then $\mathcal{C}$ is filtered? Not clear, but this could be a variation of (HTT, 4.4.2.4).

2.5. Let $f : A \to B$ be a map in $1-\mathcal{C}at$. To show that this is an equivalence, it suffices to show that for any $K \in 1-\mathcal{C}at$ the map $\mathrm{Funct}(K, A)^{Spc} \to \mathrm{Funct}(K, B)^{Spc}$ is an equivalence (they should represent the same functor).

2.5.1. Let $I \in 1-\mathcal{C}at$, assume given a diagram $h : I \times [1] \to 1-\mathcal{C}at$, $i \mapsto (\mathcal{A}_i \stackrel{\alpha_i}{\to} \mathcal{B}_i)$. Here $\alpha_i : \mathcal{A}_i \to \mathcal{B}_i$ is a functor. Set $\mathcal{A} = \lim_{i\in I} \mathcal{A}_i$, $\mathcal{B} = \lim_{i\in I} \mathcal{B}_i$. Let $\alpha : \mathcal{A} \to \mathcal{B}$ be the limit functor. Assume given an object $b \in \mathcal{B}$, write $\mathcal{A}_b = \{b\} \times_{\mathcal{B}} \mathcal{A}$. Write b_i for the image of b in $\mathcal{B}_i$.

Lemma 2.5.2. *There is a canonical isomorphism* $\mathcal{A}_b \widetilde{\to} \lim_{i\in I} \{b_i\} \times_{\mathcal{B}_i} \mathcal{A}_i$.

Proof. Transitivity of the right Kan extension. Namely, Consider the category $\mathcal{J} = \{0' \to 1 \leftarrow 0\}$, it has three objects $0, 0', 1$. Consider the functor

$$\bar{h} : I \times J \to 1-\mathcal{C}at$$

extending h such that $\bar{h}(i, 0') = * \in 1-\mathcal{C}at$, and the map $\bar{h}(i, 0') \to \bar{h}(i, 1)$ is $b_i : * \to \mathcal{B}_i$. We have the commutative diagram

$$\begin{array}{ccc} I \times J & \to & J \\ \downarrow & & \downarrow \\ I & \to & * \end{array}$$

Calculate the right Kan extension of h via both paths. □

Corollary 2.5.3. *1) Let* $F : I \to 1-\mathcal{C}at$, $i \mapsto C_i$ *be a functor, let* $C = \lim_{i\in I} C_i$. *For* $x, y \in C$ *let* $x_i, y_i \in C_i$ *be their images. One has canonically*

$$\mathrm{Map}_C(x, y) \widetilde{\to} \lim_{i\in I} \mathrm{Map}_{C_i}(x_i, y_i)$$

2) Let $I \times [1] \to 1-\mathcal{C}at$ *be the map* $i \mapsto (C_i \stackrel{f_i}{\to} D_i)$. *Assume for each* $i \in I$, f_i *is conservative. Let* $f : C \to D$ *be obtained by passing to the limit over* I. *Then* f *is conservative.*

Proof. 1) Recall that $\mathrm{Map}_C(x, y) = * \times_{C\times C} \mathrm{Funct}([1], C)$ for the map $(x, y) : * \to C \times C$. The functor $\mathrm{Funct}([1], \cdot)$ commutes with limits in the sense of Lemma 2.4.18. Apply Lemma 2.5.2 for the functor $h : I \times [1] \to 1-\mathcal{C}at$, where $h(i)$ is the functor $\mathrm{Funct}([1], C_i) \to C_i \times C_i$ and the object of $\lim_{i\in I} (C_i \times C_i) \widetilde{\to} C \times C$ is (x, y). We are done.
2) Let $\alpha : c \to c'$ be a map in C with $f(\alpha)$ isomorphism. Then for each $i \in I$ we have $\alpha_i : c_i \to c'_i$ in C_i such that $f_i(\alpha_i) : f_i(c_i) \widetilde{\to} f_i(c'_i)$. So, $\alpha_i : c_i \to c'_i$ is an isomorphism, hence α is an isomorphism. □

From (2.5.2) we get immediately the following.

Corollary 2.5.4. *Let* $I \in 1-\mathcal{C}at$ *be small. Let* $I \to 1-\mathcal{C}at$ *be a functor,* $i \mapsto \mathcal{B}_i$, *let* $\mathcal{B} = \lim_i \mathcal{B}_i$, $b \in \mathcal{B}$. *Let* $b_i \in \mathcal{B}_i$ *be the image of* b. *Then* $\mathcal{B}/b \widetilde{\to} \lim_{i\in I} \mathcal{B}_i/b_i$ *canonically.*

Corollary 2.5.5. *Let I be a set, for $i \in I$, $\mathcal{C}_i \in 1-\mathcal{C}\mathrm{at}$ and $\mathcal{C} = \prod_{i\in I} \mathcal{C}_i$. Let $K \in 1-\mathcal{C}\mathrm{at}$, assume each $\mathcal{C}_i$ admits any K-indexed colimits. Then colimits in $\mathcal{C}$ are computed pointwise. Namely, assume $p : K^{\triangleright} \to C$ is a diagram given by the collection of diagrams $p_i : K^{\triangleright} \to C_i$, here p_i is the composition of p with $C \to C_i$. Then p is a colimit diagram iff for any $i \in I$, p_i is a colimit diagram.*

Proof. The 'if' part is given by Lemma 2.2.68. Assume now p is a colimit diagram. Let us show that each p_i is a colimit diagram. For $x = (x_i), y = (y_i) \in \mathcal{C}$ we have $\mathrm{Map}_{\mathcal{C}}(x, y) \widetilde{\to} \prod_{i\in I} \mathrm{Map}_{\mathcal{C}_i}(x_i, y_i)$. Let $c = (c_i) = \mathrm{colim}_I\, p$, let $d_i = \mathrm{colim}\, p_i$ and $d = (d_i) \in \mathcal{C}$. Then for $y = (y_i) \in \mathcal{C}$ we get

$$\mathrm{Map}_{\mathcal{C}}(c, y) \widetilde{\to} \lim_{k\in K^{op}} \mathrm{Map}_{\mathcal{C}}(p(k), y) \widetilde{\to} \lim_{k\in K^{op}} \prod_{i\in I} \mathrm{Map}_{\mathcal{C}_i}(p_i(k), y_i) \widetilde{\to}$$
$$\prod_{i\in I} \lim_{k\in K^{op}} \mathrm{Map}_{\mathcal{C}_i}(p_i(k), y_i) \widetilde{\to} \prod_{i\in I} \mathrm{Map}_{\mathcal{C}_i}(\mathrm{colim}\, p_i, y_i) \widetilde{\to} \mathrm{Map}_{\mathcal{C}}(d, y)$$

So, $c \widetilde{\to} d$ in $\mathcal{C}$, and each c_i is a colimit of p_i. □

2.5.6. If $\mathcal{A}, \mathcal{B} \in 1-\mathcal{C}\mathrm{at}$ are presentable then $\mathrm{Funct}^L(\mathcal{A}, \mathcal{B}) \widetilde{\to} \mathrm{Funct}^R(\mathcal{B}, \mathcal{A})^{op}$ canonically by ([27], 5.2.6.2). Here $\mathrm{Funct}^R(\mathcal{B}, \mathcal{A}) \subset \mathrm{Funct}(\mathcal{B}, \mathcal{A})$ is the full category of functors which are right adjoints (that is, small limit-preserving and accessible), $\mathrm{Funct}^L(\mathcal{A}, \mathcal{B})$ is the category of functors, which are left adjoints.

2.5.7. Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ be idempotent complete. Pick uncountable regular cardinals $\kappa < \tau$ such that $\mathcal{C}$ is τ-small, and for each $c, d \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(c, d)$ is essentially κ-small. Then $j : \mathcal{C} \to \mathrm{Ind}_\tau(\mathcal{C})$ is an equivalence by ([27], 5.4.3.5). Since for any $c \in \mathcal{C}$, $j(c)$ is τ-compact in $\mathrm{Ind}_\tau(\mathcal{C})$ by ([27], 5.3.5.5), the full subcategory $\mathcal{C}^\tau \subset \mathcal{C}$ coincides with $\mathcal{C}$. This is used in ([28], 1.4.4.2).

2.5.8. Let $K, S, \mathcal{C} \in 1-\mathcal{C}\mathrm{at}$. Assume $\mathcal{C}$ admits K-indexed colimits. Let $\bar{h} : K^{\triangleright} \to \mathrm{Funct}(S, \mathcal{C})$ be a diagram extending $h : K \to \mathrm{Funct}(S, \mathcal{C})$. Clearly, $\bar{h}$ is a colimit of h iff $\bar{h}^{op} : (K^{op})^{\triangleleft} \to \mathrm{Funct}(S^{op}, \mathcal{C}^{op})$ is a limit of $h^{op} : K^{op} \to \mathrm{Funct}(S^{op}, \mathcal{C}^{op})$. So, by ([27], 5.1.2.3), $\mathrm{Funct}(S^{op}, \mathcal{C}^{op})$ admits K^{op}-indexed limits. Moreover, a functor $(K^{op})^{\triangleleft} \to \mathrm{Funct}(S^{op}, \mathcal{C}^{op})$ is a limit of its restriction to K^{op} iff for each $s \in S^{op}$ the induced diagram $(K^{op})^{\triangleleft} \to \mathcal{C}^{op}$ is a limit diagram.

2.5.9. For Lemma ([27], 5.5.2.3), "calculating colimits of colimits". This is a useful thing (used, for example, in [28], 3.2.3.3). In the case $\mathcal{D} = *$ it says: assume we have a functor $K \to \mathrm{Fun}(L, \mathcal{C})$ in $1-\mathcal{C}\mathrm{at}$, which is extended to a colimit diagram $f : K^{\triangleright} \to \mathrm{Fun}(L, \mathcal{C})$. So, for any $l \in L$, $f_l : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram. Assume $\mathcal{C}$ admits L-indexed colimits, so we may pick a colimit diagram $L^{\triangleright} \to \mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ extending f. Then for the corresponding functor $L^{\triangleright} \times K^{\triangleright} \to \mathcal{C}$ its restriction to the cone point of $L^{\triangleright}$ is a colimit diagram $K^{\triangleright} \to \mathcal{C}$. Is not it simply the transitivity of left Kan extensions?

2.5.10. About (HTT, 5.3.6) "adjoining colimits". The example (HTT, 5.3.6.8) seems non evident for me. Namely, let $\mathcal{C} \in 1-\mathcal{C}at$ admit finite colimits. Let $\mathcal{K}$ be the collection of ω-finite simplicial sets, $\mathcal{K}'$ the collection of all small simplicial sets. Then to show that $\operatorname{Ind}(\mathcal{C}) \widetilde{\to} \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})$, we need to prove the following. Recall that $\operatorname{Ind}(\mathcal{C})^\omega$ is the idempotent completion of $\mathcal{C}$ (HTT, 5.4.2.4). Given $\mathcal{D} \in 1-\mathcal{C}at$, say presentable, and a functor $h : \mathcal{C} \to \mathcal{D}$ preserving finite colimits, let $\bar{h} : \operatorname{Ind}(\mathcal{C}) \to \mathcal{D}$ be the continuous extension of h. We have to show that the restriction $\bar{h} : \operatorname{Ind}(\mathcal{C})^\omega \to \mathcal{D}$ preserves finite colimits. This is easy for finite direct sums: given a collection $F_1, \dots, F_n \in \operatorname{Fun}(\operatorname{Idem}, \mathcal{C})$ let $\bar{c}_i$ be the colimit of $\operatorname{Idem} \stackrel{F_i}{\to} \mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})^\omega$ then $\oplus_{i=1}^n \bar{c}_i$ is the colimit of the composition $\operatorname{Idem} \stackrel{F}{\to} \mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})^\omega$, where $F = F_1 \oplus \dots \oplus F_n$ in $\operatorname{Fun}(\operatorname{Idem}, \mathcal{C})$. So, $\bar{h}$ preserves finite direct sums. It remains to show $\bar{h}$ preserves push-out squares. This is not clear for me (one has to rewrite it as a filtered colimit in $\operatorname{Ind}(\mathcal{C})$ of some elements in $\mathcal{C}$).

For (HTT, 5.3.6.10): let $\mathcal{K} \subset \mathcal{K}'$ be two collections of ∞-categories. The functor $\mathcal{C}at^{\mathcal{K}}_\infty \to \mathcal{C}at^{\mathcal{K}'}_\infty, \mathcal{C} \mapsto \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})$ is symmetric monoidal according to (HA, 4.8.1.8). Its right adjoint is right-lax non-unital monoidal.

2.5.11. There is a notion of a *relative adjoint functor* ([28], 7.3.2) for a diagram $G : \mathcal{D} \stackrel{G}{\to} \mathcal{C} \stackrel{q}{\to} \mathcal{E}$ in $1-\mathcal{C}at$. By definition, G admits a left adjoint relative to $\mathcal{E}$ if there is a left adjoint $F : \mathcal{C} \to \mathcal{D}$ of G such that for any $c \in \mathcal{C}$ the functor q sends the unit map $c \to GFc$ to an isomorphism in $\mathcal{E}$. Let $p = Gq : \mathcal{D} \to \mathcal{E}$, in this case $pF \widetilde{\to} q$. The key thing about this seems to be ([27], 7.3.2.6): the existence of relative left adjoint functor F is equivalent to two properties: (i) for $e \in \mathcal{E}$, the map of fibres $G_e : \mathcal{D}_e \to \mathcal{C}_e$ admits a left adjoint; (ii) G sends a locally p-cartesian arrow in $\mathcal{D}$ to a locally q-cartesian arrow in $\mathcal{D}$.

2.5.12. *Groupoid objects.* This is ([27], 6.1.2). Let $\boldsymbol{\Delta}_+$ be the category of finite (possibly empty) linearly ordered sets. A simplicial object in $\mathcal{C} \in 1-\mathcal{C}at$ is a map $\boldsymbol{\Delta}^{op} \to \mathcal{C}$. An augmented simplicial object is a map $(\boldsymbol{\Delta}_+)^{op} \to \mathcal{C}$.

Definition 2.5.13. *Let $U_\bullet : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ be a simplicial object. By ([27], 6.1.2.7), $U_\bullet$ is a groupoid object of $\mathcal{C}$ iff for any $n \geq 0$ and any decomposition $[n] = S \cup S'$ such that $S \cap S' = \{s\}$ is a single element, the square in $\mathcal{C}$ is cartesian*

$$\begin{array}{ccc} U(S) & \leftarrow & U([n]) \\ \downarrow & & \downarrow \\ U(\{s\}) & \leftarrow & U(S') \end{array}$$

Let $\operatorname{Grpd}(\mathcal{C}) \subset \operatorname{Funct}(\boldsymbol{\Delta}^{op}, \mathcal{C})$ *be the full subcategory spanned by groupoid objects.*

Here for $S \subset \{0, \dots, n\}$ we view S as linearly ordered, hence an object of $\boldsymbol{\Delta}$.

For a groupoid object $U_\bullet$ we should think of $\operatorname{colim} U$ as the quotient of U_0 by the corresponding "equivalence relation" $U_1 \stackrel{0,1}{\to} U_0$.

Let $\boldsymbol{\Delta}^{\leq n}_+ \subset \boldsymbol{\Delta}_+$ be the full subcategory spanned by the objects $[k]$ for $-1 \leq k \leq n$. An augmented simplicial object $U_\bullet$ in $\mathcal{C}$ is a Cech nerve if it is a right Kan extension of its restriction to $(\boldsymbol{\Delta}^{\leq 0}_+)^{op}$. Then the underlying simplicial object $U_\bullet$ is a groupoid object, and $U_1 \widetilde{\to} U_0 \times_{U_{-1}} U_0$ (see [27], 6.1.2.11). This $U_\bullet$ is determined by the map $U_0 \to U_{-1}$ in $\mathcal{C}$.

Definition 2.5.14. *([27], 6.1.2.14). Let $U_\bullet$ be a simplicial object in $\mathcal{C}$. Then $U_\bullet$ is an effective groupoid iff it can be extended to a colimit diagram $U_\bullet^+ : (\boldsymbol{\Delta}_+)^{op} \to \mathcal{C}$ such that $U_\bullet^+$ is a Cech nerve.*

Definition 2.5.15. *([27], 7.2.2.1). Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, 1 be a final object of $\mathcal{C}$. A pointed object is a morphism $1 \to X$ in $\mathcal{C}$. A group object of $\mathcal{C}$ is a groupoid object $U_\bullet : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ such that U_0 is a final object of $\mathcal{C}$. Write $Grp(\mathcal{C})$ for the ∞-category of group objects of $\mathcal{C}$.*

For example, if $* \to c$ is a pointed object in $\mathcal{C}$, its loop space is $* \times_c * \in \mathcal{C}$. It has a structure of a group object, because this is the beginning of the Cech nerve for $* \to c$.

Definition 2.5.16. *([28], 4.1.2.2). For $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ a monoid object in $\mathcal{C}$ is a simplicial object $U_\bullet : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ such that for any $n \geq 0$ the maps $U_n \to U(\{i, i+1\})$ exibit U_n as a product $U(\{0,1\}) \times \ldots \times U(\{n-1, n\})$. Let $Mon(\mathcal{C}) \subset \mathrm{Funct}(\boldsymbol{\Delta}^{op}, \mathcal{C})$ be the full subcategory of monoid objects.*

For example, a group object of $\mathcal{C}$ is a monoid object, we have a full subcategory $\mathcal{G}\mathrm{rp}(\mathcal{C}) \subset Mon(\mathcal{C})$.

Definition 2.5.17. *Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$. A commutative monoid object in $\mathcal{C}$ is an object $\mathcal{R} \in \mathrm{Funct}(\mathcal{F}\mathrm{in}_*, \mathcal{C})$ such that for any $(* \in I)$ the induced map $\mathcal{R}(I) \to \prod_{i \in I - \{*\}} \mathcal{R}(* \in (* \sqcup i))$ is an isomorphism. Here we are using the inert maps $\rho^i : (* \in I) \to (* \in (* \sqcup i))$ sending i to itself. Let $ComMon(\mathcal{C}) \subset \mathrm{Funct}(\mathcal{F}\mathrm{in}_*, \mathcal{C})$ be the full subcategory spanned by commutative monoid objects.*

We have a map $\boldsymbol{\Delta}^{op} \to \mathcal{F}\mathrm{in}_*$ (cf. Sect. 3.3.2), in Lurie this is ([28], 4.1.2.5), it is based on the identifications of cuts of $[n]$ with the set $\langle n \rangle$. It has the property: if $f : [n] \to [m]$ is a map in $\boldsymbol{\Delta}$ whose image is convex then the induced map $f^* : \langle m \rangle \to \langle n \rangle$ is inert. Indeed, if $f(j) = i + j$ for all $0 \leq j \leq n$ with $i + n \leq m$ then the map f^* satisfies $f^*(i+r) = r$ for $r = 1, \ldots, n$ and for other elements $s \in \langle m \rangle$ we get $f^*(s) = *$. In other words, the functor $\boldsymbol{\Delta}^{op} \to \mathcal{F}\mathrm{in}_*$ sends an inert map to an inert map.

Precomposing yields $ComMon(\mathcal{C}) \to Mon(\mathcal{C})$. I think $ComGrp(\mathcal{C})$ is defined as $ComMon(\mathcal{C}) \times_{Mon(\mathcal{C})} \mathcal{G}\mathrm{rp}(\mathcal{C})$.

According to ([28], 2.4.2), $ComMon(1-\mathcal{C}\mathrm{at})$ is essentially the same thing as symmetric monoidal categories.

Remark 2.5.18. *Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit finite products, M be a monoid in $\mathcal{C}$. Then by ([28], 5.2.6.2) M is a group iff $(pr_1, m) : M \times M \to M \times M$ and $(m, \mathrm{pr}_2) : M \times M \to M \times M$ are isomorphisms.*

The above implies that $\mathcal{G}\mathrm{rp}(\mathcal{C}) \subset Mon(\mathcal{C})$ is stable under small limits (this happens at the level of the homotopy categories). Indeed, $Mon(\mathcal{C}) \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{C})$ is closed under limits, and our claim follows from my Section 2.5.31. Similarly, each of the embeddings $ComGrp(\mathcal{C}) \subset ComMon(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathcal{C})$ is closed under limits. So, the projections $\mathrm{Mon}(\mathcal{C}) \to \mathcal{C}$ and $ComMon(\mathcal{C}) \to \mathcal{C}$ preserve limits (for a strengthening of this see ([28], 3.2.2.5).

2.5.19. The notion of a factorization system in an ∞-category $\mathcal{C}$ from ([27], 5.2.8.8) has a model-independent meaning. According to ([27], 5.2.8.17), let S_L, S_R be collections of morphisms in $\mathcal{C}$ stable under equivalences in $\mathrm{Fun}([1], \mathcal{C})$ and containing every equivalence in $\mathcal{C}$. Then we may call the pair (S_L, S_R) a factorization system in $\mathcal{C}$ if the natural map $\mathrm{Fun}'([2], \mathcal{C}) \to \mathrm{Fun}(\{0,2\}, \mathcal{C})$ is an equivalence. Here $\mathrm{Fun}'([2], \mathcal{C}) \subset \mathrm{Fun}([2], \mathcal{C})$ is the full subcategory of those functors f for which $f(0 \to 1) \in S_L, f(1 \to 2) \in S_R$.

Related ([27], Lemma 5.2.8.19) is used in ([28], 2.2.4.14), it says: let $\mathcal{D} \subset \mathrm{Fun}([1], \mathcal{C})$ be the full subcategory spanned by S_R. Then the inclusion $\mathcal{D} \subset \mathrm{Fun}([1], \mathcal{C})$ has a left adjoint, so $\mathcal{D}$ is a localization of $\mathrm{Fun}([1], \mathcal{C})$. For example, inert and active morphisms give a factorization system on any ∞-operad $\mathcal{O}^{\otimes}$.

2.5.20. Let $\mathcal{P}\mathrm{r}^L$ be the 1-full subcategory of $1-\mathcal{C}\mathrm{at}$, whose objects are presentable categories, and whose morphisms are colimit preserving functors. If $\mathcal{K}$ is the collection of all small ∞-categories then $\mathcal{P}\mathrm{r}^L$ is closed under tensor products in $\overline{\mathcal{C}\mathrm{at}}_{\infty}(\mathcal{K})$ by ([28], 4.8.1.14), so $\mathcal{P}\mathrm{r}^L$ inherits a symmetric monoidal structure.

A poset A, considered as a small category, is complete (and cocomplete) iff it is a complete lattice (that is, any subset $B \subset A$ admits sup and inf). For example, $[n] \in 1-\mathcal{C}\mathrm{at}$ is cocomplete and complete (actually, presentable).

For $S \in 1-\mathcal{C}\mathrm{at}$ one has the notion of a presentable fibration $X \to S$ over S ([27], 5.5.3.2). Then $\mathrm{Fun}(S, \mathcal{P}\mathrm{r}^L)^{Spc} \widetilde{\to} (Pres/S)^{\mathrm{Spc}} \widetilde{\to} \mathrm{Fun}(S^{op}, \mathcal{P}\mathrm{r}^R)^{\mathrm{Spc}}$, here $Pres/S \subset 1-\mathcal{C}\mathrm{at}/S$ is the full category spanned by presentable fibrations ([27], 5.5.3.3).

Remark 2.5.21. *([28], 4.7.4.19) for a given functor $\chi : S \times T \to \mathcal{P}\mathrm{r}^L$ garantees under some assumptions that limit over T and colimit over S commute in $\mathcal{P}\mathrm{r}^L$.*

If I is a small category, $I \to \mathcal{P}\mathrm{r}^L$, $i \mapsto \mathcal{C}_i$ is a diagram such that for any $i_1 \to i_2$ in I the corresponding functor $\mathcal{C}_{i_1} \to \mathcal{C}_{i_2}$ admits a left adjont then for $\mathcal{D} \in \mathcal{P}\mathrm{r}^L$ one has $(\lim_I \mathcal{C}_i) \otimes \mathcal{D} \widetilde{\to} \lim_I(\mathcal{C}_i \otimes \mathcal{D})$. Here the limit is taken in $1-\mathcal{C}\mathrm{at}$ or in $\mathcal{P}\mathrm{r}^L$ or in $\mathcal{P}\mathrm{r}^R$. Indeed, this limit can be rewritten as a colimit of left adjoint functors over I^{op}.

2.5.22. *Property of n-categories.* The following is exracted from (HA, proof of 1.3.3.10). Let $\boldsymbol{\Delta}^{\le n} \subset \boldsymbol{\Delta}$ be the full subcategory spanned by $[0], \ldots, [n]$.

Lemma 2.5.23. *Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ be a n-category, that is, for $a, b \in \mathcal{C}, \mathrm{Map}_{\mathcal{C}}(a,b) \in \tau_{\le n-1}Spc$. Assume $\mathcal{C}$ has finite colimits. Then $\mathcal{C}$ admits geometric realizations of simplicial objects. For any $F : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ let $F' : (\boldsymbol{\Delta}^{\le n})^{op} \to \mathcal{C}$ be the restriction of F. Then the natural map $\mathrm{colim}\, F' \to \mathrm{colim}\, F$ is an isomorphism in $\mathcal{C}$.*

Proof. The first claim is (HA, 1.3.3.10(1)). The second part is done in *loc.cit.* also as follows. Let $\mathcal{D} = \mathrm{Ind}(\mathcal{C})$. We may and do assume $\mathcal{C}$ small, so $\mathcal{D}$ is presentable. Let $\bar{F} : \boldsymbol{\Delta}^{op} \to \mathcal{D}$ be the LKE of F' under $(\boldsymbol{\Delta}^{\le n})^{op} \subset \boldsymbol{\Delta}^{op}$. Let $\alpha : \bar{F} \to jF$ be the natural map. We have $\mathrm{colim}\, \bar{F} \widetilde{\to} \mathrm{colim}\, F' \in \mathcal{C}$, because $j : \mathcal{C} \subset \mathrm{Ind}(\mathcal{C})$ is stable under finite colimits (HTT, 5.3.5.14). So, it suffices to show that the map $\mathrm{colim}\, \alpha : \mathrm{colim}\, \bar{F} \to \mathrm{colim}(jF)$ in $\mathcal{D}$ is an isomorphism. This is done in the proof of (HA, 1.3.3.10(1)). Namely, let $L : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be the left adjoint to the inclusion $\mathcal{D} \subset \mathcal{P}(\mathcal{C})$. Write $\mid \bar{F} \mid, \mid jF \mid$ for the corresponding colimits in $\mathcal{P}(\mathcal{C})$, let $\mid \alpha \mid : \mid \bar{F} \mid \to \mid jF \mid$ be the map in $\mathcal{P}(\mathcal{C})$ induced by α. The functor L factors as $\mathcal{P}(\mathcal{C}) \stackrel{\tau_{\le n-1}}{\to} \tau_{\le n-1}\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ by (HTT, 5.5.6.22). So, it sufiices to show that the map $\tau_{\le n-1} \mid \alpha \mid : \tau_{\le n-1} \mid \bar{F} \mid \to \tau_{\le n-1} \mid jF \mid$ is an equivalence in $\tau_{\le n-1}\mathcal{P}(\mathcal{C})$. Using (HTT, 6.5.3.10), this follows from the fact that $\mid \alpha \mid$ is n-connective. □

The dual claim is as follows.

Lemma 2.5.24. *Let $\mathcal{C}$ be a n-category. Assume $\mathcal{C}$ has finite limits. Then $\mathcal{C}$ has totalizations. For any $F : \boldsymbol{\Delta} \to \mathcal{C}$ let $F' : \boldsymbol{\Delta}^{\le n} \to \mathcal{C}$ be the restriction of F. Then the natural map* $\lim F \to \lim F'$ *is an equivalence in* $\mathcal{C}$.

In [14] ch.2, 2.8.4) the limit over $\boldsymbol{\Delta}^{\le n}$ is denoted $Tot^{\le n}$.

2.5.25. *Base change of a colocalization.* Let $\mathcal{B} \in 1-\mathcal{C}\mathrm{at}$, $L : \mathcal{A} \subset \mathcal{B}$ be a full subcategory, $R : \mathcal{B} \to \mathcal{A}$ be its right adjoint. So, R is a colocalization. Let $f : \mathcal{A}' \to \mathcal{A}$ be any map in $1-\mathcal{C}\mathrm{at}$, set $\mathcal{B}' = \mathcal{A}' \times_{\mathcal{A}} \mathcal{B}$. Let $R' : \mathcal{B}' \to \mathcal{A}'$ be the projection. Then R' admits a left adjoint $L' : \mathcal{A}' \to \mathcal{B}'$ sending a' to $(a', L(f(a')))$. This follows from the calculation of the mapping spaces in $\mathcal{B}'$ given in Corollary 2.5.3. It seems R' is also a colocalization, that is, the canonical map $\mathrm{id} \to R'L'$ is the identity. This is used in the following.

Lemma 2.5.26. *Assume given a cartesian square in* $1-\mathcal{C}\mathrm{at}$

$$\begin{array}{ccc} \mathcal{C} & \to & \mathcal{C}' \\ \downarrow q & & \downarrow q' \\ \mathcal{D} & \stackrel{f}{\to} & \mathcal{D}', \end{array}$$

where f is a cocartesian fibration. If q' is cofinal then q is also cofinal.

Proof. Let $d \in \mathcal{D}$. Let us show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ is contractible. Let $d' = q'(d)$. We have a functor $R : \mathcal{D}_{d/} \to \mathcal{D}'_{d'/}$ given by composing with f. The left adjoint $L : \mathcal{D}'_{d'/} \to \mathcal{D}_{d/}$ to R is fully faithful, so R is a colocalization, see my Section 2.2.113. We have an evident functor $R' : \mathcal{C}' \times_{\mathcal{D}'} \mathcal{D}_{d/} \widetilde{\to} \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/} \to \mathcal{C}' \times_{\mathcal{D}'} \mathcal{D}'_{d'/}$. By my Section 2.5.25, this functor admits a left adjoint L'. So, the induced map $\mid R' \mid : \mid \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/} \mid \to \mid \mathcal{C}' \times_{\mathcal{D}'} \mathcal{D}'_{d'/} \mid$ is an equivalence in Spc by my Section 2.2.106. □

2.5.27. Sam says: if $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is presentable, C^{op} is essentially never presentable, as it fails to be accessible.

2.5.28. Jacob confirmed by email: in (HTT, Prop. 5.5.1.9) the condition that $\mathcal{D}$ is presentable may be relaxed, one may just require $\mathcal{D}$ cocomplete.

2.5.29. A misprint in (HTT, 5.4.1.8): if $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}$, $\mathcal{D}$ is essentially small, $\mathcal{C}$ is locally small then $\mathcal{C}^{\mathcal{D}}$ is locally small.

2.5.30. For (HTT, 5.4.7.9). Let $K, \mathcal{C} \in 1-\mathcal{C}\mathrm{at}$. Assume $\mathcal{C}$ admits all K-indexed limits. Let $\mathcal{D} \subset \mathrm{Fun}({}^{\triangleleft}K, \mathcal{C})$ be the full subcategory spanned by the limit diagrams. Then $\mathcal{D} \widetilde{\to} \mathrm{Fun}(K, \mathcal{C})$ via restriction to K. The objects of $\mathcal{D}$ are precisely the RKE from K, so the inclusion $\mathcal{D} \hookrightarrow \mathrm{Fun}({}^{\triangleleft}K, \mathcal{C})$ is a right adjoint.

2.5.31. The following is true. If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, $h : I \times [1] \to \mathcal{C}$ is a functor given by $i \mapsto (x_i \stackrel{f_i}{\to} y_i)$, assume for any $i \in I$, f_i is an isomorphism. Assume $f : x \to y$ is the limit of f_i over $i \in I$ in $\mathcal{C}$. Then f is an isomorphism. Indeed, it suffices to show that f in an isomorphism in $\mathcal{C}^{ordn}$, but x is the limit of x_i in $\mathcal{C}^{ordn}$, same for y. In the usual categories this is easy to check. This is also (HTT, 5.5.4.9).

2.5.32. (HTT, 4.3.3.8) allows sometimes to calculate the LKE with respect to a functor $\delta : \mathcal{C}^0 \to \mathcal{C}^1$, which is not necessarily fully faithful. Namely, let $f_0 : \mathcal{C}^0 \to \mathcal{D}$ be a map in $1-\mathcal{C}\mathrm{at}$, and $f_1 : \mathcal{C}^1 \to \mathcal{D}$ with $\alpha : f_0 \to f_1\delta$ be its LKE along δ. If $c \in \mathcal{C}^0$ such that for any $c' \in \mathcal{C}^0$ one has $\mathrm{Map}_{\mathcal{C}^0}(c', c) \widetilde{\to} \mathrm{Map}_{\mathcal{C}^1}(\delta(c'), \delta(c))$ then $\alpha(c) : f_0(c) \to f_1(\delta(c))$ is an isomorphism.

2.5.33. (HTT, 6.2.1.6) is useful: Let $X \in 1-\mathcal{C}\mathrm{at}$ be presentable. Every topological localization $L : X \to Y$ is accessible and left exact. (However, accessibility here is maybe problematic, not explained in HTT).

2.5.34. The pull-back of sieves from (HTT, 6.2.2.1). If $\mathcal{C} \in 1-Cat$, $f : d \to c$ is a map in $\mathcal{C}$, $\mathcal{C}^{(0)}_{/c}$ is a sieve on c then $f^*\mathcal{C}^{(0)}_{/c} \subset \mathcal{C}_{/d}$ denotes the full subcategory f those $h : d' \to d$ such that the composition $d' \stackrel{h}{\to} d \stackrel{f}{\to} c$ is in $\mathcal{C}^{(0)}_{/c}$.

2.6. If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is small, admits finite coproducts, one has the full subcategory $\mathcal{P}_\Sigma(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ from (HTT, 5.5.8.8). Note that $\mathcal{C} \subset \mathrm{Ind}(\mathcal{C}) \subset \mathcal{P}_\Sigma(\mathcal{C})$. Then $\mathcal{P}_\Sigma(\mathcal{C})$ is generated inside $\mathcal{P}(\mathcal{C})$ by $\mathcal{C}$ under sifted colimits, and $\mathcal{P}_\Sigma(\mathcal{C})$ is presentable by (HTT, 5.5.8.10(1)), its universal property of is given in (HTT, 5.5.8.15).

Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit geometric realizations of simplicial objects. Then $x \in \mathcal{C}$ is projective if the functor $\mathcal{C} \to \mathrm{Spc}$, $y \mapsto \mathrm{Map}_\mathcal{C}(x, y)$ commutes with the geometric realizations of simplicial objects (HTT, 5.5.8.18). The Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{P}_\Sigma(\mathcal{C})$ takes values in projective compact objects of $\mathcal{P}_\Sigma(\mathcal{C})$, and (HTT, 5.5.8.22) describes intrinsically the categories of the form $\mathcal{P}_\Sigma(\mathcal{C})$. For example, the full subcategory of compact projective objects of Spc is the category $\mathrm{Sets}_{<\infty}$ of finite sets. By (HTT, 5.5.8.25), the inclusion $\mathrm{Sets}_{<\infty} \subset \mathrm{Spc}$ extends to an equivalence $\mathcal{P}_\Sigma(\mathrm{Sets}_{<\infty}) \widetilde{\to} \mathrm{Spc}$.

2.7. Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, let $a : \mathcal{C}' \subset \mathcal{C}$ be a 1-full subcategory with the same class of objects as $\mathcal{C}$. Let $R : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}')$ be the restriction via $\mathcal{C}'^{op} \to \mathcal{C}^{op}$, and $L : \mathcal{P}(\mathcal{C}') \to \mathcal{P}(\mathcal{C})$ be the LKE along a. I think L is 1-replete, that is, an equivalence to a unique 1-full subcategory. We prove that L is 1-fully faithful.

Note that R preserve colimits. Let $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}'), \mathcal{P}(\mathcal{C})) \subset \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}'), \mathcal{P}(\mathcal{C}))$ be the full subcategory of colimit preserving functors. The unit $\mathrm{id} \to RL$ is a map in $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}'), \mathcal{P}(\mathcal{C}'))$. By (HTT, 5.1.5.6), $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}'), \mathcal{P}(\mathcal{C}')) \widetilde{\to} \mathrm{Fun}(\mathcal{C}', \mathcal{P}(\mathcal{C}'))$. The functor RL viewed as a functor $\mathcal{C}' \to \mathcal{P}(\mathcal{C}')$ sends $c' \in \mathcal{C}'$ to the presheaf $x \mapsto \mathrm{Map}_\mathcal{C}(x, c')$. The functor id sends $c' \in \mathcal{C}'$ to $j(c') \in \mathcal{P}(\mathcal{C}')$. So, the map $\mathrm{id}(c') \to (RL)(c')$ in $\mathcal{P}(\mathcal{C}')$ evaluated at $x \in \mathcal{C}'$ is the monomorphism $\mathrm{Map}_{\mathcal{C}'}(x, c') \hookrightarrow \mathrm{Map}_\mathcal{C}(x, c')$ of spaces. This formally implies that for any $f \in \mathcal{P}(\mathcal{C}')$ the map $\mathrm{Map}_{\mathcal{P}(\mathcal{C}')}(f, \mathrm{id}(c')) \to \mathrm{Map}_{\mathcal{P}(\mathcal{C}')}(f, (RL)(c'))$ is a full subspace. Thus, $\mathrm{id}(c') \to (RL)(c')$ is a monomorphism in $\mathcal{P}(\mathcal{C}')$.

The propety of L being 1-fully faithful is equivalent to the fact that for any $f \in \mathcal{P}(\mathcal{C}')$ the map $\mathrm{id}(f) \to RL(f)$ in $\mathcal{P}(\mathcal{C}')$ is a monomorphism in $\mathcal{P}(\mathcal{C}')$. By my Lemma 2.2.17, it suffices to check this for $f = j(c')$ with $c' \in \mathcal{C}'$. So, L is 1-fully faithful.

Assume $\mathcal{C}$ has pull-backs, and a pull-back of a map $\alpha : c_1 \to c_2$ in $\mathcal{C}'$ by any morphism $x \to c_2$ in $\mathcal{C}$ remains in $\mathcal{C}'$. Call a map $\beta : g_1 \to g_2$ in $\mathcal{P}(\mathcal{C})$ nice if for any $c \to g_2$ in $\mathcal{P}(\mathcal{C})$ with $c \in \mathcal{C}$ the base change $\bar{\beta} : g_1 \times_{g_2} c \to c$ has the property: there is a diagram $I \to \mathcal{C}', i \mapsto y_i$ such that $g_1 \times_{g_2} c$ is the colimit in $\mathcal{P}(\mathcal{C})$ of the composition $I \to \mathcal{C}' \to \mathcal{C}$, and for $i \in I$, the corresponding map $y_i \to c$ is a map in $\mathcal{C}'$. I think the composition of nice maps is nice. We should get the 1-full subcategory $\mathcal{X} \subset \mathcal{P}(\mathcal{C})$, whose objects are those $g \in \mathcal{P}(\mathcal{C})$ for which there is a diagram $I \to \mathcal{C}' \to \mathcal{C}$ such that g is the colimit of the composition, and where we restrict the morphisms to nice maps. I have not checked the details. Maybe one needs to impose a condition: given a diagram $x \overset{f}{\to} y \overset{g}{\to} z$ in $\mathcal{C}$ such that g, gf are in $\mathcal{C}'$ then f is in $\mathcal{C}'$. Maybe one also needs to require $\mathcal{C}' \to \mathcal{C}$ be left exact.

2.7.1. If $L : \mathcal{A} \to \mathcal{B}$ is left adjoint to $R : \mathcal{B} \to \mathcal{A}$, L is fully faithful, R is conservative then (L, R) are mutually inverse equivalences.

2.7.2. Let $C \in 1-\mathcal{C}\mathrm{at}$, so is small. The product of $f, g \in \mathrm{Fun}(C, 1-\mathcal{C}\mathrm{at})$ is the functor $C \to 1-\mathcal{C}\mathrm{at}$ given as the composition $C \to C \times C \overset{f \times g}{\to} 1-\mathcal{C}\mathrm{at} \times 1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$, where the last map is the cartesian product in $1-\mathcal{C}\mathrm{at}$. View $\mathrm{Fun}(C, 1-\mathcal{C}\mathrm{at})$ as equipped with Cartesian symmetric monoidal structure. Does it has inner homs? The cartesian product in $\mathrm{Fun}(C, 1-\mathcal{C}\mathrm{at})$ preserves colimits in each variable, and $\mathrm{Fun}(C, 1-\mathcal{C}\mathrm{at})$ is presentable, so $\mathrm{Fun}(C, 1-\mathcal{C}\mathrm{at})$ has the inner homs.

2.7.3. If $X \in \mathrm{Spc}, C \in 1-\mathcal{C}\mathrm{at}$ then $\lim_{x \in X} \mathcal{C} \widetilde{\to} \mathrm{Fun}(X, C)$, the limit taken in $1-\mathcal{C}\mathrm{at}$.

2.7.4. Let $L : A \leftrightarrows B : R$ be an adjoint pair in $1-\mathcal{C}\mathrm{at}$, assume $LR \to \mathrm{id}$ is an isomorphism, that is, R is fully faithful, and L is a localization. Let $\alpha : B' \to B$ be a map in $1-\mathcal{C}\mathrm{at}$, $\alpha' : A' \to A$ is obtained from α by the base change $L : A \to B$. Let $L' : A' \to B'$ be the projection. Define $R' : B' \to A'$ by $R'(b') = (R\alpha(b'), b')$ with the evident isomorphism $LR\alpha(b') \widetilde{\to} \alpha(b')$. Then R' is right adjoint to L', this is easy.

2.7.5. Let $I \to 1-\mathcal{C}\mathrm{at}$ be a diagram, $i \mapsto C_i$, with I small. Set $C = \lim_i C_i$. Assume each $ev_i : C \to C_i$ has a right adjoint $ins_i : C_i \to C$. Assume for $i \to j$ the transition map $C_i \to C_j$ has a right adjoint. Then for $c \in C$ we get a diagram $I \to C$, $i \mapsto ins_i ev_i(c)$. The natural map $c \to \lim_i ins_i ev_i(c)$ is an isomorphism in C. Indeed, for $z \in C$ we get

$$\mathrm{Map}_C(z, \lim_i ins_i ev_i(c)) \widetilde{\to} \lim_i \mathrm{Map}(z, ins_i ev_i(c)) \widetilde{\to} \lim_i \mathrm{Map}_{C_i}(ev_i(z), ev_i(c)) \widetilde{\to} \mathrm{Map}_C(z, c)$$

2.7.6. The following is due to Dima. Let $C \in 1-\mathcal{C}\mathrm{at}$ admits small colimits. Let I be a set. Let J be the partially ordered set, whose elements are nonempty subsets of I, ordered by reversed inclusion. Let $F : J \to C$ be a diagram, $I' \mapsto c_{I'}$. It gives rise to a functor $G : \boldsymbol{\Delta}^{op} \to C$ sending $[n]$ to

$$G_n = \sqcup_{i_0,\dots,i_n} c_{i_0,\dots,i_n},$$

where each i_m runs through I. Namely, given $\alpha : [m] \to [n]$ in $\boldsymbol{\Delta}$, $\{i_{\alpha(0)}, \dots, i_{\alpha(m)}\} \subset \{i_0, \dots, i_n\}$, hence a map $c_{i_0,\dots,i_n} \to c_{i_{\alpha(0)},\dots,i_{\alpha(m)}} \to G_m$, which together give a morphism $G_n \to G_m$. Then $\operatorname{colim} G \widetilde{\to} \operatorname{colim} F$ naturally.

Proof: let $K \to \boldsymbol{\Delta}^{op}$ be the category over $\boldsymbol{\Delta}^{op}$, whose objects are $[n] \in \boldsymbol{\Delta}^{op}$ and a sequence $(i_0, \dots, i_n)$ with $i_m \in I$. A morphism from $[n], (i_0, \dots, i_n)$ to $[m], (i'_0, \dots, i'_m)$ is a datum of $\alpha : [m] \to [n]$ such that

$$i'_0 = i_{\alpha(0)}, \ \dots, \ i'_m = i_{\alpha(m)}$$

Let $G' : K \to C$ send $[n], (i_0, \dots, i_n)$ to $c_{i_0,\dots,i_n}$. By transitivity of Kan extensions, $\operatorname{colim} G'$ identifies with the geometric realization of G. Consider the functor $f : K \to J$ sending $[n], (i_0, \dots, i_n)$ to $\{i_0, \dots, i_n\} \in J$. Then G' factors as $K \stackrel{f}{\to} J \stackrel{F}{\to} C$. The claim is that f is cofinal. Namely, for $I' \in J$ consider the category $K \times_J J_{I'/}$. It classifies $[n], (i_0, \dots, i_n) \in K$ such that $\{i_0, \dots, i_n\} \subset I'$. I don't see why the latter is true. Dima says this is inspired by barycentric subdivision of simplicial complexes.

2.7.7. Let I be small, filtered, $I^{op} \to 1-\mathcal{C}\mathrm{at}$ be a diagram, $i \mapsto C_i$. Assume for $i \to j$ in I, $C_j \to C_i$ is fully faithful. Assume $i_0 \in I$ is an initial object. Then $\lim_{i \in I^{op}} C_i = \cap_i C_i$ as a full subcategory of C_{i_0}. Indeed, by Lemma 2.2.17, $\lim C_i \to C_{i_0}$ is fully faithful. Since for each i we have the inclusion $\lim_j C_j \subset C_i$, we get $\lim_j C_j \subset \cap_i C_i$. The compatible system of maps $\cap_i C_i \to C_j$ must factor through $\lim_j C_j$. We are done.

2.7.8. *Right adjoint to limits of full subcategories.* Let $I \in 1-\mathcal{C}\mathrm{at}$, $I \times [1] \to 1\,\mathcal{C}\mathrm{at}$, $i \mapsto (D_i \stackrel{h_i}{\to} C_i)$ be a diagram, where each h_i is fully faithful. Assume each h_i admits a right adjoint h_i^R, so h_i^R is a colocalization functor. Let $h : D \to C$ be the map in $1-\mathcal{C}\mathrm{at}$ obtained by passing to the limit over I. For a map $i \to j$ in I write $F_{ij}^D : D_i \to D_j$ and $F_{ij}^C : C_i \to C_j$ for the transition maps. Assume F_{ij}^D, F_{ij}^C have left adjoints $\mathcal{F}_{ij}^D, \mathcal{F}_{ij}^C$, and the natural map $\mathcal{F}_{ij}^C h_j \to h_i \mathcal{F}_{ij}^D$ is an isomorphism. Then h admits a right adjoint h^R, and for any i we have $h_i^R ev_i \widetilde{\to} ev_i h^R$. Here $ev_i : C \to C_i$ and $ev_i : D \to D_i$ are the projections.

Proof: we apply ([14], ch. I.1, 2.6.4). The natural map $F_{ij}^D h_i^R \to h_j^R F_{ij}^C$ is an isomorphism, it is obtained by passing to right adjoint in the isomorphism $\mathcal{F}_{ij}^C h_j \to h_i \mathcal{F}_{ij}^D$.

2.7.9. Let $I, C \in 1-\mathcal{C}\mathrm{at}$ with I filtered. Then $\lim_{i \in I} C \widetilde{\to} C$, where the limit of the constant functor is calculated in $1-\mathcal{C}\mathrm{at}$. Indeed, $I^{op} \to \mid I^{op} \mid$ is cofinal, and I is contractible by ([28], 5.3.1.20). The same holds for $I = \boldsymbol{\Delta}$, because $\boldsymbol{\Delta}^{op}$ is contractible.

2.7.10. Let $j^* : C \leftrightarrows C_0 : j_*$ be an adjoint pair in $1-\mathcal{C}\mathrm{at}$, and assume $j^! = j^*$ admits a left adjoint $j_! : C_0 \to C$. If j_* is fully faithful then $j_!$ is also fully faithful. Indeed, pass to left adjoints in the diagram $j^* j_* \widetilde{\to} \mathrm{id}$.

2.7.11. Let $C \in 1-\mathcal{C}\mathrm{at}$ and $f : c_0 \to c_1$ is a map in C. Recall that $C_{c_0//c_1} = \mathrm{Fun}([2], C) \times_{\mathrm{Fun}([1],C)} *$, where the map $* \to \mathrm{Fun}([1], C)$ is f, and we used the map $[1] \overset{0,2}{\to} [2]$ to get the morphism $\mathrm{Fun}([2], C) \to \mathrm{Fun}([1], C)$. One has $Tw(C)_{f/} \widetilde{\to} Tw(C_{c_0//c_1})$ naturally by ([28], 5.2.1.4). If C is presentable then $C_{c_0//c_1}$ is also presentable. Indeed, $C_{c_0//c_1} \widetilde{\to} (C_{/c_1})_{c_0/}$ and apply (HTT, 5.5.3.10, 5.5.3.11).

2.7.12. If $C^0 \subset C$ is a full subcategory and the final object c of C lies in C^0 then $C^0_* \subset C_*$ is a full subcategory. Indeed, $C^0_* = C^0 \times_C C_*$, and the base change of a full embedding is a full embedding.

Lemma 2.7.13 (Nick)**.** *Let $C \in 1-\mathcal{C}\mathrm{at}$ admit finite limits and geometric realizations. Then there is an adjoint pair $B : Grp(C) \leftrightarrows C_* : \Omega$, here $C_* = Ptd(C) = C_{*/}$, where $* \in C$ is a final object. We have $B(G) \widetilde{\to} \mathrm{colim}_{[n]\in \boldsymbol{\Delta}^{op}} G^n$ taken in C_*.*

Proof. (First proof). Replacing C by C_* we may and do assume C pointed, recall that $Grp(C_*) \widetilde{\to} Grp(C)$ canonically by (HTT, 7.2.2.10). We have $Grp(\mathcal{P}(C)) \widetilde{\to} \mathrm{Fun}(C^{op}, Grp(\mathrm{Spc}))$ canonically. The Yoneda embedding $C \to \mathcal{P}(C)$ yields a fully faithful embedding $\tilde{y} : Grp(C) \hookrightarrow Grp(\mathcal{P}(C))$. Similarly, applying $\mathbb{E}_0$ to the full embedding $C \hookrightarrow \mathcal{P}(C)$, one gets a full embedding $y : C \hookrightarrow \mathcal{P}(C)_* \widetilde{\to} \mathrm{Fun}(C^{op}, \mathrm{Spc}_*)$. Since $\mathcal{P}(C)$ is a topos, we have an adjoint pair $\tilde{B} : Grp(\mathcal{P}(C)) \leftrightarrows \mathcal{P}(C)_* : \tilde{\Omega}$, where $\tilde{\Omega}(F) = * \times_F *$.

Let $y^L : \mathcal{P}(C)_* \to C$ be the partially defined left adjoint to y. Then for $c \in C, G \in Grp(C)$ we have

$$\mathrm{Map}_{Grp(C)}(G, \Omega(c)) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}(C^{op},Grp(\mathrm{Spc}))}(\tilde{y}(G), \tilde{\Omega} y(c)) \widetilde{\to} \mathrm{Map}_C(y^L \tilde{B}(\tilde{y}(G)), c)$$

provided that y^L is defined on the object $\tilde{B}(\tilde{y}(G))$. Now let $\boldsymbol{\Delta}^{op} \to C$, $[n] \mapsto c_n$ be any functor. Then for $c \in C$ we get

$$\mathrm{Map}_C(\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, c_n, c) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}(C^{op},\mathrm{Spc}_*)}(\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, y(c_n), y(c)),$$

because y is fully faithful. This means that y^L is always defined on objects of the form $\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, y(c_n)$ and sends this object to $\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, c_n$. We are done.

(Second proof) Since $\boldsymbol{\Delta}^{op}$ is contractible, $\boldsymbol{\Delta}^{op} \to *$ is cofinal. So, by ([14], ch. I.1, Lm. 2.2.2), the restriction functor $const : C \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, C)$ is fully faithful. It has a left adjoint given by the geometric realization. Let $P \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, C)$ be the full subcategory of those $f : \boldsymbol{\Delta}^{op} \to C$ such that $f(0)$ is final in C. Then $Grp(C) \subset P$ is a full subcategory. He claims the inclusion $P \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, C)$ admits a right adjoint R, and $R(const(c)) \widetilde{\to} \Omega(c)$. If yes then for $x \in C, G \in Grp(C)$

$$\mathrm{Map}_{Grp(C)}(G, \Omega(x)) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}(\boldsymbol{\Delta}^{op})}(G, const(x)) \widetilde{\to} \mathrm{Map}_C(\underset{\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, G, x)$$

The functor P sends $(\ldots x_2 \substack{\to\\ \to\\ \to} x_1 \rightrightarrows x_0)$ to $(\ldots x_1 \times_{x_0 \times x_0} * \rightrightarrows *)$. $\square$

2.7.14. *About presentability from Nick.* Fact: Let C be a presentable category and $M : C \to C$ a monad such that the underlying endo-functor is accessible. Then the category $M - Alg(C)$ of M-algebras is presentable.

This can be proved using the following basic statements about presentable categories:
1) Limits of accessible categories along accessible functors are accessible.
2) An accessible category is presentable iff it admits all limits.

This can also be used to show that if C is presentable, then the category $Grp(C)$ of group objects in C is also presentable.

2.7.15. *Fact.* For $C \in 1-\mathcal{C}\mathrm{at}$, $ComMon(\mathrm{oblv}_{Mon}) : ComMon(Mon(C)) \to ComMon(C)$ is an equivalence (related to [28], Th. 5.1.2.2). Besides, $\mathrm{oblv}_{Mon} : Mon(ComMon(C)) \to ComMon(C)$ is an equivalence.

2.7.16. Let $I \in 1-\mathcal{C}\mathrm{at}$ be filtered, $E \in 1-\mathcal{C}\mathrm{at}$ and $I \times [1] \to 1-\mathcal{C}\mathrm{at}$, $i \mapsto (C_i \subset E)$ be a diagram, where $C_i \subset E$ is a full subcategory (our diagram is constant after restriction to $I \times \{1\}$). Then the natural map $\mathrm{colim}_{i\in I} C_i \to E$ is fully faithful, here the colimit is calculated in $1-\mathcal{C}\mathrm{at}$. This follows from the description of the mapping spaces in $\mathrm{colim}_{i\in I} C_i$ from [46].

2.7.17. Let $C \in 1-\mathcal{C}\mathrm{at}$ admiiting colimits. Then it is tensored over Spc. The functor $\mathrm{Spc} \times C \to C$, $(X, c) \mapsto X \otimes c$ preserves colimits separately in each variable. This is less trivial in the first variable. Let $I \in 1-\mathcal{C}\mathrm{at}$ be small and $f : I \to \mathrm{Spc}$, $i \mapsto X_i$ be a functor. Let $\tilde{I} \to I$ be the cocartesian fibration attached to f. Recall that $\underset{i\in I}{\mathrm{colim}}\, X_i \widetilde{\to} \mid \tilde{I} \mid$ in Spc. Let $c \in C$. Now $\bar{c} := \mathrm{colim}_{i\in I} X_i \otimes c \widetilde{\to} \mathrm{colim}_{i\in I} \mathrm{colim}_{X_i} c \widetilde{\to} \underset{\tilde{I}}{\mathrm{colim}}\, c$. By Section 2.2.65, we get for $d \in C$

$$\mathrm{Map}_C(\bar{c}, d) \widetilde{\to} \lim_{(\tilde{I})^{op}} \mathrm{Map}_C(c, d) \widetilde{\to} \mathrm{Fun}((\tilde{I})^{op}, \mathrm{Map}_C(c, d)) \widetilde{\to}$$
$$\mathrm{Fun}(\mid (\tilde{I})^{op} \mid, \mathrm{Map}_C(c, d)) \widetilde{\to} \lim_{|\tilde{I}|^{op}} \mathrm{Map}_C(c, d) \widetilde{\to} \mathrm{Map}_C(\mathrm{colim}_{|\tilde{I}|} c, d)$$

We used the fact that $\mid (\tilde{I})^{op} \mid \widetilde{\to} \mid \tilde{I} \mid \widetilde{\to} \mid \tilde{I} \mid^{op}$. By ([14], ch. I.1, 2.1.6), $\mid \tilde{I} \mid \widetilde{\to} \mathrm{colim}_{i\in I} X_i =: X$ in Spc. So, $\mathrm{colim}_{i\in I} X_i \otimes c \widetilde{\to} \mathrm{colim}_X c$ in C.

2.7.18. Let $C, D : \boldsymbol{\Delta} \to 1-\mathcal{C}\mathrm{at}$ be cosimplicial categories $[n] \mapsto C_n$, $[n] \mapsto D_n$. Let $C \to D$ be a morphism of functors from $\boldsymbol{\Delta}$ to $1-\mathcal{C}\mathrm{at}$ for $[n] \in \boldsymbol{\Delta}$ given by a fully faithful functor $\alpha_n : C_n \to D_n$. Let $\mathcal{C} = \lim C, \mathcal{D} = \lim D$. Assume $\alpha_0 : C_0 \to D_0$ is an equivalence. Then the map $\bar{\alpha} : \mathcal{C} \to \mathcal{D}$ obtained by passing to the totalizations is an equivalence.

Proof: we know that $\bar{\alpha}$ is fully faithful, and its essential image is the full subcategory of spanned by collections $(d_n) \in \mathcal{D}$ such that for any $n \geq 0$, $d_n \in C_n \subset D_n$. Let $(d_n) \in \mathcal{D}$ and let $n \geq 0$. It remains to show that $d_n \in C_n$. However, d_n is the image of $d_0 \in C_0$ under $C_0 \overset{\nu}{\to} C_n \to D_n$, where say ν corresponds to $[0] \overset{0}{\to} [n]$. $\square$

2.7.19. Let I be small filtered, $I \to 1-\mathcal{C}\mathrm{at}$, $i \mapsto C_i$ be a diagram, $C = \mathrm{colim}_i\, C_i$ in $1-\mathcal{C}\mathrm{at}$. Assume each C_i admits finite colimits, and each transition functor $C_i \to C_j$ for $i \to j$, preserves finite colimits. Then using [46] one shows that C admits finite colimits, and each map $ins_i : C_i \to C$ preserves finite colimits.

2.7.20. Let $r : A \to B$ be a morphism in $1-\mathcal{C}\mathrm{at}$ of symmetric monoidal ∞-categories admitting a left adjoint $l : B \to A$. Assume l is symmetric monoidal, so r is right-lax symmetric monoidal. In particular, $r(1) \in CAlg(B)$. Then r lifts to a morphism $A \to r(1)-mod(B)$. Indeed, we have a morphism $r(1) \to \mathcal{A}$ of monads in $\mathrm{Fun}(B, B)$, where $\mathcal{A} = rl$. Since r is a $\mathcal{A}$-module in $\mathrm{Fun}(A, B)$, it is also a $r(1)$-module, apply (ch. I.1, 3.7.3).

2.7.21. Let $C \in 1-\mathcal{C}\mathrm{at}$, H be a groupoid in C acting on $S \in C$, so we have $\mathcal{X} : \boldsymbol{\Delta}^{op} \to C$, $[0] \mapsto S, [1] \mapsto H$. Let $\tau : Y \to S$ be a map in C. I propose the following definition. A lifting of the action of H on S to one on Y is a groupoid $\mathcal{X}' : \boldsymbol{\Delta}^{op} \to C$ together with a morphism $\mathcal{X}' \to \mathcal{X}$ of groupoids in C such that the following holds: $\mathcal{X}'([0]) = S$, and $\mathcal{X}'([0]) \to \mathcal{X}([0])$ is id. The map $\mathcal{X}'([1]) \to \mathcal{X}([1])$ is τ. For any $\alpha : [n] \to [m]$ in $\boldsymbol{\Delta}$, the diagram is cartesian

$$\begin{array}{ccc} \mathcal{X}([m]) & \stackrel{\mathcal{X}(\alpha)}{\to} & \mathcal{X}([n]) \\ \uparrow & & \uparrow \\ \mathcal{X}'([m]) & \stackrel{\mathcal{X}'(\alpha)}{\to} & \mathcal{X}'([n]) \end{array}$$

How good is this definition?

2.7.22. By a category object in $C \in 1-\mathcal{C}\mathrm{at}$ we mean a map $\mathcal{X} : \boldsymbol{\Delta}^{op} \to C$ such that for any $n \geq 0$ the morphisms $[1] \stackrel{i,i+1}{\to} [n]$ yield an isomorphism

$$\mathcal{X}([n]) \widetilde{\to} \mathcal{X}[1] \times_{\mathcal{X}[0]} \mathcal{X}([1]) \times_{\mathcal{X}[0]} \dots \mathcal{X}[1],$$

where $[1]$ appears n times. Then we say that $\mathcal{X}[1]$ acts on $\mathcal{X}[0]$.

Now given a map $\tau : c \to \mathcal{X}[0]$ in C, we may define the notion that the $\mathcal{X}[1]$-action on $\mathcal{X}[0]$ is extended to a right $\mathcal{X}$-action on c. This means that we get a category object $\mathcal{X}' : \boldsymbol{\Delta}^{op} \to C$ and a map $\mathcal{X}' \to \mathcal{X}$ of category objects in C such that $\mathcal{X}'[0] \to \mathcal{X}[0]$ is the map τ, and the square is cartesian

$$\begin{array}{ccc} \mathcal{X}'[0] & \leftarrow & \mathcal{X}'[1] \\ \downarrow \tau & & \downarrow \\ \mathcal{X}[0] & \stackrel{s}{\leftarrow} & \mathcal{X}[1] \end{array}$$

Here s is the source map attached to $[0] \stackrel{0}{\to} [1]$. The action map $\mathcal{X}'[1] \to c$ is then attached to $[0] \stackrel{1}{\to} [1]$.

The following is established in ([14], published version, Cor. 4.4.5 of Chapter 9). Assume C admits finite limits, and $\mathcal{X}$ is a category object in C. Then $\mathcal{X}[1] \in Alg(Corr(C))$ with the product given by the diagram $\mathcal{X}[1] \times \mathcal{X}[1] \leftarrow \mathcal{X}[1] \times_{\mathcal{X}[0]} \mathcal{X}([1]) \stackrel{m}{\to} \mathcal{X}[1]$, where m is the product map. The unit is the diagram $* \leftarrow X[0] \stackrel{u}{\to} X[1]$, where u is the unit.

2.7.23. If κ is a regular cardinal, by κ-small ∞-category I think Lurie means an ∞-category, which is κ-small as a simplicial set, that is, the set of non-degenerate simplices is $< \kappa$.

2.7.24. Decomposition of colimits into pieces is discussed in [27],[45]. Let $\mathcal{C}$ be a cocomplete ∞-category, let K be a small ∞-category written as $K \widetilde{\to} \operatorname{colim}_{i\in I} K_i$ in the category of small ∞-categories. Then for $f : K \to \mathcal{C}$, $\operatorname{colim} f \widetilde{\to} \underset{i\in I}{\operatorname{colim}}\, \underset{k\in K_i}{\operatorname{colim}} f(k)$ calculated in $\mathcal{C}$.

Proof. Consider the functor $b : I \to 1-\mathcal{C}\mathrm{at}$, $i \mapsto K_i$. Let $Y \to I$ be the cocartesian fibration attached to b. Then K is obtained from Y by inverting all the cocartesian arrows. By Remark in Section 2.4.15 the canonical arrow $Y \to K$ is cofinal. Now we can calculated the colimit of the composition $Y \to K \xrightarrow{f} C$ first taking the LKE along $Y \to I$ and then calculating the colimit over I. By ([14], I.1, 2.2.4), the value of this LKE at $i \in I$ is $\operatorname{colim}_{k\in K_i} f(k)$ as desired. □

Application: let $Y \in \mathrm{Spc}$, let G be a finite group acting on Y, write $B(G) \in \mathrm{Spc}$ for the corresponding prestack. Then

$$\operatorname{colim}_{B(G)} Y \widetilde{\to} \underset{[n]\in \boldsymbol{\Delta}^{op}}{\operatorname{colim}}\, \underset{g\in G^n}{\operatorname{colim}} Y \widetilde{\to} \underset{[n]\in \boldsymbol{\Delta}^{op}}{\operatorname{colim}} G^n \times Y \widetilde{\to} Y/G,$$

the quotient in the sense of prestacks.

We have an action of G on $B(G)$. Namely, for each $h \in G$ let $Ad_h : B(G) \to B(G)$ be the map $* \to *$ and sending a morphism $g \in G$ to hgh^{-1}. Then $Ad_{h'} Ad_h = Ad_{h'h}$, so this is an action of G on $B(G)$. Write $G-mod(\mathrm{Spc})$ for the category of spaces with a G-action. We have $G-mod(\mathrm{Spc}) \widetilde{\to} \mathrm{Spc}_{/B(G)}$, $X \mapsto X/G$, the prestack quotient. On the other hand, we have a functor $\mathrm{Spc} \to G-mod(\mathrm{Spc})$, $Y \mapsto \mathrm{Fun}(B(G), Y)$, where we view $\mathrm{Fun}(B(G), Y)$ as equipped with the G-action coming from the above G-action on Y. For $Y \in \mathrm{Spc}$ calculate $\mathrm{Fun}(B(G), Y)/G$, where the quotient is taken in PreStk. What we get?

2.7.25. Let $f : C \to D$ be a fully faithful morphism in $1-\mathcal{C}\mathrm{at}$. Then the partially defined left adjoined f^L of f is defined on objects of the form $f(c), c \in C$ by $f^L f(c) = c$.

2.7.26. Let I be finite category, $I \to 1-\mathcal{C}\mathrm{at}$, $i \mapsto C_i$ be a functor such that for $i \in I$, C_i admits filtered colimits, and the transition maps $C_i \to C_j$ for $i \to j$ in I preserve filtered colimits. Recall that $C = \lim_{i\in I} C_i$ admits filtered colimits. Let $x \in C$ given by a compatible collection $x_i \in C_i$. Assume for any $i \in I$, $x_i \in C_i^c$. Then $x \in C^c$.

Proof. Let K be small filtered, $K \to C$ given by $k \mapsto c^k$. For $i \in I$ write c_i^k for the image of c^k in C_i. Let $c = \operatorname{colim}_{k\in K} c_k$ in C. Note that the image c_i of c in C_i identifies with $\operatorname{colim}_{k\in K} c_i^k$. We get in Spc

$$\underset{k\in K}{\operatorname{colim}} \mathrm{Map}_C(x, c_k) \widetilde{\to} \underset{k\in K}{\operatorname{colim}}\, \underset{i\in I}{\lim} \mathrm{Map}_{C_i}(x_i, c_i^k) \widetilde{\to} \underset{i\in I}{\lim}\, \underset{k\in K}{\operatorname{colim}} \mathrm{Map}_{C_i}(x_i, c_i^k) \widetilde{\to}$$
$$\underset{i\in I}{\lim} \mathrm{Map}_{C_i}(x_i, \underset{k\in K}{\operatorname{colim}}\, c_i^k) \widetilde{\to} \underset{i\in I}{\lim} \mathrm{Map}_{C_i}(x_i, c_i) \widetilde{\to} \mathrm{Map}_C(x, c).$$

We used ([27], 5.3.3.3) as I is finite (in the sense of Lurie). □

3. Algebra

The notion of an ∞-operad from ([28], 2.1.1.10) makes sense in the model independent framework. We may replace the condition (3) in the def by requiring that $\mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^n$ is an equivalence. Namely,

Definition 3.0.1. *An ∞-operad is a map $p : \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ in $1-\mathcal{C}\mathrm{at}$ such that*
1) for every inert morphism $f : \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}\mathrm{in}_*$ and any $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ there is a p-cocartesian morphism $\bar{f} : C \to C'$ in $\mathcal{O}^{\otimes}$ over f;
2) Let $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, $C' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, let $f : \langle m\rangle \to \langle n\rangle$ be a morphism in $\mathcal{F}\mathrm{in}_*$, and $\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C')$ be the union of those connected components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ which lie over f. Choose p-cocartesian morphisms $C' \to C'_i$ over the inert morphisms $\rho^i : \langle n\rangle \to \langle 1\rangle$ for $1 \le i \le n$. Then the induced map

$$\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{1\le i\le n} \mathrm{Map}^{\rho^i f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$

is an isomorphism in Spc.
3) For each $n \ge 0$ the functors $\{\rho^i_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}\}_{1\le i\le n}$ determine an equivalence $\mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^n$.

If $\mathcal{O}^{\otimes}$ is an operad then $\mathcal{O}^{\otimes}_{\langle 0\rangle} \widetilde{\to} [0]$, and each object of $\mathcal{O}^{\otimes}_{\langle 0\rangle}$ is final in $\mathcal{O}^{\otimes}$.

Given two operads $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$, $\mathcal{O}'^{\otimes} \to \mathcal{F}\mathrm{in}_*$, the category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ is the full subcategory of $\mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by those functors that send inert morphisms of $\mathcal{O}^{\otimes}$ to inert morphisms of $\mathcal{O}'^{\otimes}$. Recall that a morphism in $\mathcal{O}^{\otimes}$ is inert if it is cocartesian, and its image in $\mathcal{F}\mathrm{in}_*$ is inert. The category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ is the category of ∞-operad maps from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$. (Such maps between monoidal categories are usually called right-lax non-unital monoidal functors).

For example, if $p : \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is an ∞-operad then p is a morphism of ∞-operads.

Consider the 1-full subcategory in $1-\mathcal{C}\mathrm{at}\,/\,\mathcal{F}\mathrm{in}_*$, where we keep only those objects $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$, which are ∞-operads, and only those morphisms, which are maps of ∞-operads. Nick claims it is canonically equivalent to Op_{∞} from ([28], 2.1.4.1).

Definition 3.0.2. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a cocartesian fibration. Then the composition $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is an ∞-operad iff for any $T \widetilde{\to} T_1 \oplus \ldots \oplus T_n \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ the inert morphisms $T \to T_i$ induce an equivalence $\mathcal{C}^{\otimes}_T \widetilde{\to} \prod_{i=1}^n \mathcal{C}^{\otimes}_{T_i}$. In this case we say that $\mathcal{C}^{\otimes}$ is a* $\mathcal{O}$-monoidal ∞-category.

For example, for $\mathcal{O}^{\otimes} = \mathcal{F}\mathrm{in}_*$, $\mathcal{O}$-monoidal ∞-category is also called a symmetric monoidal infinity category. Thus, a symmeric monoidal ∞-category is a cocartesian fibration $p : \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ such that for any $n \ge 0$ the functors $\rho^i_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}$ for $1 \le i \le n$ induce an equivalence $\mathcal{O}^{\otimes}_{\langle n\rangle} \widetilde{\to} \mathcal{O}^n$.

Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads. Then, in the model-independent setting, q is automatically a fibration of ∞-operads in the sense of ([28], 2.1.2.10). That is, for $C \in \mathcal{C}^{\otimes}$ and an inert morphism $f : q(C) \to X$ in $\mathcal{O}^{\otimes}$ there is an inert morphism $\bar{f} : C \to \bar{X}$ in $\mathcal{C}^{\otimes}$ with $f \widetilde{\to} q(\bar{f})$. Moreover, the inert morphisms of $\mathcal{C}^{\otimes}$ are precisely the q-cocartesian morphisms in $\mathcal{C}^{\otimes}$ whose image in $\mathcal{O}^{\otimes}$ is inert.

Notation If $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ are maps of infinity operads then $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \subset \mathrm{Funct}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ is the full subcategory spanned by maps of ∞-operads. It is called the category of $\mathcal{O}'$-algebra objects of $\mathcal{C}$. If $\alpha = \mathrm{id}$ then $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$. In the case $\mathcal{O} = \mathcal{F}\mathrm{in}_*$ Lurie denotes $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$. For $\mathcal{O} = \mathcal{O}' = \mathcal{F}\mathrm{in}_*$, he denotes $CAlg(\mathcal{C}) := \mathrm{Alg}_{\mathcal{F}\mathrm{in}_*}(\mathcal{C})$.

Example. If $p : \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is a symmetric monoidal ∞-category, then $CAlg(\mathcal{C})$ is the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of commutative algebra objects of $\mathcal{C}$. That is, the full subcategory of $\mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\mathcal{F}\mathrm{in}_*, \mathcal{C}^{\otimes})$ spanned by those functors that send inert morphisms in $\mathcal{F}\mathrm{in}_*$ to inert morphisms in $\mathcal{C}^{\otimes}$.

Definition 3.0.3. *If $\mathcal{O}^{\otimes}$ is an ∞-operad, $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ are $\mathcal{O}$-monoidal ∞-categories then $\mathrm{Funct}^{\otimes}_{\mathcal{O}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Funct}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ is the full subcategory spanned by those functors that send p-cocartesian morphisms to q-cocartesian morphisms. This is the category of $\mathcal{O}$-monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. If in addition $\mathcal{O} = \mathcal{F}\mathrm{in}_*$ then he writes $\mathrm{Funct}^{\otimes}(\mathcal{C}, \mathcal{D}) := \mathrm{Funct}^{\otimes}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$, this is the category of symmetric monoidal functors from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$.*

The ∞-category Op_{∞} of ∞-operads is defined in ([28], 2.1.4.1). For $\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes} \in Op_{\infty}$ one has $\mathrm{Map}_{Op_{\infty}}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}) \widetilde{\to} \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')^{\mathrm{Spc}}$.

Let $\mathcal{C}\mathrm{at}^{\otimes}_{\infty} \subset Op_{\infty}$ be the 1-full subcategory, whose objects are symmetric monoidal categories, and morphisms from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ are symmetric monoidal functors inside $\mathrm{Map}_{Op_{\infty}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ ([28], 2.1.4.13). This is the ∞-category of symmetric monoidal categories.

Write $\mathcal{T}\mathrm{riv} \subset \mathcal{F}\mathrm{in}_*$ for the subcategory with the same objects as $\mathcal{F}\mathrm{in}_*$, and whose morphisms are inert morphisms ([28], 2.1.1.20), this is the trivial operad. We have the functor $Op_{\infty} \to 1-\mathcal{C}\mathrm{at}$, $\mathcal{O}^{\otimes} \mapsto \mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1\rangle}$. Its left adjoint functor $q : 1-\mathcal{C}\mathrm{at} \to Op_{\infty}$ sends $\mathcal{O}$ to the operad $\mathcal{O}^{\otimes} \to \mathcal{T}\mathrm{riv}^{\otimes} \subset \mathcal{F}\mathrm{in}_*$, where $\mathcal{O}^{\otimes}_{\langle n\rangle} = \mathcal{O}^n$, and for an inert morphism $\alpha : \langle n\rangle \to \langle m\rangle$ the functor $\alpha_! : \mathcal{O}^n \to \mathcal{O}^m$ is the corresponding projection. For a morphism b in $\mathcal{F}\mathrm{in}_*$, which is not inert, there are no morphisms in $\mathcal{O}^{\otimes}$ over b.

To see this, one may show that $q(\mathcal{O})$ is the right Kan extension of the functor $\mathcal{O} : * \to 1-\mathcal{C}\mathrm{at}$ via $* \to \mathcal{T}\mathrm{riv}^{\otimes}$. The following generalizes $ComMon(\mathcal{C})$.

Definition 3.0.4. *([28], 2.4.2.1). Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, $\mathcal{O}^{\otimes}$ be an infinity operad. Then $\mathcal{O}$-monoid in $\mathcal{C}$ is a functor $M : \mathcal{O}^{\otimes} \to \mathcal{C}$ such that for any $x = x_1 \oplus \ldots \oplus x_n \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, the canonical maps $M(x) \to M(x_i)$ yield an isomorphism $M(x) \widetilde{\to} \prod_{i=1}^n M(x_i)$. Let $Mon_{\mathcal{O}}(\mathcal{C}) \subset \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{C})$ be the full subcategory of $\mathcal{O}$-monoids in $\mathcal{C}$.*

For example, for an infinity operad $\mathcal{O}^{\otimes}$, a functor $\mathcal{O}^{\otimes} \to 1-\mathcal{C}\mathrm{at}$ is a $\mathcal{O}$-monoid iff the corresponding cocartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a $\mathcal{O}$-monoidal category ([28], 2.4.2.4).

The composition with $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$, $\mathcal{C} \mapsto \mathcal{C}^{op}$ preserves the full subcategory $Mon_{\mathcal{O}}(1-\mathcal{C}\mathrm{at}) \subset \mathrm{Fun}(\mathcal{O}^{\otimes}, 1-\mathcal{C}\mathrm{at})$. If $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration of ∞-operads, let $F : \mathcal{O}^{\otimes} \to 1-\mathcal{C}\mathrm{at}$ be the corresponding functor, F' be its composition with the involution $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}$, $\mathcal{C} \mapsto \mathcal{C}^{op}$. Then $\mathcal{C}^{\otimes, op} \to \mathcal{O}^{\otimes, op}$ is a cartesian fibration corresponding to F' via the strengthening (for cartesian fibrations). We may also introduce the cocartesian fibration $\mathcal{C}^{\otimes, \overline{op}} \to \mathcal{O}^{\otimes}$ corresponding to F' via the strengthening.

Remark 3.0.5. *In ([28], Def 2.1.1.26 and 2.1.1.27) Lurie means by a fibrant simplicial colored operad a simplicial colored operad* $\mathcal{O}^{\otimes}$ *such that each* $Mul(\{X_i\}_{i\in I}, Y)$ *is a Kan complex.*

Question: is it checked that $Mon_{\mathcal{F}\mathrm{in}_*}(1-\mathcal{C}\mathrm{at})$ is equivalent to the ∞-category $\mathcal{C}\mathrm{at}_{\infty}\otimes$ from ([28], 2.1.4.13)?

Remark 3.0.6. *Let* $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a cocartesian fibration of* ∞*-operads,* $\alpha : \tilde{\mathcal{O}}^{\otimes} \to \mathcal{O}^{\otimes}$ *a map of* ∞*-operads. Then* $\tilde{\mathcal{C}}^{\otimes} := \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \tilde{\mathcal{O}}^{\otimes} \stackrel{\tilde{q}}{\to} \tilde{\mathcal{O}}^{\otimes}$ *is a cocartesian fibration of* ∞*-operads. Besides,* $\tilde{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$ *is a morphism of* ∞*-operads. The inert morphisms of* $\tilde{\mathcal{C}}^{\otimes}$ *are precisely the* $\tilde{q}$*-cocartesian morphisms whose image in* $\tilde{\mathcal{O}}^{\otimes}$ *is inert.*

Assume in addition $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ *is a map of* ∞*-operads, set* $\mathcal{C}'^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}'^{\otimes}$*, we get* $q' : \mathcal{C}'^{\otimes} \to \mathcal{O}'^{\otimes}$ *by base change. Then we have a natural functor* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\tilde{\mathcal{O}}) \to \mathrm{Alg}_{\mathcal{C}'/\mathcal{C}}(\tilde{\mathcal{C}})$ *given by base change.*

Proof. The map $\tilde{q}$ is a cocartesian fibration. For $T = T_1 \oplus \ldots \oplus T_n \in \tilde{\mathcal{O}}^{\otimes}_{\langle n\rangle}$ the inert maps $T \to T_i$ induce an equivalence $\tilde{\mathcal{C}}^{\otimes}_T \widetilde{\to} \prod_i \tilde{\mathcal{C}}^{\otimes}_{T_i}$, as $\alpha(T) \to \alpha(T_i)$ are inert and $\alpha(T) \widetilde{\to} \alpha(T_1) \oplus \ldots \oplus \alpha(T_n)$. By ([28], 2.1.2.12(b)), $\tilde{q}$ is a cocartesian fibration of ∞-operads. Now if $f \in \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\tilde{\mathcal{O}})$ let $F \in \mathrm{Fun}_{\mathcal{C}^{\otimes}}(\mathcal{C}'^{\otimes}, \tilde{\mathcal{C}}^{\otimes})$ be obtained by base change. If h is an inert arrow in $\mathcal{C}'^{\otimes}$ then $\tilde{q}F(h) = fq'(h)$ is inert in $\tilde{\mathcal{O}}^{\otimes}$. Since h is q'-cocartesian, from Remark 2.2.119 we see that $F(h)$ is $\tilde{q}$-cocartesian, so $F(h)$ is inert in $\tilde{\mathcal{C}}^{\otimes}$. So, F is a morphism of ∞-operads. □

Remark 3.0.7. *Let* $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a map of* ∞*-operads. For* $x \in \mathcal{O}$ *there is an essentially unique map* $\mathcal{T}\mathrm{riv}^{\otimes} \to \mathcal{O}^{\otimes}$ *of* ∞ *operads with* $\langle 1\rangle \mapsto x$*. Then* $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{T}\mathrm{riv}^{\otimes} \to \mathcal{T}\mathrm{riv}^{\otimes}$ *is a cocartesian fibration realizing* $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{T}\mathrm{riv}^{\otimes}$ *as an* ∞*-operad. One has canonically* $\mathrm{Alg}_{\mathcal{T}\mathrm{riv}/\mathcal{O}}(\mathcal{C}) \widetilde{\to} \mathrm{Alg}_{/\mathcal{T}\mathrm{riv}}(\mathcal{C} \times_{\mathcal{O}} \mathcal{T}\mathrm{riv}) \widetilde{\to} \mathcal{C}_x$*.*

Proof. It follows from definition of a fibration of ∞-operads that $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{T}\mathrm{riv} \to \mathcal{T}\mathrm{riv}$ is a cocartesian fibration. For any $n \geq 0$ the diagram commutes

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{\langle n\rangle} & \widetilde{\to} & \prod_{i=1}^n \mathcal{C} \\ \downarrow p & & \downarrow \prod p \\ \mathcal{O}^{\otimes}_{\langle n\rangle} & \widetilde{\to} & \prod_{i=1}^n \mathcal{O}, \end{array}$$

where the horizontal arrow are the functor $\rho^i_!$ for the inert maps $\rho^i : \langle n\rangle \to \langle 1\rangle$. So, $x \in \mathcal{O}$ yields an equivalence $\mathcal{C}^{\otimes}_{\bar{x}} \widetilde{\to} \prod_{i=1}^n \mathcal{C}_x$, where $\bar{x} = x \oplus \ldots \oplus x \in \mathcal{C}^{\otimes}_{\langle n\rangle}$. So, the conditions ([28], 2.1.2.12(b)) are satistfied, and the claim follows from ([28], 2.1.2.12). The last claim follows from ([28], 2.1.3.5). □

Remark 3.0.8. *Let* $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a cocartesian fibration of* ∞*-operads,* $\tilde{\mathcal{O}}^{\otimes} \to \mathcal{O}^{\otimes}$ *a map of* ∞*-operads,* $\tilde{p} : \tilde{\mathcal{C}}^{\otimes} \to \tilde{\mathcal{O}}^{\otimes}$ *be obtained from* p *by base change.*
i) We have $\mathrm{Alg}_{\tilde{\mathcal{O}}/\mathcal{O}}(\mathcal{C}) \widetilde{\to} \mathrm{Alg}_{/\tilde{\mathcal{O}}}(\tilde{\mathcal{C}})$ *canonically .*
ii) Assume in addition that $\tilde{\mathcal{O}}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ *factors through* $\mathcal{T}\mathrm{riv}$*. Then* $\mathrm{Alg}_{\tilde{\mathcal{O}}/\mathcal{O}}(\mathcal{C}) \widetilde{\to} \mathrm{Fun}_{\tilde{\mathcal{O}}}(\tilde{\mathcal{O}}, \tilde{\mathcal{C}})$.

Proof. i) By Remark 3.0.6, $\tilde{p}$ is a cocartesian fibration of ∞-operads, and $a : \tilde{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$ is a morphism of ∞-operads. The composition with a gives a functor $\mathrm{Alg}_{/\tilde{\mathcal{O}}}(\tilde{\mathcal{C}}) \to$

$\mathrm{Alg}_{\tilde{\mathcal{O}}/\mathcal{O}}(\mathcal{C})$. We also have the evident functor $\mathrm{Alg}_{\tilde{\mathcal{O}}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\tilde{\mathcal{O}}^{\otimes}}(\tilde{\mathcal{O}}^{\otimes}, \tilde{\mathcal{C}}^{\otimes})$, $f \mapsto \bar{f}$. To see that $\bar{f}$ is a morphism of operads, let h be an inert arrow in $\tilde{\mathcal{O}}^{\otimes}$. Then $f(h)$ is an inert arrow in $\mathcal{C}^{\otimes}$, and $\bar{f}(h) = (f(h), h)$. Since $f(h)$ is p-cocartesian, $\bar{f}(h)$ is $\tilde{p}$-cocartesian by Remark 2.2.119. Since $\tilde{p}(\bar{f}(h)) = h$ is inert, $\bar{f}(h)$ is inert. Thus, we obtained a functor $\mathrm{Alg}_{\tilde{\mathcal{O}}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\tilde{\mathcal{O}}}(\tilde{\mathcal{C}})$. They are inverse to each other.

ii) follows from ([28], 2.1.3.5). □

Proposition 3.0.9. *For maps of ∞-operads $\mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes} \xleftarrow{q} \tilde{\mathcal{O}}^{\otimes}$, $\tilde{\mathcal{C}}^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \tilde{\mathcal{O}}^{\otimes}$ is an ∞-operad, and its projections to $\mathcal{C}^{\otimes}, \tilde{\mathcal{O}}^{\otimes}$ are maps of ∞-operads.*

Proof. Any object of $\mathcal{C}^{\otimes}_{\langle n\rangle}$ writes $c_1 \oplus \ldots \oplus c_n$, $c_i \in \mathcal{C}$, its image in $\mathcal{O}^{\otimes}$ is $o_1 \oplus \ldots \oplus o_n$ with $o_i = p(c_i)$. So, any object of $\tilde{\mathcal{C}}^{\otimes}_{\langle n\rangle}$ is of the form $\tilde{c} := (\oplus_i c_i, \oplus_i \tilde{o}_i)$ with $p(c_i) \widetilde{\to} q(\tilde{o}_i)$. Given $1 \le i \le n$ for the inert map $\rho^i : \langle n\rangle \to \langle 1\rangle$ we have to show that the projection $\tilde{c} \to (c_i, \tilde{o}_i)$ is cocartesian over ρ^i. This follows from ([27], 2.4.1.3(2) and (3)).

So, for each $n \ge 0$ the maps ρ^i yield functors $\rho^i_! : \tilde{\mathcal{C}}^{\otimes}_{\langle n\rangle} \to \tilde{\mathcal{C}}$, hence a functor $\xi : \tilde{\mathcal{C}}^{\otimes}_{\langle n\rangle} \to \prod_{i=1}^n \tilde{\mathcal{C}}$. We have $\tilde{\mathcal{C}}^{\otimes}_{\langle n\rangle} \widetilde{\to} \mathcal{C}^{\otimes}_{\langle n\rangle} \times_{\mathcal{O}^{\otimes}_{\langle n\rangle}} \tilde{\mathcal{O}}^{\otimes}_{\langle n\rangle}$ and the equivalences $\tilde{\mathcal{O}}^{\otimes}_{\langle n\rangle} \widetilde{\to} \prod_{i=1}^n \tilde{\mathcal{O}}$ and similarly for $\mathcal{C}, \tilde{\mathcal{O}}$. They show that ξ is an equivalence.

It remains to check condition 2) in Definition 3.0.1. Let $\tilde{c}' = (c', \tilde{o}') \in \tilde{\mathcal{C}}^{\otimes}_{\langle m\rangle}$, so we are given $p(c') \widetilde{\to} q(\tilde{o}')$. Let $\tilde{c}_i = (c_i, \tilde{o}_i) \in \tilde{\mathcal{C}}$. Let $f : \langle m\rangle \to \langle n\rangle$ be any map in $\mathcal{F}\mathrm{in}_*$. We must show that

$$\mathrm{Map}^f_{\tilde{\mathcal{C}}^{\otimes}}(\tilde{c}', \tilde{c}) \to \prod_{i=1}^n \mathrm{Map}^{\rho^i f}_{\tilde{\mathcal{C}}^{\otimes}}(\tilde{c}', \tilde{c}_i) \tag{4}$$

is an isomorphism in Spc. The mapping spaces in the fibred product are described in my Corollary 2.5.3. We get

$$\mathrm{Map}_{\tilde{\mathcal{C}}^{\otimes}}(\tilde{c}', \tilde{c}) \widetilde{\to} \mathrm{Map}_{\mathcal{C}^{\otimes}}(c', c) \times_{\mathrm{Map}_{\mathcal{O}^{\otimes}}(o', o)} \mathrm{Map}_{\tilde{\mathcal{O}}^{\otimes}}(\tilde{o}, \tilde{o}')$$

and

$$\mathrm{Map}^f_{\tilde{\mathcal{C}}^{\otimes}}(\tilde{c}', \tilde{c}) \widetilde{\to} \mathrm{Map}^f_{\mathcal{C}^{\otimes}}(c', c) \times_{\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(o', o)} \mathrm{Map}^f_{\tilde{\mathcal{O}}^{\otimes}}(\tilde{o}, \tilde{o}')$$

We have similar decompositions for each factor of the RHS of (4). Since the maps analogous to (4) for $\mathcal{C}^{\otimes}, \mathcal{O}^{\otimes}, \tilde{\mathcal{O}}^{\otimes}$ are isomorphisms, (4) is also an isomorphism. (Isomorphisms remain isomorphisms when passing to the limit). (Jacob confirmed in email 15feb2018). □

Note that for an ∞-operad $\mathcal{C}^{\otimes}$ the category $\mathcal{C}$ could be empty. Then $\mathcal{C} \widetilde{\to} *$ is over $\langle 0\rangle \in \mathcal{F}\mathrm{in}_*$.

3.0.10. The ∞-operad $\mathcal{F}\mathrm{in}^{inj}_*$ is obtained from the colored operad, whose set of objects (or colors) is $*$. If I is a finite set, $\mid I \mid > 1$ then $Mul_I = \emptyset$, $Mul(*, *) = \{\mathrm{id}\}$, and $Mul(\emptyset, *) = *$.

3.0.11. I think the following is true. For any ∞-operad $p : \mathcal{O}^{\otimes} \to \mathcal{F}in_*$ consider the inclusion $\mathcal{O}^{\otimes} \times * \stackrel{\mathrm{id}\times\langle 1\rangle}{\to} \mathcal{O}^{\otimes} \times \mathcal{T}riv$. Then $\mathrm{id} : \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$ admits a p-right Kan extension along the above inclusion, say $a : \mathcal{O}^{\otimes} \times \mathcal{T}riv \to \mathcal{O}^{\otimes}$. Here a is a bifunctor, it sends $(x, \langle m\rangle)$ to $x \oplus \ldots \oplus x$, the sum taken m times. Moreover a realizes $\mathcal{O}^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ with $\mathcal{T}riv$. Now the unique map of operads $\mathcal{T}riv \to \mathbb{E}_0^{\otimes}$ yields for $\mathcal{O}^{\otimes} \in Op_{\infty}$ a map $\mathcal{O} \widetilde{\to} \mathcal{O} \otimes \mathcal{T}riv \to \mathcal{O} \otimes \mathbb{E}_0^{\otimes}$. This is a unit transformation for the functor $Op_{\infty} \to Op_{\infty}$, $\mathcal{O}^{\otimes} \mapsto \mathcal{O}^{\otimes} \otimes \mathbb{E}_0^{\otimes}$ realizing the latter as a localization functor (see [28], 2.3.1.8-9). The proof given in [28] depends on a model, but this does not seem very complicated. The image of this functor is the full subcategory of unital ∞-operads in Op_{∞}. It is also a colocalization of Op_{∞} in view of ([28], 2.3.1.11):

Let $\mathcal{O}^{\otimes} \in Op_{\infty}$, let $\mathcal{O}^{\otimes}_*$ be its category of pointed objects. The projection $\mathcal{O}^{\otimes}_* \to \mathcal{O}^{\otimes}$ is a map of ∞-operads, and $\mathcal{O}^{\otimes}_*$ is unital. For any unital $\mathcal{C}^{\otimes} \in Op_{\infty}$ the above projection yields an equivalence $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}_*) \widetilde{\to} \mathrm{Alg}_{\mathcal{C}}(\mathcal{O})$.

3.0.12. For a monoidal ∞-category $A^{\otimes}$ write $A^{\otimes op}$ for the opposite monoidal category. That is, the one obtained by composing $\boldsymbol{\Delta}^{op} \stackrel{A^{\otimes}}{\to} 1-\mathcal{C}at \stackrel{op}{\to} 1-\mathcal{C}at$.

For $1 \le i \le n$ write $\rho^i : [1] \to [n]$ for the map in $\boldsymbol{\Delta}$ given by $i-1, i$. We may view $\rho^i : [n] \to [1]$ as a map in $\boldsymbol{\Delta}^{op}$. Recall that a monoidal ∞-category is a cocartesian fibration $X \to \boldsymbol{\Delta}^{op}$ such that for any $x \in X$ over $[n]$, the functors $\rho^i_! : X_{[n]} \to X_{[1]}$ for $1 \le i \le n$ define an equivalence $X_{[n]} \widetilde{\to} \prod_{i=1}^n X_{[1]}$.

Right-lax monoidal functors from $A_0^{\otimes}$ to $A_1^{\otimes}$ form the full subcategory $\mathrm{Funct}_{\boldsymbol{\Delta}^{op}}(A_0^{\otimes}, A_1^{\otimes})$ spanned by those functors that send any cocartesian arrow in $A_0^{\otimes}$ over $\rho^i : [n] \to [1]$ (in $\boldsymbol{\Delta}^{op}$) to a cocartesian arrow in $A_1^{\otimes}$.

If $F : A_0^{\otimes} \to A_1^{\otimes}$ is a right-lax monoidal functor then for $x, y \in A_0$ it gives rise to morphisms $F(x) \otimes F(y) \to F(x \otimes y)$, $1 \to F(1)$.

Example: let $A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C$ be monoidal functors between monoidal ∞-categories. Let $\gamma = \beta\alpha$. Assume γ^R, β^R are the right adjoints to γ, β. We have the natural morphism of functors $\alpha(\gamma^R) \to \beta^R$. Then it is a morphism of right-lax monoidal functors in addition.

Proof given by Nick. Consider the $(\infty, 2)$-category $\mathcal{C}$, whose objects are monoidal ∞-categories, and whose morphisms are right-lax monoidal functors. If $f : A \to B$ is a 1-morphism in C, which is a strict monoidal functor and as a plain functor admits a right adjoint $f^R : B \to A$ then f^R is the right adjoint in $\mathcal{C}$ to the 1-morphism f.

Now the desired 2-morphism in $\mathcal{C}$ is defined as the composition $\alpha\gamma^R \stackrel{u}{\to} \beta^R\beta\alpha\gamma^R \stackrel{c}{\to} \beta^R$. where $u : \mathrm{id} \to \beta^R\beta$ is the unit, and $c : \gamma\gamma^R \to \mathrm{id}$ is the counit. □

3.0.13. In ([14], 3.2.3) let $A_0^{\otimes}, A_1^{\otimes} \to \boldsymbol{\Delta}^{op}$ be monoidal ∞-categories, $F : A_0 \to A_1$ a functor. The structure of a left-lax nonunital monoidal functor on F is defined as follows.

View $A_i^{\otimes}$ as functors $\boldsymbol{\Delta}^{op} \to 1-\mathcal{C}at$. Let $X_0 \to \boldsymbol{\Delta}$, $X_1 \to \boldsymbol{\Delta}$ be cartesian fibrations corresponding to $A_0^{\otimes}, A_1^{\otimes}$ respectively. Then $e \in \mathrm{Funct}_{\boldsymbol{\Delta}}(X_0, X_1)$ is left-lax nonunital monoidal iff for any injective map $f : [n] \to [m]$ with convexe image, e sends a cartesian arrow over f to a cartesian arrow.

More generally, given cocartesian fibrations of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \leftarrow \mathcal{D}^{\otimes}$ we may define a notion of a left-lax $\mathcal{O}$-monoidal functor as follows. Recall the notation $\mathcal{C}^{\otimes,\overline{op}} \to \mathcal{O}^{\otimes}$ from my Section 3.0.4. By definition, a left-lax $\mathcal{O}$-monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor F that fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes,\overline{op}} & \stackrel{F}{\to} & \mathcal{D}^{\otimes,\overline{op}} \\ & \searrow & \downarrow \\ & & \mathcal{O}^{\otimes}, \end{array}$$

such that F is a map of ∞-operads. Now

$$\mathrm{Fun}^{llax}_{\mathcal{O}}(\mathcal{C},\mathcal{D}) := (Alg_{C^{\otimes,\overline{op}}/\mathcal{O}^{\otimes}}(\mathcal{D}^{\otimes,\overline{op}}))^{op}$$

If $G : A_0 \to A_1$ is a map in $1-\mathcal{C}\mathrm{at}$, $A_i^{\otimes} \to \boldsymbol{\Delta}^{op}$ are structures of monoidal ∞-category on A_i then providing on G a structure of a right-lax nonunital monoidal functor is equivalent to providing on the corresponding functor $G : A_0^{op} \to A_1^{op}$ the structure of a left-lax nonunital monoidal functor.

([14], ch.1, Lemma 3.2.4): let $X^{\otimes} \to \boldsymbol{\Delta}^{op}, Y^{\otimes} \to \boldsymbol{\Delta}^{op}$ be monoidal ∞-categories, $X^{\otimes} \stackrel{G'}{\to} Y^{\otimes} \to \boldsymbol{\Delta}^{op}$ be a right-lax monoidal functor such that the underlying functor $G : X \to Y$ admits a left adjoint $F : Y \to X$. Then F is equipped with a structure of a left-lax monoidal as follows. For each $n \geq 0$ the fibre $G'_{[n]} : X^{\otimes}_{[n]} \to Y^{\otimes}_{[n]}$ is the functor $X^n \to Y^n, x_1 \oplus \ldots \oplus x_n \mapsto G(x_1) \oplus \ldots \oplus G(x_n)$, it has a left adjont $y_1 \oplus \ldots \oplus y_n \mapsto F(y_1) \oplus \ldots \oplus F(y_n)$. So, we may apply my Corollary 13.1.26. Let $X^{\vee,\otimes} \to \boldsymbol{\Delta}, Y^{\vee,\otimes} \to \boldsymbol{\Delta}$ be the corresponding cartesian fibrations. We get a functor $F' : Y^{\vee,\otimes} \to X^{\vee,\otimes}$ in $Cart/\boldsymbol{\Delta}$, which is a left-lax monoidal structure on F.

3.0.14. An inert morphism in $\boldsymbol{\Delta}$ is a morphism $[n] \to [m]$ that induces an isomorphism with a convex subset $[n] \widetilde{\to} \{i, i+1, \ldots, j\}$ of $[m]$. If $\mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is a symmetric monoidal ∞-category then

$$Mon(\mathcal{C}^{\otimes}) \subset \mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\boldsymbol{\Delta}^{op}, \mathcal{C}^{\otimes})$$

is a full subcategory spanned by those functors that send morphisms of the form $[1] \to [n]$, $0 \mapsto i, 1 \mapsto i+1$ in $\boldsymbol{\Delta}$ to cocartesian morphisms in $\mathcal{C}^{\otimes}$ ([28], 4.1.2.15). This is the category of associative algebras in $\mathcal{C}^{\otimes}$. Indeed, using the notations from ([28], 4.1.2.15), $\mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\boldsymbol{\Delta}^{op}, \mathcal{C}^{\otimes})$ identifies with the full subcategory of

$$\mathrm{Funct}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\boldsymbol{\Delta}^{op}, \mathcal{C}^{\otimes} \times_{\mathrm{Fun}_*} \mathcal{A}\mathrm{ss}^{\otimes})$$

spanned by functors that carry inert morphism to inert morphisms.

Equivalently, a functor $\boldsymbol{\Delta}^{op} \to \mathcal{C}^{\otimes}$ over $\mathcal{F}\mathrm{in}_*$ is an associative algebra in $\mathcal{C}^{\otimes}$ iff it sends an inert morphism to a cocartesian morphism.

Let $F : \boldsymbol{\Delta}^{op} \to \mathcal{C}^{\otimes}$ be an associative algebra in $\mathcal{C}$, set $A = F([1])$. The map $[1] \stackrel{02}{\to} [2]$ yields the multiplication $m : A \otimes A \to A$. The map $[1] \to [0]$ yields the unit $1 \to A$. The diagram

$$\begin{array}{ccc} [2] & \stackrel{013}{\to} & [3] \\ \uparrow 02 & & \uparrow 023 \\ [1] & \stackrel{02}{\to} & [2] \end{array}$$

yields the associativity axiom for m.

Lemma 3.0.15. *Let $n \geq 1$, $\mathcal{C}^{\otimes} \to \mathcal{F}in_*$ be a symmetric monoidal ∞-category, where $\mathcal{C}$ is an n-category then $CAlg(\mathcal{C})$ is also an n-category.*

Proof. Given $c \in \mathcal{C}^{\otimes}_{\langle m\rangle}$, $c' \in \mathcal{C}$, let $f : \langle n\rangle \to \langle 1\rangle$ be the active map in $\mathcal{F}in_*$. Then $\mathrm{Map}_{\mathcal{C}^{\otimes}}(c, c') \times_{\mathrm{Map}_{\mathcal{F}in_*}(\langle n\rangle,\langle 1\rangle)} \{f\} \widetilde{\to} \mathrm{Map}_{\mathcal{C}}(\otimes_i c_i, c')$, where $c = c_1 \oplus \ldots \oplus c_n$. Since $\mathcal{F}in_*$ is a 1-category, this shows that $\mathcal{C}^{\otimes}$ is a n-category. Now by (HTT, 2.3.4.8), $\mathrm{Fun}(\mathcal{F}in_*, \mathcal{C}^{\otimes})$ is also an n-category. Let $1-\mathcal{C}at^n \subset 1-\mathcal{C}at$ be the full subcategory spanned by n-categories. The full embedding $1-\mathcal{C}at^n \hookrightarrow 1-\mathcal{C}at$ admits a left adjoint, so $1-\mathcal{C}at^n$ is a localization of $1-\mathcal{C}at$. So, the full subcategory $1-\mathcal{C}at^n \subset 1-\mathcal{C}at$ is stable under all small limits (by my Lemma 4.0.45). So, $\mathrm{Fun}_{\mathcal{F}in_*}(\mathcal{F}in_*, \mathcal{C}^{\otimes})$ is a n-category. So, its full subcategory $CAlg(\mathcal{C})$ is also a n-category. □

3.0.16. *Cartesian symmetric monoidal structure.* For 3.3.3. Let $\mathcal{C} \in 1-\mathcal{C}at$ admit finite products. Their functor $(\mathcal{F}in_*)^{op} \to 1-\mathcal{C}at$ is defined as follows. It sends $(* \in I) \in \mathcal{F}in_*$ to $\mathrm{Funct}((I,*), (\mathcal{C},*))$, the full subcategory of $\mathrm{Funct}(I, \mathcal{C})$ spanned by functors sending $*$ to $*$. Here $* \in \mathcal{C}$ is the terminal object. Since $\mathrm{Funct} : 1-\mathcal{C}at^{op} \times 1-\mathcal{C}at \to 1-\mathcal{C}at$ is a functor, this is well defined. They claim further that one may pass to right adjoints, and this gives a functor $\mathcal{F}in_* \to 1-\mathcal{C}at$. The latter corresponds to a cocartesian fibration $\mathcal{C}^{\times} \to \mathcal{F}in_*$.

For I finite, the right adjoint to the diagonal map $\mathcal{C} \to \mathcal{C}^I$ is the functor $\mathcal{C}^I \to C$, $(c_i) \mapsto \prod_{i\in I} c_i$.

If $\alpha : (J,*) \to (I,*)$ is a map in $\mathcal{F}in_*$ then the restriction along α is a functor $\mathcal{C}^{I-*} \to \mathcal{C}^{J-*}$. Its right adjoint is the functor $\mathcal{C}^{J-*} \to \mathcal{C}^{I-*}$ sending $(c_j)_{j\in J-*}$ to the collection $(r_i)_{i\in I-*}$, where $r_i = \prod_{j\in\alpha^{-1}(i)} c_j$.

The category $\mathcal{C}^{\times}$ is equipped with a canonical functor $cart : \mathcal{C}^{\times} \to \mathcal{C}$, which is a Cartesian structure (in the sense of [28], 2.4.1.1). This functor for each $(I,*)$ restricts to a map $\mathcal{C}^{\times}_{(*\in I)} = \mathcal{C}^{I-*} \to \mathcal{C}$ sending (c_i) to $\prod_{i\in I-*} c_i$. For any map $\alpha : (J,*) \to (I,*)$ in $\mathcal{F}in_*$ and any $(c_j) \in \mathcal{C}^{J-*}$ the image of $\alpha_! : \mathcal{C}^{J-*} \to \mathcal{C}^{I-*}$ under $cart$ is the projection

$$\prod_{j\in J-*} c_j \to \prod_{j\in\alpha^{-1}(I-*)} c_j$$

If $\mathcal{O}^{\otimes}$ is an ∞-operad then, by ([28], 2.4.2.5), the composition with $cart$ induces an equivalence $Alg_{\mathcal{O}}(\mathcal{C}^{\times}) \widetilde{\to} Mon_{\mathcal{O}}(\mathcal{C})$.

I think if $\alpha : (J,*) \to (I,*)$ is a map in $\mathcal{F}in_*$ and $x = \underset{j\in J-*}{\oplus} c_j \in \mathcal{C}^{\times}_{(J,*)}$, $y = \underset{i\in I-*}{\oplus} c_i \in \mathcal{C}^{\times}_{(I,*)}$ then

$$\mathrm{Map}^{\alpha}_{\mathcal{C}^{\times}}(x,y) \widetilde{\to} \prod_{i\in I-*} \mathrm{Map}_{\mathcal{C}}(\prod_{j\in\alpha^{-1}(i)} c_j, c_i)$$

(I have not checked honestly).

3.0.17. Let $\mathcal{C}^{\otimes} \to \mathcal{F}in_*$ be a symmetric monoidal ∞-category. Let $\alpha : \mathcal{F}in_* \to \mathcal{C}^{\otimes}$ be a commutative algebra in $\mathcal{C}$. The relation with the classical notions: set $M = \alpha(\langle 1\rangle)$. For the unique map $b : \langle 0\rangle \to \langle 1\rangle$, $\alpha(b) : 1 \to M$ is the unit section. Both compositions $\langle 1\rangle \stackrel{i}{\to} \langle 2\rangle \stackrel{\rho^i}{\to} \langle 1\rangle$ are the identity maps, which shows that $\alpha(\langle 2\rangle) \widetilde{\to} M \times M \in \mathcal{C}^2$. The unique active map $\gamma : \langle 2\rangle \to \langle 1\rangle$ yields a diagram in $\mathcal{C}^\otimes$

$$\begin{array}{ccccc} \alpha(\langle 2\rangle) = M\times M & \to & \gamma_!(M\times M) & =: M\otimes M \\ & \searrow_{\alpha(\gamma)} & \downarrow & \\ & & M & \end{array}$$

Here the vertical arrow is given by the universal property of cocartesian arrows, it defines the product on M.

3.0.18. For ([28], 2.2.1). If $\mathcal{O}^\otimes \to \mathcal{F}\mathrm{in}_*$ is an ∞-operad, $\mathcal{D}\subset\mathcal{O}$ is a full subcategory stable under equivalences, he denotes $\mathcal{D}^\otimes\subset\mathcal{O}^\otimes$ the full subcategory spanned by objects of the form $D_1\oplus\ldots\oplus D_n$, $D_i\in\mathcal{D}$. Then $\mathcal{D}^\otimes$ is an ∞-operad, $\mathcal{D}^\otimes\to\mathcal{O}^\otimes$ is a map of ∞-operads.

([28], 2.2.1.1) evidently rewrites in the model-independent setting:

Proposition 3.0.19. *Let $p:\mathcal{C}^\otimes\to\mathcal{O}^\otimes$ be a cocartesian fibration of ∞-operads. Let $\mathcal{D}\subset\mathcal{C}$ be a full subcategory stable under equivalence. Assume for any $f\in Mul_{\mathcal{O}}(\{x_i\},y)$ the functor $\otimes_f:\prod_{i=1}^n\mathcal{C}_{x_i}\to\mathcal{C}_y$ sends $\prod_{i=1}^n\mathcal{D}_{x_i}$ to $\mathcal{D}_y$. Then*
1) the restriction map $\mathcal{D}^\otimes\to\mathcal{O}^\otimes$ is a cocartesian fibration of ∞-operads;
2) the inclusion $\mathcal{D}^\otimes\to\mathcal{C}^\otimes$ is a $\mathcal{O}$-monoidal functor;
3) Suppose, for any $x\in\mathcal{O}$, the inclusion $\mathcal{D}_x\to\mathcal{C}_x$ admits a right adjoint L_x. Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\otimes & \stackrel{L^\otimes}{\to} & \mathcal{D}^\otimes \\ & \searrow_p & \downarrow \\ & & \mathcal{O}^\otimes \end{array}$$

and a natural transformation $\alpha : L^\otimes\to\mathrm{id}_{\mathcal{C}^\otimes}$ which exhibits $L^\otimes$ as a colocalization functor (that is, admitting a fully faithful left adjoint). Besides, $L^\otimes$ is a morphism of ∞-operads.

([28], 2.2.1.2-2.2.1.3) make sense in model independent setting. 2.2.1.3 says: let $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}^{St}$ be equipped with a monoidal structure $\mathcal{C}^\otimes\to\boldsymbol{\Delta}^{op}$ such that the tensor product $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is exact in each variable. Assume $\mathcal{C}$ is equipped with a t-structure. He says *the t-structure is compatible with the monoidal structure* if the tensor product sends $\mathcal{C}_{\geq 0}\times\mathcal{C}_{\geq 0}$ to $\mathcal{C}_{\geq 0}$. In this case $\mathcal{C}_{\geq 0}$ inherits a monoidal structure, and the tensor product sends $\mathcal{C}_{\geq n}\times\mathcal{C}_{\geq m}$ to $\mathcal{C}_{\geq n+m}$ (recall Lurie uses homological conventions about t-structures!).

3.0.20. ([28], 2.2.1.9-10) seems important! That's a nice was to get a monoidal structure on a localization of a monoidal category, roughly.

The proof of ([28], 2.2.1.11) is model-independent (the reference to a categorical fibration used at the end is not needed!). But the proof of 2.2.1.9 does depend on a model of quasi-categories! A nice application is ([28], 4.8.2.7).

A comment by Lurie in his email 4/06/2017: assume we are in the situation of ([28], 2.2.1.9). Let $\mathcal{O}'\to\mathcal{O}$ be a map of ∞-operads. We get functors $L':\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})\to\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ given by composition with $L^\otimes:\mathcal{C}^\otimes\to\mathcal{D}^\otimes$ and $R':\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})\to\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ given by composition with $\mathcal{D}^\otimes\hookrightarrow\mathcal{C}^\otimes$. Then L' is left adjoint to R'.

3.0.21. ([28], Def. 2.2.2.1) has a model independent meaning: let $S\times K\stackrel{p}{\to} X\stackrel{q}{\to} S$ be a diagram in $1-\mathcal{C}at$ such that the composition is the projection. Then $X_{p_S/}$ is defined by the cartesian square in $1-\mathcal{C}at$

$$\begin{array}{ccc} S\times_{\mathrm{Funct}(K,X)}\mathrm{Funct}(K^{\triangleright},X)\times_{\mathrm{Funct}(K^{\triangleright},S)} S & \to & S\times_{\mathrm{Funct}(K,S)} S\\ \uparrow & & \uparrow\\ X_{p_S/} & \to & S\end{array}$$

the product in $1-\mathcal{C}at$. Here $p_S/$ is just a symbol, no other meaning, this is a relative over S situation! His presentation via two squares diagram is more comprehensive! This is just a family of under-categories. Compare with ([27], 4.2.2.1).

([28], Theorem 2.2.2.4) seems an important unexpected result! It says the following. If $q:\mathcal{C}^{\otimes}\to\mathcal{O}^{\otimes}$ is a map of ∞-operads, let $p:K\to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a diagram in $1-\mathcal{C}at$. Then for each $x\in\mathcal{O}$ we get a functor $p_x:K\to\mathcal{C}_x$ obtained from $p:K\times\mathcal{O}^{\otimes}\to\mathcal{C}^{\otimes}$ by restricting via $*\stackrel{x}{\to}\mathcal{O}^{\otimes}$. Then the undercategories $(\mathcal{C}_x)_{p_x/}$ (resp., overcategories $(\mathcal{C}_x)_{/p_x}$) naturally organize into a fibrations of ∞-operads $\mathcal{C}^{\otimes}_{p_{\mathcal{O}}/}\to\mathcal{O}^{\otimes}\leftarrow\mathcal{C}^{\otimes}_{/p_{\mathcal{O}}}$.

For example, the map $\mathcal{C}^{\otimes}_{p_{\mathcal{O}}/}\to\mathcal{O}^{\otimes}$ is defined as follows. Given $Y\in 1-\mathcal{C}at$ and a functor $Y\to\mathcal{O}^{\otimes}$, its lifting to a map $Y\to\mathcal{C}^{\otimes}_{p_{\mathcal{O}}/}$ is given by a commutative diagram

$$\begin{array}{ccccc} Y\times K & \to & Y\times K^{\triangleright} & \to & Y\\ \downarrow & & \downarrow & & \downarrow\\ \mathcal{O}^{\otimes}\times K & \stackrel{p}{\to} & \mathcal{C}^{\otimes} & \to & \mathcal{O}^{\otimes},\end{array}$$

where the composition in the low row is the projection. The evaluation at the cone point of $K^{\triangleright}$ gives a map $\mathcal{C}^{\otimes}_{p_{\mathcal{O}}/}\to\mathcal{C}^{\otimes}$. He then claims that an arrow in $\mathcal{C}^{\otimes}_{p_{\mathcal{O}}/}$ is inert iff its image in $\mathcal{C}^{\otimes}$ is inert.

3.0.22. *Monoidal envelopes.* ([28], 2.2.4). Let $\mathcal{C}^{\otimes}\to\mathcal{O}^{\otimes}$ be a map of ∞-operads, write $\mathrm{Act}(\mathcal{O}^{\otimes})\subset\mathrm{Fun}([1],\mathcal{O}^{\otimes})$ for the full subcategory of active morphisms. Lurie denotes $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}=\mathcal{C}^{\otimes}\times_{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})}\mathrm{Act}(\mathcal{O}^{\otimes})$, this is the monoidal envelope of $\mathcal{C}^{\otimes}$. The evaluation at 1 gives a morphism $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}\to\mathcal{O}^{\otimes}$, which is a cocartesian fibration of ∞-operads ([28], 2.2.4.4).

In particular, let $\mathcal{C}^{\otimes}_{act}\subset\mathcal{C}^{\otimes}$ be the subcategory with all objects, whose morphisms are precisely active morphisms in $\mathcal{C}^{\otimes}$. Then $\mathcal{C}^{\otimes}_{act}$ has a canonical structure of a symmetric monoidal category (this is an underlying ∞-category of a symmetric monoidal ∞-category) ([28], 2.2.4.5).

The fully faithful embedding $i:\mathcal{C}^{\otimes}\hookrightarrow\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ comes from the pull back under the map $\mathcal{O}^{\otimes}\to\mathrm{Act}(\mathcal{O}^{\otimes})$ given by constant maps $\mathcal{O}^{\otimes}\to\mathrm{Fun}([1],\mathcal{O}^{\otimes})$. The key claim here is ([28], 2.2.4.9): for any cocartesian fibration of ∞-operads $\mathcal{D}^{\otimes}\to\mathcal{O}^{\otimes}$ the inclusion i induces an equivalence of ∞-categories

$$\mathrm{Fun}^{lax}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})\widetilde{\to}\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C}),\mathcal{D})$$

Here $\mathrm{Fun}^{lax}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})\subset\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})$ is the full subcategory spanned by maps of ∞-operads.

The proof uses ([28], Lemma 2.2.4.11), its proof is model-independent!

3.0.23. The notion of a bifunctor of ∞-operads is given in ([28], 2.2.5.3). For Lurie, the lexicografical order on $\langle m\rangle^0 \times \langle n\rangle^0$ is

$$(1,1),\dots,(1,n),(2,1),\dots,(2,n),\dots,(m,1),\dots,(m,n)$$

He identifies in this way $\langle m\rangle^0 \times \langle n\rangle^0 \tilde{\to} \langle mn\rangle^0$. Then the functor $\wedge : \mathcal{F}in_* \times \mathcal{F}in_* \to \mathcal{F}in_*$ sends $\langle m\rangle, \langle n\rangle$ to $\langle mn\rangle$, and for a pair of maps $f : \langle m\rangle \to \langle m'\rangle, g : \langle n\rangle \to \langle n'\rangle$ the map $f \wedge g : \langle mn\rangle \to \langle m'n'\rangle$ is induced by the above identification $\langle m\rangle^0 \times \langle n\rangle^0 \tilde{\to} \langle mn\rangle^0$. The key property is that if both f, g are inert then $f \wedge g$ is also inert!

Given ∞-operads $\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}$ the category of bifunctors

$$BiFunc(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{O}''^{\otimes}) \subset \mathrm{Funct}_{\mathcal{F}in_*}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})$$

is a full subcategory.

Definition 3.0.24. *([28], 2.2.5.3) Let $f : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor. For any ∞-operad $\mathcal{C}^{\otimes}$ the composition with f yields a functor $\theta : \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \to BiFunc(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes})$. Then f exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ if θ is an equivalence for any ∞-operad $\mathcal{C}^{\otimes}$.*

The proof of the existence of the tensor product of ∞-operads in ([28], 2.2.5.6) depends on a model, not clear what would be a model-independent proof.

The tensor product of infinity operads actually comes from a symmetric monoidal structure on Op_{∞} ([28], 2.2.5.13).

3.0.25. *Day convolution.* ([28], 2.2.6.1-2.2.6.2) have a nice model-independent meaning. Given a map of ∞-operads $\tilde{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$ and a cocartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ this defines a notion of a *norm of* $\tilde{\mathcal{C}}^{\otimes}$ *along* p. The fact that this definition makes sense, that is, the existence of the base change functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\tilde{\mathcal{O}}) \to \mathrm{Alg}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{C}/\mathcal{C}}(\tilde{\mathcal{O}} \times_{\mathcal{O}} \mathcal{C})$ comes from my Remark 3.0.6.

Recall the ∞-category Op_{∞} of ∞-operad from Section 3.0.3. Let $\tilde{\mathcal{O}}^{\otimes} \to \mathcal{O}^{\otimes}$ be the norm of $\tilde{\mathcal{C}}^{\otimes}$ along p. It is characterized by the functorial isomorphism for $\mathcal{O}'^{\otimes} \in (Op_{\infty})_{/\mathcal{O}^{\otimes}}$

$$\mathrm{Map}_{(Op_{\infty})_{/\mathcal{O}^{\otimes}}}(\mathcal{O}'^{\otimes}, \tilde{\mathcal{O}}^{\otimes}) \tilde{\to} \mathrm{Map}_{(Op_{\infty})_{/\mathcal{C}^{\otimes}}}(\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \tilde{\mathcal{C}}^{\otimes})$$

([28], 2.2.6.4) is correct model-independent. Construction ([28], 2.2.6.7) and examples 2.2.6.9, 2.2.6.10 have model independent meaning, they seem important! The Day convolution itself is given in ([28], 2.2.6.17).

If $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration of ∞-operads, $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ is a map of ∞-operads, the map $\mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes} \to \mathcal{O}^{\otimes}$ from ([28], Construction 2.2.6.7) is characterized by: functorially for $\mathcal{O}'^{\otimes} \in (Op_{\infty})_{/\mathcal{O}^{\otimes}}$ one has

$$\mathrm{Map}_{(Op_{\infty})_{/\mathcal{O}^{\otimes}}}(\mathcal{O}'^{\otimes}, \mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes}) \tilde{\to} \mathrm{Map}_{(Op_{\infty})_{/\mathcal{O}^{\otimes}}}(\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$$

3.0.26. For *colimits of algebras* ([28], 3.2.3). The important thing here is, I think, ([28], 3.2.3.1), a model independent claim. To formulate it recall that $K \in 1-\mathcal{C}\mathrm{at}$ is called sifted iff K is not empty, and $K \to K \times K$ is cofinal.

Proposition 3.0.27 ([28], 3.2.3.1)**.** *let $K \in 1-\mathcal{C}\mathrm{at}$ be sifted, $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a cocartesian fibration of ∞-operads, which is compatible with K-indexed colimits. Then*

- $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{C}^\otimes)$ *admits K-indexed colimits;*
- *a map* $\bar{f} : K^\triangleright \to \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{C}^\otimes)$ *is a colimit diagram iff for each* $x \in \mathcal{O}^\otimes$, $\bar{f}_x : K^\triangleright \to \mathcal{C}^\otimes$ *is a colimit diagram;*
- *the full subcategories* $\mathrm{Fun}^\otimes_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) \subset \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{C}^\otimes)$ *are stable under K-indexed colimits;*
- *a map* $\bar{f} : K^\triangleright \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ *is a colimit diagram iff for each* $x \in \mathcal{O}$, $\bar{f}_x : K^\triangleright \to \mathcal{C}_x$ *is a colimits diagram*

(in the above proposition the last property implies that the forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ preserves colimits).

The following idea is hidden in some proof by Lurie, but is useful to underline (its proof uses the ∞-categorical Bar-Beck theorem). Let κ be an uncountable regular cardinal, $\mathcal{O}^\otimes$ a κ-small ∞-operad, $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ a cocartesian fibration of ∞-operads compaible with κ-small colimits. Then the forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ admits a left adjoint denoted, say F. Say for brevity that $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is free if it is in the essential image of F. Then any $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ can be presented as a geometric realization of a simplicial object $A_\bullet \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ such that for any $n \geq 0$, A_n is free.

Remark 3.0.28 ([28], 3.2.2.6)**.** *Let* $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *be a map of ∞-operads,* $\gamma : A \to A'$ *a morphism in* $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$*. If for any* $x \in \mathcal{O}$, $\gamma(x) : A(x) \to A'(x)$ *is an equivalence in* $\mathcal{C}$ *then γ is an equivalence. That is, the forgetful functor* $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ *is conservative.*

For example, if $\mathcal{C}$ is the category of vector spaces over a field k, $\mathcal{C}^\otimes$ is the usual symmetric monoidal structure on it then the coproducts in $AssAlg(\mathcal{C})$ are complicated (a description is found in wiki), and the forgetful functor $AssAlg(\mathcal{C}) \to \mathcal{C}$ does not preserve coproducts.

3.0.29. Let $\mathcal{O}^\otimes \to \mathcal{F}in_*$ be an ∞-operad. Lurie calls it unital if for any $x \in \mathcal{O}$ and (any) $* \in \mathcal{O}_{\langle 0\rangle}$, $\mathrm{Map}_{\mathcal{O}^\otimes}(*, x) \widetilde{\to} *$ in Spc.

If $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a map of ∞-operads and $\mathcal{O}^\otimes$ is unital, Lurie defines a notion of a trivial algebra object in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ in ([28], 3.2.1.7). When it exists, this is an initial object of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.

For example, if $\mathcal{C}^\otimes$ is a symmetric monoidal category then trivial algebra object in $\mathrm{Alg}(\mathcal{C})$ exists, and $\mathcal{A} \in \mathrm{Alg}(\mathcal{C})$ is initial in $\mathrm{Alg}(\mathcal{C})$ iff the unit map $1 \to A = \mathcal{A}(\langle 1\rangle)$ is an equivalence in $\mathcal{C}$ ([28], 3.2.1.9). Here $1 \in \mathcal{C}$ is the unit.

3.0.30. If $f : \mathcal{O}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$ is a bifunctor then for any $x \in \mathcal{O}$ the restriction of f to $\{x\} \times \mathcal{O}'^\otimes$ is a map of ∞-operads $\mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$. If, more generally, $x \in \mathcal{O}^\otimes$, the induced map $\mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$ is not a map of operads but sends an inert morphism to an inert morphism. Thus, f yields a functor $\mathcal{O} \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}'')$.

If $f : \mathcal{O}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$ is a bifunctor and $q : \mathcal{C}^\otimes \to \mathcal{O}''^\otimes$ is a map of ∞-operads then ([28], 3.2.4.1) actually says the following. The category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^\otimes$ is defined as the full subcategory of

$$\mathcal{O}^\otimes \times_{\mathrm{Funct}(\mathcal{O}'^\otimes, \mathcal{O}''^\otimes)} \mathrm{Funct}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$$

consisting of those objects whose projection to $\mathrm{Funct}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ is a functor sending an inert morphism to an inert morphism. (However, the latter projection is not necessarily a map of infinity operads, it may not respect the projection to $\mathcal{F}in_*$).

For any $x \in \mathcal{O}$ its image in $\mathrm{Funct}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})$ is a map of ∞-operads, and then the fibre of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}$ at x is $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})$ for this particular map of ∞-operads $\mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$.

Proposition 3.0.31. *([28], 3.2.4.3) i)* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ *is a map of* ∞*-operads. A morphism* α *in* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}$ *is inert iff its image in* $\mathcal{O}^{\otimes}$ *is inert and for any* $x \in \mathcal{O}'$ *the evaluation* $\alpha(x)$ *is inert in* $\mathcal{C}^{\otimes}$*.*
ii) If q *is a* $\mathcal{O}''$*-monoidal category then* $p : \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ *is a* $\mathcal{O}$*-monoidal category. In this case a map* α *in* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^{\otimes}$ *is* p*-cocartesian iff for any* $x \in \mathcal{O}' \subset \mathcal{O}'^{\otimes}$ *the image* $\alpha(x)$ *in* $\mathcal{C}^{\otimes}$ *is cocartesian with respect to* $q : \mathcal{C}^{\otimes} \to \mathcal{O}''^{\otimes}$*.*

Example: for an ∞-operad $q : \mathcal{C}^{\otimes} \to \mathcal{F}in_*$ and the unique bifunctor $\mathcal{F}in_* \times \mathcal{O}^{\otimes} \to \mathcal{F}in_*$ this gives the ∞-operad denoted by Lurie

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} := \mathrm{Alg}_{\mathcal{O}/\mathcal{F}in_*}(\mathcal{C})^{\otimes} \to \mathcal{F}in_* \tag{5}$$

So, $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} \subset \mathcal{F}in_* \times_{\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{F}in_*)} \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ is a full subcategory. The fibre of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ over $\langle 1\rangle$ is $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. For each $x \in \mathcal{O}$ we get the evaluation functor $e_x : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$, which is a map of ∞-operads.

If $q : \mathcal{C}^{\otimes} \to \mathcal{F}in_*$ is a symmetric monoidal category then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is a symmetric monoidal category, and for $x \in \mathcal{O}$, $e_x : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ is symmetric monoidal. This means that the tensor product here is taken pointwise. Namely, if $f_i \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ for $i = 1, \dots, n$ and $(\langle n\rangle, \oplus f_i) \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ over $\langle n\rangle \in \mathcal{F}in_*$ then for the unique active map $\alpha : \langle n\rangle \to \langle 1\rangle$ let $b : (\langle n\rangle, \oplus f_i) \to (\langle 1\rangle, g)$ be a cocartesian morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ over α. Then for each $x \in \mathcal{O}$ the map

$$b(x) : \oplus_{i=1}^n f_i(x) \to g(x)$$

is cocartesian in $\mathcal{C}^{\otimes}$ lying over α. In other words, $(\otimes_{i=1}^n f_i)(x) \widetilde{\to} \otimes_{i=1}^n f_i(x)$ in $\mathcal{C}$. By ([28], 3.2.4.7), the symmetric monoidal structure on $CAlg(\mathcal{C})^{\otimes}$ is cocartesian, so that $CAlg(\mathcal{C})$ admits finite coproducts.

By ([28], 3.2.4.5), the map $\wedge$ exhibits $\mathcal{F}in_*$ as the tensor product of $\mathcal{F}in_*$ with itself. The functor $Op_\infty \to Op_\infty$, $\mathcal{O}^{\otimes} \mapsto \mathcal{O}^{\otimes} \otimes \mathcal{F}in_*$ is a localization functor, its essential image consists precisely of cocartesian ∞-operads ([28], 3.2.4.6).

Remark 3.0.32. *Let* $\mathcal{O}^{\otimes}, B^{\otimes}, C^{\otimes}$ *be* ∞*-operads, consider* $\mathrm{Alg}_B(C)$ *as the underlying* ∞*-category of the* ∞*-operad* $\mathrm{Alg}_B(C)^{\otimes}$ *defined above. Then one has an equivalence* $Bifun(\mathcal{O}^{\otimes}, B^{\otimes}; C^{\otimes}) \widetilde{\to} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_B(C))$*.*

3.0.33. *About the cocartesian monoidal structure* Let $\mathcal{C} \in 1-\mathcal{C}at$ then we have the ∞-operad $\mathcal{C}^{\sqcup} \to \mathcal{F}in_*$ defined in ([28], 2.4.3.1). That construction is model-dependent. Assume $\mathcal{C}$ has final object $*$. To get $\mathcal{C}^{\sqcup}$ in a model-independent way, consider the functor $\mathcal{F}in_*^{op} \to 1-\mathcal{C}at$ sending $(* \in I) \in \mathcal{F}in_*$ to $\mathrm{Fun}((I, *), (\mathcal{C}, *))$, the full subcategory in $\mathrm{Fun}(I, \mathcal{C})$ sending $*$ to $*$. Let $\mathcal{C}^{\sqcup} \to \mathcal{F}in_*$ be the cartesian fibration associated to this functor. This is the desired operad. The fibre $\mathcal{C}^{\sqcup}_{\langle n\rangle} \widetilde{\to} \mathcal{C}^n$. Given a map $f : \langle n\rangle \to \langle m\rangle$

in $\mathcal{F}in_*$, $c = c_1 \oplus \ldots \oplus c_n \in \mathcal{C}^n$, $c' = c'_1 \oplus \ldots c'_m \in \mathcal{C}'^m$, the mapping space is

$$\mathrm{Map}^f_{\mathcal{C}^\sqcup}(c, c') \widetilde{\to} \prod_{i \in \langle m\rangle^0} \prod_{f(j)=i} \mathrm{Map}_{\mathcal{C}}(c_j, c'_i)$$

If $\mathcal{C}$ admits finite coproducts iff $\mathcal{C}^\sqcup \to \mathcal{F}in_*$ is a symmetric monoidal category. Then this symmetric monoidal structure is cocartesian.

Under this assumption ([28], 2.4.3.5) says the following. The projection $\Gamma^* \to \mathcal{F}in_*$ yields a map $\mathcal{C} \times \mathcal{F}in_* \to \mathcal{C}^\sqcup$ over $\mathcal{F}in_*$ sending $(c, \langle n\rangle)$ to $c \oplus \ldots \oplus c \in \mathcal{C}^n = \mathcal{C}^\sqcup_{\langle n\rangle}$. It corresponds via the bijection of ([28], 2.4.3.1) to the projection

$$(\mathcal{C} \times \mathcal{F}in_*) \times_{\mathcal{F}in_*} \Gamma^* \widetilde{\to} \mathcal{C} \times \Gamma^* \to \mathcal{C}$$

The map $h : \mathcal{C} \times \mathcal{F}in_* \to \mathcal{C}^\sqcup$ is not a morphism in $(Cart_{/\mathcal{F}in_*})_{strict}$, the image of a cartesian arrow is not always cartesian! In the model-independent setting the map h can be defined as the left Kan extension of its restriction to the full subcategory $\mathcal{C} \times \mathcal{D} \subset \mathcal{C} \times \mathcal{F}in_*$. Here $\mathcal{D} \subset \mathcal{F}in_*$ is the full subcategory spanned by $\langle 1\rangle$. The functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C}^\sqcup$ is easy to define.

For an operad $\mathcal{O}^\otimes$ the base change $\mathcal{O}^\otimes \to \mathcal{F}in_*$ gives the map $\mathcal{C} \times \mathcal{O}^\otimes \to \mathcal{C}^\sqcup \times_{\mathcal{F}in_*} \mathcal{O}^\otimes$. Let $\mathcal{A}^\otimes = \mathcal{C}^\sqcup \times_{\mathcal{F}in_*} \mathcal{O}^\otimes$, this is an operad, and the latter map yields a functor $\mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{A})$. In particular, for $\mathcal{O}^\otimes = \mathcal{F}in_*$ this gives a functor $\mathcal{C} \to CAlg(\mathcal{C})$, where we regard $\mathcal{C}$ as the underlying ∞-category of $\mathcal{C}^\sqcup$. By ([28], 2.4.3.10), the latter functor is an equivalence.

Example: for $\mathcal{C} = *$ the operad $\mathcal{C}^\sqcup$ is $\mathcal{F}in_*$.

Definition 3.0.34. *([28], 2.4.3.7) An operad is cocartesian if it is equivalent to the operad $\mathcal{C}^\sqcup$ for some ∞-category $\mathcal{C}$.*

Example: for any operad $\mathcal{C}^\otimes$ consider $CAlg(\mathcal{C})^\otimes$ given by (5), then $CAlg(\mathcal{C})^\otimes$ is a cocartesian operad by ([28], 3.2.4.10), so $CAlg(\mathcal{C}) \to CAlg(CAlg(\mathcal{C}))$ is an equivalence.

([28], 2.4.3.9, 2.4.3.18) are formulated model-independently. ([28], 2.4.3.9): Let $\mathcal{O}^\otimes$ be a unital ∞-operad and let $\mathcal{C}^\otimes$ be a coCartesian ∞-operad. Then the restriction functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ is an equivalence in $1-\mathcal{C}$at. ([28], 2.4.3.18): let $\mathcal{C} \in 1-\mathcal{C}$at, $\mathcal{D}^\otimes \in Op_\infty$. Viewing $\mathcal{C}$ as the ∞-category underlying $\mathcal{C}^\sqcup$, one has canonically $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D}) \widetilde{\to} \mathrm{Fun}(\mathcal{C}, CAlg(\mathcal{D}))$.

If $\mathcal{C}$ admits finite coproducts then $\mathcal{C}^\sqcup$ is a symmetric monoidal category ([28], 2.4.3.17).

3.0.35. The section ([28], 3.1.1) about operadic colimits diagrams contains the following useful Definition ([28], 3.1.1.18). Let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration of ∞-operads, $K \in 1-\mathcal{C}$at. Then q is *compatible with K-indexed colimits* iff the two conditions are satisfied:

- for any $x \in \mathcal{O}$, $\mathcal{C}_x$ admits K-indexed colimits;
- for every operation $f \in Mul_{\mathcal{O}}(\{x_i\}_{1\le i\le n}, y)$ the functor $\otimes_f : \prod_i \mathcal{C}_{x_i} \to \mathcal{C}_y$ preserves K-indexed colimits separately in each variable.

This notion is used, for example, in his study of colimits of algebras in ([28], 3.2.3).

3.0.36. For ([28], 3.2.1.1), initial objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads, assume $\mathcal{O}^{\otimes}$ unital. Lurie introduces the following notion. For $x \in \mathcal{O}$, a morphism $q : c_0 \to 1_x$ in $\mathcal{C}^{\otimes}$ *exhibits* $1_x \in \mathcal{C}_x$ *as a* x*-unit object* iff $c_0 \in \mathcal{C}^{\otimes}_{\langle 0\rangle}$, and q is p-cocartesian. One does not need to know the notion of "operadic p-colimit" here.

Lurie says that $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *has unit objects* iff each $x \in \mathcal{O}$ admits a x-unit object $c_0 \to 1_x$ (this is equivalent according to 3.2.1.5 to his definition ([28], 3.2.1.1)).

Main results here are ([28], 3.2.1.8, 3.1.2.9): assume $\mathcal{O}^{\otimes}$ unital, $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads, $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Then A is initial in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ iff for any $x \in \mathcal{O}^{\otimes}$ the induced map (recall that $\mathcal{O}^{\otimes}$ is pointed) $A(0) \to A(x)$ exhibits $A(x)$ as a unit object, here $0 \in \mathcal{O}^{\otimes}$ is the zero object.

3.0.37. Limits in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ are easy to calculate, see ([28], 3.2.2), even in a relative situation when we have a map of operads $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over some $\mathcal{O}^{\otimes}$, and we are interested in limits relative to the map $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$. The basic thing here is ([28], 3.2.2.4). (Fibration of ∞-operads in the model-independent setting means simply a map of ∞-operads).

3.0.38. I think a comparison of Dennis' definition of a monoidal infinity category with that of Lurie may be obtained from ([28], 4.7.1).

In ([28], 2.3.3.6) Lurie introduced a notion of a (weak) approximation to an infinity operad $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$. This is a categorical fibration $\mathcal{C} \to \mathcal{O}^{\otimes}$ with some properties. Then he introduced the following notion.

Definition 3.0.39 ([28], 2.3.3.20)**.** *Let $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$, $\mathcal{O}'^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be operads, $f : \mathcal{C} \to \mathcal{O}$ be a weak approximation to $\mathcal{O}^{\otimes}$. A functor $A : \mathcal{C} \to \mathcal{O}'^{\otimes}$ is a $\mathcal{C}$-algebra object of $\mathcal{O}'$ if two conditions hold:*
i) the diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{A} & \mathcal{O}'^{\otimes} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\otimes} & \to & \mathcal{F}\mathrm{in}_* \end{array}$$

ii) Let $c \in \mathcal{C}$ over $\langle n\rangle \in \mathcal{F}\mathrm{in}_*$, for $1 \le i \le n$ pick a locally cocartesian (over $\mathcal{F}\mathrm{in}_*$) morphism $\alpha_i : c \to c_i$ over $\rho^i : \langle n\rangle \to \langle 1\rangle$. Then $A(\alpha_i)$ is inert in $\mathcal{O}'\otimes$.

Then $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ denotes the full subcategory of $\mathrm{Funct}_{\mathcal{O}^{\otimes}}(\mathcal{C}, \mathcal{O}'^{\otimes})$ spanned by $\mathcal{C}$-algebra objects. The key thing about approximations is ([28], 2.3.3.23) saying the following in particular: let $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$, $\mathcal{O}'^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be operads, $f : \mathcal{C} \to \mathcal{O}$ be a weak approximation to $\mathcal{O}^{\otimes}$. Let $\theta : \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ be the functor given by composition with f. If f induces an equivalence $\mathcal{C}_{\langle 1\rangle} \to \mathcal{O}$ then θ is an equivalence. A version of ([28], 2.3.3.23) for monoids instead of algebras in general is ([28], 4.1.2.10).

For example, by ([28], 4.1.2.10), the functor $Cut : \boldsymbol{\Delta} \to \mathcal{A}\mathrm{ss}^{\otimes}$ is an approximation. For this reason taking into account ([28], 4.1.2.11), Dennis and Nick'a definition of a symmetric monoidal category coincides with that of Lurie ([28], 4.1.1.10).

3.0.40. If $\mathcal{C}$ is a monoidal infinity category, assume that $\mathcal{C}$ admits countable colimits and the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves countable colimits separately in each variable. Then the forgetful functor $\mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $Fr : \mathcal{C} \to \mathrm{Alg}(\mathcal{C})$, which associates to c a free algebra $\sqcup_{n\ge 0}\, c^{\otimes n}$ ([28], 4.1.1.14).

More general claims about free algebras are obtained in ([28], 3.1.3). In particular, ([28], 3.1.3.5), a model-independent claim. Its further simplification ([28], 3.1.3.6): let κ be an uncounable regular cardinal, $\mathcal{O}^{\otimes}$ a κ-small ∞-operad, $p : \mathcal{C}^{\times} \to \mathcal{O}^{\otimes}$ a cocartesian fibration of ∞-operads compatible with κ-small colimits. Then the forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ admits a left adjoint.

The construction of free algebras ([28], 3.1.3.9) uses the following idea. Let Σ_n be the symmetric group on n elements. The full subcategory of $\mathcal{T}\mathrm{riv}$ generated by the object $\langle n\rangle$ is $B(\Sigma_n)$. Given a functor $h : B(\Sigma_n) \to \mathcal{C}$ in $1-\mathcal{C}\mathrm{at}$ sending $* \mapsto c \in \mathcal{C}$, we get an action of Σ_n on c. The colimit of h should be thought of as the definition of the coinvariants of this action of Σ_n on c.

One more example is ([28], 3.1.3.14): let $\mathcal{F} : \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be a symmetric monoidal ∞-category, assume $\mathcal{C}$ admits countable colimits, and for any $x \in \mathcal{C}$ the functor $\mathcal{C} \to \mathcal{C}$, $y \mapsto x \otimes y$ preserves countable colimits. Then $CAlg(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint given by $c \mapsto \sqcup_{n\geq 0} \mathrm{Sym}^n(c)$. The notation $\mathrm{Sym}^n(c)$ is that of ([28], 3.1.3.9-10). In this case it can be interpreted as follows. Let $\mathcal{D} \subset \mathcal{T}\mathrm{riv}$ be the full subcategory spanned by $\langle n\rangle$, so $\mathcal{D} \widetilde{\to} B(\Sigma_n)$. Pick an equivalence $\mathcal{C}^{\otimes}_{\langle n\rangle} \widetilde{\to} \prod_{i=1}^n \mathcal{C}$. We get a functor $a : \mathcal{D} \to \prod_{i=1}^n \mathcal{C}$ sending $\langle 1\rangle$ to $c \oplus \ldots \oplus c$ and sending $\sigma \in \Sigma_n$ to the corresponding permutation. Let $\beta : \langle n\rangle \to \langle 1\rangle$ be the unique active map in $\mathcal{F}\mathrm{in}_*$, $f_! : \mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}$ the corresponding product functor. Let $F = f_! a : \mathcal{D} \to \mathcal{C}$, then $\mathrm{Sym}^n(c)$ is $\mathrm{colim}\, F$.

If $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration of ∞-operads and $c \in \mathcal{C}$ then there is a notion of a free $\mathcal{O}$-algebra in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ generated by c ([28], 3.1.3.12), its concrete description is given in ([28], 3.1.3.13).

3.0.41. For 3.1.5. Apply my Section 2.2.120, we get that the evaluation $\mathrm{Fun}([1], 1-\mathcal{C}\mathrm{at}) \to 1-\mathcal{C}\mathrm{at}$ at 1 is a cartesian fibration, because $1-\mathcal{C}\mathrm{at}$ admits fibered products. Let $\mathcal{X} \subset \mathrm{Fun}([1], 1-\mathcal{C}\mathrm{at})$ be the full subcategory spanned by cartesian fibrations. By Remark 3.0.42 below, this is also a cartesian fibration. By strengthening, we get a functor $\mathcal{F} : (1-\mathcal{C}\mathrm{at})^{op} \to 1-\mathcal{C}\mathrm{at}$ sending $\mathcal{C}$ to $Cart_{/\mathcal{C}}$. Restricting, we get a functor $F : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ sending $[n]$ to $Cart_{/[n]^{op}}$, and a map $\alpha : [n] \to [m]$ to the pull-back along $\alpha : [n]^{op} \to [m]^{op}$.

We have functorially $\mathcal{C}^{\mathrm{Spc}} \subset \mathcal{C}$ for $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$. We can similarly define a functor $\mathcal{F}' : (1-\mathcal{C}\mathrm{at})^{op} \to 1-\mathcal{C}\mathrm{at}$ sending $\mathcal{C}$ to $Cart_{/\mathcal{C}^{\mathrm{Spc}}}$, and a natural map $\mathcal{F} \to \mathcal{F}'$ given by the pull-back under $\mathcal{C}^{\mathrm{Spc}} \subset \mathcal{C}$. So, we get a functor $F' : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ sending $[n]$ to $Cart_{/([n]^{op})^{\mathrm{Spc}}} \widetilde{\to} 1-\mathcal{C}\mathrm{at} \times \ldots \times 1-\mathcal{C}\mathrm{at}$. We have a constant functor $G : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ with value $*$. Consider the map $G \to F'$ given for any $[n]$ by the arrow $(\mathcal{C}, \ldots, \mathcal{C}) : * \to 1-\mathcal{C}\mathrm{at} \times \ldots \times 1-\mathcal{C}\mathrm{at}$. Then $F \times_{F'} G \in \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}\mathrm{at})$ is the desired functor.

Remark 3.0.42. *Let $f : \mathcal{D} \to \mathcal{C}$ be a cartesian fibration, $\mathcal{D}^0 \subset \mathcal{D}$ a full subcategory with the property: for any $d \in \mathcal{D}^0$ and any arrow $\alpha : c \to f(d)$ in $\mathcal{C}$, let $d' \to d$ be a f-cartesian arrow in $\mathcal{D}$ over α, then $d' \in \mathcal{D}^0$. This implies that $\mathcal{D}^0 \to \mathcal{C}$ is a cartesian fibration.*

3.0.43. *Flat morphisms.* Let $f : \mathcal{X} \to \mathcal{S}$ be a map in $1-\mathcal{C}\mathrm{at}$. The property of f being flat defined in ([28], B.3.8) makes sense in a model-independent setting.

Recall first that given $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ and an arrow $c_0 \to c_1$ in $\mathcal{C}$, one has $\mathcal{C}_{c_0//c_1} \in 1-\mathcal{C}\mathrm{at}$ defined in ([14], ch.1, 1.3.7). Now ([28], B.3.2) could be used as a definition of a flat morphism $f : \mathcal{X} \to [2]$. Namely, let $\mathcal{C} = \mathcal{X}_0, \mathcal{D} = \mathcal{X}_1, \mathcal{E} = \mathcal{X}_2$ be the fibres. Then $f : \mathcal{X} \to [2]$ is flat iff for any morphism $\alpha : c \to e$ in $\mathcal{X}$ with $c \in \mathcal{C}, e \in \mathcal{E}$ the category $\mathcal{D} \times_{\mathcal{X}} \mathcal{X}_{c//e}$ is contractible, that is, $\mid \mathcal{D} \times_{\mathcal{X}} \mathcal{X}_{c//e} \mid \widetilde{\to} *$ in Spc.

Examples ([28], B.3.4-5) here are a good illustration: let $f : \mathcal{X} \to [2]$ be a map in $1 - \mathcal{C}\mathrm{at}$. Assume for any $c \in \mathcal{C}$ there is a cocartesian arrow $c \to d$ in $\mathcal{X}$ over $0 \to 1$ in $[2]$. Then f is flat. If for any $e \in \mathcal{E}$ there is a cartesian arrow $d \to e$ over $1 \to 2$ then f is flat.

Now we define a map $f : \mathcal{X} \to \mathcal{S}$ in $1 - \mathcal{C}\mathrm{at}$ to be flat iff for any morphism $[2] \to \mathcal{S}$ the induced map $\mathcal{X} \times_{\mathcal{S}} [2] \to [2]$ is flat. For example, if f is a cocartesian or cartesian fibration (in particular, if $\mathcal{S} \in \mathrm{Spc}$) then f is flat.

The flat morphisms appear in ([2], Def. 5.13) under the name of exponentiable fibration. A GREAT claim is ([2], Lemma 5.16): Let $\pi : E \to B$ be a morphism in $1 - \mathcal{C}\mathrm{at}$, the following conditions are equivalent:

- the base change functor $\pi^* : 1 - \mathcal{C}\mathrm{at}_{/B} \to 1 - \mathcal{C}\mathrm{at}_{/E}$, $K \mapsto K \times_B E$ has a right adjoint;
- the functor $\pi^* : 1 - \mathcal{C}\mathrm{at}_{/B} \to 1 - \mathcal{C}\mathrm{at}_{/E}$ preserves colimits;
- the map π is flat

Example of an application: (HTT, 3.2.2.13).

If $\pi : E \to B$ is flat and $Z \to E$ is a map in $1-\mathcal{C}\mathrm{at}$ then the functor $(1-\mathcal{C}\mathrm{at}\,/B)^{op} \to \mathrm{Spc}$, $K \mapsto \mathrm{Map}_{1-\mathcal{C}\mathrm{at}\,/E}(K \times_B E, Z)$ is representable.

3.0.44. *Coherent ∞-operads.* The definition of a coherent ∞-operad makes sense in a model-independent setting. Let $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad, $f : x_1 \oplus \ldots \oplus x_m \to y_1 \oplus \ldots \oplus y_n$ be a morphism in $\mathcal{O}^{\otimes}$. The definition of a f being semi-inert ([28], 3.3.1.1 and 3.3.1.2) makes sense in the model-independent setting.

For a unital ∞-operad $\mathcal{O}^{\otimes}$ let $\mathcal{K}_{\mathcal{O}} \subset \mathrm{Fun}([1], \mathcal{O}^{\otimes})$ be the full subcategory spanned by semi-inert morphisms, let $e_i : \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ be the evaluation at i. Now we may use ([28], 3.3.2.2) to give a model-independent definition of a coherent ∞-operad. Namely, consider a unital ∞-operad $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ such that such that $\mathcal{O} \in \mathrm{Spc}$. Then it is coherent iff $e_0 : \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ is a flat morphism. (See my Section 3.0.43 for the notion of a flat morphism).

Examples: $\mathbb{E}_0^{\otimes}, \mathbb{E}_k^{\otimes}, \mathcal{F}\mathrm{in}_*, \mathcal{A}\mathrm{ss}^{\otimes}$ are coherent.

3.0.45. For 3.4. If $\alpha : [n]^+ \to [m]^+$ is a map in $\boldsymbol{\Delta}^+$ then $\alpha^{-1}(+) = +$. This is not said explicitly in their Sect. 3.4. Let $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1 - \mathcal{C}\mathrm{at}$ be a functor lying in $1 - \mathcal{C}\mathrm{at}^{Mon^+}$. So, $A := A^{+,\otimes}([1])$ is a monoidal ∞-category, and $M := A^{+,\otimes}([0]^+)$ is a left A-module category, here $M \in 1 - \mathcal{C}\mathrm{at}$. Recall that $[0]^+ \stackrel{0+}{\to} [1]^+$ yields the action map $a : A \times M \to M$. The diagram

$$\begin{array}{ccc} [2]^+ & \stackrel{01+}{\leftarrow} & [1]^+ \\ \uparrow 02+ & & \uparrow 0+ \\ [1]^+ & \stackrel{0+}{\leftarrow} & [0]^+ \end{array}$$

yields

$$\begin{array}{ccc} A\times A\times M & \stackrel{\mathrm{id}\times a}{\to} & A\times M \\ \downarrow m\times\mathrm{id} & & \downarrow a \\ A\times M & \stackrel{a}{\to} & M \end{array}$$

Besides the composition

$$[0]^+ \stackrel{0+}{\to} [1]^+ \stackrel{00+}{\to} [0]^+$$

is the identity, and this yields the fact that the composition $M \stackrel{1\times\mathrm{id}}{\to} A\times M \stackrel{a}{\to} M$ is the identity.

The full subcategory $1-\mathcal{C}\mathrm{at}^{Mon^+}\subset \mathrm{Funct}(\boldsymbol{\Delta}^{+,op}, 1-\mathcal{C}\mathrm{at})$ is spanned by functors F such that for any $n\geq 0$ the map $F([n]^+)\to F([n])\times F([0]^+)$ given by morphisms

$$[n]\hookrightarrow [n]^+, i\mapsto i \quad \text{and} \quad [0]^+\hookrightarrow [n]^+, 0\mapsto n, +\mapsto +,$$

is an equivalence, and the restriction of F to $\boldsymbol{\Delta}^{op}$ is a monoidal category.

In the whole discussion of 3.4 we may replace $1-\mathcal{C}\mathrm{at}$ by any $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$ admitting finite products. This would produce an ∞-category $\mathcal{C}^{Mon^+}$ of left modules over a monoid in $\mathcal{C}$. If A is a monoid in $\mathcal{C}$ given by a functor $F:\boldsymbol{\Delta}^{op}\to\mathcal{C}$ with $F([1])=A$, we may also define the category $\mathcal{C}^{Mon^+}\times_{\mathcal{C}^{Mon}}\{F\}$ as in 3.4.4. This is the category $A-mod(\mathcal{C})$ of left A-modules in $\mathcal{C}$. Here $\mathcal{C}^{Mon}$ is the ∞-category of monoids in $\mathcal{C}$.

3.0.46. Lurie' version of a notion of a module over an algebra is given in ([28], 4.2.1.13). In particular, we have the operad $\mathcal{LM}^{\otimes}$ defined in ([28], 4.2.1.7). Do we have an approximation $\boldsymbol{\Delta}^{+,op}\to\mathcal{LM}^{\otimes}$? Yes, this is claimed in ([28], 4.2.2.8).

There is an isomorphism of categories $\boldsymbol{\Delta}\times[1]\widetilde{\to}\boldsymbol{\Delta}^+$ given by $([n],0)\mapsto[n], ([n],1)\mapsto [n]^+$. Now Dennis and Nick's definition of a left module from 3.4.2 becomes a particular case of definition ([28], 4.2.2.2).

3.0.47. For 3.5.1. Recall that $1-\mathcal{C}\mathrm{at}^{Mon^+}\subset(cocart/\boldsymbol{\Delta}^{+,op})_{strict}$ is a full subcategory spanned by "left modules". The category $(1-\mathcal{C}\mathrm{at}^{Mon^+})_{right-lax_{non-unit}}$ is defined as follows. It is defined as the subcategory of $cocart/\boldsymbol{\Delta}^{+,op}$ having the same objects as $1-\mathcal{C}\mathrm{at}^{Mon^+}$. For two objects $F,F'\in 1-\mathcal{C}\mathrm{at}^{Mon^+}$ corresponding to cocartesian fibrations $X\to\boldsymbol{\Delta}^{+,op}$, $X'\to\boldsymbol{\Delta}^{+,op}$, a morphism $e:X\to X'$ over $\boldsymbol{\Delta}^{+,op}$ is in

$$(1-\mathcal{C}\mathrm{at}^{Mon^+})_{right-lax_{non-unit}}$$

iff the following conditions are verified:

- for any $n\geq 0$ and the map $\rho:[1]\to[n], 0\mapsto i, 1\mapsto i+1$, $(0\leq i<n)$ in $\boldsymbol{\Delta}$, e sends a cocartesian arrow in X over ρ to a cocartesian arrow in X';
- If ρ is a morphism in $\boldsymbol{\Delta}^+$ of the form $[n]\hookrightarrow[n]^+$, $i\mapsto i$ or $[0]^+\hookrightarrow[n]^+$, $0\mapsto n, +\mapsto +$ then e sends a cocartesian arrow in X over ρ to a cocartesian arrow in X'.

I think this is equivalent to requiring that for any injective morphism ρ in $\boldsymbol{\Delta}^+$, whose image is convexe, e sends a cocartesian arrow in X over ρ to a cocartesian arrow in X'.

If $X\to\boldsymbol{\Delta}^{+,op}$ is a cocartesian fibration given by an object of $1-\mathcal{C}\mathrm{at}^{Mon^+}$, let $A=X_{[1]}$ and $M=X_{[0]^+}$ be the fibres, so that we have the multiplication $A\times M\to M$.

Let $e : X \to X'$ be a morphism in $(1-\mathcal{C}\mathrm{at}^{Mon^+})_{right-lax_{non-unit}}$ as above, set $A' = X'_{[1]}, M' = X'_{[0]^+}$. So, $e : A \to A', e : M \to M'$.

Let $a \in A, m \in M$. Let $\delta : [0]^+ \overset{0+}{\to} [1]^+$, write as in Lurie, $a \oplus m$ for the corresponding object of $X_{[1]^+} \widetilde{\to} A \times M$, so $am := \delta_!(a \oplus m)$ is the result of the action. Since $e(a \oplus m) \widetilde{\to} e(a) \oplus e(m)$, we get a canonical map

$$e(a)e(m) \to e(am) \quad \text{in} \quad M'$$

Recall also that we have for $a_i \in A$ the corresponding maps $e(a_1) \otimes e(a_2) \to e(a_1 \otimes a_2)$, $1 \to e(1)$ in A'.

Fix a monoidal category $F \in 1-\mathcal{C}\mathrm{at}^{Mon}$. We have the natural functor

$$(1-\mathcal{C}\mathrm{at}^{Mon^+})_{right-lax_{non-unit}} \to (1-\mathcal{C}\mathrm{at}^{Mon})_{right-lax_{non-unit}}$$

The category of right-lax non-unital functors between left F-modules is

$$(1-\mathcal{C}\mathrm{at}^{Mon^+})_{right-lax_{non-unit}} \times_{(1-\mathcal{C}\mathrm{at}^{Mon})_{right-lax_{non-unit}}} \{F\}$$

Let $e : X \to X'$ be a morphism in $(1-\mathcal{C}\mathrm{at}^{Mon^+})_{right-lax_{non-unit}}$ as above, let (A, M) and (A', M') be the corresponding monoidal categories and modules over them. Recall that $Alg+mod(A, M)$ is the category of right lax non-unital functors $* \to X$, its object is a pair (a, m), where $a \in Alg(A), m \in a-mod(M)$. Composing with e gives a functor $Alg + mod(A, M) \to Alg + mod(A', M')$.

3.0.48. If $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}\mathrm{at}$ is a left module M over a monoidal ∞-category A then composing $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}\mathrm{at} \overset{op}{\to} 1-\mathcal{C}\mathrm{at}$, we get a A^{op}-module structure on M^{op}.

3.0.49. Let A be a monoidal ∞-category, recall $A-mod = 1-\mathcal{C}\mathrm{at}^{Mon^+} \times_{1-\mathcal{C}\mathrm{at}^{Mon}} \{A\}$ from ([14], ch.1, 3.4.4). Given $M, M' \in A-mod$, a morphism in $A-mod$ from M to M' is what is called in ([28], 4.6.2.7) a A-linear functor $M \to M'$. Namely, for the corresponding cocartesian fibrations $X \to \boldsymbol{\Delta}^{+,op} \leftarrow X'$ this is a map $X \to X'$ in $(coCart_{/\boldsymbol{\Delta}^{+,op}})_{strict}$, whose base change by $\boldsymbol{\Delta}^{op} \hookrightarrow \boldsymbol{\Delta}^{+,op}$ is the identity. Write $X_0 = X \times_{\boldsymbol{\Delta}^{+,op}} \boldsymbol{\Delta}^{op}$, similarly for X'_0. As in ([28], 4.6.2.7), let

$$\mathrm{LinFun}_A(M, M') \subset \mathrm{Fun}_{\boldsymbol{\Delta}^{+,op}}(X, X') \times_{\mathrm{Fun}_{\boldsymbol{\Delta}^{op}}(X_0, X'_0)} \{\mathrm{id}\}$$

be the full subcategory spanned by A-linear functors.

If $F : M \to M'$ is an A-linear functor, composition with F yields a commutative diagram

$$\begin{array}{ccc} AssAlg + mod(A, M) & \to & AssAlg + mod(A, M') \\ & \searrow & \downarrow \\ & & AssAlg(A) \end{array}$$

In particular, for $\mathcal{A} \in AssAlg(A)$ a functor $\mathcal{A} - mod(M) \to \mathcal{A} - mod(M')$. For an application see Section 3.1.9.

My understanding is that $\mathrm{LinFun}_A(M, M')^{\mathrm{Spc}} \widetilde{\to} \mathrm{Map}_{A-mod}(M, M')$ for $M, M' \in A-mod$ naturally. I think there should be an $(\infty, 2)$-category, whose underlying $(\infty, 1)$-category is $A - mod$ and such that the corresponding $\mathbf{Map}(M, M')$ becomes $\mathrm{LinFun}_A(M, M')$.

Lin Chen confirms that there is an equivalent definition of $\mathrm{LinFun}_A(M, M')$ is as the relative inner hom: we have functorially in $E \in 1-\mathcal{C}\mathrm{at}$ an isomorphism

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(E, \mathrm{LinFun}_A(M, M')) \widetilde{\to} \mathrm{Map}_{A-mod}(M \times E, M')$$

Here we view $A-mod$ as a right module over $1-\mathcal{C}\mathrm{at}$ naturally. This implies easily that $A-mod \to 1-\mathcal{C}\mathrm{at}$, $M' \mapsto \mathrm{LinFun}_A(M, M')$ preserves limits.

Idea of the proof of their equivalence: Let $X, X' \to \boldsymbol{\Delta}^{+,op}$ be the cocartesian fibrations corresponding to M, M'. Let $X_0 = X \times_{\boldsymbol{\Delta}^{+,op}} \boldsymbol{\Delta}^{op}$. Then the cocartesian fibration attached to the A-module $M \times E$ is obtained as the push-out in $1-\mathcal{C}\mathrm{at}$ of the diagram

$$X \times E \leftarrow X_0 \times E \to X_0$$

Indeed, by Lemma 3.0.50 below, the desired cocartesian fibration $? \to \boldsymbol{\Delta}^{op}$ is the push-out in $1-\mathcal{C}\mathrm{at}$ of the diagram $X_0 \times E \times [1] \stackrel{\mathrm{id}\times 1}{\leftarrow} X_0 \times E \to X_1$, and our claim follows from Lemma 2.2.73.

Lemma 3.0.50. *For a functor $F : I \times [1] \to 1-\mathcal{C}\mathrm{at}$ let $X_0 \to I, X_1 \to I$ be the cocartesian fibrations attached to F_0, F_1 respectively. Then the cocartesian fibration $X \to I \times [1]$ attached for F is the push-out in $1-\mathcal{C}\mathrm{at}$ of the diagram*

$$X_0 \times [1] \stackrel{\mathrm{id}\times 1}{\leftarrow} X_0 \stackrel{f}{\to} X_1$$

Here $f : X_0 \to X_1$ is the map over I attached to F via strenthening.

Proof. Apply ([18], Th. 1.1) describing the cocartesian fibration attached to a functor $I \to 1-\mathcal{C}\mathrm{at}$ as the oplax colimit of this functor. This gives, for example, that for $F : [1] \to 1-\mathcal{C}\mathrm{at}$ given by a functor $h : F_0 \to F_1$ the corresponding cocartesian fibration is the push-out in $1-\mathcal{C}\mathrm{at}$ of

$$F_0 \times [1] \stackrel{\mathrm{id}\times 1}{\leftarrow} F_0 \stackrel{h}{\to} F_1$$

The category $Tw([1])$ is the diagram $\mathrm{id}_0 \leftarrow \alpha \to \mathrm{id}_1$, where $\alpha : 0 \to 1$ is the map in [1]. So,

$$X \widetilde{\to} \underset{(i\to i')\in Tw(I), (a\to b)\in Tw([1])}{\mathrm{colim}} I_{i'/} \times [1]_{b/} \times F(i, a)$$

Fix an element $(a \to b) \in Tw([1])$ first and calculate the corresponding colimit over $Tw(I)$, we get the diagram

$$X_0 \times [1] \stackrel{\mathrm{id}\times 1}{\leftarrow} X_0 \stackrel{f}{\to} X_1$$

indexed by $Tw([1])$. Our claim follows. □

Remark: If $f : M \to M'$ is a A-linear functor then $f^{op} : M^{op} \to M'^{op}$ is A^{op}-linear. If $\mathcal{A} \in \mathrm{coAlg}(A)$, f yields a functor $\mathcal{A}-comod(M) \to \mathcal{A}-comod(M')$.

Important generalization: let S be a symmetric monoidal ∞-category, $A \in Alg(S)$. Assume inner homs exist in S, given $M, N \in S$ write $\mathcal{H}om(M, N) \in S$ for their inner hom. Let now $M, N \in A-mod(S)$. Then $\mathcal{H}om(M, N)$ is a A-bimodule naturally. By $\mathcal{H}om_A(M, N) \in S$ we mean the object such that one has functorially on $X \in S$, $\mathrm{Map}_S(X, \mathcal{H}om_A(M, N)) \widetilde{\to} \mathrm{Map}_{A-mod(S)}(M \otimes X, N)$. Then Nick claims that $\mathcal{H}om_A(M, N) \widetilde{\to} \mathcal{H}om_{A-bimod(S)}(A, \mathcal{H}om(M, N)) \in S$, where the RHS is the relative inner hom in S of two A-bimodules. We use here the fact that $A \otimes A^{rm} - mod(S)$ is equipped with a right S-action.

3.0.51. For 3.5.2. Dennis refers here to the following ([28], 7.3.2.7): Suppose we are given a diagram of maps of ∞-operads $\mathcal{C}^{\otimes} \stackrel{F}{\to} \mathcal{D}^{\otimes} \stackrel{q}{\to} \mathcal{O}^{\otimes}$, let $p = qF$. Assume p and q are coCartesian fibrations. Assume that, for every $x \in \mathcal{O}$, the induced map of fibers $F_x : \mathcal{C}_x \to \mathcal{D}_x$ admits a right adjoint G_x. Assume that F sends a p-cocartesian arrow to a q-cocartesian arrow (this condition is missing here in ([28] 7.3.2.7), as Jacob confirmed). Then F admits a right adjoint G relative to $\mathcal{O}^{\otimes}$. Moreover, G is a map of ∞-operads.

3.0.52. For 3.5.4. Let $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1 - \mathcal{C}at$ be a left module given by a pair (A, M), here A is a monoidal ∞-category, we have the action map, say $A \times M \stackrel{\circ}{\to} M$. Let $F : *^{+,\otimes} \to A^{+,\otimes}$ be a right-lax non-unital monoidal functor. It is given on [1] by an object $\mathcal{A} \in A$ with $\mathcal{A} \in AssocAlg(A)$, and on $[0]^+$ by an object $\mathcal{M} \in M$. Applying my Section 3.0.47, we get a morphism

$$\mathcal{A} \circ \mathcal{M} \to \mathcal{M}$$

in M satisfying the usual properties of a $\mathcal{A}$-module.

3.0.53. For 3.5.5. Let $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1 - \mathcal{C}at$ be a left module given by a pair (A, M). Let $\mathcal{X} \to \boldsymbol{\Delta}^{+,op}$ be the corresponding cocartesian fibration. The category denoted $AssocAlg + mod(A, M)$ in 3.5.4 is the full subcategory of $\mathrm{Funct}_{\boldsymbol{\Delta}^{+,op}}(\boldsymbol{\Delta}^{+,op}, \mathcal{X})$ spanned by those functors, which are right-lax non-unital monoidal. Consider the forgetful functor $\theta : AssocAlg + mod(A, M) \to AssocAlg(A)$. It seems the fact that θ is a cartesian fibration (the construction of cartesian arrows) can be explained as in the next section.

A reference for the fact that this is a cartesian fibration is ([28], 4.2.3.2). Moreover, a morphism f in $AssocAlg + mod(A, M)$ is θ-cartesian iff the image of f in M is an equivalence.

In addition, ([28], 4.2.3.3) says: let $\mathcal{A}$ be an associative algebra object in A. Let K be an infinity category such that M admits K-indexed limits. Then $\mathcal{A} - mod(M)$ admits K-indexed limits. A map $K^{\triangleleft} \to \mathcal{A} - mod(M)$ is a limit diagram iff the induced map $K^{\triangleleft} \to M$ is a limit diagram.

The forgetful functor $\mathcal{A} - mod(M) \to M$ is conservative (this follows from my Sect. 2.2.99).

([28], 4.2.3.5): let $\mathcal{A}$ be an associative algebra object in A. Let K be an infinity category such that M admits K-indexed colimits, and the tensor product functor $M \to M, \mathcal{M} \mapsto a \otimes \mathcal{M}$ by any $a \in A$ preserves K-indexed colimits. Then $\mathcal{A} - mod(M)$ admits K-indexed colimits, and the forgetful functor $\mathcal{A} - mod(M) \to M$ preserves K-indexed colimits.

3.0.54. For $\mathcal{C} \in 1 - \mathcal{C}at$ admitting finite products write $Mon(\mathcal{C})$ for the category of monoids in $\mathcal{C}$, $Mon^+(\mathcal{C})$ for the category of left modules over a monoid in $\mathcal{C}$. Assume given map $f : B' \to A'$ in $Mon(\mathcal{C})$. Here $A', B' : \boldsymbol{\Delta}^{op} \to \mathcal{C}$. Write $A = A'([1]), B = B'([1])$. The corresponding map $B \to A$ is a morphism of monoids in $\mathcal{C}$. Assume

$G : \boldsymbol{\Delta}^{+,op} \to \mathcal{C}$ is a left A-module, set $M = G([0]^+)$. We want to show that M is naturally a left B-module.

We have the diagram of functors

$$\begin{array}{ccc} \boldsymbol{\Delta}^+ & \stackrel{p}{\to} & \boldsymbol{\Delta} \\ & \nwarrow & \uparrow \mathrm{id} \\ & & \boldsymbol{\Delta} \end{array}$$

Here the slanted arrow is the natural inclusion, $p([n]^+) = [n]$, and p acts naturally on morphisms.

Let $\bar{A} : \boldsymbol{\Delta}^{+,op} \to \mathcal{C}$ denote the composition $\boldsymbol{\Delta}^{+,op} \stackrel{p}{\to} \boldsymbol{\Delta}^{op} \stackrel{A'}{\to} \mathcal{C}$. Let $\bar{B}$ denote the composition $\boldsymbol{\Delta}^{+,op} \stackrel{p}{\to} \boldsymbol{\Delta}^{op} \stackrel{B'}{\to} \mathcal{C}$.

We have a natural map $p(z) \to z$ in $\boldsymbol{\Delta}^+$ functorial in $z \in \boldsymbol{\Delta}^+$. Applying G, we get a map $G(z) \to G(p(z))$ functorial in $z \in \boldsymbol{\Delta}^+$, that is, a morphism of functors $G \to \bar{A}$.

The map f gives rise by composing with p to a morphism of functors $\bar{B} \to \bar{A}$. The desired functor $\boldsymbol{\Delta}^{+,op} \to \mathcal{C}$ is obtained as $\bar{B} \times_{\bar{A}} G$, the product being taken in $\mathrm{Funct}(\boldsymbol{\Delta}^{+,op}, \mathcal{C})$.

3.0.55. Let $A^{\otimes} : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}at$ be a monoidal category. Restricting via $\boldsymbol{\Delta}^+ \to \boldsymbol{\Delta}$, $[n] \mapsto [n], [n]^+ \mapsto [n+1]$ we get $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}at$ realizing $A = A^{\otimes}([1])$ as a left A-module. Let $\mathcal{X} \to \boldsymbol{\Delta}^{+,op}$ be the cocartesian fibration associated to $A^{+,\otimes}$. Let $\bar{\mathcal{X}} \to \boldsymbol{\Delta}^{op}$ be its restriction corresponding to $A^{\otimes}$. Let $\mathcal{A} \in AssocAlg(A)$ be given by a functor $F : \boldsymbol{\Delta}^{op} \to \bar{\mathcal{X}}$. The notation $\mathcal{A} - mod := \mathcal{A} - mod(A)$ from 3.5.6 is the fibre of $AssocAlg + mod(A, A) \to AssocAlg(A)$ over F.

3.0.56. We also have ([28], 4.2.3.7) saying the following. Let $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}at$ be a left module category given by (A, M). Assume M presentable. Assume for each $a \in A$ the functor $M \to M, x \mapsto a \otimes x$ preserves small colimits. Then for any $\mathcal{A} \in Alg(A)$, $\mathcal{A} - mod(M)$ is presentable, and for any morphism $\mathcal{A} \to \mathcal{A}'$ in $Alg(A)$ the induced functor $\mathcal{A}' - mod(M) \to \mathcal{A} - mod(M)$ preserves small limits and colimits (so, admits both left and right adoints by [28], 4.2.3.8). Moreover, the forgetful functor $\theta : AssocAlg + mod(A, M) \to AssocAlg(A)$ is a presentable fibration (so, a cocartesian fibration).

There is a related claim ([28], 4.6.2.17), in which the existence of a left adjoint to the forgetful functor $\mathcal{A}' - mod(M) \to \mathcal{A} - mod(M)$ is affirmed under weaker assumptions.

3.0.57. Let $A^{\otimes} : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}at$ be a monoidal category. To better think about right A-module categories, one is tempted to introduce a category ${}^+\boldsymbol{\Delta}$. It is a full subcategory of $1-\mathcal{C}at^{ordn}$, its objects are categories of the form $[n] \in \boldsymbol{\Delta}$ and

$${}^+[n] = (+ \to 0 \to 1 \to \ldots \to n),\ n = 0, 1, \ldots$$

The morphisms are morphisms in $\boldsymbol{\Delta}$, morphisms $[n] \to {}^+[m]$ whose image does not contain $+$, and functors $f : {}^+[n] \to {}^+[m]$ such that $f^{-1}(+) = +$.

Given a monoidal category $A^{\otimes}$, a right module for it is an extension of $A^{\otimes}$ to a functor $F : {}^+\boldsymbol{\Delta}^{op} \to 1-\mathcal{C}at$ such that for any $n \geq 0$ the functor

$$F({}^+[n]) \to F({}^+[0]) \times A^{\otimes}([n])$$

given by

$$[n] \to {}^+[n], i \mapsto i \ \text{ and } \ {}^+[0] \to {}^+[n], 0 \mapsto 0, + \mapsto +$$

is an equivalence.

The map ${}^+[0] \to {}^+[1], + \mapsto +, 0 \mapsto 1$ yields an action map $M \times A \to M$.

We have the functor $p : {}^+\boldsymbol{\Delta} \to \boldsymbol{\Delta}$, ${}^+[n] \to [n+1]$, $+ \mapsto 0, i \mapsto i+1$ extending the identity $\mathrm{id} : \boldsymbol{\Delta} \to \boldsymbol{\Delta}$. For a monoidal category $A^{\otimes} : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ the composition $F = A^{\otimes} \circ p$ realizes A as a right A-module category (via the tensor product on the right).

Let $F : {}^+\boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ be a right A-module for the monoidal category $A^{\otimes}$. Let $\mathcal{X} \to {}^+\boldsymbol{\Delta}^{op}$ be the corresponding cocartesian fibration. A functor $e \in \mathrm{Funct}_{{}^+\boldsymbol{\Delta}^{op}}({}^+\boldsymbol{\Delta}^{op}, \mathcal{X})$ is "lax non-unital" if it satisfies the following:

- for any injective morphism ρ in $\boldsymbol{\Delta}$, whose image is convexe, e sends ρ to a cocartesian arrow in $\mathcal{X}$.
- for the morphism ρ of the form $[n] \to {}^+[n], i \mapsto i$ or ${}^+[0] \to +[n], + \mapsto +, 0 \mapsto 0$, e sends ρ to a cocartesian arrow in $\mathcal{X}$.

Let $M = F({}^+[0])$, we have the action functor $M \times A \to M$. Let $\mathcal{A}^{\otimes} : \boldsymbol{\Delta}^{op} \to \mathcal{X}$ be the restriction of e, this is an algebra object of A given by $\mathcal{A} = e([1]) \in A$. Let $\mathcal{M} = e({}^+[0]) \in M$. Our e yields a canonical map $\mathcal{M} \otimes \mathcal{A} \to \mathcal{M}$ in M. So, actually $\mathcal{M}$ is a right $\mathcal{A}$-module.

Let $Mod{+}AssocAlg(M, A)$ denote the category of lax non-unital functors ${}^+\boldsymbol{\Delta}^{op} \to \mathcal{X}$. This is the full subcategory of $\mathrm{Funct}_{{}^+\boldsymbol{\Delta}^{op}}({}^+\boldsymbol{\Delta}^{op}, \mathcal{X})$ spanned by lax non-unital functors. We should think of it as the category of pairs $\mathcal{A} \in AssocAlg(A)$ and $\mathcal{M} \in \mathcal{A}-mod^r(M)$. The fibre of $Mod + AssocAlg(M, A) \to AssocAlg(A)$ over $\mathcal{A}^{\otimes}$ is the category of right $\mathcal{A}$-modules in M, it should be denoted $\mathcal{A} - mod^r(M)$.

Should the above functor e be called "right-lax non-unital"? What is the good terminology? Does the notation ${}^+\boldsymbol{\Delta}$ appear somewhere in their book? We have an equivalence $\boldsymbol{\Delta} \times [1] \,\tilde{\to}\, {}^+\boldsymbol{\Delta}$, $([n], 0) \mapsto [n]$, and $([n], 1) \mapsto {}^+[n]$.

3.0.58. *Non-unital algebra objects.* Let $A^{\otimes} : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ be a monoidal ∞-category, $\mathcal{A} = A^{\otimes}([1])$. Let $\tilde{\mathcal{A}} \to \boldsymbol{\Delta}^{op}$ be the corresponding cocartesian fibration. Let $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ be the subcategory with the same objects, where we keep only injective morphisms $[n] \to [m]$. The category $Alg^{nu}(\mathcal{A})$ of non-unital associative algebras in $\mathcal{A}$ should be defined according to ([28], 5.4.3.3) as the full subcategory $Alg^{nu}(\mathcal{A}) \subset \mathrm{Funct}_{\boldsymbol{\Delta}^{op}}((\boldsymbol{\Delta}_s)^{op}, \tilde{\mathcal{A}})$ spanned by functors F that send morphisms of the form $[1] \to [n], 0 \mapsto i, 1 \mapsto i+1$ to a cocartesian arrow.

This should be equivalent to the property that F sends any inert morphism to a cocartesian arrow. Here inert in $\boldsymbol{\Delta}_s$ (by [27], 5.4.3.1) is an injective map $[n] \to [m]$ whose image is a convex subset in $\{0, \ldots, m\}$.

([28], 5.4.4.1). Let $\mathrm{Surj} \subset \mathcal{F}\mathrm{in}_*$ be the subcategory with the same objects, and a morphism $\langle n \rangle \to \langle m \rangle$ is in Surj iff it is surjective. Then $\mathrm{Surj} \to \mathcal{F}\mathrm{in}_*$ is an ∞-operad. Let $\mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be a symmetric monoidal ∞-category. Let $CAlg^{nu}(\mathcal{C}^{\otimes}) \subset \mathrm{Funct}_{\mathcal{F}\mathrm{in}_*}(\mathrm{Surj}, \mathcal{C}^{\otimes})$ be the full subcategory spanned by functors F sending inert morphisms to inert morphisms in $\mathcal{C}^{\otimes}$. This is equivalent to requiring that for $i \in I - \{*\}$ the inert map $(* \in I) \to (* \in (*, i))$, $i \mapsto i, j \mapsto *$ for $j \neq i$ is sent by F to a cocartesian arrow over $\mathcal{F}\mathrm{in}_*$.

We may also view non-unital symmetric monoidal categories as Surj-monoids in $1-\mathcal{C}at$, that is, functors $M : \mathrm{Surj} \to 1-\mathcal{C}at$ such that for any $n \geq 0$ the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ yield an isomorphism $M(\langle n \rangle) \widetilde{\to} \prod_{i=1}^n M(\langle 1 \rangle)$. Let $fSet$ be the category of finite nonempty sets and surjective morphisms. We identify $fSet$ with the subcategory of Surj, where we keep all objects and only active morphisms. Then $fSet$ is a non-unital symmetric monoidal category with respect to the disjoint union. Restriction to $fSet$ yields an equivalence $\mathrm{Mon}_{\mathrm{Surj}}(1-\mathcal{C}at) \widetilde{\to} \mathrm{Fun}^{\otimes}(fSet, 1-\mathcal{C}at)$, where $\mathrm{Mon}_{\mathrm{Surj}}(1-\mathcal{C}at)$ is the category of Surj-monoids in $1-\mathcal{C}at$. Indeed, $fSet$ is the monoidal envelope of Surj in the sense of ([28], 2.2.4.1).

3.0.59. Definition of idle map in $\mathcal{F}in_*$ in ([14], ch. 9, 1.2.3) is correct in the published version, I think. The difference between the notions of a right-lax monoidal functor and non-unital right-lax monoidal functor is also explained there.

3.0.60. For ([27], 4.4.5.2). The notion of an idempotent in a model independent setting is as follows. The ∞-category Idem from ([27], 4.4.5.2) is actually a usual category. It has one object x, and $\mathrm{Map}_{Idem}(x,x) = \{\mathrm{id}, f\}$. The composition $\mathrm{Map}_{Idem}(x,x) \times \mathrm{Map}_{Idem}(x,x) \to \mathrm{Map}_{Idem}(x,x)$ is given by $f \circ f = f$. If now $\mathcal{C} \in 1-\mathcal{C}at$ then an idempotent in $\mathcal{C}$ is a functor $Idem \to \mathcal{C}$. Then $\mathrm{Funct}(Idem, \mathcal{C})$ is the ∞-category of idempotents in $\mathcal{C}$. An idempotent $f : Idem \to \mathcal{C}$ is effective iff it has a colimit in $\mathcal{C}$. Lurie defines $\mathcal{C}$ to be idempotent complete if every idempotent is effective ([27], 4.4.5.13).

We have a fully faithful embedding $Idem \hookrightarrow Idem/x$, $x \mapsto (x \xrightarrow{f} x)$. This is not an equivalence, as $Idem$ has no final object.

Definition 3.0.61 ([28], 4.2.4.1)**.** *Let $A^{+,op} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}at$ be a left module category given by (A, M), here A is a monoidal ∞-category (Lurie's terminology is to say that M is left tensored over $A^{\otimes}$). Consider an object of $AssocAlg + mod(A, M)$ given by $(\mathcal{A}, \mathcal{M})$. Let $\lambda : \mathcal{M}_0 \to \mathcal{M}$ be a morphism in M. Then λ exhibits $\mathcal{M}$ as a free left $\mathcal{A}$-module generated by $\mathcal{M}_0$ if the composition $\mathcal{A} \otimes \mathcal{M}_0 \to \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ is an equivalence in $\mathcal{M}$. Here the second map is the action.*

The main result about free $\mathcal{A}$-modules is the following.

Proposition 3.0.62 ([28], 4.2.4.2)**.** *Let $A^{+,op} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}at$ be a left module category given by (A, M), here A is a monoidal ∞-category. Let $\mathcal{A} \in Alg(A), \mathcal{M}_0 \in M$. Then*
i) there is $\mathcal{M} \in \mathcal{A}-mod(M)$ and a map $\lambda : \mathcal{M}_0 \to \mathcal{M}$ in M, which exhibits $\mathcal{M}$ as a free left $\mathcal{A}$-module generated by $\mathcal{M}_0$.
ii) Let $\mathcal{M} \in \mathcal{A}-mod(M)$, let $\lambda : \mathcal{M}_0 \to \mathcal{M}$ exhibit $\mathcal{M}$ as a free left $\mathcal{A}$-module generated by $\mathcal{M}_0$. Let $(\mathcal{B}, \mathcal{N})$ be an object of $AssocAlg + mod(A, M)$ then the composition with λ induces an isomorphism in Spc

$$\mathrm{Map}_{AssocAlg+mod(A,M)}((\mathcal{A}, \mathcal{M}), (\mathcal{B}, \mathcal{N})) \widetilde{\to} \mathrm{Map}_{Alg(A)}(\mathcal{A}, \mathcal{B}) \times \mathrm{Map}_M(\mathcal{M}_0, \mathcal{N})$$

For example ([28], 4.2.4.6) reads: Let $A^{+,op} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}at$ be a left module category given by (A, M). Let $\mathcal{A} \in Alg(A), \mathcal{M} \in \mathcal{A}-mod(M)$. Assume $\lambda : \mathcal{M}_0 \to \mathcal{M}$ is a map in M, which exhibits $\mathcal{M}$ as a free left $\mathcal{A}$-module generated by $\mathcal{M}_0$. Then for any $\mathcal{N} \in \mathcal{A} - mod(M)$ the map $\mathrm{Map}_{\mathcal{A}-mod(M)}(\mathcal{M}, \mathcal{N}) \to \mathrm{Map}_M(\mathcal{M}_0, N)$ given by composition with λ is an isomorphism in Spc.

(HA, Corollary 4.2.4.8): let $A^{+,op} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}\mathrm{at}$ be a left module category given by (A, M). Let $\mathcal{A} \in Alg(A)$. The oblivion functor $oblv : \mathcal{A} - mod(M) \to M$ admits a left adjont sending M_0 to the free $\mathcal{A}$-module $\mathcal{A} \otimes M_0$ generated by M_0. In [14] this left adjoint is usually denoted $\mathrm{ind}_{\mathcal{A}} : M \to \mathcal{A} - mod(M)$.

3.0.63. In [28] the convention is that the set of linear ordering on an empty set is $*$. For example, for the operad $LM^{\otimes}$ there is a unique map $\mathfrak{m} \to \mathfrak{a}$ in $LM^{\otimes}$.

3.0.64. I think one may define a notion of a A-bimodule in the style of Dennis and Nick book as follows. Let ${}^+\boldsymbol{\Delta}^+$ denote the usual category, it will contain $\boldsymbol{\Delta}$ as a full subcategory. Our ${}^+\boldsymbol{\Delta}^+$ will be a full subcategory of $1-\mathcal{C}\mathrm{at}^{ordn}$ (the category of usual categories). The objects of ${}^+\boldsymbol{\Delta}^+$ are of two types:

- objects of $\boldsymbol{\Delta}$;
- for any $n, m \geq 0$ an object $[n, m]^+$, which is the category

$$-n \to -(n+1) \to \ldots \to -0 \to + \to 0 \to 1 \to \ldots \to m$$

The morphisms in ${}^+\boldsymbol{\Delta}^+$ are as follows. The morphisms in ${}^+\boldsymbol{\Delta}^+$ are as follows.

- the morphisms $[n] \to [m]$ in $\boldsymbol{\Delta}$;
- the morphisms $f : [r] \to [n, m]^+$, where the image of f is contained in $\{-n, \ldots, -0\}$ and f preserves the orders;
- the morphisms $f : [r] \to [n, m]^+$, where the image of f is contained in $\{0, \ldots, m\}$ and f preserves the orders;
- the morphisms $g : [n', m']^+ \to [n, m]^+$ such that $g^{-1}(+) = +$, and g preserves the order.

We underline that 0 and -0 here are different object of the category $[n, m]^+$.

We view ${}^+\boldsymbol{\Delta} \subset {}^+\boldsymbol{\Delta}^+$ as the full subcategory spanned by the objects $[n], [m, 0]^+$ for $n, m \geq 0$. We view $\boldsymbol{\Delta}^+ \subset {}^+\boldsymbol{\Delta}^+$ as the full subcategory spanned by the objects $[n], [0, m]^+$ for $n, m \geq 0$.

Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit finite products, we view $\mathcal{C}$ as equipped with the cartesian monoidal structure given by cartesian products. Let $A^{\otimes} : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ be an associative algebra in $\mathcal{C}$ (that is, a monoid). A $A^{\otimes}$-bimodule is a an extension $F : {}^+\boldsymbol{\Delta}^+ \to \mathcal{C}$ of the functor $A^{\otimes} : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ with the following property. For any $n, m \geq 0$ the injective morphisms

- $[n] \to [n, m]^+$ with image $\{-n, \ldots, -0\}$,
- $[m] \to [n, m]^+, i \mapsto i$;
- $[0, 0]^+ \to [n, m]^+, -0 \mapsto -0, + \mapsto +, 0 \mapsto 0$

yield an isomorphism

$$F([n, m]^+) \widetilde{\to} F([n]) \times F([0, 0]^+) \times F([m])$$

Let $A = A^{\otimes}([1])$. The object $M := F([0, 0]^+) \in \mathcal{C}$ then gets a structure of a left A-module and a right A-module, and these two actions commute. Is this a correct definition?

3.0.65. Recall the involutions $rev : \boldsymbol{\Delta} \widetilde{\to} \boldsymbol{\Delta}$ and $\sigma : \mathcal{A}\mathrm{ss}^{\otimes} \widetilde{\to} \mathcal{A}\mathrm{ss}^{\otimes}$. The map $Cut : \boldsymbol{\Delta}^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ commutes with these involutions. Recall the projections

$$\mathcal{RM}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}, \mathcal{LM}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$$

sending $(\langle n\rangle, S)$ to $\langle n\rangle$. There is a unique isomorphism $\sigma : \mathcal{LM}^{\otimes} \to \mathcal{RM}^{\otimes}$ in $1-\mathcal{C}\mathrm{at}$ sending $(\langle n\rangle, S)$ to $(\langle n\rangle, S)$ which fits into the diagram

$$\begin{array}{ccc} \mathcal{LM}^{\otimes} & \stackrel{\sigma}{\to} & \mathcal{RM}^{\otimes} \\ \downarrow & & \downarrow \\ \mathcal{A}\mathrm{ss}^{\otimes} & \stackrel{\sigma}{\to} & \mathcal{A}\mathrm{ss}^{\otimes} \end{array}$$

In ([28], 4.2.2.8) Lurie has constructed a map $\gamma : \boldsymbol{\Delta}^{op} \times [1] \to \mathcal{LM}^{\otimes}$, which is an approximation of the ∞-operad $\mathcal{LM}^{\otimes}$. Similarly, one has an approximation $\gamma^r : \boldsymbol{\Delta}^{op} \times [1] \to \mathcal{RM}^{\otimes}$ defined as follows.

Let $RCut : \boldsymbol{\Delta}^{op} \to \mathcal{RM}^{\otimes}$ be defined as the composition

$$\boldsymbol{\Delta}^{op} \stackrel{rev}{\to} \boldsymbol{\Delta}^{op} \stackrel{LCut}{\to} \mathcal{LM}^{\otimes} \stackrel{\sigma}{\to} \mathcal{RM}^{\otimes}$$

Recall that $Cut : \boldsymbol{\Delta}^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ sends $[n]$ to the set $Cut[n]$ of equivalence relations on $\{0, \dots, n\}$ with at most two equivalence classes, which are convex. The functor $LCut : \boldsymbol{\Delta}^{op} \to \mathcal{LM}^{\otimes}$ sends $[n]$ to $(LCut[n], \{[n]\})$, where $LCut[n]$ is the set of all downward-closed subsets $S \subset [n]$. The map γ is the transformation in $\mathcal{LM}^{\otimes}$ given for each $[n] \in \boldsymbol{\Delta}$ by the map

$$(LCut[n], \{[n]\}) \to (Cut[n], \emptyset)$$

sending a subset $S \subset [n]$ to the equivalence relation with equivalence classes S, $\bar{S}$ (here $\bar{S}$ is the complement of S in $\{0, \dots, n\}$).

Denote also by Cut the composition $\boldsymbol{\Delta}^{op} \stackrel{Cut}{\to} \mathcal{A}\mathrm{ss}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$. The map $\gamma \in \mathrm{Funct}(\boldsymbol{\Delta}^{op}, \mathcal{LM}^{\otimes})$ by composing gives rise to the map $\gamma^r = \sigma \circ \gamma \circ rev \in \mathrm{Funct}(\boldsymbol{\Delta}^{op}, \mathcal{RM}^{\otimes})$. Then γ^r is also an approximation, right?

Define the map $a : \boldsymbol{\Delta}^{op} \times (\boldsymbol{\Delta})^{op} \times [1] \to \mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes}$ as $(\gamma \circ \mathrm{pr}_{13}, \gamma^r \circ \mathrm{pr}_{23})$. Is it true that the composition

$$\boldsymbol{\Delta}^{op} \times (\boldsymbol{\Delta})^{op} \times [1] \stackrel{a}{\to} \mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes} \stackrel{Pr}{\to} \mathcal{BM}^{\otimes}$$

is a (maybe weak) approximation for the ∞-operad $\mathcal{BM}^{\otimes}$? Here Pr is the map defined in ([28], 4.3.2.1).

Definition from [14]: an A-bimodule is a $A \otimes A^{op}$-module.

It would be nice to have a simplicial version of a definition of the category of bimodules in terms of this approximation. Does the projection $\mathcal{BM}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ given in ([28], 4.3.1.8) lifts to a map of approximations $\boldsymbol{\Delta}^{op} \times (\boldsymbol{\Delta})^{op} \times [1] \to \boldsymbol{\Delta}^{op}$?

Let $A^{\otimes}, B^{\otimes} : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ be two monoidal ∞-categories. Can one define a notion of a $A^{\otimes}, B^{\otimes}$-bimodule category simply as an extension of the corresponding functor $\boldsymbol{\Delta}^{op} \sqcup \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ to a functor $(\boldsymbol{\Delta}^{+} \times {}^{+}\boldsymbol{\Delta})^{op} \to 1-\mathcal{C}\mathrm{at}$ with some properties? So that hopefully we would get as an answer a full subcategory in $\mathrm{Funct}((\boldsymbol{\Delta}^{+} \times {}^{+}\boldsymbol{\Delta})^{op}, 1-\mathcal{C}\mathrm{at})$? Then we could as usually define lax monoidal functors via strengtening to get usual bimodules inside for a pair $\mathcal{A} \in Alg(A), \mathcal{B} \in Alg(B)$.

3.0.66. *Approximation to $BM^{\otimes}$?* We try to construct an approximation to the ∞-operad $BM^{\otimes}$ of ([28], 4.3.1.5). Define a functor

$$F : \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \times [1] \to BM^{\otimes} \tag{6}$$

as follows. For $[n] \in \boldsymbol{\Delta}$ recall that $LCut_0([n])$ denoted the set of all downward-closed subsets $S \subset [n]$ with the distinguished point $*$ given by the empty set ([28], 4.2.2.6). We identify $LCut_0([n]) \widetilde{\to} \langle n+1 \rangle$ via the map sending $i \in \langle n+1 \rangle$ to $\{j \in [n] \mid j < i\}$ as in *loc.cit.* In other words, $LCut_0([n])$ is ordered by inclusion. For a map $\alpha : [n] \to [m]$ in $\boldsymbol{\Delta}$ the induced map $\alpha' : LCut_0([m]) \to LCut_0([n])$ sends S to $\alpha^{-1}(S)$. Given $T \in LCut_0([n])$ the set $\alpha'^{-1}(T)$ is ordered by inclusion. Let $\bar{*} \in LCut_0([n])$ denote the set $[n]$. The map α' preserves $\bar{*}$.

Write also $RCut_0([n])$ for the set of all downward-closed subsets $S \subset [n]$ with the distinguished point $*$ given by $[n]$. We view it an an object of $\mathcal{F}in_*$. For a map $\alpha : [n] \to [m]$ in $\boldsymbol{\Delta}$ the induced map $\alpha' : RCut_0([m]) \to RCut_0([n])$ sends S to $\alpha^{-1}(S)$. Given $T \in LCut_0([n])$ the set $\alpha'^{-1}(T)$ is ordered by inclusion. Write $\bar{*}$ for the element $\emptyset \in RCut_0([n])$. The map α' preserves $\bar{*}$.

For $[n], [m] \in \boldsymbol{\Delta}$ set

$$BCut([n],[m]) = LCut_0([n]) \underset{\{*,\bar{*}\}}{\sqcup} RCut_0([m])$$

We view it an an object of $\mathcal{F}in_*$ with the distinguished point $*$. Given $\alpha : [n] \to [n'], \beta : [m] \to [m']$ in $\boldsymbol{\Delta}$ the induced maps $\alpha' : LCut_0([n']) \to LCut_0([n])$ and $\beta' : RCut_0([m']) \to RCut_0([m])$ yield a morphism

$$\gamma := BCut(\alpha, \beta) : BCut([n'],[m']) \to BCut([n],[m])$$

For $S \in BCut([n],[m])$ the preimage $\gamma^{-1}(S)$ is ordered as follows. If $S \neq \bar{*}$, $S \in LCut_0([n])$ then $\gamma^{-1}(S) \subset LCut_0([n'])$ is ordered as above. Similarly if $S \neq \bar{*}$, $S \in RCut_0([m])$ then $\gamma^{-1}(S) \subset RCut_0([m'])$ is ordered as above. Finally, for $S = \bar{*} \in BCut([n],[m])$ the preimage $\gamma^{-1}(S)$ has a linear order such that the induced orders on $\gamma^{-1}(S) \cap LCut_0([n'])$ and on $\gamma^{-1}(S) \cap RCut_0([m'])$ are as above, and all the elements of $\gamma^{-1}(S) \cap LCut_0([n'])$ are less than the elements of $\gamma^{-1}(S) \cap RCut_0([m'])$.

We constructed a functor $BCut : \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \to BM^{\otimes}$. It is understood that $c_-, c_+ = 0$ (resp., 1) on elements of $LCut_0([n]) - \{*, \bar{*}\}$ (resp., $RCut_0([m]) - \{*, \bar{*}\}$). Besides, c_- and c_+ take different values $0, 1$ on $\bar{*}$.

Since $Cut : \boldsymbol{\Delta}^{op} \to Ass^{\otimes}$ is a functor, we get another functor $bCut : \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \to BM^{\otimes}$ sending $([n],[m])$ to

$$Cut([n]) \underset{*}{\sqcup} Cut([m])$$

and defined naturally on morphisms.

Finally, we have a transformation $F : BCut \to bCut$ defined as follows. On $([n],[m]) \in \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op}$ it is induced by the map

$$LCut_0([n]) \times RCut_0([m]) \to Cut([n]) \times Cut([m])$$

sending $(S,T) \in LCut_0([n]) \times RCut_0([n])$ to the pair of equivalence relations

$$(\sim_S, \sim_T) \in Cut([n]) \times Cut([m])$$

Here $\sim_S$ is the equivalence relation on $[n]$ whose classes are S and its complement, similarly for the equivalence relation $\sim_T$ on $[m]$. Our F is the desired functor (6).

Is F is an approximation of ∞-operads? No! Condition ([28], 2.3.3.6(1)) is not satisfied I think. Namely let $n, m \geq 0$, pick an equivalence relation in $Cut([n])$ whose equivalences classes are given by $(0, \ldots, i-1), (i, \ldots, n)$ for some $i > 0$. Consider the inert map $\rho : \langle n+m \rangle \to 1$ sending i to 1 and $j \neq i$ to 0. Then $([n], [m], 1) \in \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \times [1]$ is over $\langle n+m \rangle$, but there is no locally cocartesian arrow in $\boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \times [1]$ over ρ.

3.0.67. *Modules over a coherent operad.* For ([28], 3.3.3). Let $\mathcal{O}^{\otimes}$ be a coherent operad then by ([28], 3.3.2.2) the map $e_0 : \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ is flat. Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads. In ([28], 3.3.3) for an algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ Lurie defines an ∞-operad $\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^{\otimes}$. Its construction in a model-independent way uses results of my Section 3.0.43. Now $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ from ([28], 3.3.3.1) can be defined in a model-independent setting. It is given by the functorial isomorphism for $X \in 1-\mathcal{C}\mathrm{at}_{/\mathcal{O}^{\otimes}}$

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}_{/\mathcal{O}^{\otimes}}}(X, \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}) \widetilde{\to} \mathrm{Fun}_{\mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes})}(X \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}}, \mathcal{C}^{\otimes})^{\mathrm{Spc}}$$

The objects maps making the diagram commutative

$$\begin{array}{ccc} X \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}} & \to & \mathcal{C}^{\otimes} \\ \downarrow & & \downarrow \\ \mathcal{K}_{\mathcal{O}} & \stackrel{e_1}{\to} & \mathcal{O}^{\otimes} \end{array}$$

Then $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \subset \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is a full subcategory. This yields a construction of a map of ∞-operads $\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ in ([28], 3.3.3.9).

This operad is used, most importantly, for the tensor product of modules over a commutative algebra in ([28], 4.5.2.1). Namely, let $\mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be a symmetric monoidal ∞-category, $A \in CAlg(\mathcal{C})$. Since $Comm^{\otimes} = \mathcal{F}\mathrm{in}_*$ is a coherent operad, we get the ∞-operad $\mathrm{Mod}^{Comm}_A(\mathcal{C})^{\otimes}$, the underlying ∞-category is

$$\mathrm{Mod}^{Comm}_A(\mathcal{C}) \widetilde{\to} A-mod(\mathcal{C}) \widetilde{\to} A-mod^r(\mathcal{C})$$

([28], 4.5). View Ass as the underlying ∞-category of the associative operad $\mathcal{A}\mathrm{ss}^{\otimes}$. Then $Mod^{Ass}_A(\mathcal{C}) \widetilde{\to} {}_ABMod_A(\mathcal{C})$ by ([28], 4.4.1.28). Lurie writes for brevity in ([28], 4.5.1.1)

$$\mathrm{Mod}^{Comm}_A(\mathcal{C}) = \mathrm{Mod}_A(\mathcal{C}), \quad \mathrm{Mod}^{Comm}_A(\mathcal{C})^{\otimes} = \mathrm{Mod}_A(\mathcal{C})^{\otimes}$$

The key thing here is maybe ([28], 4.5.2.1): assume $\mathcal{C}$ is a symmetric monoidal ∞-category, $\mathcal{C}$ admits geometric realizations of simplicial objects, and the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Recall that $A \in CAlg(\mathcal{C})$. Then $\mathrm{Mod}_A(\mathcal{C})^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is a symmetric monoidal ∞-category, and the tensor product operation in $\mathrm{Mod}_A(\mathcal{C})^{\otimes}$ is the tensor product of A-modules over A. This is also claimed in ([14], ch.1, 4.2.4).

3.0.68. *Bimodules.* If $\mathcal{C} \to \mathcal{F}in_*$ is symmetric monoidal ∞-category, $A, B \in AssAlg(\mathcal{C})$ are associative algebras in $\mathcal{C}$ then $A-B$-bimodule in $\mathcal{C}$ can be defined as a $A\boxtimes B^{rev-mult}$-module. Here we view $\mathcal{C}$ as a module category over the monoidal ∞-category $\mathcal{C} \otimes \mathcal{C}^{rev-mult}$.

The construction of ([28], 4.3) is more general, but maybe not need. Namely, for any map of ∞-operads $\mathcal{C}^{\otimes} \to BM^{\otimes}$ with $\mathcal{M} := \mathcal{C}_{\mathfrak{m}}$ one may consider the category $BMod(\mathcal{M}) = \mathrm{Alg}_{/BM}(\mathcal{C})$ as in ([28], 4.3.1.12). We have the projection $\mathrm{pr} : BMod(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}_-) \times \mathrm{Alg}(\mathcal{C}_+)$, where $\mathcal{C}_+ = \mathcal{C}^{\otimes} \times_{BM^{\otimes}} \{\mathfrak{a}_+\}, \mathcal{C}_- = \mathcal{C}^{\otimes} \times_{BM^{\otimes}} \{\mathfrak{a}_-\}$. Recall that $\mathfrak{a}_-, \mathfrak{a}_+, \mathfrak{m}$ are the three objects of BM. The fibre of pr over A, B is denoted ${}_ABMod_B(\mathcal{M})$ in ([28], 4.3.1.12).

The map $Pr : LM^{\otimes} \times RM^{\otimes} \to BM^{\otimes}$ from ([28], 4.3.2.1) is not a bifunctor. However, it is used further in ([28], Section 4.3) as if it was a bifunctor. Namely, for a map of ∞-operads $q : \mathcal{C}^{\otimes} \to BM^{\otimes}$ the map of ∞-operads $p : LMod(\mathcal{C}_{\mathfrak{m}})^{\otimes} \to RM^{\otimes}$ defined in ([28], 4.3.2.2-5) makes sense in the model-independent setting. This construction is similar to the construction of ([28], 3.2.4.3), see also my Section 3.0.30. Main results here are ([28], 4.3.2.5-7), their formulation makes sense in the model-independent setting.

Simplified version of ([28], 4.3.2.7): let $\mathcal{C}$ be a monoidal ∞-category, $A, B \in Alg(\mathcal{C})$. One has

$${}_ABMod_B(\mathcal{C}) \widetilde{\to} B - mod^r(A - mod(\mathcal{C})),$$

here the LHS is the category of $A \boxtimes B^{rev-mult}$-modules in $\mathcal{C}$.

3.0.69. Let $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}\mathrm{at}$ be a left module category given by (A, M), where A is a monoidal ∞-category, let $K \in 1-\mathcal{C}\mathrm{at}$. Then $\mathrm{Fun}(K, M)$ is also naturally a left A-module category (in Lurie's terminology, left tensored over A).

We may assume the pair (A, M) is given by a cocartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to LM^{\otimes}$. By ([28], 2.1.3.4) then $\mathcal{D}^{\otimes} := \mathrm{Fun}(K, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(K, LM^{\otimes})} LM^{\otimes} \to LM^{\otimes}$ is also a cocartesian fibration of ∞-operads and $\mathcal{D}_{\mathfrak{m}} \widetilde{\to} \mathrm{Fun}(K, M)$. Thus the category $\mathrm{Fun}(K, A) \widetilde{\to} \mathcal{D}_{\mathfrak{a}}$ is monoidal, and $\mathrm{Fun}(K, M)$ is a module category over $\mathrm{Fun}(K, A)$. Let $\mathcal{C}_-^{\otimes} = \mathcal{C}^{\otimes} \times_{LM^{\otimes}} Ass^{\otimes}$, similarly for $\mathcal{D}_-^{\otimes}$. The diagonal functor

$$\mathcal{C}_-^{\otimes} \to \mathrm{Fun}(K, \mathcal{C}_-^{\otimes}) \times_{\mathrm{Fun}(K, Ass^{\otimes})} Ass^{\otimes} \widetilde{\to} \mathcal{D}_-^{\otimes}$$

is $Ass^{\otimes}$-monoidal. Restricting the action with respect to the later functor as in my Section 3.0.54, we get the desired A-action.

For example, this shows that $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is a left module over $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

More generally, if $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration of ∞-operads, $K \in 1-\mathcal{C}\mathrm{at}$ then we get a cocartesian fibration of operads over $\mathcal{O}^{\otimes}$ whose fibre over $X \in \mathcal{O}$ is $\mathrm{Fun}(K, \mathcal{C}_X)$ with the pointwise monoidal structure by ([28], 2.1.3.4).

3.0.70. Let A, B be monoidal ∞-categories, and $F : A \to B$ be a right-lax non-unital monoidal functor. Then we may view (A, A) and (B, B) as objects of $1-\mathcal{C}\mathrm{at}^{Mon^+}$, and F gives a morphism from (A, A) to (B, B) in $(1-\mathcal{C}\mathrm{at}^{Mon^+})_{right-lax_{non-untl}}$.

Our F induces a commutative diagram

$$\begin{array}{ccc} AssAlg + mod(A, A) & \to & AssAlg + mod(B, B) \\ \downarrow & & \downarrow \\ Alg(A) & \to & Alg(B), \end{array}$$

the low row sends $\mathcal{A}$ to $F(\mathcal{A})$. The top row sends $(\mathcal{A}, \mathcal{M})$ to $(F(\mathcal{A}), F(\mathcal{M}))$. Here the product on $F(\mathcal{A})$ is $F(\mathcal{A}) \otimes F(\mathcal{A}) \to F(\mathcal{A} \otimes \mathcal{A}) \stackrel{F(m)}{\to} F(\mathcal{A})$, where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the product on $\mathcal{A}$. The action of $F(\mathcal{A})$ on $F(\mathcal{M})$ is the composition $F(\mathcal{A}) \otimes F(\mathcal{M}) \to F(\mathcal{A} \otimes \mathcal{M}) \stackrel{F(a)}{\to} F(\mathcal{M})$, where $a : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ is the action.

In particular, for $\mathcal{A} \in \mathrm{Alg}(A)$ this induces a functor $\mathcal{A} - mod(A) \to F(\mathcal{A}) - mod(\mathcal{B})$.

3.0.71. The relative inner hom $\underline{\mathrm{Hom}}_A(m, m)$ in ([28], 4.7.1) is called an endomorphism object of m, here m is an object of a monoidal ∞-category A. By *loc.cit*, if $\mathcal{A} \in Alg(A)$ is an associative algebra in A then $\mathrm{Map}_{Alg(A)}(\mathcal{A}, \underline{\mathrm{Hom}}_A(m, m)) \widetilde{\to} \mathcal{A} - mod(A) \times_A \{m\}$.

I have not understood 3.6.5, how $\underline{\mathrm{Hom}}_{\mathbf{A},\mathcal{A}}(\mathcal{M}, \mathcal{N}) \in \mathbf{A}$ acquires a structure of a $\mathcal{A}$-module? There is however a natural map $\mathcal{A} \otimes \underline{\mathrm{Hom}}_{A,\mathcal{A}}(\mathcal{M}, \mathcal{N}) \to \underline{\mathrm{Hom}}_{A,\mathcal{A}}(\mathcal{M}, \mathcal{N})$, which probably gives the structure of a $\mathcal{A}$-module. Indeed, by definition, we have a natural map $\underline{\mathrm{Hom}}_{A,\mathcal{A}}(\mathcal{M}, \mathcal{N}) \otimes \mathcal{M} \to \mathcal{N}$ in $\mathcal{A} - mod$. The desired map corresponds via adjointness to the composition

$$\mathcal{A} \otimes \underline{\mathrm{Hom}}_{A,\mathcal{A}}(\mathcal{M}, \mathcal{N}) \otimes \mathcal{M} \to \mathcal{A} \otimes \mathcal{N} \to \mathcal{N},$$

where the second one is the action map.

From ([28], 4.7.1) one should maybe remember the following. Let $A^{+,\otimes} : \boldsymbol{\Delta}^{+,op} \to 1 - \mathcal{C}at$ be a left module category given by (A, M), where A is monoidal, and M is left tensored over A. Let $(\mathcal{A}, m)$ be an object of $AssAlg + mod(A, M)$, so $\mathcal{A} \in AssAlg(A), m \in \mathcal{A} - mod(M)$. Then ([28], 4.7.1.41) says: assume $\mathcal{A} = \underline{\mathrm{Hom}}_A(m, m)$ is the relative inner hom, let $\mathcal{B} \in AssAlg(A)$ be an associative algebra. Then

$$\mathrm{Map}_{AssAlg(A)}(\mathcal{B}, \mathcal{A}) \widetilde{\to} \mathcal{B} - mod(M) \times_M \{m\}$$

The forgetfull functor $AssAlg + mod(A, M) \times_M \{m\} \to AssAlg(A)$ is a cartesian fibration in spaces ([28], 4.7.1.42).

3.0.72. The notion of a *split simplicial object* from ([28] 4.7.2.2) is a very useful idea. To be precise, $\boldsymbol{\Delta}_{-\infty}$ is the category, whose objects are finite (possibly empty) linearly ordered sets I. A map from I to J is a morphism $\alpha : I \sqcup \{-\infty\} \to J \sqcup \{-\infty\}$ preserving the orders such that $\alpha(-\infty) = -\infty$. The full subcategory of $\boldsymbol{\Delta}_{-\infty}$ with the same set of objects, where we require $\alpha^{-1}(-\infty) = -\infty$ is $\boldsymbol{\Delta}_+$.

For example, let $\mathcal{C} \in 1 - \mathcal{C}at$ admit finite limits, $f : T \to S$ be a map in $\mathcal{C}$. Then we have the augmented simplicial object $(\boldsymbol{\Delta}_+)^{op} \to \mathcal{C}$, $I \mapsto T^I/S$, the product of $T \in \mathcal{C}_{/S}$ over the set I. This is the Cech nerve of f, it is not split in general. Consider now the augmented split simplicial object $(\boldsymbol{\Delta}_{-\infty})^{op} \to \mathcal{C}$ sending I to $(T^I_{/S}) \times_S T$. This is indeed a functor out of $(\boldsymbol{\Delta}_{-\infty})^{op}$, because this can be seen as the product of $T \in \mathcal{C}_{/S}$ over the set $I \sqcup \{-\infty\}$. So, a map $\alpha : I \sqcup \{-\infty\} \to J \sqcup \{-\infty\}$ with $\alpha(-\infty) = -\infty$

induces a morphism $(T^J_{/S}) \times_S T \to (T^I_{/S}) \times_S T$. By (HA, 4.7.2.3),

$$\operatorname*{colim}_{I\in \boldsymbol{\Delta}^{op}} (T^I_{/S}) \times_S T \widetilde{\to} T,$$

the colimit taken in $\mathcal{C}$.

3.0.73. *Split simplicial objects again.* In ([28], 4.7.2.7) there is a correction needed. The functor $\phi : \boldsymbol{\Delta}^{op}_{-\infty} \to \boldsymbol{\Delta}^{op}$ is given by $\{-\infty\}\cup[n] \mapsto \{-\infty\}\star[n]$. A map $f : \{-\infty\}\cup[n] \to \{-\infty\}\cup[m]$ in $\boldsymbol{\Delta}_{-\infty}$ is sent to the induced map $\{-\infty\}\star[n] \to \{-\infty\}\star[m]$.

So, if $\mathcal{X} : \boldsymbol{\Delta}^{op} \to C$ is a simplicial object of $C \in 1-\mathcal{C}\mathrm{at}$, it yields a split simplicial object $\boldsymbol{\Delta}^{op}_{-\infty} \stackrel{\phi}{\to} \boldsymbol{\Delta}^{op} \stackrel{\mathcal{X}}{\to} C$. Thus,

$$\operatorname*{colim}_{[n]\in \boldsymbol{\Delta}^{op}} \mathcal{X}(\{-\infty\}\star[n]) \widetilde{\to} \mathcal{X}[0]$$

in C. This is very useful!

The inclusion $[n] \hookrightarrow \{-\infty\}\star[n] \widetilde{\to} [n+1]$ sending i to $i+1$ is functorial in $[n] \in \boldsymbol{\Delta}$. This gives a morphism of simplicial diagrams $\mathcal{X}(\{-\infty\}\star[n]) \to \mathcal{X}([n])$ as $[n]$ varies in $\boldsymbol{\Delta}^{op}$. In turn, this gives passing to the colimit the morphism

$$\mathcal{X}[0] \widetilde{\to} \operatorname*{colim}_{[n]\in \boldsymbol{\Delta}^{op}} \mathcal{X}(\{-\infty\}\star[n]) \to \operatorname*{colim}_{[n]\in \boldsymbol{\Delta}^{op}} \mathcal{X}([n])$$

which is ins_0 I think.

One more point. We may also consider $\mathcal{X}$ with reversed multiplication denoted $\mathcal{X}^{rm}$, that is, the composition $\boldsymbol{\Delta}^{op} \to \boldsymbol{\Delta}^{op} \stackrel{\mathcal{X}}{\to} C$, where the first map reverses the arrows on each category. Then we can get another split simplicial object composing ϕ with $\mathcal{X}^{rm}$.

Bar construction. Let $q : \mathcal{M}^{\otimes} \to LM^{\otimes}$ be a cocartesian fibration of ∞-operads, $\mathcal{C}^{\otimes} = \mathcal{M}^{\otimes}_{\mathfrak{a}}$ the corresponding monoidal ∞-category, $\mathcal{M} := \mathcal{M}^{\otimes}_{\mathfrak{m}}$ is a module category over $\mathcal{C}^{\otimes}$. If $X : LM^{\otimes} \to \mathcal{M}^{\otimes}$ is a map of operads over $LM^{\otimes}$, let $A = X(\mathfrak{a}) \in \mathcal{C}, M = X(\mathfrak{m}) \in \mathcal{M}$. The unique active map $(\langle 2\rangle, \{2\}) \to (\langle 1\rangle, \{1\})$ in $LM^{\otimes}$ yields the action map $A\otimes M \to M$.

Let Φ be the composition $\boldsymbol{\Delta}^{op}_{-\infty} \stackrel{\phi}{\to} \boldsymbol{\Delta}^{op} \stackrel{LCut}{\to} LM^{\otimes}$, as in *loc.cit*, we get a natural transformation $\alpha : \Phi \to \Phi_0$, where $\Phi_0 : \boldsymbol{\Delta}^{op}_{-\infty} \to LM^{\otimes}$ is the constant functor with value $\mathfrak{m}$. Since q is a cocartesian fibration, we may lift α to a q-cocartesian natural transformation $\bar{\alpha} : X\circ\Phi \to X'$ for some functor $X' : \boldsymbol{\Delta}^{op}_{-\infty} \to \mathcal{M}$. Note that for any $n \geq -1$ the unique map $-\infty \hookrightarrow -\infty\cup[n]$ in $\boldsymbol{\Delta}_{-\infty}$ after applying Φ becomes an active morphism $\beta : (\langle n+2\rangle, \{n+2\}) \to (\langle 1\rangle, \{1\})$ in $LM^{\otimes}$, the corresponding order on $\beta^{-1}(1) = \{1,\ldots,n+2\}$ is the natural one. The object $X\Phi(n)$ is $A\oplus\ldots\oplus A\oplus M \in \mathcal{M}^{\otimes}$, where A appears $n+1$ times. So, $\bar{\alpha}$ pick, in particular, a cocartesian arrow $A\oplus\ldots\oplus A\oplus M \to A^{\otimes n+1}\otimes M$ in $\mathcal{M}^{\otimes}$ over β. So, $X'(n) = A^{\otimes n+1}\otimes M$ for $n \geq -1$. The restriction $X' \mid_{\boldsymbol{\Delta}^{op}}$ is the simplicial object

$$A^{\otimes 3}\otimes M \overset{\to}{\underset{\to}{\to}} A^{\otimes 2}\otimes M \rightrightarrows A\otimes M$$

For the corresponding augmented simplicial object we get that the colimit is the value $X'(-1) = M$ is the colimit of $X' \mid_{\boldsymbol{\Delta}^{op}}$.

3.0.74. *Bar-Beck-Lurie.* The last claim in ([14], ch.1, Sect. 3.7.3) is not clear. Lurie in [28] says only three lines about this, just after ([28], 4.7.3.3).

The reference to [28] in ([14], ch.1, 3.7.4) is not correct, the first claim is ([28], Lemma 4.7.3.1).

The main result in [28] about Bar-Beck-Lurie is ([28], 4.7.3.5). My understanding is that by left action of $\mathcal{E}^{\otimes}$ on $\mathcal{C}$ Lurie means there the structure of a $\mathcal{E}^{\otimes}$-left module category on $\mathcal{C}$.

It is maybe useful to remember Proposition ([28], 4.7.5.1): given a small ∞-category $\mathcal{J}$ and a diagram $q : J \to 1-\mathcal{C}\mathrm{at}$, $0 \in J$ and $\mathcal{C}^0 = q(0)$, assume $\mathcal{C} = \lim q$. It gives sufficient conditions for $\mathcal{C}$ to be monadic over $\mathcal{C}^0$. There are also ([28], 4.7.5.1-3) giving a version of descent theory in ∞-categories setting.

If A a monoidal ∞-category, a *comonoid* a in A is a monoid in A^{op}. A comodule over a is then an a-module in A^{op}. However, the subtlelty is that the category $a-comod(A)$ should be defined as $(a-mod(A^{op}))^{op}$.

Let now $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$. Then a *comonad* on $\mathcal{C}$ is a comonoid in the monoidal ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{C})$. If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is an adjoint pair of functors then $G^{op} : \mathcal{D}^{op} \rightleftarrows \mathcal{C}^{op} : F^{op}$ is an adjoint pair. Then $\mathcal{A} = FG$ is a comonoid in $\mathrm{Fun}(\mathcal{D},\mathcal{D})$, because $\mathcal{A}^{op}$ is a monoid in $\mathrm{Fun}(\mathcal{D},\mathcal{D})^{op} \widetilde{\to} \mathrm{Fun}(\mathcal{D}^{op},\mathcal{D}^{op})$. The composition $\mathcal{C}^{op} \to \mathcal{A}^{op} - mod(\mathcal{D}^{op}) \stackrel{oblv}{\to} \mathcal{D}^{op}$ is F^{op}. Passing to opposite functors, we get a diagram

$$\begin{array}{ccc} \mathcal{C} & \stackrel{F^{enh}}{\to} & (\mathcal{A}^{op} - mod(\mathcal{D}^{op}))^{op} \quad = \mathcal{A} - comod(\mathcal{D}) \\ & \searrow F & \downarrow oblv \\ & & \mathcal{D} \end{array}$$

Now F is called *comonadic* iff $F^{enh} : \mathcal{C} \to \mathcal{A} - comod(\mathcal{D})$ is an equivalence (it is equivalent to F^{op} being monadic).

If $M \in 1-\mathcal{C}\mathrm{at}$ is a left module category over a monoidal ∞-category A, and $a \in A$ is a coalgebra in A then $a - comod(M)$ is defined as $(a - mod(M^{op}))^{op}$. We use the fact that M^{op} is naturally a left A^{op}-module category.

The version of Barr-Beck-Lurie for comonads: if F is conservative and preserves totalizations then F is comonadic.

3.0.75. If $I^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is a symmetric monoidal category, consider Spc as a symmetric monoidal category $\mathrm{Spc}^{\otimes}$ with the cartesian symmetric monoidal structure. Let $F : I \to \mathrm{Spc}$ be a functor which is extended to a symmetric monoidal functor $F^{\otimes} : I^{\otimes} \to \mathrm{Spc}^{\otimes}$. Then $\mathrm{colim}_I F$ in Spc has a structure of a commutative monoid in Spc, this was used in [48].

Informal explanation: write $x_i = F(i)$ for $i \in X$, let $x = \mathrm{colim}\, F$. The product map $x \times x \to x$ is obtained as the map $x \times x \widetilde{\to} \mathrm{colim}_{i,j \in I \times I}\, x_i \times x_j \to x$. It comes from the compatible system of maps $x_i \times x_j \widetilde{\to} x_{i \otimes j} \to x$.

Similarly, if I is a non-unital symmetric monoidal category and F is a non-unital symmetric monoidal functor then $\mathrm{colim}_I F$ is naturally a non-unital commutative monoid in Spc.

In ([9], 2.2.1) the following generalization is used. Let $K \in 1-\mathcal{C}\mathrm{at}$ be a small symmetric monoidal category, $A \in 1-\mathcal{C}\mathrm{at}$ another symmetric monoidal category, and $\Psi : K \to A$ a right lax symmetric monoidal functor. Then $\mathrm{colim}_K \Psi$ is a commutative algebra object in A. My understanding is that this holds both in unital and non-unital setting. Informally, in the unital setting we have a natural map $1 \to \operatorname{colim}_{\{*\}} \Psi \to \operatorname{colim}_K \Psi$, where $\alpha : * \xrightarrow{1} K$ is the unit, and the map $1 \to \operatorname{colim}_{\{*\}} \Psi \widetilde{\to} \Psi(1)$ comes from the right lax structure of Ψ.

3.0.76. *On weak enrichments.* Recall the category $\boldsymbol{\Delta}^+$ from ([14], ch. I.1, 3.4.1). A map $[m] \to [n]$ in $\boldsymbol{\Delta}$ is inert iff it is an isomorphism of $[m]$ onto a convex subset of $[n]$. The inert maps in $\boldsymbol{\Delta}^+$ are the inert maps in $\boldsymbol{\Delta}$, the maps $[n]^+ \to [m]^+$, which identify $[n]^+$ with a convex subset of $[m]^+$, and also the maps $[n] \to [m]^+$, which identify $[n]$ with a convex subset of $[m]$. Let $X^{\otimes} \to \boldsymbol{\Delta}^{op}$ be a cocartesian fibration defining a monoidal ∞-category A. Let $M \in 1-\mathcal{C}\mathrm{at}$.

I hope the notion of a weak enrichment of M over A given in ([28], 4.2.1.12) can be equivalently reformulated as a morphism $\bar{q} : \bar{X}^{\otimes} \to \boldsymbol{\Delta}^{+,op}$ in $1-\mathcal{C}\mathrm{at}$ whose base change by $\boldsymbol{\Delta}^{op} \to \boldsymbol{\Delta}^{+,op}$ is identified with $X^{\otimes} \to \boldsymbol{\Delta}^{op}$, and such that q satisfies some *properties*, in particular:

- for any inert morphism $\alpha : a \to b$ of $\boldsymbol{\Delta}^+$ and $x \in \bar{X}^{\otimes}_b$, there is a $\bar{q}$-cocartesian morphism $x \to y$ over α.
- for any $[n]^+ \in \boldsymbol{\Delta}^+$, the inert morphisms $[n] \to [n]^+$, $[0]^+ \xrightarrow{n} [n]^+$ yields an equivalence $\bar{X}^{\otimes}_{[n]^+} \widetilde{\to} \bar{X}^{\otimes}_{[n]} \times \bar{X}^{\otimes}_{[0]^+}$

Is this true?

Let now $b : B \to A$ be a right-lax monoidal functor between monoidal ∞-categories. Assume M is left-tensored over A, so $\bar{q}$ is cocartesian fibration. Then by restriction we get a right-lax action of B on M. The following construction then should produce the corresponding enrichment of M over B.

Let $p : \boldsymbol{\Delta}^+ \to \boldsymbol{\Delta}$ be the functor from my Section 3.0.54. Let $Y^{\otimes} \to \boldsymbol{\Delta}^{op}$ be the cocartesian fibration corresponding to B, so we have the map $\beta : Y^{\otimes} \to X^{\otimes}$ over $\boldsymbol{\Delta}^{+,op}$ attached to b. Write $\tilde{Y}^{\otimes} \to \boldsymbol{\Delta}^{+,op}$ for the pullback of $Y^{\otimes} \to \boldsymbol{\Delta}^{+,op}$ by p, it corresponds to B viewed as a left B-module, we have $\tilde{X}^{\otimes} \to \boldsymbol{\Delta}^{+,op}$ defined similarly, it corresponds to A as a left A-module. The map β yields by pullback a morphism $\tilde{\beta} : \tilde{Y}^{\otimes} \to \tilde{X}^{\otimes}$ over $\boldsymbol{\Delta}^{+,op}$. We have a natural transformation $p \to \mathrm{id}$ of functors $\boldsymbol{\Delta}^{+,op} \to \boldsymbol{\Delta}^{+,op}$ from my Section 3.0.54. It yields by base change a morphism $\bar{X}^{\otimes} \to \tilde{X}^{\otimes}$ over $\boldsymbol{\Delta}^{+,op}$. Consider the product $Z^{\otimes} = \bar{X}^{\otimes} \times_{\tilde{X}^{\otimes}} \tilde{Y}^{\otimes}$ in $1-\mathcal{C}\mathrm{at}$ (equivalently, in $1-\mathcal{C}\mathrm{at}_{/\boldsymbol{\Delta}^{+,op}}$). I think $Z^{\otimes} \to \boldsymbol{\Delta}^{+,op}$ is the desired operad. The map $Z^{\otimes} \times_{\boldsymbol{\Delta}^{+,op}} \boldsymbol{\Delta}^{op} \to \boldsymbol{\Delta}^{op}$ identifies with β.

If $b : B \to A$ is left-lax instead of right-lax, then I think the construction should be the same, right?

Now the category $LMod(Z)$ is the full subcategory of $\mathrm{Fun}_{\boldsymbol{\Delta}^{+,op}}(\boldsymbol{\Delta}^{+,op}, Z^{\otimes})$ sending an inert morphism to an arrow cocartesian over $\boldsymbol{\Delta}^{+,op}$.

For $\mathcal{B} \in Alg(B)$ we can now define $\mathcal{B}-mod(M)$ along the fibre of the projection $LMod(Z) \to Alg(B)$.

3.0.77. The following is due to Lurie and generalizes Section 3.0.20.

Remark 3.0.78. *Let $C^{\otimes} \to O^{\otimes} \leftarrow D^{\otimes}$ be a diagram of cocartesian fibrations of ∞-operads. Let $A^{\otimes} \to O^{\otimes}$ be a map of ∞-operads. Assume given an adjoint pair $L : C^{\otimes} \leftrightarrows D^{\otimes} : R$ in $1-\mathcal{C}\mathrm{at}$, where L, R are maps of ∞-operads over $O^{\otimes}$. Assume that*

L is a morphism of $O^{\otimes}$-monoidal categories, that is, sends a cocartesian arrow over $O^{\otimes}$ to a cocartesian arrow over $O^{\otimes}$. Let $L' : Alg_{A/O}(C) \leftrightarrows Alg_{A/O}(D) : R'$ be obtained by composing with L and R. Then (L', R') is an adjoint pair in $1-\mathfrak{Cat}$.

Example: let A be a monoidal ∞-category, M, N be A-module categories and $L : M \to N$ a morphism of A-module categories having a right adjoint R. So, R is right-lax functor of A-module categories. Taking $A^{\otimes} = O^{\otimes}$ to be the ∞-operad classifying associative algebra + module, we get an adjoint pair

$$\bar{L} : Alg + Mod(A, M) \leftrightarrows Alg + Mod(A, N) : \bar{R}$$

over $\mathrm{Alg}(A)$. In particular, we may make a base change by $\mathcal{A} : * \to \mathrm{Alg}(A)$ by some associative algebra $\mathcal{A}$ in A, and get an adjoint pair $\mathcal{A} - mod(M) \leftrightarrows \mathcal{A} - mod(N)$.

3.1. **Duality.** First, one has to define *the exponential* as in (HA, 4.6.1.1). Namely, if A is a monoidal ∞-category, $c, m \in A$ then m^c is defined as the object that represents the functor $A^{op} \to \mathrm{Spc}$, $x \mapsto \mathrm{Map}_A(x \otimes c, m)$ (if it exists). It is equipped with the structure morphism $m^c \otimes c \to m$.

If A is presentable and the tensor product $A \times A \to A$ preserves colimits separately in each variable then the exponential always exists (HA, 4.6.1.2). Now (HA, Lemma 4.6.1.6): let $c, b \in A$ and assume given a map $e : b \otimes c \to 1$ in A. Then e extends to a duality datum on the pair (b, c) iff for any $a \in A$ the map $a \otimes b \otimes c \stackrel{\mathrm{id} \otimes e}{\to} a$ exhibits $a \otimes b$ as an exponential a^c. In other words, for any $d \in A$ one has functorially

$$\mathrm{Map}_A(d, a \otimes b) \widetilde{\to} \mathrm{Map}_A(d \otimes c, a)$$

For 4.1.1. Let A be a monoidal ∞-category, $a \in A$ admitting a right dual $a^{\vee, R}$. For $a' \in A$ the isomorphism $\mathrm{Map}_A(a \otimes a', 1) \to \mathrm{Map}_A(a', a^{\vee, R})$ opposite to that of 4.1.2 is defined as follows. It is defined as the composition

$$\mathrm{Map}_A(a \otimes a', 1) \stackrel{a^{\vee, R} \otimes}{\to} \mathrm{Map}_A(a^{\vee, R} \otimes a \otimes a', a^{\vee, R}) \to \mathrm{Map}_A(a', a^{\vee, R}),$$

where the second map is the composition with $a' \stackrel{unit \otimes \mathrm{id}}{\to} a^{\vee, R} \otimes a \otimes a'$.

Naively, the functor $(A^{right-dualizable})^{op} \to (A^{left-dualizable})^{rev-mult}$ is defined as follows. It sends a to $a^{\vee, R}$. Now a map $\phi : a \to b$ in $A^{right-dualizable}$ yields a morphism of functors of $y \in A$

$$\mathrm{Map}_A(y, b^{\vee, R}) \widetilde{\to} \mathrm{Map}_A(b \otimes y, 1) \to \mathrm{Map}_A(a \otimes y, 1) \widetilde{\to} \mathrm{Map}_A(y, a^{\vee, R})$$

represented by a morphism $\phi^{\vee, R} : b^{\vee, R} \to a^{\vee, R}$ in A.

The left dual of $a \in A$ is defined as the object $a^{\vee, L} \in A$ representing the functor $a' \mapsto \mathrm{Map}_A(a' \otimes a, 1)$ (with an additional property!). It is equipped with maps $counit : a^{\vee, L} \otimes a \to 1, unit : 1 \to a \otimes a^{\vee, L}$.

For 4.1.4. The full category $A^{right-dualizable} \subset A$ is stable under the tensor product. Namely, if $a, b \in A^{right-dualizable}$ with right duals $a^{\vee}, b^{\vee}$ then $b^{\vee} \otimes a^{\vee}$ is a right dual of $a \otimes b$. The corresponding unit and counit maps are defined as the compositions

$$a \otimes b \otimes b^{\vee} \otimes a^{\vee} \stackrel{counit_b}{\to} a \otimes a^{\vee} \stackrel{counit_a}{\to} 1$$

$$1 \stackrel{unit_b}{\to} b^{\vee} \otimes b \stackrel{unit_a}{\to} b^{\vee} \otimes a^{\vee} \otimes a \otimes b$$

For 4.1.5. Let $a_1, a_2 \in A$ with a_1 left dualizable. For $y \in A$ one gets the isomorphism $\mathrm{Map}_A(y \otimes a_1, a_2) \to \mathrm{Map}_A(y, a_2 \otimes a_1^{\vee,L})$ as follows. It is defined as the composition

$$\mathrm{Map}_A(y \otimes a_1, a_2) \stackrel{\otimes a_1^{\vee,L}}{\to} \mathrm{Map}_A(y \otimes a_1 \otimes a_1^{\vee,L}, a_2 \otimes a_1^{\vee,L}) \to \mathrm{Map}_A(y, a_2 \otimes a_1^{\vee,L}),$$

where the second map is the composition with $\mathrm{id} \otimes unit : y \to y \otimes a_1 \otimes a_1^{\vee,L}$.

Remark 3.1.1. *A property symmetric to (ch.1, 4.1.5) is missing in that section of the book: let A be a monoidal $(\infty,1)$-category, $b \in A$ be right-dualizable. Then for $d, c \in A$ we have functorial isomorphism $\mathrm{Map}_A(b \otimes d, c) \widetilde{\to} \mathrm{Map}_A(d, b^{\vee,R} \otimes c)$. It is proved similarly to (ch1, 4.1.5). This map sends $\alpha : b \otimes d \to c$ to the composition $d \stackrel{unit \otimes \mathrm{id}}{\to} b^{\vee,R} \otimes b \otimes d \stackrel{\mathrm{id} \otimes \alpha}{\to} b^{\vee,R} \otimes c$.*

One may strengthen (ch. 1, Lm. 4.1.6 a)) as follows.

Lemma 3.1.2. *Let A be a monoidal ∞-category. Assume $K \in 1-\mathcal{C}\mathrm{at}$, and A admits K-indexed limits. Let $a \in A$ be left dualizable. Then the functor $A \to A$, $d \mapsto \underline{\mathrm{Hom}}_A(a, d) \widetilde{\to} d \otimes a^{\vee,L}$ preserves K-indexed limits.*

Proof. By (ch. 1, 4.1.5), for any $d \in A$, $\underline{\mathrm{Hom}}_A(a, d)$ exists and one has $\underline{\mathrm{Hom}}_A(a, d) \widetilde{\to} d \otimes a^{\vee,L}$. Let $K \to A$ be a diagram $i \mapsto d_i$ for $i \in K$. For any $b \in A$ one has

$$\mathrm{Map}_A(b, \underline{\mathrm{Hom}}_A(a, \lim d_i)) \widetilde{\to} \mathrm{Map}_A(b \otimes a, \lim d_i) \widetilde{\to} \lim_i \mathrm{Map}_A(b \otimes a, d_i) \widetilde{\to}$$
$$\lim_i \mathrm{Map}_A(b, \underline{\mathrm{Hom}}_A(a, d_i)) \widetilde{\to} \mathrm{Map}_A(b, \lim_i \underline{\mathrm{Hom}}_A(a, d_i))$$

□

Remark 3.1.2.1 Let $A \in CAlg(1-\mathcal{C}\mathrm{at})$, $a \in A$ be dualizable. Then the dual map to the counit $c : a \otimes a^\vee \to 1_A$ identifies with the unit $u : 1_A \to a^\vee \otimes a$.

3.1.3. For 4.1.7. Let A be a monoidal ∞-category, $\mathcal{A}$ be an associative algebra in A, and $a \in A$ be left-dualizable. Then $\underline{\mathrm{Hom}}_A(a, a) \widetilde{\to} a \otimes a^{\vee,L}$ by 4.1.5. This object of A has a natural structure of an associative algebra by 3.6.6 (in [28] this is done in 4.7). This is done as usually for monads. Namely, $1 \to a \otimes a^{\vee,L}$ is the unit map, and the product

$$a \otimes a^{\vee,L} \otimes a \otimes a^{\vee,L} \to a \otimes a^{\vee,L}$$

is the map $\mathrm{id} \otimes counit \otimes \mathrm{id}$.

Now a gets a structure of a left $\underline{\mathrm{Hom}}_A(a, a)$-module by 3.6.6, where the action map $(a \otimes a^{\vee,L}) \otimes a \to a$ is $\mathrm{id} \otimes counit$. Now assume given a structure of a left $\mathcal{A}$-module on a. This is the same as a morphism of algebras $\mathcal{A} \to \underline{\mathrm{Hom}}_A(a, a)$, see my Section 3.0.71. So, to get a right $\mathcal{A}$-module structure on $a^{\vee,L}$, it suffices to do it in the case when $\mathcal{A} \to \underline{\mathrm{Hom}}_A(a, a)$ is an isomorphism.

Recall the action map $a^{\vee,L} \otimes (a \otimes a^{\vee,L}) \to a^{\vee,L}$ is the composition

$$a^{\vee,L} \otimes (a \otimes a^{\vee,L}) \stackrel{\mathrm{id} \otimes unit}{\to} a^{\vee,L} \otimes (a \otimes a^{\vee,L}) \otimes (a \otimes a^{\vee,L}) \stackrel{c \otimes c \otimes \mathrm{id}}{\to} a^{\vee,L}$$

So, this action map rewrites as

$$counit \otimes \mathrm{id} : a^{\vee,L} \otimes (a \otimes a^{\vee,L}) \to a^{\vee,L}$$

Now it is easy to see that acting twice by $a \otimes a^{\vee,L}$ on $a^{\vee,L}$ is the same as acting by their product. So, $a^{\vee,L}$ is a right $\underline{\mathrm{Hom}}_A(a,a)$-module.

About the second claim of 4.1.7. We don't assume anymore that a is left-dualizable, but assume that for $a' \in A$, $\underline{\mathrm{Hom}}_A(a,a')$ exists, and a is a left $\mathcal{A}$-module. First, we have a canonical map $\underline{\mathrm{Hom}}_A(a,a')\otimes a \to a'$ corresponding to $\mathrm{id} : \underline{\mathrm{Hom}}_A(a,a') \to \underline{\mathrm{Hom}}_A(a,a')$. Now define the action map $\underline{\mathrm{Hom}}_A(a,a') \otimes \mathcal{A} \to \underline{\mathrm{Hom}}_A(a,a')$ as the map corresponding to $\beta : \underline{\mathrm{Hom}}_A(a,a') \otimes \mathcal{A} \otimes a \to a'$. Here β is the composition

$$\underline{\mathrm{Hom}}_A(a,a') \otimes \mathcal{A} \otimes a \stackrel{\mathrm{id}\otimes\alpha}{\to} \underline{\mathrm{Hom}}_A(a,a') \otimes a \to a'$$

Here $\alpha : \mathcal{A} \otimes a \to a$ is the action map.

Addition: assume $\underline{\mathrm{Hom}}_A(a,b)$ exists, and $b \in A$ is a left $\mathcal{A}$-module. Then $\underline{\mathrm{Hom}}_A(a,b)$ is also a left $\mathcal{A}$-module. The action map corresponds to the composition

$$\mathcal{A} \otimes \underline{\mathrm{Hom}}_A(a,b) \otimes a \stackrel{\mathrm{id}\otimes\tau}{\to} \mathcal{A} \otimes b \stackrel{act}{\to} b,$$

where τ is the canonical action map, and *act* comes from the $\mathcal{A}$-module structure.

3.1.4. One may add to their Sect. 4.1.7 the following. Let A be a monoidal ∞-category, $a_i \in A$ such that $\underline{\mathrm{Hom}}_A(a_1,a_2), \underline{\mathrm{Hom}}_A(a_2,a_3), \underline{\mathrm{Hom}}_A(a_1,a_3) \in A$ exist. Then there is a natural map $\gamma : \underline{\mathrm{Hom}}_A(a_2,a_3) \otimes \underline{\mathrm{Hom}}_A(a_1,a_2) \to \underline{\mathrm{Hom}}_A(a_1,a_3)$ defined as follows. We have canonical maps $\alpha : \underline{\mathrm{Hom}}_A(a_1,a_2) \otimes a_1 \to a_2$, $\beta : \underline{\mathrm{Hom}}_A(a_2,a_3) \otimes a_2 \to a_3$. They yield the composition

$$\underline{\mathrm{Hom}}_A(a_2,a_3) \otimes \underline{\mathrm{Hom}}_A(a_1,a_2) \otimes a_1 \stackrel{\mathrm{id}\otimes\alpha}{\to} \underline{\mathrm{Hom}}_A(a_2,a_3) \otimes a_2 \stackrel{\beta}{\to} a_3$$

It corresponds to γ.

Remark Let $C \in 1-\mathcal{C}at$ be a monoidal category, $C^{left-dualizable} \subset C$ be the full subcategory of left-dualizable objects. Assume C is idempotent complete, and all the inner homs exist. Assume the tensor product preserves colimits (and limits) over Idem in each variable. Then

1) $C^{left-dualizable}$ is stable under retracts (similarly for $C^{right-dualizable}$).

2) Assume given a functor $\mathrm{Idem} \to C$ sending the unique object x to $c \in C$. Assume c is self-dual, and the duality datum $c \otimes c \to 1$ is equivariant under the diagonal $\mathcal{H}om_{\mathrm{Idem}}(x,x)$-action. Let r be a retract of c. Ther r is also self-dual.

Proof: 1) Let $c \in C$ be left dualizable. Let $f : \mathrm{Idem} \to C$ be a diagram, sending the unique object of Idem to c. Let $r = \lim f$ be the corresponding retract of c. We have the natural map $1^r \otimes r \to 1$, and we must show it extends to a duality datum on the pair $1^r, r$. For $a, d \in C$ the canonical map $1^c \otimes c \to 1$ yields an isomorphism $\mathrm{Map}(d, a \otimes (1^c)) \widetilde{\to} \mathrm{Map}(d \otimes c, a)$ which is compatible with the action of $\mathcal{H}om_C(c,c)$, hence with f. We see the latter is an isomorphism of functors $\mathrm{Idem} \to \mathrm{Spc}$. Passing to the limit over Idem, we get the desired ismorphism $\mathrm{Map}(d, a \otimes (1^r)) \widetilde{\to} \mathrm{Map}(d \otimes r, a)$. □

3.1.5. *Tensor products of modules.* For 4.2.1. I think the assumption there are: A monoidal ∞-category. The geometric realizations distribute over the monoidal operation in A should mean the following. Given $\boldsymbol{\Delta}^{op} \to A, [n] \mapsto a_n$, $b \in A$, one has $\mathrm{colim}_{\boldsymbol{\Delta}^{op}}(a_n \otimes b) \widetilde{\to} (\mathrm{colim}\, a_n) \otimes b$ naturally, and similarly for the second variable.

For 4.2.2. Dennis applies here the following. Let $\mathcal{O}^{\otimes} \to \mathcal{F}in_*$ be a symmetric monoidal ∞-category. Then $AssAlg + mod(\mathcal{O}, \mathcal{O})$ is the category $Alg_{LM}(\mathcal{O})$, that is, the category of maps of operads $LM^{\otimes} \to \mathcal{O}^{\otimes}$. It is the underlying ∞-category of the symmetric monoidal ∞-category $Alg_{LM}(\mathcal{O})^{\otimes}$, see Prop. 3.0.31. Moreover, the evaluations $e_{\mathfrak{a}}, e_{\mathfrak{m}} : Alg_{LM}(\mathcal{O})^{\otimes} \to \mathcal{O}^{\otimes}$ are symmetric monoidal. The inclusion $\mathcal{A}\mathrm{ss}^{\otimes} \subset LM^{\otimes}$ yields the restriction functor $Alg_{LM}(\mathcal{O})^{\otimes} \to \mathrm{Alg}_{Ass}(\mathcal{O})$, which is symmetric monoidal by ([28], 3.2.4.3).

For 4.2.3: the fact that this is a cocartesian fibration follows from ([28], 4.6.2.17) I think, see also my Section 3.0.56.

3.1.6. For 4.3.1. Let A be a monoidal ∞-category, $\mathcal{A} \in AssAlg(A)$. Let $N \in \mathcal{A} - mod^r, M \in \mathcal{A} - mod$ be in duality. Assume that A admits geometric realizations, and the tensor product in A preserves geometric realizations in each variable. Then we have a natural isomorphism functorial in $S \in \mathcal{A} - mod^r$

$$\mathrm{Map}_{\mathcal{A}-mod^r}(S, N) \widetilde{\to} \mathrm{Map}_{\mathcal{A}\otimes\mathcal{A}^{rm}-mod}(M \otimes S, \mathcal{A})$$

sending $\alpha : S \to N$ to the composition $M \otimes S \stackrel{\mathrm{id}\otimes\alpha}{\to} M \otimes N \stackrel{counit}{\to} \mathcal{A}$. The inverse map sends $\beta : M \otimes S \to \mathcal{A}$ to the composition

$$S \stackrel{unit\otimes\mathrm{id}}{\to} N \otimes_{\mathcal{A}} M \otimes S \stackrel{\mathrm{id}\otimes\beta}{\to} N \otimes_{\mathcal{A}} \mathcal{A} \widetilde{\to} N$$

More general claim is given in (HA, 4.6.2.1). Namely, we get an adjoint pair

$$\mathcal{F}_M : A \leftrightarrows \mathcal{A} - mod : \mathcal{F}_N,$$

where $\mathcal{F}_M(X) = M \otimes X$, and $\mathcal{F}_N(Z) = N \otimes_{\mathcal{A}} Z$.

Besides, we get an adjoint pair

$$\mathcal{F}'_N : A \leftrightarrows \mathcal{A} - mod^r : \mathcal{F}'_M,$$

where $\mathcal{F}'_N(X) = X \otimes N$ and $\mathcal{F}'_M(Z) = Z \otimes_{\mathcal{A}} M$.

For 4.3.2. Let A be a monoidal ∞-category, $\mathcal{A} \in AssAlg(A)$, $M, N \in \mathcal{A} - mod^r$. Assume M admits a dual $M^{\vee} \in \mathcal{A}-mod$. Then $\underline{\mathrm{Hom}}_{A,\mathcal{A}}(M, N) \widetilde{\to} N \otimes_{\mathcal{A}} M^{\vee}$ canonically. Indeed, given $H \in A$, the isomorphism $\mathrm{Map}_A(H, N \otimes_{\mathcal{A}} M^{\vee}) \widetilde{\to} \mathrm{Map}_{\mathcal{A}-mod^r}(H \otimes M, N)$ sends $\alpha : H \to N \otimes_{\mathcal{A}} M^{\vee}$ to the composition

$$H \otimes M \stackrel{\alpha\otimes\mathrm{id}}{\to} N \otimes_{\mathcal{A}} M^{\vee} \otimes M \stackrel{\mathrm{id}\otimes counit}{\to} N \otimes_{\mathcal{A}} \mathcal{A} = N$$

For 4.3.3: a condition is missing. They have to assume there that A admits geometric realizations, and the tensor product in A preserves geometric realizations in each variable.

3.1.7. *An application of the operad* $\mathrm{Mod}_A(\mathcal{C})^{\otimes}$. ([28], 4.8.2.10) combined with my Section 3.0.67 gives an interesting application: let $\mathcal{C}$ be a symmeric monoidal ∞-category, assume $\mathcal{C}$ has geometric realizations of simplicial objects, and the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Let A be an idempotent object of $CAlg(\mathcal{C})$, that is, the multiplication $A \otimes A \to A$ is an isomorphism. Let $L : \mathcal{C} \to \mathcal{C}$ be the functor $x \mapsto A \otimes x$, recall that L is a localization functor, $L\mathcal{C}$ inherits a symmetric monoidal structure ([28], 4.8.2.7), $L : \mathcal{C} \to L\mathcal{C}$ is symmetric monoidal functor, and $L\mathcal{C} \hookrightarrow \mathcal{C}$ is right-lax nonunital monoidal ([28], 2.2.1.9). Finally, $L\mathcal{C}$ identifies with $A-mod(\mathcal{C})$, and the symmetric monoidal structure on $L\mathcal{C}$ is the tensor product of A-modules over A according to my Section 3.0.67.

Note that ([28], 4.8.2.9) is unexpected nice result!!! It says that given $e : 1 \to E$ in $\mathcal{C}$, which exhibits E as an idempotent of $\mathcal{C}$, E automatically has an algebra structure!

3.1.8. *Relative tensor product.* In (HA, 4.4.1) there is a definition of the 'bilinear map' in the situation: let $\mathcal{C}$ be a symmetric monoidal ∞-category, $A, B, C \in \mathrm{Alg}(\mathcal{C})$, $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{C}), X \in {}_A\mathrm{BMod}_C(\mathcal{C})$. Then a map $M \otimes N \to X$ is bilinear if we are given the following. He defines the category $\mathrm{Tens}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times \boldsymbol{\Delta}^{op}$ in (HA, 4.4.4.1). Now for the map $[1] \to \boldsymbol{\Delta}^{op}$ given by $[1] \stackrel{0,2}{\to} [2]$ in $\boldsymbol{\Delta}$ he considers the base change $\mathrm{Tens}^{\otimes}_{\succ} = \mathrm{Tens}^{\otimes}_{\boldsymbol{\Delta}^{op}}[1]$. Then a map of operads $\gamma_0 : \mathrm{Tens}^{\otimes}_{\succ} \times_{[1]} \{0\} \to \mathcal{C}^{\otimes}$ is a datum of (A, B, C) as above and M, N. Besides, $\mathrm{Tens}^{\otimes}_{\succ} \times_{[1]} \{1\} \widetilde{\to} BM^{\otimes}$, and a datum of X is a datum of a map of operads $\gamma_1 : \mathrm{Tens}^{\otimes}_{\succ} \times_{[1]} \{1\} \to \mathcal{C}^{\otimes}$. Now a bilinear map $M \otimes N \to X$ that coequalizes the B-actions on M on the right and on N on the left up to coherent homotopy, is an extension of γ_0, γ_1 to a map of generalized operads $\gamma : \mathrm{Tens}^{\otimes}_{\succ} \to \mathcal{C}^{\otimes}$. This seems impossble to use in practice (except maybe for n-categories with small n)?

The functor $\phi : \boldsymbol{\Delta}^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ used by Lurie (defined in HA, 4.1.2.9) has an additional property: if $\alpha_0 : [n] \to [n']$ is a map in $\boldsymbol{\Delta}$, the induced map $\phi(\alpha_0) : \langle n' \rangle \to \langle n \rangle$ satisfies the following. If $j \in \langle n \rangle^0$ then $\phi(\alpha_0)^{-1}(j)$ is a segment $[i_0, \ldots, i_m]$. Namely, $\alpha_0(j-1) = i_0 - 1, \alpha_0(j) = i_m$.

For (HA, 4.4.1.12). Let us check that the functor $\Phi : \mathrm{Step} \to \mathrm{Tens}^{\otimes}$ from (HA, 4.4.1.12) is well-defined. If $\alpha : f' \to f$ is a map in Step given by the diagram in $\mathrm{Fun}([1], \boldsymbol{\Delta})$

$$\begin{array}{ccc} [n] & \stackrel{f}{\to} & [k] \\ \downarrow \alpha_0 & & \downarrow \alpha_1 \\ [n'] & \stackrel{f'}{\to} & [k'] \end{array}$$

then we want to check that $(\langle n' \rangle, [k'], c'_-, c'_+) \stackrel{\Phi(\alpha)}{\to} (\langle n \rangle, [k], c_-, c_+)$ is indeed a map in $\mathrm{Tens}^{\otimes}$.

For each $j \in \langle n \rangle^0$ we have $\phi(\alpha_0)^{-1}(j) = [i_0, \ldots, i_m]$ with $i_0 - 1 = \alpha_0(j-1), i_m = \alpha_0(j)$. So, $c'_-(i_0) = f'(i_0 - 1) = \alpha_1(f(j-1)) = \alpha_1 c_-(j)$, and $c'_+(i_m) = f'(i_m) = \alpha_1 f(j) = \alpha_1 c_+(j)$. The inequalities

$$c'_-(i_0) \leq c'_+(i_0) = c'_-(i_1) \leq c'_+(i_1) = c'_-(i_2) \leq \ldots \leq c'_+(i_m)$$

are clear, because $\phi(\alpha_0)^{-1}(j)$ is a segment.

Recall that $\mathrm{Tens}^{\otimes}_{[k]} := \mathrm{Tens}^{\otimes} \times_{\boldsymbol{\Delta}^{op}} \{[k]\}$. The functors $u : \boldsymbol{\Delta}^{op} \to \mathrm{Step}$, $u_+ : \boldsymbol{\Delta}^{op}_+ \to \mathrm{Step}$ are well-defined. The composition

$$U : \boldsymbol{\Delta}^{op} \stackrel{u}{\to} \mathrm{Step} \stackrel{\Phi}{\to} \mathrm{Tens}^{\otimes}$$

takes values in $\mathrm{Tens}^{\otimes}_{[2]}$, the composition

$$U_+ : \boldsymbol{\Delta}^{op}_+ \stackrel{u_+}{\to} \mathrm{Step} \stackrel{\Phi}{\to} \mathrm{Tens}^{\otimes}$$

takes values in $\mathrm{Tens}^{\otimes}_{\succ}$. The image of $([1] \stackrel{\mathrm{id}}{\to} [1]) \in \mathrm{Step}$ in $\mathrm{Tens}^{\otimes}_{[1]}$ is $\mathfrak{m}$. This is why in (HA, 4.4.2.7) we get a map of functors $U \to U'$ in $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Tens}^{\otimes}_{\succ})$. The definition of the bar construction $Bar_B(M,N)_\bullet \in \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{C}^{\otimes})$ in (HA, 4.4.2.7) is clear, this uses the notion of *cocartesian natural transformation*, see the end of this subsection. Informally, we have $Bar_B(M,N)$ given by $[n] \mapsto M \otimes B^n \otimes N$.

(HA, Example 4.4.2.11) seems sufficient for most purposes, so that I do not need to know what is "a map of generalized ∞-operads", but just accept that $\mathrm{Tens}^{\otimes}_{\succ} \to \mathrm{Tens}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ is a map of generalized operads, where the second map is the natural projection. (HA, Examples 4.4.2.11-13) are sufficient for a definition of relative tensor product in most cases I think.

Think of $\mathrm{Tens}^{\otimes}_{[2]}$ as an ∞-operad. Recall that any object of $\mathrm{Tens}_{[2]}$ is one of the following $\mathfrak{m}_{01}, \mathfrak{m}_{12}, \mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2$. The object $U([n]) \in \mathrm{Tens}^{\otimes}_{[2]}$ is $\mathfrak{m}_{01} \oplus \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_1 \otimes \mathfrak{m}_{12}$, where $\mathfrak{a}_1$ appears n times. Now the two maps $0, 1 : [0] \to [1]$ in $\boldsymbol{\Delta}$ give after applying U respectively the maps $\mathfrak{m}_{01} \oplus \mathfrak{a}_1 \oplus \mathfrak{m}_{12} \to \mathfrak{m}_{01} \oplus \mathfrak{m}_{12}$ in $\mathrm{Tens}^{\otimes}_{[2]}$, the first being the multiplication on $\mathfrak{m}_{12}$ on the left, the second being the multiplication on $\mathfrak{m}_{01}$ on the right.

For example, the map $[1] \stackrel{01}{\to} [2]$ in $\boldsymbol{\Delta}$ gives the map $(\mathrm{id}, \mathrm{id}, action) : \mathfrak{m}_{01} \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{m}_{12} \to \mathfrak{m}_{01} \oplus \mathfrak{a}_1 \oplus \mathfrak{m}_{12}$ in $\mathrm{Tens}^{\otimes}_{[2]}$. The map $[1] \stackrel{12}{\to} [2]$ in $\boldsymbol{\Delta}$ gives the map $(action, id, id) : \mathfrak{m}_{01} \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{m}_{12} \to \mathfrak{m}_{01} \oplus \mathfrak{a}_1 \oplus \mathfrak{m}_{12}$ in $\mathrm{Tens}^{\otimes}_{[2]}$. The map $[1] \stackrel{02}{\to} [2]$ in $\boldsymbol{\Delta}$ gives the map $(\mathrm{id}, mult, id) : \mathfrak{m}_{01} \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{m}_{12} \to \mathfrak{m}_{01} \oplus \mathfrak{a}_1 \oplus \mathfrak{m}_{12}$ in $\mathrm{Tens}^{\otimes}_{[2]}$, where $mult$ denotes the multiplication on $\mathfrak{a}_1$.

Key case (HA, Example 4.4.2.11): let $\mathcal{C}^{\otimes} \to \boldsymbol{\Delta}^{op}$ be a monoidal ∞-category admitting geometric realizations of simplicial objects. Assume the tensor product in $\mathcal{C}$ preserves geometric realizations of simplicial objects in each variable. Given algebras $A, B, C \in AssAlg(\mathcal{C})$, we get a functor ${}_A\mathrm{BMod}_B(\mathcal{C}) \times {}_B\mathrm{BMod}_C(\mathcal{C}) \to {}_A\mathrm{BMod}_C(\mathcal{C})$, $(M,N) \mapsto M \otimes_B N$, the relative tensor product. It is given as the geometric realization of the simplicial object $\boldsymbol{\Delta}^{op} \to \mathcal{C}$, $[n] \mapsto M \otimes B^{\otimes n} \otimes N$ by (HA, Th. 4.4.2.8). Let in addition $K \in 1-\mathcal{C}\mathrm{at}$ such that the tensor product on $\mathcal{C}$ is compatible with K-indexed colimits (that is, $\mathcal{C}$ admits K-indexed colimits and the tensor product preserves K-indexed colimits separately in each variable). Then the relative tensor product preserves the K-indexed colimits separately in each variable (HA, 4.4.2.15).

Recall that $\triangle_s^{op} \hookrightarrow \triangle^{op}$ is cofinal by (HTT, 6.5.3.7), so in the above $M \otimes_B N$ depends only on the nonunital B-module structures on M, N.

(HA, 4.4.3.12): let $\mathcal{C}^{\otimes} \to \boldsymbol{\Delta}^{op}$ be a monoidal ∞-category, admitting geometric realizations of simplicial objects. Assume the tensor product in $\mathcal{C}$ preserves geometric realizations of simplicial objects in each variable. Given $A \in AssAlg(\mathcal{C})$, the category ${}_A\mathrm{BMod}_A(\mathcal{C})$ is equipped with a monoidal structure, the tensor product is given by the relative tensor product over A.

Associativity of the relative tensor product is (HA, 4.4.3.14), Unitality of the Tensor Product in (HA, 4.4.3.16). It says that for $M \in {}_ABmod_B(\mathcal{C})$ one has $A \otimes_A M \widetilde{\to} M$, and $M \otimes_B B \widetilde{\to} M$.

Preservation of colimits separately in each variable by the relative tensor product is (HA, 4.4.2.14-15).

(HA, 4.4.2.12): Let $\mathcal{A}, \mathcal{B}$ be monoidal ∞-categories, $M \in {}_{\mathcal{A}}\operatorname{BMod}_{\mathcal{B}}(1-\mathcal{C}\mathrm{at})$. Let $A, B \in Alg(\mathcal{A}), C \in Alg(\mathcal{B})$, $M \in {}_A\operatorname{BMod}_B(\mathcal{A}), N \in {}_B\operatorname{BMod}_C(M)$. Assume $\mathcal{A}, \mathcal{B}, M$ admit geometric realizations of simplicial objects, and all the tensor product functors $\mathcal{A} \times M \to M, \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \mathcal{B} \times \mathcal{B} \to \mathcal{B}, M \times \mathcal{B} \to M$ preserve geometric realizations separately in each variable. The relative tensor product $M \otimes_B N \in {}_A\operatorname{BMod}_C(M)$ exists and is given by the bar construction: $M \otimes_B N \widetilde{\to} \operatorname{colim}_{[n] \in \boldsymbol{\Delta}^{op}} M \otimes B^n \otimes N$.

(HA, 4.4.2.13) is similar with the roles of left and rights exchanged.

The notion of a cocartesian natural transformation: given a cocartesian fibration $p : X \to S$ in $1-\mathcal{C}\mathrm{at}$, $K \in 1-\mathcal{C}\mathrm{at}$ the induced map $X^K \to S^K$ is a cocartesian fibration by ([27], 3.1.2.1). So, given $f, g \in \operatorname{Fun}(K, S)$ and a map $\alpha : f \to g$ in $\operatorname{Fun}(K, S)$ assume given also $\bar{f} \in \operatorname{Fun}(K, X)$ with an isomorphism $p\bar{f} \widetilde{\to} f$. Then there is a cocartesian arrow $\bar{\alpha} : \bar{f} \to \bar{g}$ in $\operatorname{Fun}(S, X)$ over α. Lurie refers to $\bar{\alpha}$ as a *cocartesian natural transformation.* It is used in bar construction, in particular.

3.1.9. We formulate the results from ([28], 4.8.4) in the language of ([14], ch.1). Let A be a monoidal ∞-category, $N \in A-mod$ be a left A-module category, $\mathcal{A} \in AssAlg(A)$.

([28], 4.8.4.1): Assume A, N admit geometric realizations of simplicial objects and tensor products $A \times A \to A, A \times N \to N$ preserve geometric realizations of simplicial objects separately in each variable. Then

$$\operatorname{LinFun}_A^{\mathcal{K}}(\mathcal{A} - mod^r(A), N) \widetilde{\to} \mathcal{A} - mod(N)$$

by ([28], 4.8.4.1). The notation LinFun_A is that of my Section 3.0.49, and $\operatorname{LinFun}_A^{\mathcal{K}} \subset \operatorname{LinFun}_A$ is the full subcategory of functors preserving the geometric realizations. The A-module structure on $\mathcal{A} - mod^r(A)$ is defined in ([28], 4.3.2).

Corollary: assume $A \in Alg(1-\mathcal{C}\mathrm{at})$ admits geometric realizations, and tensor products $A \times A \to A$ preserves geometric realizations of simplicial objects separately in each variable. Let $I \to A - mod$ be a diagram, $i \mapsto M_i$. Assume each M_i admits geometric relaizations, the action maps $A \times M_i \to M_i$ preserves geometric realizations of simplicial objects separately in each variable, and for $i \to j$ in I the transition functor $M_i \to M_j$ preserves geometric realizations. Let $\mathcal{A} \in Alg(A)$. Then the natural functor $\mathcal{A} - mod(\lim_{i \in I} M_i) \to \lim_{i \in I} \mathcal{A} - mod(M_i)$ is an equivalence, where in the RHS the limit is calculated in $1 - Cat$, and $\lim_{i \in I} M_i$ is calculated in $A - mod$.

Proof. Recall that for $N \in A-mod$ the functor $A-mod \to 1-\mathcal{C}\mathrm{at}$, $M' \mapsto \operatorname{LinFun}_A(N, M')$ preserves limits, cf. Section 3.0.49. Set $M = \lim_i M_i$ in $A - mod$. So,

$$\operatorname{LinFun}_A(\mathcal{A} - mod^r(A), M) \widetilde{\to} \lim_{i \in I} \operatorname{LinFun}_A(\mathcal{A} - mod^r(A), M_i)$$

For $i \in I$ the projection $M \to M_i$ preserves geometric realizations. This gives a fully faithful functor

$$a : \operatorname{LinFun}_A^{\mathcal{K}}(\mathcal{A} - mod^r(A), M) \to \lim_{i \in I} \operatorname{LinFun}_A^{\mathcal{K}}(\mathcal{A} - mod^r(A), M_i)$$

Let now $f \in \operatorname{LinFun}_A(\mathcal{A} - mod^r(A), M)$ be such that for any i its composition with $M \to M_i$ preserves geometric realizations. Then f preserves geometric realizations, so a is an equivalence. The claim follows now from ([28], 4.8.4.1). $\square$

Let $A \in Alg(1-\mathcal{C}at)$. Recall the notation $A-mod$ from my Section 3.0.49.

Claim: 1) Assume $C = \lim_{\alpha\in A} C_\alpha$ in $A-mod$ and $\mathcal{A} \in Alg(A)$. The natural functor $\mathcal{A}-mod(C) \to \lim\limits_{\alpha\in A} \mathcal{A}-mod(C_\alpha)$ is fully faithful, here lim is taken in $1-\mathcal{C}at$.
2) Assume each C_α admits geometric realizations, and each transition functor $f_{\alpha,\beta}: C_\alpha \to C_\beta$ preserves geometric realizations. Then the natural functor $\mathcal{A}-mod(C) \to \lim\limits_{\alpha\in A} \mathcal{A}-mod(C_\alpha)$ is an equivalence.

Proof. 1) Let $z, v \in \mathcal{A}-mod(C)$. For $\alpha \in A$ write z_α for the image of z in $\mathcal{A}-mod(C_\alpha)$ and also in C_α. Recall that $z \widetilde{\to} \operatorname{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \mathcal{A}^{n+1}z$ given by the bar construction ([28], 4.7.3.13). We get

$$\operatorname{Map}_{\mathcal{A}-mod(C)}(z,v) \widetilde{\to} \operatorname{Map}_{\mathcal{A}-mod(C)}(\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \mathcal{A}^{n+1}z, v) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \operatorname{Map}_{\mathcal{A}-mod(C)}(\mathcal{A}^{n+1}z, v)$$

$$\widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \operatorname{Map}_C(\mathcal{A}^n z, v) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \lim_\alpha \operatorname{Map}_{C_\alpha}(\mathcal{A}^n z_\alpha, v_\alpha) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \lim_\alpha \operatorname{Map}_{\mathcal{A}-mod(C_\alpha)}(\mathcal{A}^{n+1}z_\alpha, v_\alpha)$$

$$\widetilde{\to} \lim_\alpha \operatorname{Map}_{\mathcal{A}-mod(C_\alpha)}(\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \mathcal{A}^{n+1}z_\alpha, v_\alpha) \widetilde{\to} \lim_\alpha \operatorname{Map}_{\mathcal{A}-mod(C_\alpha)}(z_\alpha, v_\alpha) \widetilde{\to} \operatorname{Map}_{\lim_\alpha \mathcal{A}-mod(C_\alpha)}(z,v)$$

2) Consider for each α the left adjoint $free_\alpha : C_\alpha \to \mathcal{A}-mod(C_\alpha)$ of $\operatorname{oblv}_\alpha : \mathcal{A}-mod(C_\alpha) \to C_\alpha$. The functors $free_\alpha$ are compatible with the transition functors $f_{\alpha,\beta}: C_\alpha \to C_\beta$ for $\alpha \to \beta$ in A, so in the limit give a functor $L : C \to \lim_\alpha \mathcal{A}-mod(C_\alpha)$. Applying ([14], ch. I.1, 2.6.4), we see that L admits a right adjoint $R : \lim_\alpha \mathcal{A}-mod(C_\alpha) \to C$, which is the limit over $\alpha \in A$ of the functors $\operatorname{oblv}_\alpha$. The monad RL identifies with $\mathcal{A}$. Indeed, for each α we have a commutatie diagram

$$\begin{array}{ccc} C & \stackrel{RL}{\to} & C \\ \downarrow & & \downarrow \\ C_\alpha & \stackrel{z\mapsto\mathcal{A}z}{\to} & C_\alpha \end{array}$$

Thus, we get the functor $R^{enh} : \lim_\alpha \mathcal{A}-mod(C_\alpha) \to \mathcal{A}-mod(C)$. Clearly, R is conservative.

By our assumption and Lemma 2.2.69, C admits geometric realizations, and $ev_\alpha : C \to C_\alpha$ preserves geometric realizations. To show that R is monadic we apply ([28], 4.7.3.5). Let $\boldsymbol{\Delta}^{op} \to \lim_\alpha \mathcal{A}-mod(C_\alpha)$, $[n] \mapsto x_n$ be a simplicial object, which is R-split. Its image in each C_α is split, so by ([28], 4.7.3.5), $[n] \to x_{n,\alpha}$ admits a colimit in $\mathcal{A}-mod(C_\alpha)$, which is preserved by $\operatorname{oblv}_\alpha$. Here $x_{n,\alpha}$ is the image of x_n in $\mathcal{A}-mod(C_\alpha)$. We see that the transition functor $\bar f_{\alpha,\beta} : \mathcal{A}-mod(C_\alpha) \to \mathcal{A}-mod(C_\beta)$ preserves $\operatorname{colim}_{[n]\in\boldsymbol{\Delta}^{op}} x_{n,\alpha}$, because $\operatorname{oblv}_\beta$ is conservative. Thus, by Lemma 2.2.69, $\operatorname{colim}_{[n]} x_n$ in $\lim_\alpha \mathcal{A}-mod(C_\alpha)$ exists. Moreover, it is preserved by R by Lemma 2.2.68. So, R is monadic. □

([28], 4.8.4.6): TO BE INCLUDED!!!!

3.1.10. *Dualizability of colimits.* Let $A \in 1-Cat$ be symmetric monoidal such that the tensor product preserves colimits separately in each variable, and admitting small colimits. Let $I \to A^{dualizable}$ be a diagram, $i \mapsto a_i$, where $A^{dualizable} \subset A$ is the full subcategory of dualizable objects. Write $a = \operatorname{colim}_{i\in I} a_i$, the colimit in A. Passing to duals, we get a diagram $I^{op} \to A^{dualizable}$, $i \mapsto a_i^\vee$. Assume this diagram admits a limit b. Assume moreover that for any $d \in A$ the natural map $d \otimes \lim\limits_{i \in I^{op}} a_i^\vee \to \lim\limits_{i \in I^{op}} (d \otimes a_i^\vee)$ is an isomorphism. Then $a \in A^{dualizable}$, and b is isomorphic to the dual of a.

Indeed, the unit and counit map $a \otimes b \to 1$, $1 \to b \otimes a$ are constructed as in ([14], ch. I.1, 6.3.5-6.3.6).

Here is an application. Let $A \in 1-Cat$ be symmetric monoidal such that the tensor product preserves colimits separately in each variable, and admitting small colimits. Let $a \in A$ be dualizable, x be a retract of a. Then $x = \mathrm{colim}_{i \in \mathrm{Idem}}\, a_i$, where a_i are dualizable. The dual diagram over Idem^{op} admits a limit y in A. Indeed, $\mathrm{Idem}^{op} \widetilde{\to} \mathrm{Idem}$. So, by ([27], 4.4.5.14), this y can be rewritten as a colimit over Idem, its existence is garanteed because A admits small colimits. (In fact, ([27], 4.4.5.14) can be formulated more precisely saying that if a diagram $f : \mathrm{Idem} \to \mathcal{C}$ extends to a functor $\mathrm{Idem}^+ \to \mathcal{C}$ then $\mathrm{colim}_{\mathrm{Idem}} f \widetilde{\to} \lim_{\mathrm{Idem}} f$). For this reason, for any $d \in A$, the natural map $d \otimes \lim\limits_{i \in \mathrm{Idem}^{op}} a_i^\vee \to \lim\limits_{i \in \mathrm{Idem}^{op}} d \otimes a_i^\vee$ is an isomorphism. So, x is dualizable, and $y \widetilde{\to} x^\vee$.

3.1.11. Let $G^L : C \leftrightarrows D : G$ be an adjoint pair of functors, where $C, D \in 1-\mathcal{C}\mathrm{at}$. Then $\mathcal{A} := GG^L \in \mathrm{Fun}(C, C)$ is a monad. Note that $\mathrm{Fun}(C, C)$ acts on the right on $\mathrm{Fun}(C, D)$, so we have the category $\mathcal{A} - mod^r(\mathrm{Fun}(C, D))$. Note that G^L naturally lifts to an object of $\mathcal{A} - mod^r(\mathrm{Fun}(C, D))$. The action map is given by $G^L G G^L = (G^L G) G^L \stackrel{c}{\to} G^L$, where $c : G^L G \to \mathrm{id}$ is the counit of the adjunction.

3.2. **Addition about comodules.** Let $A \in 1-\mathcal{C}\mathrm{at}$ be symmetric monoidal, $\mathcal{A} \in Alg(A)$. Assume $\mathcal{A}$ is dualizable in A. Recall that $\mathcal{A}^\vee$ is naturally an $\mathcal{A}$-bimodule (ch. 1, 4.1.7). In particular, the left $\mathcal{A}$-action on $\mathcal{A}^\vee$ is defined as the composition

$$act : \mathcal{A} \otimes \mathcal{A}^\vee \stackrel{u \otimes \mathrm{id}}{\to} \mathcal{A}^\vee \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}^\vee \stackrel{\mathrm{id} \otimes m \otimes \mathrm{id}}{\to} \mathcal{A}^\vee \otimes \mathcal{A} \otimes \mathcal{A}^\vee \stackrel{\mathrm{id} \otimes c}{\to} \mathcal{A}^\vee,$$

here u is the unit, c the counit, m the multiplication.

The counit map $c : \mathcal{A} \otimes \mathcal{A}^\vee \to 1$ is $\mathcal{A}$-bilinear in the sense that given $a, x \in \mathcal{A}$, $x^* \in \mathcal{A}^\vee$, $c(xa \otimes x^*) = c(x \otimes ax^*)$, where we use the left action of $\mathcal{A}$ on $\mathcal{A}^\vee$. That is, the diagram commutes

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}^\vee & \stackrel{\mathrm{id} \otimes act}{\to} & \mathcal{A} \otimes \mathcal{A}^\vee \\ \downarrow m \otimes \mathrm{id} & & \downarrow c \\ \mathcal{A} \otimes \mathcal{A}^\vee & \stackrel{c}{\to} & 1 \end{array}$$

Note that if $\mathcal{A} \in CAlg(A)$ then the left $\mathcal{A}$-module and right $\mathcal{A}$-module structure on $\mathcal{A}^\vee$ are the same.

Lemma 3.2.1. *For any $M \in \mathcal{A} - mod$ the composition $\epsilon : M \stackrel{u \otimes \mathrm{id}}{\to} \mathcal{A}^\vee \otimes \mathcal{A} \otimes M \stackrel{\mathrm{id} \otimes act}{\to} \mathcal{A}^\vee \otimes M$ is a morphism of left $\mathcal{A}$-modules, where the $\mathcal{A}$-module structure on $\mathcal{A}^\vee \otimes M$ comes from that of $\mathcal{A}^\vee$.*

Proof. Under the duality isomorphism $\mathrm{Map}_A(\mathcal{A} \otimes M, M) \widetilde{\to} \mathrm{Map}_A(M, \mathcal{A}^\vee \otimes M)$ the map ϵ corresponds to the action map $act : \mathcal{A} \otimes M \to M$. Consider the composition $\mathcal{A} \otimes M \stackrel{\mathrm{act}}{\to} M \stackrel{\epsilon}{\to} \mathcal{A}^\vee \otimes M$. Under the duality $\mathrm{Map}_A(\mathcal{A} \otimes ?, \cdot) \widetilde{\to} \mathrm{Map}(?, \mathcal{A}^\vee \otimes \cdot)$ it corresponds to the map $\mathcal{A} \otimes \mathcal{A} \otimes M \stackrel{m \otimes \mathrm{id}}{\to} \mathcal{A} \otimes M \stackrel{\mathrm{act}}{\to} M$. Under the same duality, the composition $\mathcal{A}\otimes M \stackrel{\mathrm{id}\otimes\epsilon}{\to} \mathcal{A}\otimes\mathcal{A}^\vee\otimes M \stackrel{\mathrm{act}\otimes\mathrm{id}}{\to} \mathcal{A}^\vee\otimes M$ corresponds to $\mathcal{A}\otimes\mathcal{A}\otimes M\to M$, which is the same because of the above! □

We claim that for $M\in\mathcal{A}-mod, N\in A$ one has functorially

$$\mathrm{Map}_A(M,N)\widetilde{\to}\mathrm{Map}_{\mathcal{A}-mod}(M,\mathcal{A}^\vee\otimes N)$$

Namely, given a map $h: M\to N$ in A, it yields a morphism of left A-modules $g: M\stackrel{\epsilon}{\to} \mathcal{A}^\vee\otimes M \stackrel{\mathrm{id}\otimes h}{\to}\mathcal{A}^\vee\otimes N$. Conversely, given a map of left A-modules $g: M\to\mathcal{A}^\vee\otimes N$, we get a map $M\to N$ as $M\stackrel{1\otimes g}{\to}\mathcal{A}\otimes\mathcal{A}^\vee\otimes N\stackrel{c\otimes\mathrm{id}}{\to} N$.

So, the functor $oblv:\mathcal{A}-mod\to A$ has a right adjoint $A\to\mathcal{A}-mod$, $N\mapsto\mathcal{A}^\vee\otimes N$, where the latter is equipped with left $\mathcal{A}$-action via its action on $\mathcal{A}^\vee$.

Note that $\mathcal{A}^\vee$ is a coalgebra in A. We get the functor $\mathcal{A}-mod\to\mathcal{A}^\vee-comod$ commuting with the forgetful functors to A. Assume for simplicity that A admits totalizations. Then the above map $\mathcal{A}-mod\widetilde{\to}\mathcal{A}^\vee-comod$ is an equivalence commuting with the forgetful functors to A (Sam Raskin confirms this). This question for usual categories is discussed here [41]. Proof: the projection $oblv:\mathcal{A}-mod\to A$ is comonadic because we can apply ([14], ch. 1, 3.7.7) with the comonad $A\to A, N\mapsto\mathcal{A}^\vee\otimes N$. Indeed, we must show that the corresponding functor $(\mathcal{A}-mod)^{op}\to\mathcal{A}^\vee-mod(A^{op})$ is monadic. This is true, because the projection $(\mathcal{A}-mod)^{op}\to A^{op}$ is conservative and preserves geometric realizations (by my Section 3.0.53).

Remark 3.2.2. *Under the assumptions of Section 3.2 the map $c:\mathcal{A}^\vee\otimes\mathcal{A}\to 1$ is also $\mathcal{A}$-bilinear, that is, "given" $a,x\in\mathcal{A}, x^*\in\mathcal{A}^\vee$ one gets $c(x^*a\otimes x)=c(x^*\otimes ax)$. More precisely, the diagram commutes*

$$\begin{array}{ccc}\mathcal{A}^\vee\otimes\mathcal{A}\otimes\mathcal{A} & \stackrel{\mathrm{id}\otimes mult}{\to} & \mathcal{A}^\vee\otimes\mathcal{A}\\ \downarrow \mathrm{act}\otimes\mathrm{id} & & \downarrow c\\ \mathcal{A}^\vee\otimes\mathcal{A} & \stackrel{c}{\to} & 1\end{array}$$

3.2.3. Let A,B be symmetric monoidal $(\infty,1)$-categories with A small. Assume B admits all small limits, $f:A^{op}\to B$ is a right-lax symmetric monoidal functor. Assume the tensor product $B\times B\to B$ preserves colimits separately in each variable. By (HA, 4.8.1.12), the symmetric monoidal structure on A extends to the one on $\mathcal{P}(A)$. Let $F:\mathcal{P}(A)^{op}\to B$ be the RKE of f. Does it inherit a right-lax symmetric monoidal structure?

Sam suggest that F inherits a right-lax symmetric monoidal structure. Proof: without loss of generality, we may assume B admits small colimits. By [20], a right-lax symmetric monoidal functor f is the same as a monoid in $\mathrm{Fun}(A^{op},B)$ for the Day convolution product.

Is the restriction $\mathrm{Fun}(\mathcal{P}(A)^{op},B)\to\mathrm{Fun}(A^{op},B)$ symmetric monoidal? Then the right adjoint given by the $RKE:\mathrm{Fun}(A^{op},B)\to\mathrm{Fun}(\mathcal{P}(A)^{op},B)$ would be right-lax monoidal. So, it will sends an algebra given by f to an algebra given by F.

Let $f,g\in\mathrm{Fun}(\mathcal{P}(A)^{op},B)$, $\bar f,\bar g$ their restrictions to A^{op}. Since $\mathcal{A}^{op}\to\mathcal{P}(A)^{op}$ is symmetric monoidal, there is a natural map $\bar f\otimes\bar g\to\overline{f\otimes g}$. Namely, for $a\in A$,

$$(\bar f\otimes\bar g)(a)\widetilde{\to}\underset{a_1,a_2\in A^{op},a\stackrel{\alpha}{\to}a_1\otimes a_2}{\mathrm{colim}} f(a_1)\otimes g(a_2),$$

the map α in A. Further,

$$\overline{f\otimes g}(a)\widetilde{\to}\underset{F_1,F_2\in\mathcal{P}(A)^{op},a\overset{\alpha}{\to}F_1\otimes F_2}{colim} f(F_1)\otimes f(F_2)$$

the map α is in $\mathcal{P}(A)$. I don't think $\mathrm{Fun}(\mathcal{P}(A)^{op},B)\to\mathrm{Fun}(A^{op},B)$ is symmetric monoidal.

3.2.4. Let $C\in 1-\mathcal{C}\mathrm{at}$ be symmetric monoidal. It is known that

$$Alg(CoAlg(C))\widetilde{\to}CoAlg(Alg(C))$$

Let A be an object of this category, assume the coalgebra A is cocommutative. Consider $A-mod=A-mod(C)$. It becomes a symmetric monoidal category with the tensor product sending V_1,V_2 to $V_1\otimes V_2$, which is considered as A-module with the morphism $A\to A\otimes A$ of algebras. Note that $1\in A-mod$ with the module structure $\epsilon:A\to 1$, the counit. This is the unit object of $A-mod$.

Let $oblv: A-mod\to C$ be the projection, it is symmetric monoidal. So, if $V\in A-mod$ is dualizable then $oblv(V)$ is also dualizable. Conversely, if $\mathrm{oblv}(V)$ dualizable in C, is $V\in(A-mod)^{dualizable}$?

3.2.5. I picked the following idea from ([1], Appendix B). Let O be a symmetric monoidal category admitting geometric realizations and such that the tensor product preserves geometric realizations separately in each variable. Let $B\in ComCoAlg(O)$, which is dualizable in O, so $B^\vee$ is a commutative algebra in O. Then the functor $O\to B^\vee-mod(O)$, $M\mapsto B^\vee\otimes M$ admits a left adjoint sending N to $B\otimes_{B^\vee}N$, where we used the natural $B^\vee$-module structure on B.

Proof. Let $N\in B^\vee-mod(O)$, $M\in O$. We have $N\widetilde{\to}B^\vee\otimes_{B^\vee}N\widetilde{\to}\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}(B^\vee)^{\otimes n+1}\otimes N$ be the usul Bar complex then

$$\mathrm{Map}_{B^\vee-mod}(N,B^\vee\otimes M)\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{Map}_{B^\vee-mod}((B^\vee)^{\otimes n+1}\otimes N,B^\vee\otimes M)\widetilde{\to}$$
$$\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{Map}_O((B^\vee)^{\otimes n}\otimes N,B^\vee\otimes M)\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{Map}_O(B\otimes(B^\vee)^{\otimes n}\otimes N,M)\widetilde{\to}$$
$$\mathrm{Map}_O(\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}B\otimes(B^\vee)^{\otimes n}\otimes N,M)\widetilde{\to}\mathrm{Map}_O(B\otimes_{B^\vee}N,M)$$

One should check that the complex obtained in the last but one expression is the usual bar complex for $B\otimes_{B^\vee}N$. □

3.3. On Bar construction and Koszul duality.

3.3.1. The key results of ([28], 5.2). Let $\mathcal{C}\in Alg(1-\mathcal{C}\mathrm{at})$ be a monoidal category. Let $\mathrm{Alg}^{aug}(\mathcal{C})=\mathrm{Alg}(\mathcal{C})_{/1}$ be the ∞-category of augmented algebras. Here by Alg we mean unital associative algebra.

([28], 5.2.2.3): assume $\mathcal{C}$ admits geometric realizations of simplicial objects, and the tensor product $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Let $A\in\mathrm{Alg}^{aug}(\mathcal{C})$. Then the functor $\mathrm{oblv}:\mathcal{C}\to{}_A\mathrm{BMod}_A(\mathcal{C})$ admits a left adjoint $M\mapsto 1\otimes_A M\otimes_A 1$. For $M=A$ this left adjoint gives the bar construction $Bar(A)=1\otimes_A 1\in\mathcal{C}$. It is given by the two-sided bar construction of

Section 3.1.8 of this file: $1\otimes_A 1 \widetilde{\to} \operatorname{colim}_{[n]\in\boldsymbol{\Delta}^{op}} A^{\otimes n}$. Moreover, the following is summarized after 5.2.2.15: $Bar(A)$ is naturally a unital augmented coassociative coalgebra. The comultiplication map is

$$Bar(A)\widetilde{\to} 1\otimes_A 1\widetilde{\to} 1\otimes_A A\otimes_A 1\to 1\otimes_A 1\otimes_A 1\widetilde{\to} Bar(A)\otimes Bar(A)$$

The augmentation is the natural map

$$1\otimes 1\to 1\otimes_A 1\widetilde{\to} Bar(A)$$

appearing in the two-sided bar construction calculating $Bar(A)$. This gives a functor $Bar: \mathrm{Alg}^{aug}(\mathcal{C})^{op}\to \mathrm{Alg}^{aug}(\mathcal{C}^{op})$.

Assume in addition that $\mathcal{C}$ admits totalizations of simplicial objects (we do not need here that the tensor product in $\mathcal{C}$ preserves totalizations of simplicial obejcts separately in each variable!). Then we apply the bar construction to augmented unital associative algebra objects of $\mathcal{C}^{op}$ and get a functor $Cobar: \mathrm{Alg}^{aug}(\mathcal{C}^{op})\to Alg^{aug}(\mathcal{C})^{op}$. For $A\in \mathrm{Alg}^{aug}(\mathcal{C}^{op})$, $Cobar(A)$ is the totalization in $\mathcal{C}$ of the cosimplicial diagram

$$1 \rightrightarrows A \substack{\to\\ \to\\ \to} A\otimes A\dots$$

Moreover, we get an adjoint pair $Cobar: \mathrm{Alg}^{aug}(\mathcal{C}^{op})\leftrightarrows \mathrm{Alg}^{aug}(\mathcal{C})^{op}: Bar$. If we define $\mathrm{coAlg}^{aug}(\mathcal{C})$ as $\mathrm{Alg}^{aug}(\mathcal{C}^{op})^{op}$ then we get an adjoint pair

$$Bar: \mathrm{Alg}^{aug}(\mathcal{C})\leftrightarrows \mathrm{coAlg}^{aug}(\mathcal{C}): Cobar$$

Remark 3.3.2. *i) If $\mathcal{C}$ is presentable, $c\in\mathcal{C}$ then $\mathcal{C}_{/c}$ is presentable, and $\mathcal{C}_{/c}\to\mathcal{C}$ preserves colimits (as it is a left adjoint) and totalizations of cosimplicial obejcts. Indeed, $\boldsymbol{\Delta}$ is contractible by (HTT, 5.5.8.7), so for any $x\in\mathcal{C}$ the totalization of the constant functor $\boldsymbol{\Delta}\to *\stackrel{x}{\to}\mathcal{C}$ is x. Indeed, the map $\boldsymbol{\Delta}^{op}\to\mid\boldsymbol{\Delta}^{op}\mid = *$ is cofinal by Section 2.2.111 of this file.*
ii) Assume $\mathcal{C}\in Alg(1-\mathcal{C}\mathrm{at})$ and $\mathcal{C}$ is presentable. We have an adjoint pair $l:\mathcal{C}_{/1}\leftrightarrows \mathcal{C}_{1//1}:\mathrm{oblv}$, where l sends $(x\to 1)$ to $1\to x\sqcup 1\to 1$. Recall that $\mathcal{C}_{1//1}$ is presentable by Section 2.7.11 of this file. So, the forgetful functor $\mathcal{C}_{1//1}\to\mathcal{C}$ preserves totalizations of cosimplicial objects. As in i) one shows that $\mathcal{C}_{1//1}\to\mathcal{C}$ preserves geometric realizations of simplicial objects.

Proof. ii) if $\mathcal{D}\in 1-\mathcal{C}\mathrm{at}$ is presentable and $f:\boldsymbol{\Delta}^{op}\to *\stackrel{d}{\to}\mathcal{D}$ is a constant functor then $colim f\widetilde{\to} d$, because $\boldsymbol{\Delta}^{op}\to\mid\boldsymbol{\Delta}^{op}\mid = *$ is cofinal. For this reason the map $*\to \mathrm{Fun}([1], C)$ given by $1\stackrel{\mathrm{id}}{\to} 1$ preserves geometric realizations. Now by Lemma 2.2.69 of this file, $\mathcal{C}_{1//1}\to\mathcal{C}$ preserves geometric realizations of simplicial objects. □

3.3.3. If $\mathcal{C}\in Alg(1-\mathcal{C}\mathrm{at})$ then $C_{1//1}\in Alg(1-\mathcal{C}\mathrm{at})$ naturally, and $Alg(\mathcal{C}_{1//1})\widetilde{\to} Alg^{aug}(\mathcal{C})$ by ([28], 5.2.3.9). In addition, $(\mathcal{C}_{1//1})^{op}\widetilde{\to}(\mathcal{C}^{op})_{1//1}$ naturally as monoidal categories. This gives an equivalence $\mathrm{coAlg}(C_{1//1})\widetilde{\to}\mathrm{coAlg}^{aug}(\mathcal{C})$.

3.3.4. The above together with ([28], 5.2.2.19) gives the following simplified result: let $\mathcal{C}\in Alg(1-\mathcal{C}\mathrm{at})$ admit both geometric realizations of simplicial objects and totalizations of cosimplicial objects. Then there is an adjoint pair

$$Bar: \mathrm{Alg}^{aug}(\mathcal{C})\leftrightarrows \mathrm{coAlg}^{aug}(\mathcal{C}): Cobar$$

In particular, this applies to a monoidal ∞-category $\mathcal{C}$, which is presentable. Then $\mathcal{C}_{1//1}$ is also a presentable monoidal category.

3.3.5. For ([28], 5.2.5.2). In that definition the functors ${}_ABMod_B(C) \to LMod_A(C)$, ${}_ABMod_B(C) \to RMod_B(C)$ are just the forgerful functors, that is, commute with $\mathrm{oblv} : {}_ABMod_B(C) \to C, \mathrm{oblv} : LMod_A(C) \to C, \mathrm{oblv} : RMod_B(C) \to C$. The functors $RMod_B(C) \to C$ and $LMod_A(C) \to C$ are oblv also.

3.3.6. Nick and Dennis in ([14], ch. IV.2, 2.7.2) proposed the following general construction. Assume A is a monoidal ∞-category, $C \in 1-\mathcal{C}\mathrm{at}$, let $F : A \to \mathrm{Fun}(C, C)$ be a left-lax monoidal functor, $a \in Alg(A)$ be a unital associative algebra. They define the category $a - mod(C)$ as follows.

Heuristically, its object is $m \in C$ together with a map $\alpha : F(a) \otimes m \to m$ and compatibilities. In particular, the diagram should commute

$$\begin{array}{ccccc} F(a\otimes a)\otimes m & \to F(a)\otimes F(a)\otimes m & \stackrel{\mathrm{id}\otimes\alpha}{\to} & F(a)\otimes m \\ \downarrow mult\otimes\mathrm{id} & & & \downarrow \mathrm{id}\times\alpha \\ F(a)\otimes m & \stackrel{\alpha}{\to} & & m \end{array}$$

Besides, for the natural map $F(1) \to \mathrm{id}$ the diagram should commute

$$\begin{array}{ccc} F(1)\otimes m & \to & F(a)\otimes m \\ \downarrow & & \downarrow \alpha \\ 1\otimes m & \widetilde{\to} & m \end{array}$$

For example, if $A = *$ then the desired category $a - mod(C)$ is just C.

The definition in general is as follows. Equip $Tw(\boldsymbol{\Delta}_+)$ with the monoidal structure such that the product of $(I \to J)$ with $(I' \to J')$ is $(I \sqcup I' \to J \sqcup J')$. It is understood that the order on $I \sqcup I'$ is such that any element of I is smaller than any element of I' (the lexicographical one). We have a strictly monoidal functor $f : Tw(\boldsymbol{\Delta}_+) \to \mathrm{Fun}(C, C)$ sending $(I \to J)$ to $\otimes_{j\in J} F(a^{\otimes I_j})$. Note that for $I \in \boldsymbol{\Delta}_+$, and a map $I \to A, i \mapsto a_i$ the product $\otimes_i a_i$ makes sense using the order on I. For a map

$$\begin{array}{ccc} I & \to & J \\ \downarrow & & \downarrow \\ I' & \to & J' \end{array}$$

the map $\underset{j\in J}{\otimes} F(a^{\otimes I_j}) \to \underset{j'\in J'}{\otimes} F(a^{\otimes I'_{j'}})$ is as follows. The left-lax monoidal structure on F gives a map $F(a^{I_j}) \to \underset{j'\in J'_j}{\otimes} F(a^{\otimes I_{j'}})$, the desired map is the composition

$$\underset{j\in J}{\otimes} F(a^{\otimes I_j}) \to \underset{j\in J}{\otimes} \underset{j'\in J'_j}{\otimes} F(a^{I_{j'}}) = \underset{j'\in J'}{\otimes} F(a^{I_{j'}}) \to \underset{j'\in J'}{\otimes} F(a^{\otimes I'_{j'}}),$$

where the last map is given by the products in the algebra a.

Now they apply the following general construction. Let $\mathcal{A}$ be a monoidal ∞-category, $f : \mathcal{A} \to \mathrm{Fun}(C, C)$ a strict monoidal functor. Then they define the category $f - alg(C)$ as follows. Let $a \in \mathrm{Fun}(C, C)$ be a colimit of f assuming it exists. Then a is canonically an algebra object in $\mathrm{Fun}(C, C)$, and they define $f - alg(C)$ as $a - mod(C)$.

Applying this to $f : Tw(\boldsymbol{\Delta}_+) \to \mathrm{Fun}(C, C)$ they get the category $a - mod(C) := f - alg(C)$.

Note that if $b \to a$ is a map in $Alg(A)$ let f_a, f_b be the corresponding functors as above. The natural morphism of functors $f_b \to f_a$ gives a morphism of monads $\mathrm{colim}\, f_b \to \mathrm{colim}\, f_a$ in $\mathrm{Fun}(C, C)$, hence the restriction functor $a - mod(C) \to b - mod(C)$.

Remark: assume $F, G : A \to \mathrm{Fun}(C, C)$ are left-lax monoidal functors, and $\beta : F \to G$ is a left-max monoidal natural transformation. Let $a \in Alg(A)$. Let $f_F, f_G : Tw(\boldsymbol{\Delta}_+) \to \mathrm{Fun}(C, C)$ be the corresponding functors, we get a natural transformation $\bar{\beta} : f_F \to f_G$ coming from β. This gives $\mathrm{colim}\, f_F \to \mathrm{colim}\, f_G$ in $Alg(\mathrm{Fun}(C, C))$. This gives a restriction functor $a - mod^G(C) = \mathrm{colim}\, f_G - mod(C) \to \mathrm{colim}\, f_F - mod(C) = a - mod^F(C)$.

Definition: let A be a monoidal category, $A \to \mathrm{Fun}(C, C)$ be a right-lax monoidal functor, $a \in coAlg(A)$. Then $a - comod(C)$ is defined as $(a - mod(C^{op}))^{op}$.

Lemma 3.3.7. *If $A \to \mathrm{Fun}(C, C)$ is actually monoidal functor then $a - mod(C)$ in both senses are the same.*

Proof. We have an adjoint pair $l : \boldsymbol{\Delta}_+ \leftrightarrows Tw(\boldsymbol{\Delta}_+) : r$, where $r(I \to J) = I$, and $l(I) = (I \to [0])$. So, for any functor $e : \boldsymbol{\Delta}_+ \to D$, $D \in 1 - \mathcal{C}\mathrm{at}$ the functor $LKE(e) : Tw(\boldsymbol{\Delta}_+) \to D$ along $l : \boldsymbol{\Delta}_+ \to Tw(\boldsymbol{\Delta}_+)$ is $LKE(e) = e \circ r$. So, $\underset{Tw(\boldsymbol{\Delta}_+)}{\mathrm{colim}}\, e \circ r \widetilde{\to} \underset{\boldsymbol{\Delta}_+}{\mathrm{colim}}\, e$ assuming it exists.

The functor $f : Tw(\boldsymbol{\Delta}_+) \to \mathrm{Fun}(C, C)$ is the composition $e \circ r$, where $e : \boldsymbol{\Delta}_+ \to \mathrm{Fun}(C, C)$ sends I to $F(a)^I$, and e sends $(I \to J)$ to the product map $F(a)^I \to F(a)^J$ along $I \to J$ for the algebra $F(a) \in Alg(\mathrm{Fun}(C, C))$. So, $\mathrm{colim}\, f \widetilde{\to} \mathrm{colim}_{\boldsymbol{\Delta}_+}\, e \widetilde{\to} F(a)$, because $[0] \in \boldsymbol{\Delta}_+$ is a final object. So, $a - mod(C)$ in the new sense is just $F(a) - mod(C)$, so the same as in the old sense. □

3.4. Let $C \in 1 - \mathcal{C}\mathrm{at}$ admit fibred products and a final object. Let $f : (\boldsymbol{\Delta}^+)^{op} \to C$ be module over a monoid $\mathbb{G} : \boldsymbol{\Delta}^{op} \to C$ in C, so $G = \mathbb{G}$ acts on $M = f([0]^+)$. Consider the restriction $F : \boldsymbol{\Delta}^{op} \to C$ of f under $\boldsymbol{\Delta} \times \{1\} \hookrightarrow \boldsymbol{\Delta} \times [1] \widetilde{\to} \boldsymbol{\Delta}^+$. Assume F is extended to an augmented simplicial object $F_+ : (\boldsymbol{\Delta}_+)^{op} \to C$ with $F(\emptyset) = c$. In other words, the map $\alpha : M \to c$ is G-invariant. Here $\boldsymbol{\Delta}_+$ is the category defined in (HTT, 6.1.2.2), namely this is the category of finite (possibly empty) linearly ordered sets. Consider the augmented simplicial object $[n] \mapsto Y^{\times (n+1)_c}$, which is the Cech nerve of α. Then we get a morphism of augmented simplicial objects $(\boldsymbol{\Delta}_+)^{op} \to C$, sending $[n]$ to

$$G^n \times M \to M_c^{\times n+1} := M \times_c M \times_c \ldots \times_c M, (g_1, \ldots, g_n, m) \mapsto (g_1 \ldots g_n m, g_2 \ldots g_n m, \ldots, m)$$

where in the RHS, M appears $n + 1$ times. Indeed, since the RHS is the right Kan extension from $(\boldsymbol{\Delta}_+^{\leq 0})^{op}$, by functoriality of the right Kan extension this morphism is determined by the corresponding diagram over $(\boldsymbol{\Delta}_+^{\leq 0})^{op}$, where it comes from the map $G \times M \to M \times_Q M$, $(g, m) \mapsto (gm, m)$.

3.4.1. Let $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be an ∞-operad. The following holds. Consider the category $\mathrm{Mon}_{\mathcal{O}}(1 - \mathcal{C}\mathrm{at})$ of $\mathcal{O}$-monoidal categories. By definition, $\mathrm{Mon}_{\mathcal{O}}(1 - \mathcal{C}\mathrm{at}) \subset \mathrm{Fun}(\mathcal{O}^{\otimes}, 1 - \mathcal{C}\mathrm{at})$ is a full subcategory given in Definition 3.0.4. Then $\mathrm{Mon}_{\mathcal{O}}(1 - \mathcal{C}\mathrm{at})$ is naturally a

2-category. Its objects are cocartesian fibrations $C^{\otimes} \to \mathcal{O}^{\otimes}$ such that the composition $C^{\otimes} \to \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ is an ∞-operad. The category of maps $\mathbf{Map}(C^{\otimes}, D^{\otimes})$ is the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(C^{\otimes}, D^{\otimes})$ classifying functors sending $\mathcal{O}^{\otimes}$-cocartesian arrows to $\mathcal{O}^{\otimes}$-cocartesian arrows.

Taking $\mathcal{O} = \mathrm{Surj}$, we get the 2-category $CAlg^{nu}(1-\mathcal{C}\mathrm{at})$. Given $E, E' \in CAlg^{nu}(1-\mathcal{C}\mathrm{at})$, the mapping category in $CAlg^{nu}(1-\mathcal{C}\mathrm{at})$ from E to E' is $\mathrm{Fun}^{\otimes}(E, E')$. Namely, if $E^{\otimes} \to \mathrm{Surj}, E'^{\otimes} \to \mathrm{Surj}$ are the cocartesian fibrations corresponding to E, E' then $\mathrm{Fun}^{\otimes}(E, E') \subset \mathrm{Fun}_{\mathrm{Surj}}(E^{\otimes}, E'^{\otimes})$ is the full subcategory of functors sending Surj-cocartesian arrows to cocartesian arrows.

3.4.2. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, $\mathcal{C}$ be a $\mathcal{O}$-monoidal category. Then $Alg_{\mathcal{O}}(\mathcal{C}), CoAlg_{\mathcal{O}}(\mathcal{C})$ inherit $\mathcal{O}$-monoidal structures (given by pointwise tensr product).

Recall that $\mathcal{P}\mathrm{r}^L$ is the category of presentable ∞-categories and colimit-preserving functors (endowed with its symmetric monoidal structure given by Lurie product).

Definition 3.4.3. *An ∞-category C is said to be presentably $\mathcal{O}$-monoidal if C is $\mathcal{O}$-monoidal, for each object $X \in \mathcal{O}^{\otimes}$, the fiber $C^{\otimes}_X$ is presentable, and for every morphism $f : X \to Y$ in $\mathcal{O}^{\otimes}$, the associated functor $f_! : C^{\otimes}_X \to C^{\otimes}_Y$ preserves small colimits.*

The following is ([44], Pp. 2.8): Let $\mathcal{O}^{\otimes}$ be an essentially small ∞-operad. Let C be a presentably $\mathcal{O}$-monoidal ∞-category. Then $CoAlg_{\mathcal{O}}(C)$ is a presentably $\mathcal{O}$-monoidal ∞-category. In particular, it is presentable.

3.4.4. Let $B \subset C$ be a full subcategory, $B, C \in 1-\mathcal{C}\mathrm{at}$. Let $\mathcal{A}$ be a monad on C preserving the full subcategory B. Then $\mathcal{A}-mod(B) \to \mathcal{A}-mod(C)$ is a full embedding.

Proof. Let $\mathrm{ind}_C : C \to \mathcal{A} - mod(C)$ be the left adjoint to $\mathrm{oblv} : \mathcal{A} - mod(C) \to C$. It is given by $c \mapsto \mathcal{A}(c)$ informally. Let $\mathrm{ind}_B : B \to \mathcal{A} - mod(B)$ be the left adjoint to $\mathrm{oblv} : \mathcal{A} - mod(B) \to B$. Set $\mathcal{X} = \{c \in \mathcal{A} - mod(C) \mid \mathrm{oblv}(c) \in B\}$, so $\mathcal{X} \subset \mathcal{A} - mod(C)$ is a full subcategory, and we have the evident functor $f : \mathcal{A} - mod(B) \to \mathcal{X}$. We must show f is an equivalence. Let $\alpha : \mathcal{X} \to B$ be the restriction of $\mathrm{oblv} : \mathcal{A} - mod(C) \to C$. We claim that $f \circ \mathrm{ind}_B$ is the left adjoint to α. Indeed, for $c \in \mathcal{X}$, $b \in B$ wee get $\mathrm{Map}_{\mathcal{A}-mod(C)}(f\, \mathrm{ind}_B(b), c) \widetilde{\to} \mathrm{Map}_C(b, \mathrm{oblv}(c)) \widetilde{\to} \mathrm{Map}_B(b, \alpha(c))$. Clearly, α is conservative, and $\alpha\alpha^L : B \to B$ coincides with $\mathcal{A} : B \to B$. Let $X^{\bullet}$ be a simplicial object of $\mathcal{X}$, which is α-split. Since $\mathcal{X} \subset \mathcal{A} - mod(C)$ is full, $X^{\bullet}$ admits a colimit in $\mathcal{X}$, and α preserves this colimit. So, $\mathcal{X} \widetilde{\to} \mathcal{A} - mod(B)$, hence f is an equivalence. □

4. Stable categories

4.0.1. See ([28], ch. 1). For a pointed category $\mathcal{C}$, we have the suspension functor $\mathcal{C} \to \mathcal{C}$, $X \mapsto \sum X$, and the loop functor $\mathcal{C} \to \mathcal{C}$, $X \mapsto \Omega X$. Moreover, Ω is right adjoint to $\sum$.

If $\mathcal{C}$ is stable then $\sum$ and Ω are mutually inverse equivalences. In this case for $n \geq 0$ we write $X[n] := \sum^n X$. For $n \geq 0$, $X[n] = \Omega^{-n} X$.

The definition of a stable ∞-category in ([14], ch. I.1, 5.1.1) is different from the definition of Lurie ([28], 1.1.1.9). However, they are equivalent because of ([28], 1.1.3.4).

If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is stable, $x, y \in \mathcal{C}$ then the natural map $x \sqcup y \to x \times y$ is an isomorphism ([28], 1.1.3.5), and this object is denoted $x \oplus y$.

The category denoted $1-\mathcal{C}\mathrm{at}^{St}$ in 5.1.3 admits small limits. Moreover, the inclusion $1-\mathcal{C}\mathrm{at}^{St}\to 1-\mathcal{C}\mathrm{at}$ preserves limits ([28], 1.1.4.4). This was claimed in 5.1.3 without a reference. Besides, $1-\mathcal{C}\mathrm{at}^{St}$ admits small filtered colimits, and the inclusion $1-\mathcal{C}\mathrm{at}^{St}\to 1-\mathcal{C}\mathrm{at}$ preserves small filtered colimits ([28], 1.1.4.6).

If $\mathcal{C}$ is stable then for $x,y\in\mathcal{C}$, $n\geq 0$ one has $\pi_0\operatorname{Map}_{\mathcal{C}}(x[n],y)\widetilde{\to}\pi_n\operatorname{Map}_{\mathcal{C}}(x,y)$ by ([28], 1.1.2.18). Indeed, $\Omega^i\operatorname{Map}_{\mathcal{C}}(x,y)\widetilde{\to}\operatorname{Map}_{\mathcal{C}}(\sum^i x,y)$ for $i\geq 0$. Notation: for $x,y\in\mathcal{C}$ Lurie writes in ([28], 1.1.2.18) $\operatorname{Ext}^n_{\mathcal{C}}(x,y)=\operatorname{Hom}_{\mathcal{C}^{ordn}}(x[-n],y)$.

For $A,B\in 1-\mathcal{C}\mathrm{at}^{St}$, $\operatorname{Fun}_{ex}(A,B)^{op}\widetilde{\to}\operatorname{Fun}_{ex}(A^{op},B^{op})$ naturally. The embedding $\operatorname{Fun}_{ex}(A,B)\subset\operatorname{Fun}(A,B)$ is stable under finite limits and colimits.

For 5.1.2: If $c_1\to c_2\to c_3$ is a fibre sequence in a stable category C then the boundary map $\delta: c_3\to c_1[1]$ can also be defined as the map $0\sqcup_{c_1}c_3\to 0\sqcup_{c_1}0$ coming from $c_3\to 0$.

4.0.2. If $\mathcal{C}$ is stable, $x\to y\to x$ a diagram in $\mathcal{C}$, the composition being id_x then $y\widetilde{\to}x\oplus z$, where $z\to y\to x$ is a fibre sequence. Indeed, use axion TR3 from (HA, Definition 1.1.2.5 of a triangulated category). It gives a morphism from the triangle $z\to z\oplus y\to y$ to $z\to x\to y$ such that the exteriour maps are isomorphisms, hence the middle map is also an isomorphism.

4.0.3. For 5.1.5. I think Dennis uses here ([28], 1.4.4.1-2).

Remark: let O be a stable category. Equip it with the cocartesian (=cartesian) symmetric monoidal structure. Since $Assoc^{\otimes}$ is a unital operad (defined in [28], 4.1.1.3), the functor oblv : $\operatorname{Mon}(O)\widetilde{\to}Alg(O)\to O$ is an equivalence by ([28], 2.4.3.9). The inclusion $\mathcal{G}\mathrm{rp}(O)\hookrightarrow Mon(O)$ is also an equivalence by Section9.5.7, 1) of this file.

4.0.4. For 5.1.7. Let C,D be stable ∞-categories with D cocomplete (presentable). The fact that $\operatorname{Funct}_{ex}(C,D)$ is cocomplete follows from my Sect. 2.2.37. Besides, $\operatorname{Fun}_{ex}(C,D)\subset\operatorname{Fun}(C,D)$ is stable under small limits and colimits. However, it is not clear if $\operatorname{Funct}_{ex}(C,D)$ is presentable in this case.

If C,D are stable cocomplete presentable then $\operatorname{Funct}_{ex,cont}(C,D)$ is stable and cocomplete. Besides, $\operatorname{Funct}_{ex,cont}(C,D)$ is presentable by ([27], 5.5.3.8). The subcategory $\operatorname{Funct}_{ex,cont}(C,D)\subset\operatorname{Funct}_{ex}(C,D)$ is clearly closed under colimits, and for $F\in\operatorname{Funct}_{ex,cont}(C,D)$ the functor ΩF is also continuous. Indeed, this functor is the composition $C\stackrel{F}{\to}D\stackrel{\Omega}{\to}D$ of two continuous functors. So, the full subcategory $\operatorname{Funct}_{ex,cont}(C,D)\subset\operatorname{Funct}_{ex}(C,D)$ is stable under translations, hence is a stable subcategory.

4.0.5. More about stable cocomplete categories is found in [29]. For example, if $\mathcal{C}$ is stable cocomplete and $x\in\mathcal{C}$ then x is compact iff the following holds: for any map $f: x\to\sqcup_{i\in I}y_i$ in $\mathcal{C}$ there is a finite subset $I_0\subset I$ such that f factors through $x\to\sqcup_{i\in I_0}y_i$.

4.0.6. If $\mathcal{C}$ is stable and $x'\to x\to x''$ is a fibre sequence in $\mathcal{C}$ then for $y\in\mathcal{C}$, $\operatorname{Map}_{\mathcal{C}}(x'',y)\to\operatorname{Map}_{\mathcal{C}}(x,y)\to\operatorname{Map}_{\mathcal{C}}(x',y)$ is a fibre sequence of spaces, so one has the long exact sequence

$$\begin{aligned}&\to \mathrm{Ext}^0(x'', y) \to \mathrm{Ext}^0(x, y) \to \mathrm{Ext}^0(x', y) \to \mathrm{Ext}^1_{\mathcal{C}}(x'', y) \to \dots \\ &\to \mathrm{Ext}^{-1}(x'', y) \to \mathrm{Ext}^{-1}(x, y) \to \mathrm{Ext}^{-1}(x', y) \\ &\to \dots\end{aligned}$$

going from low rows to top ones. Here $\mathrm{Ext}^{-n}_{\mathcal{C}}(x, y) = \pi_n \mathrm{Map}_{\mathcal{C}}(x, y)$ for $n \geq 0$. Similarly, $\mathrm{Map}_{\mathcal{C}}(y, x') \to \mathrm{Map}_{\mathcal{C}}(y, x) \to \mathrm{Map}_{\mathcal{C}}(y, x'')$ is a fibre sequence in spaces, and we get an exact sequence

$$\dots \to \mathrm{Ext}^{-1}_{\mathcal{C}}(y, x'') \to \mathrm{Ext}^0_{\mathcal{C}}(y, x') \to \mathrm{Ext}^0_{\mathcal{C}}(y, x) \to \mathrm{Ext}^0_{\mathcal{C}}(y, x'') \to \mathrm{Ext}^1_{\mathcal{C}}(y, x'') \to \dots$$

In ([28], 1.2.1.11) for a stable category $\mathcal{C}$ with a t-structure Lurie defines the functor $\mathcal{C} \to \mathcal{C}^{\heartsuit}, x \mapsto \pi_n(x)$. It factors through $\mathcal{C}^{ordn} \to \mathcal{C}^{\heartsuit}$. For a fibre sequence $x' \to x \to x''$ in $\mathcal{C}$ the sequence in the abelian category $\mathcal{C}^{\heartsuit}$ is exact

$$\dots \pi_0(x') \to \pi_0(x) \to \pi_0(x'') \to \pi_{-1}(x') \to \pi_{-1}(x) \to \pi_{-1}(x'') \to \dots$$

If $x \in \mathcal{C}_{\geq 0}, y \in \mathcal{C}_{\leq -1}$ then $\mathrm{Map}_{\mathcal{C}}(x, y) = *$ in Spc (with homological indexing conventions) ([28], 1.2.1.5).

Lemma 4.0.7. *Let $\mathcal{C}$ be stable with a t-structure. Then for $x \in \mathcal{C}$, $n \in \mathbb{Z}$ we have canonically $(\tau_{\leq n-1}x)[1] \widetilde{\to} \tau_{\leq n}(x[1])$ and $(\tau_{\geq n}x)[1] \widetilde{\to} \tau_{\geq n+1}(x[1])$ in the notations of ([28], 1.2.1).*

Proof. We have the fibre sequence $\tau_{\geq n}x \to x \to \tau_{\leq n-1}x$, hence a fibre sequence $(\tau_{\geq n}x)[1] \to x[1] \to (\tau_{\leq n-1}x)[1]$. Since $(\tau_{\leq n-1}x)[1] \in \mathcal{C}_{\leq n}$, $(\tau_{\geq n}x)[1] \in \mathcal{C}_{\geq n+1}$, this is the unique fibre sequence $\tau_{\geq n+1}(x[1]) \to x[1] \to \tau_{\leq n}(x[1])$. □

If $\mathcal{C}$ is stable with a t-structure then the category $\mathcal{C}_{\leq n}$ admits all finite limits and finite colimits. If $F : I \to \mathcal{C}_{\leq n}$ is a finite diagram, let $\bar{F}$ be the composition $I \stackrel{F}{\to} \mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$. Let $\bar{F}^{\triangleright} : I^{\triangleright} \to \mathcal{C}$ be a colimit diagram then the composition $I^{\triangleright} \stackrel{\bar{F}^{\triangleright}}{\to} \mathcal{C} \stackrel{\tau_{\leq n}}{\to} \mathcal{C}_{\leq n}$ is a colimit diagram extending F. Similarly, $\mathcal{C}_{\geq n}$ admits all finite limits and colimits.

4.0.8. Let $\mathcal{C}$ be stable with a t-structure. If $f : a \to b$ is a map in $\mathcal{C}^{\heartsuit}$, let $x = fib(f)$. Then $x \in \mathcal{C}_{[-1,0]}$ and the sequence in $\mathcal{C}^{\heartsuit}$ is exact $0 \to \pi_0(x) \to a \to b \to \pi_{-1}(x) \to 0$. For example, if f is surjective in the abelian category $\mathcal{C}^{\heartsuit}$ then $x \in \mathcal{C}^{\heartsuit}$, and x is the kernel of f in the usual sense in $\mathcal{C}^{\heartsuit}$. We see in particular, that if $0 \to x \to a \to b \to 0$ is exact in $\mathcal{C}^{\heartsuit}$ then $x \to a \to b$ is a fibre sequence in $\mathcal{C}$.

Remark 4.0.9. *If $\mathcal{C}$ is stable ∞-category, $0 \to x \stackrel{f}{\to} y$ is a fibre sequence then f is an isomorphism. Indeed, this property is holds already on the level of the triangulated category $\mathcal{C}^{ordn}$. It follows from the fact that in a triangulated category each map can be inserted into a fibre sequence in a unique (up to isomorphism) way.*

4.0.10. Let $\mathcal{C} \in 1-\mathcal{C}at$ be a stable with a t-structure. Then on $\mathcal{C}^{op}$ we get a t-structure given by $(\mathcal{C}^{op})_{\leq n} = (\mathcal{C}_{\geq -n})^{op}$ and $(\mathcal{C}^{op})_{\geq n} = (\mathcal{C}_{\leq -n})^{op}$. The functor $\Omega : \mathcal{C} \to \mathcal{C}$ induces $\Omega^{op} : \mathcal{C}^{op} \to \mathcal{C}^{op}$, which coincides with $\Sigma_{(\mathcal{C}^{op})}$. The functor $\tau_{\geq n} : \mathcal{C} \to \mathcal{C}_{\geq n}$ induces a functor $(\tau^{\mathcal{C}}_{\geq n})^{op} : \mathcal{C}^{op} \to (\mathcal{C}^{op})_{\leq -n}$, which is isomorphic to $\tau^{\mathcal{C}^{op}}_{\leq -n}$. Similarly, the functor $(\tau^{\mathcal{C}}_{\leq n})^{op} : \mathcal{C}^{op} \to (\mathcal{C}^{op})_{\geq -n}$ is the functor $\tau^{\mathcal{C}^{op}}_{\geq -n}$. Indeed, if $L : \mathcal{A} \to \mathcal{B}$, $R : \mathcal{B} \to \mathcal{A}$ are adjoint functors, L is left adjoint to R then for $L^{op}: \mathcal{A}^{op} \to \mathcal{B}^{op}, R: \mathcal{B}^{op} \to \mathcal{A}^{op}$, L^{op} is the right adjoint to R^{op}.

If $\hat{\mathcal{C}}$ is the left completion of $\mathcal{C}$ then $(\hat{\mathcal{C}})^{op}$ should be the right completion of $\mathcal{C}^{op}$. This forces the definition: the right completion of $\mathcal{C}$ is the limit over $\mathbb{Z}$ of the sequence

$$\dots \mathcal{C}_{\geq -2} \stackrel{\tau_{\geq -1}}{\to} \mathcal{C}_{\geq -1} \stackrel{\tau_{\geq 0}}{\to} \mathcal{C}_{\geq 0} \to \dots$$

Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ be presentable stable category with an accessible t-structure. Then $\mathcal{C}$ is right complete iff $\mathrm{colim}_{n\in\mathbb{Z}}\, \mathcal{C}_{\geq -n} \to \mathcal{C}$ is an equivalence, where the colimit is calculated in $\mathcal{P}\mathrm{r}^L$. Indeed, $\mathcal{C}_{\leq m}$ is presentable for any m (HA, 1.4.4.13), so we use ([14], I.1, 2.5.7). Note that if we calculate the latter colimit in $1-\mathcal{C}\mathrm{at}$, we get $\mathcal{C}^- = \cup_n \mathcal{C}_{\geq n}$ by [46].

If $\mathcal{C}$ as above is right complete then for any $z \in \mathcal{C}$ we get $z \widetilde{\to} \mathrm{colim}_n\, \tau^{\leq n} z$ by my Section 2.2.121. Conversely, assume $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ with an accessible t-structure. Assume that for any $z \in \mathcal{C}$ the natural map $\mathrm{colim}_n\, \tau^{\leq n} z \to z$ is an isomorphism, where the colimit is calculated in $\mathcal{C}$. Then $\mathcal{C}$ is right complete for this t-structure. Indeed, first $\mathcal{C} \to \lim \mathcal{C}^{\leq n}$ is fully faithful: for $z, y \in \mathcal{C}$ we have

$$\mathrm{Map}_{\mathcal{C}}(z, y) \widetilde{\to} \lim_{n\in\mathbb{Z}^{op}} \mathrm{Map}_{\mathcal{C}}(\tau^{\leq n} z, y) \widetilde{\to} \lim_{n\in\mathbb{Z}^{op}} \mathrm{Map}_{\mathcal{C}^{\leq n}}(\tau^{\leq n} z, \tau^{\leq n} y)$$

We get a diagram $\mathrm{colim}_{n\in\mathbb{Z}}\, \mathcal{C}^{\leq n} \to \mathcal{C} \to \lim_{n\in\mathbb{Z}^{op}} \mathcal{C}^{\leq n}$, where the colimit is calculated in $\mathcal{P}\mathrm{r}^L$. Recall that $\mathrm{colim}_{n\in\mathbb{Z}}\, \mathcal{C}^{\leq n} \to \lim_{n\in\mathbb{Z}^{op}} \mathcal{C}^{\leq n}$ is an equivalence. So, any $z \in \lim_{n\in\mathbb{Z}^{op}} \mathcal{C}^{\leq n}$ is an image of a suitable element from $\mathcal{C}$. We are done.

Corollary 4.0.11. *Let $C \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ with an accessible t-structure, which is right complete. Let $x \in C^c$ then there is n such that $x \in C^{\leq n}$.*

Proof. Since $x \widetilde{\to} \mathrm{colim}_n\, \tau^{\leq n} x$, $\mathrm{id}: x \to x$ factors through $\tau^{\leq n} x$ for some n. □

In the above corollary, $x \in C^c$ need not lie in $C^{\geq n}$ for some n. For example, if $S = \mathrm{Spec}\, A$ is a derived scheme, which is not eventually coconnective then A is not bounded from below, however, A is compact in $A-mod$.

Remark: i) let C be a stable category, which is left complete. Then for $c \in C$ the natural map $c \to \lim\limits_{n\in\mathbb{Z}^{op}} \tau^{\geq -n} c$ is an isomorphism; ii) Assume $C \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ such that $C^{\leq 0} \subset C$ is stable under the countable products. Assume that for $c \in C$ the natural map $c \to \lim\limits_{n\in\mathbb{Z}^{op}} \tau^{\geq -n} c$ is an isomorphism. Then C is left complete.

Proof. i) By assumption, $F: C \to \lim\limits_{n\in\mathbb{Z}^{op}} C^{\geq -n}$ is an equivalence. So, for $x, c \in C$ we get

$$\mathrm{Map}_C(x, \lim_{n\in\mathbb{Z}^{op}} \tau^{\geq -n} c) \widetilde{\to} \lim_{n\in\mathbb{Z}^{op}} \mathrm{Map}_C(x, \tau^{\geq -n} c) \widetilde{\to} \lim_{n\in\mathbb{Z}^{op}} \mathrm{Map}_C(\tau^{\geq -n} x, \tau^{\geq -n} c)$$
$$\widetilde{\to} \mathrm{Map}_{\lim_{n\in\mathbb{Z}^{op}} C^{\geq -n}}(F(x), F(c))$$

So, c and $\lim_{n\in\mathbb{Z}^{op}} \tau^{\geq -n} c$ represent the same functor.
ii) the argument as in ([28], 1.2.1.19) apply here. □

More generally, if C is stable and $\hat{C}$ its left completion, we have a natural functor $C \to \hat{C}$. If C admits limits, the functor $\hat{C} \to C, \{c^n \in C^{\geq -n}\} \mapsto \lim_n c^n$ is its right adjoint. Then $C \to \hat{C}$ is fully faithful iff for any $c \in C$ the map $c \to \lim_{n\in\mathbb{Z}} \tau^{\geq -n} c$ is an isomorphism in C.

4.0.12. If $\mathcal{C}$ is a stable category then for $x, y \in \mathcal{C}$ on $\mathrm{Map}_{\mathcal{C}}(x,y)$ we have an operation. Recall that $x \oplus y = x \sqcup y \widetilde{\to} x \times y$ canonically. Given $f, g \in \mathrm{Map}_{\mathcal{C}}(x,y)$, the composition $x \to x \times x \stackrel{f\times g}{\to} y \times y \widetilde{\to} y \sqcup y \to y$ should be denoted $f+g$. It is maybe not uniquely defined, but up to a contractible space of choices. The map $-\mathrm{id} \in \mathrm{Map}_{\mathcal{C}}(x,x)$ should be understood as any morphism over $-\mathrm{id} \in \mathrm{Map}_{\mathcal{C}^{ordn}}(x,x) = \pi_1 \mathrm{Map}_{\mathcal{C}}(x, x[1])$, it is defined up to isomorphism I think. Now for example, given a map $f : x \to y$ in $\mathcal{C}$, we may consider the fibre of $f - \mathrm{id} : x \oplus y \to y$.

For $x \in \mathcal{C}$ consider the functor $\mathcal{C} \to \mathrm{Spc}$, $y \mapsto \mathrm{Map}_{\mathcal{C}}(x,y)$. Since $\mathcal{C}$ is stable, by ([14], ch. 1, 5.1.10), it factors naturally as the composition $\mathcal{C} \to CGrp(\mathrm{Spc}) \to \mathrm{Spc}$, so $\mathrm{Map}_{\mathcal{C}}(x,y)$ has a structure of a commutative group in spaces.

Recall the equivalence $CGrp(\mathrm{Spc}) \widetilde{\to} \mathrm{Sptr}^{\leq 0}$ of ([14], ch. 1, 6.2.9). We see that for $G, H \in CGrp(\mathrm{Spc})$, $\mathrm{Map}_{CGrp(\mathrm{Spc})}(H,G)$ has a natural structure of a commutative group in Spc. Note that $CGrp(\mathrm{Spc})$ is presentable, because $\mathrm{Sptr}^{\leq 0}$ is presentable by (HA, 1.4.3.4). For $G, H \in CGrp(\mathrm{Spc})$ consider the functor $CGrp(\mathrm{Spc})^{op} \to \mathrm{Spc}$, $K \mapsto \mathrm{Map}_{CGrp(\mathrm{Spc})}(K \times H, G)$. It preserves limits, hence is representable by (HTT, 5.5.2.2). Is it representable by $\mathrm{Map}_{CGrp(\mathrm{Spc})}(H,G)$ with the above structure of a commutative group in spaces?

More generally, assume $\mathcal{C}$ has countable products. Given a diagram $\ldots \to x_n \stackrel{f_n}{\to} x_{n-1} \stackrel{f_{n-1}}{\to} \to \ldots$ in $\mathcal{C}$, we may construct its limit as the fibre of the map $\prod_n x_n \to \prod_n x_n$, where the map $\prod_n x_n \to x_m$ is the composition

$$\prod_n x_n \stackrel{\mathrm{pr}}{\to} x_{m+1} \oplus x_m \stackrel{f_{m+1}-\mathrm{id}}{\to} x_m$$

This is used in ([28], 1.2.1.19).

For the proof of ([28], 1.2.1.19): to see that the functor $\hat{\mathcal{C}} \to \mathcal{C}, f \mapsto \lim_{\mathbb{Z}} f$ is the right adjoint to $\theta : \mathcal{C} \to \hat{\mathcal{C}}$, note that for $c \in \mathcal{C}, f \in \hat{\mathcal{C}}$ we get

$$\mathrm{Map}_{\hat{\mathcal{C}}}(\theta(c), f) \widetilde{\to} \lim_{i\in\mathbb{Z}} \mathrm{Map}_{\mathcal{C}_{\leq -i}}(\tau_{\leq -i} c, f(i)) \widetilde{\to} \lim_{i\in\mathbb{Z}} \mathrm{Map}_{\mathcal{C}}(c, f(i)) \widetilde{\to} \mathrm{Map}_{\mathcal{C}}(c, \lim_{i\in\mathbb{Z}} f(i))$$

4.0.13. Let $\mathcal{C}$ be stable, $v_1 \stackrel{a_1}{\leftarrow} v \stackrel{a_2}{\to} v_2$ be a diagram in $\mathcal{C}$, let v' be the colimit of this diagram in $\mathcal{C}$. It can be calculated as the cofibre of $v \stackrel{a_1, -a_2}{\to} v_1 \oplus v_2$, see the proof of (HA, 1.1.3.4). Dually, the product $v_1 \times_v v_2$ is the fibre of $v_1 \oplus v_2 \stackrel{a_1, -a_2}{\to} v$.

4.0.14. Let $\mathcal{C} \in 1-\mathcal{C}at$ be stable with a t-structure. Not only $\mathcal{C}_{\leq n}$ is stable under extensions (by [27], 1.2.1.16), but $\mathcal{C}_{\geq n}$ is also stable under extensions. Indeed, if $x \to y \to z$ is fibre sequence, $x, z \in \mathcal{C}_{\geq n}$ then y is the cofibre of $z[-1] \to x$. Since $z[-1] \in \mathcal{C}_{\geq n-1}$ and $\mathcal{C}_{\geq r}$ is closed under colimits in $\mathcal{C}$, we get $y \in \mathcal{C}_{\geq n-1}$. The exact sequence $\pi_{n-1} x \to \pi_{n-1} y \to \pi_{n-1} z$ in $\mathcal{C}^{\heartsuit}$ shows that $\pi_{n-1} y \widetilde{\to} 0$, so $y \in \mathcal{C}_{\geq n}$.

4.0.15. Let $a \to b \stackrel{h}{\to} c$ be a fibre sequence in C, where C is stable, $N \in \mathbb{Z}$. Let $K \to \tau^{\geq N} b \stackrel{\tau^{\geq N} h}{\to} \tau^{\geq N} c$ be a fibre sequence in C. We get a natural map $\tau^{\geq N} a \to K$. The induced map $\tau^{>N} a \to \tau^{>N} K$ is an isomorphism.

4.0.16. For ([28], 1.2.2) spectral sequences. If $\mathcal{C} \in 1-\mathcal{C}at$ is stable and $F \in Gap(\mathbb{Z}, \mathcal{C})$ be a $\mathbb{Z}$-complex in $\mathcal{C}$ then for $i \leq j \leq k$ one has a commutative square

$$\begin{array}{ccc} F(j,k) & \to & F(i,j)[1] \\ \uparrow & & \uparrow \\ F(i,k) & \to & F(i,i)[1] \end{array}$$

This is used after Rem. 1.2.2.3, so $(C_\bullet, d)$ is a chain complex in $\mathcal{C}^{ordn}$.

For ([28], 1.2.2.3 and 1.2.2.4). Let $\mathcal{C}$ be a stable category, and $Y_0 \stackrel{f_1}{\to} Y_1 \stackrel{f_2}{\to} Y_2 \to \dots$ be a diagram in it. Set $C_n = cofib(f_n)[-n]$. Then we get a chain complex $\dots C_1 \stackrel{d_1}{\to} C_0 \to \dots$ in $\mathcal{C}$ constructed in ([28], Remark 1.2.2.3). Namely, first for $i \leq j$, $i, j \in \mathbb{Z}$ let $F(i,j)$ be the cofibre of the composed map $Y_i \to Y_j$. Then $F \in Gap(\mathbb{Z}, \mathcal{C})$, and we get the complex $C_\bullet$.

For ([28], 1.2.2.13): Let $\mathcal{C} \in 1 - \mathcal{C}at$ be stable with a t-structure. Let $\mathcal{C}$ admit sequential colimits, assume t-structure is compatible with the sequential colimits. Then for any n, $\mathcal{C}_{\leq n}$ is stable under the sequential colimits. Since $\mathcal{C}_{\geq 0}$ is stable under any colimits, $\mathcal{A}$ is stable under the sequential colimits. To show that $\pi_n : \mathcal{C} \to \mathcal{A}$ preserves the sequential colimits, let $\mathbb{Z}_+ \to \mathcal{C}, i \mapsto c_i$ be a functor. It suffices to prove this for $n = 0$. Since $\tau_{\leq 0}$ preserves colimits, we may assume $c_i \in \mathcal{C}_{\leq 0}$. For each i we have the fibre sequence $\pi_0(c_i) \to c_i \to \tau_{\leq -1} c_i$. The colimit diagram $\operatorname{colim} \pi_0(c_i) \to \operatorname{colim} c_i \to \operatorname{colim} \tau_{\leq -1} c_i$ is the cofibre sequence (hence, a fibre sequence). Since $\mathcal{C}_{\leq -1}$ and $\mathcal{A}$ are stable under the sequential colimits, the latter is the fibre sequence $\pi_0(\operatorname{colim} c_i) \to \operatorname{colim} c_i \to \tau_{\leq -1}(\operatorname{colim} c_i)$.

Remark: let $C \in 1 - \mathcal{C}at$ say admitting colimits, $f : \boldsymbol{\Delta}^{op} \to C$ be a functor. Let $f_n : \boldsymbol{\Delta}^{op} \to C$ be the n-skeleton, and $c_n = \operatorname{colim} f_n$. Offen there is n such that $c := \operatorname{colim} f \stackrel{\sim}{\to} \operatorname{colim} f_n := c_n$. Since we have a diagram $c_1 \to c_2 \to \dots \to c_n = c$ in C, we get a filtration in C on c with the succesive quotients $cofib(c_k \to c_{k+1})$. What is it on the relative tensor product?

4.0.17. For Dold-Kan correspondence ([28], 1.2.3). Let $d^i : [n-1] \to [n]$ be the unique injective map in $\boldsymbol{\Delta}$ whose image does not contains i. For a semisimplicial object $A_\bullet$ of an additive category let $d_i : A_n \to A_{n-1}$ be the corresponding face map. One has $d^i d^j = d^{j+1} d^i$ for $i \leq j$. It easily follows that $d_{j-1} d_i = d_i d_j$ for $i < j$. As in ([28], 1.2.3.8) let $A_\bullet$ be a semisimplicial object of an additive category. let $d(n) : A_n \to A_{n-1}$ be $d(n) = \oplus_{i=0}^n (-1)^i d_i$. Then $d(n-1)d(n) = 0$ for $n \geq 2$. Indeed,

$$d(n-1)d(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} d_i d_j = \sum_{i<j} (-1)^{i+j} d_i d_j + \sum_{i \geq j} (-1)^{i+j} d_i d_j$$

In the first sum using the equality $d_{j-1} d_i = d_i d_j$ for $i < j$ replace $d_i d_j$ by $d_{j-1} d_i$. Then the two summands are opposite, so the sum is zero.

For ([28], 1.2.3.9): if $\mathcal{A}$ is an abelian category, $A_\bullet$ is a simplicial object in $\mathcal{A}$ then the map $d_0 : N_n(A) \to A_{n-1}$ takes values in $N_{n-1}(A)$, because for $j \geq 0$ we have $d_j d_0 = d_0 d_{j+1}$. Moreover, we get $d_0^2 = 0$, so that $\dots N_2(A) \to N_1(A) \to N_0(A) \to 0$ is a chain complex.

If $A_\bullet$ is a nonpositively graded chain complex with values in an abelian category $\mathcal{A}$ then $DK_\bullet(A)$ has the following property. Given a surjection $\alpha : [n] \to [k]$ in $\boldsymbol{\Delta}$ and a map $\beta : [n'] \to [n]$ in $\boldsymbol{\Delta}$, the composition $A_k \subset DK_n(A) \stackrel{\beta^*}{\to} DK_{n'}(A) = \oplus_{\alpha' : [n'] \to [s]} A_s$ takes values in at most one of the summands. Here A_k is the summand indexed by α. More precisely, if $\alpha\beta$ is surjective, it takes values in A_k correspnding to $\alpha\beta$; if $\mathrm{Im}(\alpha\beta) = \{1, \dots, k\}$ then there is a unique $\alpha' : [n'] \to [k-1]$ such that $d^0\alpha' = \alpha\beta$, and it takes values in A_{k-1} indexed by α'; otherwise it vanishes. This is used in ([28], 1.2.3.12).

Explanation for the end of the proof of ([28], 1.2.3.13) with Lurie's notations: in the last paragraph of the proof let $x \in A(i)_n$, $x' = \alpha^*(x) \in A_n$. To see that $x - x' \in A(i-1)_n$ note the following. First, $d^i\alpha = d^i$, so $d_i(x - x') = 0$. Second, if $j > i$ then there is $\beta : [n-1] \to [n-1]$ such that $\alpha d^j = d^j \beta$, so $d_j(x') = \beta^*(d_j x) = 0$. Since $d_j x = 0$ also, we get $d_j(x - x') = 0$ finally. We are done.

Note that an abelian category is idempotent complete so that for an abelian category Dold-Kan correspondence gives an equivalence ([28], 1.2.3.7).

For ([28], 1.2.3.17). If $\mathcal{A}$ is any additive category and $B \in Ch(\mathcal{A})_{\geq 0}$ let $A = DK(B)$. Then inside $C(A)$ we have a chain subcomplex whose n-th component is $\oplus_{\alpha : [n] \to [k]} B_k$, the sum over all surjections α with $k < n$. That is, the differential

$$\oplus_{i=0}^n (-1)^n d_i : C_n(A) \to C_{n-1}(A)$$

preserves the corresponding subobjects. This subcomplex is precisely $\mathrm{Ker}\, v : C(A) \to N(A)$ in the notations of ([28], 1.2.3.16).

The definition of the differential in the tensor product of complexes (and generalizations) appear in ([28], 1.2.3.21).

4.0.18. Explanation for Alexander-Whitney maps defined in ([28], 1.2.3.22). It is not evident to check that the map $\overline{AW} : C(F(A^1, \dots, A^n) \to Ch(F)_{\geq 0}(C(A^1), \dots, C(A^n))$ commute with differentials, so defines a map of chain complexes. Let us check this is the simplest case $n = 2$. For simplicity, we pretend that F is the "tensor product", for more general F the argument is the same. Let A^1, A^2 be simplicial objects in $\mathcal{A}^1, \mathcal{A}^2$. Given $p \geq 0$, we have to show that the diagram commutes

$$\begin{array}{ccc} A^1_p \otimes A^2_p & \to & \bigoplus\limits_{p_1+p_2=p} A^1_{p_1} \otimes A^2_{p_2} \\ \downarrow & & \downarrow \\ A^1_{p-1} \otimes A^2_{p-1} & \to & \bigoplus\limits_{q_1+q_2=p-1} A^1_{q_1} \otimes A^2_{q_2} \end{array}$$

Here the left vertical map is $\oplus_{j=0}^p (-1)^j d_j$, where $d_j : A^k_p \to A^k_{p-1}$ is the standard face map corresponding to $d^j : [p-1] \to [p]$ whose image does not contain j. For a partition $p = p_1 + p_2$, $p_i \geq 0$ denote by $\alpha_1 : [p_1] \to [p], \alpha_2 : [p_2] \to [p]$ the "beginning" and "end" convex parts, both maps are inclusions. Similarly, $\beta_1 : [q_1] \to [p-1], \beta_2 : [q_2] \to [p-1]$ are "beginning" and "end" parts.

The key thing is the following

Lemma 4.0.19. *Let $0 \leq j \leq p$, let $q = q_1 + q_2$ with $q_i \geq 0$. Let $\beta_i : [q_i] \to [p-1]$ be as above.*
1) If $j > q_1$ then the composition $[q_1] \stackrel{\beta_1}{\to} [p-1] \stackrel{d^j}{\to} [p]$ is $\alpha_1 : [q_1] \to [p]$, and the composition $[q_2] \stackrel{\beta_2}{\to} [p-1] \stackrel{d^j}{\to} [p]$ is $[q_2] \stackrel{d^{j-q_1}}{\to} [q_2+1] \stackrel{\alpha_2}{\to} [p]$.
2) If $j \leq q_1$ then the composition $[q_1] \stackrel{\beta_1}{\to} [p-1] \stackrel{d^j}{\to} [p]$ is $[q_1] \stackrel{d^j}{\to} [q_1+1] \stackrel{\alpha_1}{\to} [p]$, and the composition $[q_2] \stackrel{\beta_2}{\to} [p-1] \stackrel{d^j}{\to} [p]$ is $\alpha_2 : [q_2] \to [p]$.

Assume $q_1, q_2 < p$ first. Let us pretend that each map is applied to $x^1 \otimes x^2$, where $x^i \in A^i_p$. Fix a partition of q as in the above lemma. We check that (q_1, q_2)-components of the answer are the same. The partitions of p that contribute are only (q_1+1, q_2) and (q_1, q_2+1). We need to prove the following equality, where the two summands in parenthesis correspond to the above two partitions of p

$$\sum_{j=0}^{p}(-1)^j \beta_1^*(d_j x^1) \otimes \beta_2^*(d_j x^2) = (\sum_{j_1=0}^{q_1+1}(-1)^{j_1} d_{j_1}\alpha_1^* x^1 \otimes \alpha_2^* x^2) + ((-1)^{q_1}\alpha_1^* x_1 \otimes \sum_{j_2=0}^{q_2+1}(-1)^{j_2} d_{j_2}\alpha_2^* x^2)$$

To obtain it, rewrite the left sum as a sum $\sum_{j=0}^{q_1} + \sum_{j=q_1+1}^{p}$. In the second sum make a change of variables $j_2 = j - q_1$, in the first sum denote $j_1 = j$ and apply the lemma. This gives the desired result because the terms corresponding to $j_1 = q_1 + 1$ and $j_2 = 0$ compensate:

$$(-1)^{q_1+1} d_{q_1+1}\alpha_1^* x^1 \otimes \alpha_2^* x^2 + (-1)^{q_1}\alpha_1^* x_1 \otimes d_0 \alpha_2^* x^2 = 0$$

Indeed, $[q_1] \stackrel{d_{q_1+1}}{\to} [q_1+1] \stackrel{\alpha_1}{\to} [p]$ coincides with α_1 and $[q_2] \stackrel{d_0}{\to} [q_2+1] \stackrel{\alpha_2}{\to} [p]$ coincides with α_2.

The remaining cases $q_1 = p$ or $q_2 = p$ are easier.

4.0.20. For ∞-categorical Dold-Kan correspondence ([28], 1.2.4.1). Let $\mathcal{C} \in 1-\mathcal{C}at$ be stable, $F : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ be a simplicial object in $\mathcal{C}$. Let $\boldsymbol{\Delta}_{\leq n} \subset \boldsymbol{\Delta}$ be the full subcategory spanned by the objects $[m]$ with $m \leq n$. By n-skeleton of a functor $G : \boldsymbol{\Delta}^{op}_{\leq n} \to \mathcal{C}$ one usually means the LKE of G under $\boldsymbol{\Delta}^{op}_{\leq n} \to \boldsymbol{\Delta}^{op}$.

The $\mathbb{Z}_+$-filtered object associated to F in ([28], 1.2.4.1) is $D(0) \to D(1) \to D(2) \to \ldots$, where $D(n)$ is the colimit of the composition $\boldsymbol{\Delta}^{op}_{\leq n} \to \boldsymbol{\Delta}^{op} \stackrel{F}{\to} \mathcal{C}$.

4.0.21. For DG-categories in ([28], 1.3.1). Let k be a commutative ring. Let $Ch(k)$ be the category of unbounded chain complexes of k-modules. Let $k - mod$ be the abelian category of k-modules. The functor $Ch(k) \to k - mod$, $A \mapsto Ker d_0$ is right-lax monoidal. Here $\ldots A_1 \stackrel{d_1}{\to} A_0 \stackrel{d_0}{\to} A_{-1} \to A_{-2} \to$. This is used in ([28], 1.3.1.4).

Besides, the functor $Ch(k) \to k - mod$, $A \mapsto H_0(A)$ is also right-lax monoidal. This allows to associate to a DG-category its homotopy category.

4.0.22. Derived ∞-categories. In ([28], 1.3.2.7) by $\mathcal{A}_{proj}$ he means the full subcategory of the abelian category $\mathcal{A}$ spanned by the projective objects. Explanation for ([28], 1.3.2.11). Recall that for a simplicial set K Lurie denotes $\mathbb{Z}K$ for the simplicial abelian group, where $(\mathbb{Z}K)_m$ is the free abelian group with base K_m. Further, for a simplicial set K, $N_*(K)$ denotes the normalized chain complex of $\mathbb{Z}K$. The complex $N_*(\Delta^n)$ is as follows. For $m \geq 0$, $N_m(\Delta^n) = \oplus_{\alpha:[m]\to[n]}\mathbb{Z}_\alpha$, the sum over α injective with $\mathbb{Z}_\alpha = \mathbb{Z}$. The differential in the chain complex $N_*(\Delta^n)$ is the map $\oplus_{j=0}^m(-1)^j d_j : N_m(\Delta^n) \to N_{m-1}(\Delta^n)$. This is in fact a subcomplex of $C(\mathbb{Z}\ \Delta^n)$ and also a quotient complex of $C(\mathbb{Z}\ \Delta^n)$. Namely, $(\mathbb{Z}\ \Delta^n)_m = \oplus_{\alpha:[m]\to[n]}\mathbb{Z}_\alpha$ with $\mathbb{Z}_\alpha = \mathbb{Z}$, the sum over all maps $\alpha : [m] \to [n]$ in $\boldsymbol{\Delta}$, and

$$\sum_{\alpha\ not\ inj} \mathbb{Z}_\alpha \subset (\mathbb{Z}\ \Delta^n)_m = C_m(\mathbb{Z}\ \Delta^n)$$

form a subcomplex of $C_*(\mathbb{Z}\ \Delta^n)$. The corresponding quotient is $N_*(\Delta^n)$.

Fix $0 \leq i \leq n$. Similarly, for any $m \geq 0$, $(\mathbb{Z}\Lambda^n_i)_m = \oplus_{\alpha:[m]\to[n]}\mathbb{Z}_\alpha$ with $\mathbb{Z}_\alpha = \mathbb{Z}$, the sum over all maps α such that $\operatorname{Im}(\alpha) \cup \{i\} \neq [n]$. One gets $N_m(\mathbb{Z}\Lambda^n_i) = \sum_{\alpha\ inj} \mathbb{Z}_\alpha$, the sum over all $\alpha : [m] \to [n]$ injective such that $\operatorname{Im}(\alpha) \cup \{i\} \neq [n]$. Actually, $N_*(\mathbb{Z}\Lambda^n_i) \subset C_*(\mathbb{Z}\Lambda^n_i)$ is a subcomplex, it is also realized as a quotient of $C_*(\mathbb{Z}\Lambda^n_i)$.

Let $E(n) = (\mathbb{Z} \stackrel{\mathrm{id}}{\to} \mathbb{Z})$ in degrees $n, n-1$ as in Lurie. This immediately gives an isomorphism of chain complexes $N_*(\Lambda^n_i) \oplus E(n) \widetilde{\to} N_*(\Delta^n)$, they are placed in degrees between 0 and n.

4.0.23. For ([28], 1.3.2.17). Let $\mathcal{A}$ be an additive category. If $f : M \to M'$ is a map in $Ch(\mathcal{A})$ then the mapping cone $C(f)$ is the object on $Ch(\mathcal{A})$ given by $C(f)_n = M'_n \oplus M_{n-1}$ with the differential

$$\begin{array}{ccc} M'_{n+1} & \stackrel{\partial'}{\to} & M'_n \\ \oplus & \nearrow f & \oplus \\ M_n & \stackrel{-\partial}{\to} & M_{n-1} \end{array}$$

For ([28], 1.3.2.10). If $\mathcal{A}$ is an additive category, $M \in Ch(\mathcal{A})$, the suspension functor $\Sigma : N_{dg}(Ch(\mathcal{A})) \to N_{dg}(Ch(\mathcal{A}))$ sends M to the complex M', where $M'_n = M_{n-1}$ and the differential $M'_n \to M'_{n-1}$ is $-d : M_{n-1} \to M_{n-2}$. See the proof of ([28], 1.3.2.10).

4.0.24. If $\mathcal{A}$ is an abelian category with enough projective objects, $M, N \in \mathrm{D}^-(\mathcal{A})$ then $\operatorname{Map}_{D^-(\mathcal{A})}(M, N) \in \mathrm{Spc}$ is represented by the Kan complex $DK(\tau_{\geq 0} \operatorname{Map}_{Ch^-(\mathcal{A})}(M, N))$. In particular, $\pi_0 \operatorname{Map}_{D^-(\mathcal{A})}(M, N) \widetilde{\to} \mathrm{H}_0 \operatorname{Map}_{Ch^-(\mathcal{A})}(M, N)$.

For example, if $M \in \mathcal{A}$ is projective and $N \in \mathrm{D}^-_{\geq 0}(\mathcal{A})$ and $i > 0$ then

$$\operatorname{Ext}^i_{\mathrm{D}^-(\mathcal{A})}(M, N) \widetilde{\to} \mathrm{H}_0 \operatorname{Map}_{Ch^-(\mathcal{A})}(M[-i], N) = 0$$

The chain complex of abelian groups $\operatorname{Map}_{Ch^-(\mathcal{A})}(M[-i], N)$ can have homologies in degrees $\geq i$.

4.0.25. If $\mathcal{A}$ is an abelian category with enough projective objects, $P, P' \in \mathrm{D}^-(\mathcal{A})$. A morphism $f : P \to P'$ in $\mathrm{D}^-(\mathcal{A})$ is precisely a morphism $P \to P'$ in $Ch^-(\mathcal{A}_{proj})$. Here $\mathcal{A}_{proj} \subset \mathcal{A}$ is the full subcategory spanned by projective objects. If f is a quasi-isomorphism then f is an isomorphism in $\mathrm{D}^-(\mathcal{A})$. Indeed, this is a chain homotopy equivalence.

For ([28], 1.3.2.22). By an ∞-category underlying a simplicial ∞-category Lurie means the simplicial nerve.

In ([28], 1.3.3.7) by "t-exact functor" he means a functor $F : \mathrm{D}^-(\mathcal{A}) \to \mathcal{C}$ which is both left and right t-exact.

4.0.26. Universal property of $\mathrm{D}^-(\mathcal{A})$. For ([28], 1.3.3.11). If $\mathcal{C}$ is a stable ∞-category with a t-structure then for $a \le b$ the category $\mathcal{C}_{[a,b]} = (\mathcal{C}_{\ge a})_{\le b}$ admits all finite colimits. Indeed, if $I \to \mathcal{C}_{[a,b]}$ is a finite diagram let $c = colim_{i \in I} c_i$, the colimit in $\mathcal{C}$. Then $c \in \mathcal{C}_{\ge a}$ ([28], 1.2.1.6), and $\tau_{\le b} c$ will be the colimit in $\mathcal{C}_{[a,b]}$, because $\tau_{\le b}$ preserves small colimits. If $a \le b \le c$ then the functor $\mathcal{C}_{[a,c]} \to \mathcal{C}_{[a,b]}$, $x \mapsto \tau_{\le b} x$ is right exact.

If $a \le b$ then for any $x, y \in \mathcal{C}_{[a,b]}$, $\mathrm{Map}_{\mathcal{C}}(x, y)$ is $b - a$-truncated space. So, $\mathcal{C}_{[a,b]}$ is equivalent to n-category with $n = 1 + b - a$ (in the terminolofy of Lurie [28], 2.3.4.1).

For ([28], 1.3.3.8). If $\mathcal{A}$ is an abelian category with enough projective objects then $\mathrm{D}^-(\mathcal{A})$ is left complete, so $\mathrm{D}^-_{\ge 0}(\mathcal{A})$ admits geometric realizations of simplicial objects by ([28], 1.3.3.11(2)).

If $\mathcal{C}$ is a stable ∞-category with a left complete t-structure, let $\mathcal{C}^\heartsuit$ be its heart. Assume that $\mathcal{C}^\heartsuit$ has enough projectives. Then there is a canonical right t-exact functor $c : \mathrm{D}^-(\mathcal{C}^\heartsuit) \to \mathcal{C}$ such that the composition $\mathcal{C}^\heartsuit \hookrightarrow \mathrm{D}^-(\mathcal{C}^\heartsuit) \xrightarrow{c} \mathcal{C} \xrightarrow{\tau^{\le 0}} \mathcal{C}$ is $\mathrm{id} : \mathcal{C}^\heartsuit \to \mathcal{C}^\heartsuit$. The universal property of $\mathrm{D}^-(\mathcal{A})$ (HA, 1.3.3.2) generalizes this claim. The above functor c is actually t-exact, so its restriction to $\mathcal{C}^\heartsuit$ is $\mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}$.

In (HA, 1.3.3.5) we get the functor: $\mathrm{D}^-(\mathcal{A}\mathrm{b}) \to \mathbb{S}\mathrm{p}$ of the generalized Eilenberg-MacLane spectrum, it is t-exact and extends the canonical inclusion $\mathcal{A}\mathrm{b} \hookrightarrow \mathbb{S}\mathrm{p}$.

Remark: if $\mathcal{A}$ is an abelian category with enough projectives, the proof of (HA, 1.3.3.7) depends on a model: namely, if $X, Y \in \mathcal{A}$ with X projective, one has to show that $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) = 0$ for $i > 0$. Lurie's proof of this uses a model instead of the universal property.

4.0.27. A version of the universal property for $\mathrm{D}^+(A)$, where A is an abelian category with enough injective objects is an analog of (HA, 1.3.3.2): the category $\mathrm{D}^+(A)$ is defined as $(\mathrm{D}^-(A^{op}))^{op}$ by (HA, 1.3.2.8). Let C be a stable ∞-category with a right complete t-structure. Let $\mathcal{E} \subset \mathrm{Fun}(\mathrm{D}^+(A), C)$ be the full subcategory spanned by those left t-exact functors that carry injective objects into $C^\heartsuit$. The construction $F \mapsto \tau^{\le 0}(F \mid_{(\mathrm{D}^+(A))^\heartsuit})$ gives an equivalence from $\mathcal{E}$ to the category of left exact functors $A \to C^\heartsuit$.

4.0.28. Let $\mathcal{C}$ be a stable category. A notion of a generator of $\mathcal{C}$ from ([28], 1.4.4.1) is correct. One could give also the following *different* definition: an object $x \in \mathcal{C}$ is a generator iff for $y \in \mathcal{C}$ the condition $\mathrm{Map}_{\mathcal{C}}(x, y) = *$ implies $y = 0$.

([28], 1.4.4.2) claims: Let $\mathcal{C}$ be a stable category. Then $\mathcal{C}$ is presentable iff

- $\mathcal{C}$ admits all small coproducts

- $\mathcal{C}^{ordn}$ is locally small in the sense of [28]
- there is a regular cardinal κ and a κ-compact generator $x \in \mathcal{C}$

4.0.29. For 5.1.9. Let $\mathcal{C} \in 1-\mathcal{C}at$ be stable. Let M be a commutative monoid in $\mathcal{C}$. By (5.1), we may assume that the product map $m : M \oplus M \to M$ is $(\mathrm{id}, \mathrm{id})$. Recall that $\mathrm{Map}_{\mathcal{C}}(M, M)$ is an abelian group. To check that M is a group, it suffices by Remark 2.5.18 to check that maps $(\mathrm{pr}_1, m) : M \oplus M \to M \oplus M$ and $(m, \mathrm{pr}_2) : M \oplus M \to M \oplus M$ are isomorphisms. For example, the inverse to (pr_1, m) is the map (pr_1, f), where $f : M \oplus M \to M$ is the map $(-\mathrm{id}, \mathrm{id})$. Similarly for (m, pr_2).

Lemma 4.0.30. *Let $\mathcal{C}$ be a stable category, $F^R : \mathcal{C} \to \mathcal{D}$ a map in $1-\mathcal{C}at$, which admits a left adjoint $F : \mathcal{D} \to \mathcal{C}$. Then F^R is conservative iff (for $c \in \mathcal{C}$ the condition $F^R(c) \widetilde{\to} F^R(0)$ implies $c \widetilde{\to} 0$).*

Proof. Assume our condition. Let us show that F^R is conservative. Let $\alpha : a \to b$ be a map in $\mathcal{C}$ such that $F^R(a) \to F^R(b)$ is an isomorphism, let $c = 0 \times_b a$. Since F^R preserves small limits, $F^R(c) \widetilde{\to} F^R(0) \times_{F^R(b)} F^R(a)$. For $d \in \mathcal{D}$ this gives a cartesian square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}}(d, F^R(c)) & \to & \mathrm{Map}_{\mathcal{D}}(d, F^R(a)) \\ \downarrow & & \downarrow \\ * & \to & \mathrm{Map}_{\mathcal{D}}(d, F^R(b)) \end{array}$$

So, the map $F^R(c) \to F^R(0)$ is an isomorphism in $\mathcal{D}$, because Yoneda is fully faithful. By assumption, $c \widetilde{\to} 0$, so α is an isomorphism. □

4.0.31. A proof of ([14], 5.3.4). Recall that $\mathcal{P}r^L$ admits all limits and colimits ([27], 5.5.3.13, 5.5.3.18). Let $F : I \to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ be a diagram. Let $\bar{F}$ be the functor F composed with $1-\mathcal{C}at^{St,cocmpl}_{cont} \to \mathcal{P}r^L$, here $\mathcal{P}r^L$ is the notation from ([27], 5.5.3.1). Let $\mathcal{C} = \mathrm{colim}\, \bar{F}$. Let $F' : I^{op} \to 1-\mathcal{C}at^{St,cocmpl}$ be obtained from F by passing to right adjoints. Let $\bar{F}' : I^{op} \to 1-\mathcal{C}at$ be the composition of F' with the inclusion $1-\mathcal{C}at^{St,cocmpl} \hookrightarrow 1-\mathcal{C}at$. Recall that $\mathcal{C} \widetilde{\to} \lim \bar{F}'$ canonically ([14], ch 1, Prop. 2.5.7). Recall also that $\mathcal{P}r^R \hookrightarrow 1-\mathcal{C}at$ preserves limits, so $\lim \bar{F}'$ could also be taken in $\mathcal{P}r^R$. Let us show that $\mathcal{C}$ is stable. We know already that for $i \in I^{op}$ the projection $\mathcal{C} \to F'(i)$ lies in $\mathcal{P}r^R$, that is, is accessible and limit-preserving.

For any map $i \to j$ in I^{op} the corresponding functor $F'(i) \to F'(j)$ preserves all limits, hence is exact. Since $\mathcal{C}$ is presentable, it has all limits and colimits. If c is a final object of $\mathcal{C}$ then for any $i \in I^{op}$ its image in $F'(i)$ is zero. Since each of the functors $F'(i) \to F'(j)$ preserves finite colimits, from my Lemma 2.2.70 we see that $c \in \mathcal{C}$ is initial.

Consider a cartesian square σ in $\mathcal{C}$. For any $i \in I$ its image in $F'(i)$ is a cartesian square, hence a cocartesian square as $F'(i)$ is stable. Since each transition functor $F'(i) \to F'(j)$ preserves finite colimits, σ is a cocartesian square by Lemma 2.2.69. Similarly, if σ is cocartesian square in $\mathcal{C}$, use Lemma 2.2.69 2) and Lemma 2.2.68 2). They show that σ is cartesian square. So, $\mathcal{C}$ is stable, and for each $i \in I^{op}$ the forgetful functor $\mathcal{C} \to F'(i)$ preserves all limits. Write $1-\mathcal{C}at^{St,cocmpl}_{lim}$ for the 1-full subcategory of $1-\mathcal{C}at^{St,cocmpl}$, where we keep only limit-preserving accessible functors. We have checked that the inclusion $1-\mathcal{C}at^{St,cocmpl}_{lim} \subset \mathcal{P}r^R$ preserves all limits. This implies that $C = \operatorname{colim} F$, because $(1-\mathcal{C}at^{St,cocmpl}_{cont})^{op} \widetilde{\to} 1-\mathcal{C}at^{St,cocmpl}_{lim}$.

We have shown also that the inclusion $1-\mathcal{C}at^{St,cocmpl}_{cont} \to \mathcal{P}r^L$ is stable under all colimits.

Let us show now that $1-\mathcal{C}at^{St,cocmpl}_{cont} \subset \mathcal{P}r^L$ is stable under all limits. Let $F : I \to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ be a diagram, $\bar{F}$ the composition

$$I \stackrel{F}{\to} 1-\mathcal{C}at^{St,cocmpl}_{cont} \to \mathcal{P}r^L$$

Let $\mathcal{D} = \lim \bar{F}$. By (HTT, 5.5.3.13), this is also a limit in $1-\mathcal{C}at$. Since $\mathcal{D}$ is presentable, it has all finite limits and finite colimits. For any map $i \to j$ in I the transition functor $F(i) \to F(j)$ preserves all colimits. From Lemma 2.2.69 we see that $\mathcal{D}$ is pointed, and for any $i \in I$ the projection $\mathcal{D} \to F(i)$ preserves all colimits. Let σ be a cocartesian square in $\mathcal{D}$. Then for any $i \in I$ its image in $F(i)$ is a cocartesian square, hence a cartesian square. Since each transition functor $F(i) \to F(j)$ is exact, applying again Lemma 2.2.69 we see that σ is cartesian in $\mathcal{D}$, and each projection $\mathcal{D} \to F(i)$ preserves finite limits. Now one shows that if σ is a cartesian square in $\mathcal{D}$ then it is a cocartesian square again by Lemma 2.2.69. So, $\mathcal{D}$ is stable, and the diagram ${}^{\triangleleft}I \to \mathcal{P}r^L$ realizing $\mathcal{D}$ as a limit of $\bar{F}$ lies actually in $1-\mathcal{C}at^{St,cocmpl}_{cont}$. Thus, $1-\mathcal{C}at^{St,cocmpl}_{cont} \subset \mathcal{P}r^L$ is stable under all limits. Recall also that the inclusion $\mathcal{P}r^L \hookrightarrow 1-\mathcal{C}at$ preserves all small limits ([27], 5.5.3.13).

To see that $1-\mathcal{C}at^{St,cocmpl}_{cont} \to 1-\mathcal{C}at^{St,cocmpl}$ preserves limits, note that $1-\mathcal{C}at^{St,cocmpl}_{cont} \to 1-\mathcal{C}at$ preserves limits by the above. Since $1-\mathcal{C}at^{St}$ admits small limits, and $1-\mathcal{C}at^{St} \to 1-\mathcal{C}at$ preserves limits, we see that $1-\mathcal{C}at^{St,cocmpl}_{cont} \to 1-\mathcal{C}at^{St}$ preserves limits. It actually takes values in the full subcategory $1-\mathcal{C}at^{St,cocmpl}$, so $1-\mathcal{C}at^{St,cocmpl}_{cont} \to 1-\mathcal{C}at^{St,cocmpl}$ preserves limits.

Note that we showed that $1-\mathcal{C}at^{St,cocmpl}_{cont} \to \mathcal{P}r^L$ preserves colimits.

4.0.32. *Stability of module categories.* Let $\boldsymbol{\Delta}^{+,op} \to 1-\mathcal{C}at$ be a left module category given by (A, M), where A is monoidal ∞-category. Assume M stable, let $\mathcal{A} \in AssAlg(A)$. Assume that for any $a \in A$ the functor $M \to M$, $x \mapsto a \otimes x$ preserves small colimits. The forgetful functor $\mathcal{A}-mod(M) \to M$ is conservative by ([28], 3.2.2.6). Using ([28], 4.2.3.3, 4.2.3.5) we conclude that $\mathcal{A}-mod(M)$ is stable.

The presentability of $\mathcal{A}-mod(M)$ is discussed in ([28], 4.2.3.7). So, if in addition M is presentable, and for any $a \in A$ the functor $M \to M, x \mapsto a \otimes x$ preserves small colimits then $\mathcal{A}-mod(M) \in 1-\mathcal{C}at^{St,cocmpl}$.

4.0.33. For 5.3.5. Let $C \in 1-\mathcal{C}at^{St,cocmpl}$. Recall that $\operatorname{Fun}_{ex,cont}(C,C)$ is a monoidal ∞-category, and C is a left module category over $\operatorname{Fun}_{ex,cont}(C,C)$. If $\mathcal{A}$ is an exact continuous monad, that is, $\mathcal{A} \in AssAlg(\operatorname{Fun}_{ex,cont}(C,C))$ then, by my Section 4.0.32, $\mathcal{A}-mod(C) \in 1-\mathcal{C}at^{St,cocmpl}$. The forgetful functor $\mathcal{A}-mod(C) \to C$ preserves colimits by ([28], 4.2.3.5), so is a map in $1-\mathcal{C}at^{St,cocmpl}_{cont}$, as well as its left adjoint $ind_{\mathcal{A}}$.

For 5.3.8. Let $G : D \to C$ be a map in $1-\mathcal{C}at^{St,cocmpl}_{cont}$ admitting a left adjoint $G^L : C \to D$. Suppose G does not send a nonzero object of D to zero. From Lemma 4.0.30

we see that G is conservative. It also preserves small colimits, so G is monadic by (ch. 1, 3.7.7).

4.0.34. ([14], ch1, Lemma 5.4.3) easily follows from my Lemma 4.0.30.

4.0.35. For the proof of 5.4.5. The full subcategory $C' \subset C$ is stable under colimits in C, stable, containing c_α, and the smallest with this propeties. The fact that C' is presentable follows from ([28], 1.4.4.2). So, $i : C' \to C$ indeed admits a right adjoint.

A notion of kernel makes sense in any pointed infinity category. The category $1-\mathcal{C}at^{St,cocmpl}$ is pointed, the zero object is $*$. Indeed, for $\mathcal{E} \in 1-\mathcal{C}at^{St}$, an exact functor $* \to \mathcal{E}$ sends $*$ to 0. So, for a map $F : \mathcal{C}_1 \to \mathcal{C}_2$ in $1-\mathcal{C}at^{St,cocmpl}$, we have $\mathrm{Ker}(f) = \mathcal{C}_1 \times_{\mathcal{C}_2} *$. This is the full subcategory of objects $x \in \mathcal{C}_1$ such that $F(x) \widetilde{\to} 0$.

Since $i^R : C \to C'$ preserves small limits, i^R is exact, hence a map in $1-\mathcal{C}at^{St,cocmpl}$. Recall that C contains all small limits, because C is presentable, so C'' admits all small limits (in fact, $C'' \subset C$ is stable under all limits). Since i^R commutes with translations, C'' is stable under translations, so C'' is stable (as in [28], 1.1.3.3). In fact, the limits of the diagram $C \overset{i^R}{\to} C' \overset{0}{\leftarrow} *$ can be calculated in $\mathcal{P}r^R$, recall that $\mathcal{P}r^R \hookrightarrow 1-\mathcal{C}at$ preserves limits, so C'' is presentable.

For any $c \in C, i^R i i^R(c) \to i^R(c)$ is an isomorphism (see next section). Therefore, for $c \in C$,

$$j^L(c) = cofib(ii^R(c) \to c) \in C''$$

So, $j^L : C \to C''$ is well-defined. If $z \in C'', c \in C$ then, since $\mathrm{Map}_C(\cdot, z)$ preserves colimits, the square is cartesian

$$\begin{array}{ccc} \mathrm{Map}_C(j^L(c), z) & \to & \mathrm{Map}_C(c, z) \\ \downarrow & & \downarrow \\ * & \to & \mathrm{Map}_C(ii^R(c), z) \end{array}$$

In addition, $\mathrm{Map}_C(ii^R(c), z) \widetilde{\to} \mathrm{Map}_C(i^R(c), i^R(z)) \widetilde{\to} *$, because $i^R(z) \widetilde{\to} 0$. So,

$$\mathrm{Map}_C(j^L(c), z) \widetilde{\to} \mathrm{Map}_C(c, z)$$

naturally for $z \in C'', c \in C$, so that j^L is indeed the left adjoint to j.

The category $(C')^\perp$ is by definition $\{z \in C \mid \text{for any } y \in C', \mathrm{Map}_C(y, z) \widetilde{\to} *\}$, a full subcategory of C. It coincides with C''.

The nontrivial part of the proof: if $C'' = \{0\}$, why $i^R : C \to C'$ is an equivalence? It is essentially surjective, because $i^R(z) \widetilde{\to} z$ for $z \in C'$. Now the fact that i^R is fully faithful means that for any $y_1, y_2 \in C$ one has

$$\mathrm{Map}_C(ii^R(y_1), y_2) \widetilde{\to} \mathrm{Map}_C(y_1, y_2)$$

Indeed, the natural map $ii^R(y_1) \to y_1$ is an isomorphism, because its cofibre is $j^L(y_1) \widetilde{\to} 0$.

4.0.36. Let $C \in 1-\mathcal{C}at$, $i : C' \subset C$ be a full subcategory. Assume there is a right adjoint $i^R : C \to C'$ to i. Then for $z \in C'$, the natural map $z \to i^R i(z)$ is an isomorphism. Indeed, both object represent the same functor on $\mathcal{C}'$.

4.0.37. For 5.4.7, its formulation is not precise. It actually says the following. Let $F : D \to C$ be a map in $1 - \mathcal{C}at^{St,cocmpl}_{cont}$. The essential image of F generated C iff for any morphism $G : C \to C'$ in $1 - \mathcal{C}at^{St,cocmpl}_{cont}$ such that $GF = 0$ one has $G = 0$.

In their proof of 5.4.7, the 'if' direction: the map $j^L : C \to C''$ is a map in $1 - \mathcal{C}at^{St,cocmpl}_{cont}$. For any $z \in C'$ the map $ii^R(z) \to z$ is an isomorphism, because of Section 4.0.36. Therefore, $j^L F = 0$, so j^L is zero by assumption. So, $C'' = 0$. As in the proof of 5.4.5 this implies that $C' = C$.

4.0.38. *Symmetric monoidal structure on* $\mathcal{P}r^L$. It is given by tensor product defined in ([28], 4.8.1.15). It has the following property ([30], Lemma 4.1.5): for $\mathcal{C}, \mathcal{D} \in \mathcal{P}r^L$ let $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the full subcategory of colimit preserving functors, write $\mathcal{C} \otimes \mathcal{D}$ for the tensor product in $\mathcal{P}r^L$. For $\mathcal{C}_i, \mathcal{D} \in \mathcal{P}r^L$ one has an equivalence

$$\mathrm{Fun}^L(\mathcal{C}_1 \otimes \ldots\ldots \otimes \mathcal{C}_n, \mathcal{D}) \widetilde{\to} \mathrm{Fun}_{\mathcal{R}}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n, \mathcal{D}),$$

where the RHS is the full subcategory of $\mathrm{Fun}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n, \mathcal{D})$ consisting of functors preserving colimits separately in each variable.

Important remark is ([30], Remark 4.2.5): if $n > 0$ and $\mathcal{C}_i \in \mathcal{P}r^L$ are stable in addition then $\mathcal{C}_1 \otimes \ldots \otimes \mathcal{C}_n$ is stable (for $n = 0$ this is wrong, the unit object of $\mathcal{P}r^L$ is Spc, it is not stable).

For $\mathcal{D}, \mathcal{C} \in 1 - \mathcal{C}at^{St,cocmpl}_{cont}$ we get $\mathcal{C} \otimes \mathcal{D} \widetilde{\to} \mathrm{Fun}^R(\mathcal{C}^{op}, \mathcal{D})$ by ([28], 4.8.1.17), where $\mathrm{Fun}^R \subset \mathrm{Fun}$ is the full subcategory of those functors, which are right adjoints (equivalently, preserving limits, for the equivalence of the two definitions see my Section 5.1.8).

For $c \in \mathcal{C}, d \in \mathcal{D}$ denote $c \boxtimes d$ the image of (c, d) under $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$. We underline that the latter functor preserves colimits separately in each variable!

4.0.39. *About* $1 - \mathcal{C}at^{St,cocmpl}_{cont}$. Let $\mathcal{C}_i, \mathcal{D} \in 1 - \mathcal{C}at^{St,cocmpl}_{cont}$. Write $\mathcal{C}_1 \otimes \ldots\ldots \otimes \mathcal{C}_n$ for the tensor product in $1 - \mathcal{C}at^{St,cocmpl}_{cont}$. As in ([28], proof of 4.8.1.3) one has an equivalence

$$\mathrm{Fun}^L(\mathcal{C}_1 \otimes \ldots\ldots \otimes \mathcal{C}_n, \mathcal{D}) \widetilde{\to} \mathrm{Fun}_{\mathcal{R}}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n, \mathcal{D})$$

where $\mathrm{Fun}^L \subset \mathrm{Fun}$ is the full subcategory of functors preserving colimits (we assume that "cocomplete" means, in particular, presentable, so this is equivalent to being left adjoint). Here the RHS is the full subcategory of $\mathrm{Fun}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n, \mathcal{D})$ consisting of functors preserving colimits separately in each variable. This is a consequence of Section 4.0.38.

We have $\mathrm{Map}_{1-\mathcal{C}at^{St,cocmpl}_{cont}}(\mathcal{E}, \mathcal{D}) = \mathrm{Fun}^L(\mathcal{E}, \mathcal{D})^{\mathrm{Spc}} = \mathrm{Fun}_{ex,cont}(\mathcal{E}, \mathcal{D})^{\mathrm{Spc}}$.

The fact the tensor product in $1 - \mathcal{C}at^{St,cocmpl}_{cont}$ preserves colimits separately in each variables is similar to ([28], 4.8.1.24). Proof: given a diagram $I \to 1 - \mathcal{C}at^{St,cocmpl}_{cont}$, $i \mapsto C_i$, let $C = \mathrm{colim}\, C_i$, $\mathcal{D}, \mathcal{E} \in 1 - \mathcal{C}at^{St,cocmpl}_{cont}$. Then

$$\begin{aligned}\mathrm{Map}(C \otimes \mathcal{D}, \mathcal{E}) \widetilde{\to} \mathrm{Map}(\mathcal{C}, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E})) \widetilde{\to} \lim_{I^{op}} \mathrm{Map}(C_i, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E})) \\ \widetilde{\to} \lim_{I^{op}} \mathrm{Map}(C_i \otimes \mathcal{D}, \mathcal{E}) \widetilde{\to} \mathrm{Map}(\mathrm{colim}_I (C_i \otimes \mathcal{D}), \mathcal{E}),\end{aligned}$$

here for brevity $\mathrm{Map} = \mathrm{Map}_{1-\mathcal{C}at^{St,cocmpl}_{cont}}$.

Recall that $\mathcal{P}r^L$ has a symmetric monoidal structure given by tensor product ([28], 4.8.1.15), and the full subcategory inclusion $1-\mathcal{C}at^{St,cocmpl}_{cont}\subset\mathcal{P}r^L$ preserves tensor products of $n>0$ terms (but not the units!). By ([28], 4.8.1.23) for $\mathcal{C}\in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ we get $\mathcal{C}\otimes\mathrm{Sptr}\,\widetilde{\to}\,\mathcal{C}$, where Sptr is the ∞-category of spectra, because of ([28], 1.4.2.21).

The category $1-\mathcal{C}at^{St,cocmpl}_{cont}$ is denoted in ([28], 4.8.2) by $\mathcal{P}r^{St}$.

For $C_i\in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ and $c_i\in C_i$ we have $0\boxtimes c_2=0$ in $C_1\otimes C_2$ and

$$(c_1[n])\boxtimes c_2\,\widetilde{\to}\,(c_1\boxtimes c_2)[n]\,\widetilde{\to}\,c_1\boxtimes(c_2[n])$$

for $n\in\mathbb{Z}$.

Sam Raskin says $1-\mathcal{C}at^{St,cocmpl}_{cont}$ is not presentable (similarly to $\mathcal{P}r^L$).

4.0.40. For ([14], ch. I.1, 6.2.1). Since Sptr is a unit object of $1-\mathcal{C}at^{St,cocmpl}_{cont}$, the tensor product $\mathrm{Sptr}\times\mathrm{Sptr}\to\mathrm{Sptr}\otimes\mathrm{Sptr}\,\widetilde{\to}\,\mathrm{Sptr}$ gives a structure of a monoidal ∞-category on Sptr. By ([28], 4.8.2), Sptr is a symmetric monoidal ∞-category.

For ([14], ch I.1, 6.2.7). The categories Spc, Sptr are presentable, and $\sum^\infty:\mathrm{Spc}\to\mathrm{Sptr}$ preserves small colimits, so admits a right adjoint. Note that $1_{\mathrm{Sptr}}\in\mathrm{Sptr}^{\le 0}$, see Sect 4.0.71, and Σ^∞ factors as $\mathrm{Spc}\to\mathrm{Sptr}^{\le 0}\hookrightarrow\mathrm{Sptr}$. So, Ω^∞ is the composition $\mathrm{Sptr}\stackrel{\tau^{\le 0}}{\to}\mathrm{Sptr}\to\mathrm{Spc}$.

The key things in ([28], 4.8.2) are 4.8.2.18, 4.8.2.19, very important!! It affirms that $(\mathrm{Sptr},1_{\mathrm{Sptr}})$ is idempotent in $\mathcal{P}r^L$ in the sense of ([28], 4.8.2.10). The forgetful functor $\mathrm{Sptr}-mod(\mathcal{P}r^L)\to\mathcal{P}r^L$ is fully faithful, its essential image is the full subcategory $\mathcal{P}r^{St}=1-\mathcal{C}at^{St,cocmpl}_{cont}$. This also gives the symmetric monoidal structure on Sptr ([28], 4.8.2.19).

Any map $f:\mathcal{C}\to\mathcal{D}$ in $1-\mathcal{C}at^{St,cocmpl}_{cont}$ is a map of Sptr-modules: for $z\in\mathrm{Sptr}, c\in\mathcal{C}$, $f(z\otimes c)\,\widetilde{\to}\,z\otimes f(c)$.

4.0.41. For ([28], 1.2.3.8). Let $\boldsymbol{\Delta}_+\subset\boldsymbol{\Delta}$ be the full subcategory with the same objects, where we keep only those morphisms $[n]\to[m]$, which are injective. (A conflict of notations with the category of possibly empty finite sets from my Section 2.5.12). A semisimplicial object of a category $\mathcal{C}$ is a functor $\boldsymbol{\Delta}_+^{op}\to\mathcal{C}$.

4.0.42. About the notion of *a left completion of a stable category with a t-structure* from ([28], 1.2.1.16). By $\mathbb{Z}$ Lurie means the category associated with the linearly ordered set $\mathbb{Z}$, where $n<m$ is the usual order. If $\mathcal{C}$ is a stable ∞-category with a t-structure then the description of $\hat{\mathcal{C}}=\lim_{n\in\mathbb{Z}}\mathcal{C}_{\le -n}$ defined in ([28], 1.2.1.16) follows from my Proposition 2.2.66. Here for $m<n$ the transition map $\mathcal{C}_{\le -m}\to\mathcal{C}_{\le -n}$ is the functor $\tau_{\le -n}$.

Recall that for $\mathcal{C}\in 1-\mathcal{C}at$ stable Lurie says that $\mathcal{C}$ is left bounded iff $\mathcal{C}^+=\mathcal{C}$ ([28], 1.2.1.16), $\mathcal{C}$ is left complete iff $\mathcal{C}\to\hat{\mathcal{C}}$ is an equivalence. If $\mathcal{C}\in 1-\mathcal{C}at$ is stable then $\mathcal{C}^+$ is left bounded, and $\hat{\mathcal{C}}$ is left complete (by [28], 1.2.1.18).

Remark 4.0.43. *Let $\mathcal{D}\in 1-\mathcal{C}at$, $\ldots\subset\mathcal{D}_1\subset\mathcal{D}_2\subset\ldots\mathcal{D}$ be a filtration by full subcategories. Assume that $\mathcal{D}_i\hookrightarrow\mathcal{D}$ admits a left adjoint $\tau_{\le_i}:\mathcal{D}\to\mathcal{D}_i$ for any $i\in\mathbb{Z}$. Let $\mathcal{X}\subset\mathbb{Z}\times\mathcal{D}$ be the full subcategory spanned by objects (i,x) such that $x\in\mathcal{D}_{-i}$. Then* *the projection $\mathcal{X} \to \mathbb{Z}$ is a cocartesian fibration. For $i \leq j$ in $\mathbb{Z}$ given $x \in \mathcal{D}_{-i}$ the natural map $x \to \tau_{\leq -j}x$ is cocartesian over $i \to j$ in $\mathbb{Z}$.*

4.0.44. About ([28], 1.2.1.6). Let $\mathcal{C}$ be a stable ∞-category with a t-structure. In Lurie's notation, $\tau_{\leq n}$ is a left adjoint to $\mathcal{C}_{\leq n} \subset \mathcal{C}$, and $\tau_{\geq n}$ is a right adjoint to $\mathcal{C}_{\geq n} \subset \mathcal{C}$. If we are given a diagram $p : I \to \mathcal{C}_{\leq -1}$, let c be the limit of the composition of p with $i : \mathcal{C}_{\leq -1} \to \mathcal{C}$. To check that $c \in \mathcal{C}_{\leq -1}$, it suffices according to ([28], 1.2.1.3) to show that for any $y \in \mathcal{C}_{\geq 0}$, $\mathrm{Hom}_{\mathcal{C}^{ordn}}(y, c) = 0$. The latter identifies with $\lim_{i \in I} \mathrm{Hom}_{\mathcal{C}^{ordn}}(y, p(i)) = 0$. So, $\mathcal{C}_{\leq -1}$ is closed under limits. Thus, $\mathcal{C}_{\leq n}$ is closed under all limits which exist in $\mathcal{C}$. Similarly, $\mathcal{C}_{\geq n}$ is closed under all colimits which exist in $\mathcal{C}$.

More generally, one has the following.

Remark 4.0.45. *Let $A \in 1-\mathcal{C}\mathrm{at}$, $j : B \subset A$ be a full subcategory such that j admits a left adjoint (that is, B is a localization of A). Then B is stable under all limits which exist in A.*

Proof. An element $x \in A$ lies in the essential image of $L : A \to B$ iff for any $y \in A$, $\mathrm{Map}_A(Ly, x) \to \mathrm{Map}_A(y, x)$ is an isomorphism in Spc (see [27], 5.5.4.2(1)). This property is preserved under passing to a limit. □

4.0.46. The Brown representability theorem is ([28], 1.4.1.2). For a pointed ∞-category $\mathcal{C}$ admitting small colimits one defines a notion of cohomoogy theory on $\mathcal{C}$ as in ([28], 1.4.1.6). The main application of Brown representability theorem seems to be ([28], Cor. 1.4.1.10). It says: if $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is presentable pointed, assume $\mathcal{C}$ is generated under colimits by compact objects which are cogroup objects in $\mathcal{C}^{ordn}$. Let $\{H^n, \delta^n\}$ be a cohomology theory on $\mathcal{C}$. Then for each $n \in \mathbb{Z}$ the functor $H^n : (\mathcal{C}^{ordn})^{op} \to Sets$ is representable by some object $E(n) \in \mathcal{C}$.

4.0.47. For ([28], 1.4.2). Lurie defines a notion of an excisive functor in ([28], 1.4.2.1): let $F : \mathcal{C} \to \mathcal{D}$ be a map in $1-\mathcal{C}\mathrm{at}$ and $\mathcal{C}$ admit push-outs. Then F is called excisive if F carries pushout squares in $\mathcal{C}$ to pull-back squares in $\mathcal{D}$. Assume also that $\mathcal{C}$ admits a final object $*$ then he calls F reduced if $F(*)$ is a final object of $\mathcal{D}$.

About ([28], 1.4.2.3): let $K, \mathcal{D}, \mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, assume $\mathcal{D}$ admits K-indexed limits, and $\mathcal{C}$ admits push-outs. Recall that $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ admits K-indexed limits and they are calculated pointwise. Then the full subcategory of excisive functors $Exc(\mathcal{C}, \mathcal{D}) \subset \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is closed under K-indexed limits. This comes from my Section 2.2.37. Namely, let J be the category $\{0' \to 1 \leftarrow 0\}$. If $c = \mathrm{colim}_{j \in J^{op}}\ c_j$ is a push-out in $\mathcal{C}$ and $K \to Exc(\mathcal{C}, \mathcal{D}), i \mapsto F_i$ is a functor let $F = \lim_{i \in K} F_i$. Then

$$F(c) \widetilde{\to} \lim_{i \in K} F_i(c) \widetilde{\to} \lim_{i \in K} \lim_{j \in J} F_i(c_j)$$

and we may permute the limits.

4.0.48. As in ([27], 5.5.4.16), for $\mathcal{D} \in 1-\mathcal{C}\mathrm{at}$ presentable and its full subcategory $\mathcal{D}_0 \subset \mathcal{D}$, we say that $\mathcal{D}_0$ is *strongly reflexive* iff $\mathcal{D}_0$ is presentable, stable under equivalences in $\mathcal{D}$, and the inclusion $\mathcal{D}_0 \subset \mathcal{D}$ admits a left adjoint.

Let $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}$ with $\mathcal{D}$ presentable. Recall that $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is presentable. For $c \in \mathcal{C}$ viewed a a functor $c : * \to \mathcal{C}$ we get the restriction functor $R : \mathrm{Funct}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}, F \mapsto F(c)$. This restriction functor commutes with limits and colimits, so admits left and right adjoints. Assume in addition that c is a final object of $\mathcal{C}$. As in ([28], 1.4.2.1) we denote by $\mathrm{Funct}_*(\mathcal{C},\mathcal{D}) \subset \mathrm{Funct}(\mathcal{C},\mathcal{D})$ the full subcategory of reduced functors (that is, given by condition that $F(c)$ is final in $\mathcal{D}$). To see that $\mathrm{Funct}_*(\mathcal{C},\mathcal{D}) \subset \mathrm{Funct}(\mathcal{C},\mathcal{D})$ is strongly reflexive, apply ([27], 5.5.4.17). Namely, the full subcategory $\mathcal{D}_0 \subset \mathcal{D}$ spanned by final objects is strongly reflexive, see ([27], 5.5.4.19). So, $\mathrm{Funct}_*(\mathcal{C},\mathcal{D})$ is the full subcategory of F such that $R(F) \in \mathcal{D}_0$, and now ([27], 5.5.4.17) yields the desired claim. This is used in ([28], 1.4.2.4).

4.0.49. In ([28], 1.4.2.5) Lurie defines the full subcategory $\mathrm{Spc}^{fin} \subset \mathrm{Spc}$ as the smallest full subcategory that contains $*$ and is stable under finite colimits. It is called ∞-category of finite spaces. Let $\mathrm{Spc}^{fin}_* \subset \mathrm{Spc}_*$ be the ∞-category of pointed objects of Spc^{fin}. (Lurie denotes it differently). The inclusion $\mathrm{Spc}^{fin}_* \subset \mathrm{Spc}_*$ is stable under push-outs, and the projection $\mathrm{Spc}_* \to \mathrm{Spc}$ preserves push-outs. So, the projection $\mathrm{Spc}^{fin}_* \subset \mathrm{Spc}^{fin}$ preserves push-outs.

By definition, $C \in \mathrm{Spc}$ lies in Spc^{fin} if there is a simplicial set K with a finite number of nondegenerate simplices such that C is the colimit of the constant functor $K \to \mathrm{Spc}$ with value $*$. That is, C is obtained from K by inverting all morphisms. Then by Section 2.2.111, $K \to C$ is cofinal.

Lemma 4.0.50. *If $\mathcal{D} \in 1-\mathcal{C}\mathrm{at}$ admits finite colimits then any functor $* \stackrel{d}{\to} \mathcal{D}$ admits a LKE along $* \hookrightarrow \mathrm{Spc}^{fin}$.*

Proof. Let $C \in \mathrm{Spc}^{fin}$. Pick a simplicial set K with a finite number of nondegenerate simplicies such that $C \widetilde{\to} \mid K \mid$. We have to show that the functor

$$C \widetilde{\to} * \times_{\mathrm{Spc}^{fin}} \mathrm{Spc}^{fin}/C \to * \stackrel{d}{\to} \mathcal{D}$$

admits a colimit. Since $K \to C$ is cofinal, we are done. □

For example, let $\mathcal{C}$ be a usual groupoid (not an infinity one), so an object of $1-\mathcal{C}\mathrm{at}^{ordn} \cap \mathrm{Spc}$. The colimit of the functor $\mathcal{C} \to \mathrm{Spc}$ with constant value $*$ is $\mid \mathcal{C} \mid \widetilde{\to} \mathcal{C}$. So, if in addition $\mathcal{C}$ has finite set of isomorphism classes and finite group of automorphisms of an object then $\mathcal{C} \in \mathrm{Spc}^{fin}$ (I don't really why if this is true!). So, Spc^{fin} is not contained in *Sets*.

Question: Let for $n = 1$ G be a finite group object in *Sets*, for $n \geq 2$ let G be a commutative group object in *Sets*. Let X be the Eilenberg-MacLane object in Spc equipped with $\pi_n(X) \widetilde{\to} G$ (in the sense of [27], 7.2.2.12). Is it true that the image $\bar{X}$ of X in Spc satisfies $\bar{X} \in \mathrm{Spc}^{fin}$? For $n = 1$ this is true by the previous paragraph.

4.0.51. If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is pointed, admitting finite limits and colimits then $\mathcal{C}^{op}$ satisfies the same property, and $\Sigma^{op} : \mathcal{C}^{op} \to \mathcal{C}^{op}$ is the functor $\Omega_{\mathcal{C}^{op}}$. This is used in ([28], 1.4.2.11).

4.0.52. If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ has a terminal object $*$ and admits finite colimits then the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ admits a left adjoint $\mathcal{C} \to \mathcal{C}_*$ given by $c \mapsto c \sqcup *$.

4.0.53. According to ([28], 1.4.2.20), for $n \geq 0$ the n-sphere S^n is an object of the category Spc^{fin} of finite spaces. By ([27], 6.5.1.1) as a simplicial set it is defined as $S^n = \partial \vartriangle^{n+1}$. It is known that for the suspension functor $\Sigma : \mathrm{Spc}_*^{fin} \to \mathrm{Spc}_*^{fin}$ one has $\Sigma S^n \widetilde{\to} S^{n+1}$ in Spc_*^{fin}, and S^0 is the union of two points.

Question: is S^n a usual category?

Note that if $a \to b \leftarrow c$ is a diagram in Spc such that $a \times_b c \to c$ is an isomorphism in Spc then $a \to b$ is not always the isomorphism. For example, this is not true for usual groupoids. However, this will be true if $c \to b$ is an effective epimorphism, I think.

4.0.54. Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ admit finite limits, the category $Sp(\mathcal{C}) = Exc_*(\mathrm{Spc}_*^{fin}, \mathcal{C})$ is the full subcategory of $\mathrm{Funct}(\mathrm{Spc}_*^{fin}, \mathcal{C})$ consisting of reduced excisive functors ([28], 1.4.2.8). The functor $\Omega^\infty : Sp(\mathcal{C}) \to \mathcal{C}$ defined in ([28], 1.4.2.20) is the evaluation at $* \hookrightarrow * \sqcup * = S^0$. It preserves finite limits, so is left exact. The zero object of $Sp(\mathcal{C})$ is the constant functor $\mathrm{Spc}_*^{fin} \to \mathcal{C}$ with value $*$, here $* \in \mathcal{C}$ is the final object.

The category Spc_*^{fin} admits finite colimits, so one has the suspension functor $\Sigma : \mathrm{Spc}_*^{fin} \to \mathrm{Spc}_*^{fin}$, $\Sigma(x) = * \sqcup_x *$. It is known that $\Sigma(S^n) \widetilde{\to} S^{n+1}$ in Spc_*^{fin} for $n \geq 0$.

The inclusion $Sp(\mathcal{C}) \to \mathrm{Funct}(\mathrm{Spc}_*^{fin}, \mathcal{C})$ preserves limits.

Assume $\mathcal{C}$ pointed in addition. The functor $F \mapsto \Omega F = F[-1]$ on $Sp(\mathcal{C})$ is as follows: if $S \in \mathrm{Spc}_*^{fin}$ then $(\Omega F)(S) = \Omega_{\mathcal{C}}(F(S))$. For any $F \in Sp(\mathcal{C})$ the functor $\mathrm{Spc}_*^{fin} \to \mathcal{C}$, $S \mapsto F(\Sigma S)$ lies in $Sp(\mathcal{C})$ and is naturally isomorphic to the functor ΣF. Indeed, $\Omega : Sp(\mathcal{C}) \to Sp(\mathcal{C})$ is an equivalence, and after applying Ω, these functors become naturally isomorphic.

By definition, $\Omega^{\infty-n} : Sp(\mathcal{C}) \to \mathcal{C}$ is the functor $F \mapsto (\Sigma^n F)(S^0) \widetilde{\to} F(\Sigma^n S^0)$ for $n \geq 0$. So,

$$\Omega^\infty(F) \widetilde{\to} \Omega(\Omega^{\infty-1}(F))$$

for $F \in Sp(\mathcal{C})$. For an object $c \in \mathcal{C}$ the condition that c lies in the image of $\Omega^\infty : Sp(\mathcal{C}) \to \mathcal{C}$ seems very strong, because for any $n \geq 0$ there is $x \in \mathcal{C}$ with $c \widetilde{\to} \Omega^n(x)$. Such c has a natural structure of a commutative group object in $\mathcal{C}$, see (Ch. I.1, Sect. 6.2.7, [14]). So, Ω^∞ factors as $Sp(\mathcal{C}) \to ComGrp(\mathcal{C}) \to \mathcal{C}$.

The ∞-category of spectra $Sp(\mathrm{Spc}_*) \widetilde{\to} Sp(\mathrm{Spc})$ is presentable by ([28], 1.4.2.4).

If $\mathcal{C}$ is presentable pointed then for any $n \geq 0$, $\Omega^{\infty-n} : Sp(\mathcal{C}) \to \mathcal{C}$ preserves limits. Indeed, by ([28], 1.4.2.3), $Sp(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{Spc}_*^{fin}, \mathcal{C})$ is closed under limits, and the evaluation $\mathrm{Fun}(\mathrm{Spc}_*^{fin}, \mathcal{C}) \to \mathcal{C}$ at S^n preserves limits. So, $\Omega^{\infty-n}$ admits a left adjoint, denote it by $\Sigma^{\infty-n}$. Since $\Omega^\infty = \Omega^n \circ \Omega^{\infty-n}$ as functors $Sp(\mathcal{C}) \to \mathcal{C}$, we get Σ^∞ is isomorphic to the composition $\mathcal{C} \stackrel{\Sigma^n}{\to} \mathcal{C} \stackrel{\Sigma^{\infty-n}}{\to} Sp(\mathcal{C})$.

For the functor $\Sigma^\infty : \mathrm{Spc}_* \to \mathrm{Sptr}$ left adjoint to Ω^∞ we have $\Sigma^\infty(S^0) \widetilde{\to} 1_{\mathrm{Sptr}}$. For the functor $\Omega^{\infty-n} : \mathrm{Sptr} \to \mathrm{Spc}_*$ and its left adjoint $\Sigma^{\infty-n} : \mathrm{Sptr} \to \mathrm{Spc}_*$ this gives $\Sigma^{\infty-n}(S^0) \widetilde{\to} 1_{\mathrm{Sptr}}[-n]$, because $\Sigma^{\infty-n}$ preserves colimits.

4.0.55. Let $\mathcal{D}^0, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}$ admit finite limits, $\mathcal{D}^0 \subset \mathcal{D}$ a full embedding which is left exact. Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ be pointed and admitting finite colimits. Then $Exc_*(\mathcal{C}, \mathcal{D}^0) = Exc_*(\mathcal{C}, \mathcal{D}) \cap \mathrm{Funct}(\mathcal{C}, \mathcal{D}^0)$ is a full subcategory of those functors which take values in $\mathcal{D}^0$.

Let now $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}at$, $\mathcal{C}$ be pointed admitting finite colimits, $\mathcal{D}$ admits finite limits. Recall that $Exc_*(\mathcal{C}, \mathcal{D}) \subset \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is a full subcategory closed under finite limits. So, $Sp(Exc_*(\mathcal{C}, \mathcal{D})) \subset Sp(\mathrm{Funct}(\mathcal{C}, \mathcal{D}))$ is a full subcategory.

4.0.56. In some places in [28] (and maybe [27]) Lurie says colimits/limits parametrized by "weakly contractible simplicial set". This means in a model independent setting colimit/limit over a contractible ∞-category (for example, in [28], 1.4.2.26).

If $S \in 1-\mathcal{C}at$ is contractible, and S' is a retract of S in $1-\mathcal{C}at$ then S' is also contractible.

4.0.57. For the proof of ([28], 1.4.2.24). Let $\bar{\mathcal{C}}$ be the limit of the tower $\ldots \to \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C}$. Write this tower as $\ldots \mathcal{C}_2 \to \mathcal{C}_1 \to \mathcal{C}_0$, where $\mathcal{C}_i = \mathcal{C}$. For $x \in \bar{\mathcal{C}}$ let x_i be its image in $\mathcal{C}_i$. By my Lemma 2.2.69, the projection $\bar{\mathcal{C}} \to \mathcal{C}_i$ preserves finite limits. So, the functor $\Omega_{\bar{\mathcal{C}}}$ sends x to $\Omega_{\bar{\mathcal{C}}}x$, where $(\Omega_{\bar{\mathcal{C}}}x)_i \widetilde{\to} \Omega_{\mathcal{C}}(x_i)$. The functor $\bar{\mathcal{C}} \to \bar{\mathcal{C}}$ inverse to $\Omega_{\bar{\mathcal{C}}}$ sends x to y, where $y_i = x_{i+1}$ for $i \geq 0$.

The functor $G : \bar{\mathcal{C}} \to \mathcal{C}$ appearing in the proof is the projection on $\mathcal{C}_0 = \mathcal{C}$.

The functor $G' : \bar{\mathcal{C}} \to Sp(\mathcal{C})$ in the proof has the following property. If $x \in \bar{\mathcal{C}}$ and $F = G'(x)$ then $F(S^i) \widetilde{\to} x_i$ for all $i \geq 0$. In particular, we see that an excisive reduced functor $F : \mathrm{Spc}_*^{fin} \to \mathcal{C}$ is completely defined by its restriction to the collection of objects $\{S^n\}_{n \geq 0}$ together with isomorphisms $\Omega_{\mathcal{C}}(F(S^{n+1})) \widetilde{\to} F(S^n)$ in $\mathcal{C}$.

4.0.58. In the proof of ([28], 1.4.4.11) Lurie uses the term colocalization. The definition is as follows. Let $f : \mathcal{A} \to \mathcal{B}$ be a map in $1-\mathcal{C}at$ then f is a colocalization if f admits a fully faithful left adjoint. This is equivalent to the property that $f^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ admits a fully faithful right adjoint.

Then ([27], 5.2.7.8) has an analog for colocalizations. Namely, let $\mathcal{C}^0 \subset \mathcal{C}$ be a full subcategory, $\mathcal{C} \in 1-\mathcal{C}at$. By definition, for $c \in \mathcal{C}$, a morphism $f : d \to c$ in $\mathcal{C}$ exhibits d as a $\mathcal{C}^0$-colocalization of c iff $d \in \mathcal{C}^0$ and composition with f induces an isomorphism

$$\mathrm{Map}_{\mathcal{C}}(e, d) \to \mathrm{Map}_{\mathcal{C}}(e, c)$$

for each $e \in \mathcal{C}^0$. This is equivalent to requiring that $f : d \to c$ is a final object of $\mathcal{C}/_c \times_{\mathcal{C}} \mathcal{C}^0$. Indeed, viewing $f \in \mathcal{C}/_c$, we have $(\mathcal{C}/_c)/_f \widetilde{\to} \mathcal{C}/_f$, here we denote by $\mathcal{C}/_f$ the overcategory of $\mathcal{C}$ over the functor $f : [1] \to \mathcal{C}$. Let us show that the natural map $\mathcal{C}/_f \to \mathcal{C}/_d$ is an equivalence. This follows from a dual version of ([27], 4.1.1.7):

Lemma 4.0.59. *If $v : K' \to K$, $p : K \to \mathcal{D}$ are maps in $1-\mathcal{C}at$, and $v^{op} : K'^{op} \to K^{op}$ is cofinal then $\mathcal{D}/_p \to \mathcal{D}/_{pv}$ is an equivalence of right fibrations over $\mathcal{D}$.*

Proof. For any functor $p : K \to \mathcal{C}$, one has $(\mathcal{C}_{p/})^{op} \widetilde{\to} \mathcal{C}^{op}/_{p^{op}}$ for $p^{op} : K^{op} \to \mathcal{C}^{op}$. □

For the inclusion $v : 0 \to [1]$ the map $v^{op} : 0^{op} \to [1]^{op}$ is cofinal, our claim follows. Now f is a final object of $\mathcal{C}/_c$ iff the natural map $\mathcal{C}/_d \to \mathcal{C}/_c$ is an equivalence.

([27], 5.2.7.12) has an analog for colocalizations also.

4.0.60. By ([28], 1.4.2.24), if $\mathcal{C} \in 1-\mathcal{C}at$ is pointed and admitting finite limits then the functor $\Omega^\infty : Sp(\mathcal{C}) \to \mathcal{C}$ can be lifted to an equivalence $Sp(\mathcal{C}) \widetilde{\to} \bar{\mathcal{C}}$ for the diagram

$$\ldots \to \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C}.$$

Here $\bar{\mathcal{C}}$ is the limit of this tower. An object of $\bar{\mathcal{C}}$ is a collection $\{x_i\}$ for $i \geq 0$, where $x_i \in \mathcal{C}$ are equipped with $\Omega(x_{i+1}) \widetilde{\to} x_i$ for $i \geq 0$. We visualize this as a diagram

$$\ldots \to x_2 \to x_1 \to x_0$$

Lurie constructs a functor $G' : \bar{\mathcal{C}} \to Sp(\mathcal{C})$ is the proof of ([28], 1.4.2.24), which is shown to be an equivalence.

We restrict an excisive reduced functor $f : \mathrm{Spc}_*^{fin} \to \mathcal{C}$ to the collection of objects $\{S^n\}_{n\geq 0}$. This gives a collection $\{x_i\}_{i\geq 0}$, where $x_i = \Omega^{\infty-i}(f) \in \mathcal{C}$. In view of the isomorphisms $\Omega(x_{i+1}) \widetilde{\to} x_i$ for $i \geq 0$, we get an object $\bar{x} \in \bar{\mathcal{C}}$. This defines a functor $\xi : Sp(\mathcal{C}) \to \bar{\mathcal{C}}$. Then ξ an inverse to the functor $G' : \bar{\mathcal{C}} \to Sp(\mathcal{C})$ from the proof of ([28], 1.4.2.24).

The functor $\Sigma : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ sends $\{x_n\}$ to the collection $\{y_n\}_{n\geq 0}$, where $y_i = x_{i+1}$.

With the above notations, assuming $\mathcal{C}$ presentable the definition of the full subcategory $Sp(\mathcal{C})_{\leq -1} \subset Sp(\mathcal{C})$ from ([28], 1.4.3.4) becomes: x_0 is final in $\mathcal{C}$.

According to the proof of ([28], 1.4.3.6), for $x = \{x_n\}_{n\geq 0} \in Sp$ and $m \geq 0$ we get $x \in Sp_{\leq -1-m}$ iff $x_m \in \mathrm{Spc}$ is final. For $m \geq 0$ we should get $x \in Sp_{\leq m}$ iff $\Omega^{m+1}(x_0)$ is final. To formulate this uniformly, for $m \in \mathbb{Z}$, $x \in Sp$ we have $x \in Sp_{\leq m}$ iff (for $n \geq 0$ we have $\Omega^{m+n+1}(x_n) \widetilde{\to} *$).

Notation: for $X \in Sp$ Lurie denotes $\pi_n(X) = \tau_{\geq n}\tau_{\leq n}(X)$ in the proof of ([28], 1.4.3.6) for the t-structure on Sp. So, $\pi_n(X) \in Ab$, here Ab is the category of abelian groups.

4.0.61. To verify for ([28], 1.4.3.6): Write Spc_* for the category of pointed spaces.

Lemma 4.0.62. *For $n \geq -1$ and $X \in \mathrm{Spc}_*$ we have $\Omega(\tau_{\leq n}X) \widetilde{\to} \tau_{\leq n-1}(\Omega X)$. Here $\tau_{\leq n} : \mathrm{Spc}_* \to \mathrm{Spc}_*$ is the truncation functor.*

Proof. Since $\Omega(\tau_{\leq n}X) \in \tau_{\leq n-1}(\mathrm{Spc}_*)$, the natural map $\Omega(X) \to \Omega(\tau_{\leq n}X)$ yields a morphism $\tau_{\leq n-1}(\Omega X) \to \Omega(\tau_{\leq n}X)$. It suffices to show that this morphisms induces isomorphisms on all the homology groups. Note also that $\Omega(X)$ (resp., $\Omega(\tau_{\leq n}X)$) has a structure of a group, so all of its components are isomorphic. The corresponding homology groups of are calculated using (HTT, 6.5.1.9). □

Lemma 4.0.63. *Let $n \geq 0$ and $(* \to X) = X' \in \mathrm{Spc}_*$. The two conditions*
i) $X' \in \tau_{\leq n}(\mathrm{Spc}_*)$, that is, every connected component of X is n-truncated;
ii) $\Omega^{n+1}(X')$ is final in Spc_*
are not equivalent in general. However, they are equivalent if $X' = \Omega(Y, *)$ for some $(Y, *) \in \mathrm{Spc}_*$.

Proof. X' has a group structure, so all the connected components of X are isomorphic to each other in Spc. □

Because of the above lemma, for $X \in Sp$ we have $X \in Sp_{\leq m}$ iff each X_n is $n+m$-truncated. This is used in ([28], 1.4.3.6).

Lemma 4.0.64. *Let* $n \geq -1$, $X' = (* \to X) \in \mathrm{Spc}_*$. *Then* $X' \in \tau_{\leq n}\mathrm{Spc}_*$ *iff* $X \in \tau_{\leq n}\mathrm{Spc}$.

Proof. Assume $X' \in \tau_{\leq n}\mathrm{Spc}_*$. The functor $\mathrm{Spc}_* \to \mathrm{Spc}$ forgetting the point has a left adjoint sending Y to $Y \sqcup *$. If for any $Y \in \mathrm{Spc}$, $\mathrm{Map}_{\mathrm{Spc}_*}(Y \sqcup *, X') \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(Y, X) \in \tau_{\leq n}\mathrm{Spc}$ then X is n-truncated.

Now assume $X \in \tau_{\leq n}\mathrm{Spc}$. The projection $\mathrm{Spc}_* \to \mathrm{Spc}$ is the universal cocartesian fibration in spaces. By ([27], 5.5.6.15), the diagonal map $X \to X \times X$ is $n-1$-truncated, so $\mathrm{Map}_X(y, x)$ are $n-1$-truncated spaces for any points $x, y \in X$. So, $* \to X$ is a $n-1$-truncated object of the fibre $\mathrm{Spc}_* \to \mathrm{Spc}$ over X (this fibre identifies with X). Now ([27], 5.5.6.6) shows that $(* \to X) \in \tau_{\leq n}\mathrm{Spc}_*$. □

The above claim generalizes as follows.

Lemma 4.0.65. *Let* $n \geq -1$. *Let* $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ *admit finite colimits and a final object* $*$. *Let* $X' = (* \stackrel{a}{\to} X) \in \mathcal{C}_*$. *Then* $X' \in \tau_{\leq n}\mathcal{C}_*$ *iff* $X \in \tau_{\leq n}\mathcal{C}$.

Proof. The projection $\mathcal{C}_* \to \mathcal{C}$ has a left adjoint given by $y \mapsto y \sqcup *$. So, if $X' \in \tau_{\leq n}\mathcal{C}_*$ then $X \in \tau_{\leq n}\mathcal{C}$ as above.

Let now $X \in \tau_{\leq n}\mathcal{C}$. The projection $\mathcal{C}_* = \mathcal{C}_{*/} \to \mathcal{C}$ is a cocartesian fibration. The fibre over X is $\mathrm{Map}_{\mathcal{C}}(*, X)$, it is a n-truncated space. So, the diagonal map $\mathrm{Map}_{\mathcal{C}}(*, X) \to \mathrm{Map}_{\mathcal{C}}(*, X) \times \mathrm{Map}_{\mathcal{C}}(*, X)$ is $n-1$-truncated, so a is a $n-1$-truncated object of $\mathrm{Map}_{\mathcal{C}}(*, X)$. Applying ([27], 5.5.6.6) we see that $X' \in \tau_{\leq n}\mathcal{C}_*$. □

The following is also used in ([28], 1.4.3.6) without an explanation:

Lemma 4.0.66. *The functor* $\Omega^\infty : Sp \to \mathrm{Spc}_*$ *preserves* ω*-filtered colimits.*

Proof. (This follows from ([28], 1.4.3.9)). We claim that the functor $\Omega : \mathrm{Spc}_* \to \mathrm{Spc}_*$ commutes with ω-filtered colimits. Indeed, the inclusion $\mathrm{Spc}_* \subset \mathrm{Funct}([1], \mathrm{Spc})$ is stable under filtered colimits and limits. For filtered colimits this follows from the fact that each filtered category is weakly contractible ([27], 5.3.1.18). For limits this is because the inclusion admits a left adjoint, and Spc_* is presentable. Let J be a small ω-filtered category, $p : J \to \mathrm{Spc}_*$ a diagram. We have a natural map $\mathrm{colim}_{j \in J}\, \Omega p(j) \to \Omega(\mathrm{colim}\, p)$ in Spc_*. To show this is an isomorphism, it suffices to check that its composition with the projection $\mathcal{C}_* \to \mathcal{C}$ is an isomorphism in $\mathcal{C}$. This follows from ([27], 5.3.3.3).

Now each transition map in the diagram $\ldots \mathrm{Spc}_* \stackrel{\Omega}{\to} \mathrm{Spc}_* \stackrel{\Omega}{\to} \mathrm{Spc}_*$ commutes with filtered colimits. By Lemma 2.2.69, Sp admits filtered colimits (this is automatic, as it is presentable), and the evaluation functor $\Omega^\infty : Sp \to \mathrm{Spc}_*$ preserves filtered colimits. □

For 6.2.7: the projection $\mathrm{Spc}_* \to \mathrm{Spc}$ preserves filtered colimits and limits. Indeed, this is a composition $\mathrm{Spc}_* \hookrightarrow \mathrm{Fun}([1], \mathrm{Spc}) \to \mathrm{Spc}$, and each functor preserves filtered colimits and limits. The projection $\mathrm{Spc}_* \to \mathrm{Spc}$ is conservative, hence reflects limits.

4.0.67. From (HA, 1.4.3.6) we also learn the following: the homotopy functors $\pi_n : \mathrm{Sptr} \to Ab$ preserve products and coproducts, besides $\pi_n : \mathrm{Sptr} \to Ab$ commutes with filtered colimits. Each of the functors $\mathrm{Sptr} \stackrel{\Omega^\infty}{\to} \mathrm{Spc}$, $\mathrm{Spc} \stackrel{\pi_0}{\to} \mathrm{Sets}$ preserves products and filtered colimits! (Here $\pi_0 : \mathrm{Spc} \to \tau_{\leq 0}\mathrm{Spc}$ is a left adjoint). Consider an object $X \in$

Sptr viewed as a collection $(X_n)_{n\ge 0}$ with $X_n\in \mathrm{Spc}_*$, $\Omega(X_{n+1})\widetilde{\to} X_n$. For $m\in\mathbb{Z}$ the truncation $\tau_{\le m}X$ in the sense of the t-structure on Sptr is the collection $(\tau_{\le n+m}X_n)_{n\ge 0}$. Besides, $X\in \mathrm{Sptr}_{\ge m}$ iff each X_n is $(n+m)$-connective. The truncation $\tau_{\ge m}X$ in the sense of the t-structure on Sptr is obtained as the fibre of $X\to\tau_{\le m-1}X$. So, for $n\ge 0$ the term $(\tau_{\ge m}X)_n$ is the fibre of the natural map $X_n\to\tau_{\le n+m-1}X_n$.

Remark 4.0.68. *The composition* $\mathrm{Sptr}\overset{\Omega^\infty}{\to}\mathrm{Spc}_*\overset{\pi_0}{\to}$ *Sets equals* $\mathrm{Sptr}\overset{\pi_0}{\to}$ *Ab* $\to$ *Sets, where* π_0 *is the homology for the t-structure. The functor* $\pi_n:\mathrm{Sptr}\to$ *Ab preserves products.*

Proof. Write $X\in\mathrm{Sptr}$ as a collection $(X_n)_{n\ge 0}$ with $X_n\in\mathrm{Spc}_*$, $\Omega(X_{n+1})\widetilde{\to}X_n$. Then $\pi_0(X)\in\mathrm{Sptr}$ is the fibre of the natural map $\tau_{\le 0}X\to\tau_{\le -1}X$, where τ is the truncation in the sense of the t-structure on Sptr. So, the abelian group $\pi_0(X)$ is simply the fibre of the map $\tau_{\le 0}(X_0)=(\tau_{\le 0}X)_0\to(\tau_{\le -1}X)_0=\tau_{\le -1}(X_0)=*$ in Spc_*. So, $\pi_0(X)$ in the sense of the t-structure on Sptr identifies with $\pi_0(X_0)$.

The last claim is written in (HA, 1.4.3.6). It comes from the fact that $\pi_0:\mathrm{Spc}\to$ *Sets* preserves products. □

The above remark implies that the functor $\tau^{\ge n}:\mathrm{Sptr}\to\mathrm{Sptr}$ preserves products.

4.0.69. Let $\mathcal{C}$ be presentable, let $f:\mathcal{C}\to\mathrm{Funct}([1],\mathcal{C})$ be the functor restriction via $[1]\to *$. Its right adjoint f^* sends $c_1\to c_2$ to c_1. Let $\mathcal{C}_0\subset\mathcal{C}$ be the full subcategory spanned by final objects. Then $\mathcal{C}_0\subset\mathcal{C}$ is strongly reflective. Now $\mathcal{C}_*\subset\mathrm{Funct}([1],\mathcal{C})$ is the full subcategory of $c_1\to c_2$ such that c_1 is final in $\mathcal{C}$. Now by ([27], 5.5.4.17) we see that $\mathcal{C}_*$ is a strongly reflective subcategory of $\mathrm{Funct}([1],\mathcal{C})$. Besides, $\mathcal{C}_*$ is stable under limits in $\mathrm{Funct}([1],\mathcal{C})$. Indeed, the functor $*\to\mathcal{C}$ sending $*$ to the final objects preserves limits, and the category $*$ admits limits. (See also HTT, 1.2.13.8).

If $K\to\mathcal{C}_*$ is a diagram in $\mathcal{C}_*$, it has a colimit in $\mathrm{Funct}([1],\mathcal{C})$. If K is weakly contractible then this colimit actually lies in $\mathcal{C}_*$. (Indeed, if $c\in\mathcal{C}$ is final then the colimit of the functor $K\to *\overset{c}{\to}\mathcal{C}$ can be calculated as $F(\operatorname{colim} h)$, where $F:\mathrm{Spc}\to\mathcal{C}$ is colimit-preserving with $F(*)=c$, and h is the composition $K\to *\to\mathrm{Spc}$). In particular, this holds for κ-filtered colimits. So, $\mathcal{C}_*\subset\mathrm{Funct}([1],\mathcal{C})$ is stable under κ-filtered colimits. This implies that the forgetful functor $\mathcal{C}_*\to\mathcal{C}$ preserves κ-filtered colimits (actually, reflects κ-filtered colimits). See also ([28], proof of 1.4.4.4).

For example, let K be the usual category $(1\leftarrow 0\to 2)$, it has 3 objects. Then K is contractible, because K has an initial object. So, by the above, $\mathcal{C}_*$ admits push-outs, and $\mathcal{C}_*\subset\mathrm{Funct}([1],\mathcal{C})$ is stable under push-outs, and the forgetful functor $\mathcal{C}_*\to\mathcal{C}$ preserves push-outs.

4.0.70. ([28], 1.4.4.5) may be formulated more precisely: let $\mathcal{C},\mathcal{D}\in 1-\mathcal{C}at$ be presentable, $\mathcal{D}$ stable. Then composition with $\Omega^\infty: Sp(\mathcal{C})\to\mathcal{C}$ induces a commutative diagram, where the horizontal maps are equivalences

$$\begin{array}{ccc}\mathrm{Fun}'(\mathcal{D},Sp(\mathcal{C})) & \widetilde{\to} & \mathrm{Fun}'(\mathcal{D},\mathcal{C})\\ \uparrow & & \uparrow\\ \mathrm{Fun}^R(\mathcal{D},Sp(\mathcal{C})) & \widetilde{\to} & \mathrm{Fun}^R(\mathcal{D},\mathcal{C})\end{array}$$

Here in the first line $'$ means that we take full subcategory of left exact functors (as in [28], 1.4.2.23), and R means the full subcategory of functors, which are right adjoints (equivalently, limit preserving).

4.0.71. Let $\Sigma^\infty_+ : \mathrm{Spc} \to Sp$ be the left adjoint to $\Omega^\infty : Sp \to \mathrm{Spc}$. The sphere spectrum in Sp is defined ([28], 1.4.4.5) as the image of $* \in \mathrm{Spc}$ under Σ^∞_+ (I think there is a misprint there in [28], namely Σ^∞ should be replaced by Σ^∞_+).

Difference in NOTATIONS: if $\mathcal{C} \in 1-\mathcal{C}at$ has finite limits, the functor $\Omega^\infty_* : Sp(\mathcal{C}) \to \mathcal{C}_*$ takes values in the pointed category $\mathcal{C}_*$, its left adjoint is denoted $\Sigma^\infty : \mathcal{C}_* \to Sp(\mathcal{C})$. The left adjoint to $\Omega^\infty : Sp(\mathcal{C}) \to \mathcal{C}$ is denoted $\Sigma^\infty_+ : \mathcal{C} \to Sp(\mathcal{C})$ ([28], 1.4.4.4).

If $X \in \mathrm{Spc}_*$ then $\Sigma^\infty(X) \in Sp_{\geq 0}$. Indeed, for any $Y \in Sp_{\leq -1}$ we get

$$\mathrm{Map}_{Sp}(\Sigma^\infty(X), Y) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}_*}(X, \Omega^\infty(Y)) \widetilde{\to} *$$

What are the compositions $\mathrm{Spc}_* \stackrel{\Sigma^\infty}{\to} Sp(\mathcal{C}) \stackrel{\Omega^{\infty-n}}{\to} (\mathrm{Spc}_*)$? How the functor Σ^∞ interacts with the t-structure on $Sp(\mathcal{C})$? That is, what are the homotopy groups of $\Sigma^\infty(X)$ for $X \in \mathrm{Spc}_*$ for the standard t-structure on Sp? See further in [28]?

4.0.72. Question: let $\mathcal{C}$ be an ∞-topos, let $Disc(\mathcal{C})$ be the category of discrete objects of $\mathcal{C}$. Since $\mathcal{C}$ is presentable, we have the t-structure on $Sp(\mathcal{C})$ defined in ([28], 1.4.3.4). What is the heart of this t-structure, is it equivalent to the category of abelian groups in $Disc(\mathcal{C})$? Toen says yes.

4.0.73. *Grothendieck abelian categories.* For ([14], Ch. I.1), 10.1.2). If R is a ring then the category of R-modules is a Grothendieck abelian category $\mathcal{A}$, so one has the unbounded derived category $\mathrm{D}(\mathcal{A})$ defined in ([28], 1.3.5.8). It is equipped with a right complete t-structure. By ([28], 1.3.5.24), we have a full embedding $\mathrm{D}^-(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$, whose image is $\cup_n \mathrm{D}(\mathcal{A})_{\geq -n}$. So, $\mathrm{D}(\mathcal{A})$ is the right completion of $\mathrm{D}^-(\mathcal{A})$.

This category has an additional property that for any n the functor $\pi_n : \mathrm{D}(\mathcal{A}) \to \mathcal{A}$ commutes with filtered colimits, and the category $\mathrm{D}(\mathcal{A})_{\leq 0}$ is closed under filtered colimits proved in (HA, 1.3.5.21).

Lurie introduces a general notion: given a stable ∞-category $\mathcal{C}$ with a t-structure, *the t-structure is compatible with filtered colimits* iff $\mathcal{C}_{\leq 0}$ is closed under filtered colimits in $\mathcal{C}$. In such a category the functors $\tau_{\geq 0}, \tau_{\leq 0}$ commute with filtered colimits. Indeed, if $K \widetilde{\to} \mathrm{colim}_{i\in I} K_i$ in $\mathcal{C}$ with I filtered then for each i we have a fibre sequence $\tau_{\geq 0}K_i \to K_i \to \tau_{\leq -1}K_i$ in $\mathcal{C}$, hence the sequence $\mathrm{colim}(\tau_{\geq 0}K_i) \to \mathrm{colim}\, K_i \to \mathrm{colim}(\tau_{\leq -1}K_i)$ is also a cofibre sequence in $\mathcal{C}$, here all the colimits are taken in $\mathcal{C}$. We know that $\mathrm{colim}(\tau_{\geq 0}K_i) \in \mathcal{C}_{\geq 0}$, as $\mathcal{C}_{\geq 0}$ is closed under all colimits that exists in $\mathcal{C}$, and by assumption $\mathrm{colim}(\tau_{\leq -1}K_i) \in \mathcal{C}_{\leq -1}$. So, $\tau_{\geq 0}K \widetilde{\to} \mathrm{colim}(\tau_{\geq 0}K_i)$ and $\tau_{\leq -1}K \widetilde{\to} \mathrm{colim}(\tau_{\leq -1}K_i)$.

In particular this holds for the category $\mathrm{Vect} \widetilde{\to} \mathrm{D}(k)$ from (ch. 1, 10.1.1).

4.0.74. Let $\mathcal{C} \in 1-\mathcal{C}at$ be stable with a t-structure, which is right complete. Assume $\mathcal{C}^\heartsuit$ has enough injective objects. By (HA, 1.3.3.2), there is a natural t-exact functor $f : \mathrm{D}^+(\mathcal{C}) \to \mathcal{C}$ extending the identity on $\mathcal{C}^\heartsuit$. We claim that f induces isomorphisms for $a, b \in \mathcal{C}^\heartsuit$ and $i = 0, 1$

$$\mathrm{Ext}^i_{\mathcal{C}^\heartsuit}(a, b) \to \mathrm{Ext}^i_{\mathcal{C}}(a, b) \tag{7}$$

For $i=0$ this is by definition. For $i=1$ to see the injectivity, consider an exact sequence $0\to b\to x\to a\to 0$ in $\mathcal{C}^{\heartsuit}$. Then $b\to x\to a$ is a fibre sequence in $\mathcal{C}$, hence yields a morphism $a\stackrel{\gamma}{\to} b[1]$ in $\mathcal{C}$. If $\gamma=0$ in $\pi_0\operatorname{Map}_{\mathcal{C}}(a,b[1])=\operatorname{Ext}^1_{\mathcal{C}}(a,b)$ the fibre sequence $x\to a\to b[1]$ is isomorphic to $a\oplus b\to a\stackrel{0}{\to} b[1]$, and the initial exact sequence splits. Indeed, a a map $a\to x$ in C giving the splitting is actually a morphism in $C^{\heartsuit}$.

Let now $\bar\gamma\in\operatorname{Ext}^1_{\mathcal{C}}(a,b)$ by represented by a map $\gamma: a\to b[1]$ in $\mathcal{C}$. Let x be the fibre of γ. So, we get a fibre sequence $b\to x\to a$ in $\mathcal{C}$. Then $x\in\mathcal{C}^{\heartsuit}$, and the sequence $0\to b\to x\to a\to 0$ in $\mathcal{C}^{\heartsuit}$ is exact. The corresponding element of $\operatorname{Ext}^1_{\mathcal{C}^{\heartsuit}}(a,b)$ goes to $\bar\gamma$. We checked the surjectivity.

Let us show that (7) is injective for $i=2$. Pick an exact sequence $0\to b\to b_0\to b_0/b\to 0$ in $\mathcal{C}^{\heartsuit}$ with b_0 injective. We get a diagram

$$\begin{array}{ccccccc}
\operatorname{Ext}^1_{\mathcal{C}^{\heartsuit}}(a,b_0) & \to & \operatorname{Ext}^1_{\mathcal{C}^{\heartsuit}}(a,b_0/b) & \to & \operatorname{Ext}^2_{\mathcal{C}^{\heartsuit}}(a,b) & & \to 0\\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Ext}^1_{\mathcal{C}}(a,b_0) & \to & \operatorname{Ext}^1_{\mathcal{C}}(a,b_0/b) & \to & \operatorname{Ext}^2_{\mathcal{C}}(a,b) & & \to \operatorname{Ext}^2_{\mathcal{C}}(a,b_0)
\end{array}$$

where the left two vertical arrows are isomorphisms. The diagram chase implies the desired claim.

4.0.75. If $C\in 1-\mathcal{C}at$ is stable then a set of objects $c_i\in C$, $i\in I$ is a set of co-generators iff for any $x\in C$ the assumption $\operatorname{Map}_C(x,c_i[n])=*$ for all n,i implies $x=0$. In other words, c_i is a set of generators for C^{op}.

4.1. For ([14], ch. I.1, 6.2.10). Let $\mathbf{C}\in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ and $c_0,c_1\in\mathbf{C}$. Recall they denote $\mathcal{M}aps_{\mathbf{C}}(c_0,c_1)=\underline{\operatorname{Hom}}_{\operatorname{Sptr}}(c_0,c_1)$. This relative inner hom always exists. Indeed, the functor $\operatorname{Sptr}^{op}\to\operatorname{Spc}$, $x\mapsto\operatorname{Map}_{\mathbf{C}}(x\boxtimes c_0,c_1)$ preserves limits, because the tensor product $\operatorname{Sptr}\times\mathbf{C}\to\operatorname{Sptr}\otimes\mathbf{C}\widetilde{\to}\mathbf{C}$ preserves colimits separately in each variable.

For the sphere spectrum $\Sigma^\infty(*)=1_{\operatorname{Sptr}}\in\operatorname{Sptr}$ we get

$$\operatorname{Map}_{\mathbf{C}}(c_0,c_1)\widetilde{\to}\operatorname{Map}_{\mathbf{C}}(1_{\operatorname{Sptr}}\otimes c_0,c_1)\widetilde{\to}\operatorname{Map}_{\operatorname{Sptr}}(1_{\operatorname{Sptr}},\mathcal{M}aps_{\mathbf{C}}(c_0,c_1))\widetilde{\to}$$
$$\operatorname{Map}_{\operatorname{Spc}}(*,\Omega^\infty\mathcal{M}aps_{\mathbf{C}}(c_0,c_1))\widetilde{\to}\Omega^\infty\mathcal{M}aps_{\mathbf{C}}(c_0,c_1)$$

4.1.1. In ([14], ch. I.1, 6.3.4) by $\mathbf{C}_*$ they denote the colimit of $\mathbf{C}_I: I\to 1-\mathcal{C}at^{St,cocmpl}_{cont}$. If $I^{\triangleright}\to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ is the colimit diagram for $\mathbf{C}_*$ then it is claimed that $\mathbf{C}_*$ is dualizable, and the dual diagram ${}^{\triangleleft}(I^{op})\to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ is a limit diagram in $1-\mathcal{C}at^{St,cocmpl}_{cont}$, hence also in $1-\mathcal{C}at$.

The following is used without explanation in 6.3.6 (compare it with Lemma 2.2.56):

Lemma 4.1.2. *Let $f: A\to B$ be a morphism in $1-\mathcal{C}at^{St,cocmpl}_{cont}$ whose right adjoint $g: B\to A$ is continuous. Assume A,B dualizable. Then $g^\vee: A^\vee\to B^\vee$ is left adjoint to $f^\vee: B^\vee\to A^\vee$.*

Proof. Consider the $(\infty,2)$-category $\mathbf{1\text{-}Cat}^{St,cocmpl}_{cont}$ from 5.2.1. Inside we have the full subcategory of dualizable objects $(\mathbf{1\text{-}Cat}^{St,cocmpl}_{cont})^{dualizable}$. The dualization functor should be a functor between $(\infty,2)$-categories

$$((\mathbf{1\text{-}Cat}^{St,cocmpl}_{cont})^{dualizable})^{op}\to(\mathbf{1\text{-}Cat}^{St,cocmpl}_{cont})^{dualizable}$$

Since our monoidal category is symmetric, we may not add $rev - mult$ in the RHS. Such a functor sends adjoint 1-morphisms to adjoint 1-morphisms. (Check this!!) $\square$

For the proof of 6.3.4. In 6.3.5 they denote by $\lim_{I^{op}} \mathbf{C}^{\vee}_{I^{op}}$ the limit in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Then it is also the limit in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ by Cor. 5.3.4. For this reason in 6.3.6 we may rewrite this limit as a colimit according to Cor. 5.3.4.

4.1.3. If $f : C \to D$ is a map in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, C, D dualizable, let $f^{\vee} : D^{\vee} \to C^{\vee}$ be the dual map. Then the dagrams commute

$$\begin{array}{ccc} C^{\vee}\otimes C & \stackrel{f}{\to} & C^{\vee}\otimes D \\ \uparrow \mu & & \uparrow f^{\vee} \\ \mathrm{Sptr} & \stackrel{\mu}{\to} & D^{\vee}\otimes D \end{array} \qquad \begin{array}{ccc} C\otimes D^{\vee} & \stackrel{f^{\vee}}{\to} & C\otimes C^{\vee} \\ \downarrow f & & \downarrow ev \\ D\otimes D^{\vee} & \stackrel{ev}{\to} & \mathrm{Sptr} \end{array}$$

where μ is the unit map, ev is the evaluation map. This was used in 6.3.5-6.

Actually this holds for any symmetric monoidal ∞-category $\mathcal{C}$ and a map $f : C \to D$ in $\mathcal{C}$ between dualizable objects.

4.1.4. For the proof of 6.4.2. The functor $j^L : C \to C''$ is continuous between stable cocomplete categories. The universal property that they mention is that such a functor is completely determined by the composition $C_1 \times C_2 \to C_1 \otimes C_2 = C \stackrel{j^L}{\to} C''$, and this composition is exact and continuous in each variable. We need to show the latter composition is zero. It factors through $C' \stackrel{i}{\to} C \stackrel{j^L}{\to} C''$. The composition $j^L i = 0$, because $j^L(c) = cofib(ii^R(c) \to c)$ and $i^R i \widetilde{\to} \mathrm{id}$.

4.1.5. For Proposition 6.4.3. It is understood that $F_i : D_i \to C_i$ are morphisms in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$.

4.1.6. Let $F_i : C_i \leftrightarrows D_i : G_i$ be adjoint functors, maps in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Then $F_1 \otimes F_2 : C_1 \otimes C_2 \leftrightarrows D_1 \otimes D_2 : G_1 \otimes G_2$ is also an adjoint pair.

Indeed, if $\mu_i : \mathrm{id} \to G_i F_i$ is the unit, $c_i : F_i G_i \to \mathrm{id}$ the counit of the adjunction then $\mu_1 \otimes \mu_2 : \mathrm{id} \to (G_1F_1) \otimes (G_2F_2) = (G_1 \otimes G_2)(F_1 \otimes F_2)$ will be the unit, and $c_1 \otimes c_2 : (F_1G_1) \otimes (F_2G_2) = (F_1 \otimes F_2)(G_1 \otimes G_2) \to \mathrm{id}$ the counit of the new adjuntion. The reason: tensor product of identity maps is an identity map. To prove this use ([14], ch. I, 4.4 and 5.3.2), that is, we use the $(\infty, 2)$-categorical enhancement of $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, and the fact that (F_i, G_i) is a dual pair in the sense of that $(\infty, 2)$-category.

This is used in the proof of ([14], ch1, 6.4.5), namely $ind_{\mathcal{A}_1} \otimes ind_{\mathcal{A}_2}$ is the left adjoint to $oblv_{\mathcal{A}_1} \otimes oblv_{\mathcal{A}_2}$.

If $C_i \in 1-\mathcal{C}\mathrm{at}^{St,cocmp}_{cont}$ then there is a natural morphism

$$\mathrm{Fun}_{ex,cont}(C_1, C_1) \otimes \mathrm{Fun}_{ex,cont}(C_1, C_1) \to \mathrm{Fun}_{ex,cont}(C_1 \otimes C_2, C_1 \otimes C_2)$$

in $Alg(1-\mathcal{C}\mathrm{at}^{St,cocmp}_{cont})$.

4.1.7. All the exponentials for $1-\mathcal{C}at^{St,cocmpl}_{cont}$ exist. Namely, if $D, E \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ then $E^D = \mathrm{Fun}_{ex,cont}(D, E)$. So, if $D \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ is dualizable then

$$D^\vee \widetilde{\to} \mathrm{Sptr}^D \widetilde{\to} \mathrm{Fun}_{ex,cont}(D, \mathrm{Sptr})$$

We always have the natural map $e : D \otimes \mathrm{Fun}_{ex,cont}(D, \mathrm{Sptr}) \to \mathrm{Sptr}$. By (HA, 4.6.1.6), it extends to a duality datum on D iff for any $A \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ the above map e induces an isomorphism

$$\mathrm{Fun}_{ex,cont}(X, A \otimes D)^{\mathrm{Spc}} \widetilde{\to} \mathrm{Fun}_{ex,cont}(X \otimes \mathrm{Fun}_{ex,cont}(D, \mathrm{Sptr}), A)^{\mathrm{Spc}}$$

4.2. **Compactly generated stable categories.** For 7.1. Let $C \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$, $c, x \in C$. Let $\Sigma^{\infty-n} : \mathrm{Spc}_* \to \mathrm{Sptr}$ be the left adjoint to $\Omega^{\infty-n} : \mathrm{Sptr} \to \mathrm{Spc}_*$ as in my Section 4.0.54. Recall that $\Sigma^{\infty-n}(S^0) \widetilde{\to} 1_{\mathrm{Sptr}}[-n]$. We get

$$\Omega^{\infty-n}\mathcal{M}aps_C(c, x) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}_*}(S^0, \Omega^{\infty-n}\mathcal{M}aps_C(c, x)) \widetilde{\to}$$
$$\mathrm{Map}_{\mathrm{Sptr}}(\Sigma^{\infty-n}(S^0), \mathcal{M}aps_C(c, x)) \widetilde{\to} \mathrm{Map}_C(1_{\mathrm{Sptr}}[-n] \otimes c, x) \widetilde{\to} \mathrm{Map}_C(c, x[n])$$

Lemma 4.2.1. *Let $C \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$, $c \in C$. The functor $C \to \mathrm{Spc}$, $x \mapsto \mathrm{Map}_C(c, x)$ preserves filtered colimits iff $C \to \mathrm{Sptr}$, $x \mapsto \mathcal{M}aps_C(c, x)$ preserves filtered colimits.*

Proof. The if direction follows from my Lemma 4.0.66. Now assume $C \to \mathrm{Spc}$, $x \mapsto \mathrm{Map}_C(c, x)$ preserves filtered colimits. Theh $C \to \mathrm{Spc}_*$, $x \mapsto \mathrm{Map}_C(c, x)$ also preserves filtered colimits. Let $x = \mathrm{colim}\, x_i$ be a filtered colimit in C. Since $C \to C, x \mapsto x[n]$ preserves colimits, $x[n] \widetilde{\to} \mathrm{colim}_i\, x_i[n]$. So, $\mathrm{Map}_C(c, x[n]) \widetilde{\to} \mathrm{colim}_i\, \mathrm{Map}_C(c, x_i[n])$. So, the functor $C \to \mathrm{Spc}_*$, $x \mapsto \Omega^{\infty-n}\mathcal{M}aps_C(c, x)$ preserves filtered colimits. It follows now from my Lemma 2.2.68 that $x \mapsto \mathcal{M}aps_C(c, x)$ preserves filtered colimits. □

Lemma 4.2.2. *Let $C \in 1-\mathcal{C}at^{St,cocmpl}$ then C^c is stable.*

Proof. Clearly, $0 \in C$ is compact. From ([27], 5.3.4.15) it follows that $C^c \subset C$ is stable under finite colimits, so C^c admits finite colimits. It also shows that for $z \in C^c, z[1] \in C^c$. Since $C \to C, x \mapsto x[1]$ preserves colimits, for $z \in C^c$ the functor $x \mapsto \mathrm{Map}_C(z, x[1]) = \mathrm{Map}_C(z[-1], x)$ preserves filtered colimits, so $z[-1] \in C^c$. Thus, $C^c \subset C$ is a stable subcategory by ([28], 1.1.3.3). □

4.2.3. Proof of ([14], Lemma 7.1.5). Let $F : C \to D$ be a map in $1-\mathcal{C}at^{St,cocmpl}_{cont}$ having a right adjoint F^R. Assume F^R continuous. Then for $z \in C^c$ and $d = \mathrm{colim}_I\, d_i$ in D with I filtered we get

$$\mathrm{Map}_D(F(z), d) \widetilde{\to} \mathrm{Map}_C(z, F^R(d)) \widetilde{\to} \mathrm{Map}_C(z, \mathrm{colim}_I\, F^R(d_i)) \widetilde{\to}$$
$$\mathrm{colim}_I\, \mathrm{Map}_C(z, F^R(d_i)) \widetilde{\to} \mathrm{colim}_I\, \mathrm{Map}_D(F(z), d_i)$$

So, $F(C^c) \subset D^c$. Conversely, assume $F(C^c) \subset D^c$. Let $d = \mathrm{colim}_J\, d_j$ in D, where J is filtered. It suffices to show that the natural map $\mathrm{colim}_J\, F^R(d_j) \to F^R(d)$ is an isomorphism in C. For this it suffices to show that for any $z \in C^c$ the induced map

$$\mathrm{Map}_C(z, \mathrm{colim}_J\, F^R(d_j)) \to \mathrm{Map}_C(z, F^R(d)) \widetilde{\to} \mathrm{Map}_D(F(z), d)$$

is an isomorphism. Since $z \in C^c$, this map rewrites

$$\operatorname{colim}_J \operatorname{Map}_D(F(z), d_j) \widetilde{\to} \operatorname{colim}_J \operatorname{Map}_C(z, F^R(d_j)) \to \operatorname{Map}_D(F(z), d)$$

Since $F(z) \in D^c$, the latter map is an isomorphism. We are done. (cf. HTT, 5.5.7.2).

(ch. 1, Lemma 7.1.5) can be strengthened as follows: let $F : C \to D$ be a map in $1-\mathcal{C}at^{St,cocmpl}_{cont}$, C compactly generated. Assume $C_0 \subset C^c$ is a full subcategory such that $\operatorname{Ind}(C_0) \widetilde{\to} C$ and $f(C_0) \subset D^c$. Then the right adjoint F^R is continuous.

4.2.4. In (ch. 1, 7.2.1) a condition is missing. If C_0 is stable, $f : C_0 \to \mathrm{Spc}$ preserves fibred products, it is not necessarily left exact. Indeed, for such a functor and $Z \in \mathrm{Spc}$ the functor $C_0 \to \mathrm{Spc}, x \mapsto f(x) \times Z$ preserves fibred products.

On the other hand, if $f : C_0 \to \mathrm{Spc}$ preserves fibred products and final objects then by ([27], 4.4.2.5) it preserves finite limits (that is, is left exact). If C_0 is stable then $\operatorname{Ind}(C_0) \subset \mathcal{P}(C_0)$ is the full subcategory of left exact functors $f : C_0^{op} \to \mathrm{Spc}$.

In Dennis' notations, $\tau_{\le 0} : \mathrm{Sptr} \to \mathrm{Sptr}^{\le 0}$ is the right adjoint to the inclusion $\mathrm{Sptr}^{\le 0} \hookrightarrow \mathrm{Sptr}$, so $\tau_{\le 0}$ preserves all limits. So, composition with $\tau_{\le 0}$ yields a functor $\mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Sptr}) \to \mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Sptr}^{\le 0})$, where Lex stands for the full subcategories of left exact functors. The projection $ComGrp(\mathrm{Spc}) \to \mathrm{Spc}$ preserves limits, this follows from my Section 2.5.18 and ([28], 3.2.2.5). So, this composition with this functor yields a map $\mathrm{Fun}^{Lex}(C_0^{op}, ComGrp(\mathrm{Spc})) \to \mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Spc})$. Note that $\mathrm{Sptr}^{\le 0}$ admits limits, because $ComGrp(\mathrm{Spc})$ admits small limits. By ([28], 1.4.2.23) the composition $\mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Sptr}) \to \mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Spc})$ is an equivalence.

Recall that $\mathrm{Sptr}^{\le 0} \subset \mathrm{Sptr}$ is stable under all small colimits. If a functor $f : C^{op} \to \mathrm{Sptr}^{\le 0}$ is right exact then it is also left exact. Indeed, we have $f(0) = 0$. If $b' = b \sqcup_a a'$ in C^{op} then $f(b') \widetilde{\to} f(b) \sqcup_{f(a)} f(a')$ in $\mathrm{Sptr}^{\le 0}$, hence also in Sptr. So, $f(a) \widetilde{\to} f(b) \times_{f(b')} f(a')$ in Sptr. Since $\tau_{\le 0} : \mathrm{Sptr} \to \mathrm{Sptr}^{\le 0}$ preserves limits, $f(a) \widetilde{\to} f(b) \times_{f(b')} f(a')$ in $\mathrm{Sptr}^{\le 0}$.

The inclusion $\mathrm{Sptr}^{\le 0} \hookrightarrow \mathrm{Sptr}$ preserves products, but does not preserves finite limits. Namely, $\mathrm{Sptr}^{\heartsuit} = Ab$, if $y \to z$ is a map in the category Ab of abelian groups, let $x \to y \to z$ be a fiber sequence. Then $0 \to \pi_0(x) \to y \to z \to \pi_{-1}(x) \to 0$ is exact in Ab. Here π_i is Lurie's notation for homotopy groups, so it corresponds to homological indexing conventions (in cohomological conventions this means that $\mathrm{H}^1(x)$ could be nonzero).

If a functor $f : C^{op} \to \mathrm{Sptr}^{\le 0}$ is left exact, it is not necessarily right exact. For example, take $C = \mathrm{Sptr}^{op}$. In Dennis' notations (cohomological indexing conventions) the functor $\tau_{\le 0} : \mathrm{Sptr} \to \mathrm{Sptr}^{\le 0}$ is left exact, but not right exact. Indeed, consider a fiber sequence $x \to y \to z$ in Sptr with $y, z \in \mathrm{Sptr}^{\heartsuit}$. Its image by $\tau_{\le 0}$ is $\pi_0(x) \to y \to z$. If $\pi_{-1}(x) \neq 0$ in our above notations then the latter triangle is not a fibre sequence in Sptr, so

$$\begin{array}{ccc} \pi_0(x) & \to & y \\ \downarrow & & \downarrow \\ 0 & \to & z \end{array}$$

is not cocartesian in $\mathrm{Sptr}^{\le 0}$, because $\mathrm{Sptr}^{\le 0} \hookrightarrow \mathrm{Sptr}$ preserves colimits.

Let us show the above functors

$$\mathrm{Fun}_{ex}(C_0^{op}, \mathrm{Sptr}) \to \mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Sptr}^{\le 0}) \to \mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Spc})$$

are equivalences. For the composition this is precisely ([28], 1.4.2.23), as $\mathbb{S}\mathrm{p}(\mathrm{Spc}) \widetilde{\to} \mathrm{Sptr}$. Since $\mathrm{Sptr}^{\le 0}$ is pointed, by ([28], 1.4.2.24), $\mathbb{S}\mathrm{p}(\mathrm{Sptr}^{\le 0})$ identifies with the right completion of $\mathrm{Sptr}^{\le 0}$.

This is a general fact: if C is a stable ∞-category, $\mathbb{S}\mathrm{p}(C^{\le 0})$ identifies with the right completion of C, as the diagram $\dots C^{\le 0} \stackrel{\Omega}{\to} C^{\le 0}$ identifies with

$$\dots \to C^{\le 2} \stackrel{\tau^{\le 1}}{\to} C^{\le 1} \stackrel{\tau^{\le 0}}{\to} C^{\le 0}$$

Since Sptr is right complete, we get $\mathbb{S}\mathrm{p}(\mathrm{Sptr}^{\le 0}) \widetilde{\to} \mathrm{Sptr}$. So, the first isomorphism also follows from ([28], 1.4.2.23) applied to $\mathrm{Sptr}^{\le 0}$.

4.2.5. *Compactly generated category.* In addition to ([14], ch.1, Def. 7.1.3), where the property of being compactly generated is introduced for a stable cocomplete category, there is a more general definition ([27], 5.5.7.1). Namely, $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is compactly generated iff C is presentable and accessible.

Let $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ be small. Then $\mathcal{P}(\mathcal{C})$ is compactly generated. Indeed, we invoke ([27], 5.3.5.12): for $\mathcal{D} = \mathcal{P}(\mathcal{C})^c$ we get $\mathrm{Ind}(\mathcal{D}) \widetilde{\to} \mathcal{P}(\mathcal{C})$. By ([27], 5.3.4.15) the inclusion $\mathcal{P}(\mathcal{C})^c \subset \mathcal{P}(\mathcal{C})$ is stable under finite colimits. We also formally use ([27], 5.4.2.2(3)).

We may also invoke ([27], Example 5.4.2.7), it says that for any small $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, $\mathcal{P}(\mathcal{C})$ is accessible.

By ([27], 5.3.4.17), each $h \in \mathcal{P}(\mathcal{C})^c$ is a retract of some $f \in \mathcal{P}(\mathcal{C})^c$, where $f = \mathrm{colim}\, \bar{p}$ for the composition $\bar{p} : K \stackrel{p}{\to} \mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$, and K is an ∞-category coming from a ω-small simplicial set.

FACT ([28], 1.4.3.7): let $C \in 1-\mathcal{C}\mathrm{at}$ be compactly generated. Then $Sp(C)$ is compactly generated.

The category Sptr is compactly generated, and its compact objects are described in ([29], 9.7).

4.2.6. If $\mathcal{D}$ is stable, d' is a retract of d then d' is a direct summand of d. Indeed, this happens already on the level of the underlying triangulated category (see Stack project, Lemma 13.4.10).

4.2.7. For 7.2.3. If C_0 is stable then the composition with Ω^∞ yields an equivalence $\mathrm{Fun}_{ex}(C_0^{op}, \mathrm{Sptr}) \widetilde{\to} \mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Spc})$, see ([28], 1.4.2.23). By definition, $\mathrm{Ind}(C_0) = \mathrm{Fun}^{Lex}(C_0^{op}, \mathrm{Spc})$.

For the proof of 7.2.4: the essential image of $C_0 \to \mathrm{Ind}(C_0)$ generates C_0 by ([14], ch.1, Lemma 5.4.5). Indeed, by ([27], 5.3.5.4), $\mathrm{Ind}(C_0)$ is obtained from C_0 by adjoining filtered colimits. The category $\mathrm{Ind}(C^0)^c$ is described in (HTT, 5.4.2.4). If c is a compact object of $\mathrm{Ind}(C_0)$ then its image in $\mathcal{P}(C_0) = \mathrm{Fun}(C_0^{op}, \mathrm{Spc})$ is compact in $\mathcal{P}(C_0)$. Indeed, this follows from the fact that $\mathrm{Ind}(C_0) \subset \mathcal{P}(C_0)$ is stable under filtered colimits ([27], 5.3.5.3). To describe compact objects of $\mathrm{Ind}(C_0)$ it seems useful to use ([27], 5.5.7.3). Namely, according to my Section 2.2.40, the inclusion $\mathrm{Ind}(C_0) \hookrightarrow \mathcal{P}(C_0)$ admits a left adjoint $\bar{\Phi} : \mathcal{P}(C_0) \to \mathrm{Ind}(C_0)$, which is the LKE of the inclusion $C_0 \to \mathrm{Ind}(C_0)$, so $\bar{\Phi}$ is a localization functor. The functor $\bar{\Phi}$ is continuous. So, by ([27], 5.5.7.3) any compact object of $\mathrm{Ind}(C_0)$ is a direct summand of $\bar{\Phi}(x)$ for some compact object $x \in \mathcal{P}(\mathcal{C})^c$. Recall that $\mathcal{P}(\mathcal{C})^c$ is described in ([27], 5.3.4.17). Since $\bar{\Phi}$ preserves small colimits, we may now invoke ([27], 5.3.5.14) showing that $j : C_0 \to \mathrm{Ind}(C_0)$ preserves finite colimits. We see that each compact object of $\mathrm{Ind}(C_0)$ is a direct summand of the image of some $j(z)$, where $z \in C_0$. We amy also use ([27], 5.4.2.4) simply.

The part (2) of Lemma 7.2.4 can be derived from ([27], 5.5.1.9 and 5.3.5.10). Warning here: a continuous functor (in Lurie's sense) between stable ∞-categories is not necessarily exact, because there are finite colimits, which are not filtered! This is why we write $\mathrm{Fun}_{ex,cont}$. A suttle point here, if $f : C^0 \to C$ is exact, let $\bar{f} : \mathrm{Ind}(C^0) \to C$ be its continuous extension given by (HTT, 5.3.5.10). Then the restriction $\bar{f} : \mathrm{Ind}(C^0)^c \to C$ preserves finite colimits! This is affirmed in (HTT, Example 5.3.6.8), and also follows from (HTT, 5.5.1.9).

In part (3) of Lemma 7.2.4 one does not need to assume C^0 stable, just a full subcategory. Part (3) of Lemma 7.2.4 follows from ([27], 5.3.5.11), and (3') from ([27], 5.5.7.1).

4.2.8. For 7.2.5. Let $C_I : I \to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ be a functor, assume each C_i compactly generated, and each transition functor $C_i \to C_j$ preserving compactness. Then by C_* they mean the colimit of C_I. Note also that $C^R_{I^{op}} : I^{op} \to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ obtained by passing to right adjoints takes values in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$.

The colimit $\operatorname*{colim}_I C^c_I$ in $1-\mathcal{C}\mathrm{at}^{St}$ is not known to exist apriori.

To correct the argument, use the fact that $\mathcal{P}\mathrm{r}^L_\omega$ from ([27], 5.5.7.7) admits all colimits by ([27], 5.5.7.6). Namely, C_I becomes a functor $\tilde{C}_I : I \to \mathcal{P}\mathrm{r}^L_\omega$, let $\mathcal{D} = \operatorname{colim} \tilde{C}_I$. Then $\mathcal{D}$ is compactly generated, presentable. Passing to right adjoints, we get a functor $\tilde{C}^R_{I^{op}} : I^{op} \to \mathcal{P}\mathrm{r}^R_\omega$, and $\mathcal{D} = \lim \tilde{C}^R_{I^{op}}$ because of the equivalence $\mathcal{P}\mathrm{r}^L_\omega \widetilde{\to} (\mathcal{P}\mathrm{r}^R_\omega)^{op}$ from ([27], 5.5.7.7). Since the map $\mathcal{P}\mathrm{r}^R_\omega \hookrightarrow 1-\mathcal{C}\mathrm{at}$ preserves limits, $\mathcal{D}$ is also a limit of the composition $I^{op} \to \mathcal{P}\mathrm{r}^R_\omega \to 1-\mathcal{C}\mathrm{at}$, hence also of the composition $I^{op} \to \mathcal{P}\mathrm{r}^R_\omega \to \mathcal{P}\mathrm{r}^R$. We see that $\mathcal{D}$ identifies with the colimit C_* of the composition $I \to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont} \to \mathcal{P}\mathrm{r}^L$. Now as in my Section 4.0.31 one shows that $\mathcal{D}$ is stable, and $\mathcal{D}$ is the colimit of $C_I : I \to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. We see that $\mathcal{D}$ is compactly generated, and we get a diagram

$$\bar{C}_I : I^{\triangleright} \to 1 - Cat^{St}$$

by applying the functor $\mathcal{X} \mapsto \mathcal{X}^c$ to the colimit diagram $I^{\triangleright} \to \mathcal{P}\mathrm{r}^L_\omega$ for C_I.

Instead of showing that $\bar{C}_I$ is a colimit diagram, consider the subcategory $\mathcal{E} \subset 1-\mathcal{C}\mathrm{at}$, where we restrict objects to idempotent complete categories, which admit finite colimits, and morphisms to those which preserve finite colimits. By (HTT, 5.5.7.8), we have an equivalence $\mathcal{P}\mathrm{r}^L_\omega \widetilde{\to} \mathcal{E}$, $X \mapsto X^c$. So, the colimit diagram $I^{\triangleright} \to \mathcal{P}\mathrm{r}^L_\omega$ by composing with the above equivalence $\mathcal{P}\mathrm{r}^L_\omega \widetilde{\to} \mathcal{E}$ gives a colimit diagram. Finally, apply the functor Ind to the obtained diagram $I^{\triangleright} \to \mathcal{E}$.

We could want to show that $\bar{C}_I$ is a colimit diagram. Pick $E \in 1-\mathcal{C}\mathrm{at}^{St}$. Note that $\mathrm{Fun}_{ex}(D^c, E) \subset \mathrm{Fun}_{ex}(D^c, \mathrm{Ind}(E))$ is a full subcategory. We used the fact that

$E \hookrightarrow \mathrm{Ind}(E)$ preserves finite colimits (HTT, 5.3.5.14), so is exact. Recall that

$$\mathrm{Map}_{1-\mathcal{C}at^{St,cocmpl}_{cont}}(D, \mathrm{Ind}(E)) \widetilde{\to} \mathrm{Fun}_{ex,cont}(D, \mathrm{Ind}(E))^{Spc} \widetilde{\to} \mathrm{Fun}_{ex}(D^c, \mathrm{Ind}(E))^{Spc} \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}at^{St}}(D^c, \mathrm{Ind}(E))$$

We get

$$\mathrm{Map}_{1-\mathcal{C}at^{St}}(D^c, \mathrm{Ind}(E)) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}at^{St,cocmpl}_{cont}}(D, \mathrm{Ind}(E)) \widetilde{\to} \lim_{I^{op}} \mathrm{Map}_{1-\mathcal{C}at^{St,cocmpl}_{cont}}(C_i, \mathrm{Ind}(E)) \widetilde{\to} \lim_{I^{op}} \mathrm{Map}_{1-\mathcal{C}at^{St}}(C_i^c, \mathrm{Ind}(E))$$

For each i we have a full subspace $\mathrm{Map}_{1-\mathcal{C}at^{St}}(C_i^c, E) \subset \mathrm{Map}_{1-\mathcal{C}at^{St}}(C_i^c, \mathrm{Ind}(E))$, so $\lim_{I^{op}} \mathrm{Map}_{1-\mathcal{C}at^{St}}(C_i^c, E) \subset \lim_{I^{op}} \mathrm{Map}_{1-\mathcal{C}at^{St}}(C_i^c, \mathrm{Ind}(E))$ is a full subspace by my Lemma 2.2.17. This gives a full embedding $\mathrm{Map}_{1-\mathcal{C}at^{St}}(D^c, E) \subset \lim_{I^{op}} \mathrm{Map}_{1-\mathcal{C}at^{St}}(C_i^c, E)$. Why it is essentially surjective? Any object of $\lim_{I^{op}} \mathrm{Map}_{1-\mathcal{C}at^{St}}(C_i^c, E)$ comes from an exact continuous functor $D \xrightarrow{\epsilon} \mathrm{Ind}(E)$ sending C_i^c to E. Since $D = \mathrm{colim}\, \tilde{C}_I$, the map ϵ is a map in $\mathcal{P}r^L_\omega$, so $\epsilon(D^c) \subset \mathrm{Ind}(E)^c$. However, it is not clear if $\epsilon(D^c) \subset E$, because E may be not idempotent complete.

Part (b) of their Cor. 7.2.7 is not clear.

4.2.9. *The dual of a compactly generated category.* Explanation for 7.3.1. Let $C \in 1-\mathcal{C}at^{St,cocmpl}$. To see that $C^{op} \times C \to \mathrm{Sptr}$, $(c, c') \mapsto \mathcal{M}aps_C(c, c')$ is exact in each variable, by my Lemma 2.2.68 it suffices to show that for any $n \geq 0$ the functor $C^{op} \times C \to \mathrm{Spc}, (c, c') \mapsto \Omega^{\infty-n}\mathcal{M}aps_C(c, c') \widetilde{\to} \mathrm{Map}_C(c, c'[n])$ preserves finite limits in each variable. The latter isomorphism is given in Section 4.2. This claim is clear.

For 7.3.2. Assume $C_0 \in 1-\mathcal{C}at^{St}$ and $C = \mathrm{Ind}(C_0)$. The restriction of the above functor $C_0^{op} \times C \to \mathrm{Sptr}$ is exact in each variable, and continuous with respect to the second variable, because $C_0 \subset C^c$. So, this is the left Kan extension of its restriction to $C_0^{op} \times C_0$. The corresponding functor $C \to \mathrm{Fun}_{ex}(C_0^{op}, \mathrm{Sptr})$ is exact and continuous (actually equivalence). So, their functor $\mathrm{Ind}(C_0^{op}) \times C \to \mathrm{Sptr}$ can be seen as the LKE of its restriction $C_0^{op} \times C \to \mathrm{Sptr}$ given by $(c, c') \mapsto \mathcal{M}aps_C(c, c')$. One more way, the corresponding functor $C_0^{op} \to \mathrm{Fun}_{ex,cont}(C, \mathrm{Sptr})$ is exact, so extends unquely to a functor $\mathrm{Ind}(C_0^{op}) \to \mathrm{Fun}_{ex,cont}(C, \mathrm{Sptr})$ by ([27], 5.3.5.10), which is exact and continuous. Actually, the latter is the identity functor (after the identification $\mathrm{Fun}_{ex,cont}(C, \mathrm{Sptr}) \widetilde{\to} \mathrm{Fun}_{ex}(C_0, \mathrm{Sptr})$). Its exactness follows from ([27], 5.5.1.9). So, the functor $\mathrm{Ind}(C_0^{op}) \times C \to \mathrm{Sptr}$ is exact and continuous in each variable, hence gives rise to a functor

$$\mathrm{Ind}(C_0^{op}) \otimes C \to \mathrm{Sptr},$$

where the tensor product is for the symmetric monoidal structure on $1-\mathcal{C}at^{St,cocmpl}_{cont}$.

Another way to think about this: the exponent Sptr^C identifies canonically with $\mathrm{Fun}_{ex,cont}(C, \mathrm{Sptr}) \widetilde{\to} \mathrm{Ind}(C_0^{op})$. It remains to show that the natural map $C \otimes \mathrm{Sptr}^C \to \mathrm{Sptr}$ extends to a duality datum on this pair (as in HA, 4.6.1.6).

For $D \in 1-\mathcal{C}at^{St,cocmpl}$ the functor $\mathrm{Ind}(C_0^{op}) \otimes D \to \mathrm{Fun}_{ex,cont}(C, D)$ corresponds to the composition $\mathrm{Ind}(C_0^{op}) \otimes C \otimes D \to \mathrm{Sptr} \otimes D \widetilde{\to} D$ via the fact that $\mathrm{Fun}_{ex,cont}(C, D)$ is the inner hom object in $1-\mathcal{C}at^{St,cocmpl}_{cont}$ (see [14], ch1, Sect. 6.1.7).

Another way to say, in view of the equivalence $\mathrm{Fun}_{ex,cont}(\mathrm{Ind}(C_0),\mathrm{Sptr})\widetilde{\to}\mathrm{Ind}(C_0^{op})$, this is the tautological functor $\mathrm{Fun}_{ex,cont}(\mathrm{Ind}(C_0),\mathrm{Sptr})\otimes D\to\mathrm{Fun}_{ex,cont}(C,D)$ coming from $\mathrm{Sptr}\otimes D\widetilde{\to}D$.

To prove ([14], ch1, Prop. 7.3.2), it suffices indeed to show that for any $D\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, their functor (7.4) induces as equivalence

$$\mathrm{Ind}(C_0^{op})\otimes D\to\mathrm{Fun}_{ex,cont}(C,D),$$

because the RHS is the exponent of D by C in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. I think their proof actually gives an equivalence of categories, not only of the underlying spaces: if $E\in 1-\mathcal{C}\mathrm{at}^{St,cocompl}$ then

$$\mathrm{Fun}_{ex,cont}(\mathrm{Fun}_{ex}(C_0,D),E)\widetilde{\to}\mathrm{Fun}_{ex}(C_0^{op},\mathrm{Fun}_{ex,cont}(D,E))$$

There is a problem in the proof of Pp. 7.3.2: they write

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(E^{op},\mathrm{Fun}_{ex}(C_0,D)^{op})$$

for example. The problem is that if E is presentable, E^{op} does not need to be presentable (it is not in most cases), so their convention of footnote 7 in Sect 5.1.5 is broken, is it?

In the setting of 7.3.4 if $F_0: C_0\to D_0$ is a map in $1-\mathcal{C}\mathrm{at}^{St}$ and $F: C\to D$ is its ind-completion then the right adjoint $F^R: D\to C$ sends each left exact functor $D_0^{op}\to\mathrm{Spc}$ to the composition $C_0^{op}\to D_0^{op}\to\mathrm{Spc}$ by (HTT, 5.3.5.13). I think for this reason (ch. 1, Prop. 7.3.5) is correct, the functor F^R is continuous there (cf. my Sect. 4.2.3). Namely, the functor $F^\vee:\mathrm{Fun}^{Lex}(D_0,\mathrm{Spc})\to\mathrm{Fun}^{Lex}(C_0,\mathrm{Spc})$ is the composition with $C_0\to D_0$. Now its left adjoint is described by (HTT, 5.3.5.13). On the other hand we know from Lemma 4.1.2 that the left adjoint to $F^\vee$ is $(F^R)^\vee$.

4.2.10. **TO MEMORIZE**: for $C,D\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ with C compactly generated we have canonically by (ch. 1, 4.1.5)

$$C^\vee\otimes D\widetilde{\to}\mathrm{Fun}_{ex,cont}(C,D) \tag{8}$$

One also has $\mathrm{Fun}_{ex,cont}(\mathrm{Sptr},D)\widetilde{\to}D$ by ([28], 1.4.4.6).

TO MEMORIZE: let $E,D,C\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ with C compactly generated then one has canonically

$$\mathrm{Fun}_{ex,cont}(E\otimes C,D)\widetilde{\to}\mathrm{Fun}_{ex,cont}(E,C^\vee\otimes D) \tag{9}$$

Proof. the LHS is $\mathrm{Fun}^{bi-L}(E\times C,D)\widetilde{\to}\mathrm{Fun}^L(E,\mathrm{Fun}^L(C,D))\widetilde{\to}\mathrm{Fun}_{ex,cont}(E,C^\vee\otimes D)$. Here L stands for the colimit-preserving functors, and $bi-L$ for colimit preserving functors separately in each variable. □

TO MEMORIZE: if $C,D\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ are compactly generated then the dual to a map $F: C\to D$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ can be obtained by applying the functor $\mathrm{Fun}_{ex,cont}(?,\mathrm{Sptr})$ to F. (actually, this holds for any C,D dualizable).

4.2.11. For 7.4. Recall that the full subcategory $(1-\mathcal{C}at^{St,cocmpl}_{cont})^{dualizable}$ is stable under the tensor product. So, if $C, D \in 1-\mathcal{C}at^{St,cocmpl}$ are compactly generated then $C\otimes D\in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ is dualizable. By (9) and (8),

$$\mathrm{Fun}_{ex,cont}(C^\vee\otimes D^\vee, \mathrm{Sptr})\widetilde{\to}\mathrm{Fun}^L(C^\vee, D)\widetilde{\to}\mathrm{Fun}^L(\mathrm{Sptr}, C\otimes D)\widetilde{\to} C\otimes D,$$

where L stands for the colimit-preserving fucntors. Recall that $\mathrm{Fun}^L(D^\vee, \mathrm{Sptr})\widetilde{\to} D$ canonically. In 7.4 they prove additional properties of the last displayed isomorphism.

Assume $C=\mathrm{Ind}(C_0), D=\mathrm{Ind}(D_0)$ with $C_0, D_0\in 1-\mathcal{C}at^{St}$. The equivalence from 7.4.5 is clear.

To explain a version of Yoneda used in 7.4.6, it seems natural to establish the following.

I think the following should be true. Let $C_0\in 1-\mathcal{C}at^{St}$, $C=\mathrm{Ind}(C_0)\in 1-\mathcal{C}at^{St,cocmpl}$. We have the Yoneda embedding $j: C_0\to \mathrm{Fun}_{ex}(C_0^{op}, \mathrm{Sptr})=C$ sending x to the functor $y\mapsto \mathcal{M}aps_C(y,x)$. For $F\in \mathrm{Fun}_{ex}(C_0^{op}, \mathrm{Sptr})=C$, $c_0\in C_0$ we have $\mathcal{M}aps_C(c_0, F)=F(c_0)$.

4.3. **Question.** Let $A\in AssAlg(1-\mathcal{C}at^{St,cocmpl}_{cont})$, $M\in A-mod^r(1-\mathcal{C}at^{St,cocmpl}_{cont})$, $N\in A-mod(1-\mathcal{C}at^{St,cocmpl}_{cont})$. Can we describe $M\otimes_A N$ as a category of functors $M^{op}\to N$ with some properties? Or maybe instead of functors some suitable "inner hom". I am asking about the analog of ([28], 4.8.1.17) in this setting.

Lemma 4.3.1. *Let $E\in 1-\mathcal{C}at^{St,cocmpl}_{cont}$. The functor $1-\mathcal{C}at^{St,cocmpl}_{cont}\to 1-\mathcal{C}at^{St,cocmpl}_{cont}$, $D\mapsto \mathrm{Fun}_{ex,cont}(E,D)$ preserves small limits.*

Proof. Let $C\in 1-\mathcal{C}at^{St,cocmpl}_{cont}$. Let $I\to 1-\mathcal{C}at^{St,cocmpl}_{cont}$, $i\mapsto D_i$ be a diagram. For brevity in this proof write Map for $\mathrm{Map}_{1-\mathcal{C}at^{St,cocmpl}_{cont}}$. We get

$$\mathrm{Map}(C, \mathrm{Fun}_{ex,cont}(E, \lim_i D_i))\widetilde{\to}\mathrm{Map}(C\otimes E, \lim_i D_i)\widetilde{\to}\lim_i \mathrm{Map}(C\otimes E, D_i)\widetilde{\to}$$
$$\lim_i \mathrm{Map}(C, \mathrm{Fun}_{ex,cont}(E, D_i))\widetilde{\to}\mathrm{Map}(C, \lim_i \mathrm{Fun}_{ex,cont}(E, D_i))$$

□

4.3.2. Given $A, C\in 1-\mathcal{C}at^{St,cocmpl}_{cont}$, we have the natural map $C\otimes \mathrm{Fun}_{ex,cont}(A, \mathrm{Sptr})\to \mathrm{Fun}_{ex,cont}(A, C)$. It is not always an equivalence! Namely, let A be non dualizable. This means by definition that there is $C\in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ such that the natural map $C\otimes \mathrm{Fun}_{ex,cont}(A,\mathrm{Sptr})\otimes A\to C$ does not realize $C\otimes\mathrm{Fun}_{ex,cont}(A,\mathrm{Sptr})$ as the exponential $\mathrm{Fun}_{ex,cont}(A, C)$, see my Sect. 3.1.

4.3.3. Let $C\in 1-\mathcal{C}at^{St,cocompl}$. Assume for $i\in I$, $K_i\to L_i\to M_i$ is a fibre sequence in C. Then $\oplus_{i\in I}K_i\to \oplus_{i\in I}L_i\to\oplus_{i\in I}M_i$ is also a fibre sequence.

Assume C is equipped with an accessible t-structure, $L_i\in C^{\le 0}$ for $i\in I$. We claim that for $n\le 0$ one has $\mathrm{H}^n(\oplus_{i\in I}L_i)\widetilde{\to}\oplus_{i\in I}\mathrm{H}^n(L_i)$.

4.3.4. If $C\in 1-\mathcal{C}at$ is stable the forgetful functor $\mathrm{Mon}(C)\to C$ is an equivalence. (Indeed, for any $\mathcal{D}\in 1-\mathcal{C}at$ admitting finite products, $Mon(ComMon(\mathcal{D})\to ComMon(\mathcal{D})$ is an equivalence). Moreover $Grp(C)\to \mathrm{Mon}(C)$ is an equivalence (see my Section 9.5.7).

4.3.5. If C, D are stable categories with t-structures, let $f : C \to D$ be an exact functor, which is t-exact. Let $\hat{C}, \hat{D}$ be the left completions. Then the induced functor $\hat{f} : \hat{C} \to \hat{D}$ is also exact.

Indeed, let us use the cohomological indexing conventions. For each n, $\tau^{\geq n} : \hat{C} \to C^{\geq n}$ preserves colimits, same for D. So, it suffices to show that for $n \in \mathbb{Z}$ the functor $f : C^{\geq n} \to D^{\geq n}$ preserves finite colimits. Let $I \to C^{\geq n}$ be a finite diagram, then its colimit in $C^{\geq n}$ is $\tau^{\geq n}(\operatorname{colim}_i c_i)$, where $\operatorname{colim}_i c_i$ is calculated in C. We get $f(\tau^{\geq n}(\operatorname{colim}_i c_i)) \tilde{\to} \tau^{\geq n} f(\operatorname{colim}_i c_i) \tilde{\to} \tau^{\geq n}(\operatorname{colim}_i f(c_i))$. Since $\tau^{\geq n}(\operatorname{colim}_i f(c_i))$ is the colimit of the diagram $i \mapsto f(c_i)$ in $D^{\geq n}$, we are done.

5. For tensor product of ∞-categories

5.1. As in ([28], 4.8.1.1) let $\mathcal{K}$ be a collection of small ∞-categories, $\mathcal{C}at_\infty(\mathcal{K}) \subset 1-\mathcal{C}at$ be the 1-full subcategory of those categories that admit $\mathcal{K}$-indexed colimits and those functors which preserve $\mathcal{K}$-indexed colimits.

If $\mathcal{C}, \mathcal{D} \in \mathcal{C}at_\infty(\mathcal{K})$ let $\mathcal{E} \subset \operatorname{Funct}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by functors preserving $\mathcal{K}$-indexed colimits. Then $\mathcal{E} \in \mathcal{C}at_\infty(\mathcal{K})$ and the inclusion $\mathcal{E} \subset \operatorname{Funct}(\mathcal{C}, \mathcal{D})$ is stable under $\mathcal{K}$-indexed colimits.

([28], 4.8.1.6) is clear and could be strengthen as follows I think: (the full subcategory of $\operatorname{Funct}(\mathcal{T}, \mathcal{E})$ spanned by functors preserving $\mathcal{K}$-indexed colimits) equals the full subcategory of $\operatorname{Funct}(\mathcal{C} \times \mathcal{T}, \mathcal{D})$ spanned by functors which preserve $\mathcal{K}$-indexed colimits separately in each variable.

Given $\mathcal{C}_i \in \mathcal{C}at_\infty(\mathcal{K})$ the tensor product $\mathcal{C}_1 \otimes \ldots \mathcal{C}_n \in \mathcal{C}at_\infty(\mathcal{K})$ in $\mathcal{C}at_\infty(\mathcal{K})$ defined in ([28], 4.8.1.4) is the category $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n)$ in the notations of the proof of ([28], 4.8.1.4). That is, it satisfies the following universal property from ([27], 5.3.6.2). There is a functor $j : \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \to \mathcal{C}_1 \otimes \ldots \mathcal{C}_n$, the category $\mathcal{C}_1 \otimes \ldots \otimes \mathcal{C}_n$ admits all $\mathcal{K}$-indexed colimits, that is, lies in $\mathcal{C}at_\infty(\mathcal{K})$. For any $\mathcal{D} \in \mathcal{C}at_\infty(\mathcal{K})$, composition with j induces an equivalence

$$\operatorname{Funct}_{\mathcal{K}}(\mathcal{C}_1 \otimes \ldots \otimes \mathcal{C}_n, \mathcal{D}) \tilde{\to} \operatorname{Funct}_{\mathcal{R}}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n, \mathcal{D})$$

Here the subscript $\mathcal{K}$ means that we take the full subcategory spanned by functors preserving $\mathcal{K}$-indexed colimits, and the subscript $\mathcal{R}$ means that we take the full subcategory of functors preserving $\mathcal{K}$-indexed colimits separately in each variable.

Let $1 - \mathcal{C}at^\times \to \mathcal{F}in_*$ be the cartesian monoidal structure on $1 - \mathcal{C}at$, $\mathcal{C}at_\infty(\mathcal{K})^\otimes \to \mathcal{F}in_*$ the symetric monoidal category defined in ([28], 4.8.1.4). The inclusion $\mathcal{C}at_\infty(\mathcal{K})^\otimes \to 1 - \mathcal{C}at^\times$ is right-lax monoidal ([28], 4.8.1.4).

5.1.1. Explanation for ([28], 4.8.1.9). Let $\mathcal{K} \subset \mathcal{K}'$ be collections of simplicial sets (∞-categories). Then the inclusion $\mathcal{C}at_\infty(\mathcal{K}')^\otimes \subset \mathcal{C}at_\infty(\mathcal{K})^\otimes$ is a right-lax monoidal functor. Indeed, for each inert map $\rho^i : \langle n \rangle \to \langle 1 \rangle$, the image of the cocartesian arrow $\oplus_{j=1}^n \mathcal{C}_j \to \mathcal{C}_i$ is the same arrow $\oplus_{j=1}^n \mathcal{C}_j \to \mathcal{C}_i$. On the other hand, in ([28], 4.8.1.8) we have obtained a symmetric monoidal functor $\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}} : \mathcal{C}at_\infty(\mathcal{K})^\otimes \to \mathcal{C}at_\infty(\mathcal{K}')^\otimes$. From ([27], 5.3.6.2) it follows that $\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}} : \mathcal{C}at_\infty(\mathcal{K}) \to \mathcal{C}at_\infty(\mathcal{K}')$ is left adjoint to the inclusion $\mathcal{C}at_\infty(\mathcal{K}') \subset \mathcal{C}at_\infty(\mathcal{K})$.

Note that if $\mathcal{D} \in \mathcal{C}at_\infty(\mathcal{K}'), \mathcal{Y} \in \mathcal{C}at_\infty(\mathcal{K})$ then $\operatorname{Funct}_{\mathcal{K}}(\mathcal{Y}, \mathcal{D}) \in \mathcal{C}at_\infty(\mathcal{K}')$.

Lemma 5.1.2. *The above functor* $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}: \mathcal{C}\mathrm{at}_\infty(\mathcal{K})^\otimes \to \mathcal{C}\mathrm{at}_\infty(\mathcal{K}')^\otimes$ *is left adjoint to the inclusion* $\mathcal{C}\mathrm{at}_\infty(\mathcal{K}')^\otimes \subset \mathcal{C}\mathrm{at}_\infty(\mathcal{K})^\otimes$.

Proof. Let us verify the assumptions of ([28], 2.2.1.9). Write L for the functor $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}: \mathcal{C}\mathrm{at}_\infty(\mathcal{K}) \to \mathcal{C}\mathrm{at}_\infty(\mathcal{K}')$. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism in $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ such that $L(\mathcal{X}) \to L(\mathcal{Y})$ is an equivalence in $\mathcal{C}\mathrm{at}_\infty(\mathcal{K}')$, let $\mathcal{Z} \in \mathcal{C}\mathrm{at}_\infty(\mathcal{K})$. We must show that $L(\mathcal{X} \otimes \mathcal{Z}) \to L(\mathcal{Y} \otimes \mathcal{Z})$ is an equivalence in $\mathcal{C}\mathrm{at}_\infty(\mathcal{K}')$. Here the tensor product is taken in $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$. It suffices to show that for any $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty(\mathcal{K}')$ the restriction

$$\mathrm{Funct}_{\mathcal{K}'}(L(\mathcal{Y} \otimes \mathcal{Z}), \mathcal{D}) \to \mathrm{Funct}_{\mathcal{K}'}(L(\mathcal{X} \otimes \mathcal{Z}), \mathcal{D})$$

is an equivalence. By ([27], 5.3.6.2), the latter map identifies with

$$\mathrm{Funct}_{\mathcal{K}}(\mathcal{Y} \otimes \mathcal{Z}, \mathcal{D}) \to \mathrm{Funct}_{\mathcal{K}}(\mathcal{X} \otimes \mathcal{Z}, \mathcal{D})$$

which in turn identifies with

$$\mathrm{Funct}_{\mathcal{K}}(\mathcal{Y}, \mathrm{Funct}_{\mathcal{K}}(\mathcal{Z}, \mathcal{D})) \to \mathrm{Funct}_{\mathcal{K}}(\mathcal{X}, \mathrm{Funct}_{\mathcal{K}}(\mathcal{Z}, \mathcal{D}))$$

and in turn with

$$\mathrm{Funct}_{\mathcal{K}'}(L(\mathcal{Y}), \mathrm{Funct}_{\mathcal{K}}(\mathcal{Z}, \mathcal{D})) \to \mathrm{Funct}_{\mathcal{K}}(L(\mathcal{X}), \mathrm{Funct}_{\mathcal{K}}(\mathcal{Z}, \mathcal{D}))$$

Our result follows now from ([28], 2.2.1.9). □

My understanding is that under the assumptions of ([28], 2.2.1.9) if $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ is a morphism of ∞-operads then the following holds. Let $\mathcal{L}: \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ be the composition with $L^\otimes$, let $\mathcal{R}: \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ be the composition with the inclusion $\mathcal{D}^\otimes \subset \mathcal{C}^\otimes$. Then $\mathcal{L}$ is left adjoint to $\mathcal{R}$ (confirmed by Jacob in an email).

5.1.3. (HA, 4.8.1.10) says the following. Let $\mathcal{K} \subset \mathcal{K}'$ be collections of simplicial sets, $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration of ∞-operads such that the $\mathcal{O}$-monoidal structure on $\mathcal{C}$ is compatible with $\mathcal{K}$-indexed colimits. For every $X \in \mathcal{O}$ consider $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C}_X)$. As X varies in $\mathcal{O}$ these categories form a category $\mathcal{D}$, which is naturally a $\mathcal{O}$-monoidal category (its monoidal structure is compatible with $\mathcal{K}'$-indexed collimits). The natural functor $\mathcal{C}_X \to \mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C}_X)$ extends to a $\mathcal{O}$-monoidal functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$.

5.1.4. For (HA, 4.8.1.14). I think there is a misprint in the formulation. Namely, at the end the arrow $\mathrm{Fun}^\otimes(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ should be replaced by $\mathrm{Fun}^\otimes(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^\otimes(\mathcal{C}, \mathcal{D})$. Indeed, the restriction of a symetric monoidal functor via $\mathcal{C}^\otimes \xrightarrow{a} \mathrm{Ind}(\mathcal{C})^\otimes$ will be symmetric monoidal, because a itself is symmetric monoidal.

5.1.5. For (HA, 4.8.1.8): let $\mathcal{K}$ be the collection of all small simplicial sets. The functor $\hat{\mathcal{C}\mathrm{at}}_\infty \to \hat{\mathcal{C}\mathrm{at}}_\infty(\mathcal{K})$, $\mathcal{C} \mapsto \mathcal{P}(\mathcal{C})$ is symmetric monoidal. In particular, Spc is the unit object of $\hat{\mathcal{C}\mathrm{at}}_\infty(\mathcal{K})$, and $\mathcal{P}(\mathcal{C}_1) \otimes \mathcal{P}(\mathcal{C}_2) \tilde{\to} \mathcal{P}(\mathcal{C}_1 \times \mathcal{C}_2)$.

5.1.6. For ([28], 4.8.1.15). Let $\mathcal{D} \in 1-\mathcal{C}\mathrm{at}$ be presentable. Let $\mathcal{E} \subset \mathrm{Funct}([1], \mathcal{D})$ be the full subcategory spanned by equivalences. Then $\mathcal{E}$ is an accessible localization of $\mathrm{Funct}([1], \mathcal{D})$. Indeed, the functor $\mathcal{E} \to \mathcal{D}$ sending an arrow to its end is an equivalence, hence $\mathcal{E}$ is presentable. This inclusion clearly admits a left adjoint.

5.1.7. For ([28], 4.8.1.20). Let $\mathcal{C}, \mathcal{D}$ be presentable, then $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}_*) \widetilde{\to} \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})_*$ canonically. This yields an equivalence $\mathrm{Funct}^R(\mathcal{C}^{op}, \mathcal{D}_*) \widetilde{\to} \mathrm{Funct}^R(\mathcal{C}^{op}, \mathcal{D})_*$ of full subcategories. Indeed, for $F : \mathcal{C}^{op} \to \mathcal{D}_*$ the condition of being limit preserving is equivalent to the property that the composition $\mathcal{C}^{op} \xrightarrow{F} \mathcal{D}_* \to \mathcal{D}$ is limit preserving.

5.1.8. For ([28], 4.8.1.16). Let $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}at$ be presentable, $f : \mathcal{D}^{op} \to \mathcal{C}$ be a limit preserving functor. Then f has a right adjoint. In other words, $\mathrm{Funct}^R(\mathcal{D}^{op}, \mathcal{C}) \subset \mathrm{Funct}(\mathcal{D}^{op}, \mathcal{C})$ is the full subcategory of limit preserving functors.

Indeed, $f^{op} : \mathcal{D} \to \mathcal{C}^{op}$ is colimit preserving. Since $\mathcal{D}$ is presentable, from ([27], 5.5.2.9 and 5.5.2.10) we see that f^{op} is a left adjoint. So, f is a right adjoint.

This is used in the proof of ([28], 4.8.1.23): the functor $\otimes : \mathcal{P}\mathrm{r}^L \times \mathcal{P}\mathrm{r}^L \to \mathcal{P}\mathrm{r}^L$ preserves small colimits separately in each variable.

Proof. Let $f : I \to \mathcal{P}\mathrm{r}^L$, $i \mapsto C_i$ be a diagram, I small, $\mathcal{C} = \mathrm{colim}\, f$. Let $f^R : I^{op} \to \mathcal{P}\mathrm{r}^R$ be obtained from f by passing to the right adjoints, let $\bar{f}^R$ be the composition $I^{op} \xrightarrow{f^R} \mathcal{P}\mathrm{r}^R \hookrightarrow 1-\mathcal{C}at$. Recall that $\mathcal{C} = \lim \bar{f}^R$. Let $\mathcal{D} \in \mathcal{P}\mathrm{r}^L$. Then $\mathcal{D} \otimes \mathcal{C} \widetilde{\to} \mathrm{Funct}^R(\mathcal{D}^{op}, \mathcal{C})$. We have $\mathrm{Funct}(\mathcal{D}^{op}, \mathcal{C}) \widetilde{\to} \lim_{i \in I^{op}} \mathrm{Funct}(\mathcal{D}^{op}, \mathcal{C}_i)$. In the projective system $\bar{f}^R : I^{op} \to 1-\mathcal{C}at$ all the transition functors are limit preserving, so our Lemma 2.2.69 applies. For $\mathcal{F} \in \mathrm{Funct}(\mathcal{D}^{op}, \mathcal{C})$ the condition of being limit preserving is equivalent to being right adjoint, in turn it is equivalent to the property that each $\mathcal{D}^{op} \to \mathcal{C}_i$ is limit preserving, that is, each $\mathcal{D}^{op} \to \mathcal{C}_i$ is right adjoint. So, the above equivalence restricts to an equivalence of full subcategories $\mathrm{Funct}^R(\mathcal{D}^{op}, \mathcal{C}) \widetilde{\to} \lim_{i \in I^{op}} \mathrm{Funct}^R(\mathcal{D}^{op}, \mathcal{C}_i)$. The latter identifies with $\lim_{i \in I^{op}}(\mathcal{D} \otimes \mathcal{C}_i) \widetilde{\to} \mathrm{colim}_{i \in I}(\mathcal{D} \otimes \mathcal{C}_i)$, where the colimit is taken in $\mathcal{P}\mathrm{r}^L$. □

If $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathcal{P}\mathrm{r}^L$ then $\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \widetilde{\to} \mathrm{Fun}^L(\mathcal{C}, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E}))$ (cf. HA, proof of 4.8.1.17). Here $\mathrm{Fun}^L(\mathcal{D}, \mathcal{E})) \subset \mathrm{Fun}(\mathcal{D}, \mathcal{E})$ is the full subcategory of colimit-preserving functors (equivalently, left adjoints). The tensor product here is in $\mathcal{P}\mathrm{r}^L$, cf. Sect. 4.0.38.

5.1.9. Question: recall that $[n] \in \mathcal{P}\mathrm{r}^L$. Let $\mathcal{C} \in \mathcal{P}\mathrm{r}^L$, what is $\mathcal{C} \otimes [n] \widetilde{\to} \mathrm{Fun}^R([n]^{op}, \mathcal{C})$?

6. Algebra in stable categories

6.0.1. For 8.1.3: $oblv_A : A-mod^{St,cocmpl}_{cont} \to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ preserves colimits (and even reflects colimits), and category $A-mod^{St,cocmpl}_{cont}$ admits all limits and colimits as explained in my Section 3.0.53 and (HA, 4.2.3.5).

Let $A \in AssAlg(1-\mathcal{C}at^{St,cocmpl}_{cont})$. For $D \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ consider the functor $(A-mod^r)^{St,cocmpl}_{cont} \to (A-mod^r)^{St,cocmpl}_{cont}$, $M \mapsto D \otimes M$. It preserves colimits, because $oblv_A : (A-mod^r)^{St,cocmpl}_{cont} \to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ reflects colimits.

In (ch. 1, 8.2.1) they mean that $M, N \in (A-mod^r)^{St,cocmpl}_{cont}$, and view $(A-mod^r)^{St,cocmpl}_{cont}$ as a module over $1-\mathcal{C}at^{St,cocmpl}_{cont}$ by tensor product on the left. Then the action functor $1-\mathcal{C}at^{St,cocmpl}_{cont} \times (A-mod^r)^{St,cocmpl}_{cont} \to (A-mod^r)^{St,cocmpl}_{cont}$ preserves colimits separately in each variable.

Sam claims for $M, N \in (A-mod^r)^{St,cocmpl}_{cont}$ the inner hom $\mathrm{Fun}_A(M, N) \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ always exists, it is calculated as some totalization of functor categories.

The difficulty here is that $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ is not presentable, so it is not garanteed that a functor $(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})^{op}\to \mathrm{Spc}$ preserving limits is representable. In practice, the corresponding representing object is usually constructed by hands.

If $M,N\in A-mod^{St,cocmpl}_{cont}$ then $\mathrm{Fun}_A(M,N)\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ is also defined as a relative inner hom. Namely, by the universal property: for $D\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, $\mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(D,\mathrm{Fun}_A(M,N))\widetilde{\to}\mathrm{Map}_{A-mod^{St,cocmpl}_{cont}}(M\otimes D,N)$.

There is a natural map $\mathrm{Fun}_A(M,N)\to \mathrm{LinFun}_A(M,N)$, I think in general it is not an isomorphism: for example, for $M,N\in \mathrm{DGCat}_{cont}$ and $A=\mathrm{Vect}$, an object of $\mathrm{LinFun}_A(M,N)$ is a functor, which is not necessarily continuous, it could be a map in DGCat with additional properties I think.

6.0.2. For $M,N\in A-mod^{St,cocmpl}_{cont}$ we have the natural functor $\mathrm{Fun}_A(M,N)\to \mathrm{Fun}_{ex,cont}(M,N)$. Indeed, for any $D\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ we get a diagram

$$\begin{array}{ccc}\mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(D,\mathrm{Fun}_A(M,N)) & \widetilde{\to} & \mathrm{Map}_{A-mod^{St,cocmpl}_{cont}}(M\otimes D,N)\\ \downarrow & & \downarrow\\ \mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(D,\mathrm{Fun}_{ex,cont}(M,N)) & \widetilde{\to} & \mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(M\otimes D,N),\end{array}$$

where the right vertical arrow comes from the forgetful functor $A-mod^{St,cocmpl}_{cont}\to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Is it a full subcategory?

6.0.3. (ch. 1, 8.2.2) follows from (HA, 4.2.4.6), see my Section 3.0.62. Namely, for $X\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}, M\in A-mod^r(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$, one has

$$\mathrm{Map}_{A-mod^r(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})}(X\otimes A,M)\widetilde{\to}\mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(X,M)$$

6.0.4. In (ch. 1, 8.4.1), $F^R: N\to M$ is a right-lax functor between A-module categories lying in $A-mod^{St,cocmpl}_{cont}$. They mean that F^R is strict if F^R is a map in $A-mod^{St,cocmpl}_{cont}$.

6.0.5. If $A\in AssAlg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ then $A-mod^{St,cocmpl}_{cont}$ admits all limits and colimits, this follows from my Section 3.0.53.

In (ch. 1, 8.4.2) they consider a functor $C_I: I\to A-mod^{St,cocmpl}_{cont}$ for some associative algebra A in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. If for any map $i\to j$ in I the right adjoint to $C_i\to C_j$ is a map in $A-mod^{St,cocmpl}_{cont}$, one gets a functor $C^R_{I^{op}}: I^{op}\to A-mod^{St,cocmpl}_{cont}$. Let now $C_*=\mathrm{colim}_I\, C_I\in A-mod^{St,cocmpl}_{cont}$, let $\bar C=\lim_{I^{op}} C^R_{I^{op}}$ in $A-mod^{St,cocmpl}_{cont}$. It has to be explaned that the right adjoint to each $C_i\to C_*$ is a not only a right-lax functor of A-module categories, but it is strict.

The projection $A-mod^{St,cocmpl}_{cont}\to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ preserves limits and colimits, so $C_*\widetilde{\to}\mathrm{colim}_{i\in I}\, C_i$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, and $\bar C=\lim_{i\in I^{op}} C_i$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Recall also that $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}\to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ preserves limits, and the natural map $C_*\to\bar C$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ is an isomorphism by (ch. 1, 5.3.4). So, the map $C_*\to\bar C$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ is also an isomorphism. So, the right adjoint to $C_i\to C_*$ is the projection $\bar C\to C_i$, hence this is a morphism in $A-mod^{St,cocmpl}_{cont}$. So, in the whole colimit diagram $C_I^{\triangleright}: I^{\triangleright}\to A-mod^{St,cocmpl}_{cont}$ we can pass to right adjoints in the $(\infty,2)$-category $A-mod^{St,cocmpl}_{cont}$.

6.0.6. Recall that if $A\in CAlg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ then $A-mod^{St,cocmpl}_{cont}$ is naturally a 2-category ([14], ch. I.1, 8.3). Let $A\to B$ be a map in $CAlg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$. The functor $A-mod^{St,cocmpl}_{cont}\to B-mod^{St,cocmpl}_{cont}$, $M\mapsto M\otimes_A B$ is a map in $2-\mathcal{C}\mathrm{at}$?

Given $M_i\in A-mod^{St,cocmpl}_{cont}$, the natural map $\mathrm{Fun}_A(M_1,M_2)\otimes_A M_1\to M_2$ in $A-mod^{St,cocmpl}_{cont}$ yields after extension of scalars by $A\to B$ a map

$$(\mathrm{Fun}_A(M_1,M_2)\otimes_A B)\otimes_B(M_1\otimes_A B)\to M_2\otimes_A B$$

Here by Fun_A we mean the inner hom in $A-mod^{St,cocmpl}_{cont}$. By definition of the inner hom, it gives a map

$$\mathrm{Fun}_A(M_1,M_2)\otimes_A B\to \mathrm{Fun}_B(M_1\otimes_A B, M_2\otimes_A B)$$

in $B-mod^{St,cocmpl}_{cont}$. This is why the answer is yes, as we have the corresponding morphisms of the mapping categories.

Consider the forgetful functor $B-mod^{St,cocmpl}_{cont}\to A-mod^{St,cocmpl}_{cont}$. We claim this is a morphism of 2-categories also. Indeed, given $N,N'\in B-mod^{St,cocmpl}_{cont}$, we have canonical morphisms

$$\mathrm{Fun}_B(N,N')\otimes_A N\to \mathrm{Fun}_B(N,N')\otimes_B N\to N',$$

the first is a morphism in $A-mod^{St,cocmpl}_{cont}$, the second in $B-mod^{St,cocmpl}_{cont}$. This gives the desired map

$$\mathrm{Fun}_B(N,N')\to \mathrm{Fun}_A(N,N')$$

in $A-mod^{St,cocmpl}_{cont}$. It should be compatible with compositions. So, the adjoint pair

$$A-mod^{St,cocmpl}_{cont}\leftrightarrows B-mod^{St,cocmpl}_{cont}$$

is a diagram in $2-Cat$.

Assume in addition that A,B are rigid, then B is dualizable in $A-mod^{St,cocmpl}_{cont}$. So, the forgetful functor $B-mod^{St,cocmpl}_{cont}\to A-mod^{St,cocmpl}_{cont}$ has a right adjoint by my Section 3.2. In this case we claim that for $M\in B-mod^{St,cocompl}_{cont}, N\in A-mod^{St,cocmpl}_{cont}$ one has canonically

$$\mathrm{Fun}_A(M,N)\widetilde{\to}\mathrm{Fun}_B(M,N\otimes_A B)$$

in $A-mod^{St,cocmpl}_{cont}$. Since the forgetful functor $A-mod^{St,cocmpl}_{cont}\to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ is conservative, this follows from isomorphisms for any $D\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(D,\mathrm{Fun}_A(M,N))\widetilde{\to}\mathrm{Map}_{A-mod^{St,cocmpl}_{cont}}(M\otimes D,N)\widetilde{\to}$$
$$\mathrm{Map}_{B-mod^{St,cocmpl}_{cont}}(M\otimes D,N\otimes_A B)\widetilde{\to}\mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(D,\mathrm{Fun}_B(M,N\otimes_A B))$$

Now given $M_i\in A-mod^{St,cocmpl}_{cont}$, we get a map $\mathrm{Fun}_A(M_1,M_2)\to\mathrm{Fun}_A(M_1\otimes_A B,M_2)$ in $A-mod^{St,cocmpl}_{cont}$.

We claim now that for $C,C'\in A-mod^{St,cocmpl}_{cont}$, one has canonically

$$\mathrm{Fun}_B(C\otimes_A B,C'\otimes_A B)\widetilde{\to}\mathrm{Fun}_A(C,C')\otimes_A B$$

Indeed, we get by the above

$$\mathrm{Fun}_A(C,C')\otimes_A B\widetilde{\to}\mathrm{Fun}_B(B,\mathrm{Fun}_A(C,C')\otimes_A B)\widetilde{\to}\mathrm{Fun}_A(B,\mathrm{Fun}_A(C,C'))$$
$$\widetilde{\to}\mathrm{Fun}_A(B\otimes_A C,C')\widetilde{\to}\mathrm{Fun}_B(B\otimes_A C,B\otimes_A C')$$

We used the fact that for $X\in A-mod$, $M\in B-mod$, $\mathrm{Fun}_B(X\otimes_A B,M)\widetilde{\to}\mathrm{Fun}_A(X,M)$ canonically.

6.0.7. For (ch. 1, 8.4.4). If $A\in Alg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$, $C\in A-mod^{St,cocmpl}_{cont}$, and $\mathcal{B}$ is an algebra in $\mathbf{Map}_{A-mod^{St,cocmpl}_{cont}}(C,C)=\mathrm{Fun}_A(C,C)$ then it is nontrivial that $\mathcal{B}-mod(C)$ has a structure of an object of $A-mod^{St,cocmpl}_{cont}$, what is the reference?

6.0.8. For (ch. 1, 8.5.2). $Alg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ has a structure of a symmetric monoidal $(\infty,1)$-category by (ch. 1, 3.3.5), so $A_1\otimes A_1\in Alg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$. The projection $Alg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})\to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ is symmetric monoidal, see my Section 3.0.31. Since $AssAlg+mod(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ is symmetric monoidal by (ch. 1, 4.2.2), given $(A_1,M_2),(A_2,M_2)\in AssAlg+mod(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$, $M_1\otimes M_2$ is a $A_1\otimes A_2$-module.

Note that $1-\mathcal{C}\mathrm{at}^{Mon}$ from (ch. 1, 3.1.4) is a symmetric monoidal category. This is in fact a corollary of Proposition 3.0.31. So, for two monoidal categories given by functors $F,G:\boldsymbol{\Delta}^{op}\to 1-\mathcal{C}\mathrm{at}$ the functor $[n]\mapsto F_n\times G_n$ is also a monoidal category.

Now if $A_1,A_2\in Alg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ then the natural functor $A_1\times A_2\to A_1\otimes A_2$ is monoidal. Now given $\mathcal{A}_i\in Alg(A_i)$, $(\mathcal{A}_1,\mathcal{A}_2)\in \mathrm{Alg}(A_1\times A_2)$ naturally, hence its image $\mathcal{A}_1\boxtimes\mathcal{A}_2\in A_1\otimes A_2$ is an associative algebra.

For (ch. 1, 8.5.4). Recall the adjoint pair $\mathrm{ind}_{\mathcal{A}_i}:M_i\leftrightarrows\mathcal{A}_i-mod(A_i):oblv_{\mathcal{A}_i}$ for $i=1,2$. We see that the functor $G^L:=\mathrm{ind}_{\mathcal{A}_1}\otimes\mathrm{ind}_{\mathcal{A}_2}:M_1\otimes_A M_2\to\mathcal{A}_1-mod(M_1)\otimes_A\mathcal{A}_2-mod(M_2)$ is left adjoint to the forgetful functor $G:\mathcal{A}_1-mod(M_1)\otimes_A\mathcal{A}_2-mod(M_2)\to M_1\otimes_A M_2$. Compare with the idea from my Section 4.1.6. To prove (ch. 1, Prop. 8.5.4) by (ch. 1, Corollary 5.3.8) it remains to show that the functor G does not send a nonzero object to zero. By my Lemma 4.0.30, this is equivalent to requiring that G is conservative. Now the proof is finished as in (ch. 1, Lemma 6.4.5).

Let A be a monoidal $(\infty,1)$-category and $\mathcal{A}\in Alg(A)$. Then by definition $\mathcal{A}-mod^r(A)\widetilde{\to}\mathcal{A}^{rev-mult}-mod(A^{rev-mult})$. In (ch. 1, 8.5.8) misprints, they take

$$(A_1,\mathcal{A}_1,M_1)=(A,\mathcal{A},A),\qquad (A_2,\mathcal{A}_2,M_2)=(A^{rev-mult},\mathcal{A}^{rev-mult},A^{rev-mult})$$

(ch. 1, Cor 8.5.9) reads

$$\mathcal{A}-mod\otimes_A\mathcal{A}-mod^r\widetilde{\to}\mathcal{A}-mod(A)\otimes_A\mathcal{A}^{rm}-mod(A^{rm})\widetilde{\to}(\mathcal{A}\boxtimes\mathcal{A}^{rm})-mod(A\otimes_A A^{rm})$$

We have an isomorphism $A\otimes_A A^{rm}\widetilde{\to}A$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Viewing A as a $A\otimes A^{rev-mult}$-module, where A acts by multiplication on the left and $A^{rev-mult}$ on the right, we get $\mathcal{A}\boxtimes\mathcal{A}^{rev-mult}\in\mathrm{Alg}(A\otimes A^{rev-mult})$. By one of the definitions of a bimodule, $(\mathcal{A}\boxtimes\mathcal{A}^{rm})-mod(A)$ is the category of $\mathcal{A}-\mathcal{A}$-bimodules in A.

6.0.9. Precision for (ch. 1, 8.5.10). Given a stable monoidal category A, the functor $Alg(A)^{op} \to 1-\mathcal{C}\mathrm{at}$, $\mathcal{A} \mapsto \mathcal{A}-mod$ lifts naturally to a functor $Alg(A)^{op} \to A^{rev-mult}-mod^{St,cocmpl}_{cont}$. Namely, if $\mathcal{A} \in Alg(A)$ then $\mathcal{A}-mod$ is stable cocomplete, and the action map $(\mathcal{A}-mod) \times A \to \mathcal{A}-mod$ given by the tensor product is exact and continuous in each variable. Moreover, if $\mathcal{A} \to \mathcal{B}$ is a map in $Alg(A)$ then the restriction functor $\mathcal{B}-mod \to \mathcal{A}-mod$ is a map in $A^{rev-mult}-mod^{St,cocmpl}_{cont}$.

If in addition A is stable symmetric monoidal then $A^{rev-mult}$ coincides with A, and the above yields a functor $Alg(A)^{op} \to A-mod^{St,cocmpl}_{cont}$. Moreover, in this case $Alg(A)$ is a symmetric monoidal ∞-category, and the above functor is symmetric monoidal: given $\mathcal{A}_i \in Alg(A)$, one has by (ch. 1, 8.5.4)

$$\mathcal{A}_1-mod \otimes_A (\mathcal{A}_2-mod) \tilde{\to} (\mathcal{A}_1 \otimes \mathcal{A}_2)-mod$$

6.0.10. In (ch. 1, 8.6.3) the assumption is $\mathcal{A} \in Alg(A)$.

In (ch. 1, 8.6.4) the assumptions: $M \in A-mod^{St,cocmpl}_{cont}$, $A \in Alg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$, and $\mathcal{A} \in Alg(A)$. To prove this Corollary 8.6.4, taking into account the equivalence $\mathcal{A}-mod \otimes_A M \tilde{\to} \mathcal{A}-mod(M)$ of (ch. 1, 8.5.7), one needs to establish an isomorphism functorial in $D \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}}(D, \mathcal{A}-mod \otimes_A M) \tilde{\to} \mathrm{Map}_{A-mod^{St,cocmpl}_{cont}}((\mathcal{A}-mod^r(A)) \otimes D, M)$$

This is done as in (ch. 1, 4.3.2). Namely, the above map sends a functor $\alpha : D \to \mathcal{A}-mod \otimes_A M$ to the composition

$$(\mathcal{A}-mod^r(A)) \otimes D \stackrel{\mathrm{id}\otimes\alpha}{\to} (\mathcal{A}-mod^r(A)) \otimes (\mathcal{A}-mod \otimes_A M) \stackrel{counit\otimes\mathrm{id}}{\to} A \otimes_A M = M$$

Another way: (ch. 1, 8.6.4) follows from ([28], 4.8.4.1).

6.0.11. In their (ch. 1, Prop. 8.7.2) an assumption is missing. One needs to assume that the unit functor $\mathrm{Sptr} \to A$ admits a continuous right adjoint. Here A is an algebra in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$.

6.0.12. Proof of (ch. 1, 8.7.4). From (ch. 1, 8.7.2) we see that $N \otimes M \to N \otimes_A M$ sends compact objects to compact ones. Besides, $N \otimes M$ is compactly generated. By (ch. 1, Lemma 8.2.6), the essential image of $N \otimes M \to N \otimes_A M$ generates the target. Now $N \otimes_A M$ is compactly generated by remark below.

Remark 6.0.13. *Given $A, B \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}$ with A compactly generated let $f : A \to B$ be a map in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ sending compact objects to compact ones. Assume the essential image of f generates B. Then $f(A^c)$ generates B. So, B is compactly generated.*

Proof. $\mathrm{Ind}(A^c) \tilde{\to} A$, so each b in the essenitial image $\mathrm{Im}(f)$ of f writes as a filtered colimit of objects from $f(A^c)$. If $B' \subset B$ is a cocomplete stable subcategory containing B^c then $\mathrm{Im}(f) \subset B'$, because $f(A^c) \subset B^c$. So, $B' = B$ by (ch. 1, 5.4.5). □

6.0.14. For (ch. 1, 8.8.4) Note that for any $m, m' \in M$ the relative inner hom $\underline{\mathrm{Hom}}_A(m, m')$ exists, because A is presentable. Proof of part (a): assume $m \in M^c$. Let I be a filtered $(\infty, 1)$-category and $z = \mathrm{colim}_{i\in I}\, z_i$ in M. We want to check that the natural map $\mathrm{colim}_{i\in I}\, \underline{\mathrm{Hom}}_A(m, z_i) \to \underline{\mathrm{Hom}}_A(m, z)$ is an isomorphism in A. Since A^c is stable, it suffices to show that for any $a \in A^c$ the induced map

$$\mathrm{Map}_A(a, \mathrm{colim}_{i\in I}\, \underline{\mathrm{Hom}}_A(m, z_i)) \to \mathrm{Map}_A(a, \underline{\mathrm{Hom}}_A(m, z))$$

is an isomorphism in Spc. This map rewrites as

$$\mathrm{colim}_{i\in I}\, \mathrm{Map}_M(a \otimes m, z_i) \to \mathrm{Map}_M(a \otimes m, z)$$

It is an isomorphism, because $a \otimes m \in M^c$ by assumption.

6.0.15. For (ch. 1, 8.8.5). If $A \in Alg(1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ then $A^{left-dualizable} \subset A$ is stable under translations. Namely, if $counit : a \otimes b \to 1$ extends to a duality datum then $counit : a[1] \otimes b[-1] \widetilde{\to} a \otimes b \to 1$ extends to a duality datum.

Proof of (ch. 1, 8.8.6). Let $a \in A$ be compact relative to A. We have to show that their map $a' \otimes \underline{\mathrm{Hom}}_A(a, 1) \to \underline{\mathrm{Hom}}_A(a, a')$ given by (8.5) is an isomorphism, provided that a' is left-dualizable. Let $b = a'^{\vee, L}$, so $a' = b^{\vee, R}$. To do so, we will show that for any $d \in A$ the induced map

$$\mathrm{Map}_A(d, b^{\vee,R} \otimes \underline{\mathrm{Hom}}_A(a, 1)) \to \mathrm{Map}_A(d \otimes a, b^{\vee,R})$$

is an isomorphism. The RHS identifies with $\mathrm{Map}_A(b \otimes d \otimes a, 1) \widetilde{\to} \mathrm{Map}_A(b \otimes d, \underline{\mathrm{Hom}}_A(a, 1))$. The desired isomophism follows now from my Remark 3.1.1.

6.1. For (ch. 1, 9.1.1).

Lemma 6.1.1. *Let $f : \mathrm{Sptr} \to \mathcal{C}$ be a map in $1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Assume $f(1_{\mathrm{Sptr}}) \in \mathcal{C}^c$. Then $f(\mathrm{Sptr}^c) \subset \mathcal{C}^c$. So, the right adjoint $f^R : \mathcal{C} \to \mathrm{Sptr}$ is continuous.*

Proof. By ([29], 9.7), every compact object of Sptr is a retract of $\Sigma^{\infty-n}(Y)$ for some $Y \in \mathrm{Spc}^{fin}_*$ and some $n \geq 0$. Here $\Sigma^{\infty-n} : \mathrm{Spc}_* \to \mathrm{Sptr}$ is the left adjoint to $\Omega^{\infty-n}$. We have $\Sigma^{\infty-n}(S^0) \widetilde{\to} 1_{\mathrm{Sptr}}[-n]$. Besides, $\mathcal{C}^c$ is stable and $\mathcal{C}^c \subset \mathcal{C}$ is closed under finite colimits and retracts. Our claim follows now from the fact that $\mathrm{Spc}^{fin}_* \subset \mathrm{Spc}_*$ is the smallest full subcategory which contains S^0 and is stable under finite colimits. □

6.1.2. Let $F : A \to B$ be a map in $CAlg(1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$, $\mathcal{A} \in CAlg(A)$, $\mathcal{B} \in CAlg(B)$. Assume given a map $F(\mathcal{A}) \to \mathcal{B}$ in $CAlg(B)$. Consider the functor $\alpha : \mathcal{A} - mod(A) \to \mathcal{B} - mod(B)$, $M \mapsto F(M) \otimes_{F(\mathcal{A})} \mathcal{B}$. What is its right adjoint? Our α is the composition

$$\mathcal{A} - mod(A) \stackrel{\alpha_1}{\to} F(\mathcal{A}) - mod(B) \stackrel{\alpha_2}{\to} \mathcal{B} - mod(B),$$

where α_1 sends M to $F(M)$, and the second is the extension of scalars. Now α_2 has a right adjoint $\alpha_2^R = \mathrm{Res} : \mathcal{B} - mod(B) \to F(\mathcal{A}) - mod(B)$, here Res is continuous (recall that $oblv : \mathcal{B} - mod(B) \to B$ preserves colimits). Assume F has a continuous right adjoint $F^R : B \to A$. Then α_1 has the following right adjoint.

Since $F^R : B \to A$ is right-lax monoidal, it induces a functor $F(\mathcal{A}) - mod(B) \to F^R F(\mathcal{A}) - mod(A)$. Restricting the scalars further via $\mathcal{A} \to F^R F(\mathcal{A})$, we get the functor $\alpha_1^R : F(\mathcal{A}) - mod(B) \to \mathcal{A} - mod(A)$, which is the right adjoint to α_1. Note that α_1^R is continuous. So, $\alpha^R = \alpha_1^R \alpha_2^E$ is continuous.

Generalizing the previous, one has the following.

Lemma 6.1.3. *Let $F : A \to B$ be a map in $CAlg(1-\mathcal{C}at^{St,cocmpl}_{cont})$, $\mathcal{A} \in CAlg(A)$. We get a functor $q : \mathcal{A}-mod(A) \to F(\mathcal{A})-mod(B)$, $M \mapsto F(M)$. Let $F^R : B \to A$ be the right adjoint to F, it is right-lax monoidal, so induces a functor $F(\mathcal{A})-mod(B) \to F^RF(\mathcal{A})-mod(A)$. Let q^R denote the composition*

$$F(\mathcal{A})-mod(B) \to F^RF(\mathcal{A})-mod(A) \to \mathcal{A}-mod(A),$$

where the second arrow is the restriction of scalars via the natural map $\mathcal{A} \to F^RF(\mathcal{A})$. Then q^R is the right adjoint of q.

Proof. 1) Let $\mathcal{B} \in Alg(B)$. The functor $e : \mathcal{B}-mod(B) \to F^R(\mathcal{B})-mod(A)$, $M \mapsto F^R(M)$ admits a left adjoint $\mathcal{L}$, because e preserves limits.

Our F is a map of right A-module categories, so F^R is a right-lax map of right A-module categories. Namely, given $b \in B, a \in A$, we have the natural map $F^R(b)\otimes a \to F^R(b)\otimes F^R(F(a)) \to F^R(b\otimes F(a))$. So, e is a right-lax functor of A-module categories. For this reason, $\mathcal{L}$ is a left-lax functor of A-module categories. We claim that $\mathcal{L}$ is a strict functor of A-module categories.

Indeed, the essential image of the induction functor $A \to F^R(\mathcal{B})-mod(A)$, $M \mapsto F^R(\mathcal{B})\otimes M$ generates $F^R(\mathcal{B})-mod(A)$ under colimits, as its right adjoint *oblv* : $F^R(\mathcal{B})-mod(A) \to A$ is conservative. Now given $M \in \mathcal{B}-mod(B), N \in A$ we have

$$\begin{aligned}\mathrm{Map}_{\mathcal{B}-mod(B)}(\mathcal{L}(F^R(\mathcal{B})\otimes N), M) \widetilde{\to} \mathrm{Map}_{F^R(\mathcal{B})-mod(A)}(F^R(\mathcal{B})\otimes N, F^R(M)) \widetilde{\to} \\ \mathrm{Map}_A(N, F^R(M)) \widetilde{\to} \mathrm{Map}_B(F(N), M) \widetilde{\to} \mathrm{Map}_{\mathcal{B}-mod(B)}(\mathcal{B}\otimes F(N), M)\end{aligned}$$

So, $\mathcal{B}\otimes F(N) \widetilde{\to} \mathcal{L}(F^R(\mathcal{B})\otimes N)$ in $\mathcal{B}-mod(B)$. So, on objects of the form $F^R(\mathcal{B})\otimes N$ with $N \in A$, the functor $\mathcal{L}$ is strict functor of A-module categories. Since $\mathcal{L}$ preserves colimits, it is strict.

2) The functor q^R by 1) admits a left adjoint $\bar{\mathcal{L}}$, which is a strict functor of A-module categories. Note that q is also a strict functor of right A-module categories. Now it suffices to show that $\bar{\mathcal{L}}(\mathcal{A}) \widetilde{\to} F(\mathcal{A})$ in $F(\mathcal{A})-mod(B)$. This is easy. □

6.1.4. For 9.1.2. Note that mult^R is a functor of A-bimodules catgeories iff for $a \in A$ the natural maps are isomorphisms

$$\mathrm{mult}^R(a) \leftarrow (a\boxtimes 1)\otimes \mathrm{mult}^R(1), \quad \mathrm{mult}^R(a) \leftarrow \mathrm{mult}^R(1)\otimes(1\boxtimes a)$$

For 9.1.3. Let $\mathcal{A} \in CAlg(\mathrm{Sptr})$. Let $\mathcal{A}-mod = \mathcal{A}-mod(\mathrm{Sptr})$. Let us check that $\mathcal{A}-mod \in CAlg(1-\mathcal{C}at^{St,cocmpl}_{cont})$ is rigid. For $M \in \mathcal{A}-mod$ we have

$$\mathrm{Map}_{\mathcal{A}-mod}(\mathcal{A}, M) \widetilde{\to} \mathrm{Map}_{\mathrm{Sptr}}(1_{\mathrm{Sptr}}, M)$$

The projection $\mathcal{A}-mod \to \mathrm{Sptr}$ preserves colimits (see my Section 3.0.53). Since $1_{\mathrm{Sptr}} \in \mathrm{Sptr}^c$, $\mathcal{A}$ is compact in $\mathcal{A}-mod$. Now $\mathcal{A}-mod\otimes\mathcal{A}-mod \widetilde{\to} (\mathcal{A}\otimes\mathcal{A})-mod$, where we used the symmetric monoidal structure on $CAlg(\mathrm{Sptr})$. The multiplication map $mult : \mathcal{A}-\mathrm{mod}\otimes\mathcal{A}-mod \to \mathcal{A}-mod$, $(M,N) \mapsto M\otimes_{\mathcal{A}} N$ identifies with the extension of scalars $(\mathcal{A}\otimes\mathcal{A})-mod \to \mathcal{A}-mod$ via $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$. Here m is the product in the algebra $\mathcal{A}$. So, the right adjoint $mult^R : \mathcal{A}-mod \to (\mathcal{A}\otimes\mathcal{A})-mod$ is the restriction of scalars via m, it is continuous.

The functor $mult^R$ is a functor $\mathcal{A}-mod$-bimodule categories, as far as I understand, because of the following. Given $M, M_i \in \mathcal{A}-mod$, consider $M_1 \otimes M_2 \in \mathcal{A}\otimes\mathcal{A}-mod$. Then $(M_1 \otimes M_2)\otimes_{\mathcal{A}\otimes\mathcal{A}} M \widetilde{\to} ((M_1\otimes M_2)\otimes_{\mathcal{A}\otimes\mathcal{A}}\mathcal{A})\otimes_{\mathcal{A}} M$, and

$$(M_1 \otimes M_2)\otimes_{\mathcal{A}\otimes\mathcal{A}}\mathcal{A} \widetilde{\to} M_1 \otimes_{\mathcal{A}} M_2$$

More generally, let $A \in CAlg(1-\mathcal{C}at^{St,cocmpl}_{cont})$ be rigid, $\mathcal{A} \in CAlg(A)$. Let us show that $\mathcal{A}-mod = \mathcal{A}-mod(A)$ is rigid. Since for $M \in \mathcal{A}-mod$,

$$\mathrm{Map}_{\mathcal{A}-mod}(\mathcal{A}, M) \widetilde{\to} \mathrm{Map}_A(1_A, M),$$

$\mathcal{A}$ is compact in $\mathcal{A}-mod$. The multiplication functor is the composition $(\mathcal{A}\boxtimes\mathcal{A})-mod(A\otimes A) \stackrel{\alpha}{\to} (\mathcal{A}\otimes\mathcal{A})-mod(A) \stackrel{\beta}{\to} \mathcal{A}-mod$, where β is the extension of scalars via the product $p : \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$, and α is the functor $M \mapsto m(M)$. Here $m : A\otimes A \to A$ is the multiplication functor. The right adjoint β^R of β is the restriction of scalars via p, it is continuous. The right adjoint to α is the composition

$$(\mathcal{A}\otimes\mathcal{A})-mod(A) \stackrel{\gamma_1}{\to} m^R(\mathcal{A}\otimes\mathcal{A})-mod(A\otimes A) \stackrel{\gamma_2}{\to} (\mathcal{A}\boxtimes\mathcal{A})-mod(A\otimes A)$$

where γ_2 is the restriction of scalars via $\mathcal{A}\boxtimes\mathcal{A}\to m^R(m(\mathcal{A}\boxtimes\mathcal{A}))$, and γ_1 is the functor $H \mapsto m^R(H)$. It exists because m^R is right-lax symmetric monoidal, this is in turn because $m : A\otimes A\to A$ is symmetric monoidal. We see that the composition $\gamma_2\gamma_1\beta^R$ is continuous.

Why $\gamma_2\gamma_1\beta^R$ is a functor of $(\mathcal{A}-mod)$-bimodule categories? We have already seen this for β^R above. We have to show that given $M, N_i \in \mathcal{A}-mod$,

$$m^R(N_1\otimes_{\mathcal{A}} M\otimes_{\mathcal{A}} N_2) \widetilde{\to} m^R(M)\otimes_{\mathcal{A}\boxtimes\mathcal{A}}(N_1\boxtimes N_2)$$

I think this is proved using the fact that m^R is continuous and writing the bar resolution of $N_1\otimes_{\mathcal{A}} M\otimes_{\mathcal{A}} N_2$.

6.1.5. For (ch. 1, proof of 9.1.5). They say that if $A \in Alg(1-\mathcal{C}at^{St,cocmpl}_{cont})$ and $F : M\to N$ is a right-lax functor between A-modules, where $M, N \in A-mod^{St,cocmpl}_{cont}$, assume A is compactly generated and for any $a\in A^c, m\in M$, $a\otimes F(m)\to F(a\otimes m)$ is an isomorphism. Then F is strict.

Suppose A rigid. Then for $a \in A^c$ the functor $R : A \to \mathrm{Sptr}, x\mapsto \mathcal{M}aps_A(a,x)$ is continuous, its left adjoint is the functor $L : \mathrm{Sptr}\to A, z\mapsto z\otimes a$. Using my Section 4.1.6, we see that the functor $\mathrm{id}\otimes R : A\otimes A\to A$ is right adjoint to $\mathrm{id}\otimes L : A\to A\otimes A$. Here $\mathrm{id}\otimes L$ sends b to $b\boxtimes a$. So, the functor $A\to A$, $a'\mapsto a'\otimes a$ admits a right adjoint. The left dual $a^{\vee,L}$ to $a\in A^c$ is calculated as $(\mathrm{id}\otimes R)m^R(1)$. Here $m^R : A\to A\otimes A$ is the right adjoint to $m : A\otimes A\to A$. The right dual $a^{\vee,R}$ to $a\in A^c$ is calculated as $(R\otimes\mathrm{id})m^R(1)$.

Remark Let $A\in Alg(1-\mathcal{C}at^{St,cocmpl}_{cont})$ be rigid then $m^R : A\to A\otimes A$ is not necessarly fully faithful. For an example of such a DG-category, take $A = \mathrm{Rep}(\check{G})$, where $\check{G}$ is a reductive group. Then for $\vartriangle: B(\check{G})\to B(\check{G}\times\check{G})$ the map $\vartriangle^*\vartriangle_* e\to e$ is not the isomorphism, where e is the trivial representation of $\check{G}$.

6.1.6. For 9.2.1. Let us check that the composition $A \overset{\mathrm{id}\otimes u}{\to} A\otimes A\otimes A \overset{c\otimes\mathrm{id}}{\to} A$ is isomorphic to id. This uses the fact that (in any monoidal category) $1^{\vee,L}\widetilde{\to}1\widetilde{\to}1^{\vee,R}$. We have to show that for $a\in A$ the image of $(a\boxtimes 1)\otimes m^R(1)\widetilde{\to}m^R(a)\widetilde{\to}m^R(1)\otimes(1\boxtimes a)\in A\otimes A$ under $R_1\otimes\mathrm{id}: A\otimes A\to A$ is a. Here $R_1: A\to \mathrm{Sptr}$ sends b to $\mathcal{M}aps_A(1,b)$. This follows from the fact that $(R_1\otimes\mathrm{id})m^R(1)\widetilde{\to}1^{\vee,R}\widetilde{\to}1$, see the previous section.

Similarly, the composition $A \overset{u\otimes\mathrm{id}}{\to} A\otimes A\otimes A \overset{\mathrm{id}\otimes c}{\to} A$ is the identity, this follows from $(\mathrm{id}\otimes R_1)m^R(1)\widetilde{\to}1$.

The isomorphism $\phi_A: A\to A^\vee = \mathrm{Fun}_{ex,cont}(A,\mathrm{Sptr})$ sends a to the functor $A\to \mathrm{Sptr}, b\mapsto \mathcal{M}aps_A(1,b\otimes a)$. The functor $\phi_A^{-1}: A^\vee\to A$ sends $f: A\to\mathrm{Sptr}$ to the composition $(\mathrm{id}\otimes f)m^R(1)$, here $\mathrm{id}\otimes f: A\otimes A\to A$ is the functor obtained from f by tensoring with A on the left. Indeed, this map sends R_1 to 1, and is a map of left A-modules.

6.1.7. For 9.2.3. Recall that $\mathrm{Fun}_{ex,cont}(A,A)$ is the relative inner hom from A to A in $1-\mathcal{C}at^{St,cocmpl}_{cont}$, see my Section 4.1.7. The right action of $h\in \mathrm{Fun}_{ex,cont}(A,A)$ on $g\in\mathrm{Fun}_{ex,cont}(A,\mathrm{Sptr})$ is gh. The homomorphism $A^{rm}\to\mathrm{Fun}_{ex,cont}(A,A)$ sends a to the functor $A\to A, x\mapsto x\otimes a$, and this gives a left action of A on $A^\vee$. Then ϕ_A is compatible with the left A-module structure. The isomorphism ϕ_A is not compatible with the right A-module structures. Indeed, this is because A is not symmetric monoidal in general: given $a,x,b\in A$, $\mathcal{M}aps_A(1,a\otimes b\otimes x)$ is different from $\mathcal{M}aps_A(1,a\otimes x\otimes b)$.

6.1.8. For 9.2.6. Let $L: A\to B$, $R: B\to A$ be maps in $1-\mathcal{C}at^{St,cocmpl}_{cont}$. Assume L is left adjoint to R. Assume A,B dualizable, so we get the dual functors $L^\vee: B^\vee\to A^\vee$, $R^\vee: A^\vee\to B^\vee$. The dualization extends to a functor of $(\infty,2)$-categories? Sam says the answer is yes. So, $R^\vee$ is left adjoint to $L^\vee$.

The map $m^\vee: A^\vee\to A^\vee\otimes A^\vee$ is a map of left A-modules, so in their diagram 9.2.6(a) all the maps are maps of left A-modules. It suffices to check that the images of 1_A are the same. One has $\phi_A(1)=R_1$, this is the functor $A\to\mathrm{Sptr}, x\mapsto\mathcal{M}aps_A(1,x)$. We may think of $A^\vee$ as a free left A-module over A with generator R_1. Commutativity of the diagram 9.2.6(a) is obtained as follows: It suffices to show that the diagram commutes

$$\begin{array}{ccc} A^\vee & \overset{(m^R)^\vee}{\leftarrow} & A^\vee\otimes A^\vee \\ \uparrow \phi_A & & \uparrow \phi_A\otimes\phi_A \\ A & \overset{m}{\leftarrow} & A\otimes A \end{array}$$

Let $h\in A^\vee$. Since all the maps are maps of left A-modules, it suffices to show that both images of $R_1\boxtimes h$ coincide in A. We have seen above that $(R_1\otimes\mathrm{id})m^R(1)\widetilde{\to}1$. So, for $a\in A$, $(R_1\boxtimes h)m^R(a)=(\mathrm{id}\boxtimes h)(R_1\boxtimes\mathrm{id})(m^R(1)\otimes(1\boxtimes a))\widetilde{\to}h(a)$. So, h is the image of $R_1\otimes h$ by $(m^R)^\vee$. Further, the image of h by ϕ_A^{-1} is $(\mathrm{id}\otimes h)m^R(1)\in A$, see my Section 6.1.6. The image of $R_1\otimes h$ under $\phi_A^{-1}\otimes\phi_A^{-1}$ is $1\otimes((\mathrm{id}\otimes h)m^R(1))\in A\otimes A$. We are done.

6.1.9. For 9.2.6(b). Let A_2 act on $A_2^\vee$ on the left, so $a\in A_2$ sends h to the functor $A_2\to\mathrm{Sptr}, z\mapsto h(z\otimes a)$. Similarly, A_1 acts on the left on $A_1^\vee$. Since $F: A_1\to A_2$ is monoidal, $F^\vee: A_2^\vee\to A_1^\vee$ is a morphism of left A_1-modules, where A_1 acts on $A_2^\vee$

on the left via composing with $F : A_1 \to A_2$. So, if 9.2.6(b) is true, the map F^R should also be a morphism of left A_1-modules. This means that for $a_i \in A_i$ we should have $F^R(F(a_1) \otimes a_2) \widetilde{\to} a_1 \otimes F^R(a_2)$. In the other direction, this property would imply 9.2.6(b). Indeed, given $a_2 \in A_2, x \in A_1$, we have to show that

$$\mathcal{M}aps_{A_2}(1, F(x) \otimes a_2) \widetilde{\to} \mathcal{M}aps_{A_1}(1, x \otimes F^R(a_2))$$

This is obtained as follows: for $z \in \mathrm{Sptr}$,

$$\mathrm{Map}_{\mathrm{Sptr}}(z, \mathcal{M}aps_{A_2}(1, F(x) \otimes a_2)) \widetilde{\to} \mathrm{Map}_{A_2}(z \otimes 1, F(x) \otimes a_2) \widetilde{\to}$$
$$\mathrm{Map}_{A_1}(z \otimes 1, F^R(F(x) \otimes a_2)) \widetilde{\to} \mathrm{Map}_{A_1}(z \otimes 1, x \otimes F^R(a_2)) \widetilde{\to} \mathrm{Map}_{\mathrm{Sptr}}(z, \mathcal{M}aps_{A_1}(1, x \otimes F^R(a_2)))$$

Clearly, F is a morphism of A_1-modules, so F^R is a right-lax map of A_1-modules. Their (ch. 1, 9.3) claims that any right-lax morphism of A_1-modules is strict. So, we have reduced 9.2.6(b) to (ch. 1, 9.3), that is, their Lemma 9.3.6.

6.1.10. For 9.3.2. Explanation for the proof: the transformation from the identity functor $\mathrm{id} : A \otimes M \to A \otimes M$ to the functor

$$A \otimes M \stackrel{m^R \otimes \mathrm{id}}{\to} A \otimes A \otimes M \stackrel{\mathrm{id} \otimes \mathrm{act}}{\to} A \otimes M$$

uses the natural map $1 \boxtimes 1 \to m^R(1)$ in $A \otimes A$. It gives functorially in $a \in A, m \in M$ the map $(a \boxtimes 1) \to (a \boxtimes 1) \otimes m^R(1) \widetilde{\to} m^R(a)$.

For (ch. 1, 9.3.3). Let $A \in Alg(1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ be rigid, $M \in A - mod, N \in A^{rm} - mod$. One may strengthen (ch. 1, 9.3.3) as follows. Consider the dual pair $l : N \otimes M \leftrightarrows N \otimes_A M : r$ in $1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. The functor r is monadic, and gives an equivalence $N \otimes_A M \widetilde{\to} \mathcal{A} - mod(N \otimes M)$, where $\mathcal{A} = rl$ is the corresponding monad. This follows from (HA, 4.7.5.1). Indeed, we may pass to right adjoint in the diagram $N \otimes_A M \widetilde{\to} \underset{[n] \in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, N \otimes A^n \otimes M$ and get $N \otimes_A M \widetilde{\to} \underset{[n] \in \boldsymbol{\Delta}}{\lim}\, N \otimes A^n \otimes M$ in $1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Besides, $1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont} \to 1 - \mathcal{C}\mathrm{at}$ preserves limits.

6.1.11. For (ch. 1, 9.3.4). The nontrivial part is: let $m \in M$ be compact then m is compact relative to A. To prove this let $a \in A$. We show that the functor $M \to \mathrm{Spc}, m' \mapsto \mathrm{Map}_M(a \otimes m, m')$ is continuous. The functor $M \to M, m \mapsto a \otimes m$ has a right adjoint given as the composition $M \stackrel{\mathrm{act}^R}{\to} A \otimes M \stackrel{f \otimes \mathrm{id}}{\to} M$, where $f : A \to \mathrm{Sptr}$ is the functor $f(z) = \mathcal{M}aps_A(a, z)$. Note that f is not necessarily continuous! However, $(f \otimes \mathrm{id})\,\mathrm{act}^R : M \to M$ is continous, because this is the functor $m' \mapsto u \otimes m'$ of action by u, where $u \in A$ is the element $(f \otimes \mathrm{id}) m^R(1)$. Here $m^R : A \to A \otimes A$ is the right adjoint to $m : A \otimes A \to A$.

6.1.12. For the proof of (ch. 1, 9.3.6): there is a misprint, (9.5) maps to (9.2). This map uses in addition the right-lax structure on F. Namely, given $m \in M$, one gets the map $\mathrm{act}(\mathrm{id} \otimes F)(\mathrm{id} \otimes \mathrm{act})(mult^R(1) \otimes m) \to F(\mathrm{act}(\mathrm{id} \otimes \mathrm{act})(mult^R(1) \otimes m)) = F(\mathrm{act}(\mathrm{mult}\,\mathrm{mult}^R(1) \otimes m)) \to F(\mathrm{act}(1 \otimes m)) = F(m)$.

The desired map is a map of left A-modules, so it is easy to add $a \in A$ in the above.

6.1.13. For (ch. 1, 9.4.4). Let $A \in \mathrm{Alg}(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ be dualizable. The functor $oblv : A-mod^{St,cocmpl}_{cont} \to 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ has a right adjoint by my Section 3.2. This right adjoint $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont} \to A-mod^{St,cocmpl}_{cont}$ sends C to $A^\vee \otimes C$, where the A-module structure is given by the left A-action on $A^\vee$. This formally implies an equivalence

$$\mathrm{Fun}_{ex,cont}(M,N) \widetilde{\to} \mathrm{Fun}_A(M, A^\vee \otimes N) \tag{10}$$

in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ for $M \in A-mod^{St,cocmpl}_{cont}, N \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Here the A-module structure on $A^\vee \otimes N$ comes from the left A-action on $A^\vee$.

Note that $\mathrm{Fun}_{ex,cont}(M,N)$ is naturally a right A-module, view $\mathrm{Fun}_A(M, A^\vee \otimes N)$ as a right A-module via the right A-action on $A^\vee$. Then (10) is an isomorphism in $(A-mod^r)^{St,cocmpl}_{cont}$, this follows from Remark 3.2.2.

Assume now $A \in CAlg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$, $M \in A-mod^{St,cocmpl}_{cont}$. Assume M dualizable in the symmetric monoidal category $A-mod^{St,cocmpl}_{cont}$. Then for $N \in A-mod^{St,cocmpl}_{cont}$ we get

$$\mathrm{Fun}_A(M,N) \widetilde{\to} M^\vee \otimes_A N$$

in $A-mod^{St,cocmpl}_{cont}$, see (ch. 1, 4.3.2). So, for $D, C \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ we get

$$D \otimes \mathrm{Fun}_{ex,cont}(M,C) \widetilde{\to} D \otimes M^\vee \otimes_A (A^\vee \otimes C) \widetilde{\to} M^\vee \otimes_A (A^\vee \otimes C \otimes D),$$

because the tensor product over A is a map of $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$-categories (see ch. 1, 4.2.1).

The same idea is used in ([14], ch. I.1, 9.4.8).

Lemma 6.1.14. *Let $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, $c \in \mathcal{C}^c, d \in \mathcal{D}^c$. Then $c \boxtimes d \in (\mathcal{C} \otimes \mathcal{D})^c$.*

Proof. Consider the maps $f : \mathrm{Sptr} \to \mathcal{C}$, $g : \mathrm{Sptr} \to \mathcal{D}$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ given by $f(1) = c, f(1) = d$. By my Lemma 6.1.1, they admit continuous right adjoints f^R, g^R. By Section 4.1.6, $f^R \otimes g^R : \mathcal{C} \otimes \mathcal{D} \to \mathrm{Sptr}$ is right adjoint to $f \otimes g : \mathrm{Sptr} \to \mathcal{C} \otimes \mathcal{D}$. Since $f^R \otimes g^R$ is continuous, $c \boxtimes d$ is compact by (ch. 1, 7.1.5). Recall that Sptr is compactly generated. □

6.1.15. Let I be a small set, for $i \in I$ let $C_i \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Then $\sqcup_{i\in I} C_i \widetilde{\to} \prod_{i\in I} C_i$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Indeed, given $D \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, one has

$$\mathrm{Fun}_{ex,cont}(\prod_{i\in I} C_i, D) \widetilde{\to} \mathrm{Fun}^R(D, \prod_{i\in I} C_i)^{op} \widetilde{\to} \prod_{i\in I} \mathrm{Fun}^R(D, C_i)^{op} \widetilde{\to} \prod_{i\in I} \mathrm{Fun}_{ex,cont}(C_i, D),$$

here Fun^R denotes the category of functors, which are right adjoints.

6.1.16. If $A, B \in Alg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$ are rigid then $A \otimes B$ is also rigid. Indeed, $1 \in A \otimes B$ is compact by Lemma 6.1.14. Let $m^R : A \to A \otimes A$ and $\bar{m}^R : B \to B \otimes B$ be right adjoint to $m : A \otimes A \to A$ and $\bar{m} : B \otimes B \to B$ respectively. Since $m^R, \bar{m}^R$ are continuous, $m^R \otimes \bar{m}^R$ is also continuous, and similarly, $m^R \otimes \bar{m}^R$ is $A \otimes B$-bilinear.

6.1.17. Given $A \in Alg(1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont})$, $M \in A-mod^{St,cocmpl}_{cont}$, $N \in (A-mod^r)^{St,cocmpl}_{cont}$, their cotensor product is $M \otimes^A N \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ is given by the property: functorially in $D \in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, one has

$$\mathrm{Map}_{1-\mathcal{C}\mathrm{at}^{St,ccompl}_{cont}}(D, M \otimes^A N) \widetilde{\to} \mathrm{Map}_{A\otimes A^{rm}-mod^{St,cocmpl}_{cont}}(A \otimes D, M \otimes N)$$

It is understood that the $A \otimes A^{rm}$-module structure on $A \otimes D$ comes from that on A. This cotensor product clearly always exists, $M \otimes^A N \widetilde{\to} \mathrm{Fun}_{A\otimes A^{rm}}(A, M \otimes N)$.

Besides,

$$N \otimes_A M \widetilde{\to} A \otimes_{A\otimes A^{rm}} (M \otimes N),$$

here A is viewed as a right $A \otimes A^{rm}$-module. Indeed, write $A \widetilde{\to} \underset{[n]\in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, A^{\otimes n+2}$ as the usual bar complex in $A\otimes A^{rm}-mod$. The terms of this colimit are free $A\otimes A^{rm}$-modules, so

$$\underset{[n]\in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, A^{\otimes n+2} \otimes_{A\otimes A^{rm}} (M \otimes N) \widetilde{\to} \underset{[n]\in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, A^{\otimes n} \otimes M \otimes N \widetilde{\to} N \otimes_A M,$$

we used that the last colimit is the usual bar complex calculating $N \otimes_A M$ as in Section 3.1.8.

6.1.18. The following idea is due to Lin Chen (email 29dec2019), but the proof is wrong!!! The problem is that the equivalence $C \otimes D \widetilde{\to} \mathrm{RFun}(C^{op}, D)$ is ill-behaved in functoriality.

Lemma 6.1.19. *Let $A \in Alg(1-\mathcal{C}\mathrm{at}^{Stcocmpl}_{cont})$, let $C, D \in 1-\mathcal{C}\mathrm{at}$ be small. The natural functor*

$$\mathrm{Fun}(C, A) \otimes_A \mathrm{Fun}(D, A) \widetilde{\to} \mathrm{Fun}(C \times D, A)$$

is an equivalence. Here the tensor product is the relative rensor product in $1-\mathcal{C}\mathrm{at}^{Stcocmpl}_{cont}$ (equivalently, in $\mathcal{P}\mathrm{r}^L$).

Proof. The LHS identifies with $\mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \mathrm{Fun}(C, A) \otimes A^{\otimes n} \otimes \mathrm{Fun}(D, A)$. By (HA, 4.8.1.17),

$$\mathrm{Fun}(C, A) \otimes A^{\otimes n} \otimes \mathrm{Fun}(D, A) \widetilde{\to} \mathrm{RFun}((\mathrm{Fun}(C, A) \otimes A^{\otimes n})^{op}, \mathrm{Fun}(D, A)) \widetilde{\to}$$
$$\mathrm{Fun}(D, \mathrm{RFun}((\mathrm{Fun}(C, A) \otimes A^{\otimes n})^{op}, A) \widetilde{\to} \mathrm{Fun}(D, \mathrm{Fun}(C, A) \otimes A^{\otimes n} \otimes A)$$

Here $\mathrm{RFun}(-,-)$ denotes the full subcategory of Fun spanned by functors which are right adjoints (equivalently, preserving small limits and accessible).

The colimits diagram $\mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \mathrm{Fun}(C, A)\otimes A^{\otimes n}\otimes A \widetilde{\to} \mathrm{Fun}(C, A)$ is an augmented split simplicial object (HA, Def. 4.7.2.2). By (HA, Remark 4.7.2.4), we get

$$\underset{[n]\in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, \mathrm{Fun}(D, \mathrm{Fun}(C, A) \otimes A^{\otimes n} \otimes A) \widetilde{\to} \mathrm{Fun}(D, \mathrm{Fun}(C, A))$$

□

6.1.20. if A is an algebra in $\mathbb{S}\mathrm{p}$ then we have the category $A-mod(\mathbb{S}\mathrm{p})$ of A-modules in $\mathbb{S}\mathrm{p}$. By Section 4.0.32, $A-mod(\mathbb{S}\mathrm{p})\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. The t-structure on $\mathbb{S}\mathrm{p}$ is compatible with filtered colimits by Lemma 4.0.66. Recall that $\mathbb{S}\mathrm{p}^{\le 0}\otimes\mathbb{S}\mathrm{p}^{\le 0}\widetilde{\to}\mathbb{S}\mathrm{p}^{\le 0}$, where the tensor product is taken in the sense of $\mathcal{P}\mathrm{r}^L$, see ([33], C.4.1). Assume $A\in\mathbb{S}\mathrm{p}^{\le 0}$. Then we define the t-structure on $A-mod(\mathbb{S}\mathrm{p})$ so that $A-mod(\mathbb{S}\mathrm{p})^{\le 0}$ is the preimage of $\mathbb{S}\mathrm{p}^{\le 0}$ under $\mathrm{oblv}: A-mod(\mathbb{S}\mathrm{p})\to\mathbb{S}\mathrm{p}$. This is an accessible t-structure by ([28], 1.4.4.11), and $A-mod(\mathbb{S}\mathrm{p})$ is compactly generated by A ([28], 7.1.2.1). We have $\mathcal{M}aps_{A-mod(\mathbb{S}\mathrm{p})}(A,x)\widetilde{\to}\mathrm{oblv}(x)$ in $\mathbb{S}\mathrm{p}$ for $x\in A-mod(\mathbb{S}\mathrm{p})$. Here for $C\in 1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ and $c,c'\in C$ we write $\mathcal{M}aps_C(c,c')\in\mathbb{S}\mathrm{p}$ for the relative inner hom. The t-structure on $A-mod(\mathbb{S}\mathrm{p})$ is compactly generated, in the sense that $A-mod(\mathbb{S}\mathrm{p})^{\le 0}$ is generated under filtered colimits by $A-mod(\mathbb{S}\mathrm{p})^{\le 0}\cap A-mod(\mathbb{S}\mathrm{p})^c$. Now as in Lemma 9.3.5, the t-structure on $A-mod(\mathbb{S}\mathrm{p})$ is compatible with filtered colimits.

As in Lemma 9.3.12, one shows that $A-mod(\mathbb{S}\mathrm{p})^{>0}=\mathrm{oblv}^{-1}(\mathbb{S}\mathrm{p}^{>0})$, so $\mathrm{oblv}: A-mod(\mathbb{S}\mathrm{p})\to\mathbb{S}\mathrm{p}$ is t-exact. By my Section 4.0.10, $A-mod(\mathbb{S}\mathrm{p})$ is right complete, because $\mathrm{oblv}: A-mod(\mathbb{S}\mathrm{p})\to\mathbb{S}\mathrm{p}$ preserves colimits. Then $(A-mod(\mathbb{S}\mathrm{p}))^{\heartsuit}$ identifies with $\mathrm{H}^0(A)-mod(\mathbb{S}\mathrm{p}^{\heartsuit})$ by ([33], C.1.4.6). Note also that $\mathbb{S}\mathrm{p}^{\le 0}\subset\mathbb{S}\mathrm{p}$ is stable under products by Section 4.2.4. Since the t-structure on $\mathbb{S}\mathrm{p}$ is left complete, my Remark after Cor. 4.0.11 shows now that the t-structure on $A-mod(\mathbb{S}\mathrm{p})$ is left complete.

For example this holds, for $A=\mathbb{Q}$. Let $\mathrm{Vect}_{\mathbb{Q}}$ be the DG-category of vector spaces over $\mathbb{Q}$, namely the (left and right completion) of the derived category attached to the abelian category of $\mathbb{Q}$-vector spaces.

Lemma 6.1.21. *The category* $\mathrm{Vect}_{\mathbb{Q}}$ *identifies with* $\mathbb{Q}-mod(\mathbb{S}\mathrm{p})$*, where* $\mathbb{S}\mathrm{p}$ *is the category of spectra.*

Proof. The category $\mathbb{Q}-mod(\mathbb{S}\mathrm{p}^{\heartsuit})$ has enough injective objects, so there is a canonical functor $\mathrm{D}(\mathbb{Q}-mod(\mathbb{S}\mathrm{p}^{\heartsuit}))^+\to\mathbb{Q}-mod(\mathbb{S}\mathrm{p})^+$ given by the universal property of the derived category (HA, 1.3.3.2). We want to apply ([14], ch. I.3, 2.4.5) with the correction from my Section 10.2.8. The category $\mathbb{Q}-mod(\mathbb{S}\mathrm{p}^{\heartsuit})$ is that of $\mathbb{Q}$-vector spaces, every its object is injective. To apply ([14], ch. I.3, 2.4.5) it suffices to check that for $x,y\in\mathbb{Q}-mod(\mathbb{S}\mathrm{p}^{\heartsuit})$ and $n>0$ we have $\mathrm{Hom}_{\mathbb{Q}-mod(\mathbb{S}\mathrm{p})}(x,y[n])=0$. Writing x as a colimit of finite-dimensional vector spaces, we may assume $\dim_{\mathbb{Q}}x<\infty$ and the in turn, $x=\mathbb{Q}$. In the latter case we have $\mathcal{M}aps_{\mathbb{Q}-mod(\mathbb{S}\mathrm{p})}(\mathbb{Q},y[n])\widetilde{\to}y[n]$ in $\mathbb{S}\mathrm{p}$. So, $\mathrm{H}^0(\mathcal{M}aps_{\mathbb{Q}-mod(\mathbb{S}\mathrm{p})}(\mathbb{Q},y[n])=0$ for $n>0$. Thus,

$$\mathrm{D}(\mathbb{Q}-mod(\mathbb{S}\mathrm{p}^{\heartsuit}))^+\widetilde{\to}\mathbb{Q}-mod(\mathbb{S}\mathrm{p})^+$$

is an equivalence.

It remains to check that $\mathbb{Q}-mod(\mathbb{S}\mathrm{p})$ is left complete. The functor $\mathrm{oblv}:\mathbb{Q}-mod(\mathbb{S}\mathrm{p})\to\mathbb{S}\mathrm{p}$ reflects limits. So, for $x\in\mathbb{Q}-mod$ the natural map $x\to\lim_n\tau^{\ge -n}x$ is an isomorphism, because $\mathbb{S}\mathrm{p}$ is left complete. Now apply Remark after Corollary 4.0.11, it is applicable, because $\mathbb{S}\mathrm{p}^{\le 0}\subset\mathbb{S}\mathrm{p}$ is stable under products by Remark 4.0.68. This shows that $\mathbb{Q}-mod(\mathbb{S}\mathrm{p})$ is left complete. ☐

It is known that $\mathbb{Q}-mod(\mathbb{S}\mathrm{p})\otimes\mathbb{Q}-mod(\mathbb{S}\mathrm{p})\widetilde{\to}\mathbb{Q}-mod(\mathbb{S}\mathrm{p})$, where the tensor product is taken in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$ and in $\mathcal{P}\mathrm{r}^L$. So, $\mathbb{S}\mathrm{p}\to\mathrm{Vect}_{\mathbb{Q}}$ defined an idempotent in $1-\mathcal{C}at^{St,cocmpl}_{cont}$, and

$$\mathrm{Vect}_{\mathbb{Q}}-mod(1-\mathcal{C}at^{St,cocmpl}_{cont})\subset 1-\mathcal{C}at^{St,cocmpl}_{cont}$$

is a full subcategory.

7. Truncations and homotopy groups

7.1. For $n\geq 0$ a space $X\in \mathrm{Spc}$ is called n-truncated iff $\pi_i(X,x)=0$ for all $x\in X$, $i>n$. We say that X is -1-truncated if it is empty or contractible, X is -2-truncated if X is contractible ([27], preface). Recall that $\pi_i(X,x)$ is defined as $\pi_0(\Omega^i(X,x))$ for the i-th iterated loop space $\Omega^i(X,x)$.

Whitehead theorem: if $X\in\mathrm{Spc}$ and $\pi_i(x,X)=0$ for all $x\in X, i\geq 0$ then $X\widetilde{\to}*$ in Spc.

In fact, for $n\geq -1$ a space $X\in\mathrm{Spc}$ is n-truncated iff for any $x,y\in X$, $\mathrm{Map}_X(x,y)$ is $n-1$-truncated ([27], proof of Cor. 2.3.4.19).

For $n\geq 0$, $X\in\mathrm{Spc}$ is n-connective iff X is nonempty and for any $x\in X,\pi_i(X,x)$ vanish for $i<n$. By definition, any $X\in\mathrm{Spc}$ is -1-connective ([27], preface). A space X is 0-connective iff X is nonempty.

If $f:X\to Y$ is a morphism in Spc then f is a bicartesian fibration in spaces. Let $x\in X$, $y=f(x)$ and $X_y=X\times_Y y$ then there is a long exact sequence of groups (at the end of pointed sets)

$$\pi_{n+1}(Y,y)\to\pi_n(X_y,x)\to\pi_n(X,x)\to\pi_n(Y,y)\to\pi_{n-1}(X_y,x)\to\dots$$
$$\to\pi_1(Y,y)\to\pi_0(X_y)\to\pi_0(X)\to\pi_0(Y)$$

The full subcategory $\tau_{\leq n}\mathrm{Spc}\subset\mathrm{Spc}$ is stable under filtered colimits. Since Idem is filtered, $\tau_{\leq n}\mathrm{Spc}\subset\mathrm{Spc}$ is stable under retracts.

For $\mathcal{C}\in 1-\mathcal{C}at$ let $\mathcal{P}_{\leq n}(\mathcal{C})=\mathrm{Funct}(\mathcal{C}^{op},\tau_{\leq n}\mathrm{Spc})$. If $\mathcal{C}$ is small and equivalent to n-category then $\mathrm{Ind}(\mathcal{C})\subset\mathcal{P}_{\leq n-1}(\mathcal{C})$, and $\mathrm{Ind}(\mathcal{C})$ is also equivalent to n-category ([27], 5.3.5.6). So, if $\mathcal{C}$ is ordinary then $\mathrm{Ind}(\mathcal{C})$ is also ordinary category. If $fSets$ is the category of finite (possibly empty) sets and any morphisms then $\mathrm{Ind}(fSets)$ is the category $\tau_{\leq 0}\mathrm{Spc}$ of sets.

The functor $\pi_i:\mathrm{Spc}_*\to\mathrm{Sets}$ preserves filtered colimits. Indeed, if $X\widetilde{\to}\mathrm{colim}_{i\in I}X_i$ with I filtered in Spc_* then $\Omega(X)\widetilde{\to}\mathrm{colim}_{i\in I}\Omega(X_i)$ by (HTT, 5.3.3.3). Besides, π_0 preserves all colimits, and π_i preserves finite products.

The functor $\pi_0:\mathrm{Spc}\to\mathrm{Sets}$ preserves all products. This gives that $\pi_i:\mathrm{Spc}_*\to\mathrm{Sets}$ preserves all products.

Question: let $X\in\tau_{\leq n}(\mathrm{Spc}_*)$, set $A=\pi_n(X)$, let $K(A,n)\in\mathcal{E}M_n(\mathrm{Spc})$ be the corresponding Eilenberg-MacLane object. Do we have a natural map $K(A,n)\to X$ inducing an isomorphism on π_n? Motivation: if it was an object of Sptr then yes, this would be the map $\tau_{\geq n}X\to X$ in Lurie's notations.

7.1.1. By (HTT, 5.5.8.13) any $G\in\mathrm{Spc}$ can be written as a geometric realization of sets. In turn, every set is a filtered colimit of finite sets. Thus, Spc is generated by finite sets under sifted colimits.

7.2. Let $\mathcal{C} \in 1-\mathcal{C}at$, $X \in \mathcal{C}$. By definition, if $n \geq -1$ then X is n-truncated iff for any $Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is n-truncated. We say that X is -2-truncated iff X is final in $\mathcal{C}$, X is discrete iff X is 0-truncated. Denote by $\tau_{\leq n}\mathcal{C} \subset \mathcal{C}$ the full subcategory spanned by n-truncated objects ([27], 5.5.6.1).

The category $\tau_{\leq 0}\mathcal{C}$ is canonically equivalent to the nerve of its homotopy category and denoted $Disc(\mathcal{C})$, the category of discrete objects. So, $Disc(\mathcal{C})$ is a usual category.

Recall that $\pi_0(X)$ is a set, $\pi_1(X, x)$ is a group, and for $n \geq 2$, $\pi_i(X, x)$ are abelian groups.

The category $\tau_{\leq 0}\,\mathrm{Spc} \subset \mathrm{Spc}$ is precisely the category of sets. For $X \in \mathrm{Spc}$, $\tau_{\leq 0}X \widetilde{\to} \pi_0(X)$ canonically in Spc. The category $\tau_{\leq 1}\,\mathrm{Spc}$ is the category of usual groupoids. For $X \in \mathrm{Spc}$ we have $\tau_{\leq -1}(X) = \emptyset$ for $X = \emptyset$, and $\tau_{\leq -1}(X) = *$ for $X \neq \emptyset$.

If $X \in \mathrm{Spc}$ then $X^{ordn} \in \tau_{\leq 1}\,\mathrm{Spc}$. Do we have $X^{ordn} \widetilde{\to} \tau_{\leq 1}X$ naturally?

7.2.1. A map $f : X \to Y$ is Spc is k-truncated iff any fibre X_y for any $y \in Y$ is k-truncated. A map $f : X \to Y$ in Spc is -2-truncated iff f is an equivalence (iff any fibre is isomorphic to $*$).

If $\mathcal{C} \in 1-\mathcal{C}at$, $f : c \to d$ a morphism in $\mathcal{C}$. Then f is k-truncated iff for any $e \in \mathcal{C}$ the map $\mathrm{Map}_{\mathcal{C}}(e, c) \to \mathrm{Map}_{\mathcal{C}}(e, d)$ is k-truncated. The latter is equivalent to saying that $f \in \mathcal{C}/d$ is k-truncated ([27], 5.5.6.8; 5.5.6.10).

The k-truncated morphisms in $\mathcal{C}$ are preserved under passing to pull-backs ([27], 5.5.6.12).

If $f : x \to y$ is a map in $\mathcal{C} \in 1-\mathcal{C}at$, it is said to be a monomorphism iff for any $z \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(z, x) \to \mathrm{Map}_{\mathcal{C}}(z, y)$ is a full subspace (equivalently, f is -1-truncated).

The claim ([27], 5.5.6.15): let $\mathcal{C} \in 1-\mathcal{C}at$ admit finite limits and $k \geq -1$. A morphism $f : c \to c'$ in $\mathcal{C}$ is k-truncated iff the diagonal map $c \to c \times_{c'} c$ is $(k-1)$-truncated.

For $\mathcal{C} \in 1-\mathcal{C}at$, ${}_{\tau_{\leq -2}}\mathcal{C} \subset \mathcal{C}$ is the full subcategory of final objects. If $\mathcal{C}$ admits a final object 1 then ${}_{\tau_{\leq -1}}\mathcal{C} \subset \mathcal{C}$ is the full subcategory spanned by the subobjects of 1.

7.2.2. If $\mathcal{C}, K \in 1-\mathcal{C}at$. Assume $\tau_{\leq n}\mathcal{C} = \mathcal{C}$ then $\mathrm{Funct}(K, \mathcal{C})$ has the same property ([27], 2.3.4.20). This implies that the canonical inclusion $\mathrm{Fun}(\mathcal{C}, \tau_{\leq n}\,\mathrm{Spc}) \hookrightarrow \tau_{\leq n}\mathrm{Fun}(\mathcal{C}, \mathrm{Spc})$ is an equivalence. Indeed, given $Y \in \tau_{\leq n}\mathcal{P}(\mathcal{C})$ for $c \in \mathcal{C}$ we get $\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(c, Y) \widetilde{\to} Y(c)$. So, $Y(c)$ is n-truncated. The truncation functor

$$\tau_{\leq n} : \mathrm{Fun}(\mathcal{C}, \mathrm{Spc}) \to \mathrm{Fun}(\mathcal{C}, \tau_{\leq n}\,\mathrm{Spc})$$

is obtained from $\tau_{\leq n} : \mathrm{Spc} \to \tau_{\leq n}\,\mathrm{Spc}$ applying $\mathrm{Fun}(\mathcal{C}, \cdot)$.

7.2.3. Recall that $1-\mathcal{C}at^{ordn} \subset 1-\mathcal{C}at$ denotes the full subcategory of ordinary categories. Is it $\tau_{\leq 1}(1-\mathcal{C}at)$? If $\mathcal{A}, \mathcal{B} \in 1-\mathcal{C}at^{ordn}$ then $\mathrm{Funct}(\mathcal{A}, \mathcal{B}) \in 1-\mathcal{C}at^{ordn}$ and $\mathrm{Map}_{1-\mathcal{C}at}(\mathcal{A}, \mathcal{B}) \in \tau_{\leq 1}\,\mathrm{Spc}$. So, $\mathcal{B}$ is 1-truncated in $1-\mathcal{C}at$. We have shown that $1-\mathcal{C}at^{ordn} \subset \tau_{\leq 1}(1-\mathcal{C}at)$ is a full subcategory.

Given $\mathcal{B} \in 1-\mathcal{C}at$, we have $\mathcal{B} \in \tau_{\leq 0}(1-\mathcal{C}at)$ iff $\mathcal{B}$ is a usual category such that for any $b \in \mathcal{B}$, $\mathrm{Aut}_{\mathcal{B}}(b)$ is trivial. Indeed, since $\mathcal{B}^{\mathrm{Spc}}$ is 0-truncated and $\mathrm{Map}_{1-\mathcal{C}at}([1], \mathcal{B})$ is 0-truncated, we see that each $\mathrm{Map}_{\mathcal{B}}(b, b')$ is a set. Conversely, for $\mathcal{B}$ as above and any $\mathcal{A} \in 1-\mathcal{C}at$, $\mathrm{Map}_{1-\mathcal{C}at}(\mathcal{A}, \mathcal{B}) \widetilde{\to} \mathrm{Funct}(\mathcal{A}^{ordn}, \mathcal{B})^{\mathrm{Spc}}$ is a set.

7.2.4. If $\mathcal{C}$ is presentable, the inclusion $\tau_{\leq n}\mathcal{C} \subset \mathcal{C}$ admits an (accessible) left adjoint $\tau_{\leq n} : \mathcal{C} \to \tau_{\leq n}\mathcal{C}$ ([27], 5.5.6.18), and $\tau_{\leq n}\mathcal{C}$ is also presentable ([27], 5.5.6.21).

If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is presentable, a simplicial resolutions in $\mathcal{C}$ is an augmented simplicial object $U^+_\bullet : (\boldsymbol{\Delta}_+)^{op} \to \mathcal{C}$, which is a colimit of the underlying simplicial object $U_\bullet : \boldsymbol{\Delta}^{op} \to \mathcal{C}$. Let $\mathrm{Res}(\mathcal{C}) \subset \mathrm{Funct}((\boldsymbol{\Delta}_+)^{op}, \mathcal{C})$ be the full subcategory spanned by simplicial resolutions. Then $\mathrm{Res}(\mathcal{C}) \to \mathrm{Funct}(\boldsymbol{\Delta}^{op}, \mathcal{C})$ is an equivalence ([27], after 6.1.4.3).

For a topos (actually, for a semi-topos) $\mathcal{C}$, Lurie defines a notion of effective epimorphism. Namely, let $\mathrm{Res}_{Eff}(\mathcal{C}) \subset \mathrm{Funct}((\boldsymbol{\Delta}_+)^{op}, \mathcal{C})$ be the full subcategory spanned by Cech nerves, which are simplicial resolutions. A map $f : U \to X$ in $\mathcal{C}$ is an effective epimorphism iff the Cech nerve $\check{C}(f)$ is a simplicial resolution. The restriction functor $\mathrm{Funct}((\boldsymbol{\Delta}_+)^{op}, \mathcal{C}) \to \mathrm{Funct}([1], \mathcal{C})$, where $[1] = (\boldsymbol{\Delta}_{+,\leq 0})^{op}$, identifies $\mathrm{Res}_{Eff}(\mathcal{C})$ with the full subcategory of $\mathrm{Funct}([1], \mathcal{C})$ spanned by effective epimorphisms ([27], 6.2.3.5).

For example, a map $f : X \to Y$ in Spc is an effective epimorphism iff $\pi_0(X) \to \pi_0(Y)$ is surjective ([27], 7.2.1.15).

More generally, for an ∞-topos $\mathcal{X}$ a morphism $\phi : u \to x$ in $\mathcal{X}$ is an effective epimorphism iff $\tau_{\leq 0}\phi : \tau_{\leq 0}u \to \tau_{\leq 0}x$ is an effective epimorphism in the ordinary topos $\tau_{\leq 0}\mathcal{X}$ ([27], 7.2.1.14). So, for $C \in 1-\mathcal{C}\mathrm{at}$ a map $f \to g$ in $\mathrm{Fun}(C, \mathrm{Spc})$ is an effective epimorphism iff for any $c \in C$, $\pi_0(f(c)) \to \pi_0(g(c))$ is surjective.

If $\mathcal{C}$ is an ∞-topos then the functor $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}$ preserves finite products (HTT, 6.5.1.2).

7.2.5. If $\mathcal{C}$ is κ-compactly generated then $c \in \mathcal{C}$ is n-truncated iff for any $d \in \mathcal{C}^\kappa$, $\mathrm{Map}_{\mathcal{C}}(d, c)$ is n-truncated (cf. HTT, proof of 5.5.7.4).

7.2.6. Let x be an object in an ∞-topos $\mathcal{X}$ and $n \geq -1$. We say that x is n-connective iff $\tau_{\leq n-1}x$ is a final object of $\mathcal{X}$ ([27], 6.5.1.12). Every object of $\mathcal{X}$ is (-1)-connective. Equivalently, x is $(n+1)$-connective iff the natural map $\mathrm{Map}_{\mathcal{X}}(1, y) \to \mathrm{Map}_{\mathcal{X}}(x, y)$ is an equivalence for all n-truncated objects y in $\mathcal{X}$ (after 6.5.1.13).

For $x \in \mathcal{X}$ we say that x is *connected* iff x is 1-connective, that is, $\tau_{\leq 0}x$ is a final object of $\mathcal{X}$. This is equivalent to the property that any map $1 \to x$ is an effective epimorphism (here 1 is a final object of $\mathcal{X}$), see the proof of ([27], 7.2.2.11).

The homotopy group of $X \in \mathcal{X}$ are defined in ([27], 6.5.1.1). Namely, if S^n is n-sphere with a based point, then $* \to S^n$ yields a map $s : X^{S^n} \to X$ in $\mathcal{X}$, so $s \in \mathcal{X}/X$. Then $\pi_n(X) = \tau_{\leq 0}s \in \mathcal{X}/X$.

Now if $f : X \to Y$ is a morphism in the ∞-topos $\mathcal{X}$ then for $0 \leq n \leq \infty$ Lurie says that f is n-connective iff it is an effective epimorphism and $\pi_k(f) \widetilde{\to} *$ for $0 \leq k < n$. Every morphism is -1-connective ([27], 6.5.1.10). In fact, $f : X \to Y$ is n-connective iff f is n-connective in the ∞-topos $\mathcal{X}/Y$, this is equivalent to the property that the $n-1$-truncation of f is an equivalence $X' \widetilde{\to} Y$ in $\mathcal{X}$.

Example: let $X \to Y$ be a map in an ∞-topos $\mathcal{X}$, $y \in Y$, X_y the fibre at y. Let $1 \in \mathcal{X}$ be a final object. Let $n \geq 0$. If $\tau_{\leq 0}X \to 1$ is an effective epimorphism in $\tau_{\leq 0}\mathcal{X}$ and X_y, Y are n-connective then X is n-connective.

Remark 7.2.7. *Let $f : x \to y$ be a map in an ∞-topos $\mathcal{X}$. If f is n-connective then $\tau_{\leq n-1}x \to \tau_{\leq n-1}y$ is an isomorphism in $\mathcal{X}$.*

Proof. Let $z \in \tau_{\leq n-1}\mathcal{X}$. It suffices to show that the natural map $\mathrm{Map}_{\mathcal{X}}(y,z) \to \mathrm{Map}_{\mathcal{X}}(x,z)$ is an isomorphism. The functor $\mathcal{X} \to \mathcal{X}_{/y}, h \mapsto h \times y$ is a right adjoint, hence left exact, so sends r-truncated objects to r-truncated. Now apply (HTT, 6.5.1.14) for the $(n-1)$-truncated object $z \times y \to y$ of $\mathcal{X}_{/y}$. □

7.2.8. Let $\mathcal{X}$ be a topos. The category $\mathcal{E}\mathrm{M}_n(\mathcal{X})$ of Eilenberg-Maclane objects of degree n in $\mathcal{X}$ is defined in ([27], 7.2.2.1). This is the full subcategory of $\mathrm{Funct}([1], \mathcal{X})$ classifying pointed objects $1 \to x, x \in \mathcal{X}$ such that 1 is final in $\mathcal{X}$, and x is both n-truncated and n-connective. This makes sense at least for $n \geq 0$.

For example, $\mathcal{E}\mathrm{M}_0(\mathrm{Spc})$ is the category of nonempty pointed sets $* \to x$, where $x \in \mathrm{Sets}$.

For $\mathcal{X}$ an ∞-topos, $\mathcal{E}\mathrm{M}_0(\mathcal{X}) \widetilde{\to} Disc(\mathcal{X})_*$ is the category of pointed objects of $Disc(\mathcal{X})$ ([27], 7.2.2.12). If $x \in Disc(\mathcal{X})$, 1 is a final object of $\mathcal{X}$, then $x \to 1$ is an effective epimorphism in $\mathcal{X}$.

For a given ∞-topos $\mathcal{X}$ starting from $\mathcal{E}\mathrm{M}_0(\mathcal{X})$ one may recover $\mathcal{E}\mathrm{M}_n(\mathcal{X})$ for $n \geq 0$ using ([27], 7.2.2.11). Namely, let $\mathcal{X}^0 \subset \mathcal{X}$ be the full subcategory of connected objects, $Ptd(\mathcal{X}^0)$ be the pointed category of $\mathcal{X}^0$. Then for $n \geq 1$ we have a fully faithfull embedding $\mathcal{E}\mathrm{M}_n(\mathcal{X}) \subset Ptd(\mathcal{X}^0)$, which identifies under the equivalence $Ptd(\mathcal{X}^0) \widetilde{\to} \mathcal{G}\mathrm{rp}(\mathcal{X})$ with the full subcategory $\mathcal{G}\mathrm{rp}(\mathcal{E}\mathrm{M}_{n-1}(\mathcal{X})) \subset \mathcal{G}\mathrm{rp}(\mathcal{X}_*) \widetilde{\to} \mathcal{G}\mathrm{rp}(\mathcal{X})$.

7.2.9. If $S \in 1-\mathcal{C}\mathrm{at}$ then the final object of $\mathcal{P}(S) = \mathrm{Funct}(S^{op}, \mathrm{Spc})$ is the constant presheaf with value $*$. An object $F \in \mathcal{P}(S)$ is -1-truncated iff it is a subobject of the constant sheaf $*$. There could me many of them. For example, if Z is a topological space and S is the category of open subsets in Z then any $u \in S$ gives a -1-truncated object $\mathrm{Map}_S(\cdot, u)$. What is the description of (-1)-truncated objects in $\mathcal{P}(\mathcal{C})$, where $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is arbitrary?

If $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$, $F \in \mathcal{P}(\mathcal{C})$, $c \in \mathcal{C}$ then $F(c) \widetilde{\to} \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(c, F)$, where we identified c with the image of its Yoneda embedding ([27], 5.5.2.1). So, if $F \in Disc(\mathcal{P}(\mathcal{C}))$ then for any $c \in \mathcal{C}$, $F(c)$ is a set, and $F : \mathcal{C} \to \mathrm{Sets}$ is a functor. So, F factors canonically through a functor $\bar{F} : \mathcal{C}^{ordn} \to \mathrm{Sets}$. We constructed a functor $Disc(\mathcal{P}(\mathcal{C})) \to \mathrm{Funct}(\mathcal{C}^{ordn}, \mathrm{Sets})$. This is an equivalence. The subcategory $\tau_{\leq -1}(\mathcal{P}(\mathcal{C})) \subset \mathrm{Funct}(\mathcal{C}^{ordn}, \mathrm{Sets})$ is the full subcategory spanned by the subobjects of the terminal object(=constant presheaf with value $*$).

If now $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ is equipped with a Grothendieck topology, consider the category $Shv(\mathcal{C})$ of sheaves on $\mathcal{C}$ with respect to this topology ([27], 6.2.2.6). Is it true that $Shv(\mathcal{C}^{ordn}, \mathrm{Sets})$ is canonically equivalent to $Disc(Shv(\mathcal{C}))$? For any $c \in \mathcal{C}$ write $\bar{c}$ for the sheafification of c. For $F \in Shv(\mathcal{C})$ we get $\mathrm{Map}_{Shv(\mathcal{C})}(\bar{c}, F) \widetilde{\to} \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(c, F) \widetilde{\to} F(c)$. If $F \in Disc(Shv(\mathcal{C}))$ then $F(c)$ has to be a set for any $c \in \mathcal{C}$. So, $F : \mathcal{C} \to \mathrm{Spc}$ factors through $\mathrm{Sets} \subset \mathrm{Spc}$, hence also factors through $\bar{F} : \mathcal{C}^{ordn} \to \mathrm{Sets}$. The resulting $\bar{F}$ has to be a sheaf in this Grothendieck topology (which is really given on $\mathcal{C}^{ordn}$).

7.2.10. (HTT, 6.2.3.20) is model independent, could be useful. For an ∞-topos $\mathcal{X}$ and a small $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ with a Grothendieck topology, it describes the left exact colimit-preserving functors $Shv(\mathcal{C}) \to \mathcal{X}$ as some full subcategory in $\mathrm{Fun}(\mathcal{C}, \mathcal{X})$.

7.2.11. If $\mathcal{C}$ is a usual category admitting finite products, write $\mathcal{G}\mathrm{rp}(\mathcal{C})$ for the category of groups in $\mathcal{C}$, $Ab(\mathcal{C})$ for the category of abelian group in $\mathcal{C}$. It is easy to see that $\mathcal{G}\mathrm{rp}(\mathcal{G}\mathrm{rp}(\mathcal{C}))\widetilde{\to} Ab(\mathcal{C})$ canonically. Besides, $\mathcal{G}\mathrm{rp}(Ab(\mathcal{C}))\widetilde{\to} Ab(\mathcal{C})$ canonically (used in [27], 7.2.2.12).

7.2.12. If $\mathcal{C}$ is an ∞-topos, 1 the final object of $\mathcal{C}$. Lurie mentions that $Disc(\mathcal{C})$ is also a topos. By definition $x \in \mathcal{C}$ is connected iff $\tau_{\le 0} x$ is a final object in $\mathcal{C}$. Denote by $Ptd(\mathcal{C})$ the ∞-category of pointed objects $1 \to x, x \in \mathcal{C}$. Denote by $\mathcal{C}^0 \subset \mathcal{C}$ the full subcategory spanned by connected objects. Then the inclusion $Ptd(\mathcal{C}^0) \hookrightarrow Ptd(\mathcal{C})$ admits a right adjoint F, and F sends $1 \to x$ to $1 \to x^0$. Here $1 \widetilde{\to} \tau_{\le 0} 1 \to \tau_{\le 0} x$ is the induced map, and $x^0 = x \times_{\tau_{\le 0}(x)} 1$. For $x \in \mathrm{Ptd}(\mathcal{C})$ we have a canonical map in $\mathrm{Ptd}(\mathcal{C})$

$$x \times_{\tau_{\le 0}(x)} 1 \to x$$

Indeed, for any $y \in \mathcal{X}$ has has the evident map $\mathrm{Map}(y, x \times_{\tau_{\le 0}(x)} 1) \to \mathrm{Map}(y, x)$ functorial in $y \in \mathcal{C}$. I think it induces an isomorphism $\Omega(x \times_{\tau_{\le 0}(x)} 1) \widetilde{\to} \Omega(x)$, but I have not checked this.

Consider the functor $\Omega : Ptd(\mathcal{C}) \to \mathcal{G}\mathrm{rp}(\mathcal{C})$. We may now derive the existence of the left adjoint B to this functor from ([27], 7.2.2.11). Moreover, we see that B is the composition

$$\mathcal{G}\mathrm{rp}(\mathcal{C}) \widetilde{\to} Ptd(\mathcal{C}^0) \hookrightarrow Ptd(\mathcal{C}),$$

it sends G to $B(G) \widetilde{\to} \mathrm{colim}_{[n] \in \boldsymbol{\Delta}^{op}} G^n$ taken in C. The inclusion $Ptd(\mathcal{C}^0) \hookrightarrow Ptd(\mathcal{C})$ is stable under finite products. So, the functor $B : \mathcal{G}\mathrm{rp}(\mathcal{C}) \to \mathrm{Ptd}(\mathcal{C})$ preserves finite products. By (HTT, 7.2.2.5), B sends $\mathcal{G}\mathrm{rp}(\mathcal{G}\mathrm{rp}(\mathcal{C}))$ to $\mathcal{G}\mathrm{rp}(\mathrm{Ptd}(\mathcal{C}))$.

Let $H \in \mathrm{ComGrp}(\mathcal{C})$ act on some $G \in \mathcal{G}\mathrm{rp}(\mathcal{C})$ via a group homomorphism $H \to G$, which is "central". To be precise, we will mean by this that the diagram $\boldsymbol{\Delta}^{op} \to \mathcal{C}$, $[n] \mapsto H^n \times G$ defining the H-action on G, is actually a diagram $\boldsymbol{\Delta}^{op} \to \mathcal{G}\mathrm{rp}(\mathcal{C})$. Applying the functor B to this diagram, we get an action of $B(H)$ on $B(G)$. In particular, the action map $H \times G \to G$ is a morphism in $\mathcal{G}\mathrm{rp}(\mathcal{C})$, so we may apply B to this map.

Question: assume $f : G \to K$ is a morphism in $\mathcal{G}\mathrm{rp}(\mathcal{C})$, which is an effective epimorphism, and H is the fibre of G. Assume $H \in ComGrp(\mathcal{C})$ and H is "central" in G. Under these assumptions, we would like to conclude that the quotient of $B(G)$ by the action of $B(H)$ is $B(K)$. That is, we get a map $\beta : B(K) \to B^2(H)$ in $\mathrm{Ptd}(\mathcal{C})$, which by adjuntion corresponds to $K \to B(H)$. Can we conclude that the fibre of β is $B(G)$?

7.2.13. If $\mathcal{X}, \mathcal{Y}$ are ∞-topoi, a geometric morphism from $\mathcal{X}$ to $\mathcal{Y}$ is a functor $f_* : \mathcal{X} \to \mathcal{Y}$ which admits a left exact left adjoint (denoted f^*). The left exactness of f^* means that f^* preserves finite limits.

7.2.14. Let $\mathcal{X}$ be an ∞-topos, $\mathcal{G} \in \mathcal{G}\mathrm{rp}(\mathcal{X})$ and $G = \mathcal{G}([1]) \in \mathcal{X}$. Let $P \in \mathcal{X}$. An ∞-action of G on P is defined as follows. Write $\mathcal{X}^{Mon^+}$ for the category of left modules over a monoid in $\mathcal{X}$ defined in [14]. Let $\mathcal{X}^{Mon}$ be the category of monoids in $\mathcal{X}$. Let $\mathcal{M} \in \mathcal{X}^{Mon^+}$ be such that the underlying monoid is $\mathcal{G}$, and $\mathcal{M}([0]^+) = P$. So, G acts on

P on the left. Recall that $\boldsymbol{\Delta}^+ \widetilde{\to} \boldsymbol{\Delta} \times [1]$, let $j : \boldsymbol{\Delta} \to \boldsymbol{\Delta}^+$ be the map sending $[n]$ to $[n]^+$ and defined naturally on morphisms. In other words,

$$j : \boldsymbol{\Delta} \widetilde{\to} \boldsymbol{\Delta} \times \{1\} \hookrightarrow \boldsymbol{\Delta} \times [1] \widetilde{\to} \boldsymbol{\Delta}^+$$

Then $\mathcal{M} \circ j : \boldsymbol{\Delta}^{op} \to \mathcal{X}$ is a groupoid. The colimit of $\mathcal{M} \circ j$ is called the quotient P/G of the action of G on P. We have a natural map $P/G \to B(G)$. Recall that $G - mod$ is defined as $\mathcal{X}^{Mon^+} \times_{\mathcal{X}^{Mon}} \{\mathcal{G}\}$. Here is the corresponding diagram

$$\begin{array}{ccccccc} G \times G \times P & \substack{\to\\\to\\\to} & G \times P & \substack{\to\\\to} & P & \to & P/G \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G \times G \times 1 & \substack{\to\\\to\\\to} & G \times 1 & \substack{\to\\\to} & 1 & \to & B(G) \end{array}$$

Here $1 \in \mathcal{X}$ is a final object. The two maps $G \times P \to P$ are the action and the projection. In the above diagram both rows are Cech nerves. Moreover, the top row is obtained by the base change $P/G \to B(G)$ from the low row.

So, a G-action on P is simply a datum of an object $Q \to B(G)$ together with an isomorphism $P \widetilde{\to} 1 \times_{B(G)} Q$.

A principal G-bundle over some $Y \in \mathcal{X}$ is a G-action on some P and an isomorphism $P/G \widetilde{\to} Y$. In other words, the ∞-category $GBun(Y)$ of principal G-bundles on Y is defined as $G - mod \times_{\mathcal{X}} \{Y\}$, where the map $G - mod \to \mathcal{X}$ sends a left module (G, P) to P/G.

We have a natural map $\mathrm{Map}_{\mathcal{X}}(Y, B(G)) \to GBun(Y)$ sending $Y \to B(G)$ to its fibre. The fact that P indeed is a principal G-bundle over Y follows from the fact that the colimits are universal. Now ([43], 3.17) says that the above map is an equivalence

$$GBun(Y) \widetilde{\to} \mathrm{Map}_{\mathcal{X}}(Y, B(G))$$

in $1 - \mathcal{C}\mathrm{at}$. In particular, $GBun(X)$ is a space. The map in the opposite direction is given in ([43], 3.13).

COMPARE with the results of ([27], 7.2.2.25 and around)! Related exposition is in `https://ncatlab.org/nlab/show/principal+infinity-bundle`

The projection $G - mod \to \mathcal{X}$, $(G, P) \mapsto P$ preserves small limits. Suppose $P_1 \to P \leftarrow P_2$ is a diagram in $G - mod$ admitting the fibre product $P_1 \times_P P_2$ in $G - mod$ (this fibred product always exists). The functor $F : G - mod \to \mathcal{X}$ sending (G, Q) to the quotient Q/G preserves this particular limit. Indeed, $G - mod \widetilde{\to} \mathcal{X}_{/B(G)}$. Our claim follows from the fact that the projection $\mathcal{X}_{/B(G)} \to \mathcal{X}$ preserves the fibred products $\bar{P}_1 \times_{\bar{P}} \bar{P}_2$. Note that this projection does not preserve all limits!

7.2.15. For $n \geq 1$ let $\mathcal{Y} \subset 1 - \mathcal{C}\mathrm{at}$ be the full subcategory spanned by categories $\mathcal{C}$ such that for any $x, y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(x, y)$ is $n - 1$-truncated. In Lurie's terminology ([27], 2.3.4.18), $\mathcal{C}$ is equivalent to an n-category. Then the inclusion $\mathcal{Y} \hookrightarrow 1 - \mathcal{C}\mathrm{at}$ admits a left adjoint $1 - \mathcal{C}\mathrm{at} \to \mathcal{Y}$ by ([27], 2.3.4.14). In ([28], 5.1.1.7) it appears under the name the "homotopy n-category".

7.2.16. Let $G \in \mathcal{G}\mathrm{rp}(\mathrm{Spc})$, $B(G) \in Ptd(\mathrm{Spc})$ be the corresponding classifying space. Then viewing $B(G)$ as an object of Spc, one has $\mathrm{Map}_{B(G)}(1, 1) \widetilde{\to} G$ for its point 1 :

$* \to B(G)$. Indeed, if $H \subset G$ is a subgroup then $B(H) \times_{B(G)} * \widetilde{\to} G/H$. Apply this to the diagonal embedding $G \subset G \times G$, we get $\mathrm{Map}_{B(G)}(1,1) \widetilde{\to} B(G) \times_{B(G)\times B(G)} * \widetilde{\to} G$.

7.2.17. Let $\mathcal{X}$ be an ∞-topos, $G \in \mathcal{G}\mathrm{rp}(\mathcal{X})$. For any map $Y_1 \to Y_2$ in $G-mod(\mathcal{X})$, the square is cartesian

$$\begin{array}{ccc} Y_1 & \to & Y_2 \\ \downarrow & & \downarrow \\ Y_1/G & \to & Y_2/G \end{array}$$

This should be by definition. Namely, $G-mod$ is the category $\mathcal{X}_{/B(G)}$, so a map in $G-mod$ is by definition a map $Y_1/G \to Y_2/G$ over $B(G)$. So, making the base change by $* \to B(G)$, we get the above cartesian square.

7.2.18. Let $f : G \to H$ be a map in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$ which is an effective epimorphism, that is, $\pi_0(G) \to \pi_0(H)$ is surjective. Let K be the fibre of f in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$. We show that $B(K)$ is the fibre of the natural map $B(G) \to B(H)$.

Write Z for this fibre, we have an exact sequence

$$\ldots \to \pi_1(Z) \to \pi_1(B(G)) \to \pi_1(B(H)) \to \pi_0(Z) \to \pi_0(B(G)) \widetilde{\to} *$$

here $\pi_1(B(G)) \widetilde{\to} \pi_0(G), \pi_1(B(H)) \widetilde{\to} \pi_0(H)$, so Z is connected. Let $\mathrm{Spc}^0 \subset \mathrm{Spc}$ be the full subcategory of connected spaces. Recall that $Ptd(\mathrm{Spc}^0) \widetilde{\to} \mathcal{G}\mathrm{rp}(\mathrm{Spc}), U \mapsto \Omega U$ is an equivalence. So, Z is recovered from ΩZ. We have an isomorphism $K \widetilde{\to} G \times_H * \widetilde{\to} \Omega(Z)$ in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$, hence also in Spc, as the projection $\mathcal{G}\mathrm{rp}(\mathrm{Spc}) \to \mathrm{Spc}$ preserves limits. On the other hand, $\Omega B(K) \widetilde{\to} K$ in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$, so $Z \widetilde{\to} B(K)$ indeed.

We get a left action of K on G by left translation by restricting the diagram

$$\ldots G \times G \times G \overset{\to}{\underset{\to}{\to}} G \times G \rightrightarrows G$$

to $\ldots K \times K \times G \overset{\to}{\underset{\to}{\to}} K \times G \rightrightarrows G$. My understanding is that the quotient of G by this action is H, so we get a map $H \to B(K)$ whose fibre is G.

My understanding is that for any $Y \in \mathrm{Spc}$ with a G-action, one has canonically $Y/K \widetilde{\to} Y/G \times_{B(G)} B(K)$ in Spc. If yes then for any G-morphism $Y \to Y'$ in Spc, the square is cartesian

$$\begin{array}{ccc} Y/G & \to & Y'/G \\ \uparrow & & \uparrow \\ Y/K & \to & Y'/K \end{array}$$

If now $Y \in \mathrm{Spc}$ is equipped with a G-action then Y/K is equipped with a H-action such that $(Y/K)/H \widetilde{\to} Y/G$. This follows from the diagram, where both squares are cartesian

$$\begin{array}{ccccc} Y/G & \to & B(G) & \to & B(H) \\ \uparrow & & \uparrow & & \uparrow \\ Y/K & \to & B(K) & \to & pt \end{array}$$

Assume in addition that $K \in \mathrm{ComGrp}(\mathrm{Spc})$. Then $B(K) \in \mathrm{ComGrp}(\mathrm{Spc})$. By definition, we say that the extension is central if the map $\alpha : H \to B(K)$ is a morphism in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$. In this case applying B it yields a morphism $\bar{\alpha} : B(H) \to B^2(K)$. Let Z be the fibre of $\bar{\alpha}$. The exact sequence $\pi_1(B^2(K)) \to \pi_0(Z) \to \pi_0(B(H))$ shows that Z is connected. Further, $\Omega Z \widetilde{\to} H \times_{B(K)} * \widetilde{\to} G$, so $Z \widetilde{\to} B(G)$. This means that $B(G) \to B(H)$ is a $B(K)$-torsor.

Is it true that H is the cofibre of $K \to G$ in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$?

Remark: if $K \in E_2^{grp-like}(\mathrm{Spc})$ then we may still define a central extension of H by K as a morphism $\alpha : H \to B(K)$ in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$.

7.2.19. If $\mathcal{X}$ is an ∞-topos then the filtered colimits in $\mathcal{X}$ are left exact (HTT, 7.3.4.7).

7.2.20. Let $\mathcal{X}$ be an ∞-topos, $G, H \in \mathcal{G}\mathrm{rp}(\mathcal{X})$. Then $B(G \times H) \widetilde{\to} B(G) \times B(H)$ in $\mathcal{X}$. Indeed, $B(G) \widetilde{\to} \operatorname{colim}_{[n] \in \boldsymbol{\Delta}^{op}} G([n])$ in $\mathcal{X}$, so $B(G \times H) \widetilde{\to} \operatorname{colim}_{[n] \in \boldsymbol{\Delta}^{op}} G([n]) \times H([n]) \widetilde{\to} \operatorname{colim}_{[n],[m] \in \boldsymbol{\Delta}^{op}} G([n]) \times H([m]) \widetilde{\to} B(G) \times B(H)$. We used that $\boldsymbol{\Delta}^{op}$ is sifted, and the colimits are universal.

7.2.21. Let $f : G \to H$ be a morphism in $\mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, assume for each $S \in \mathrm{Sch}$, $G(S) \to H(S)$ is an effective epimorphism in Spc, that is, $\pi_0(G(S)) \to \pi_0(H(S))$ is surjective. Let K be the fibre of f in $\mathcal{G}\mathrm{rp}(\mathrm{PreStk})$. Then $B(K)$ is the fibre of the map $B(G) \to B(H)$ in PreStk.

Indeed, we have to show that for any $S \in \mathrm{Sch}$, $B(K(S))$ is the fibre of $B(G(S)) \to B(H(S))$ in Spc, where $B(G(S))$ is the classifying space of $G(S)$ in Spc and same for $B(H(S))$. However, $K(S)$ is the fibre of $G(S) \to H(S)$ in $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$, and our claim follows from Section 7.2.18.

7.3. Comment to [26]. Consider the full subcategory of $\mathcal{G}\mathrm{rp}(\mathrm{Spc})$ spanned by $\mathcal{G} \in \mathcal{G}\mathrm{rp}(\mathrm{Spc})$ such that $\pi_i(\mathcal{G}) = 0$ for $i > 1$. In other words, this are groups in usual groupoids. This $(\infty, 1)$-category is described in [26] essentially. More precisely, for $\mathcal{G}$ in that category, $G := \pi_0(\mathcal{G})$ is a group, and $M := \pi_1(\mathcal{G})$ is a G-module. Now for a given group G and a G-module M they describe the category of $\mathcal{G} \in \mathcal{G}\mathrm{rp}(\mathrm{Spc})$ with $\pi_i(\mathcal{G}) = 0$ for $i > 1$ and given isomorphisms $\pi_0(\mathcal{G}) \widetilde{\to} G$, $\pi_1(\mathcal{G}) \widetilde{\to} M$ as G-modules. The answer is the 2-category denoted $\mathcal{H}^3$ in ([26], Section 6), essentially given in terms of $\mathrm{H}^3(G, M)$.

For example, for any abelian group M, $\mathrm{H}^3(\mathbb{Z}, M) = 0$ and $\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, M) \widetilde{\to} M_n = \{m \in M \mid nm = 0\}$.

7.3.1. Assume now in addition that G, M are abelian groups, and M is the trivial G-module. They they define "abelian cohomology" group $\mathrm{H}^3_{ab}(G, M)$. It is shown that $\mathrm{H}^3_{ab}(G, M) \widetilde{\to} \mathrm{Quad}(G, M)$ is the group of M-valued quadratic forms on G. They define a 2-category $\mathcal{H}^3_{ab}$, which is a kind of categorification of this $\mathrm{H}^3_{ab}(G, M)$. Let $\mathbb{E}_2^{grp-like}(\mathrm{Spc})_{G,M}$ be the category of $\mathcal{G} \in \mathbb{E}_2^{grp-like}(\mathrm{Spc})$ equipped with isomorphisms of abelian groups $\pi_0(\mathcal{G}) \widetilde{\to} G$, $\pi_1(\mathcal{G}) \widetilde{\to} M$ and such that $\pi_i(\mathcal{G}) = 0$ for $i > 1$. They define an equivalence

$$\mathcal{H}^3_{ab} \to \mathbb{E}_2^{grp-like}(\mathrm{Spc})_{G,M}$$

There is a complete description of $\mathcal{H}^3_{ab}$ in their Proposition 15. The set of isomorphism classes of $\mathbb{E}_2^{grp-like}(\mathrm{Spc})_{G,M}$ is given by $\mathrm{Quad}(G, M)$.

Similarly, one may define the category $\mathcal{H}^2_{ab}(G, M)$, it is equivalent to the usual groupoid of extensions $0 \to M \to ? \to G \to 0$ (in the category of abelian groups).

We know by ([28], 5.2.6.15) that $\mathbb{E}_2^{grp-like}(\mathrm{Spc})\tilde{\to}(\mathrm{Spc}_*)^{\geq 2}$, the category of 2-connective pointed spaces. How the two claims are related? As far as I understand, the functor Ω^2 will give an equivalence

$$\{\mathcal{G}\in(\mathrm{Spc}_*)^{\geq 2}\mid \pi_2(\mathcal{G})=G, \pi_3(\mathcal{G})=M, \pi_i(\mathcal{G})=0 \text{ for } i>3\}\tilde{\to}\mathbb{E}_2^{grp-like}(\mathrm{Spc})_{G,M}$$

Recall that, using the cohomological grading conventions, $\mathrm{Sptr}^{\leq 0}\tilde{\to}\mathrm{ComGrp}(\mathrm{Spc})$ canonically, the RHS is the category of commutative groups in Spc. Given abelian groups G, M in Sets, let $\mathrm{ComGrp}(\mathrm{Spc})_{G,M}$ be the category of $\mathcal{G}\in\mathrm{ComGrp}(\mathrm{Spc})$ equipped with $\pi_0(\mathcal{G})\tilde{\to}G, \pi_1(\mathcal{G})\tilde{\to}M$ such that $\pi_i(\mathcal{G})=0$ for $i>1$. The isomorphism classes of $\mathrm{ComGrp}(\mathrm{Spc})_{G,M}$ are $\mathrm{Ext}^2_{\mathrm{Sptr}}(G,M)$.

Question 1. Do we have canonically $\mathrm{Ext}^2_{\mathrm{Sptr}}(G,M)\tilde{\to}\mathrm{Hom}(G,M_2)$, where $M_2=\{m\in M\mid 2m=0\}$? What is the reference?

Question 2. Consider the forgetful functor $\mathrm{ComGrp}(\mathrm{Spc})_{G,M}\to\mathbb{E}_2^{grp-like}(\mathrm{Spc})_{G,M}$. Is it true that on the level of isomorphism classes it induces the natural inclusion $\mathrm{Hom}(G,M_2)\to\mathrm{Quad}(G,M)$?

Question 3. It seems there should be some intermediate object between the two corresponding to the subgroup $\mathrm{Quad}(G,M_2)\subset\mathrm{Quad}(G,M)$. The latter subgroup contains $\mathrm{Hom}(G,M_2)$, but is strictly bigger in general! What is it?

7.3.2. The projection $\mathcal{G}rp(\mathrm{Spc})\to\mathrm{Spc}$ preserves limits. Let $G\to H$ be a morphism in $\mathcal{G}rp(\mathrm{Spc})$, let Z be its fibre in $\mathcal{G}rp(\mathrm{Spc})$, hence also in Spc. Write G/Z for the quotient of G by Z say acting by right translations (quotient in the sense of the topos Spc). We have the induced map $f: G/Z\to H$. We claim that f is a monomorphism of spaces.

Proof: Since colimits in Spc are universal,

$$(G/Z)\times_H(G/Z)\tilde{\to}\underset{[n],[m]\in\boldsymbol{\Delta}^{op}\times\boldsymbol{\Delta}^{op}}{\mathrm{colim}}(Z^n\times G)\times_H(Z^m\times G)$$

Since $\boldsymbol{\Delta}^{op}$ is sifted by (HTT, 5.5.8.4), this rewrites as $\mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}}(Z^n\times G)\times_H(Z^n\times G)$. Using the isomorphism $G\times_H G\tilde{\to}G\times Z$, the latter rewrites as the quotient of $G\times Z$ by the action of $Z\times Z$, which gives G/Z. So, the diagonal map $G/Z\to(G/Z)\times_H(G/Z)$ is an isomorphism.

Lemma 7.3.3. *Let $X^\bullet:\boldsymbol{\Delta}^{op}\to\mathrm{Spc}$ be a groupoid. Assume that for any $n\geq 0$, $X^n\in\tau_{\leq m}\mathrm{Spc}$. Let X be the geometric realization of $X^\bullet$. Then $X\in\tau_{\leq m+1}\mathrm{Spc}$.*

Proof. Let $X^\bullet:\boldsymbol{\Delta}_+^{op}$ be the augmented simplicial groupoid, which is a colimit diagram. Recall that it is a Cech nerve, and the square is cartesian

$$\begin{array}{ccc} X^1 & \stackrel{f}{\to} & X^0\\ \downarrow & & \downarrow g\\ X^0 & \stackrel{g}{\to} & X\end{array}$$

Since X^1, X^0 are m-truncated, f is m-truncated (because $\tau_{\leq m}\mathrm{Spc}\subset\mathrm{Spc}$ is closed under limits). By (HTT, 6.2.3.17), since g is an effective epimorphism, g is also m-truncated. Consider now for any point $*\to x$ the fibre $y\to x_0\to x$ over this point, note that $y\in\tau_{\leq m}\mathrm{Spc}$. The corresponding long exact sequence of homotopy groups gives for $n\geq m+2$, $\pi_n(x_0)\to\pi_n(x)\to\pi_{n-1}(y)$ is exact. Since $\pi_n(x_0),\pi_{n-1}(y)$ are trivial, $\pi_n(x)$ is trivial. $\square$

7.3.4. Let $Sets$ be the category of sets, $I \in 1-\mathcal{C}$at be small filtered, $f : I \to Sets$ be a functor. Then $X = \operatorname{colim}_{i\in I} f(i)$ calculated in Spc or $1-\mathcal{C}$at lies in Sets, and any its object comes from some element of $f(i)$ for some $i \in I$ by Lemma 13.1.14. My understanding is that X is the quotient of $\sqcup_{i\in I} f(i)$ by the equivalence relation: $x_i \in f(i)$ and $x_j \in f(j)$ are equivalent if there is a diagram $i \to i' \leftarrow j$ in I such that the images of x_i, x_j in $f(i')$ coincide.

8. Little cube operads

8.0.1. Let $\mathcal{C} \in 1-\mathcal{C}$at admit finite limite, set $\mathbb{E}_0(\mathcal{C}) := \mathrm{Ptd}(\mathcal{C}) = \mathcal{C}_{*/}$. For $k \geq 1$ set $\mathbb{E}_k(\mathcal{C}) = \mathrm{Mon}(\mathbb{E}_{k-1}(\mathcal{C}))$. Let $\mathbb{E}_k^{grp-like} \subset \mathbb{E}_k(\mathcal{C})$ be the full subcategory of group-like objects, defined as the preimage of $\mathcal{G}\mathrm{rp}(\mathcal{C}) \subset \mathrm{Mon}(\mathcal{C})$ under any of k forgetful functors $\mathbb{E}_k(\mathcal{C}) \to \mathbb{E}_1(\mathcal{C})$.

If $f : \mathcal{C} \to \mathcal{D}$ is left exact then f induces a functor $\mathbb{E}_k(\mathcal{C}) \to \mathbb{E}_k(\mathcal{D})$ and $\mathbb{E}_k^{grp-like}(\mathcal{C}) \to \mathbb{E}_k^{grp-like}(\mathcal{D})$.

8.0.2. The ∞-operads $\mathbb{E}_0^{\otimes}$, $\mathrm{Assoc}^{\otimes}$ are defined in [28]. For a monoidal category $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ one has $Alg_{\mathbb{E}_0/\mathrm{Assoc}}(\mathcal{C}) \widetilde{\to} \mathcal{C}_{1/}$ by ([28], 5.2.2.10).

9. DG-categories

9.1. The category Vect can be defined as $\mathrm{D}(\mathcal{A})$, where $\mathcal{A}$ is the abelian category of k-vector spaces, this is the definition from (HA, 1.3.5.8). The t-structure on Vect is compatible with filtered colimits, and each $H^n : \mathrm{Vect} \to \mathrm{Vect}^{\heartsuit}$ preserves filtered colimits by (HA, 1.3.5.21). So, the functors $\tau^{\leq n}, \tau^{\geq n} : \mathrm{Vect} \to \mathrm{Vect}$ commute with filtered colimits (see my Section 4.0.73). They also preserve products (as for Sptr, see my Section 4.0.68).

For 10.1.3. To calculate Vect^c use ([28], Prop. 1.4.4.1). If $V \in \mathrm{Vect}^c$ then let $V' = \sum_{n\in\mathbb{Z}} \pi_n(V)[-n]$. The natural map $V \to \pi_n(V)$ for each n gives taking their sum a map $V \to V'$ (we used remark at the end of this subsection). By ([28], Prop. 1.4.4.1) the latter should factor through a sum over a finite subset in $\mathbb{Z}$, so $\pi_n(V) = 0$ for all but finite number of $n \in \mathbb{Z}$. Pick $n \in \mathbb{Z}$. Let us show that $\pi_n(V)$ is finite-dimensional. Pick a base $\{e_i\}_{i\in I}$ in V. Consider the map $V \to \pi_n(V) \to \oplus_{i\in I} ke_i$. Again, by ([28], Prop. 1.4.4.1) it should factor through $\oplus_{i\in J} ke_i$ for some finite subset $J \subset I$. So, $\dim \pi_n(V) < \infty$.

The opposite inclusion follows from the general observation: let $\mathcal{C}$ be a stable presentable category. Let $K_1, K_2 \in \mathcal{C}^c$ and $K_1 \to K \to K_2$ be a fibre sequence in $\mathcal{C}$. Then $K \in \mathcal{C}^c$. Indeed, $\mathcal{C}^c$ is stable under cofibres by [27], 5.3.4.15 and 5.5.1.1), and $K_2[-1] \to K_1 \to K$ is a fibre sequence. Besides, $\mathcal{C}^c$ is stable under translations (a translation preserves colimits, because it is an equivalence).

Vect is compactly generated (cf. ch. 1, 10.3).

For $n \in \mathbb{Z}$ the functors $\mathrm{H}^n : \mathrm{Vect} \to \mathrm{Vect}^{\heartsuit}$ and $\mathrm{H}^n : \mathrm{Vect}^{\leq 0} \to \mathrm{Vect}^{\heartsuit}$ preserve products. Indeed, we may assume $n \leq 0$. The functor $\tau^{\leq 0} : \mathrm{Vect} \to \mathrm{Vect}^{\leq 0}$ preserves limits. For $V \in \mathrm{Vect}^{\leq 0}$ we have $\mathrm{H}^n(V) \widetilde{\to} \pi_{-n}(Dold-Kan(V))$, where $Dold-Kan : \mathrm{Vect}^{\leq 0} \to \mathrm{Spc}_*$. Since $Dold-Kan : \mathrm{Vect}^{\leq 0} \to \mathrm{Spc}_*$ preserves all limits, and π_i :

$\mathrm{Spc}_* \to Sets$ preserves all products, we see that $\mathrm{H}^n : \mathrm{Vect}^{\leq 0} \to \mathrm{Vect}^{\heartsuit}$ preserves all products. We are done.

Remark: given $A \in Alg(\mathrm{Vect}^{\heartsuit})$, $M \in \mathrm{Vect}^{\heartsuit}$ a right A-module, $N_i \in \mathrm{Vect}^{\heartsuit}$ a left A-module, the natural map $M \otimes_A \prod_i N_i \to \prod_i M \otimes_A N_i$ is not always an isomorphism (but it is, if M is finitely presented A-module). Here we mean products and tensor products in $\mathrm{Vect}^{\heartsuit}$.

Remark. Let $M \in \mathrm{Vect}$, $M_0 = \tau^{\geq 0}M$, and $M_i = \mathrm{H}^i(M)[i]$ for $i < 0$. Then for $N \leq 0$ we have $\tau^{\geq N}M \widetilde{\to} \oplus_{i=-N}^{0} M_i$. One has $M \widetilde{\to} \lim_N \tau^{\geq -N}M$. The natural map $\oplus_{i\leq 0}M_i \to \lim_N \tau^{\geq -N}M \widetilde{\to} M$ is an isomorphism. Indeed, for any j, $\tau^{\geq j}(\oplus_{i\leq 0}M_i) \widetilde{\to} \tau^{\geq j}M$ is an isomorphism, because $\tau^{\geq j} : \mathrm{Vect} \to \mathrm{Vect}^{\geq j}$ preserves colimits.

9.1.1. From the explicit description of Vect in (HA, 1.3.5), we see that the functors $\tau^{\geq n}, \tau^{\leq n} : \mathrm{Vect} \to \mathrm{Vect}$ commute with direct sums, and $\mathrm{H}^n : \mathrm{Vect} \to \mathrm{Vect}^{\heartsuit}$ commutes with direct sums (recall that $\mathrm{Vect}^{\heartsuit}$ is also presentable).

DGCat_{cont} admits all limits and colimits, see my Section 6.0.1.

9.2. For 10.1.5. Since $\mathrm{Vect}^{f.d.}$ is a symmetric monoidal ∞-category, the dualization functor $(\mathrm{Vect}^{f.d.})^{op} \widetilde{\to} \mathrm{Vect}^{f.d.}$ is an equivalence of monoidal categories.

For (ch.1, 10.2). Recall that in a stable category $\mathcal{C}$ with a t-structure (and cohomological conventions) for $X \in \mathcal{C}^{\leq 0}, Y \in \mathcal{C}^{\geq 1}$ we have $\mathrm{Map}_{\mathcal{C}}(X,Y) \widetilde{\to} *$. Let $K \in \mathrm{Vect}$ then $\mathrm{Map}_{\mathrm{Vect}}(k,K) \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}}(k, \tau^{\leq 0}K)$. Since $Dold - Kan : \mathrm{Vect}^{\leq 0} \to \mathrm{Spc}$ is the right-adjoint to the composition $\mathrm{Spc} \stackrel{\Sigma^\infty}{\to} \mathrm{Sptr} \to \mathrm{Vect}$, we get

$$\mathrm{Map}_{\mathrm{Vect}}(k, \tau^{\leq 0}K) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(*, Dold - Kan(\tau^{\leq 0}K)) \widetilde{\to} Dold - Kan(\tau^{\leq 0}K)$$

So, for $i \geq 0$, $\mathrm{H}^{-i}(\tau^{\leq 0}K) \widetilde{\to} \pi_i \mathrm{Map}_{\mathrm{Vect}}(k,K)$. We have seen above that $\mathrm{H}^{-i} : \mathrm{Vect} \to \mathrm{Vect}$ and $\tau^{\leq 0} : \mathrm{Vect} \to \mathrm{Vect}$ preserve filtered colimits, so the functor $\mathrm{Vect} \to \mathrm{Spc}$, $K \mapsto \mathrm{Map}_{\mathrm{Vect}}(k,K)$ preserves filtered colimits. So, the unit object of Vect is compact.

The functor $Dold - Kan^{\mathrm{Sptr}} : \mathrm{Vect} \to \mathrm{Sptr}$ is t-exact, and for $V \in \mathrm{Vect}^{heart}$ we have, according to ([14], I.1, 10.2.3), $Dold - Kan^{\mathrm{Sptr}}(V) = V \in \mathrm{Sptr}^{\heartsuit}$, here we view V just as an abelian group. Thus, $Dold - Kan^{\mathrm{Sptr}}$ is obtained from the universal property ([28], 1.3.3.2). Namely, the forgetful functor $\mathrm{Vect}^{\heartsuit} \to Ab$ is exact, hence extends first to a t-exact functor $\mathrm{D}^-(\mathrm{Vect}^{\heartsuit}) \to \mathrm{Sptr}$. Since Sptr is right complete for its t-structure, passing to the completion, we get a functor $\mathrm{Vect} \to \mathrm{Sptr}$. It must coincide with $Dold - Kan^{\mathrm{Sptr}} : \mathrm{Vect} \to \mathrm{Sptr}$, because $Dold - Kan^{\mathrm{Sptr}}$ is continuous. Recall that for $v \in V$ we have $v \widetilde{\to} \mathrm{colim}_n \tau^{\leq n}v$ in Vect, so the above functor $\mathrm{Vect}^- \to \mathrm{Sptr}$ extends uniquely by continuity to the functor $Dold - Kan^{\mathrm{Sptr}}$.

The functor $Dold - Kan^{\mathrm{Sptr}}$ is conservative, because the image of the sphere spectrum under its left adjoint $\mathrm{Sptr} \to \mathrm{Vect}$ is k, and k generates Vect.

For (ch. I.1, 10.3.1): if $f : C \to D$ is a map in $Vect^{fd} - mod(1 - \mathcal{C}\mathrm{at})$, where C, D are stable then f is exact. Indeed, by ([28], 1.4.2.14), it suffices to show that $f(0) \widetilde{\to} 0$ and the natural map $f(x)[1] \to f(x[1])$ is an isomorphism for $x \in C$. This is true, because for $0 \in \mathrm{Vect}, x \in C$, $0 \otimes x \widetilde{\to} 0$.

9.2.1. For (ch. 1, 10.3.5). If $D, C \in \mathrm{DGCat}^{non-cocmpl}$ then

$$\mathrm{Funct}_k(D,C) := \underline{\mathrm{Hom}}_{1-\mathcal{C}\mathrm{at}, \mathrm{Vect}^{fd}}(D,C) \in \mathrm{Vect}^{fd} - mod(1 - \mathcal{C}\mathrm{at})$$

is defined by the property that functorially for $X \in 1-\mathcal{C}at$

$$\mathrm{Funct}(X, \mathrm{Funct}_k(D,C))^{\mathrm{Spc}} \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}^{fd}-mod}(X \times D, C),$$

here Vect^{fd} acts on $X \times D$ via its right action on D. It exists by my Section 3.0.49. Moreover, this category acquires a Vect^{fd}-module structure by ([14], I.1, 3.6.5), roughly via the action of Vect^{fd} on C. We have a natural map $\mathrm{Vect}^{fd} \to \mathrm{Funct}_k(C,C)$, and $\mathrm{Funct}_k(C,C)$ acts on $\mathrm{Funct}_k(D,C)$ on the left.

The action map $1-\mathcal{C}at \times \mathrm{Vect}^{fd}-mod(1-\mathcal{C}at) \to \mathrm{Vect}^{fd}-mod(1-\mathcal{C}at)$ commutes with colimits in the first variable. This follows from the fact that the forgetful functor $\mathrm{Vect}^{fd}-mod(1-\mathcal{C}at) \to 1-\mathcal{C}at$ is conservative. So, if $I \in 1-\mathcal{C}at$ is small, $I \to 1-\mathcal{C}at$ is a diagram and $X = \mathrm{colim}_{i\in I} X_i$ in $1-\mathcal{C}at$ then

$$\mathrm{Map}_{\mathrm{Vect}^{fd}-mod}(X \times D, C) \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}^{fd}-mod}(\mathrm{colim}_{i\in I}(X_i \times D), C) \widetilde{\to} \lim_{i \in I^{op}} \mathrm{Map}_{\mathrm{Vect}^{fd}-mod}(X_i \times D, C)$$

Since $1-\mathcal{C}at$ is presentable, we conclude that the functor $(1-\mathcal{C}at)^{op} \to \mathrm{Spc}$, $X \mapsto \mathrm{Map}_{\mathrm{Vect}^{fd}-mod}(X \times D, C)$ is representable ([27], 5.5.2.2).

If $C, D \in \mathrm{DGCat}^{non-cocmpl}$ with C cocomplete then $\mathrm{Fun}_k(D,C)$ is cocomplete by (HA, Lemma 4.8.4.13).

For 10.3.6. If $D, C \in \mathrm{DGCat}_{cont}$ then the DG-category $\mathrm{Funct}_{k,cont}(D,C)$ of continuous exact k-linear functors is defined by the property: functorially on $X \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$

$$\mathrm{Funct}_{ex,cont}(X, \mathrm{Funct}_{k,cont}(D,C))^{\mathrm{Spc}} \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}-mod(1-\mathcal{C}at^{St,cocmpl}_{cont})}(X \otimes D, C)$$

As above, the tensor product functor $1-\mathcal{C}at^{St,cocmpl}_{cont} \times \mathrm{Vect}-mod(1-\mathcal{C}at^{St,cocmpl}_{cont}) \to \mathrm{Vect}-mod(1-\mathcal{C}at^{St,cocmpl}_{cont})$ preserves colimits in the first variable, because $oblv$: $\mathrm{Vect}-mod(1-\mathcal{C}at^{St,cocmpl}_{cont}) \to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ is conservative. The category $1-\mathcal{C}at^{St,cocmpl}_{cont}$ is not presentable, but cocomplete. The representability of

$$(1-\mathcal{C}at^{St,cocmpl}_{cont})^{op} \to \mathrm{Spc}, \quad X \mapsto \mathrm{Map}_{\mathrm{Vect}-mod(1-\mathcal{C}at^{St,cocmpl}_{cont})}(X \otimes M, N)$$

is a particular case of a more general claim from (ch. 1, 8.2.1): for any associative algebra $A \in Alg(1-\mathcal{C}at^{St,cocmpl}_{cont})$ and $M, N \in A-mod^{St,cocmpl}_{cont}$ the relative inner hom $\underline{\mathrm{Hom}}_{1-\mathcal{C}at^{St,cocmpl}_{cont},A}(M,N)$ exists. By (ch. 1, 8.2.4), $\mathrm{Funct}_{k,cont}(D,C) \in \mathrm{DGCat}_{cont}$ is the inner hom in DGCat_{cont}.

9.2.2. For $D, C \in \mathrm{DGCat}_{cont}$ the embedding $\mathrm{Fun}_{k,cont}(D,C) \to \mathrm{Fun}_k(D,C)$ is defined as follows. If $X \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ we have the full embedding

$$\mathrm{Map}_{\mathrm{Vect}-mod(1-\mathcal{C}at^{St,cocmpl}_{cont})}(X \otimes D, C) \subset \mathrm{Map}_{\mathrm{Vect}^{fd}-mod}(X \times D, C),$$

whose image consists of functors $f : X \times D \to C$ exact and continuous in each variable. (The action of Vect^{fd} automatically extends to that of Vect by (ch. 1, Lm 10.3.4)).

Remark The category $\mathrm{DGCat}^{non-cocmpl}$ admits limits. Indeed, $\mathrm{Vect}^{fd}-mod(1-\mathcal{C}at)$ admits limits by ([28], 4.2.3.3) and the projection $\mathrm{Vect}^{fd}-mod(1-\mathcal{C}at) \to 1-\mathcal{C}at$ preserves limits. Now $\mathrm{DGCat}^{non-cocompl} \subset \mathrm{Vect}^{fd}-mod(1-\mathcal{C}at)$ is stable under limits.

In fact, $\mathrm{DGCat}^{non-cocmpl}$ admits filtered colimits, and the projection $\mathrm{DGCat}^{non-cocmpl} \to 1-\mathcal{C}\mathrm{at}$ preserves filtered colimits.

9.2.3. For 10.3.7. Let $C \in \mathrm{DGCat}^{non-cocmpl}$, $c_0, c_1 \in C$. The functor $(\mathrm{Vect}^{fd})^{op} \to \mathrm{Spc}$, $V \mapsto \mathrm{Map}_C(V \otimes c_0, c_1)$ is representable because of the following. It clearly preserves finite limits. Let $F : \mathrm{Vect}^{op} \to \mathrm{Spc}$ be its RKE along $(\mathrm{Vect}^{fd})^{op} \to \mathrm{Vect}^{op}$. Consider $F^{op} : \mathrm{Vect} \to \mathrm{Spc}^{op}$, it is continuous by (HTT, 5.3.5.8). By (HTT, 5.5.2.2) it suffices to show that F preserves small limits. Thus, it remains to show F preserves finite limits. The category Spc^{op} is not presentable probably, but in (HTT, Prop. 5.5.1.9) the condition that $\mathcal{D}$ is presentable may be relaxed, one may just require $\mathcal{D}$ cocomplete! (Jacob confirmed by email). Then the desired claim follows from (HTT, 5.5.1.9).

Important addition: for $c_i \in C, V \in \mathrm{Vect}^{fd}$ one has naturally

$$\mathrm{Map}_C(V \otimes c_0, c_1) \widetilde{\to} \mathrm{Map}_C(c_0, V^\vee \otimes c_1),$$

where $V^\vee$ is the dual of V by ([28], 4.6.1.5, a version with right-tensored replaced by left-tensored).

Assume in addition $C \in \mathrm{DGCat}$. For $c_i \in C$ write $\mathcal{H}om_C(c_0, c_1) \in \mathrm{Vect}$ for the inner hom with respect to the Vect-action on C. Then for $V \in \mathrm{Vect}^c$ one has

$$\mathcal{H}om_C(V \otimes c_0, c_1) \widetilde{\to} \mathcal{H}om_C(c_0, V^\vee \otimes c_1)$$

Besides, for $V \in \mathrm{Vect}$ there is a canonical map $V \otimes \mathcal{H}om_C(c_0, c_1) \to \mathcal{H}om_C(c_0, V \otimes c_1)$ in Vect, which is an isomorphism for $V \in \mathrm{Vect}^c$. It comes from the canonical map $\mathcal{H}om_C(c_0, c_1) \to \mathcal{H}om_C(V \otimes c_0, V \otimes c_1) \widetilde{\to} \mathcal{H}om_{\mathrm{Vect}}(V, \mathcal{H}om(c_0, V \otimes c_1))$ in Vect.

Besides, if $c_0 \in C^c$ then the above map $V \otimes \mathcal{H}om_C(c_0, c_1) \to \mathcal{H}om_C(c_0, V \otimes c_1)$ is an isomorphism (but not in general). For example, for $C = \mathrm{Vect}$, $c_0 = W, V$ infinite-dimensional vector spaces the latter map is not an isomorphism for $c_1 = e$.

Another addition Assume in addition C is compactly generated, and $f : (C^c)^{op} \to \mathrm{Spc}$ a functor, which preserves finite limits. Then there is $x \in C$ such that for $c \in C^c$ one has $f(c) \widetilde{\to} \mathrm{Map}_C(c, x)$.

Proof: let $\bar{f} : C^{op} \to \mathrm{Spc}$ be the RKE of f along $(C^c)^{op} \to C^{op}$. Then $\bar{f}$ preserves filtered limits by (HTT, 5.3.5.8). Now by (HTT, Prop. 5.5.1.9) we see that $\bar{f}$ preserves limits. By (HTT, 5.5.2.2), $\bar{f}$ is representable. □

A version for Spc **replaced by** Vect. We assume C compacly generated. Let $g : (C^c)^{op} \to \mathrm{Vect}$ be a functor which preserves finite limits and satisfies: for $V \in \mathrm{Vect}^c$, $g(V \otimes c) \widetilde{\to} V^\vee \otimes g(c)$ in a way compatible with tensor structure on Vect^c. Then there is $x \in C$ such that one has $g(c) \widetilde{\to} \mathcal{H}om_C(c, x)$ functorially in $c \in C^c$.

Proof: let $\bar{g} : C^{op} \to \mathrm{Vect}$ be the RKE of g along $(C^c)^{op} \to C^{op}$. Then $\bar{g}$ preserves filtered limits by (HTT, 5.3.5.8). Now by (HTT, Prop. 5.5.1.9) we see that $\bar{g}$ preserves limits. So, $(\bar{g})^{op}$ admits a right adjoint $h^{op} : \mathrm{Vect}^{op} \to C$ by (HTT, 5.5.2.9). This means that $h : \mathrm{Vect} \to C^{op}$ is the left adjoint to $\bar{g}$. So, for $c \in C^c, V \in \mathrm{Vect}$ we get

$$\mathrm{Map}_{\mathrm{Vect}}(V, g(c)) \widetilde{\to} \mathrm{Map}_C(c, h(V)) \tag{11}$$

For $W \in \mathrm{Vect}, V \in \mathrm{Vect}^c$ we have naturally $h(V \otimes W) \widetilde{\to} V^\vee \otimes h(W)$. Indeed, for $c \in C^c$ it suffices to establish an isomorphism

$$\mathrm{Map}_C(c, V^\vee \otimes h(W)) \widetilde{\to} \mathrm{Map}_C(c, h(V \otimes W)),$$

which follows from the above adjointness.

Now (11) for $V \in \mathrm{Vect}^c$ gives

$$\mathrm{Map}_{\mathrm{Vect}}(V, g(c)) \widetilde{\to} \mathrm{Map}_C(V \otimes c, h(e)) \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}}(V, \mathcal{H}om_C(c, h(e)))$$

We are done. □

9.2.4. For (ch. 1, 10.3.8). If $C \in \mathrm{DGCat}_{cont}$ then restricting the Vect-module structure on C via the symmetric monoidal functor $\mathrm{Sptr} \to \mathrm{Vect}$, we get the canonical Sptr-module structure on C. This immediately gives the fact that for $c_0, c_1 \in C$, $\mathcal{M}aps_C(c_0, c_1) \widetilde{\to} Dold - Kan^{\mathrm{Sptr}}(\mathcal{M}aps_{k,C}(c_0, c_1))$ in Sptr.

So, $\mathrm{Map}_C(c_0, c_1) \widetilde{\to} \Omega^\infty Dold - Kan^{\mathrm{Sptr}}(\mathcal{M}aps_{k,C}(c_0, c_1))$ in Spc. For $K \in \mathrm{Vect}$ we get

$$\Omega^\infty Dold - Kan^{\mathrm{Sptr}}(K) \widetilde{\to} \Omega^\infty Dold - Kan^{\mathrm{Sptr}}(\tau^{\leq 0} K) \widetilde{\to} Dold - Kan(\tau^{\leq 0} K),$$

where $Dold - Kan$ is that of ([14], ch. I.1, 10.2.3).

9.2.5. For 10.4.2. If $M \in \mathrm{DGCat}_{cont}$ is dualizable then the dual of M identifies with $\mathrm{Fun}_{k,cont}(M, \mathrm{Vect})$, because of (ch. 1, Prop. 9.4.4). For $B, C \in \mathrm{DGCat}_{cont}$ a map $f : B \otimes C \to \mathrm{Vect}$ extends to a duality datum for B, C iff for any $D \in \mathrm{DGCat}_{cont}$ the functor $\mathrm{id} \otimes f : D \otimes B \otimes C \to D$ yields an equivalence $D \otimes B \widetilde{\to} \mathrm{Fun}_{k,cont}(C, D)$.

9.2.6. Let $F : I \to \mathrm{DGCat}_{cont}$ be a functor, $i \mapsto C_i$, $C = \mathrm{colim}_i\, C_i$ in DGCat_{cont}. This is also a colimits of $\bar{F} : I \to 1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. Let $F' : I^{op} \to 1 - \mathcal{C}\mathrm{at}^{St,cocmpl}$ be obtained from $\bar{F}$ by passing to right adjoints. Recall that $\lim F' \widetilde{\to} C$, the limit is calculated in $1 - \mathcal{C}\mathrm{at}^{St,cocmpl}$. For each i we get evaluation functors $ev_i : C \to C_i$. It is the right adjoint to $C_i \to C$. If $i \to j$ is a map in I and $F_{ij} : C_i \to C_j$ is the corresponding transition functor then its right adjoint $C_j \to C_i$ is a strict functor of Vect-module categories in $1 - \mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, that is, a map in DGCat, because Vect is rigid (ch. 1, 9.3.6).

An improvement, let $F^R : I^{op} \to \mathrm{DGCat}$ be obtained from F by passing to right adjoints. Then $\lim F^R \widetilde{\to} C$ (see [12], Lm. 1.3.3). Here the limit is calculated in DGCat.

If $G : I \to \mathrm{DGCat}_{cont}$ is such that for any $i \to j$ in I the corresponding functor $G_i \to G_j$ admits a left adjoint then we may pass to left adjoint and get a functor $G^L : I^{op} \to \mathrm{DGCat}_{cont}$. Then $\mathrm{colim}\, G^L \widetilde{\to} \lim G$, where both limit and colimit are calculated in DGCat_{cont}. Indeed, the projection $\mathrm{DGCat}_{cont} \to \mathrm{DGCat}$ preserves limits by ([12], Lm. 1.3.1). In this setting ([16], 0.8.5) claims the following. Let $\mathcal{C} = \lim G$ in DGCat_{cont}, let $ev_i : \mathcal{C} \to G_i$ be the projection, and $ins_i : G_i \to \mathcal{C}$ be the functor obtained from $\mathrm{colim}_{i \in I^{op}}\, G_i \widetilde{\to} \mathcal{C}$. Then for any $c \in \mathcal{C}$, the natural map $\mathrm{colim}_{i \in I^{op}}\, ins_i(ev_i(c)) \to c$ is an isomorphism in $\mathcal{C}$ (this even holds if $G : I \to \mathrm{DGCat}$ is only obtained by passing to right adjoints from G^L).

Proof. For any $y \in \mathcal{C}$ by my Corollary 2.5.3, one gets

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{i \in I^{op}}\, ins_i(ev_i(c)), y) \widetilde{\to} \lim_{i \in I} \mathrm{Map}_{\mathcal{C}}(ins_i(ev_i(c)), y) \widetilde{\to}$$

$$\lim_{i \in I} \mathrm{Map}_{\mathcal{C}_i}(ev_i(c), ev_i(y)) \widetilde{\to} \mathrm{Map}_{\mathcal{C}}(c, y)$$

□

Assume in addition $R : D \to \mathcal{C} = \lim G$ is a map in DGCat_{cont}. Denote by f_i the composition $D \stackrel{R}{\to} \mathcal{C} \stackrel{ev_i}{\to} G_i$ for $i \in I$. Assume each f_i has a left adjoint $g_i : G_i \to D$, so g_i is a map in DGCat_{cont}. Let $L : \mathrm{colim}_{I^{op}} G^L \to D$ be the functor coming from the compatible system of functors $g_i : G_i \to D$, $i \in I$. Then L is left adjoint to R.

Proof. For $c \in \mathcal{C}, d \in D$ one has

$$\mathrm{Map}_{\mathcal{C}}(c, R(d)) \widetilde{\to} \lim_{i\in I} \mathrm{Map}_{G_i}(ev_i(c), f_i(d)) \widetilde{\to} \lim_{i\in I} \mathrm{Map}_D(g_i(ev_i(c), d)) \widetilde{\to} \mathrm{Map}_D(\mathrm{colim}_{i\in I^{op}} L(ins_i(ev_i(c))), d)$$

Since L preserves colimits, and $\mathrm{colim}_{i\in I^{op}} ins_i(ev_i(c)) \widetilde{\to} c$, the latter identifies with $\mathrm{Map}_D(L(c), d)$. We also used the fact that D admits colimits. □

A generalization: let $A \in Alg(\mathrm{DGCat}_{cont})$, I small, $F : I \to A-mod(\mathrm{DGCat}_{cont})$ a functor such that for any $i \to j$ in I the functor $F(i) \to F(j)$ admits a continuous A-linear right adjoint $F(j) \to F(i)$ in $A-mod(\mathrm{DGCat}_{cont})$. Let $F^R : I^{op} \to A-mod(\mathrm{DGCat}_{cont})$ be obtained from F by passing to right adjoints. Then $\mathrm{colim}\, F \widetilde{\to} \lim F^R$, where both are calculated in $A-mod(\mathrm{DGCat}_{cont})$. Indeed, $\mathrm{oblv} : A-mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves limits and colimits.

Let in addition $R : D \to \lim_{i\in I} F(i)$ a map in $A-mod$, where the limit is taken in $A-mod$. Assume for each i the composition $D \stackrel{R}{\to} \lim_{i\in I} F(i) \to F(i)$ admits a left adjoint $g_i : F(i) \to D$ in $A-mod$. Let $L : \mathrm{colim}_{i\in I} F(i) \to D$ be a map in $A-mod$ obtained from the compatible system of maps g_i, here the colimit is understood in $A-mod$. Then L is left adjoint to R.

Corollary Let $I_0 \subset I$ be a full subcategory with I small. Let $I \to \mathrm{DGCat}_{cont}$ be a functor $i \mapsto C_i$. Assume each transition functor $C_i \to C_j$ has a continiuous right adjoint $C_j \to C_i$. We have natural functors $\mathcal{L} : \mathrm{colim}_{i\in I_0} C_i \to \mathrm{colim}_{i\in I} C_i$ and $\mathcal{R} : \lim_{i\in I^{op}} C_i \to \lim_{i\in I_0^{op}} C_i$ in DGCat_{cont}. Then $(\mathcal{L}, \mathcal{R})$ is an adjoint pair in DGCat_{cont}. □

9.2.7. For $D \in \mathrm{DGCat}_{cont}$ one has $\mathrm{Fun}_{k,cont}(\mathrm{Vect}, D) \widetilde{\to} D$ by (ch. 1, 8.2.2). Dennis says in the conventions of [15] that for $C, D \in \mathrm{DGCat}_{cont}$, $\mathrm{Fun}_{k,cont}(C, D)$ is the inner hom from C to D in DGCat_{cont}.

Remark: let $I \to \mathrm{DGCat}_{cont}$ be a diagram such that for $\alpha : i \to j$ in I the corresponding functor $f_\alpha : C_i \to C_j$ has a continuous right adjoint. Assume I filtered and for each $\alpha : i \to j$, f_α is fully faithful. Let $C = \mathrm{colim}_{i\in I} C_i$ in DGCat_{cont}. Then $ins_i : C_i \to C$ is fully faithful. This follows from ([12], Lm. 1.3.6).

9.2.8. Let $A \in 1-\mathcal{C}at$ be small, $F : A \to \mathrm{DGCat}_{cont}$ be a diagram, $a \mapsto C_a$. Let $C = \mathrm{colim}_{a\in A} C_a$ in DGCat_{cont}, let $\bar{F} : A^{\triangleright} \to \mathrm{DGCat}_{cont}$ be the corresponding colimit diagram. Assume for any $\alpha : a \to b$ in A the corresponding functor $f_\alpha : C_a \to C_b$ admits a left adjoint $h_\alpha : C_b \to C_a$ in DGCat_{cont} (that is, preserves limits). Let $H : A^{op} \to \mathrm{DGCat}_{cont}$ be the functor obtained by passing to left adjoints from F. Let $\bar{C} = \lim_{a\in A^{op},h} C_a$ be the limit in DGCat_{cont}. Let $\bar{H} : {}^{\triangleleft}(A^{op}) \to \mathrm{DGCat}_{cont}$ be the corresponding limit diagram, let $\bar{H}^R : A^{\triangleright} \to \mathrm{DGCat}$ be obtained from $\bar{H}$ by passing to right adjoints. The projection $\mathrm{DGCat}_{cont} \to \mathrm{DGCat}$ does not preserve colimits, is there any relation between C and $\bar{C}$? When they are isomorphic?

9.2.9. In DGCat_{cont} coproducts coincide with products (as in Section 6.1.15 for $1 - \mathcal{C}at^{St,cocmpl}_{cont}$). We may use ([12], Lm. 1.1.1), which says that for $C, D \in \mathrm{DGCat}_{cont}$, $\mathrm{Fun}_{k,cont}(C, D) \widetilde{\to} (\mathrm{Fun}_{k,cocont}(D, C))^{op}$. Here *cocont* means limit preserving functors. Then repeat the same proof. Note that if J is a small set, $C_j \in \mathrm{DGCat}_{cont}$ for $j \in J$ then for any $c \in \prod C_j$ let c_j be its projection on C_j. Then the natural map $\underset{j\in J}{\oplus} c_j \to c$ is an isomorphism in $\prod C_j$.

Let in addition $D \in \mathrm{DGCat}_{cont}$ and $f_i \in \mathrm{Fun}_{k,cont}(C_j, D)$ giving rise to the functor $f : \oplus_j C_j \to D$. Then the right adjoint to f is the functor $f^R : D \to \prod_j C_j$ given as $\prod_j f^R_j$.

9.2.10. Let $A, B \in \mathrm{DGCat}^{SymMon}_{cont}$, let $B \to B'$ be a map in $\mathrm{DGCat}^{SymMon}_{cont}$. By $A \otimes B - mod$ we understand $A \otimes B - mod(\mathrm{DGCat}_{cont})$, the tensor product being taken over Vect. Given $C \in A \otimes B - mod$, one has $C \otimes_{A\otimes B} A \otimes B' \widetilde{\to} C \otimes_B B'$ in $B' - mod$. I don't know a reference, but this is a base change.

I think the following holds. Given $A, A_i, B, B_i \in CAlg(\mathrm{DGCat}_{cont})$ and diagrams $A_1 \leftarrow A \to A_2$, $B_1 \leftarrow B \to B_2$ in $CAlg(\mathrm{DGCat}_{cont})$, one has canonically.

$$(A_1 \otimes B_1) \otimes_{A\otimes B} (A_2 \otimes B_2) \widetilde{\to} (A_1 \otimes_A A_2) \otimes (B_1 \otimes_B B_2)$$

Indeed, the diagonal map $\boldsymbol{\Delta}^{op} \hookrightarrow \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op}$ is cofinal. A variant of this: let $C \in A^{rm} \otimes B^{rm} - mod(\mathrm{DGCat}_{cont})$. Then

$$C \otimes_{A\otimes B} A_1 \otimes B_1 \widetilde{\to} (C \otimes_A A_1) \otimes_B B_1$$

9.2.11. If $A \to B$ is a map in $\mathrm{Alg}(\mathrm{DGCat}_{cont})$, $M \in B - mod^r$, $N \in A - mod$ then $M \otimes_A N \widetilde{\to} M \otimes_B (B \otimes_A N)$ naturally. If the map $A \to B$ is a map in $CAlg(\mathrm{DGCat}_{cont})$ then this is an isomorphism in $A - mod$.

9.2.12. If $A \to B$ is a map in $\mathrm{DGCat}^{SymMon}_{cont}$, that is, map of commutative algebras then the restriction functor $B - mod \to A - mod$ is conservative (by my Section 3.0.53).

9.2.13. Let $C \in 1 - \mathcal{C}at, D \in \mathrm{DGCat}_{cont}$. I think then $\mathrm{Fun}(C, D)$ is naturally an object of DGCat_{cont}. Indeed, by (HA, 1.1.3.1), it is stable. By (HTT, 5.5.3.6) it is presentable. We have the natural map $\mathrm{Vect} \to \mathrm{Fun}(D, D)$ in $Alg(1 - \mathcal{C}at^{St,cocmpl}_{cont})$. The action of $\mathrm{Fun}(D, D)$ on $\mathrm{Fun}(C, D)$ yields now the Vect-action on $\mathrm{Fun}(C, D)$.

If $Y_i \in 1-\mathcal{C}at$ are small, is it true that the natural map $\mathrm{Fun}(Y_1, \mathrm{Vect}) \otimes \mathrm{Fun}(Y_2, \mathrm{Vect}) \to \mathrm{Fun}(Y_1 \times Y_2, \mathrm{Vect})$ is an equivalence? My understanding is that DGCat_{cont} has a final object given by $* = \mathrm{Fun}(\emptyset, \mathrm{Vect})$.

9.2.14. If $C \in \mathrm{DGCat}^{non-ccmpl}$ in the sense of ([14], ch. I.1, 10.3.1), C is small then $\mathrm{Ind}(C)$ gets a Vect-action, and becomes an object of DGCat_{cont}.

9.2.15. If $C, D \in 1 - \mathcal{C}at$ are small then $\mathrm{Fun}(C, \mathrm{Vect}) \otimes \mathrm{Fun}(D, \mathrm{Vect}) \to \mathrm{Fun}(C \times D, \mathrm{Vect})$ is an equivalence? (Here the tensor product is taken in DGCat_{cont}). My Lemma 6.1.19 is wrong, so this is not clear.

9.2.16. Let $C \in \mathrm{DGCat}_{cont}$ be dualizable, $A \in CAlg(\mathrm{DGCat}_{cont})$. Then $C \otimes A$ is dualizable in $A - mod(\mathrm{DGCat}_{cont})$, and its dual in this category is $C^\vee \otimes A$. Here $C^\vee = \mathrm{Fun}_{k,cont}(C, \mathrm{Vect})$ is the dual of C in DGCat_{cont}. Indeed, the functor of extension of scalars is symmetric monoidal.

9.2.17. The following is similar to Lemma 4.1.2 in the setting of DG-categories. Let $f : A \to B$ be a morphism in DGCat_{cont} whose right adjoint $g : B \to A$ is continuous. Assume A, B dualizable in DGCat_{cont}. Then $g^\vee : A^\vee \to B^\vee$ is left adjoint to $f^\vee : B^\vee \to A^\vee$. Compare also with (Lm. 2.2.2, [12]).

Lemma 9.2.18. *Let $\mathcal{C} \in \mathrm{DGCat}_{cont}$. Then $\mathcal{C}$ admits a compact generator iff there is an algebra $A \in \mathrm{Vect}$ and an isomorphism $\mathcal{C} \widetilde{\to} A^{op} - mod(\mathrm{Vect})$.*

Proof. Let first $A \in Alg(\mathrm{Vect})$. Then $oblv : A^{op} - mod(\mathrm{Vect}) \to \mathrm{Vect}$ preserves colimits, hence A^{op} is compact in $A^{op} - mod(\mathrm{Vect})$ (use my Section 3.0.62). Consider the functor $\mathrm{ind} : \mathrm{Vect} \to A^{op} - mod$, $M \mapsto A^{op} \otimes M$. Its right adjoint $oblv : A^{op} - mod \to \mathrm{Vect}$ is conservative, so the essential image of ind generates $A^{op} - mod$ by ([14], ch. I.1, 5.4.3). For $M \in A^{op} - mod$, $i \geq 0$ we get

$$\mathrm{Map}_{A^{op}-mod}(A[-i], M) \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}}(k, M[i]) \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}}(k, \tau^{\leq 0}(M[i]))$$

Assume for any $i \geq 0$, $\mathrm{Map}_{A^{op}-mod}(A[-i], M) \widetilde{\to} *$. Then, as in my Section 9.2, we get $\tau^{\leq 0}(M[i]) \widetilde{\to} 0$ in Vect for any $i \geq 0$. Since Vect is both left and right complete, this gives $M \widetilde{\to} 0$ in Vect, hence also in $A^{op} - mod$. Thus, A is a compact generator of $A^{op} - mod$.

Conversely, assume $\mathcal{C} \in \mathrm{DGCat}_{cont}$ has a compact generator $c \in \mathcal{C}$. Let $f : \mathrm{Vect} \to \mathcal{C}$ be the continuous k-linear functor with $f(k) = c$. Let $f^R : \mathcal{C} \to \mathrm{Vect}$ be its right adjoint. Since the essential image of f generates $\mathcal{C}$, f^R is conservative. We have $f(\mathrm{Vect}^c) \subset \mathcal{C}^c$. Indeed, $\mathcal{C}^c$ is stable under finite colimits and given a fibre sequence $K_1 \to K \to K_2$ in $\mathcal{C}$ with $K_i \in \mathcal{C}^c$ we have $K \in \mathcal{C}^c$. Now by ([14], ch. I.1, 7.1.5), f^R is continuous, so the adjoint pair $f : \mathrm{Vect} \leftrightarrows \mathcal{C} : f^R$ takes place in DGCat_{cont}.

Let $B = f^R(c) = f^R f(k)$ and $\mathcal{B} = f^R \circ f \in \mathrm{Fun}_{k,cont}(\mathrm{Vect}, \mathrm{Vect}) \widetilde{\to} \mathrm{Vect}$. The monoidal category structure on $\mathrm{Fun}_{k,cont}(\mathrm{Vect}, \mathrm{Vect})$ comes from the symmetric monoidal structure on Vect. Thus, $\mathcal{B} \in Alg(\mathrm{Vect})$, we identify it with B. In fact, $B \widetilde{\to} \mathcal{H}om_{\mathcal{C}}(c, c)$ is the inner hom with respect to Vect-action on $\mathcal{C}$. The functor f^R canonically extends to a functor $(f^R)^{enh} : \mathcal{C} \to B - mod$. It is not clear apriopri that $(f^R)^{enh}$ is k-linear. However, f^R is conservative and continuous, so by ([14], ch. I.1, 3.7.7), $(f^R)^{enh}$ is an equivalence. Set $A = B^{op}$, we are done. □

9.2.19. If $\mathcal{C} \in \mathrm{DGCat}_{cont}$, $f : \mathrm{Vect} \xrightarrow{c} \mathcal{C}$ is an element of $\mathrm{Fun}_{k,cont}(\mathrm{Vect}, \mathcal{C})$, let $f^R : \mathcal{C} \to \mathrm{Vect}$ be the right adjoint to f, it is Vect-linear, so f^R is a map in DGCat. Then $f^R \circ f \in Alg(\mathrm{Fun}_k(\mathrm{Vect}, \mathrm{Vect}))$ is a monad. However, $f^R \circ f$ is not necessarily continuous. Set $B = f^R f(k)$. Then $f^R(B \otimes c) \widetilde{\to} B \otimes f^R(c) \widetilde{\to} B \otimes B$, so we get a morphism $B \otimes B \to B$ is Vect. Clearly, $B \widetilde{\to} \mathcal{M}aps_{k,\mathcal{C}}(c, c)$, hence B is an algebra. Is it true that $B - mod(\mathrm{Vect})$ identifies with $\mathcal{A} - mod(\mathrm{Vect})$ for the monad $\mathcal{A} = f^R \circ f$?

9.2.20. Since DGCat_{cont} is cocomplete, it is tensored over Spc. Denote for $X \in \mathrm{Spc}, \mathcal{D} \in \mathrm{DGCat}_{cont}$ the corresponding category by $X \otimes \mathcal{D}$. Equip Spc with the cartesian monoidal structure, DGCat_{cont} with the Lurie tensor product (over Vect). Let us show that the functor $f : \mathrm{Spc} \to \mathrm{DGCat}_{cont}, X \mapsto X \otimes \mathrm{Vect}$ is symmetric monoidal.

For $X \in \mathrm{Spc}$ the category $X \otimes \mathcal{C}$ is characterized by

$$\mathrm{Map}_{\mathrm{DGCat}_{cont}}(X \otimes \mathcal{C}, \mathcal{D}) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(X, \mathrm{Fun}_{k,cont}(\mathcal{C}, \mathcal{D})^{\mathrm{Spc}})$$

for $\mathcal{D} \in \mathrm{DGCat}_{cont}$. For $X, Y \in \mathrm{Spc}$, we get functorially in $\mathcal{D} \in \mathrm{DGCat}_{cont}$

$$\mathrm{Map}_{\mathrm{DGCat}_{cont}}(X \otimes \mathrm{Vect}, \mathcal{D}) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(X, \mathcal{D}^{\mathrm{Spc}}) \widetilde{\to} \mathrm{Fun}(X, \mathcal{D}^{\mathrm{Spc}})$$

So,

$$\mathrm{Map}_{\mathrm{DGCat}_{cont}}((X \times Y) \otimes \mathrm{Vect}, \mathcal{D}) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(X \times Y, \mathcal{D}^{\mathrm{Spc}}) \widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(X, \mathrm{Fun}(Y, \mathcal{D}^{\mathrm{Spc}}))$$
$$\widetilde{\to} \mathrm{Map}_{\mathrm{Spc}}(X, \mathrm{Fun}_{k,cont}(Y \otimes \mathrm{Vect}, \mathcal{D})^{\mathrm{Spc}}) \widetilde{\to} \mathrm{Fun}_{k,cont}(X \otimes \mathrm{Vect}, \mathrm{Fun}_{k,cont}(Y \otimes \mathrm{Vect}, \mathcal{D}))^{\mathrm{Spc}}$$
$$\widetilde{\to} \mathrm{Fun}_{k,cont}((X \otimes \mathrm{Vect}) \otimes_{\mathrm{Vect}} (Y \otimes \mathrm{Vect}), \mathcal{D})^{\mathrm{Spc}}$$

This gives an isomorphism $(X \times Y) \otimes \mathrm{Vect} \widetilde{\to} (X \otimes \mathrm{Vect}) \otimes_{\mathrm{Vect}} (Y \otimes \mathrm{Vect})$ in DGCat_{cont}. Besides, $* \otimes \mathrm{Vect} \widetilde{\to} \mathrm{Vect}$.

Thus, f sends algebras to algebras. Let us show that f preserves colimits. For $X \widetilde{\to} \mathrm{colim}_{i \in I} X_i$ in Spc and $\mathcal{D} \in \mathrm{DGCat}_{cont}$ we get

$$\mathrm{Map}_{\mathrm{DGCat}_{cont}}(\mathrm{colim}_{i \in I}(X_i \otimes \mathrm{Vect}), \mathcal{D}) \widetilde{\to} \lim_{i \in I^{op}} \mathrm{Fun}(X_i, \mathcal{D}^{\mathrm{Spc}})$$
$$\widetilde{\to} \mathrm{Fun}(X, \mathcal{D}^{\mathrm{Spc}}) \widetilde{\to} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(X \otimes \mathrm{Vect}, \mathcal{D})$$

So, f preserves colimits. If X is a finite set then $X \otimes \mathrm{Vect} \widetilde{\to} \sqcup_{x \in X} \mathrm{Vect} \widetilde{\to} \prod_{x \in X} \mathrm{Vect}$.

9.2.21. Let $I \in 1-\mathcal{C}\mathrm{at}$, $I \times [1] \to \mathrm{DGCat}_{cont}$ be a functor sending i to $f_i : C_i \to D_i$. For $i \to j$ in I let $F^C_{ij} : C_i \to C_j$, F^D_{ij} be the corresponding transition functors. Assume each f_i, F^C_{ij}, F^D_{ij} admit left adjoints. Let $g_i : D_i \to C_i$ be a left adjoint to f_i. Assume for any $i \to j$ in I the natural transformation $g_j F^D_{ij} \to F^C_{ij} g_i$ is an isomorphism. Let $C = \lim_{i \in I} C_i$, $D = \lim_{i \in I} D_i$ taken in DGCat_{cont} or DGCat or $1-\mathcal{C}\mathrm{at}$, this is the same. (Recall that the functors $\mathrm{DGCat}_{cont} \to \mathrm{DGCat}$ and $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}\mathrm{at}$ preserve limits). Let $f : C \to D$ be $f = \lim_{i \in I} f_i$. Recall that by Lemma 2.4.1, f admits a left adjoint $g : D \to C$ and for any $i \in I$ the natural transformation $g_i ev^D_i \to ev^C_i g$ is an isomorphism.

By Section 9.2.6, $C \widetilde{\to} \mathrm{colim}_{i \in I^{op}} C_i$, $D \widetilde{\to} \mathrm{colim}_{i \in I^{op}} D_i$. We may pass to the left adjoints in the initial diagram $I \times [1] \to \mathrm{DGCat}_{cont}$. Denote by $\bar{g} : D \to C$ the functor obtained by passing to the colimit, that is, $\bar{g} = \mathrm{colim}_{i \in I^{op}} g_i$. Then $g \widetilde{\to} \bar{g}$ naturally. Indeed, using again Section 9.2.6, note that for $d \in D$ one has $d \widetilde{\to} \underset{i \in I^{op}}{\mathrm{colim}}\, ins_i ev_i(d)$. Besides, by construction $\bar{g}(ins_i(x) \widetilde{\to} ins_i(g_i(x))$. So,

$$\bar{g}(d) \widetilde{\to} \underset{i \in I^{op}}{\mathrm{colim}}\, \bar{g}(ins_i ev_i(d)) \widetilde{\to} \underset{i \in I^{op}}{\mathrm{colim}}\, ins_i(g_i ev_i(d)) \widetilde{\to} \underset{i \in I^{op}}{\mathrm{colim}}\, ins_i(ev_i(g(d)) \widetilde{\to} g(d)$$

A strengthened version is given in Section 9.2.39.

9.2.22. Let $A \in Alg(\mathrm{DGCat}_{cont})$, $M, N \in A-mod(\mathrm{DGCat}_{cont})$. Then there is a cosimplicial category $\boldsymbol{\Delta} \to \mathrm{DGCat}_{cont}$, $[n] \mapsto \mathrm{Fun}_{k,cont}(A^{\otimes n} \otimes M, N)$ whose totalization identifies with $\mathrm{Fun}_A(M,N) \in \mathrm{DGCat}_{cont}$ canonically. Here $\mathrm{Fun}_A(M,N) \in \mathrm{DGCat}_{cont}$ is the category of A-linear functors as in ([14], I.1, 8.2.1).

The two maps $\mathrm{Fun}_{k,cont}(M,N) \to \mathrm{Fun}_{k,cont}(A \otimes M, N)$ correspond to the two compositions

$$\mathrm{Fun}_{k,cont}(M,N) \otimes A \otimes M \stackrel{\mathrm{id}\otimes \mathrm{act}}{\to} \mathrm{Fun}_{k,cont}(M,N) \otimes M \stackrel{\circ}{\to} N$$
$$\mathrm{Fun}_{k,cont}(M,N) \otimes M \otimes A \stackrel{\circ \otimes \mathrm{id}}{\to} N \otimes A \stackrel{\mathrm{act}}{\to} N$$

More generally, for the injective map $[n] \to [n+1]$ the corresponding map $\mathrm{Fun}_{k,cont}(A^{\otimes n} \otimes M, N) \to \mathrm{Fun}_{k,cont}(A^{\otimes n+1} \otimes M, N)$ appears as follows: it may send f to the map

$$a_1 \otimes \ldots \otimes a_{n+1} \otimes m \to a_1 f(a_2 \otimes \ldots \otimes a_{n+1} \otimes m)$$

or

$$a_1 \otimes \ldots \otimes a_{n+1} \otimes m \to f(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} m)$$

or

$$a_1 \otimes \ldots \otimes a_{n+1} \otimes m \to f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1} \otimes m)$$

The order of the sequence $a_1, \ldots, a_{n+1}$ never changes, as our A is not necessarily symmetric.

Idea of the proof. Let $\mathcal{C}^{\otimes} \to \boldsymbol{\Delta}^{+,op}$ be the cocartesian fibration corresponding to DGCat_{cont} viewed as a left module over itself. Then M, N are given by right-lax functors $f, g : \boldsymbol{\Delta}^{+,op} \to \mathcal{C}^{\otimes}$ over $\mathrm{id} : \boldsymbol{\Delta}^{+,op} \to \boldsymbol{\Delta}^{+,op}$, whose restrictions f_0, g_0 to $\boldsymbol{\Delta}^{op}$ correspond to A. Let $\mathcal{C}^{0,\otimes} \to \boldsymbol{\Delta}^{op}$ be the restriction of $\mathcal{C}^{\otimes}$. By definition $\mathrm{Map}_{A-mod}(M,N)$ is the space

$$\{\mathrm{id}\} \times_{\mathrm{Map}_{\mathrm{Fun}(\boldsymbol{\Delta}^{+,op}, \boldsymbol{\Delta}^{+,op})}(\mathrm{id},\mathrm{id})} \mathrm{Map}_{\mathrm{Fun}(\boldsymbol{\Delta}^{+,op}, \mathcal{C}^{\otimes}))}(f,g) \times_{\mathrm{Map}_{\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{C}^{0,\otimes})}(f_0,g_0)} \{\mathrm{id}\}$$

Now $\mathrm{Map}_{\mathrm{Fun}(\boldsymbol{\Delta}^{+,op}, \mathcal{C}^{\otimes}))}(f,g)$ is described via ([18], Pp. 5.1) as

$$\lim_{(x \to y) \in Tw(\boldsymbol{\Delta}^{+,op})^{op}} \mathrm{Map}_{\mathcal{C}^{\otimes}}(f(x), g(y))$$

This is obtained as a particular case of the claim from Section 9.2.36 of this file.

9.2.23. For 10.5.3. If $C \in \mathrm{DGCat}^{non-cocmpl}$, $D \in 1-\mathcal{C}at^{St}$ then $\mathrm{Fun}_{ex}(C,D)$ gets a structure of a Vect^{fd}-module. The map $\mathrm{Fun}_{ex}(C,D) \times \mathrm{Vect}^{fd} \to \mathrm{Fun}_{ex}(C,D)$ corresponds to the map $\mathrm{Fun}_{ex}(C,D) \times \mathrm{Vect}^{fd} \times C \to D$ sending (f, V, c) to $f(V \otimes c)$. The induced map $\mathrm{Fun}_{ex}(C,D) \times \mathrm{Vect}^{fd} \to \mathrm{Fun}(C,D)$ takes values in $\mathrm{Fun}_{ex}(C,D)$.

9.2.24. Let $\mathcal{A} \in CAlg(\mathrm{Vect}^{\le 0})$. Then $\mathcal{A}-mod = \mathcal{A}-mod(\mathrm{Vect})$ is dual to $\mathcal{A}-mod^r$ in DGCat_{cont}. This follows from ([14], ch. I.1, 8.6.3 and 4.3.3). The counit map $(\mathcal{A}-mod^r) \otimes (\mathcal{A}-mod) \to \mathrm{Vect}$ is $(M,N) \mapsto M \otimes_{\mathcal{A}} N$. In addition, by ([14], ch. I.1, 8.6.4), for $M \in \mathrm{DGCat}_{cont}$ we get $\mathrm{Fun}_{k,cont}(\mathcal{A}-mod^r, M) \widetilde{\to} \mathcal{A}-mod(M)$.

9.2.25. *Gluing of categories.* Let $C \in \mathrm{DGCat}_{cont}$, $(j_i)_* : C_i \to C$ be fully faithful functors in DGCat_{cont} for $i = 1, 2$ admitting left adjoints $j_i^* : C \to C_i$. We view C_1, C_2 as full subcategories of C, let $C_{12} = C_1 \cap C_2$. Define D as colimit in DGCat_{cont} of the diagram $C_1 \overset{j_1^*}{\leftarrow} C \overset{j_2^*}{\to} C_2$. Passing to the right adjoints, we get $D = C_{12}$. Let $(j_{12})_* : C_{12} \to C$ be the embedding, $j_{12}^* : C \to C_{12}$ its left adjoint.

The functor $j_{12}^*(j_i)_*$ is left adjoint to the embedding $(j_{12,i})_* : C_{12} \hookrightarrow C_i$. We get the functor $f : C \to C_1 \times_{C_{12}} C_2$ sending F to

$$(j_1^* F, j_2^* F, j_{12}^* F)$$

It has a right adjoint $g : C_1 \times_{C_{12}} C_2 \to C$ sending (F_1, F_2, F_{12}) to K, where K is the fibre of $(j_1)_* F_1 \oplus (j_2)_* F_2 \to (j_{12})_* F_{12}$ in C.

The the natural map $fg \to \mathrm{id}$ is an isomorphism of functors? In general g is not fully faithful. Here is an example, where j_1^*, j_2^* are two localization functors, which do not commute. Take $C = Shv(\mathbb{A}^1)$ in the constructible context, $C_1 = Shv(\{0\})$, $C_2 = Shv(\mathbb{G}_m)$ with the inclusions $(j_2)* : Shv(\mathbb{G}_m) \to Shv(\mathbb{A}^1)$ and $(j_1)_* : Shv(\{0\}) \to Shv(\mathbb{A}^1)$.

In general, consider the following cosimplicial category. For $n \geq 0$ set $E^n = \prod_{i_0,\ldots,i_n} C_{i_0,\ldots,i_n}$, here $1 \leq i_j \leq 2$. We have denoted $C_{i_0,\ldots,i_n} = C_{i_0} \cap \ldots \cap C_{i_n}$. We have a natural map $Tot(E^\bullet) \to C_1 \times_{C_{12}} C_2$ in DGCat_{cont}. The latter map an equivalence (see my Section 2.7.6).

9.2.26. Let $f : C \leftrightarrows D : f^R$ be an ajoint pair in DGCat_{cont}. Then for $d \in D$, $f^R(d)$ is a direct summand in $f^R f(f^R(d))$. Indeed, $f^R(d) \to f^R f(f^R(d)) \to f^R(d)$ is the identity (general pattern of duality, ch. I.1, 4.4.1), so $f^R(d)$ is a retract of $f^R f(f^R(d))$.

9.2.27. Let $C \in \mathrm{DGCat}_{cont}$ then $(C^c)^{op}$ is idempotent complete. Indeed, by (HTT, 4.4.5.15), this follows from the fact that any diagram $\mathrm{Idem} \to C^c$ admits a colimit, and $\mathrm{Idem}^{op} \widetilde{\to} \mathrm{Idem}$. Besides, $C^c \subset C$ is stable under retracts by (HTT, 5.3.4.16).

9.2.28. Let $C_i \in \mathrm{DGCat}_{cont}$ be compactly generated for $i \in I$, I is a set. Then $E = \prod_{i \in I} C_i$ is compactly generated.

Proof: If $x \in C_i^c$ for some i then $x \in E^c$. Indeed, for $c \in E$ let $c_i \in C_i$ be its image in C_i. Then $\mathrm{Map}(x, c) \widetilde{\to} \prod_j \mathrm{Map}(x_j, c_j) \widetilde{\to} \mathrm{Map}_{C_i}(x, c_i)$, because $ev_j(x) = 0$ for $j \neq i$. The projection $ev_i : E \to C_i$ preserves colimits, so $x \in E^c$.

The collection $\sqcup_i C_i^c$ generates E. Indeed, if $c \in E$ and $\mathrm{Map}_E(x, c) = *$ for any $x \in C_i^c$ then $ev_i(c) \widetilde{\to} 0$. So, $x \widetilde{\to} 0$.

Consider the smallest stable subcategory $\mathcal{E} \subset E$ containing C_i^c for all i and idempotent complete. We claim that $\mathcal{E} = E^c$. Indeed, $\mathcal{E} \subset E^c$, because E^c is stable (and idempotent complete by HTT, 5.3.4.16). By ([14], ch. I.1, 7.2.4(3)), $\mathrm{Ind}(\mathcal{E}) \widetilde{\to} E$. Now by (HTT, 5.4.2.4), $E^c = \mathcal{E}$, because $\mathcal{E}$ is idempotent complete.

In particular, if $c = (c_i) \in E^c$ then $c_i = 0$ for all but finite number of $i \in I$.

9.2.29. The map $\mathrm{DGCat}_{cont} \to 1 - \mathcal{C}\mathrm{at}$ does not preserve filtered colimits, and $1 - \mathcal{C}\mathrm{at}_{cont}^{St,cocompl} \to 1 - \mathcal{C}\mathrm{at}^{St}$ does not preserve filtered colimits. Indeed, for an ind-scheme Y, the dualizing sheaf is not supported on some closed subscheme, though $Shv(Y) \widetilde{\to} \mathrm{colim}_i\, Shv(Y_i)$ for $Y \widetilde{\to} \mathrm{colim}\, Y_i$.

9.2.30. Let $A \in Alg(\mathrm{DGCat}_{cont})$, $M \in A{-}mod(\mathrm{DGCat}_{cont})$. Assume given a monoidal functor $A \to \mathrm{Vect}$, so making Vect a right A-module. Consider $\mathrm{Vect} \otimes_A M$, where we use the right A-module structure on Vect. Then $\mathrm{Fun}_{k,cont}(\mathrm{Vect} \otimes_A M, \mathrm{Vect})$ becomes the totalization of $\mathrm{Fun}_{k,cont}(A^{\otimes \bullet} \otimes M, \mathrm{Vect})$. The latter calculates $\mathrm{Fun}_A(M, \mathrm{Vect})$, where we view M and Vect as left A-modules. So, $\mathrm{Fun}_A(M, \mathrm{Vect}) \widetilde{\to} \mathrm{Fun}_{k,cont}(\mathrm{Vect} \otimes_A M, \mathrm{Vect})$.

More generally, for a morphism $A \to B$ in $Alg(\mathrm{DGCat}_{cont})$, $M \in A - mod$, $N \in B - mod$ we get $\mathrm{Fun}_A(M, N) \widetilde{\to} \mathrm{Fun}_B(B \otimes_A M, N)$ by adjointness.

Remark 9.2.31. *Let $D \subset C$ be a full subcategory, a morphism in* DGCat_{cont}. *Let $d_i \in D$, write $\mathcal{H}om_D(d_1, d_2) \in \mathrm{Vect}$ for the inner hom with respect to the* Vect*-action. Then $\mathcal{H}om_D(d_1, d_2) \widetilde{\to} \mathcal{H}om_C(d_1, d_2)$ naturally.*

9.2.32. Let $i : B \rightleftarrows A : R$ be an adjoint pair in DGCat_{cont}, where $A, B \in Alg(\mathrm{DGCat}_{cont})$, and R is monoidal. Assume i fully faithful, so B is a colocalization of A. Assume in addition that B is stable under left and right actions by A. Then B becomes a A-bimodule, and the above adjointness takes places in $A - mod - A$, the category of A-bimodules.

Now B is a retract of A in $A - mod(\mathrm{DGCat}_{cont})$. So, $B \in A - mod$ is dualizable and its dual is $B \in A - mod^r$ via the right translations by A (in the sense of [14], ch. I.1, 8.6.1). So, at least in the case when $A \in CAlg(\mathrm{DGCat}_{cont})$ the restriction functor $\tilde{R} : B - mod \to A - mod$ along $R : A \to B$ admits a right adjoint given by $A - mod \xrightarrow{\tilde{L}} B - mod$, $M \mapsto B \otimes_A M$, see my Section 3.2. Thus, the right and left adjoint to $\tilde{R}$ are both isomorphic to $\tilde{L}$ in this case.

Claim. For $C \in A - mod$ the category $B \otimes_A C$ is a colocalization of C, the biggest full subcategory on which the A-action factors through $R : A \to B$.

Proof. Tensor the adjoint pair $i : B \rightleftarrows A : R$ in $A - mod^r$ by $C \in A - mod$. We get an adjoint pair $\bar{i} : B \otimes_A C \rightleftarrows C : \bar{R}$ in DGCat_{cont}. Since $Ri \widetilde{\to} \mathrm{id}$, we get $\bar{R}\bar{i} \widetilde{\to} \mathrm{id}$. □

The above also shows that the functor $\tilde{R} : B - mod \to A - mod$ is fully-faithful. Indeed, $B \otimes_A B \widetilde{\to} B$. For this reason for $M \in B - mod$, $N \in B - mod^r$ one has $N \otimes_A M \widetilde{\to} N \otimes_B M$.

Assume in addition that $A, B \in CAlg(\mathrm{DGCat}_{cont})$ and $i : B \rightleftarrows A : R$ be an adjoint pair in $CAlg(\mathrm{DGCat}_{cont})$. Then the functor $\tilde{R}$ is non-unital symmetric monoidal, and $\tilde{L}$ is symmetric monoidal. In addition, if $M \in B - mod$ is dualizable in $B - mod$ then $\tilde{R}(M)$ is dualizable in $A - mod$. Namely, if $u : B \to M \otimes_B M^\vee$, $c : M \otimes_B M^\vee \to B$ is a duality datum for M then the maps $A \xrightarrow{R} B \xrightarrow{u} M \otimes_A M^\vee$, $M \otimes_A M^\vee \xrightarrow{c} B \xrightarrow{i} A$ form a duality datum for $\tilde{R}(M)$.

Version for localizations. Let $L : A \leftrightarrows B : R$ be an adjoint pair in DGCat_{cont} with $A, B \in Alg(\mathrm{DGCat}_{cont})$, where L is monoidal. So, R is right-lax non-unital monoidal. Assume that R is a map of A-bimodules naturally. Then this adjoint pair takes place in $A - bimod(\mathrm{DGCat}_{cont})$. Again, B is a retract of A in $A - mod(\mathrm{DGCat}_{cont})$, so B is dualizable in $A - mod$.

Let $\tilde{L} : B - mod \to A - mod$ be the restriction of scalars along L, let $\tilde{R} : A - mod \to B - mod$ be the functor $C \mapsto B \otimes_A C$. Then $\tilde{R}$ is left adjoint to $\tilde{L}$.

For $C \in A-mod$ one gets by applying $\cdot \otimes_A C$ the adjoint pair $l : C \leftrightarrows B \otimes_A C : r$, in which the right adjoint is fully faithful. In particular, the map $r : B \otimes_A B \to B$ is fully faithful. It is clearly essentially surjective, as $r(1 \otimes b) = b$. So, $r : B \otimes_A B \to B$ is an equivalence. This implies that $\tilde{L}$ is fully faithful.

This implies that for $M \in B-mod^r, N \in B-mod$ one has $M \otimes_B N \widetilde{\to} M \otimes_A N$ canonically. Indeed,

$$M \otimes_A N \widetilde{\to} (M \otimes_B B) \otimes_A (B \otimes_B N) \widetilde{\to} M \otimes_B (B \otimes_A B) \otimes_B N.$$

9.2.33. The category DGCat_{cont} is not stable: a retract of $C \in \mathrm{DGCat}_{cont}$ is not always given by a direct summand. For example, for a sheaf theory Shv and a scheme of finite type Z with a closed immersion $i : Y \to Z$ we have $i_! : Shv(Y) \to Shv(Z)$ is left adjoint to $i^!$ and $i_!$ is fully faithful. However, usually $Shv(Y)$ is not a direct summand.

9.2.34. Let $A \in Alg(\mathrm{DGCat}_{cont})$. Let $h : C^0 \to C$ be a fully faithful morphism in $A-mod$, write C/C^0 for the cofibre of h in $A-mod$ and $p : C \to C/C^0$ for the projection. The canonical map $C^0 \to \mathrm{Ker}\, p$ in $A-mod$ is an equivalence. Since $A-mod \to \mathrm{DGCat}_{cont}$ preserves limits and colimits, this follows from ([34], Lm. 0.2.8).

Let $A \in Alg(\mathrm{DGCat}_{cont})$. Then A generates $A-mod = A-mod(\mathrm{DGCat}_{cont})$ under colimits? Sam says no!

9.2.35. Consider a diagram $C \xrightarrow{f_1} C_1 \xrightarrow{h} C_2$ in DGCat_{cont}, set $f_2 = hf_1$. Assume f_i has a fully faithful left adjoint $L_i : C_i \to C$. Then $L_2 : C_2 \to C$ factors through the full subcategory $L_1 : C_1 \hookrightarrow C$. The resulting functor $L : C_2 \to C_1$ is left adjoint to h.

Proof. we have to show that the natural map $L_1 f_1 L_2 \to L_2$ is an isomorphism. Write f_1^R for the (maybe discontinuous) right adjoint to f_1. By passing to right adjoints, it is enough to show that the map $f_2 \xrightarrow{unit} f_2 f_1^R f_1$ is an isomorphism. Write $f_2 = hf_1$ and consider the diagram $hf_1 \xrightarrow{unit} hf_1 f_1^R f_1 \xrightarrow{counit} hf_1$, the composition is an isomorphism. However, $f_1 f_1^R \to \mathrm{id}$ is also an isomorphism, because the corresponding map of left adjoints $\mathrm{id} \to f_1 L_1$ is an isomorphism. Thus, $L_2 : C_2 \to C$ factors through $C_1 \hookrightarrow C$. $\square$

A more general claim in ([47], Lemma 2.15.1): we don't need in the above that L_2 be fully faithful, only existence of L_2 suffices. Then $L : C_2 \to C_1$ is given as $f_1 L_2$.

9.2.36. Let $B \to A$ be a map in $Alg(\mathrm{DGCat}_{cont})$, $M, N \in A-mod(\mathrm{DGCat}_{cont})$. In addition to the isomorphism $\mathrm{Fun}_A(M,N) \widetilde{\to} Tot(\mathrm{Fun}(A^{\otimes \bullet} \otimes M, N)$, one has a similarly defined isomorphism $\mathrm{Fun}_A(M,N) \widetilde{\to} Tot(\mathrm{Fun}_B(A \otimes_B A \otimes_B \ldots \otimes_B A \otimes_B M, N))$, where all the tensor products inside are taken over B.

This is due to Sam: let $M \in A-mod$. Since $A \in Alg(B \otimes B^{rev}-mod(\mathrm{DGCat}_{cont})$, we may write $M \widetilde{\to} A \otimes_A M$ and rewrite the latter relative tensor product as

$$M \widetilde{\to} \underset{[n]\in \boldsymbol{\Delta}^{op}}{\mathrm{colim}} (A_B^{n+1}) \otimes_B M,$$

here $A_B^n = A \otimes_B A \otimes_B \ldots \otimes_B A$, the product taken n times. Now

$$\mathrm{Fun}_A(M,N) \widetilde{\to} \lim_{[n]\in \boldsymbol{\Delta}} \mathrm{Fun}_A((A_B^{n+1}) \otimes_B M, N) \widetilde{\to} \lim_{[n]\in \boldsymbol{\Delta}} \mathrm{Fun}_B((A_B^n) \otimes_B M, N)$$

We used the fact that the functor $B-mod \to A-mod$, $D \mapsto D \otimes_B A$ is left adjoint to the forgetful functor $A-mod \to B-mod$.

Remark: the above totalization can be rewritten as the limit over $\boldsymbol{\Delta}_s$, which is the subcategory of $\boldsymbol{\Delta}$, where we keep all objects and only injective morphisms. Indeed, by (HTT, 6.5.3.7), $\boldsymbol{\Delta}_s^{op} \hookrightarrow \boldsymbol{\Delta}^{op}$ is cofinal. In this sense the category $\mathrm{Fun}_A(M, N)$ depends only on the non-unital A-module structures on M, N.

9.2.37. A comment on ([14], ch. I.1, 6.3.4). Let $I \in 1-\mathcal{C}\mathrm{at}$ be a small category, $C_I : I \to \mathrm{DGCat}_{cont}$ be a functor such that for any $i \in I$, C_i is dualizable, and for any $i \to j$ in I the right adjoint to the transition functor $C_i \to C_j$ is continuous. Let $C^R_{I^{op}} : I^{op} \to \mathrm{DGCat}_{cont}$ be obtained from C_I by passing to right adjoints, $C^\vee_{I^{op}} : I^{op} \to \mathrm{DGCat}_{cont}$ be obtained by passing to the duals. Let $(C^R_I)^\vee : I \to \mathrm{DGCat}_{cont}$ be obtained from $C^R_{I^{op}}$ by passing to duals. Let $C := \operatorname{colim} C_I$ in DGCat_{cont}. Recall that by ([14], ch. I.1, 6.3.4), $C^\vee \widetilde{\to} \lim C^\vee_{I^{op}}$ naturally, the limit in DGCat_{cont}. We also have $\operatorname{colim} C_I \widetilde{\to} \lim C^R_{I^{op}}$ and $\operatorname{colim}(C^R_I)^\vee \widetilde{\to} \lim C^\vee_{I^{op}}$ by ([14], ch. I.1, 5.3.4). Here if $i \in I$ and $ins_i : C_i \to C$ is the natural functor coming from C_I then its right adjoint is $ev_i : C \to C_i$, the projection functor coming from the projective system $C^R_{I^{op}}$. If we pass to the duals in the adjoint pair $ins_i : C_i \leftrightarrows C : ev_i$, we get the corresponding functors for the projective and injective systems $C^\vee_{I^{op}}$ and $(C^R_I)^\vee$ respectively.

9.2.38. The natural map $\mathrm{DGCat}^{non-cocompl} \to 1-\mathcal{C}\mathrm{at}^{St}$ preserves filtered colimits. Since $1-\mathcal{C}\mathrm{at}^{St} \to 1-\mathcal{C}\mathrm{at}$ preserves filtered colimits by ([28], 1.1.4.6), the same holds for $\mathrm{DGCat}^{non-cocompl} \to 1-\mathcal{C}\mathrm{at}$.

Proof: Let I be small filtered, $f : I \to \mathrm{DGCat}^{non-cocompl}$ be a functor $i \mapsto C_i$. Let $C = \operatorname{colim}_{i\in I} C_i$ taken in $1-\mathcal{C}\mathrm{at}$ (or $1-\mathcal{C}\mathrm{at}^{St}$). We equip C with the Vect^{fd}-action obtained by passing to the colimit in the action maps $\mathrm{Vect}^{fd} \times C_i \to C_i$. The resulting functor $\mathrm{Vect}^{fd} \times C \to C$ is exact in each variable (this uses Cor. 13.1.14 of this file).

9.2.39. Let $I \in 1-\mathcal{C}\mathrm{at}$, $I \times [1] \to \mathrm{DGCat}_{cont}$ be the functor $i \mapsto (C_i \stackrel{f_i}{\to} D_i)$. For $i \to j$ in I let $F^C_{ij} : C_i \to C_j$, $F^D_{ij} : D_i \to D_j$ be the transition functors. Assume we may pass to continuous right adjoints in the diagram $I \times [1] \to \mathrm{DGCat}_{cont}$, so we get $f^R_i : D_i \to C_i$ and $(F^C_{ij})^R : C_j \to C_i, (F^D_{ij})^R : D_j \to D_i$ in DGCat_{cont}. Let $f : C \to D$ in DGCat_{cont} be obtained by passing to the colimit over I in DGCat_{cont}. Recall that $C \widetilde{\to} \lim_{i\in I^{op}} C_i$, $D \widetilde{\to} \lim_{i\in I^{op}} D_i$ in DGCat_{cont} with respect to $(F^C_{ij})^R, (F^D_{ij})^R$ respectively. Let $g : D \to C$ be obtained by passing to the limit in DGCat_{cont} over $i \in I^{op}$ in the diagram $f^R_i : D_i \to C_i$. Then g is right adjoint to f.

Proof: let $c \in C, d \in D$. We have $c \widetilde{\to} \operatorname*{colim}_{i\in I} ins_i ev_i(c)$. So, $f(c) \widetilde{\to} \operatorname*{colim}_{i\in I} ins_i f_i(ev_i(c))$. So,

$$\mathrm{Map}_D(f(c), d) \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Map}(ins_i f_i(ev_i(c)), d)$$

Now, $ins_i : D_i \to D$ is left adjoint to $ev_i : D \to D_i$, so

$$\begin{aligned}\mathrm{Map}(ins_i f_i(ev_i(c)), d) &\widetilde{\to} \mathrm{Map}(f_i(ev_i(c)), ev_i(d)) \widetilde{\to} \mathrm{Map}(ev_i(c), f^R_i ev_i(d)) \\ &\widetilde{\to} \mathrm{Map}(ev_i(c), ev_i g(d)) \widetilde{\to} \mathrm{Map}(ins_i ev_i(c), g(d))\end{aligned}$$

This gives in turn

$$\mathrm{Map}_D(f(c), d) \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Map}(ins_i ev_i(c), g(d)) \widetilde{\to} \mathrm{Map}(c, g(d))$$

We are done. In fact, we did not need here the continuity of $(F^C_{ij})^R, (F^D_{ij})^R$, f_i^R, so no assumptions on right adjoints are needed (in that case we should understand the corresponding limits as those in DGCat).

A generalization of this in the case of I filtered is ([50], Lm. 6.5.2). It suffices to require that each f_i admits a continuous right adjoint, no need to require these right adjoints to be compatible with the transition functors F^C_{ij}, F^D_{ij}. Then f has a continuous right adjont given by the formula of Sam.

One more point: in the situation of my Section 9.2.39 assume in addition that for any map $i \to j$ in I we have $F^C_{ij} f_i^R \widetilde{\to} f_j^R F^D_{ij}$ naturally, and each f_i^R is continuous. Let $\tilde{g} : D \to C$ be the functor in DGCat_{cont} obtained by passing to the colimit over $i \in I$ in $f_i^R : D_i \to C_i$. Then $\tilde{g} = g$.

Proof: let $d \in D$, we have $g(d) \widetilde{\to} \operatorname{colim}_{i\in I} ins_i ev_i g(d)$ in C. Now $f_i^R ev_i(d) \widetilde{\to} ev_i g(d)$ in C_i. Besides, $ins_i \circ f_i^R \widetilde{\to} \tilde{g} \circ ins_i$ as functors $D_i \to C$. So,

$$g(d) \widetilde{\to} \operatorname{colim}_{i\in I} \tilde{g}(ins_i ev_i(d)) \widetilde{\to} \tilde{g}(\operatorname{colim}_{i\in I} ins_i ev_i(d)) \widetilde{\to} \tilde{g}(d)$$

We are done.

9.2.40. Let $\mathcal{J} \in 1-\mathcal{C}\mathrm{at}$ be small, $\mathcal{J}^0 \subset \mathcal{J}$ a full subcategory, $F : \mathcal{J} \to \mathrm{DGCat}_{cont}$ a functor sending $i \to j$ to $F(i) \to F(j)$, and $F^0 : \mathcal{J}^0 \to \mathrm{DGCat}_{cont}$ its restriction. We have a natural map $R : \lim F \to \lim F^0$ in DGCat_{cont}. Assume that for any $i \to j$ in $\mathcal{J}^0$ the transition functor $F(i) \to F(j)$ admits a left adjoint, let $(F^0)^L : (\mathcal{J}^0)^{op} \to \mathrm{DGCat}_{cont}$ be obtained from F by passing to left adjoints. Recall that this gives $\lim F^0 \widetilde{\to} \operatorname{colim}(F^0)^L$, where the colimit is calculated in DGCat_{cont}.

For $i \in \mathcal{J}^0$ the composition $\lim F \to \lim F^0 \to F(i)$ is canoncal projection $ev_i : \mathcal{D} := \lim F \to F(i)$. Assume that for $i \in \mathcal{J}^0$ the functor $ev_i : \mathcal{D} \to F(i)$ admits a left adjoint $ins_i : F(i) \to \mathcal{D}$. Let $L : \operatorname{colim}_{i\in\mathcal{J}^0} F(i) \to \mathcal{D}$ be the functor coming from a compatible system of functors $ins_i : F_i \to D$. By Section 9.2.6 of this file, L is left adjoint to R.

For example, it suffices to require that for any $i \to j$ in $\mathcal{J}$ the transition functor $F(i) \to F(j)$ admits a left adjoint. Then each $ev_i : \mathcal{D} \to F(i)$ admits a left adjoint, and the above claim holds. Moreover, L is a natural functor $\operatorname{colim}(F^0)^L \to \operatorname{colim} F^L$, where $F^L : \mathcal{J}^{op} \to \mathrm{DGCat}_{cont}$ is obtained from F by passing to left adjoints.

9.2.41. Let $I \in CAlg(1-\mathcal{C}\mathrm{at})$ be small, $C \in CAlg(\mathrm{DGCat}_{cont})$. Then $\mathrm{Fun}(I, C) \in \mathrm{DGCat}_{cont}$ is equipped with the symmetric monoidal structure given by the Day convolution ([28], 2.2.6.17). One checks that $\mathrm{Fun}(I, C) \in CAlg(\mathrm{DGCat}_{cont})$, that is, the tensor product is Vect-linear and preserves colimits separately in each variable.

A version: let $C \in Alg(\mathrm{DGCat}_{cont})$. Then $\mathrm{Fun}(I, C) \in Alg(\mathrm{DGCat}_{cont})$.

9.2.42. For (ch. I.1, 10.5.4). Let $C_0 \in \mathrm{DGCat}^{non-cocmpl}$. For the map $C_0 \to \mathrm{Ind}(C_0) = \mathrm{Fun}_{ex}(C_0^{op}, \mathrm{Sptr})$ to be a morphism of Vect^{fd}-categories, one has to define the action of $V \in \mathrm{Vect}^{fd}$ on $f \in \mathrm{Fun}_{ex}(C_0^{op}, \mathrm{Sptr})$ so that the resulting object of $\mathrm{Fun}_{ex}(C_0^{op}, \mathrm{Sptr})$ sends c_0 to $f(V^\vee \otimes c_0)$. Indeed, for $c \in C_0$ let $h_c \in \mathrm{Ind}(C_0)$ be the functor $c_0 \mapsto$

$\mathcal{M}aps(c_0, c)$. Then $h_{V\otimes c} \widetilde{\to} V \otimes h_c$ for $V \in \mathrm{Vect}^{fd}$ by my Section 9.2.3. The indeed the equivalence

$$\mathrm{Fun}_{ex,cont}(\mathrm{Ind}(C_0), C) \widetilde{\to} \mathrm{Fun}_{ex}(C_0, C)$$

is a morphism of Vect^{fd}-bimodule categories. We then extend the action of Vect^{fd} to that of Vect by continuity.

By Section 3.0.50, we may apply the functor $\mathrm{Fun}_{\mathrm{Vect}-bimod(1-\mathcal{C}at^{St,cocmpl}_{cont})}(\mathrm{Vect}, \bullet)$ to the above equivalence and get the equivalence

$$\mathrm{Fun}_{k,cont}(\mathrm{Ind}(C_0), C) \widetilde{\to} \mathrm{Fun}_k(C_0, C)$$

Moreover, since Vect is rigid, Vect is self-dual in the category $\mathrm{Vect}-bimod(1-\mathcal{C}at^{St,cocmpl}_{cont})$. Therefore, for $E, E' \in \mathrm{DGCat}$,

$$\mathrm{Fun}_{k,cont}(E, E') \widetilde{\to} \mathrm{Vect} \otimes_{(\mathrm{Vect}\otimes\mathrm{Vect})} \mathrm{Fun}_{ex,cont}(E, E')$$

in addition (this is as in ch. I.1, 9.4.4-9.4.8). This follows from my Section 9.2.45.

9.2.43. Let $A \in CAlg(\mathrm{DGCat}_{cont})$ be rigid, $M, N \in A-mod(\mathrm{DGCat}_{cont})$. Assume M is dualizable in $1-\mathcal{C}at^{St,cocmpl}_{cont}$, hence also in $A-mod(\mathrm{DGCat}_{cont})$ by ([14], ch. I.1, 9.4.4). By my Section 6.1.13, $\mathrm{Fun}_A(M, N) \widetilde{\to} M^\vee \otimes_A N$, where

$$M^\vee \widetilde{\to} \mathrm{Fun}_A(M, A) \widetilde{\to} \mathrm{Fun}_{ex,cont}(M, \mathrm{Sptr})$$

9.2.44. Let $M, L \in \mathrm{DGCat}_{cont}$, $N \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$. Then one has naturally $M \otimes_{\mathrm{Vect}} (L \otimes N) \widetilde{\to} (M \otimes_{\mathrm{Vect}} L) \otimes N$, where $\otimes$ is the tensor product in $1-\mathcal{C}at^{St,cocmpl}_{cont}$, and $\otimes_{\mathrm{Vect}}$ is the tensor product in DGCat_{cont}.

For this reason we have the following. Let $A \in Alg(\mathrm{DGCat}_{cont})$, $M, N \in A-mod(\mathrm{DGCat}_{cont})$, $L \in \mathrm{DGCat}_{cont}$. Then one has naturally

$$\mathrm{Fun}_{k,cont}(L, \mathrm{Fun}_A(M, N)) \widetilde{\to} \mathrm{Fun}_A(M \otimes L, N),$$

where the tensor product $M \otimes L$ is taken in DGCat_{cont}. So we may think of $\mathrm{Fun}_A(M, N)$ as the relative inner hom $\underline{\mathrm{Hom}}_{\mathrm{DGCat}_{cont},A}(M, N)$, where we view $A-mod(\mathrm{DGCat}_{cont})$ as a right DGCat_{cont}-module category.

Proof: for $S \in 1-\mathcal{C}at^{St,cocmpl}_{cont}$ we have

$$\mathrm{Map}_{1-\mathcal{C}at^{St,cocmpl}_{cont}}(S, \mathrm{Fun}_{k,cont}(L, \mathrm{Fun}_A(M, N))), \widetilde{\to} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(L \otimes S, \mathrm{Fun}_A(M, N))$$
$$\widetilde{\to} \mathrm{Map}_{A-mod(\mathrm{DGCat}_{cont})}(M \otimes_{\mathrm{Vect}} (L \otimes S), N) \widetilde{\to} \mathrm{Map}_{A-mod^{St,cocompl}_{cont}}((M \otimes_{\mathrm{Vect}} L) \otimes S, N)$$
$$\widetilde{\to} \mathrm{Map}_{1-\mathcal{C}at^{St,cocmpl}_{cont}}(S, \mathrm{Fun}_A(M \otimes_{\mathrm{Vect}} L, N)$$

We used the fact that $A-mod^{St,cocompl}_{cont} \widetilde{\to} A-mod(\mathrm{DGCat}_{cont})$ naturally.

We could also write $M \widetilde{\to} \mathrm{colim}_{[n]\in \boldsymbol{\Delta}^{op}} A^{\otimes(n+1)} \otimes M$ as usually, so

$$\mathrm{Fun}_A(M, N) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}_{k,cont}(A^{\otimes n} \otimes M, N)$$

and plug it in the desired formula, this would lead to a proof.

9.2.45. Let $A \in CAlg(\mathrm{DGCat}_{cont})$ and $M, N \in A-mod(\mathrm{DGCat}_{cont})$. One has canonically $\mathrm{Fun}_{A\otimes A}(A, \mathrm{Fun}_{k,cont}(M,N)) \widetilde{\to} \mathrm{Fun}_A(M,N)$.

Proof: consider the bar complex that calculates $A\otimes_A A$. By Section 3.1.8, it gives $A \widetilde{\to} A\otimes_A A \widetilde{\to} \mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}} A^{\otimes n+2}$ in $A\otimes A-mod$, the tensor product being taken over Vect. So,

$$\mathrm{Fun}_{A\otimes A}(A, \mathrm{Fun}_{k,cont}(M,N)) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}_{A\otimes A}(A^{\otimes n+2}, \mathrm{Fun}_{k,cont}(M,N)) \widetilde{\to}$$
$$\lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}_{k,cont}(A^{\otimes n}, \mathrm{Fun}_{k,cont}(M,N)) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}_{k,cont}(A^{\otimes n}\otimes M, N) \widetilde{\to} \mathrm{Fun}_A(M,N)$$

Here $A^{\otimes 0} \widetilde{\to}$ Vect. The last limit we got is that of Section 9.2.22.

9.2.46. Consider the adjoint pair $l : \mathrm{Spc} \leftrightarrows \mathrm{DGCat}_{cont} : r$, where $r(E) = E^{\mathrm{Spc}}$, and $l(Y) = Y \otimes \mathrm{Vect}$. Here we use the fact that DGCat_{cont} is tensored over spaces. By Section 9.2.20, l is symmetric monoidal, where Spc is equipped with the cartesian symmetric monoidal structure. We see that r is right-lax symmetric monoidal. By ([35], 1.2.9), we get an adjoint pair $L : CAlg(\mathrm{Spc}) \leftrightarrows CAlg(\mathrm{DGCat}_{cont}) : R$ given by composing with l, r. Similarly for Alg instead of $CAlg$.

9.2.47. Let $I \in 1-\mathcal{C}at$ be small filtered, $I\times[1] \to \mathrm{DGCat}_{cont}$ be a functor $i \mapsto (C_i \xrightarrow{f_i} D_i)$. For $i\to j$ in I write $\phi^C_{ij} : C_i \to C_j$, $\phi^D_{ij} : D_i \to D_j$ for the transition functors. Assume each C_i, D_i compactly generated and the functors ϕ^C_{ij}, ϕ^D_{ij} admit continuous right adjoints. Assume each f_i fully faithful. Let $f : C \to D$ in DGCat_{cont} be obtained by passing to the colimit over I. Recall that each compact object of C is of the form $ins_i(c)$ for some $c \in C^c_i$ by ([7], 1.9.5). Then f is fully faithful.

Proof: It suffices to show that for $c, c' \in C^c$, the map $\mathrm{Map}_C(c,c') \to \mathrm{Map}_D(f(c), f(c'))$ is an isomorphism. By the above, $c \widetilde{\to} ins_i(x)$, $c' \widetilde{\to} ins_{i'}(x')$ for some $x \in C^c_i, x' \in C^c_{i'}$. By ([7], 1.9.5) we get

$$\mathrm{Map}_C(ins_i(x), ins_{i'}(x')) \widetilde{\to} \operatorname*{colim}_{i\xrightarrow{\alpha}j, i'\xrightarrow{\beta}j, j\in I} \mathrm{Map}_{C_j}(\phi^C_{ij}(x), \phi^C_{i'j}(x'))$$

and

$$\mathrm{Map}_{C_j}(\phi^C_{ij}(x), \phi^C_{i'j}(x')) \widetilde{\to} \mathrm{Map}_{D_j}(\phi^D_{ij}(f_i(x)), \phi^D_{i'j}(f_{i'}(x')))$$

Besides,

$$\mathrm{Map}_D(ins_i(f_i(x)), ins_{i'}(f_{i'}(x'))) \widetilde{\to} \operatorname*{colim}_{i\xrightarrow{\alpha}j, i'\xrightarrow{\beta}j, j\in I} \mathrm{Map}_{D_j}(\phi^D_{ij}(f_i(x)), \phi^D_{i'j}(f_{i'}(x')))$$

Passing to the colimit in the above isomoprhisms we get the desired isomorphism

$$\mathrm{Map}_C(ins_i(x), ins_{i'}(x')) \widetilde{\to} \mathrm{Map}_D(ins_i(f_i(x)), ins_{i'}(f_{i'}(x')))$$

9.2.48. Let $C \in \mathrm{DGCat}_{cont}$ with an accessible t-structure, which is compatible with filtered colimits. Then $C^{\heartsuit}$ is a Grothendieck abelian category by ([27], 1.3.5.23). Let now $\mathcal{A}$ be a Grothendieck abelian category and $K \in \mathcal{A}$ be written as $K \widetilde{\to} \mathrm{colim}_{i\in I} K_i$ in $\mathcal{A}$, where I is small filtered. Let for $i \in I$, $\bar K_i$ be the image of K_i in K, we get the natural maps $\mathrm{colim}_{i\in I} K_i \to \mathrm{colim}_{i\in I} \bar K_i \xrightarrow{b} K$, whose composition is id. Since $\bar K_i \to K$ is injective, b is also injective, so b is an isomorphism.

9.2.49. Let $I \to \mathrm{DGCat}_{cont}$, $i \mapsto C_i$ be a diagram and $C = \lim_{i\in I} C_i$ taken in DGCat_{cont}. Let $x, y \in C$, and for $i \in I$ let $x_i, y_i \in C$ their images. Then

$$\mathcal{H}om_C(x,y) \widetilde{\to} \lim_{i\in I} \mathcal{H}om_{C_i}(x_i, y_i)$$

in Vect, where $\mathcal{H}om_C(x,y) \in \mathrm{Vect}$ denotes the inner hom with respect to the Vect-action. Same for DGCat_{cont} replaced by DGCat.

9.2.50. Let $C \in \mathrm{DGCat}_{cont}$ be compactly generated, I be a finite category, J a small filtered category, let $I \times J \to C$, $(i,j) \mapsto c_{i,j}$ be a diagram. Then one has naturally $\mathrm{colim}_{j\in J} \lim_i c_{i,j} \widetilde{\to} \lim_i \mathrm{colim}_{j\in J} c_{i,j}$. Indeed, it suffices to show that applying $\mathrm{Map}_C(x,\cdot)$ to both sides, one gets an isomorphism for any $x \in C^c$, which reduces to the same claim in Spc proved in HTT.

9.2.51. Let $L : C \leftrightarrows D : R$ be an adjoint pair in $\mathrm{DGCat}^{non-cocmpl}$. Then $\mathrm{Ind}(L) : \mathrm{Ind}(C) \leftrightarrows \mathrm{Ind}(D) : \mathrm{Ind}(R)$ is a dual pair in DGCat_{cont}.

9.2.52. The functor $\mathrm{Ind} : \mathrm{DGCat}^{non-cocmpl} \to \mathrm{DGCat}_{cont}$ admits a right adjoint $oblv : \mathrm{DGCat}_{cont} \to \mathrm{DGCat}^{non-cocmpl}$. Moreover, for $C \in \mathrm{DGCat}^{non-cocmpl}, D \in \mathrm{DGCat}_{cont}$, $\mathrm{Fun}_{k,cont}(\mathrm{Ind}(C), D) \widetilde{\to} \mathrm{Fun}_{k,ex}(C,D)$ via restriction (passing to Spc on both sides, we get the adjointness). Now if I is small filtered and $I \to \mathrm{DGCat}^{non-cocmpl}$ is a diagram $i \mapsto C_i$ then let $C = \mathrm{colim}_i C_i$ in $\mathrm{DGCat}^{non-cocmpl}$. Then $\mathrm{Ind}(C) \widetilde{\to} \mathrm{colim}_i \mathrm{Ind}(C_i)$ in DGCat_{cont}.

9.2.53. Let $f : A \to B$ is a map in $\mathrm{Alg}(\mathrm{DGCat}_{cont})$, $C \in A-mod$. Then $\mathrm{Fun}_A(B,C)$ is naturally a left B-module. Indeed, we have a map $B^{rm} \to \mathrm{Fun}_A(B,B)$ in $Alg(\mathrm{DGCat}_{cont})$, $a \mapsto (b \mapsto b \otimes a)$, and $\mathrm{Fun}_A(B,B)$ acts naturally on the right on $\mathrm{Fun}_A(B,C)$.

9.2.54. Let $s : B \to A$ be a map in $Alg(\mathrm{DGCat}_{cont})$, where $B \in CAlg(\mathrm{DGCat}_{cont})$. Then given $M, C \in A-mod$, $\mathrm{Fun}_A(M,C) \in B-mod$ naturally. Besides, for $N \in B-mod$ we get functorially $\mathrm{Fun}_A(M \otimes_B N, C) \widetilde{\to} \mathrm{Fun}_B(N, \mathrm{Fun}_A(M,C))$.

So, the functor $A-mod \to B-mod$, $C \mapsto \mathrm{Fun}_A(M,C)$ has a left adjoint $N \mapsto M \otimes_B N$.

Assume in addition given a map $f : A \to B$ in $Alg(\mathrm{DGCat}_{cont})$ with $fs \widetilde{\to} \mathrm{id}$. Taking $M = N$ we conclude from the above that the restrictioin functor $res : B-mod \to A-mod$ along f has a right adjoint $A-mod \to B-mod$, $C \mapsto \mathrm{Fun}_A(B,C)$.

A generalization is given in Section 9.2.56.

9.2.55. Let $I \in 1-\mathcal{C}at$ be small sifted, $A \widetilde{\to} \mathrm{colim}_{i\in I} A_i$ in $Alg(\mathrm{DGCat}_{cont})$. Then for M, N in $A-mod(\mathrm{DGCat}_{cont})$ one has

$$\mathrm{Fun}_A(M,N) \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Fun}_{A_i}(M,N),$$

where we view M, N as A_i–modules via restriction through $A_i \to A$. This follows from the fact that I is sifted. Indeed,

$$\mathrm{Fun}_A(M,N) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}(A^{\otimes n} \otimes M, N) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}(\mathrm{colim}_{i\in I} A_i^{\otimes n} \otimes M, N)$$

$$\lim_{[n]\in\boldsymbol{\Delta}, i\in I^{op}} \mathrm{Fun}(A_i^{\otimes n} \otimes M, N) \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Fun}_{A_i}(M,N)$$

In particular,

$$\mathrm{Map}_{A-mod}(M,N)\widetilde{\to}\mathrm{Fun}_A(M,N)^{\mathrm{Spc}}\widetilde{\to}\lim_{i\in I^{op}}\mathrm{Fun}_{A_i}(M,N)^{\mathrm{Spc}}\widetilde{\to}$$
$$\lim_{i\in I^{op}}\mathrm{Map}_{A_i-mod}(M,N)\widetilde{\to}\mathrm{Map}_{\lim\limits_{i\in I^{op}}A_i-mod}(M,N)$$

Here the limit of $\mathrm{Fun}_{A_i}(M,N)^{\mathrm{Spc}}$ is taken in Spc.

This shows that the natural functor $A-mod\to\lim\limits_{i\in I^{op}}A_i-mod$ is fully faithful. It is actually an equivalence. Indeed, given $M\in\lim\limits_{i\in I^{op}}A_i-mod$ restricting to $\mathrm{Vect}\to A$ we get the underlying object of DGCat_{cont} denoted also by M by abuse of notations. Our datum is then a compatible family of maps $A_i\to\mathrm{Fun}(M,M)$ in $\mathrm{Alg}(\mathrm{DGCat}_{cont})$. Passing to the colimit, it gives the desired morphism $A\to\mathrm{Fun}(M,M)$ in $\mathrm{Alg}(\mathrm{DGCat}_{cont})$.

9.2.56. Let $A,B\in Alg(\mathrm{DGCat}_{cont})$, $M\in A\otimes B^{rm}-mod$, $C\in A-mod$, $D\in B-mod$. Then one has canonically in DGCat_{cont}

$$\mathrm{Fun}_A(M\otimes_B D,C)\widetilde{\to}\mathrm{Fun}_B(D,\mathrm{Fun}_A(M,C))$$

Here rm means reversed multiplication.

Proof. (sketch). We have $M\otimes_B D\widetilde{\to}\operatorname*{colim}\limits_{[m]\in\boldsymbol{\Delta}^{op}}M\otimes B^{\otimes m}\otimes D$. So,

$$\mathrm{Fun}_A(M\otimes_B D,C)\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{Fun}(A^{\otimes n}\otimes M\otimes_B D,C)$$
$$\widetilde{\to}\lim_{[n],[m]\in\boldsymbol{\Delta}\times\boldsymbol{\Delta}}\mathrm{Fun}(A^{\otimes n}\otimes M\otimes B^{\otimes m}\otimes D,D)$$

Similarly,

$$\mathrm{Fun}_B(D,\mathrm{Fun}_A(M,C))\widetilde{\to}\lim_{[m]\in\boldsymbol{\Delta}}\mathrm{Fun}(B^{\otimes m}\otimes D,\mathrm{Fun}_A(M,C))\widetilde{\to}$$
$$\lim_{[m]\in\boldsymbol{\Delta}}\mathrm{Fun}(B^{\otimes m}\otimes D,\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{Fun}(A^{\otimes n}\otimes M,C))\widetilde{\to}\lim_{[n],[m]\in\boldsymbol{\Delta}\times\boldsymbol{\Delta}}\mathrm{Fun}(A^{\otimes n}\otimes M\otimes B^{\otimes m}\otimes D,D)$$

One should verify that the corresponding transition maps are the same in both inverse systems. □

We obtained an adjoint pair $L:B-mod\leftrightarrows A-mod:R$, where $L(D)=M\otimes_B D$, and $R(C)=\mathrm{Fun}_A(M,C)$.

Addition: if $A\in CAlg(\mathrm{DGCat}_{cont})$ then the above gives the following. For $M,D,C\in A-mod(\mathrm{DGCat}_{cont})$,

$$\mathrm{Fun}_A(M\otimes_A D,C)\widetilde{\to}\mathrm{Fun}_A(D,\mathrm{Fun}_A(M,C))$$

So, $\mathrm{Fun}_A(M,C)\in A-mod$ is the relative inner hom in $A-mod$.

Claim Let $A,B\in Alg(\mathrm{DGCat}_{cont})$, $M\in A\otimes B^{rm}-mod$, $N\in B\otimes A^{rm}-mod$. Assume a map $B\to N\otimes_A M$ of (B,B)-bimodules in DGCat_{cont} equibits M as a right dual of N in the sense of (A,B)-bimodules as in ([4], A.2.1) or equivalently ([28], 4.6.2.3). Then for any $C\in A-mod$ we have $\mathrm{Fun}_A(M,C)\widetilde{\to}N\otimes_A C$ in $B-mod$. Besides, for $X\in A^{rm}-mod$ one has $\mathrm{Fun}_{A^{rm}}(N,X)\widetilde{\to}X\otimes_A M$ in $B^{rm}-mod$.

Proof. This is ([4], A.2.6). We can derive it from the above result as follows: for any $D \in B-mod$ we get

$$\mathrm{Fun}_B(D, N\otimes_A C)\widetilde{\to}\mathrm{Fun}_A(M\otimes_B D, C)$$

by ([28], 4.6.2.1), which also identifies with $\mathrm{Fun}_B(D, \mathrm{Fun}_A(M, C))$ by the above. Since this equivalence in functorial in D, the first claim follows. The second one in given in ([4], A.2.6). □

Claim 2 ([4], A.3.8). Let $A\to B$ is a map in $Alg(\mathrm{DGCat}_{cont})$, $M\in B-mod$. Assume M is left-dualizable as a B-module, let $M^{\vee,B}\in B^{rm}-mod$ be its dual. Suppose B is dualizable as a (A,B)-bimodule DG-category, write $B^{\vee,A}$ for its dual, it is (B,A)-bimodule DG-category. Then M is left-dualizable as a A-module category and its dual is the right A-module $M^{\vee,B}\otimes_B B^{\vee,A}$. □

9.2.57. Let us rewrite ([37], Section 1.0.8) for DG-categories. Consider the forgetful functor $\Phi: \mathrm{DGCat}_{cont}\to 1-\mathcal{C}\mathrm{at}$, it is right-lax symmetric monoidal. The right-lax structure is given by the maps $C\times D\to C\otimes D$, $(c,d)\mapsto c\boxtimes d$ and $*\stackrel{e}{\to}\mathrm{Vect}$. Let $\mathcal{X}\to\mathrm{DGCat}_{cont}$ be the cocartesian fibration attached to Φ. By ([48], 5.15), $\mathcal{X}$ is a symmetric monoidal category, and the projection $\mathcal{X}\to\mathrm{DGCat}_{cont}$ is symmetric monoidal. By definition, for $(C,c),(D,d)\in\mathcal{X}$ we have $(C,c)\otimes(D,d)=(C\otimes D, c\boxtimes d)$.

Let $\mathcal{X}'\subset\mathcal{X}$ be the 1-full subcategory, where we keep all objects and only those arrows, which are cocartesian over DGCat_{cont}. So, $\mathcal{X}'\to\mathrm{DGCat}_{cont}$ is a cocartesian fibration in spaces, and $\mathcal{X}'$ inherits a symmetric monoidal structure.

Now consider the functor $\mathcal{F}:\mathcal{X}'\to\mathrm{Alg}(\mathrm{Vect})$ sending (C,c) to $\mathcal{H}om_C(c,c)$. The latter is the inner hom in Vect. We view Alg(Vect) as symmetric monoidal with the pointwise tensor product monoidal structure. Then $\mathcal{F}$ is right-lax symmetric monoidal. Indeed, given $(C,c),(D,d)\in\mathcal{X}'$ one has the natural map

$$\mathcal{H}om_C(c,c)\otimes\mathcal{H}om_D(d,d)\to\mathcal{H}om_{C\otimes D}(c\otimes d, c\otimes d)$$

functorial in $(C,c),(D,d)\in\mathcal{X}'$. Besides, the natural map $e\to\mathcal{H}om_{\mathrm{Vect}}(e,e)$ is an isomorphism. Thus, $\mathcal{F}$ induces a functor $\mathrm{Alg}(\mathcal{X}')\to\mathrm{Alg}_{E_2}(\mathrm{Vect})$, where $\mathrm{Alg}_{E_2}(\mathrm{Vect})$ denotes the category of E_2-algebras in Vect.

Now if $C\in\mathrm{Alg}(\mathrm{DGCat}_{cont})$ then $(C,1)\in\mathrm{Alg}(\mathcal{X}')$ with the product $(C\otimes C, 1_C\boxtimes 1_C)\to(C,1_C)$ given by the multiplication $m: C\otimes C\to C$. Here $\mathrm{Vect}\stackrel{1_C}{\to}C$ is the map $V\mapsto V\otimes 1_C$. Applying $\mathcal{F}$, we see that $\mathcal{H}om_C(1_C,1_C)\in\mathrm{Alg}_{E_2}(\mathrm{Vect})$.

Example: take $E\in\mathrm{DGCat}_{cont}$ and $C=\mathrm{Fun}_{k,cont}(E,E)$. Then $\mathcal{H}om_C(\mathrm{id},\mathrm{id})\in\mathrm{Alg}_{E_2}(\mathrm{Vect})$.

9.2.58. Let $A,B\in Alg(\mathrm{DGCat}_{cont})$. The functor $A\times B\to A\otimes B$, $(a,b)\mapsto a\boxtimes b$ is monoidal, that is, a map in $Alg(1-\mathcal{C}\mathrm{at})$. So, passing to the opposite categories, we get a monoidal functor $A^{op}\times B^{op}\to(A\otimes B)^{op}$, hence a functor $\mathrm{Alg}(A^{op}\times B^{op})\to Alg((A\otimes B)^{op})$. Passing to the opposite categories once again, we get a functor $coAlg(A)\times coAlg(B)\to coAlg(A\otimes B)$ sending (a,b) to $a\boxtimes b$.

Note also that the map $A\to A\otimes A$, $a\mapsto a\boxtimes a$ is a map in $\mathrm{Alg}(1-\mathcal{C}\mathrm{at})$, so yields a map $coAlg(A)\to coAlg(A\otimes A)$.

9.2.59. Let $C \in \mathrm{DGCat}_{cont}$ and $\mathcal{A} \in \mathrm{Fun}_{k,cont}(C,C)$ be a comonad. **Question**. Under which additional assumptions one has $\mathcal{A}-comod(C) \in \mathrm{DGCat}_{cont}$?

By ([28], 4.2.3.3), $\mathcal{A}-mod(C^{op})$ admits limits, and $\mathrm{oblv}^{op}: \mathcal{A}-mod(C^{op}) \to C^{op}$ reflects limits. Besides, oblv^{op} is conservative (cf. Section 3.0.53). So, $\mathrm{oblv}: \mathcal{A}-comod(C) \to C$ is conservative, $\mathcal{A}-comod(C)$ admits colimits, and oblv reflects colimits. Let $\mathrm{coind}: C \to \mathcal{A}-comod(C)$ be the right adjoint to oblv. We see that coind preserves colimits. Informally, $\mathrm{coind}(c) = \mathcal{A}(c)$.

Is $\mathcal{A}-mod(C^{op})$ stable? Apply ([28], 4.2.3.5) with $\mathrm{Fun}_{k,cont}(C,C)^{op}$ acting on C^{op}. Since each $b \in \mathrm{Fun}_{k,cont}(C,C)$ is exact, $b(0)=0$, so $0 \in \mathcal{A}-mod(C^{op})$ is initial, hence a zero object of $\mathcal{A}-mod(C^{op})$. If $\alpha: x \to y$ is a morphism in $\mathcal{A}-mod(C^{op})$, let $x \stackrel{\alpha}{\to} y \to z$ be the cofibre of α in C^{op}. Then for any $\mathcal{B}^{op} \in \mathrm{Fun}_{k,cont}(C,C)$, $\mathcal{B}(x) \to \mathcal{B}(y) \to \mathcal{B}(z)$ is a cofibre sequence in C^{op}, as $\mathcal{B}$ is exact. By ([28], 4.2.3.5), now α admits a cofibre in $\mathcal{A}-mod(C^{op})$, and $\mathrm{oblv}: \mathcal{A}-mod(C^{op}) \to C^{op}$ preserves this cofibre.

Let $x \stackrel{\alpha}{\to} y \to z$ be a triangle in $\mathcal{A}-mod(C^{op})$. If it is a fibre sequence then it is also a fibre sequence in C^{op}, hence also a cofibre sequence in C^{op}. We know already that α admits a cofibre in $\mathcal{A}-mod(C^{op})$ preserved by oblv. Since oblv is conservative, we see that $x \stackrel{\alpha}{\to} y \to z$ is a cofibre sequence in $\mathcal{A}-mod(C^{op})$.

Let $x \stackrel{\alpha}{\to} y \to z$ be a cofibre sequence in $\mathcal{A}-mod(C^{op})$, then it is also a cofibre sequence on C^{op}, as we have seen, hence a fibre sequence in C^{op}. Since $\mathcal{A}-mod(C^{op}) \to C^{op}$ reflects limits, $x \stackrel{\alpha}{\to} y \to z$ is a fibre sequence in $\mathcal{A}-mod(C^{op})$. By definition ([28], 1.1.1.9), $\mathcal{A}-mod(C^{op})$ is stable.

Sam claims $\mathcal{A}-comod(C)$ is presentable, so an object of DGCat_{cont}. Idea: for a regular uncountable cardinal, $K \in \mathcal{A}-comod(C)$ should be κ-compact iff $\mathrm{oblv}(K) \in C$ is κ-compact. Thus, the adjoint pair $\mathrm{oblv}: \mathcal{A}-comod(C) \leftrightarrows C : coind$ takes place in DGCat_{cont}.

Important phenomenon here: for any $c \in \mathcal{A}-comod(C)$, one has $c \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathcal{A}^{n+1}(c)$ in $\mathcal{A}-comod(C)$. Namely, in $\mathcal{A}-mod(C^{op})$ one has $c \widetilde{\to} \mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \mathcal{A}^{n+1}(c)$ by ([28], 4.7.2.7), as this is actually a split simplicial object in $\mathcal{A}-mod(C^{op})$, the bar construction. The tensor product in C^{op} here does not preserve the geometric realizations separately in each variable, so we really need the splitness of this simplicial object!

A version: assume $C \in CAlg(\mathrm{DGCat}_{cont})$, and $\mathcal{A} \in coAlg(C)$, this means unital coalgebra. The functor of tensoring by $\mathcal{A}$ is a comonad in $\mathrm{Fun}_{k,cont}(C,C)$, so $\mathcal{A}-comod(C) \in \mathrm{DGCat}_{cont}$. Then moreover $\mathcal{A}-comod(C) \in C-mod$, and the adjoint pair $\mathrm{oblv}: \mathcal{A}-comod(C) \leftrightarrows C : coind$ is in $C-mod$.

9.2.60. Let $A, A' \in \mathrm{Alg}(\mathrm{DGCat}_{cont})$, $M, N \in A-mod(\mathrm{DGCat}_{cont}), M', N' \in A'-mod(\mathrm{DGCat}_{cont})$. Then there is a natural continuous functor

$$\mathrm{Fun}_A(M,N) \otimes \mathrm{Fun}_{A'}(M',N') \to \mathrm{Fun}_{A\otimes A'}(M\otimes M', N\otimes N')$$

Indeed, we have the natural A-linear functor $M \otimes \mathrm{Fun}_A(M,N) \to N$ and a natural A'-linear functor $M' \otimes \mathrm{Fun}_{A'}(M',N') \to N'$. Their tensor, product

$$M \otimes M' \otimes \mathrm{Fun}_A(M,N) \otimes \mathrm{Fun}_{A'}(M',N') \to N \otimes N'$$

is $A\otimes A'$-linear, so by the universal property of the RHS given in Section 9.2.44, yields the desired functor.

Remark: the functor $ev^0 : \mathrm{Fun}_A(M,N) \to \mathrm{Fun}_{k,cont}(M,N)$ of evaluations in the corresponding cosimplicial diagram is comonadic by ([28], 4.7.5.1).

9.2.61. Let $C, D \in \mathrm{DGCat}_{cont}$. Recall the isomorphism $\mathrm{Fun}_{k,cont}(C,D) \tilde{\to} \mathrm{Fun}^R(D,C)^{op}$ sending f to f^R, where Fun^R denotes the category of functors, which are right adjoints (of continuous k-linear functors). The induced isomorphism

$$\mathrm{Fun}_{k,cont}(C,C) \tilde{\to} \mathrm{Fun}^R(C,C)^{op}$$

preserves the monoidal structures. In particular, if $\mathfrak{a}$ is a continuous comonad on C then $\mathfrak{a}^R$ is a monad on C. If $\mathfrak{a}^R$ is a comonad on C then $\mathfrak{a}$ is a continuous monad on C.

Let now $\mathfrak{a}^R \in coAlg(\mathrm{Fun}^R(C,C))$ be a comonad on C. Then we get an equivalence $\mathfrak{a}^R{-}comod(C) \tilde{\to} \mathfrak{a}{-}mod(C)$ commuting with oblivion functors to C. Indeed, if $c \to \mathfrak{a}^R c$ is the coaction map then it corresponds under the adjointness to $\mathfrak{a}c \to c$, which is the action map. In particular, this shows that $\mathfrak{a}^R - comod(C) \in \mathrm{DGCat}_{cont}$.

9.2.62. Let $A \in Alg(\mathrm{DGCat}_{cont})$, $D, E \in \mathrm{DGCat}_{cont}$, and $N \in A - mod(\mathrm{DGCat}_{cont})$. Assume N is dualizable in DGCat_{cont}. Recall that $N^\vee = \mathrm{Fun}_{k,cont}(N, \mathrm{Vect})$ is naturally a right A-module category. We claim the equivalence

$$\mathrm{Fun}_{k,cont}(D, E \otimes N) \tilde{\to} \mathrm{Fun}_{k,cont}(D \otimes N^\vee, E)$$

in DGCat_{cont} lifts to one in $A - mod(\mathrm{DGCat}_{cont})$, that is, respects the A-actions.

9.2.63. Let $A \in Alg(\mathrm{DGCat}_{cont})$, $M \in A - mod(\mathrm{DGCat}_{cont})$ dualizable with its dual $L \in A - mod^r(\mathrm{DGCat}_{cont})$ in the sense of ([14], I.1, 4.3.1). From Section 3.1.6 one derives an equivalence functorial in $D \in \mathrm{DGCat}_{cont}, Z \in A - mod(\mathrm{DGCat}_{cont})$

$$\mathrm{Fun}_A(M \otimes D, Z) \tilde{\to} \mathrm{Fun}_{k,cont}(D, L \otimes_A Z)$$

From Section 9.2.56 we see also $L \otimes_A Z \tilde{\to} \mathrm{Fun}_A(M, Z)$ canonically.

Assume in addition $N \in A - mod^r(\mathrm{DGCat}_{cont})$ is such that N is dualizable in DGCat_{cont}. Then for $D, E \in \mathrm{DGCat}_{cont}$ we get combining the above with Section 9.2.56

$$\mathrm{Fun}_{k,cont}(N \otimes_A M \otimes D, E) \tilde{\to} \mathrm{Fun}_A(M \otimes D, \mathrm{Fun}_{k,cont}(N, E)) \tilde{\to} \mathrm{Fun}_{k,cont}(D, L \otimes_A (N^\vee \otimes E))$$

This shows that $N \otimes_A M$ is dualizable in DGCat_{cont}, and its dual identifies canonically with $L \otimes_A (N^\vee)$.

9.2.64. Let $A \in \mathrm{Alg}(\mathrm{DGCat}_{cont})$, $M, C_i \in A - mod(\mathrm{DGCat}_{cont})$. Let $f : C_1 \to C_2$ in $A - mod$ be fully faithful. Then $\mathrm{Fun}_A(M, C_1) \to \mathrm{Fun}_A(M, C_2)$ is also fully faithful (use Lemma 2.2.16).

9.2.65. Let $\mathcal{O}^\otimes$ be an essentially small ∞-operad, let $\mathcal{O}^\otimes \to \mathrm{DGCat}_{cont}$ be an $\mathcal{O}$-monoid, so the corresponding cocartesian fibration $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a $\mathcal{O}$-monoidal category. Then, in particular, $\mathcal{C}$ is presentably $\mathcal{O}$-monoidal, so $CoAlg_{\mathcal{O}}(\mathcal{C})$ is also presentably $\mathcal{O}$-monoidal by ([44], Prop. 2.8).

Example: take $\mathcal{O}^\otimes$ to be Surj. We get that given $C \in CAlg^{nu}(\mathrm{DGCat}_{cont})$, the category $ComCoAlg^{nu}(C)$ is presentable.

9.2.66. Let $f_i : C_i \to D_i$ be maps in DGCat_{cont} with f_i conservative. If D_1, C_2 are dualizable then $f_1 \otimes f_2 : C_1 \otimes C_2 \to D_1 \otimes D_2$ is conservative. Indeed, it is the composition $C_1 \otimes C_2 \to D_1 \otimes C_2 \to D_1 \otimes D_2$ of conserative functors, as the first rewrites as $\mathrm{Fun}_{k,cont}(C_2^\vee, C_1) \to \mathrm{Fun}_{k,cont}(C_2^\vee, D_1)$, and similarly for the second.

9.2.67. *A generalization of Day convolution.* Let $A, D \in Alg(\mathrm{DGCat}_{cont})$. Assume that the product map $m : A \otimes A \to A$ admits a continuous right adjoint m^R, $\mathrm{Vect} \stackrel{1_A}{\to} A$ admits a continuous right adjoint $\mathcal{H}om_A(1_A, \cdot)$. Assume A dualizable. Then (A, m^R) is naturally a coalgebra in DGCat_{cont}, and $A^\vee \in Alg(\mathrm{DGCat}_{cont})$ with the product $(m^R)^\vee : A^\vee \otimes A^\vee \to A^\vee$. Since $Alg(\mathrm{DGCat}_{cont})$ is symmetric monoidal,

$$A^\vee \otimes D \widetilde{\to} \mathrm{Fun}_{e,cont}(A, D) \in Alg(\mathrm{DGCat}_{cont}).$$

This structure looks like the Day convolution. Namely, given $f_i \in \mathrm{Fun}_{e,cont}(A, D)$ their product $f_1 * f_2$ is obtained from $A \otimes A \stackrel{f_1 \otimes f_2}{\to} D \otimes D \stackrel{\otimes D}{\to}$ by applying the functor $\mathrm{Fun}_{e,cont}(A \otimes A, D) \to \mathrm{Fun}_{e,cont}(A, D)$ left adjoint to the functor $\mathrm{Fun}_{e,cont}(A, D) \to \mathrm{Fun}_{e,cont}(A \otimes A, D)$ given by composing with m.

The unit of $\mathrm{Fun}_{e,cont}(A, D)$ is obtained from $1_D : \mathrm{Vect} \to D$ by applying the functor $\mathrm{Fun}_{e,cont}(\mathrm{Vect}, D) \to \mathrm{Fun}_{e,cont}(A, D)$ left adjoint to the functor $\mathrm{Fun}_{e,cont}(A, D) \to \mathrm{Fun}_{e,cont}(\mathrm{Vect}, D)$ given by the composition with $1_A : \mathrm{Vect} \to A$.

The key thing here is that

$$\mathrm{Fun}^{rlax}_{e,cont}(A, D) \widetilde{\to} \mathrm{Alg}(\mathrm{Fun}_{e,cont}(A, D)),$$

where $\mathrm{Fun}_{e,cont}(A, D)$ is equipped with the above monoidal structure.

As for the Day convolution, we may replace here associative algebras operad by other ones.

9.2.68. Let $(C, *) \in CAlg(\mathrm{DGCat}_{cont})$. Since C is presentable, for $c, d \in C$ there is the inner hom $\underline{\mathrm{Hom}}(c, d) \in C$, (HTT, 5.5.2.2). Equip C^{op} with the induced symmetric monoidal structure. Then the functor $\underline{\mathrm{Hom}}(\cdot, 1) : C^{op} \to C$ is right-lax nonunital symmetric monoidal. Namely, if $x_i \in C$ for $i \in I$ a finite nonepty set then $\underset{i\in I}{*} \underline{\mathrm{Hom}}(x_i, 1) \to \underline{\mathrm{Hom}}(\underset{i\in I}{*} x_i, 1)$ corresponds to the morphism

$$(\underset{i\in I}{*} x_i) * (\underset{i\in I}{*} \underline{\mathrm{Hom}}(x_i, 1)) \to 1$$

which is the product in C of the morphisms $x_i * \mathrm{Hom}(x_i, 1) \to 1$. The morphism $1 \to \underline{\mathrm{Hom}}(1, 1)$ is attached to $\mathrm{id} : 1 * 1 \to 1$.

Let now $c \in coAlg(C)$ then $\underline{\mathrm{Hom}}(c, 1) \in Alg(C)$.

9.2.69. The following example is due to Sam. Let $C \in CAlg(\mathrm{DGCat}_{cont})$. Let I be a small category assumed contractible. Let $I \to coAlg(C)$ be a functor $i \mapsto A_i$. Let $A = \mathrm{colim}_{i\in I} A_i$ calculated in $coAlg(C)$ or equivalently in C. We get a map $I \to C - mod/C$, $i \mapsto A_i - comod(C)$, where the transition functors are extensions of scalars via $A_i \to A_j$ for $i \to j$. Let $D = \mathrm{colim}_{i\in I} A_i - comod(C)$ calculated in $C - mod$ or equivalently in DGCat_{cont}. Then the natural functor $D \to A - comod(C)$ is not an equivalence.

Example: $C = \mathrm{Vect}$. For $n \in \mathbb{N}$ take $A_n = e[\mathbb{G}_a^n]$, the space of functions on the group $\mathbb{G}_a^n$. Our field of cooefficients e is of characteristic zero and algtebraically closed. Consider the homomoprhism of groups $\mathbb{G}_a^{n+1} \to \mathbb{G}_a^n$ forgetting the last factor, it gives a map of coalgebras $A_n \to A_{n+1}$. We take $I = \mathbb{N}$ with the usual order, the map $\mathbb{N} \to coAlg(\mathrm{Vect})$ sends n to A_n with the above transition maps. So, $A = \mathrm{colim}_{n\in\mathbb{N}} A_n \widetilde{\to} k[G]$. Here $G = \prod_{n\in\mathbb{N}} \mathbb{G}_a$, the product being calculated in the category of affine schemes. So, $A = e[x_1, x_2, \ldots]$.

He claims $\underset{n\in\mathbb{N}}{\mathrm{colim}}\, A_n - comod(\mathrm{Vect}) \to A - comod(\mathrm{Vect})$ is not an equivalence. Namely, the natural t-structure on $A - comod(\mathrm{Vect})$ is left separated, while he claims this is not the case for the LHS.

9.2.70. Let I be a small category, $C_I : I \to \mathrm{DGCat}_{cont}$, $i \mapsto C_i$ be a diagram, assume each C_i dualizable. Let $C^\vee_{I^{op}} : I^{op} \to \mathrm{DGCat}_{cont}$ be the functor obtained by passing to the duals. When $\mathrm{colim}_{i\in I} C_i$ and $\lim_{i\in I^{op}} C_i^\vee$ are mutually dual? More precisely, there is a canonical Vect-valued pairing between them, and we ask when it realizes the two categories as mutually dual. For this one needs that for any $D \in \mathrm{DGCat}_{cont}$, the natural map $D \otimes \lim_{i\in I^{op}} C_i^\vee \to \lim_{i\in I^{op}}(D \otimes C_i^\vee)$ is an isomorphism.

A nice example of this is ([49], Lemma A.2.1).

A point related to his ([49], Lemma A.6.1). Here is a simplified version:

Lemma 9.2.71. *Let $C, E \in \mathrm{DGCat}_{cont}$. Then the natural map*

$$\mathrm{Fun}_{k,cont}(\mathrm{Fun}([1], C), E) \to \mathrm{Fun}([1], \mathrm{Fun}_{k,cont}(C, E)) \tag{12}$$

is an equivalence.

Proof. Write an object of $\mathrm{Fun}([1], C)$ as (c_0, c_1, α), where $c_i \in C$ and $\alpha : c_0 \to c_1$ in C. For $c \in C$ we have a fibre sequence in $\mathrm{Fun}([1], C)$

$$(0, c, 0) \to (c, c, \mathrm{id}) \to (c, 0, 0) \overset{\delta}{\to} (0, c[1], 0),$$

here δ is the boundary morphism, it depends on α. Let $\Theta : \mathrm{Fun}([1], C) \to E$ be a map in DGCat_{cont}. It gives the functors $\xi_i : C \to E$ for $i = 0, 1$ given by $\xi_0(c) = \Theta(c, 0, 0)$, $\xi_1(c) = \Theta(0, c[1], 0)$ and a map $\eta : \xi_0 \to \xi_1$ in $\mathrm{Fun}_{k,cont}(C, E)$ given by applying Θ to δ.

The inverse map to (12) is as follows. For any object $(c_0, c_1, \alpha) \in \mathrm{Fun}([1], C)$ we have a commutative diagram, where the arrows are fibre sequences in $\mathrm{Fun}([1], C)$

$$\begin{array}{ccccccc}
(0, c_1, 0) & \to & (c_0, c_1, \alpha) & \to & (c_0, 0, 0) & \overset{\bar{\alpha}}{\to} & (0, c_1[1], 0) \\
\downarrow \mathrm{id} & & \downarrow \alpha\times\mathrm{id} & & \downarrow \alpha & & \downarrow \mathrm{id} \\
(0, c_1, 0) & \to & (c_1, c_1, \mathrm{id}) & \to & (c_1, 0, 0) & \overset{\delta}{\to} & (0, c_1[1], 0)
\end{array}$$

Now α yields maps

$$\xi_0(c_0) \overset{\xi_0(\alpha)}{\to} \xi_0(c_1) \overset{\eta(c_1)}{\to} \xi_1(c_1),$$

and $\Theta(c_0, c_1, \alpha) \in E$ is recovered as the fibre in E of $\eta(c_1)\xi_0(\alpha) : \xi_0(c_0) \to \xi_1(c_1)$. □

Corollary 9.2.72. *For any $D, C \in \mathrm{DGCat}_{cont}$ the natural map $\mathrm{Fun}([1], C) \otimes D \to \mathrm{Fun}([1], C \otimes D)$ is an equivalence.*

Proof. For $E \in \mathrm{DGCat}_{cont}$ we check that the composition

$$\mathrm{Fun}_{k,cont}(\mathrm{Fun}([1], C \otimes D), E) \to \mathrm{Fun}_{k,cont}(\mathrm{Fun}([1], C) \otimes D, E)$$

is an equivalence. By the previous lemma, the LHS is

$$\mathrm{Fun}([1], \mathrm{Fun}_{k,cont}(C \otimes D, E))$$

and the RHS is

$$\mathrm{Fun}_{k,cont}((\mathrm{Fun}([1], C), \mathrm{Fun}_{k,cont}(D, E)) \widetilde{\to} \mathrm{Fun}([1], \mathrm{Fun}_{k,cont}(C, \mathrm{Fun}_{k,cont}(D, E))$$

Both sides are the same. □

For I, J small categories and $C, D \in \mathrm{DGCat}_{cont}$, one has a canonical map $\mathrm{Fun}(I, C) \otimes \mathrm{Fun}(J, D) \to \mathrm{Fun}(I \times J, C \otimes D)$. Sam says that for I, J finite categories this should be an equivalence, but probably not in general.

9.2.73. Let $A \in CAlg(\mathrm{DGCat}_{cont})$, let $\mathcal{A} \in CAlg(A)$ be an idempotent commutative algebra in the sense of ([28], 4.8.2.8). Then $\mathrm{ind} : A \leftrightarrows \mathcal{A} - mod(A) : \mathrm{oblv}$ is an adjoint pair in $A - mod$. By ([28], 4.8.2.10), ind is a localization functor, and oblv is fully faithful. Moreover, for any $M \in A - mod$ tensoring the above adjoint pair by M, one gets an adjoint pair $\mathrm{ind} : M \leftrightarrows \mathcal{A} - mod(M) : \mathrm{oblv}$ in $A - mod$. Here we used the equivalence $\mathcal{A} - mod(A) \otimes_A M \widetilde{\to} \mathcal{A} - mod(M)$ from ([14], I.1, 8.5.7). So, $\mathrm{oblv} : \mathcal{A} - mod(M) \to M$ is fully faithful, and its image is the image of $\mathcal{A} : M \to M$.

9.2.74. Let $A \in CAlg(\mathrm{DGCat}_{cont})$, assume A dualizable in DGCat_{cont}, recall that $A^\vee$ is naturally an A-module. We claim that for $M \in A - mod(\mathrm{DGCat}_{cont}), N \in \mathrm{DGCat}_{cont}$ one has canonically

$$\mathrm{Fun}_A(M, A^\vee \otimes N) \widetilde{\to} \mathrm{Fun}_{k,cont}(M, N)$$

Proof: the functor $\mathrm{oblv} : A - mod \to \mathrm{DGCat}_{cont}$ has a right adjoint given by $N \mapsto A^\vee \otimes N$. So, for $E \in \mathrm{DGCat}_{cont}$ one has

$$\begin{aligned}\mathrm{Map}_{\mathrm{DGCat}_{cont}}(E, \mathrm{Fun}_A(M, A^\vee \otimes N)) &\widetilde{\to} \mathrm{Fun}_{k,cont}(E, \mathrm{Fun}_A(M, A^\vee \otimes N))^{\mathrm{Spc}} \widetilde{\to} \\ \mathrm{Fun}_A(M \otimes E, A^\vee \otimes N))^{\mathrm{Spc}} &\widetilde{\to} \mathrm{Map}_{A-mod}(M \otimes E, A^\vee \otimes N) \widetilde{\to} \\ \mathrm{Map}_{\mathrm{DGCat}_{cont}}(M \otimes E, N) &\widetilde{\to} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(E, \mathrm{Fun}_{k,cont}(M, N))\end{aligned}$$

□

If in addition $A^\vee \widetilde{\to} A$ is given such that the counit map $c : A \otimes A \to \mathrm{Vect}$ is A-bilinear in the sense that $c(a_1 b \boxtimes a_2) \widetilde{\to} c(a_1 \boxtimes b a_2)$ as functors $A \otimes A \otimes A \to \mathrm{Vect}$ then the isomorphism $A \widetilde{\to} A^\vee$ is an isomorphism in $A - mod$. In this case for $M \in A - mod$ we get $\mathrm{Fun}_A(M, A) \widetilde{\to} \mathrm{Fun}_A(M, A^\vee) \widetilde{\to} \mathrm{Fun}_{k,cont}(M, \mathrm{Vect})$, this is an isomorphism in $A - mod(\mathrm{DGCat}_{cont})$.

9.2.75. Let $A \to B$ be a map in $Alg(\mathrm{DGCat}_{cont})$, $C \in A - mod$. Consider the functor $\gamma : \mathrm{Fun}_A(C, C) \to \mathrm{Fun}_B(B \otimes_A C, B \otimes_A C)$ sending $f : C \to C$ to $\mathrm{id} \otimes f : B \otimes_A C \to B \otimes_A C$. It is monoidal. In particular, if $\mathcal{A} \in \mathrm{Alg}(\mathrm{Fun}_A(C, C))$ is an A-linear continuous monad on C then $\mathrm{id} \otimes \mathcal{A} \in Alg(\mathrm{Fun}_B(B \otimes_A C, B \otimes_A C))$ is a B-linear continuous monad, and the same for comonads.

Recall that $C \in \mathrm{Fun}_A(C, C) - mod$ and $B \otimes_A C \in \mathrm{Fun}_B(B \otimes_A C, B \otimes_A C) - mod$. The natural functor $\alpha : C \widetilde{\to} A \otimes_A C \to B \otimes_A C$ coming from $A \to B$ is $\mathrm{Fun}_A(C, C)$-linear, where on the RHS it acts through γ.

Let now $\mathcal{A} \in Alg(\mathrm{Fun}_A(C,C))$. Then α induces a functor $\mathcal{A}-mod(C) \to \mathcal{A}-mod(B\otimes_A C)$. Here

$$\mathcal{A}-mod(B\otimes_A C)\widetilde{\to}(\mathrm{id}\otimes\mathcal{A})-mod(B\otimes_A C)$$

naturally with $\gamma(\mathcal{A})=\mathrm{id}\otimes\mathcal{A}$.

Note that $C\in \mathrm{Fun}_A(C,C)\otimes A-mod$, because these two actions commute. For this reason, $\mathcal{A}-mod(C)\in A-mod$ naturally. Similarly, $(\mathrm{id}\otimes\mathcal{A})-mod(B\otimes_A C)\in B-mod$. The above functor

$$\mathcal{A}-mod(C)\to(\mathrm{id}\otimes\mathcal{A})-mod(B\otimes_A C)$$

is A-linear, where on the RHS it acts through $A\to B$. By adjunction, this gives a continuous B-linear functor

$$B\otimes_A(\mathcal{A}-mod(C))\to(\mathrm{id}\otimes\mathcal{A})-mod(B\otimes_A C)$$

Let now $\mathcal{L}\in CoAlg(\mathrm{Fun}_A(C,C))$. As above we get $\mathcal{L}-comod(C)\in A-mod$. Since $\mathrm{id}\otimes\mathcal{L}:=\gamma(\mathcal{L})\in CoAlg(\mathrm{Fun}_B(B\otimes_A C, B\otimes_A C))$, we get

$$\mathcal{L}-comod(B\otimes_A C)\widetilde{\to}\gamma(\mathcal{L})-comod(B\otimes_A C)\in B-mod.$$

The functor α induces an A-linear functor $\mathcal{L}-comod(C)\to\mathcal{L}-comod(B\otimes_A C)$, where on the RHS A acts through $A\to B$. By adjointness, this gives a continuous B-linear functor

$$B\otimes_A(\mathcal{L}-comod(C))\to\gamma(\mathcal{L})-comod(B\otimes_A C).$$

The adjoint pair $\mathrm{oblv}:\mathcal{L}-comod(C)\leftrightarrows C:\mathrm{coind}$ takes place in $A-mod$.

9.2.76. Lurie's result ([28], 4.7.4.19) provides the following corollary:

Claim 1 Let $S,T\in 1-\mathfrak{C}\mathrm{at}$. Let $\chi: S\times T\to \mathrm{DGCat}_{cont}$ be a functor, $(s,t)\mapsto\chi(s,t)$. For $\alpha: s\to s'$ in S, $\beta: t\to t'$ in T consider the diagram

$$\begin{array}{ccc} \chi(s,t) & \overset{F_{\alpha,t}}{\to} & \chi(s',t) \\ \downarrow F_{s,\beta} & & \downarrow F_{s',\beta} \\ \chi(s,t') & \overset{F_{\alpha,t'}}{\to} & \chi(s',t') \end{array} \tag{13}$$

Assume it is right adjointable. This means that there are right adjoints $F^R_{\alpha,t}, F^R_{\alpha,t'}$, and the induced map $F_{s,\beta}F^R_{\alpha,t}\to F^R_{\alpha,t}F_{s',\beta}$ is an isomorphism for any choices of α,β as above. Then the canonical map

$$\underset{s\in S}{\mathrm{colim}}\lim_{t\in T}\chi(s,t)\to\lim_{t\in T}\underset{s\in S}{\mathrm{colim}}\chi(s,t)$$

is an equivalence in DGCat_{cont}.

Proof. Apply Claim from Section 10.1.1 and ([28], 4.7.4.19) □

Claim 2(version for "left adjointable diagrams"). If we replace in the above formulation right adjoints $F^R_{\alpha,t}$ by left adjoints $F^L_{\alpha,t}$ and "right adjointable" by "left adjointable" then the conclusion is that the canonical map

$$\underset{t\in T}{\mathrm{colim}}\lim_{s\in S}\chi(s,t)\to\lim_{s\in S}\underset{t\in T}{\mathrm{colim}}\chi(s,t)$$

is an equivalence in DGCat_{cont}.

Proof. Assume. $F^L_{\alpha,t}, F^L_{\alpha,t'}$ exists for any data as above. Note that $F^R_{s,\beta}, F^R_{s',\beta}$ exists automatically, though they are maybe not continuous. Since the diagram (13) is assumed left adjointable, it follows that the transposed diagram

$$\begin{array}{ccc} \chi(s,t) & \overset{F_{s,\beta}}{\to} & \chi(s,t') \\ \downarrow F_{\alpha,t} & & \downarrow F_{\alpha,t'} \\ \chi(s',t) & \overset{F_{s',\beta}}{\to} & \chi(s',t') \end{array}$$

is right adjointable. The result follows now from Claim 1. □

9.2.77. Let $C_i, D_i \in \mathrm{DGCat}_{cont}$ for $i \in I$, where I is a small set. Let $\alpha_i : C_i \to D_i$ be a map in DGCat_{cont} such that $\prod_i \alpha_i : \prod_i C_i \to \prod_i D_i$ is an equivalence. Then each α_i is an equivalence. Indeed, let α_i^R be the right adjoint to α_i. Then $\prod_i \alpha_i^R$ is the right adjoint to $\prod_i \alpha_i$. The canonical map $\mathrm{id} \to (\prod_i \alpha_i^R) \circ (\prod_i \alpha_i)$ is an equivalence, and similarly for $(\prod_i \alpha_i) \circ (\prod_i \alpha_i^R) \to \mathrm{id}$. The claim follows.

9.2.78. The following is actually proved in ([50], 6.5.2). Let $I \in 1-\mathcal{C}\mathrm{at}$ small such that for any diagram A of the form $i_1 \leftarrow j \to i_2$, $I_{A/}$ is contractible. This is precisely the condition saying that for any map $\beta : k \to i$ in I the functor $I_{i/} \to I_{k/}$ given by

$$(i \overset{\alpha}{\to} j) \mapsto (k \overset{\alpha\beta}{\to} j)$$

is contractible.

Let now $I \times [1] \to \mathrm{DGCat}_{cont}$ be a functor sending i to $F_i : C_i \to D_i$. For a map $\alpha : i \to j$ in I write $\phi_\alpha : D_i \to D_j$ and $\psi_\alpha : C_i \to C_j$ for the transition functors. Let $C = \mathrm{colim}_i C_i, D = \mathrm{colim}_i D_i$ in DGCat_{cont}. Write $\phi_i : D_i \to D$, $\psi_i : C_i \to C$ for the natural functors. Assume each right adjoint $G_i : D_i \to C_i$ of F_i is continuous. Let $F = \mathrm{colim}_i F_i : C \to D$ be the induced functor.

Then F has a contiuous right adjoint G. Moreover, for any $i \in I$ the composition $G\phi_i$ identifies with

$$\underset{(i \overset{\alpha}{\to} j) \in I_{i/}}{\mathrm{colim}} \psi_j G_j \phi_\alpha$$

taken in $\mathrm{Fun}_{e,cont}(D_i, C)$.

9.3. **About t-structures.**

9.3.1. Let $C \in \mathrm{DGCat}_{cont}$ with a t-structure. The t-structure is called right separated if $\cap_{n\in\mathbb{Z}} C^{\geq n} = 0$. It is left separated if $\cap_{n\in\mathbb{Z}} C^{\leq n} = 0$. It is sometimes called *non-degenerate* if it is both left and right separated. Recall that

$$C^{>0} = \{c \in C \mid \text{for all } c_0 \in C^{\leq 0}, \mathrm{Map}_C(c_0, c) \widetilde{\to} *\}$$

Note that for $c_0 \in C^{\leq 0}, V \in \mathrm{Vect}^{\leq 0}$ one has $V \otimes c_0 \in C^{\leq 0}$. Indeed, it suffices to check this for $V \in \mathrm{Vect}^{\leq 0} \cap \mathrm{Vect}^c$, where it is clear.

Lemma 9.3.2. *1) One has* $C^{>0} = \{c \in C \mid \textit{for all}\ c_0 \in C^{\leq 0}, \mathcal{M}aps_{k,C}(c_0, c) \in \mathrm{Vect}^{>0}\}$.
2) One has $C^{\leq 0} = \{c_0 \in C \mid \textit{for all}\ c \in C^{>0}, \mathcal{M}aps_{k,C}(c_0, c) \in \mathrm{Vect}^{>0}\}$.

Proof. 1) Let $c \in C^{>0}$, $c_0 \in C^{\leq 0}$. Then for $V \in \mathrm{Vect}^{\leq 0}$ one has

$$\mathrm{Map}_{\mathrm{Vect}}(V, \mathcal{M}aps_{k,C}(c_0, c)) \widetilde{\to} \mathrm{Map}_C(V \otimes c_0, c) \widetilde{\to} *$$

in Spc, because $V \otimes c_0 \in C^{\leq 0}$. So, $\mathcal{M}aps_{k,C}(c_0, c) \in \mathrm{Vect}^{>0}$.

Conversely, let $c \in C$ and for any $c_0 \in C^{\leq 0}$, $\mathcal{M}aps_{k,C}(c_0, c) \in \mathrm{Vect}^{>0}$. Then for $c_0 \in C^{\leq 0}$ we get $\mathrm{Map}_C(c_0, c) \widetilde{\to} \mathrm{Map}_{\mathrm{Vect}}(k, \mathcal{M}aps_{k,C}(c_0, c_1)) \widetilde{\to} *$ in Spc. So, $c \in C^{>0}$.
2) is similar. □

Remark: let $C \in \mathrm{DGCat}$ with a t-structure compatible with filtered colimits. Then $\tau^{\leq 0} : C \to C^{\leq 0}$ preserves filtered colimits. Indeed, let $c \widetilde{\to} \operatorname{colim} c_i$ with I small filtered, the colimit in C. For each i we have a fibre sequence $\tau^{\leq 0}c_i \to c_i \to \tau^{>0}c_i$. Passing to the colimit, we get a fibre sequence $\operatorname{colim} \tau^{\leq 0}c_i \to c \to \operatorname{colim} \tau^{>0}c_i$, which identifies with $\tau^{\leq 0}c \to c \to \tau^{>0}c$.

9.3.3. By ([14], ch. I.1, 7.1.1), $c \in C$ is compact iff the functor $\mathcal{M}aps_C(c, \cdot) : C \to \mathrm{Sptr}$ preserves filtered colimits. Since $Dold - Kan^{\mathrm{Sptr}} : \mathrm{Vect} \to \mathrm{Sptr}$ is continuous and conservative, this is also equivalent to the property that $\mathcal{M}aps_{k,C}(c, \cdot) : C \to \mathrm{Vect}$ preserves filtered colimits.

9.3.4. Let $C \in \mathrm{DGCat}_{cont}$ be compactly generated, assume given a t-structure on C. This t-structure on C is called *compactly generated* if $C^{\leq 0}$ is generated under filtered colimits by $C^{\leq 0} \cap C^c$. In this case $\mathrm{Ind}(C^{\leq 0} \cap C^c) \to C^{\leq 0}$ is an equivalence by ([27], 5.3.5.11). Note that $C^{\leq 0} \cap C^c$ admits finite colimits, so $C^{\leq 0}$ is presentable by ([27], 5.5.1.1), that is, the t-structure is accessible.

Lemma 9.3.5. *Let $C \in \mathrm{DGCat}_{cont}$ be compactly generated with a compactly generated t-structure. Then one has the following.*
i) The t-structure on C is compatible with filtered colimits.
ii) We have $C^{>0} = \{c \in C \mid \text{for all } x \in C^c \cap C^{\leq 0}, \mathcal{M}aps_{k,C}(x, c) \in \mathrm{Vect}^{>0}\}$.

Proof. i) Let I be small filtered, $I \to C^{>0}$ be a functor $i \mapsto c_i$, set $c = \operatorname{colim}_i c_i$ taken in C. By Lemma 9.3.2, it suffices to show that if $d \in C^{\leq 0}$ then $\mathcal{M}aps_{k,C}(d, c) \in \mathrm{Vect}^{>0}$. Pick J small filtered and a presentation $d \widetilde{\to} \operatorname{colim}_{j \in J} d_j$ with $d_j \in C^c \cap C^{\leq 0}$. We get $\mathcal{M}aps_{k,C}(d, c) \widetilde{\to} \lim_{j \in J^{op}} \mathcal{M}aps_{k,C}(d_j, c)$. For each $j \in J$,

$$\mathcal{M}aps_{k,C}(d_j, c) \widetilde{\to} \operatorname{colim}_i \mathcal{M}aps_{k,C}(d_j, c_i)$$

in Vect. Since $\mathcal{M}aps_{k,C}(d_j, c_i) \in \mathrm{Vect}^{>0}$, and the t-structure on Vect is compatible with filtered colimits, we get $\mathcal{M}aps_{k,C}(d_j, c) \in \mathrm{Vect}^{>0}$. Since $\mathrm{Vect}^{>0} \subset \mathrm{Vect}$ is closed under limits, we get $\mathcal{M}aps_{k,C}(d, c) \in \mathrm{Vect}^{>0}$.

ii) We check that the RHS is contained in $C^{>0}$. Let c lie in the RHS, $y \in C^{\leq 0}$. Write $y \widetilde{\to} \operatorname{colim}_{i \in I} y_i$ with I filtered, $y_i \in C^c \cap C^{\leq 0}$. Then $\mathcal{M}aps_{k,C}(y, c) \widetilde{\to} \lim_{i \in I^{op}} \mathcal{M}aps_{k,C}(y_i, c)$ in Vect. Since $\mathrm{Vect}^{>0} \subset \mathrm{Vect}$ is stable under limits, we get $\mathcal{M}aps_{k,C}(y, c) \in \mathrm{Vect}^{>0}$. So, $c \in C^{>0}$ by Lemma 9.3.2. □

9.3.6. Let $C, D \in \mathrm{DGCat}_{cont}$ be compactly generated equipped with t-structures. Assume the t-structures compactly generated for C, D.

We equip $C \otimes D$ with the t-structure declaring $(C \otimes D)^{\leq 0}$ to be the smallest full subcategory containing $c \boxtimes d$ for $c \in C^c \cap C^{\leq 0}, d \in D^c \cap D^{\leq 0}$, closed under extensions and small colimits. This is indeed an accessible t-structure by ([28], 1.4.4.11). Moreover, $C \otimes D$ is compactly generated by objects of the form $c \boxtimes d$ with $c \in C^c, d \in D^c$ by ([14], ch. I.1, 7.4.2).

Lemma 9.3.7. *Let $C, D \in \mathrm{DGCat}_{cont}$ be compactly generated equipped with compactly generated t-structures. Then the t-structure on $C \otimes D$ is compactly generated.*

Proof. We have an equivalence $h : C \otimes D \widetilde{\to} \mathrm{Fun}_{bi-ex,k}((C^c)^{op} \times (D^c)^{op}, \mathrm{Vect})$, where the RHS is the category of functors, which are exact and Vect^{fd}-linear in each variable. Here k stands for the field of coefficients of our DG-categories. The equivalence is obtained using ([14], ch. I.1, 10.5.6) as

$$C \otimes D \widetilde{\to} \mathrm{Fun}_{k,cont}(C^\vee, D) \widetilde{\to} \mathrm{Fun}_{ex,k}((C^c)^{op}, \mathrm{Fun}_{k,cont}(D^{op}, \mathrm{Vect})) \widetilde{\to}$$
$$\mathrm{Fun}_{ex,k}((C^c)^{op}, \mathrm{Fun}_{ex,k}((D^c)^{op}, \mathrm{Vect})) \widetilde{\to} \mathrm{Fun}_{bi-ex,k}((C^c)^{op} \times (D^c)^{op}, \mathrm{Vect})$$

For $c \in C, d \in D$ the functor $h(c \boxtimes d) \in \mathrm{Fun}_{bi-ex,k}((C^c)^{op} \times (D^c)^{op}, \mathrm{Vect})$ is the functor

$$(c_0, d_0) \mapsto \mathcal{M}aps_{k,C}(c_0, c) \otimes \mathcal{M}aps_{k,D}(d_0, d),$$

see ([14], ch. I.1, 10.5.8). Note also that for this functor to be Vect^{fd}-linear in each variable, the action of $V \in \mathrm{Vect}^{fd}$ on $c_0 \in (C^c)^{op}$ is defined as $V^\vee \otimes c_0$, and similarly on the second variable. For $f \in \mathrm{Fun}_{bi-ex,k}((C^c)^{op} \times (D^c)^{op}, \mathrm{Vect})$ and $c \in C^c, d \in D^c$ we get

$$\mathcal{M}aps_{k,C\otimes D}(h(c \boxtimes d), f) \widetilde{\to} f(c \times d)$$

So, $f \in \mathrm{Fun}_{bi-ex,k}((C^c)^{op} \times (D^c)^{op}, \mathrm{Vect})^{>0}$ iff for any $c \in C^c \cap C^{\leq 0}, d \in D^c \cap D^{\leq 0}$ one has $f(c \times d) \in \mathrm{Vect}^{>0}$. This means that $(C \otimes D)^{\leq 0}$ is generated by objects of the form $c \boxtimes d$ with $c \in C^c \cap C^{\leq 0}$ and $d \in D^c \cap D^{\leq 0}$. To finish, apply the lemma below taking $E' \subset (C \otimes D)^{\leq 0}$ the smallest full subcategory containing $c \boxtimes d$ for $c \in C^c \cap C^{\leq 0}, d \in D^c \cap D^{\leq 0}$ and closed under finite colimits. Note that $E' \subset (C \otimes D)^c$ by ([27], 5.3.4.15). If $z \in (C \otimes D)^{\leq 0}$ satisfies $\mathrm{Map}_{C\otimes D}(v, z) \widetilde{\to} *$ for any $v \in E'$ then $z = 0$ by the definition of $(C \otimes D)^{\leq 0}$. We are done.

If $f \widetilde{\to} \mathrm{colim}_{j\in J} f_j$ in $C \otimes D$, where J is small filtered then

$$f(c \times d) \widetilde{\to} \mathrm{colim}_{j\in J} f_j(c \times d),$$

where the colimit is calculated in Vect. Since the t-structure on Vect is compatible with filtered colimits, we see that the t-structure on $C \otimes D$ is also compatible with filtered colimits. □

Lemma 9.3.8. *Let E be ω-compactly generated in the sense of ([27], 5.5.7.1) and $E' \subset E^c$ is a full subcategory. Assume that if $z \in E$ and $\mathrm{Map}_E(e', z) \widetilde{\to} *$ for any $e' \in E'$ then z is isomorphic to the final object of E. Then the natural map $\mathrm{Ind}(E') \to E$ is an equivalence.*

Proof. Sam claims this is true. Proof under an additional assumption that $E' \subset E^c$ is stable under finite colimits, which is sufficient for Lemma 9.3.7. Since $E' \subset E^c$, we conclude that $h : \mathrm{Ind}(E') \to E$ is fully faithful by ([27], 5.3.5.11), and $\mathrm{Ind}(E')$ is presentable. By adjoint functor theorem, h admits a right adjoint h^R. By our assumption, the map $E \to \mathcal{P}(E')$ is fully faithful (find a reference!), and $\mathrm{Ind}(E') \subset \mathcal{P}(E')$ is fully faithful.

Let $z \in E$ then the natural map $hh^R(z) \to z$ indices an isomorphism of their images in $\mathcal{P}(E')$ by assumption. □

Remark 9.3.9. *Let $C \in \mathrm{DGCat}_{cont}$ be compactly generated and equipped with a t-structure compatible with filtered colimits. Then $C^{>0}$ is compactly generated in the sense of ([27], 5.5.7.1) by ([27], 5.5.7.3). Moreover, any compact object of $C^{>0}$ is a retract of $\tau^{>0}c$ for some $c \in C^c$.*

9.3.10. Let $f : C_1 \to C_2$ be a map in DGCat_{cont}. Let $D \in \mathrm{DGCat}_{cont}$ be equipped with a compactly generated t-structure. Assume C_i equipped with t-structures and f is t-exact. We equip $C_i \otimes D$ with t-structures as in Section 9.3.6.

Then $f \otimes \mathrm{id} : C_1 \otimes D \to C_2 \otimes D$ is t-exact. Indeed, the left t-exactness is ([47], Lemma B.2.4). The right t-exactness follows by definition of t-structures.

Lemma 9.3.11. *Let $C \in \mathrm{DGCat}_{cont}$ be equipped with a t-structure, $\mathcal{A} \in Alg(\mathrm{Fun}_{k,cont}(C, C))$ be a monad on C, so the underlying functor $f : C \to C$ is a map in DGCat_{cont}. Assume f is t-exact. Then $\mathcal{A} - mod(C)$ admits a unique t-structure such that both $oblv : \mathcal{A} - mod(C) \to C$ and $\mathrm{ind} : C \to \mathcal{A} - mod(C)$ are t-exact.*

Proof. We have $\mathcal{A} - mod(C) \in 1 - \mathcal{C}\mathrm{at}^{St,cocmpl}$ by Section 4.0.32. Note that $oblv : \mathcal{A} - mod(C) \to C$ reflects limits. Set

$$\mathcal{A} - mod(C)^{\leq 0} = oblv^{-1}(C^{\leq 0}),\ \ \mathcal{A} - mod(C)^{>0} = oblv^{-1}(C^{>0})$$

We check that this defines a t-structure on $\mathcal{A} - mod(C)$. Given $x \in \mathcal{A} - mod(C)^{\leq 0}, y \in \mathcal{A} - mod(C)^{>0}$, the bar construction ([28], 4.7.2.7) gives $x \widetilde{\to} \mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \mathcal{A}^{n+1}(x)$, the colimit calculated in $\mathcal{A} - mod(C)$ (we may also refer to [28], 4.7.3.14). So,

$$\mathrm{Map}_{\mathcal{A}-mod(C)}(x, y) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Map}_{\mathcal{A}-mod(C)}(\mathcal{A}^{n+1}(x), y) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Map}_C(\mathcal{A}^n(x), oblv(y)) \widetilde{\to} *,$$

because $\mathrm{Map}_C(\mathcal{A}^n(x), oblv(y)) \widetilde{\to} *$ for each n.

Consider now the full subcategory $\mathrm{Fun}_{k,cont}(C, C)^t \subset \mathrm{Fun}_{k,cont}(C, C)$ spanned by t-exact functors. It is stable under composition and inherits a monoidal structure from $\mathrm{Fun}_{k,cont}(C, C)$. Moreover, $\mathcal{A} \in Alg(\mathrm{Fun}_{k,cont}(C, C)^t)$. Now $C^{\leq 0}, C^{>0}$ are naturally module categories over $\mathrm{Fun}_{k,cont}(C, C)^t$. Moreover, the functors $\tau^{\leq 0} : C \to C^{\leq 0}$, $\tau^{>0} : C \to C^{>0}$ are naturally $\mathrm{Fun}_{k,cont}(C, C)^t$-linear functors in the sense of Section 3.0.49 of this file and ([28], 4.6.2.7). So, as in Section 3.0.49, they upgrade to functors

$$\tau^{\leq 0} : \mathcal{A} - mod(C) \to \mathcal{A} - mod(C^{\leq 0}),\ \ \tau^{>0} : \mathcal{A} - mod(C) \to \mathcal{A} - mod(C^{>0})$$

Moreover, the inclusions $C^{>0} \hookrightarrow C \leftarrow C^{\leq 0}$ are $\mathrm{Fun}_{k,cont}(C, C)^t$-linear functors. Now given $z \in \mathcal{A} - mod(C)$,we get the triangle $\tau^{\leq 0}z \to z \to \tau^{>0}z$ in $\mathcal{A} - mod(C)$. To see that this is a fibre sequence it suffices to check this after applying $oblv$, as $oblv$ reflects limits. The t-structure is constructed.

To see uniqueness, let $D^{\leq 0} \subset \mathcal{A}-mod(C)$ be a full subcategory defining another t-structure with the required properties. Then $D^{\leq 0} \subset oblv^{-1}(C^{\leq 0}) = \mathcal{A}-mod(C)^{\leq 0}$, because *oblv* is t-exact. On the other hand, for any $z \in \mathcal{A}-mod(C)^{\leq 0}$ the colimit $z \widetilde{\to} \operatorname{colim}_{[n]\in \boldsymbol{\Delta}^{op}} \mathcal{A}^{n+1}(z)$ taken in $\mathcal{A}-mod(C)$ must be in $D^{\leq 0}$, because ind is t-exact and $D^{\leq 0}$ is closed under colimits. Thus, $D^{\leq 0} = \mathcal{A}-mod(C)^{\leq 0}$. □

Remark 9.3.12. *i) Assume* $C \in \mathrm{DGCat}_{cont}$ *with a t-structure,* $\mathcal{A} \in Alg(\mathrm{Fun}_{k,cont}(C,C))$ *such that the underlying functor* $f : C \to C$ *is right t-exact. Then there is a t-structure on* $\mathcal{A}-mod(C)$ *such that* $oblv : \mathcal{A}-mod(C) \to C$ *is t-exact, and* $\mathrm{ind} : C \to \mathcal{A}-mod(C)$ *is right t-exact.* [1]

Any other t-structure on $\mathcal{A}-mod(C)$*, for which* $\mathrm{oblv} : \mathcal{A}-mod(C) \to C$ *and* $\mathrm{ind} : C \to \mathcal{A}-mod$ *are both right t-exact coincides with the above one.*
ii) Assume in addition $\mathcal{A} \in \mathrm{Alg}(\mathrm{Vect}^{\leq 0})$ *giving a monad in* C*, assume the t-structure on* C *accessible. Then under the equivalence* $\mathcal{A}-mod(C) \widetilde{\to} (\mathcal{A}-mod) \otimes C$ *the above t-structure corresponds to the tensor product t-structure on the right hand side.*

Proof. By Section 4.0.32, $\mathcal{A}-mod(C) \in 1-\mathfrak{Cat}^{St,cocmpl}$.
i) Write $\mathrm{Fun}^{rex}_{k,cont}(C,C) \subset \mathrm{Fun}_{k,cont}(C,C)$ for the full subcategory of right t-exact functors. It is stable under compositions, so inherits the monoidal structure from $\mathrm{Fun}_{k,cont}(C,C)$. View C and $C^{\leq 0}$ as modules over $\mathrm{Fun}^{rex}_{k,cont}(C,C)$. Then the functor $\tau^{\leq 0} : C \to C^{\leq 0}$ is a right lax functor of $\mathrm{Fun}^{rex}_{k,cont}(C,C)$-module categories, because its left adjoint is $\mathrm{Fun}^{rex}_{k,cont}(C,C)$-linear. Namely, for $f \in \mathrm{Fun}^{rex}_{k,cont}(C,C)$ applying f to the natural map $\tau^{\leq 0}c \to c$, one gets a morphism $f(\tau^{\leq 0}c) \to f(c)$, which factors as $f(\tau^{\leq 0}c) \to \tau^{\leq 0}(f(c))$. Further, $\mathcal{A} \in Alg(\mathrm{Fun}^{rex}_{k,cont}(C,C))$, so $\tau^{\leq 0}$ induces a morphism $\mathcal{A}-mod(C) \to \mathcal{A}-mod(C^{\leq 0})$. Given $c \in \mathcal{A}-mod(C)$, the action of $\mathcal{A}$ on $\tau^{\leq 0}c$ is the composition

$$\mathcal{A}(\tau^{\leq 0}c) \to \tau^{\leq 0}(\mathcal{A}c) \stackrel{\tau^{\leq 0}(act)}{\to} \tau^{\leq 0}c$$

The inclusion $i : C^{\leq 0} \to C$ is a morphism of $\mathrm{Fun}^{rex}_{k,cont}(C,C)$-module categories. We get an adjoint pair $i : \mathcal{A}-mod(C^{\leq 0}) \leftrightarrows \mathcal{A}-mod(C) : \tau^{\leq 0}$ by Section 3.0.78.

Define $\mathcal{A}-mod(C)^{\leq 0}, \mathcal{A}-mod(C)^{>0}$ as in the proof of Lemma 9.3.11. As in *loc.cit*, we have for $x \in \mathcal{A}-mod(C)^{\leq 0}, y \in \mathcal{A}-mod(C)^{>0}$, $\mathrm{Map}_{\mathcal{A}-mod(C)}(x,y) \widetilde{\to} *$. Let $c \in \mathcal{A}-mod(C)$, so $\tau^{\leq 0}c$ is an object of $\mathcal{A}-mod(C^{\leq 0}) \subset \mathcal{A}-mod(C)$ as above. Consider the exact triangle $\tau^{\leq 0}c \to c \to z$ in $\mathcal{A}-mod(C)$. Since $\mathrm{oblv} : \mathcal{A}-mod(C) \to C$ is exact, this is an exact triangle in C, so z if the cofibre of the canonical map $\tau^{\leq 0}c \to c$, hence $z \widetilde{\to} \tau^{>0}c$ and $z \in \mathcal{A}-mod(C)^{>0}$. By definition, this is a t-structure on $\mathcal{A}-mod(C)$, and $\mathrm{oblv} : \mathcal{A}-mod(C) \to C$ is t-exact.

To see uniqueness, let $D^{\leq 0} \subset \mathcal{A}-mod(C)$ be another full subcategory defining a t-structure with required properties. Then $D^{\leq 0} \subset \mathrm{oblv}^{-1}(C^{\leq 0})$, because oblv is right t-exact. For $a \in \mathcal{A}-mod(C)^{\leq 0}$ the colimit $z \widetilde{\to} \operatorname{colim}_{[n]\in \boldsymbol{\Delta}^{op}} \mathcal{A}^{n+1}(z)$ taken in $\mathcal{A}-mod(C)$ must be in $D^{\leq 0}$, because ind is right t-exact.

ii) The functor $(\mathcal{A}-mod)\otimes C \to \mathcal{A}-mod(C)$, $V \boxtimes c \mapsto V \otimes c$ is right t-exact by definition. It is an equivalence by ([14], ch. I.1, 8.5.7). Since $\mathrm{ind} : \mathrm{Vect} \otimes C \to (\mathcal{A}-mod) \otimes C$ is right t-exact, our claim follows from the uniqueness in i). □

[1] Dennis claims the existence of t-structure on $\mathcal{A}-mod(C)$ for $\mathcal{A}$ left t-exact, but I don't see that.

Lemma 9.3.13. *Let $C \in \mathrm{DGCat}_{cont}$ be compactly generated with a compactly generated t-structure. Let $\mathcal{A} \in Alg(\mathrm{Fun}_{k,cont}(C, C))$ be such that the underlying functor $C \to C$ is t-exact. Then $\mathcal{A} - mod(C)$ is compactly generated, and the t-structure on $\mathcal{A} - mod(C)$ defined in Lemma 9.3.11 is compactly generated.*

Proof. Let $\mathrm{ind} : C \to \mathcal{A} - mod(C)$ be the left adjoint to $\mathrm{oblv} : \mathcal{A} - mod(C) \to C$. Then ind preserves compact objects, so $\mathcal{A} - mod(C)$ is compactly generated by objects of the form $\mathrm{ind}(c)$ for $c \in C^c$.

If $c \in C^c \cap C^{\leq 0}$ then $\mathrm{ind}(c) \in \mathcal{A} - mod(C)^c \cap \mathcal{A} - mod(C)^{\leq 0}$. Let $z \in \mathcal{A} - mod(C)^{\leq 0}$. Assume that for any $c \in C^c \cap C^{\leq 0}$, $\mathrm{Map}_{\mathcal{A}-mod(C)}(\mathrm{ind}(c), z) \widetilde{\to} *$, that is, $\mathrm{Map}_C(c, \mathrm{oblv}(z)) \widetilde{\to} *$. Since the t-structure on C is compactly generated, this gives $\mathrm{oblv}(z) \widetilde{\to} 0$ in $C^{\leq 0}$, hence $z \widetilde{\to} 0$, because oblv is conservative. □

9.3.14. Let $C \in \mathrm{DGCat}_{cont}$ with a t-structure, $c_i \in C$ be infinitely connective objects for $i \in I$, here I is a small set. Let $C' \subset C$ be the smallest cocomplete stable subcategory containing all c_i (and stable under Vect-action also). Then any object of C' is infinitely connective. Indeed, for any $n \in \mathbb{Z}$, $C^{\leq n}$ is stable under all colimits.

9.3.15. Recall that we have the involution $C \mapsto C^{op}$ on $\mathrm{DGCat}^{non-cocompl}$ by ([14], ch. I.1, 10.3.2). For $C \in \mathrm{DGCat}^{non-cocompl}$ one defines $Pro(C) = (\mathrm{Ind}(C^{op}))^{op}$. By ([14], ch. I.1, 10.5.5) we have $Pro(C) \widetilde{\to} \mathrm{Fun}_k(C, \mathrm{Vect})^{op}$. Here $\mathrm{Fun}_k(C, \mathrm{Vect})$ is defined in ([14], ch. I.1, 10.3.5).

The following was used in ([6], Appendix A), and it follows from ([27], 5.3.5.13). Let $G : C' \to C''$ be an exact functor, a map in $\mathrm{DGCat}^{non-cocompl}$, so it is Vect^{fd}-linear. It gives a functor $G : Pro(C') \to Pro(C'')$, namely the corresponding functor $\mathrm{Ind}(C'^{op}) \to \mathrm{Ind}(C''^{op})$ is the left Kan extension of the composition $C'^{op} \to C''^{op} \to \mathrm{Ind}(C''^{op})$ along $C'^{op} \to \mathrm{Ind}(C'^{op})$. Then G admits a left adjoint $G^L : Pro(C'') \to Pro(C')$ sending a functor $f : C'' \to \mathrm{Vect}$ to the composition $C' \stackrel{G}{\to} C'' \stackrel{f}{\to} \mathrm{Vect}$. So, in some sense the left adjoint to G always exists as a functor $C'' \to Pro(C')$, namely the composition $C'' \to Pro(C'') \stackrel{G^L}{\to} Pro(C')$.

9.3.16. Let $f : C \to \mathrm{Vect}$ be a map in $CAlg(\mathrm{DGCat}_{cont})$, so a symmetric monoidal functor. Assume C equipped with a t-structure, f conservative and t-exact. Then the t-structure on C is accessible, compatible with filtered colimits, and C is right complete.

Proof: We have $C^{\leq 0} = \{c \in C \mid f(c) \in \mathrm{Vect}^{\leq 0}\}$ and $C^{\geq 0} = \{c \in C \mid f(c) \in \mathrm{Vect}^{\geq 0}\}$. Since $\mathcal{P}\mathrm{r}^L \to 1-\mathcal{C}\mathrm{at}$ preserves limits, $C^{\leq 0}$ is presentable, so the t-structure is accessible. For $z \in C$ the natural map $\mathrm{colim}_n \tau^{\leq n} z \to z$ is an isomorphism in C, because it becomes an isomorphism after applying f. So, by Section 4.0.10, C is right complete.

9.3.17. If $C, D \in \mathrm{DGCat}_{cont}$ with accessible t-structures then $C \otimes D$ also acquires an accessible t-structure defined as follows. It is known that any presentable ∞-category E admits a small set of objects S that generates E under small colimits. Pick small sets of objects $C' \subset C^{\leq 0}$ that generate $C^{\leq 0}$ under small colimits, and similarly for $D' \subset D^{\leq 0}$. Then the collection $\mathcal{E} = \{c \boxtimes d \mid c \in C', d \in D'\}$ is a small set of objects in $C \otimes D$. Let $(C \otimes D)^{\leq 0} \subset C \otimes D$ be the smallest full subcategory containing $\mathcal{E}$ and closed under extensions and small colimits. Then $(C\otimes D)^{\leq 0}$ is presentable by ([28], 1.4.4.11), and defines an accessible t-structure on $C\otimes D$.

Assume in addition that the t-structures on C, D are compatible with filtered colimits. Then the t-structure on $C\otimes D\in \mathrm{DGCat}_{cont}$ is compatible with filtered colimits also.

Idea of proof: (cf. [33], C.4.2.2 for details). Write $C\otimes_{\mathrm{Vect}} D$ for the tensor product over Vect, and simply $C\otimes D$ for the tensor product in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. By [[51], Remark C.4.2.2], $C\otimes D$ is equipped with a t-structure compatible with filtered colimits. Consider the adjoint pair $m:\mathrm{Vect}\otimes\mathrm{Vect}\leftrightarrows\mathrm{Vect}:m^R$ in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$, where the tensor product is in $1-\mathcal{C}\mathrm{at}^{St,cocmpl}_{cont}$. I imagine, m^R is monadic. Now tensoring by $C\otimes D$ over $\mathrm{Vect}\otimes\mathrm{Vect}$, we also get an adjoint pair $\bar m: C\otimes D\leftrightarrows (C\otimes D)\otimes_{\mathrm{Vect}\otimes\mathrm{Vect}}\mathrm{Vect}\,\widetilde{\to}\, C\otimes_{\mathrm{Vect}} D:\bar m^R$ with $\bar m^R$ monadic and monad $\mathcal{A}$. The monad $\mathcal{A}$ should be right t-exact Sam says. Namely, I have to check that $\mathrm{Vect}\,\widetilde{\to}\, k-mod(\mathrm{Sptr})$, so $\mathrm{Vect}\otimes\mathrm{Vect}\,\widetilde{\to}\,(k\otimes k)-mod(\mathrm{Sptr})$, and the monad should be the tensoring with the $k\otimes k$-module k. So, the monad $\mathcal{A}$ is right t-exact. As in Remark 9.3.12 now, $\mathcal{A}-mod(C\otimes D)$ acquires a t-structure such that $\mathrm{oblv}:\mathcal{A}-mod(C\otimes D)\to C\otimes D$ is t-exact and conservative. This implies that the t-structure on $C\otimes_{\mathrm{Vect}} D$ is compatible with filtered colimits.

9.3.18. Let $C_0\in\mathrm{DGCat}^{non-cocmpl}$ with a t-structure. Let $C=\mathrm{Ind}(C_0)$. Recall that by ([14], II.1, 1.2.4), C is equipped with a t-structure compatible with filtered colimits and accessible such that $C_0\to C$ is t-exact. Assume that the t-structure on C_0 is bounded. Then the t-structure on C is right complete.

Proof: C is stable by (HA, 1.1.3.6), it is also presentable, because C_0 admits finite colimits. The t-structure on C is accessible by ([14], II.1, Lemma 1.2.4). So, by my Section 4.0.10, it suffices to show that for any $K\in C$, the natural map $K\to \mathrm{colim}_{n\in\mathbb{Z}}\,\tau^{\leq n}K$ is an isomorphism, where the colimit is understood in C.

Pick a presentation $K\,\widetilde{\to}\,\mathrm{colim}_{i\in I} K_i$ in C, where $K_i\in C_0$ and I is small filtered. The functors $\tau^{\leq n}: C\to C$ preserve filtered colimits, so

$$\mathrm{colim}_{m\in\mathbb{Z}}\,\tau^{\leq m}K\,\widetilde{\to}\,\mathrm{colim}_{m\in\mathbb{Z}}\,\mathrm{colim}_{i\in I}\,\tau^{\leq m}K_i\,\widetilde{\to}\,\mathrm{colim}_{i\in I}(\underset{m\in\mathbb{Z}}{\mathrm{colim}}\,\tau^{\leq m}(K_i))$$
$$\widetilde{\to}\,\mathrm{colim}_{i\in I} K_i\,\widetilde{\to}\, K$$

We used the fact that K_i is bounded, so $\underset{m\in\mathbb{Z}}{\mathrm{colim}}\,\tau^{\leq m}(K_i)\,\widetilde{\to}\, K_i$ in C, because I is contractible. Thus, C is right complete.

9.3.19. Let $I\in 1-\mathcal{C}\mathrm{at}$ be small filtered and we are given $I\to\mathrm{DGCat}_{cont}$, $i\mapsto C_i$. Assume each C_i is equipped with an accessible t-structure, and for $\alpha: i\to j$ in I the map $h_{ij}=h_\alpha: C_i\to C_j$ is t-exact. Note that the right adjoint $h^R_\alpha: C_j\to C_i$ is left t-exact. Set $C=\mathrm{colim}_{i\in I} C_i$ in DGCat_{cont}. We have $C\,\widetilde{\to}\,\lim_{i\in I^{op}} C_i$ with respect to the functors h^R_α, where the limit is calculated in DGCat. Assume for simplicity that h^R_α are continuous. Set $C^{>0}=\lim_{i\in I^{op}} C_i^{>0}$ in $\mathcal{P}\mathrm{r}^L$, so $C^{>0}$ is presentable, and this is a full subcategory in $C\,\widetilde{\to}\,\lim_{i\in I^{op}} C_i$, where the limit is calculated in DGCat_{cont}. We also have $C^{>0}=\lim_{i\in I^{op}} C_i^{>0}$ in $1-\mathcal{C}\mathrm{at}$.

Assume each h^R_α fully faithful. Let $i_0 \in I$ be an initial object. Then by my Section 2.7.7 we have $C = \cap C_i$ as full subcategories of C_{i_0}. The inclusion $C \hookrightarrow C_{i_0}$ has a left adjoint L given by the formula from ([36], Lm. 1.2.15). Namely, for $c \in C_{i_0}$, $L(c) \widetilde{\to} \operatorname{colim}_{j\in I} h_{i_0 j}(c)$, where the colimit is calculated in C_{i_0}. We also have $C^{>0} = \cap_i C_i^{>0}$ as full subcategory of C_{i_0}.

Let us also define $C^{\le 0}$ as the image of $C_{i_0}^{\le 0}$ under L (equivalently, for any i as the image of $C_i^{\le 0}$ under the restriction $L : C_i \to C$ of L). Assume that the t-structure on each C_i is compatible with filtered colimits. Then we claim that $(C^{\le 0}, C^{>0})$ define a t-structure on C. Indeed, let $x \in C$. Let $\tau_i^{\le 0} : C_i \to C_i^{\le 0}$, $\tau_i^{>0} : C_i \to C_i^{>0}$ be the truncation functors. We have the fibre sequence $L(\tau_{i_0}^{\le 0} x) \to x \to L(\tau_{i_0}^{>0}(x))$ in C and $L(\tau_{i_0}^{\le 0} x) \in C^{\le 0}$. We claim that $L(\tau_{i_0}^{>0}(x)) \in C^{>0}$.

Indeed, recall that $L(\tau_{i_0}^{>0}(x)) \widetilde{\to} \operatorname{colim}_{j\in I} h_{i_0 j}(\tau_{i_0}^{>0}(x))$, the colimit being calculated in C_{i_0}. It suffices to show that this is an object of $C_j^{>0}$ for any $j \in J$. For this we may replace I in the colimit by $I_{j/}$. Then for each $j \to j'$ in I, $h_{i_0 j'}(\tau_{i_0}^{>0}(x)) \in C_j^{>0}$, because $h^R_{jj'}$ is left t-exact. Since $C_j^{>0}$ is stable under filtered colimits, the colimit remains in $C_j^{>0}$. Thus, we constructed a fibre sequence for x, which is $\tau_C^{\le 0} x \to x \to \tau_C^{>0}(x)$. The rest is easy. So, this is indeed a t-structure. Note that $C^{>0}$ is presentable, so this t-structure is accessible. It is also compatible with filtered colimits: if J is filtered and $x \widetilde{\to} \operatorname{colim}_{j\in J} x_j$ with $x_j \in C^{>0}$, where the colimit is calculated in C then this is also the colimit in C_{i_0}, but for any i, $C_i^{>0}$ is stable under filtered colimits, so this colimit is in $C_i^{>0}$. Since i is arbitrary, $x \in C^{>0}$.

9.3.20. Let $C \in \mathrm{DGCat}_{cont}$ with an accessible t-structure. Let $\hat{C}$ denote the left completion of C, it is equipped with the induced t-structure. The t-structure on $\hat{C}$ is accessible, because $C^{\ge 0} \widetilde{\to} (\hat{C})^{\ge 0}$ is presentable (see [28], 1.4.4.13). Note that $C^{\ge n}$ is presentable for each n. For each n the functor $\tau^{\ge -n} : C^{\ge -n-1} \to C^{\ge -n}$ preserves colimits, as it is a left adjoint. So, $\lim_{n\in\mathbb{Z}^{op}} C^{\ge -n}$ can be understood in the category $\mathcal{P}^L$ of presentable ∞-categories and colimit preserving functors, so $\hat{C}$ is presentable.

Assume in addition the t-structure on C is compatible with filtered colimits. Then for $V \in \mathrm{Vect}^{\ge 0}, c \in C^{\ge -n}$ we have $V \otimes c \in C^{\ge -n}$. Indeed, it suffices to check this for $V \in \mathrm{Vect}^c \cap \mathrm{Vect}^{\ge 0}$. In this case for $y \in C^{<-n}$ we have

$$\mathrm{Map}_C(y, V \otimes c) \widetilde{\to} \mathrm{Map}(V^\vee \otimes y, c) \widetilde{\to} *,$$

so $V \otimes c \in C^{\ge -n}$. So, for $V \in \mathrm{Vect}^\heartsuit$ the functor $C \to C, c \mapsto V \otimes c$ is t-exact. We define the $\mathrm{Vect}^\heartsuit$-action on $\hat{C}$ by the formula: for $x := (c_n) \in \hat{C}$, where $c_n \in C^{\ge -n}$ and $\tau^{\ge -n} c_{n+1} \widetilde{\to} c_n$ we let $V \otimes x \in \hat{C}$ be given by the collection $(V \otimes c_n)$. This extends to an action of Vect^c first, and then by continuity, to an action of Vect, because $\hat{C}$ is presentable. The natural functor $C \to \hat{C}$ preserves filtered colimits. It is also exact by ([28], 1.2.1.17), so is a map in DGCat_{cont}.

Let $f : C \to D$ be a map in DGCat_{cont}, D is equipped with an accessible t-structure compatible with filtered colimits also. Assume f is t-exact. Let $\hat{f} : \hat{C} \to \hat{D}$ ne obtained by passing to left completions. Then is exact by Section 4.3.5. Besides, $\hat{f}$ is continuous, because for each n, $f : C^{\geq -n} \to D^{\geq -n}$ preserves filtered colimits (as $C^{\geq -n} \subset C$ is closed under filtered colimits). Our $\hat{f}$ is Vect-linear, so is a map in DGCat_{cont}.

9.3.21. Let $C \in \mathrm{DGCat}_{cont}$ with an accessible t-structure. Let $\hat{C}$ denote the right completion of C, it is equipped with the induced t-structure. For each n, $C^{\leq n}$ is presentable. Assume the t-structure on C is compatible with filtered colimits. Then the functor $\tau^{\leq n} : C^{\leq n+1} \to C^{\leq n}$ preserves filtered colimits (so is accessible) and all limits. By ([27], 5.5.3.18) we may understand $\lim_{n \in \mathbb{Z}^{op}} C^{\leq n}$ in $\mathcal{P}\mathrm{r}^R$, the category of presentable categories, where the morphisms are limit-preserving accessible functors. So, $\hat{C}$ is presentable. Now $C^{\leq 0} \widetilde{\to} (\hat{C})^{\leq 0}$, so the t-structure on $\hat{C}$ is accessible. The Vect-action on $\hat{C}$ is obtained as in the previous section, so $\hat{C} \in \mathrm{DGCat}_{cont}$. The natural functor $C \to \hat{C}$ preserves filtered colimits by construction (Lemma 2.2.68) and is exact by ([28], 1.2.1.17), so is a map in DGCat_{cont}.

Let $f : C \to D$ be a map in DGCat_{cont}, where D is equipped with an accessible t-structure compatible with filtered colimits. Assume f is t-exact. Let $\hat{f} : \hat{C} \to \hat{D}$ be obtained by passing to right completions. Then $\hat{f}^{op} : \hat{C}^{op} \to \hat{D}^{op}$ is exact by Section 4.3.5, so $\hat{f}$ is also exact. Since each $f : C^{\leq n} \to D^{\leq n}$ preserves filtered colimits, we conclude that $\hat{f}$ is continuous, so $\hat{f}$ is a map in DGCat_{cont}.

9.3.22. *On comodule categories.* Let $C \in \mathrm{DGCat}_{cont}$, $\mathcal{A} \in \mathrm{Fun}_{k,cont}(C, C)$ be a k-linear continious comonad on C. Recall the adjoint pair $\mathrm{oblv} : \mathcal{A} - comod(C) \leftrightarrows C : coind$ in DGCat_{cont}. Assume C is equipped with an accessible t-structure. We equip $\mathcal{A} - comod(C)$ with the t-structure characterized by $\mathcal{A} - comod(C)^{\leq 0} = \mathrm{oblv}^{-1}(C^{\leq 0})$. By ([28], 1.4.4.11), this is an accessible t-structure on $\mathcal{A} - comod(C)$.

Lemma 9.3.23. *Assume in the situation of the previous subsection that $\mathcal{A} : C \to C$ is t-exact. Then*

i) both functors in the adjoint pair $\mathrm{oblv} : \mathcal{A} - comod(C) \leftrightarrows C : coind$ *are t-exact.*

ii) If the t-structure on C is right complete then the t-structure on $\mathcal{A} - comod(C)$ is also right complete.

Proof. i) By Section 9.2.59, each $c \in \mathcal{A} - comod(C)$ writes as $c \widetilde{\to} \lim_{[n] \in \boldsymbol{\Delta}} \mathcal{A}^{n+1}(c)$, the limit calculated in $\mathcal{A} - comod(C)$.

We claim that the pair of subcategories $(\mathrm{oblv}^{-1}(C^{\leq 0}), \mathrm{oblv}^{-1}(C^{\geq 0}))$ define a t-structure on $\mathcal{A} - comod(C)$. First, for $x \in \mathrm{oblv}^{-1}(C^{\leq 0}), y \in \mathrm{oblv}^{-1}(C^{>0})$ one has

$$\mathrm{Map}_{\mathcal{A}-comod(C)}(x, y) \widetilde{\to} \lim_{[n] \in \boldsymbol{\Delta}} \mathrm{Map}_{\mathcal{A}-comod(C)}(x, \mathcal{A}^{n+1}(y)) \widetilde{\to} \lim_{[n] \in \boldsymbol{\Delta}} \mathrm{Map}_C(\mathrm{oblv}(x), \mathcal{A}^n(y)) \widetilde{\to} *$$

as $\mathrm{Map}_C(\mathrm{oblv}(x), \mathcal{A}^n(y)) \widetilde{\to} *$ for each n.

Let $\mathrm{Fun}_{k,cont}(C, C)^t \subset \mathrm{Fun}_{k,cont}(C, C)$ be the full subcategory of t-exact functors. It inherits a monoidal structure from $\mathrm{Fun}_{k,cont}(C, C)$. The adjoint pairs $\tau^{\leq 0} : C \leftrightarrows C^{\leq 0} : j$ and $i : C^{\leq 0} \leftrightarrows C : \tau^{\leq 0}$ take place in $\mathrm{Fun}_{k,cont}(C, C)^t$-modules. So, they induces the corresponding functors $\tau^{\leq 0} : \mathcal{A} - comod(C) \to \mathcal{A} - comod(C^{\leq 0}), \tau^{\geq 0} : \mathcal{A} - comod(C) \to \mathcal{A} - comod(C^{\geq 0})$ and so on.

For $c \in \mathcal{A} - comod(C)$ we get a triangle $\tau^{\leq 0}(c) \to c \to \tau^{>0}(c)$ in $\mathcal{A} - comod(C)$. To see that this is a cofibre sequence, recall that $\mathrm{oblv} : \mathcal{A} - comod(C) \to C$ reflects colimits. It becomes a cofibre sequence in C, hence also in $\mathcal{A} - comod(C)$. Since $\mathrm{oblv} : \mathcal{A} - comod(C) \to C$ preserves the shift [1], and the latter is an equivalence, it also preserves its inverse $[-1]$. So, this is indeed a t-structure, which coincides with that of Section 9.3.22. So, oblv is t-exact.

Now $\mathrm{coind} : C \to \mathcal{A} - comod(C)$ is informally given by $c \mapsto \mathcal{A}(c)$, it is t-exact.

ii) Let $z \in \mathcal{A} - comod(C)$. By Section 4.0.10, it suffices to show that the natural map $\mathrm{colim}_n \tau^{\leq n} z \to z$ in $\mathcal{A} - comod(C)$ is an isomorphism. Since oblv is t-exact an conservative, our claim follows again by Section 4.0.10. □

Remark 9.3.24. *In the situation of Section 9.3.22 if the t-structure on C is left separated then the t-structure on $\mathcal{A} - comod(C)$ is also left separated.*

9.3.25. Let $p_! : C \to D$ be a map in DGCat_{cont}. Assume D is equipped with an accessible t-structure. Let $C^{\leq 0} = p_!^{-1}(D^{\leq 0})$. Clearly, $C^{\leq 0}$ is closed under colimits and extensions, it is presentable. By ([28], 1.4.4.11), this defines an accessible t-structure on C. Assume $p_!$ has a right adjoint $p^! : D \to C$, which is fully faithful. Then $p^!$ is t-exact.

Lemma 9.3.26. *In the situation of Section 9.3.22 assume that $\mathcal{A} : C \to C$ is left t-exact. Then $\mathrm{oblv}^{-1}(C^{>0}) = \mathcal{A} - comod(C)^{>0}$, and $\mathrm{oblv} : \mathcal{A} - comod(C) \to C$ is t-exact. Besides, $\mathrm{coind} : C \to \mathcal{A} - comod(C)$ is left t-exact.*

Any other t-structure on $\mathcal{A} - comod(C)$, for which both oblv, coind are left t-exact, coincides with the above one.

Proof. By Section 9.2.59, each $c \in \mathcal{A} - comod(C)$ writes as $c \widetilde{\to} \lim_{[n]\in \boldsymbol{\Delta}} \mathcal{A}^{n+1}(c)$, the limit calculated in $\mathcal{A} - comod(C)$.

We claim that the pair of subcategories $(\mathrm{oblv}^{-1}(C^{\leq 0}), \mathrm{oblv}^{-1}(C^{\geq 0}))$ define a t-structure on $\mathcal{A} - comod(C)$. If $x \in \mathrm{oblv}^{-1}(C^{\leq 0}), y \in \mathrm{oblv}^{-1}(C^{>0})$ as in the proof of Lemma 9.3.23 one shows that $\mathrm{Map}_{\mathcal{A}-comod(C)}(x, y) \widetilde{\to} *$.

Let $\mathrm{Fun}^{lex}_{k,cont}(C, C) \subset \mathrm{Fun}_{k,cont}(C, C)$ be the full subcategory of left t-exact functors. It inherits a monoidal structure from $\mathrm{Fun}_{k,cont}(C, C)$. Since the inclusion $i : C^{\geq 0} \to C$ is $\mathrm{Fun}^{lex}_{k,cont}(C, C)$-linear, $\tau^{\geq 0} : C \to C^{\geq 0}$ is a left-lax functor of $\mathrm{Fun}^{lex}_{k,cont}(C, C)$-module categories. Now $\mathcal{A}$ is a comonoid in $\mathrm{Fun}^{lex}_{k,cont}(C, C)$, so $\tau^{\geq 0}$ induces a functor

$$\tau^{\geq 0} : \mathcal{A} - comod(C) \to \mathcal{A} - comod(C^{\geq 0})$$

We get an adjoint pair $\tau^{\geq 0} : \mathcal{A} - comod(C) \leftrightarrows \mathcal{A} - comod(C^{\geq 0}) : i$.

For $a \in \mathrm{Fun}^{lex}_{k,cont}(C, C), c \in C$ we get a natural map $\tau^{\geq 0}(ac) \to a\tau^{\geq 0}(c)$. Given $c \in \mathcal{A} - comod(C)$, the coaction of $\mathcal{A}$ on $\tau^{\geq 0}(c)$ is the composition

$$\tau^{\geq 0}(c) \stackrel{\tau^{\geq 0}(coact)}{\to} \tau^{\geq 0}(\mathcal{A}c) \to \mathcal{A}(\tau^{\geq 0}c).$$

Given $c \in \mathcal{A} - comod(C)$, we have $\tau^{\geq 0}(c) \in \mathcal{A} - comod(C)$. Consider the exact triangle $z \to c \to \tau^{\geq 0}c$ in $\mathcal{A} - mod(C)$. Since oblv is exact, this is an exact triangle in C also, so $z \widetilde{\to} \tau^{<0}(c)$ in C. Thus, $z \in \mathcal{A} - comod(C)^{<0}$. By definition, this is a t-structure on $\mathcal{A}-comod(C)$, and it coincides with that of Section 9.3.22. First claim follows.

Let now $D^{\geq 0}\subset \mathcal{A}-comod(C)$ determine a t-structure such that both coind, *oblv* are left t-exact. Then $D^{\geq 0}\subset \mathrm{oblv}^{-1}(C^{\geq 0})$. Let now $z\in \mathrm{oblv}^{-1}(C^{\geq 0})$. We have $z\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\mathcal{A}^{n+1}(c)$ in $\mathcal{A}-comod(C)$. For any $n\geq 0$, $\mathcal{A}^{n+1}(c)\in D^{\geq 0}$, hence $z\in D^{\geq 0}$ also. Thus, $D^{\geq 0}=\mathrm{oblv}^{-1}(C^{\geq 0})$. □

9.4. **Symmetric sequences.**

9.4.1. They are discussed in ([14], vol. 2, ch. 5, 1.1), there is also a nlab page "symmetric sequence". The collection $\{S_n\}_{n\geq 1}$, where $S_0=*$ form a graded semigroup in the category of groups. That is, for $n,m\geq 1$ we have a homomorphism $S_n\times S_m\to S_{n+m}$ given by the usual order $\{1,\dots,n\}\sqcup\{n+1,\dots,n+m\}$. They are associative.

9.4.2. If G is a finite group then $\mathrm{QCoh}(B(G))\widetilde{\to}\mathrm{Fun}(B(G),\mathrm{Vect})$ naturally. Indeed, $B(G)=\mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}}G^n$ in PreStk, and

$$\mathrm{Fun}(B(G),\mathrm{Vect})\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{Fun}(G^n,\mathrm{Vect})\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\mathrm{QCoh}(G^n)\widetilde{\to}\mathrm{QCoh}(B(G))$$

If $a:G\to H$ is a homomorphism of finite groups, we have an adjoint pait $\mathrm{Ind}_G^H:\mathrm{Fun}(B(G),\mathrm{Vect})\leftrightarrows\mathrm{Fun}(B(H),\mathrm{Vect}):\mathrm{Res}$, where Res is the composition with $\bar{a}:B(G)\to B(H)$, and Ind_G^H is the LKE along $\bar{a}$. The functor Res is $\bar{a}^*$. By Section 10.2.12, $\mathrm{QCoh}(B(G))\widetilde{\to}B-comod(\mathrm{Vect})$, where $B=\mathrm{H}^0(G,\mathcal{O})$ is a coalgebra in Vect. The natural map $\mathrm{H}^0(H,\mathcal{O})\to\mathrm{H}^0(G,\mathcal{O})$ is a map of coalgebras in Vect, it gives a morphism $\mathrm{Res}:\mathrm{H}^0(H,\mathcal{O})-comod(\mathrm{Vect})\to\mathrm{H}^0(G,\mathcal{O})-comod(\mathrm{Vect})$. If $M\in\mathrm{H}^0(H,\mathcal{O})-comod(\mathrm{Vect})$ then the composition $M\to\mathrm{H}^0(H,\mathcal{O})\otimes M\to\mathrm{H}^0(G,\mathcal{O})\otimes M$ is the coaction for $\mathrm{Res}(M)$. The functor $\mathrm{oblv}:\mathrm{H}^0(G,\mathcal{O})-comod(\mathrm{Vect})\to\mathrm{Vect}$ is the $*$-restriction along $\mathrm{Spec}\,k\to B(G)$.

Recall that $\mathrm{H}^0(G,\mathcal{O})-comod(\mathrm{Vect})\widetilde{\to}\mathrm{H}^0(G,\mathcal{O})^\vee-mod(\mathrm{Vect})$ canonically by Section 3.2.1 in a way commuting with oblivion functors to Vect. So, in the adjoint pair $\mathrm{Ind}_G^H:\mathrm{H}^0(G,\mathcal{O})^\vee-mod(\mathrm{Vect})\leftrightarrows\mathrm{H}^0(H,\mathcal{O})^\vee-mod(\mathrm{Vect}):\mathrm{Res}$ the functor Res is given by restriction along the morphism of algebras $\mathrm{H}^0(G,\mathcal{O})^\vee\to\mathrm{H}^0(H,\mathcal{O})^\vee$, hence its left adjoint sends $M\in\mathrm{H}^0(G,\mathcal{O})^\vee-mod(\mathrm{Vect})$ to

$$\mathrm{H}^0(H,\mathcal{O})^\vee\otimes_{\mathrm{H}^0(G,\mathcal{O})^\vee}M \tag{14}$$

The coproduct in $\mathrm{H}^0(G,\mathcal{O})$ sends g to $\sum\limits_{x,y\in G,xy=g}x\otimes y$. So, $\mathrm{H}^0(G,\mathcal{O})^\vee$ is precisely the group algebra of G. So, (14) is the classical formula for the induced representation.

Recall that $\mathrm{QCoh}(B(G))$ has a t-structure (as for any Artin stack): $M\in\mathrm{H}^0(G,\mathcal{O})-comod(\mathrm{Vect})$ is connective (resp. coconnective) iff $\mathrm{oblv}(M)\in\mathrm{Vect}$ has the same property. We see that $\mathrm{Res},\mathrm{Ind}_G^H$ are t-exact for these t-structures.

The trivial action of G on *pt* defines the augmentation $\mathrm{H}^0(G,\mathcal{O})^\vee\to e$, a homomorphism of algebras. Given $V\in\mathrm{H}^0(G,\mathcal{O})^\vee-mod(\mathrm{Vect})$, $V_G=e\otimes_{\mathrm{H}^0(G,\mathcal{O})^\vee}V$ are the coinvariants.

For G,H finite groups the above map $\bar{a}:B(G)\to B(H)$ is pseudo-proper in the sense of ([16], 1.5.3), so $\bar{a}_!$ is defined in any context, even for $\mathcal{D}$-modules. Do we have $\bar{a}_!=\bar{a}_*$? Note that $\mathrm{H}^0(H,\mathcal{O})^\vee$ is dualizable in $\mathrm{H}^0(G,\mathcal{O})^\vee-mod$, because it is given by a constructible object in $Shv(B(G))$. So, by my Section 3.2, the restriction functor $\mathrm{H}^0(H,\mathcal{O})^\vee - mod \to \mathrm{H}^0(G,\mathcal{O})^\vee - mod$ admits a right adjoint given by $M \mapsto \mathrm{H}^0(H,\mathcal{O}) \otimes_{\mathrm{H}^0(G,\mathcal{O})^\vee} M$. Here we identified the dual of $\mathrm{H}^0(H,\mathcal{O})^\vee$ in the category $\mathrm{H}^0(G,\mathcal{O})^\vee$ with $\mathrm{H}^0(H,\mathcal{O})$. Actually, this module is self-dual, so we get that this right adjoint coincides with (14), and $\bar{a}_! = \bar{a}_*$.

If H is reductive then $\mathrm{Rep}(H) \widetilde{\to} \prod_{V\in Irrep(H)} \mathrm{Vect}$. This is obtained from ([14], ch. I.3, 2.4.2) by taking left completions on both sides. Indeed, $\mathrm{QCoh}(H)$ is left-complete by ([14], ch. I.3, 1.5.7). In particular, this holds for a finite group G, namely $\mathrm{Rep}(G) \widetilde{\to} \prod_{V\in Irrep(G)} \mathrm{Vect}$.

9.4.3. Let Σ be the groupoid of finite nonempty sets and bijections. So, $\Sigma = \underset{n\geq 1}{\sqcup} B(S_n)$. Define Vect^Σ as $\prod_{n\geq 1} \mathrm{Rep}(S_n) \widetilde{\to} \mathrm{Fun}(\Sigma, \mathrm{Vect})$.

Now $\Sigma \in \mathrm{Spc}$ is nonunital symmetric monoidal with the operation given by the disjoint union. By ([28], 2.2.6.17), $\mathrm{Fun}(\Sigma, \mathrm{Vect})$ is equipped with the Day convolution nonunital symmetric monoidal structure.

Given $f \in \mathrm{Fun}(\Sigma, \mathrm{Vect})$, it gives as a collection of functors $f(I) : B(S(I)) \to \mathrm{Vect}$ for any finite set $I \in \Sigma$. We have denoted by $S(I)$ the group of automorphisms of a finite set I. Given $I \in \Sigma$, the category $(\Sigma \times \Sigma) \times_\Sigma \Sigma_{/I}$ is a set of decompositions of I as $I = I_1 \sqcup I_2$ of two non empty subsets. By definition of the Day convolution, we get for $f, g \in \mathrm{Fun}(\Sigma, \mathrm{Vect})$

$$(f \otimes g)(I) = \underset{I_1\sqcup I_2 = I}{\oplus} f(I_1) \otimes g(I_2)$$

the sum is taken over all decompostions of I into an ordered pair of disjoint non empty subsets. The action of $S(I)$ is seen in the following formula

Given $\{V_n\}, \{U_n\} \in \mathrm{Vect}^\Sigma$ with $V_n \in \mathrm{Rep}(S_n)$ we get

$$(V \otimes U)_n = \underset{p+q=n,\ p>0, q>0}{\oplus} \mathrm{Ind}_{S_p\times S_q}^{S_n}(V_p \boxtimes U_q) \in \mathrm{Rep}(S_n),$$

here we use the inclusion $S_p \times S_q \to S_n$ via $\{1,\dots,p\} \sqcup \{p+1,\dots,p+q\} = \{1,\dots,n\}$. An approach to ∞-operads via symmetric sequences is also discussed in [25].

9.4.4. If $C \in CAlg^{nu}(\mathrm{DGCat}_{cont})$, then we define the functor $S(\cdot,\cdot) : \mathrm{Vect}^\Sigma \times C \to C$ sending $\{V_n\} \in \mathrm{Vect}^\Sigma, c \in C$ to

$$S(V, c) = \underset{n\geq 1}{\oplus} (V_n \otimes c^{\otimes n})_{S_n},$$

where the subscript S_n stands for the coinvariants. Here

$$(V_n \otimes c^{\otimes n})_{S_n} := \mathrm{colim}_{B(S_n)}(V_n \otimes c^{\otimes n})$$

taken in C. For $V \in \mathrm{Vect}^\Sigma$, $S(V,\cdot) : C \to C$ is continuous, but not exact, I think. It is important that $S(V,\cdot)$ preserves sifted colimits.

For $c \in C$ the obtained functor $S(\cdot, c) : \mathrm{Vect}^\Sigma \to C$ is nonunital symmetric monoidal and continuous, where Vect^Σ is equipped with the Day convolution symmetric monoidal structure, see ([10], 2.1.5). This, I think, comes from the isomorphism for $n, m \geq 1$, $V, W \in \mathrm{Vect}^{\Sigma}, c \in C$

$$(V_n \otimes c^{\otimes n})_{S_n} \otimes (W_m \otimes c^{\otimes m})_{S_m} \widetilde{\to} ((\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V_n \otimes W_m) \otimes c^{n+m})_{S_{n+m}}$$

It may help that $\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V_n \otimes W_m)$ should be the LKE along $B(S_n) \times B(S_m) \to B(S_{n+m})$ of the functor $B(S_n) \times B(S_m) \stackrel{V_n \otimes W_m}{\to} \mathrm{Vect}$. Write $\Sigma_n \subset \Sigma$ for the groipoid of finite sets of order n.

Proof. Given $f, g \in \mathrm{Vect}^{\Sigma}$, $c \in C$ we have

$$\begin{aligned} S(f \otimes g, c) = & \underset{n \geq 1}{\oplus} \underset{I \in \Sigma_n}{\mathrm{colim}} (f \otimes g)(I) \otimes c^{\otimes I} \widetilde{\to} \\ & \underset{n \geq 1}{\oplus} \underset{n_1 + n_2 = n}{\oplus} \underset{I_1 \in \Sigma_{n_1}, I_2 \in \Sigma_{n_2}}{\mathrm{colim}} f(I_1) \otimes c^{\otimes I_1} \otimes g(I_2) \otimes c^{\otimes I_2} \widetilde{\to} \\ & \underset{n_1, n_2 \geq 1}{\oplus} (\underset{I_1 \in \Sigma_{n_1}}{\mathrm{colim}} f(I_1) \otimes c^{\otimes I_1}) \otimes (\underset{I_2 \in \Sigma_{n_2}}{\mathrm{colim}} g(I_2) \otimes c^{\otimes I_2}) \widetilde{\to} S(f, c) \otimes S(g, c) \end{aligned}$$

as desired. □

Let $\mathbb{U} \in \mathrm{Vect}^{\Sigma}$ be the object given by $\mathbb{U}(*) = e$ and $\mathbb{U}(I) = 0$ for $\mid I \mid > 1$. Then $S(\mathbb{U}, \cdot) : C \to C$ is the identity functor. So, for $r \geq 1$ we get

$$S(\mathbb{U}^{\otimes r}, c) \widetilde{\to} c^{\otimes r}$$

In the next subsection we use this action of Vect^{Σ} on $C = (\mathrm{Vect}^{\Sigma}$, *Day convolution*).

9.4.5. There is another nonsymmetric monoidal structure on Vect^{Σ} called *composition monoidal structure*, where for $X = \{X_n\}, Y = \{Y_n\} \in \mathrm{Vect}^{\Sigma}$ the product is given by

$$X \circ Y = S(X, Y) = \underset{k \geq 1}{\oplus} (X_k \otimes Y^{\otimes_{Day} k})_{S_k} \quad \in \mathrm{Fun}(\Sigma, \mathrm{Vect}) \tag{15}$$

Here by $Y^{\otimes_{Day} k}$ we mean the k-th tensor power of Y in $\mathrm{Fun}(\Sigma, \mathrm{Vect})$ with respect to the Day convolution. The subscript S_k means S_k-coinvariants. We used the fact that $\mathrm{Fun}(\Sigma, \mathrm{Vect}) \in \mathrm{DGCat}_{cont}$, so Vect acts on it. We may also rewrite

$$X \circ Y = \underset{I \in \Sigma}{\mathrm{colim}}\, X(I) \otimes Y^{\otimes_{Day} I},$$

the colimit taken in Vect^{Σ}. Explicitly, for $I \in \Sigma$,

$$(Y^{\otimes_{Day} k})(I) = \underset{I_1 \sqcup \ldots \sqcup I_k = I}{\oplus} Y(I_1) \otimes \ldots \otimes Y(I_k)$$

the sum over the discrete category $(\Sigma^{\times k}) \times_{\Sigma} \Sigma_{/I}$, that is, over all possible decompositions of I into a disjoint union of nonempty subsets I_i indexed by $\{1, \ldots, k\}$.

Let $\underline{k} = \{1, \ldots, k\}$. Thus finally

$$(X \circ Y)_n = \underset{k \geq 1}{\oplus} \left(X_k \otimes (\underset{I_1 \sqcup \ldots \sqcup I_k = \underline{n}}{\oplus} Y(I_1) \otimes \ldots \otimes Y(l_k)) \right)_{S_k} \quad \in \mathrm{Vect}$$

Here the subscript S_k denotes the coinvariants.

9.4.6. If $V_n \in \mathrm{Rep}(S_n)$ then for the coinvariants we get naturally in Vect

$$(\mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(V_n \boxtimes V_m))_{S_{n+m}} \widetilde{\to} (V_n)_{S_n} \otimes (V_m)_{S_m}$$

Write $e[S_n]$ for the regular representation of S_n. If $V \in \mathrm{Rep}(S_n)$ then $(V \otimes e[S_n])_{S_n} \widetilde{\to} V$ naturally in $\mathrm{Rep}(S_n)$, here we used both left and right actions of S_n on $e[S_n]$.

This shows that for $V \in \mathrm{Vect}^\Sigma$ we have an isomorphism in Vect^Σ

$$S(V, \mathbb{U}) = \underset{r\geq 1}{\oplus} (V_r \otimes \mathbb{U}^{\otimes r})_{S_r} \widetilde{\to} V$$

Here we have taken $\mathcal{C} := \mathrm{Vect}^\Sigma$ equipped with the Day convolution as an object of $CAlg^{nu}(\mathrm{DGCat}_{cont})$.

9.4.7. My understanding is that (15) can be rewritten as follows. For a finite nonempty set I let $Q(I)$ denote the set of equivalence relations on I. We write $J \in Q(I)$ meaning a surjection $\phi : I \to J$. Then for $I \in \Sigma$ we get

$$(X \circ Y)(I) = \underset{J\in Q(I)}{\oplus} X(J) \otimes (\underset{j\in J}{\otimes} Y(I_j))$$

Here we do not have to take the coinvariants with respect to the symmetric group anymore. This formula appears in ([5], Def. 2.2.5).

The composition monoidal structure is unital: the unit is the symmetric sequence $\mathbb{U}$ given by $\mathbb{U}(I) = e$ for $\mid I \mid = 1$ and $\mathbb{U}(I) = 0$ otherwise.

Proposition 9.4.8 ([10], Pp. 2.2.1)**.** *Let $\mathcal{C} \in CAlg(\mathrm{DGCat}_{cont})$, write $\mathrm{Fun}_{cont}(\mathcal{C}, \mathcal{C})$ for the category of all continuous functors (not necessarily exact nor e-linear). Then the functor $S : (\mathrm{Vect}^\Sigma, \circ) \to (\mathrm{Fun}_{cont}(\mathcal{C}, \mathcal{C}), \circ)$, $X \mapsto S(X, \cdot)$ is naturally monoidal. Here $\mathrm{Fun}_{cont}(\mathcal{C}, \mathcal{C})$ is equipped with the composition monoidal structure. In fact, for $X \in \mathrm{Vect}^\Sigma$, $S(X, \cdot) : \mathcal{C} \to \mathcal{C}$ preserves sifted colimits. Besides, S preserves colimits.*

Note that for $X, Y \in \mathrm{Vect}^\Sigma$ in the above proposition we get

$$S(X \circ Y) = \underset{r\geq 1}{\oplus} (X_r \otimes S(Y)^{\otimes r})_{S_r} \widetilde{\to} S(X) \circ S(Y)$$

9.4.9. The distribution relation in Vect^Σ relating the Day tensor product $\otimes$ and the composition: for $M, V, W \in \mathrm{Vect}^\Sigma$ one has

$$(M \otimes V) \circ W \widetilde{\to} (M \circ W) \otimes (V \circ W)$$

by Section 9.4.4.

9.4.10. Now (Vect-enriched) operad is a unital associative algebra in Vect^Σ with respect to the composition monoidal structure. An augmented operad is a map $X \to \mathbb{U}$ in $Alg(\mathrm{Vect}^\Sigma, \circ)$, that is, a map of operads. An operad is by definition equipped with a map $\mathbb{U} \to X$ of operads, we call it *reduced* iff the natural map $\mathbb{U}(pt) \to X(pt)$ is an isomorphism.

9.4.11. Let $\mathcal{C} \in CAlg(\mathrm{DGCat}_{cont})$. By Proposition 9.4.8, there is a left action of $(\mathrm{Vect}^{\Sigma}, \circ)$ on $\mathcal{C}$. Namely, $\mathcal{C}$ is a $\mathrm{Fun}_{cont}(\mathcal{C}, \mathcal{C})$-module category, hence also a $(\mathrm{Vect}^{\Sigma}, \circ)$-module category.

As in ([14], ch. 5, 1.1.2), we should consider only reduced operads. Now for such $P \in Alg(\mathrm{Vect}^{\Sigma}, \circ)$ we get the category $P - mod(\mathcal{C})$ of left P-modules in $\mathcal{C}$.

By ([28], 4.2.4.8), $\mathrm{oblv} : P - mod(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $free_P$ sending M to $S(P, M) = P \circ M$.

Lemma 9.4.12. *$P - mod(\mathcal{C})$ is presentable.*

Proof. We can not apply ([28], 4.2.3.7). The problem is that given $X \in \mathrm{Vect}^{\Sigma}$, the functor $\mathcal{C} \to \mathcal{C}$, $M \mapsto X \circ M$ preserves sifted colimits, but maybe not all colimits.

Nick: for any $M \in P - mod(\mathcal{C})$, $M \widetilde{\to} \underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, P^{n+1} \circ M$ by ([28], 4.7.2.7). Now for I small and $I \to P - mod(\mathcal{C})$, $i \mapsto M_i$, we get $\mathrm{colim}_{i\in I} M_i \widetilde{\to} \underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, (\underset{i\in I}{\mathrm{colim}}\, P^{n+1} \circ M_i)$, where the colimit inside is the functor $free_P$ applied to $(\mathrm{colim}_{i\in I} P^n \circ M_i) \in \mathcal{C}$. Indeed, since $free_P$ is a left adjoint, it preserves colimits. To check that this is a colimit diagram, for $N \in P - mod(\mathcal{C})$ we get

$$\mathrm{Map}_{P-mod(\mathcal{C})}(\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, (\underset{i\in I}{\mathrm{colim}}\, P^{n+1} \circ M_i), N) \widetilde{\to} \underset{[n]\in\boldsymbol{\Delta}}{\lim}\, \mathrm{Map}_{\mathcal{C}}(\underset{i\in I}{\mathrm{colim}}\, P^n \circ M_i, N) \widetilde{\to}$$
$$\underset{[n]\in\boldsymbol{\Delta}}{\lim}\, \underset{i\in I^{op}}{\lim}\, \mathrm{Map}_{\mathcal{C}}(P^n \circ M_i, N) \widetilde{\to} \underset{[n]\in\boldsymbol{\Delta}}{\lim}\, \underset{i\in I^{op}}{\lim}\, \mathrm{Map}_{P-mod(\mathcal{C})}(P^{n+1} \circ M_i, N) \widetilde{\to}$$
$$\underset{i\in I^{op}}{\lim}\, \mathrm{Map}_{P-mod(\mathcal{C})}(\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, P^{n+1} \circ M_i, N) \widetilde{\to} \underset{i\in I^{op}}{\lim}\, \mathrm{Map}_{P-mod(\mathcal{C})}(M_i, N)$$

So, $P - mod(\mathcal{C})$ admits small colimits. Since $\mathrm{oblv}_P : P - mod(\mathcal{C}) \to \mathcal{C}$ preserves κ-filtered colimits, $free_P : \mathcal{C} \to P - mod(\mathcal{C})$ preserves κ-compact objects for any infinite regular cardinal κ. Pick κ such that κ-compact objects generate $\mathcal{C}$. We may assume that $\boldsymbol{\Delta}^{op}$ is κ-filtered. Then for $c \in \mathcal{C}^{\kappa}$, the objects of the form $free_P(c)$ generate $P - mod(C)$ under the geometric relaizations, hence under small κ-filtered colimits. By (HTT, 5.4.2.2), $P - mod(\mathcal{C})$ is accessible. □

By ([28], 4.2.3.5), $P - mod(\mathcal{C})$ admits sifted colimits and $\mathrm{oblv} : P - mod(\mathcal{C}) \to \mathcal{C}$ reflects sifted colimits, we used here that $S(V, \cdot)$ preserves sifted colimits. By ([28], 4.2.3.3), $P - mod(\mathcal{C})$ admits all small limits and $\mathrm{oblv} : P - mod(\mathcal{C}) \to \mathcal{C}$ reflects limits. This is used in the proof of ([14], vol. 2, ch. 5, 1.2.6).

They write $\mathcal{P}$ rather for an operad. The adjoint pair $free_P : O \leftrightarrows \mathcal{P} - Alg(O)$ shows that $\mathcal{P} - Alg(O)$ is pointed: we have $free_P(0) = 0$, and 0 is the final object of $\mathcal{P} - Alg(O)$.

The paper [9] claims in addition that $\mathrm{triv}_P : \mathcal{C} \to P - mod(\mathcal{C})$ preserves colimits. This is not true. For example, take P to be the operad of augmented commutative algebras. Then $\mathrm{triv}_P : \mathcal{C} \to CAlg^{nu}(\mathcal{C})$ does not preserve finite coproducts. If $B_i \in CAlg^{nu}(\mathcal{C})$ then $B_1 \sqcup B_2$ in $CAlg^{nu}(\mathcal{C})$ is $B_1 \oplus B_2 \oplus B_1 \otimes B_2$, see [35].

Remark 9.4.13. *The notion of operad from our Section 9.4.10 gives only unital operads in the sense of Lurie ([28], 2.3.1.1), as Σ does not contain the empty set.*

9.4.14. Let $O, O' \in CAlg^{nu}(\mathrm{DGCat}_{cont})$ and $f : O \to O'$ be a right-lax non-unital symmetric monoidal with $f \in \mathrm{Fun}_{e,cont}(O, O')$. Then $f : O \to O'$ is a right-lax functor of $(\mathrm{Vect}^{\Sigma}, \circ)$-module categories. Indeed, for $x \in O$, $\mathcal{P} \in \mathrm{Vect}^{\Sigma}$ we have the natural map

$$\mathcal{P} \star f(x) = \underset{I \in \Sigma}{\mathrm{colim}} \mathcal{P}(I) \otimes f(x)^{\otimes I} \to \underset{I \in \Sigma}{\mathrm{colim}} \mathcal{P}(I) \otimes f(x^{\otimes I}) \widetilde{\to} f(\underset{I \in \Sigma}{\mathrm{colim}} \mathcal{P}(I) \otimes x^{\otimes I}) = f(\mathcal{P} \star x)$$

functorial in $\mathcal{P}$ and x, and compatible with the constraints.

Now for $\mathcal{P} \in \mathrm{Vect}^{\Sigma}$ a reduced operad we get the map $\bar{f} : \mathcal{P} - alg(O) \to \mathcal{P} - alg(O')$ such that the diagram commutes

$$\begin{array}{ccc} \mathcal{P} - alg(O) & \stackrel{\bar{f}}{\to} & \mathcal{P} - alg(O') \\ \downarrow & & \downarrow \\ O & \stackrel{f}{\to} & O' \end{array}$$

Remark: if $L : O \leftrightarrows O' : R$ is an adjoint pair in DGCat_{cont}, where L is non-unital symmetric monoidal, so R is right-lax non-unital symmetric monoidal then we get an adjoint pair $\bar{L} : \mathcal{P} - alg(O) \leftrightarrows \mathcal{P} - alg(O') : \bar{R}$.

9.4.15. For ([14], vol. 2, ch. 5, 1.2.4). Let $O \in CAlg(\mathrm{DGCat}_{cont})$ and $A \in CAlg(O)$. Then $ind : O \to A - mod(O)$ is symmetric monoidal functor, its right adjoint is $\mathrm{oblv} : A - mod(O) \to O$.

Let $A \in CAlg(O)$ and $\mathcal{P}$ be a reduced operad as above. The functor $\mathcal{P} - mod(O) \to \mathcal{P} - mod(O)$, $x \mapsto A \otimes x$ defined in ([14], vol. 2, ch. 5, 1.2.5) is explicitly as follows. Let $\mathcal{P}$ be given by the symmetric sequence $\{V_n\}$. For a finite nonempty set I, $x \in \mathcal{P} - mod(O)$ we get a map $(A \otimes x)^{\otimes I} \to A \otimes (x^{\otimes I})$, which is $S(I)$-equivariant. So, for each $n \geq 1$ a map $(V_n \otimes (A \otimes x)^{\otimes n})_{S_n} \to (V_n \otimes A \otimes (x^{\otimes n}))_{S_n} \to A \otimes x$, where the second map comes from the $\mathcal{P}$-module structure on x just by tensoring with A.

9.4.16. The denote by $Com^{aug} \in Alg(\mathrm{Vect}^{\Sigma})$ the operad given by $Com^{aug}(I) = k$ for any $I \in \Sigma$. Since only nonempty finite sets appear in Σ, $Com^{aug} - Alg(O)$ is the category of nounital commutative algebras in O, equivalently, augmented commutative algebras.

9.5. **Filtered and graded objects.**

9.5.1. For ([14], vol.2, ch. 5, 1.3). If $C \in \mathrm{DGCat}_{cont}$ then $C^{Filt}, C^{Filt, \geq 0}, C^{Filt, \leq 0}, C^{gr}$ lie in DGCat_{cont} by Section 9.2.13. Recall that $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, C) \widetilde{\to} \mathrm{Fun}(\boldsymbol{\Delta}^{op}, C)$ by ([28], 1.2.4.1).

9.5.2. For ([14], vol.2, ch. 5, 1.3.5). Let $O \in CAlg(\mathrm{DGCat}_{cont})$. Recall that $O^{Filt} = \mathrm{Fun}(\mathbb{Z}, O)$, where $\mathbb{Z}$ is viewed as an ordered set, hence a category. The symmetric monoidal structure on O^{Filt} is given by the Day convolution. Namely, $\mathbb{Z}$ is symmetric monoidal with respect to the operation sum $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. So, for $f_i \in O^{Filt}$, $f_1 \otimes f_2 \in O^{Filt}$ is given on $n \in \mathbb{Z}$ by

$$(f_1 \otimes f_2)(n) = \underset{n_1 + n_2 \leq n, \ n_i \in \mathbb{Z}}{\mathrm{colim}} f_1(n_1) \otimes f_2(n_2)$$

The latter colimit is not a direct sum over pairs $(n_1, n_2) \in \mathbb{Z}^2$ such that $n_1 + n_2 = n$. Indeed, given $n_1 + n_2 = n = m_1 + m_2$ there are pairs $(s_1, s_2) \in \mathbb{Z}^2$ with $s_i \neq n_i, s_i \neq m_i$ for all i. So, this is a complicated colimit.

The category $\mathbb{Z}^{\mathrm{Spc}}$ is a set, it is also symmetric monoidal with respect to the sum, and the functor $\mathbb{Z}^{\mathrm{Spc}} \to \mathbb{Z}$ is symmetric monoidal. We similarly equip $O^{gr} = \mathrm{Fun}(\mathbb{Z}^{\mathrm{Spc}}, O)$ with the symmetric monoidal structure given by the Day convolution product, so for $V = \{V_n\}, U = \{U_n\} \in O^{gr}$,

$$(U + V)_n = \bigoplus_{n_1+n_2=n} U_{n_1} \otimes V_{n_2}$$

Thus, $O^{Filt}, O^{gr} \in CAlg(\mathrm{DGCat}_{cont})$ by Section 9.2.41.

We get the adjoint pait $(gr \to Filt) : O^{gr} \leftrightarrows O^{Filt} : Rees$ in DGCat_{cont}, where $(gr \to Filt)$ is symmetric monoidal by ([35], 1.2.8), and $Rees$ is the restriction along $\mathbb{Z}^{\mathrm{Spc}} \to \mathbb{Z}$. So, $Rees$ is right-lax symmetric monoidal.

The unit of O^{gr} is the collection $\{V_n\}$ with $V_0 = 1_O$ and $V_n = 0$ for $n \neq 0$. The unit of O^{Filt} is the collection $\{U_n\}$ with $U_n = 1_O$ for $n \geq 0$ and $U_n = 0$ for $n < 0$. We see that $Rees$ is not symmetric monoidal, as the map $V \to Rees(U)$ is not an isomorphism (assuming $1_O \neq 0$).

The functor $\mathrm{oblv}_{Filt} : O^{Filt} \to O$ sends f to $\mathrm{colim}_{\mathbb{Z}} f$. It is symmetric monoidal. Indeed, given $f, g \in O^{Filt}$ we have

$$\underset{n\in\mathbb{Z}}{\mathrm{colim}}\ \underset{n_1+n_2\leq n}{\mathrm{colim}}\ f(n_1) \otimes g(n_2) \widetilde{\to} \underset{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}}{\mathrm{colim}} f(n_1) \otimes f(n_2) \widetilde{\to} (\mathrm{colim}\, f) \otimes (\mathrm{colim}\, g)$$

by Section 2.2.111 of this file.

For $m \in \mathbb{Z}, a \in O$ the *step-sequence* $\langle m, a\rangle$ is the object of O^{Filt} sending n to 0 for $n < m$, and constant with value a for $n \geq m$. By ([24], 2.24), step-sequences form a system of generators of O^{Filt}. For $m \in \mathbb{Z}$ the functor $O \to O^{Filt}$ sending a to $\langle m, a\rangle$ is the LKE of $* \stackrel{a}{\to} O$ along $* \stackrel{m}{\to} \mathbb{Z}$. So, for $f \in O^{Filt}$,

$$\mathrm{Map}_{O^{Filt}}(\langle m, a\rangle, f) \widetilde{\to} \mathrm{Map}_O(a, f(m))$$

Besides, $\langle m, a\rangle \otimes \langle m', a'\rangle \widetilde{\to} \langle m + m', a \otimes a'\rangle$ in O^{Filt}.

The above is used to show that $ass - gr : O^{Filt} \to O^{gr}$ is symmetric monoidal, see ([24], 2.26). The reason is that step-sequences generate O^{Filt} under colimits in the sense of ([27], 5.1.5.7). Another argument is to use ([14], IV.5, Proposition-Construction 1.3.3), and the fact that the restriction functor $\mathrm{QCoh}(\mathbb{A}^1)^{\mathbb{G}_m} \to \mathrm{QCoh}(\{0\})^{\mathbb{G}_m}$ is symmetric monoidal.

Note that $O^{Filt,\geq 0} \subset O^{Filt}$ is closed under the tensor product, so inherits a symmetric monoidal structure from O^{Filt}.

9.5.3. Example: let $O = \mathrm{Vect} \in CAlg(\mathrm{DGCat}_{cont})$ with its usual t-structure. Let $f \in \mathrm{Vect}^{Filt}$ be such that for any $n < 0$, $f(n) = 0$, and for $n \geq 0$ we have $f(n) \in O^{\heartsuit}$ such that $f(n-1) \to f(n)$ is injective in $O^{\heartsuit}$. Let $g \in \mathrm{Vect}^{Filt}$ satisfy the same property. Then I think $f \otimes g$ satisfies the same property, and for any $n \geq 0$,

$$(f \otimes g)(n) = \sum_{n_1+n_2=n} f(n_1) \otimes g(n_2),$$

the usual sum of vector spaces taken inside the vector space $f(n)\otimes g(n)$. Is this correct? Looks plausible.

9.5.4. *Adding a filtration.* ([14], vol.2, ch. 5, 1.4.1). Recall that $O^{Filt,\geq 0}\subset O^{Filt}$ is closed under the tensor product (given by the Day convolution), so $O^{Filt,\geq 0}$ is symmetric monoidal, and we may consider $\mathcal{P}-Alg(O^{Filt,\geq 0})$ for a reduced operad $\mathcal{P}$.

The following holds actually. Let $A\in CAlg(\mathrm{Vect}^{Filt,\geq 0})$. It gives a functor $O\to O^{Filt,\geq 0}$, $B\mapsto A\otimes B$, where the filtration on $A\otimes B$ is induced by the one on A. So, $(A\otimes B)_n=A_n\otimes B$ for $n\geq 0$. This functor is right-lax symmetric monoidal: given $B_i\in O$, the map $(A\otimes B_1)\otimes(A\otimes B_2)\to A\otimes(B_1\otimes B_2)$ is as follows (here the tensor product $(A\otimes B_1)\otimes(A\otimes B_2)$ is taken in $O^{Filt,\geq 0}$). For $n\geq 0$ we have to specify a map

$$\operatorname*{colim}_{n_1+n_2\leq n}(A\otimes B_1)_{n_1}\otimes(A\otimes B_2)_{n_2}\to A_n\otimes(B_1\otimes B_2)$$

It comes from a compatible system of maps $(product\otimes\mathrm{id}):(A_{n_1}\otimes A_{n_2})\otimes(B_1\otimes B_2)\to A_n\otimes(B_1\otimes B_2)$. So, our functor $O\to O^{Filt,\geq 0}$ is a right-lax functor of $(\mathrm{Vect}^{\Sigma},\circ)$-module categories, hence induces a functor

$$\mathcal{P}-Alg(O)\to\mathcal{P}-Alg(O^{Filt,\geq 0})$$

The functor $\mathrm{oblv}_{Filt}:O^{Filt,\geq 0}\to O$ is symmetric monoidal, so induces a functor $\mathcal{P}-Alg(O^{Filt,\geq 0})\to\mathcal{P}-Alg(O)$. For A as above and $B\in\mathcal{P}-Alg(O)$ we get

$$\mathrm{oblv}_{Filt}(A\otimes B)\widetilde{\to}(\mathrm{oblv}(A))\otimes B$$

in $\mathcal{P}-Alg(O)$, where we used the fact that $\mathrm{oblv}(A)\in CAlg(\mathrm{Vect})$. Indeed, the projection $\mathcal{P}-Alg(O)\to O$ preserves filtered colimits.

The following diagram commutes

$$\begin{array}{ccc}\mathcal{P}-Alg(O) & \stackrel{A\otimes\bullet}{\to} & \mathcal{P}-Alg(O^{Filt,\geq 0})\\ \uparrow triv_{\mathcal{P}} & & \uparrow triv_{\mathcal{P}}\\ O & \stackrel{A\otimes\bullet}{\to} & O^{Filt,\geq 0}\end{array}$$

Let now $A=k\oplus k$ viewed as commutative algebra in Vect (functions on union of two points). They let $A_0=k$ included diagonally, $A_n=A$ for $n\geq 1$. View A as augmented via the projection on the first copy.

View k as filtered namely as the object $(gr\to Filt)(k^{deg=0})$. Recall that the functor $(gr\to Filt)\circ(deg=0):\mathrm{Vect}\to\mathrm{Vect}^{Filt}$ is symmetric monoidal, so sends algebras to algebras. So, the augmentation on A is a map $A\to k$ in $CAlg(\mathrm{Vect}^{Filt,\geq 0})$. By functoriality, it induces a natural transformation $A\otimes B\to B\otimes k=B$ of functors $\mathcal{P}-Alg(O)\to\mathcal{P}-Alg(O^{Filt,\geq 0})$. Then they define $AddFil:\mathcal{P}-Alg(O)\to\mathcal{P}-Alg(O^{Filt,\geq 0})$ by

$$AddFil(B)=Fib(A\otimes B\to B)=(A\otimes B)\times_B 0$$

Recall that since $O^{Filt,\geq 0}\in CAlg(\mathrm{DGCat}_{cont})$, $\mathcal{P}-Alg(O^{Filt,\geq 0})$ admits all small limits by my Section 9.4.11. The above is also a product in $O^{Filt,\geq 0}$. Recall that the limits in $O^{Filt,\geq 0}$ as in the category of functors are calculated pointwise, so for $n\geq 0$ we have $AddFil(B)_n=(A_n\otimes B)\times_B 0$, the product taken in O. So, $AddFil(B)_0=0$ and for $n\geq 1$ we get $AddFil(B)_n\widetilde{\to}B$ as mere objects of O. For $1\leq n\leq m$ the transition map $AddFil(B)_n \to AddFil(B)_m$ in the filtration is id : $B \to B$. This implies that $\text{oblv}_{Filt}\, AddFilt(B) \widetilde{\to} B$ in O.

Since the functor $\text{oblv}_{Filt} : O^{Filt,\geq 0} \to O$ is symmetric monoidal, it induces a functor $\text{oblv}_{Filt} : \mathcal{P} - Alg(O^{Filt,\geq 0}) \to \mathcal{P} - Alg(O)$ commuting with $\text{oblv}_{\mathcal{P}} : \mathcal{P} - Alg(O) \to O$ and $\text{oblv}_{\mathcal{P}} : \mathcal{P} - Alg(O^{Filt,\geq 0}) \to O^{Filt,\geq 0}$.

By the above we know that $\text{oblv}(A \otimes B) \widetilde{\to} (\text{oblv}(A)) \otimes B \widetilde{\to} B \times B$ in $\mathcal{P} - Alg(O)$.

9.5.5. For ([14], vol.2, ch. 5, 1.4.6). The functor $ass - gr : O^{Filt,\geq 0} \to O^{gr,\geq 0}$ is symmetric monoidal, so yields a functor $\mathcal{P} - Alg(O^{Filt,\geq 0}) \to \mathcal{P} - Alg(O^{gr,\geq 0})$.

The functor $ass - gr : O^{Filt,\geq 0} \to O^{gr,\geq 0}$ preserves finite limits? The following is just the transitivity of left Kan extension: let $x \to y$ be a map in O^{Filt}, let $z = Cofib(x \to y)$ in O^{Filt}. Then $ass - gr(z) \widetilde{\to} Cofib(ass - gr(x) \to ass - gr(y))$ naturally. This was used in the proof of 1.4.6: $ass - gr(B^{Fil}[1]) \widetilde{\to} Cofib(ass - gr(A \otimes B) \to B)$, and also $ass - gr(A \otimes B) \widetilde{\to} ass - gr(A) \otimes B$. For this reason

$$ass - gr(B^{Fil}) \widetilde{\to} Fib(ass - gr(A) \otimes B \to B)$$

9.5.6. For ([14], vol.2, ch. 5, 1.5.2). The functor $\text{oblv}_{Fil} : O^{Filt,\geq 0} \to O$ is symmetric monoidal, so for their natural transformation Φ the diagram commutes

$$\begin{array}{ccc} \mathcal{P} - Alg(O^{Filt,\geq 0}) & \to & \mathcal{C}(O^{Filt,\geq 0}) \\ \downarrow \text{oblv}_{Filt} & & \downarrow \text{oblv}_{Filt} \\ \mathcal{P} - Alg(O) & \to & \mathcal{C}(O) \end{array}$$

The get (1.11) from the fact that $ass - gr : O^{Filt,\geq 0} \to O^{gr,\geq 0}$ is symmetric monoidal, so the diagram commutes

$$\begin{array}{ccc} \mathcal{P} - Alg(O^{Filt,\geq 0}) & \to & \mathcal{C}(O^{Filt,\geq 0}) \\ \downarrow ass-gr & & \downarrow ass-gr \\ \mathcal{P} - Alg(O^{gr,\geq 0}) & \to & \mathcal{C}(O^{gr,\geq 0}) \end{array}$$

The fact that $gr - Fil : O^{gr} \to O^{Filt}$ is symmetric monoidal gives the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{P} - Alg(O^{Filt}) & \to & \mathcal{C}(O^{Filt}) \\ \uparrow gr-Fil & & \uparrow gr-Fil \\ \mathcal{P} - Alg(O^{gr}) & \to & \mathcal{C}(O^{gr}) \end{array}$$

The diagram also commutes

$$\begin{array}{ccc} O^{gr,\geq 0} & \stackrel{gr-Fil}{\to} & O^{Fil,\geq 0} \\ \downarrow triv_{\mathcal{P}} & & \downarrow triv_{\mathcal{P}} \\ \mathcal{P} - Alg(O^{gr,\geq 0}) & \stackrel{gr-Fil}{\to} & \mathcal{P} - Alg(O^{Fil,\geq 0}) \end{array}$$

This together with their Section 1.4.4 gives the commutatitivy of the last diagram in their Section 1.5.2.

9.5.7. For ([14], vol.2, ch. 5, 1.6.2).

1) Let $C \in 1-\mathcal{C}at$ be pointed admitting finite products. Assume that for a map $c_1 \to c_2$ in C if $c_1 \times_{c_2} * \to *$ is an isomorphism then $c_1 \to c_2$ is an isomorphism. Then $Grp(C) \to Mon(C)$ is an equivalence, use Remark 2.5.18 of this file.

2) The category $\mathcal{P} - Alg(O)$ is pointed by my Section 9.4.11 and satisfies 1) above. Indeed, $\text{oblv}_{\mathcal{P}} : \mathcal{P} - Alg(O) \to O$ preserves limits, and O is stable.

9.5.8. For ([14], vol.2, ch. 5, 1.6.3). By my Lemma 2.7.13, the left adjoint to $\Omega_{\mathcal{P}} : \mathcal{P} - Alg(O) \to Grp(\mathcal{P} - Alg(O))$ exists.

I think ([14], vol.2, ch. 5, 1.8.4) means that applying $\mathbb{S}\text{p}$ to the functor $coPrym_{\mathcal{P}} : \mathcal{P} - Alg(O) \to O$, one gets an equivalence $\mathbb{S}\text{p}(\mathcal{P} - Alg(O)) \to \mathbb{S}\text{p}(O) \widetilde{\to} O$.

In fact, let C be a stable category. The composition with $CMon(\mathcal{P}-Alg(O)) \stackrel{\text{oblv}_{CMon}}{\to} \mathcal{P} - Alg(O)$ induces an equivalence

$$\text{Fun}_{ex}(C, CMon(\mathcal{P} - Alg(O))) \to \text{Fun}^{lex}(C, \mathcal{P} - Alg(O)),$$

where Fun^{lex} stands for the category of left exact functors. Indeed, oblv_{CMon} preserves limits, so this functor is well-defined. By ([14], ch. I.1, 5.1.10), any left exact functor $f : C \to \mathcal{P} - Alg(O)$ factors canonically as $C \stackrel{\bar{f}}{\to} CMon(\mathcal{P} - Alg(O)) \stackrel{\text{oblv}_{CMon}}{\to} \mathcal{P} - Alg(O)$. Moreover, $\bar{f}$ preserves finite limits, because of ([28], 3.2.2.5 applied to the commutative operad $\mathcal{O}^{\otimes} = \mathcal{F}\text{in}_*$). This defines a functor $\text{Fun}^{lex}(C, \mathcal{P} - Alg(O)) \to \text{Fun}_{ex}(C, CMon(\mathcal{P} - Alg(O)))$. My understanding is that they are inverse to each other, is this correct?

9.6. **On Koszul duality.**

9.6.1. For ([14], vol.2, ch. 5, 2.1.2). I think $\text{Vect}^{\Sigma}_{fd}$ should be the full subcategory of those $\mathcal{P} \in \text{Vect}^{\Sigma}$ such that for any $I \in \Sigma$, $\mathcal{P}(I) \in \text{Vect}^c$, so also bounded. Then $(\text{Vect}^{\Sigma}_{fd}, \circ) \subset (\text{Vect}^{\Sigma}, \circ)$ is a full monoidal subcategory, and the term-wise dualization is a monoidal equivalence $(\text{Vect}^{\Sigma}_{fd})^{op} \widetilde{\to} \text{Vect}^{\Sigma}_{fd}$. In the version of this chapter of 9 Dec. 2021 they precised the definition of $\text{Vect}^{\Sigma}_{fd}$, where it means that for any $I \in \Sigma, n \in \mathbb{Z}$, $H^n(f(I))$ is finite-dimensional.

Just to underline, cooperads are objects of $CoAlg(\text{Vect}^{\Sigma}, \circ) \widetilde{\to} (Alg((\text{Vect}^{\Sigma})^{op}))^{op}$. They consider only reduced cooperads $\mathcal{Q}$, that is, those for which the counit map gives an isomorphism $Q(1) \widetilde{\to} k$.

By definition, $Assoc^{aug} \in \text{Vect}^{\Sigma}$ is the operad sending $I \in \Sigma$ to $k^{ord(I)}$, where $ord(I)$ is the set of linear orders on I, the composition is given by the lexicographical order as in [28].

9.6.2. Let Q be a co-operad. If we have a cosimplicial object in $Q - Coalg^{ind-nilp}(O)$, which is $\text{oblv}^{ind-nilp}_Q$-split then it admits a totalization in $Q - Coalg^{ind-nilp}(O)$, and $\text{oblv}^{ind-nilp}_Q$ preserves this totalization (by [28], 4.7.3.5).

Recall that $\mathbb{U}$ is the unit of $(\text{Vect}^{\Sigma}, \circ)$. The augmentation on Q is a map $\mathbb{U} \to Q$ of coalgebras in $(\text{Vect}^{\Sigma}, \circ)$.

9.6.3. For ([14], vol.2, ch. 5, 2.3.1). By *Operads* they mean the category of unital associative algebras in Vect^{Σ}, which are reduced, that is, $k \to \mathcal{P}(1)$ is an isomorphism (in particular, augmented). By *coOperads* they mean the category of counital coassociative coalgebras in Vect^{Σ}, which are reduced.

9.6.4. For ([14], vol.2, ch. 5, 2.3.3). For $\mathcal{P} \in AssocAlg(\mathrm{Vect}^{\Sigma}, \circ)$ they mean by $\mathcal{P}[-1]$ the following, as Nick explains. Call *Operads* the category of those $\mathcal{P} \in AssocAlg(\mathrm{Vect}^{\Sigma}, \circ)$, which are reduced, that is, $k \to \mathcal{P}(1)$ is an isomorphism. Then *Operads* is pointed. Indeed, we may view it as the category of associative algebras in $(\mathrm{Vect}^{\Sigma})_{1//1}$ as in my Section 3.3.4. There is a functor $Operads \to Operads$, $\mathcal{P} \mapsto \mathcal{P}[n]$ such that for $x \in O$, $\mathcal{P}[n]$-algebra structure on x is the same as $\mathcal{P}$-algebra structure on $x[n]$.

It is given by $X[n](I) = X(I) \otimes (e[n])^{\otimes I} \otimes e[-n]$, where S_I acts diagonally. The multiplication on $X[n]$ is given by the natural map for a finite nonempty set I

$$\begin{array}{cccc} (X[n] \circ X[n])(I) = & \underset{J \in Q(I)}{\oplus} X[n](J) \otimes (\underset{j\in J}{\otimes} X[n](I_j)) & \to & X[n](I) \\ & || & & || \\ & \underset{J \in Q(I)}{\oplus} X(J) \otimes \frac{(e[n])^{\otimes J}}{e[n]} \otimes (\underset{j\in J}{\otimes} X(I_j) \otimes \frac{(e[n])^{\otimes I_j}}{e[n]}) & \to & X(I) \otimes \frac{e[n]^{\otimes I}}{e[n]} \end{array}$$

Now for $y \in O$ a map $X[n] \circ y \to y$ is the same as a map $X \circ y[n] \to y[n]$ in O.

9.6.5. For ([14], vol.2, ch. 5, 2.4.1). The category $\mathcal{P} - Alg(O)$ is presentable by Lemma 9.4.12 of this file. This is why $\mathrm{triv}_{\mathcal{P}} : O \to \mathcal{P} - Alg(O)$ has a left adjoint (apply [28], 4.6.2.17). This left adjoint $coPrim_{\mathcal{P}}$ is given as a particular case of ([28], 4.4.2.12). Namely, it sends $x \in \mathcal{P} - Alg(O)$ to $\mathbb{U} \otimes_{\mathcal{P}} x = \underset{[n]\in \boldsymbol{\Delta}^{op}}{\mathrm{colim}}\, \mathbb{U} \circ \mathcal{P}^n \circ x$ taken in O.

The comonad on O given by $x \mapsto \mathcal{P}^{\vee} \circ x$ is $x \mapsto (\mathbb{U} \otimes_{\mathcal{P}} \mathbb{U}) \circ x$. The fact that $coPrim_{\mathcal{P}} : \mathcal{P} - Alg(O) \to O$ lifts to a functor

$$coPrim_{\mathcal{P}}^{enh,ind-nilp} : \mathcal{P} - Alg(O) \to Q - Coalg^{ind-nilp}(O)$$

with $Q = \mathcal{P}^{\vee}$ follows from my Section 3.0.74.

For their formula (2.3): given $x \in O$, we have $coPrim_{\mathcal{P}} free_{\mathcal{P}}(x) \widetilde{\to} \mathbb{U} \otimes_{\mathcal{P}} (\mathcal{P} \circ x) \widetilde{\to} x$, however the structure of a Q-comodule on it comes from the augmentation $\mathbb{U} \to Q$.

9.6.6. the map $coPrym_{\mathcal{P}} : \mathcal{P} - Alg(O) \to O$ can be seen as a natural transformation of functors $\mathrm{DGCat}_{cont}^{SymMon} \to 1 - \mathcal{C}\mathrm{at}$ indeed. If $\mathcal{F} : O \to O'$ is a map in $CAlg(\mathrm{DGCat}_{cont})$ then the diagram commutes

$$\begin{array}{ccc} \mathcal{P} - Alg(O) & \overset{coPrym_{\mathcal{P}}}{\to} & O \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \mathcal{P} - Alg(O') & \overset{coPrym_{\mathcal{P}}}{\to} & O' \end{array}$$

Similarly, $coPrym_{\mathcal{P}}^{enh,ind-nilp} : \mathcal{P} - Alg(O) \to \mathcal{P}^{vee} - Coalg^{ind-nilp}(O)$ is a natural transformation of functors $\mathrm{DGCat}_{cont}^{SymMon} \to 1 - \mathcal{C}\mathrm{at}$.

Their isomorphism (2.4) in ([14], vol.2, ch. 5, 2.5.2) comes from their Section 1.5.

9.6.7. For ([14], vol.2, ch. 5, 2.6.1). If Q is a co-operad,

$$\mathrm{oblv}_Q^{ind-nilp} : Q - Coalg^{ind-nilp}(O) \to O$$

preserves colimits (this happens for any comonad acting on a category). So, $\mathrm{triv}_Q^{ind-nilp}$ indeed has a right adjoint $Prim_Q^{ind-nilp}$. We don't know if $Q - Coalg^{ind-nilp}(O)$ is presentable, and we do not need this by ([27], 5.5.2.10), because $Q - Coalg^{ind-nilp}(O)$ is locally small.

9.6.8. For ([14], vol.2, ch. 5, 2.6.2). For a co-operad Q and $x \in Q - Coalg^{ind-nilp}(O)$ they get $Prym_Q^{ind-nilp}(x) = \mathbb{U} \otimes_Q x$ taken in O^{op}, that is, this is a totalization of the cobar complex $[x \rightrightarrows Q \circ x \substack{\to\\\to\\\to} Q^2 \circ x \dots]$ in O.

They further use the fact that if $\mathcal{M} \to \mathcal{M}'$ is a morphism of monads on $C \in 1 - \mathcal{C}at$ then we get the oblivion functor $\mathcal{M}' - mod(C) \to \mathcal{M} - mod(C)$. This gives rise to the functor $Prym_Q^{enh,ind-nilp} : Q - Coalg^{ind-nilp}(O) \to \mathcal{P} - Alg(O)$, where $\mathcal{P} = Q^\vee$.

Recall that $Q^\vee \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} Q^n$ in Vect^Σ. The monad $Prym_Q^{ind-nilp}\,\mathrm{triv}_Q^{ind-nilp} : O \to O$ sends x to $\lim_{[n]\in\boldsymbol{\Delta}} Q^n \circ x$ taken in O. The morphism of monads

$$(Q^\vee \circ \bullet) \to Prym_Q^{ind-nilp}\,\mathrm{triv}_Q^{ind-nilp}$$

comes from the natural morphism $(\lim\limits_{[n]\in\boldsymbol{\Delta}} Q^n) \circ x \to \lim\limits_{[n]\in\boldsymbol{\Delta}} Q^n \circ x$. It is not necessarily an isomorphism: already for $V \in \mathrm{Vect}$ I think for $x \in O$ the functor $V \to O, V \mapsto V \otimes x$ does not maybe preserve totalizations, and does not commute with needed colmits in O.

The **key point** is their adjoint pair (2.6):

$$coPrym_{\mathcal{P}}^{enh,ind-nilp} : \mathcal{P} - Alg(O) \leftrightarrows Q - Coalg^{ind-nilp}(O) : Prym_Q^{enh,ind-nilp}$$

with $Q = \mathcal{P}^\vee$.

9.6.9. For ([14], vol.2, ch. 5, 2.7.1). For the $*$-action, as opposed to $\star$-action, they replace S_n-coinvariants by S_n-invariants.

Definition 9.6.10. *If $C \in 1 - \mathcal{C}at$, G is a finite group and $f : B(G) \to C$ is a functor viewed as $c \in C$ equipped with G-action then c^G is defined as* $\lim f$.

Let $O \in CAlg(\mathrm{DGCat}_{cont})$ and $f, h : B(G) \to O$ functors. Then we have $f \otimes h : B(G) \to O$, which is the composition $B(G) \to B(G) \times B(G) \stackrel{f\times h}{\to} O \times O \stackrel{\otimes}{\to} O$. There is a natural map $(\lim f) \otimes (\lim h) \to \lim(f \otimes h)$.

If $H \subset G$ is a subgroup, we have an adjoint pair $\mathrm{Ind}_H^G : \mathrm{Fun}(B(H), O) \leftrightarrows \mathrm{Fun}(B(G), O) : \mathrm{Res}$ in DGCat_{cont}, where Ind_H^G is the LKE along $B(H) \to B(G)$. If c is the unique object of $B(G)$ then the category $B(H) \times_{B(G)} B(G)_{/c}$ identifies with the set G/H. In fact, $B(G)_{/c} \widetilde{\to} *$.

Let $k[G] \in \mathrm{Vect}$ be the group algebra of G. For $H = \{1\}$ the functor $\mathrm{Res} : \mathrm{Fun}(B(G), O) \to O$ is monadic, and the corresponding monad is $O \to O, x \mapsto k[G] \otimes x$. Indeed, Res is continuous and conservative, and its left adjoint is Ind_1^G. So,

$\mathrm{Fun}(B(G),O)\tilde{\to} k[G]-mod(O)$. By ([14], 8.5.7) we have $k[G]-mod(\mathrm{Vect})\otimes O\tilde{\to} k[G]-mod(O)$ canonically, and $k[G]-mod(\mathrm{Vect})\tilde{\to}\mathrm{QCoh}(B(G))$ by my Section 9.4.2. So,

$$\mathrm{Fun}(B(G),\mathrm{Vect})\otimes O\tilde{\to}\mathrm{Fun}(B(G),O),$$

where the tensor product is in DGCat_{cont}. Recall that $\mathrm{QCoh}(B(G))\tilde{\to}\prod_{Irr(G)}\mathrm{Vect}$ by my Section 9.4.2. So,

$$\mathrm{Fun}(B(G),O)\tilde{\to}\prod_{Irr(G)} O$$

naturally. That is, each $x\in O$ with a G-action writes canonically as

$$\underset{V\in Irr(G)}{\oplus} V\otimes x_V\in O, \tag{16}$$

where $x_V\in O$, and the G-action comes from that on V on each summand.

Lemma 9.6.11. *The functor* $\mathrm{Fun}(B(G),O)\to O, x\mapsto x_G=\mathrm{colim}_{B(G)}\, x$ *sends (16) to* x_{triv}*, where* $triv=e$ *is the trivial* G*-module. The functor* $\mathrm{Fun}(B(G),O)\to O, x\mapsto x^G=\lim_{B(G)} x$ *is canonically identifies with the previous one.*

Proof. ii) For $q: B(G)\to *$ we have an adjoint pair $q^*=\mathrm{Res}:\mathrm{Vect}\leftrightarrows\mathrm{Fun}(B(G),\mathrm{Vect}):\lim=q_*$, where q_* is the projection on the *triv*-component. Tensoring by O, one gets the desired claim.
i) We have the adjoint pair $\mathrm{colim}:\mathrm{Fun}(B(G),\mathrm{Vect})\leftrightarrows\mathrm{Vect}:\mathrm{Res}$, where $\mathrm{colim}:\prod_{V\in Irr(G)}\mathrm{Vect}\to\mathrm{Vect}$ is the projection on the *triv*-component The same with Vect replaced by O. □

9.6.12. By right-lax action of Vect^Σ on O in ([14], vol.2, ch. 5, 2.7.1) they mean a right-lax monoidal functor $\mathrm{Vect}^\Sigma\to\mathrm{Fun}(O,O)$. Note that for $x\in O$, $\mathbb{U}*x\tilde{\to}x$. The right-lax structure means that for $V,U\in\mathrm{Vect}^\Sigma$ we have to define a natural map

$$V\circ(U*x)\to(V\circ U)*x \tag{17}$$

Key case: assume that U is supported on $B(S_n)$ and V on $B(S_m)$ for some $n,m\ge 1$. Then $V\circ U$ is supported on $B(S_{nm})$, and we need to define the morphism

$$(V(m)\otimes((U(n)\otimes x^{\otimes n})^{S_n})^{\otimes m})^{S_m}\to((V\circ U)(nm)\otimes x^{\otimes nm})^{S_{nm}}$$

In this case $U*x=U\circ x$, and $V\circ(U\circ x)\tilde{\to}(V\circ U)\circ x$. So, $V\circ(U*x)\tilde{\to}(V\circ U)*x$ is the desired map.

Let $\overline{nm}=\{1,\ldots,nm\}$. Let $Q_n(\overline{nm})$ be the set of equivalence relations on $\overline{nm}$, whose each equivalence class has n elements. Then

$$(V\circ U)(nm)=\underset{J\in Q_n(\overline{nm})}{\oplus} V(J)\otimes(\underset{j\in J}{\otimes} U(\overline{nm}_j))$$

By Lemma 9.6.11, in their definition of $*$-action as compared with the $\star$-action the only difference is that $\oplus_{n\ge 1}$ is replaced by $\prod_{n\ge 1}$.

Definition in general: given $U,V\in\mathrm{Vect}^\Sigma$, $x\in O$, let $V^{\le m}\in\mathrm{Vect}^\Sigma$ denote the extension on $V\mid_{\Sigma_{\le m}}$ by zero to Σ. Here $\Sigma_{\le m}\subset\Sigma$ is the subgroupoid of sets of order $\le m$. So,

$$U^{\le m}*x=U^{\le m}\circ x=\prod_{n=1}^{m}(V(n)\otimes x^{\otimes n})_{S_n}$$

Note that $(V^{\le m}\circ U^{\le m})^{\le m}\widetilde{\to}(V\circ U)^{\le m}$, so we have the projection $\epsilon : V^{\le m}\circ U^{\le m}\to (V\circ U)^{\le m}$. Besides, $U * x = \lim_{m\in\mathbb{N}^{op}} U^{\le m} * x$.

Define now the map $\delta_m : V\circ (U*x)\to (V\circ U)^{\le m} * x$ as the composition

$$V\circ (U*x)\to V\circ (U^{\le m}\circ x)\to V^{\le m}\circ (U^{\le m}\circ x)\stackrel{\epsilon}{\to} (V\circ U)^{\le m} * x$$

The maps δ_m are compatible with the transition maps in $\lim_{m\in\mathbb{N}^{op}} (V\circ U)^{\le m} * x\widetilde{\to}(V\circ U)*x$, so define the desired map (17).

We used the fact that in any $\mathcal{C}\in 1-\mathcal{C}at$ containing all limits, given $x_n\in\mathcal{C}$ one has $\prod_{n\ge 1} x_x\widetilde{\to}\lim_{n\in\mathbb{N}^{op}}\prod_{i=1}^n x_i$, which follows from [45]. Indeed, $\mathbb{N}\widetilde{\to}\operatorname{colim}_{n\in\mathbb{N}}\mathbb{N}_{\le n}$, where $\mathbb{N}_{\le n} = \{1,\dots,n\}$ is a set, and the latter colimit is taken in the category of small categories.

For ([14], IV.2, 2.7.2) they use the notion of a comodule which is given in my Section 3.3.6.

9.6.13. For $m\ge 1$ the functor $(\mathrm{Vect}^{\Sigma},\circ)\to(\mathrm{Vect}^{\Sigma},\circ)$, $V\mapsto V^{\le m}$ is right-lax monoidal: for $V, U\in\mathrm{Vect}^{\Sigma}$ the map ϵ from the previous subsection defines this structure. So, we get a functor $Alg(\mathrm{Vect}^{\Sigma})\to Alg(\mathrm{Vect}^{\Sigma})$, $V\mapsto V^{\le m}$. So, for a reduced operad $\mathcal{P}$, $\mathcal{P}^{\le m}$ is also a reduced operad. Moreover, the natural map $\mathcal{P}\to\mathcal{P}^{\le m}$ is a morphism of reduced operads.

We may equip $\mathrm{Vect}^{\Sigma_{\le m}} = \mathrm{Fun}(\Sigma_{\le m},\mathrm{Vect})$ with a monodal structure sending (P,Q) (viewed as object of Vect^{Σ} extended by zero from $\Sigma_{\le m}$) to $(P\circ Q)^{\le m}$. The restriction functor $\mathrm{Res} : \mathrm{Vect}^{\Sigma}\to\mathrm{Vect}^{\Sigma_{\le m}}$, $V\mapsto V^{\le m}$ is monoidal. Its left adjoint $LKE : \mathrm{Vect}^{\Sigma_{\le m}}\to\mathrm{Vect}^{\Sigma}$ is given by the extension by zero along $\Sigma_{\le m}\to\Sigma$, so LKE is left-lax monoidal. The functor $RKE : \mathrm{Vect}^{\Sigma_{\le m}}\to\mathrm{Vect}^{\Sigma}$ coincides with the LKE, so it also has another right-lax monoidal structure.

The above map $\mathcal{P}\to\mathcal{P}^{\le m}$ of algebras comes from the adjoint pair $\mathrm{Res} : Alg(\mathrm{Vect}^{\Sigma},\circ)\leftrightarrows Alg(\mathrm{Vect}^{\Sigma_{\le m}}) : RKE$.

Now given $O\in CAlg^{nu}(\mathrm{DGCat}_{cont})$, we consider the action of $(\mathrm{Vect}^{\Sigma},\circ)$ on it given by $\circ$. Let $\mathcal{P}\in Operad$ be a reduced operad. The restriction $j_m : \mathcal{P}^{\le m}-Alg(O)\to\mathcal{P}-Alg(O)$ along $\mathcal{P}\to\mathcal{P}^{\le m}$ is a fully faithful functor, it is the full subcategory of those $x\in\mathcal{P}-Alg(O)$ such that for the action map $\mathcal{P}\circ x\to x$ the maps $\mathcal{P}(n)\circ x\to x$ are zero for $n>m$. The functor j_m has a left adjoint L_m given by $x\mapsto\mathcal{P}^{\le m}\otimes_{\mathcal{P}} x$ by ([28], 4.6.2.17).

In ([9], 3.4.3) they introduce the subcategory $\mathcal{P}-Alg^{nil}(O)\subset\mathcal{P}-Alg(O)$ as the smallest full subcategory of $\mathcal{P}-Alg(O)$ containing each $\mathcal{P}^{\le m}-Alg(O)$ and closed under limits. Why this $\mathcal{P}-Alg^{nil}(O)$ is a localization of $\mathcal{P}-Alg(O)$? Francis says this is wrong.

A possible idea here: let S_m be the set of L_m-equivalences. It is strongly saturated in the sense of ([27], 5.5.4.5). Then $S := \cap_m S_m$ is also strongly saturated by ([27], 5.5.4.7). **Question**: is $\mathcal{P}-Alg^{nil}(O)$ just the subcategory of S-local objects? Probably no.

John Francis says there is a mistake in the definition of $\mathcal{P}-Alg^{nil}(O)$, it is not a localization of $\mathcal{P}-Alg(O)$ in general. Consider O as equipped with the right-lax $*$-action of Vect^{Σ}. Then $\mathcal{P}-Alg^{nil}(O)$ should be defined as the category of $\mathcal{P}$-algebras with respect to the $*$-action of Vect^{Σ} on O.

Then for $\mathcal{Q}$ a reduced cooperad, $Q-coalg(O)^{op}$ is defined as $Q-alg(O^{op})$ in the sense of my Section 3.3.6. Namely, we have the left-lax action of $(\mathrm{Vect}^{\Sigma})^{op}$ on O^{op}, so that construction applies for $Q\in Alg((\mathrm{Vect}^{\Sigma})^{op},\circ)$.

Maybe $\mathcal{P}-Alg^{nil}(O)$ could also be define it as $\lim\limits_{m\in\mathbb{N}^{op}}\mathcal{P}^{\le m}-Alg(O)$, where the transition maps are $L_m:\mathcal{P}^{\le m+1}-Alg(O)\to\mathcal{P}^{\le m}-Alg(O)$?

Remark 9.6.14. *If $A\in Alg(1-\mathcal{C}\mathrm{at})$, $C\in 1-\mathcal{C}\mathrm{at}$, assume given a right-lax monoidal functor $A\to\mathrm{Fun}(C,C)$. Then C is weakly enriched over A in the sense of ([28], 4.2.1.12). So, for an algebra $\mathcal{A}\in Alg(A)$ we have the category $\mathcal{A}-mod(C)$. However, for a coalgebra $\mathcal{B}\in coAlg(A)$, $\mathcal{B}-comod(C)$ is also defined via my Section 3.3.6. If $A\to\mathrm{Fun}(C,C)$ is left-lax monoidal then C does not get a structure of a weakly enriched category over A as Jacob confirms, however.*

If $f:A\to\mathrm{Fun}(C,C)$ and $g:A\to\mathrm{Fun}(C,C)$ are right-lax monoidal functors, and $h:f\to g$ is a right-lax monoidal natural transformation then given $\mathcal{A}\in coAlg(A)$, we should get a functor $\mathcal{A}-mod^g(C)\to\mathcal{A}-mod^f(C)$, which sends $c\in\mathcal{A}-mod^g(C)$ to c with the new action given by $f(\mathcal{A},c)\to g(\mathcal{A},c)\stackrel{act}{\to}c$.

9.6.15. Definition ([9], 4.1.1) is nice. We especially want to use it for $\mathcal{X}=\mathrm{Vect}$, in this case $C_i\in CAlg^{nu}(\mathrm{DGCat}_{cont})$, and $\lim_{i\in\mathbb{N}^{op}}C_i$ is taken in $CAlg^{nu}(\mathrm{DGCat}_{cont})$. Their definition garantees that for any $i\ge 1$, the product $C_i^{\otimes i+1}\to C_i$ is zero.

For ([9], 4.1.4). If $C\in CAlg^{nu}(\mathrm{DGCat}_{cont})$ is written as

$$C\widetilde{\to}\lim_{\alpha\in A}C_\alpha$$

in $CAlg^{nu}(\mathrm{DGCat}_{cont})$ then for any $\mathcal{O}\in\mathrm{Alg}(\mathrm{Vect}^{\Sigma})$ teh natural map $\mathcal{O}-Alg(C)\to\lim_{\alpha\in A}\mathcal{O}-Alg(C_\alpha)$ is an equivalence. Indeed, apply Corollary from my Section 3.1.9. It is applicable because the composition $\circ:\mathrm{Vect}^{\Sigma}\times\mathrm{Vect}^{\Sigma}\to\mathrm{Vect}^{\Sigma}$ preserves geometric realization separately in each variable, and the same for the action map $\mathrm{Vect}^{\Sigma}\times D\to D$ for any $D\in\mathrm{DGCat}_{cont}$. Besides, $\mathrm{oblv}:CAlg^{nu}(\mathrm{DGCat}_{cont})\to 1-\mathcal{C}\mathrm{at}$ preserves limits.

Their second isomorphism $\mathcal{O}^\vee-coalg^{ind-nilp}(C)\widetilde{\to}\lim_\alpha\mathcal{O}^\vee-coalg^{ind-nilp}(C_\alpha)$ follows from Claim in my Section 3.1.9. Namely, for any diagram $A\to\mathrm{DGCat}_{cont}$, $\alpha\mapsto C_\alpha$, where the transition functors preserve totalizations, this holds.

9.6.16. Let Vect^{Σ}_r be the category classifying $V\in\mathrm{Vect}^{\Sigma}$ and a map $\mathbb{U}\to V$ such that $\mathbb{U}(1)\to V(1)$ is an isomorphism. This is naturally a monoidal category. Let $\mathrm{Vect}^{\Sigma}_{>}\subset\mathrm{Vect}^{\Sigma}$ be the full subcategory of V such that $V(1)=0$. This is a monoidal subcategory, and $\mathrm{Vect}^{\Sigma}_{>}$ is pronilpotent.

View $\mathbb{U}$ also as a unit in the monoidal category $\mathrm{Vect}^{\Sigma_{\le m}}$. Let $\mathrm{Vect}_r^{\Sigma_{\le m}}$ be the category classifying $V\in\mathrm{Vect}^{\Sigma_{\le m}}$ with a map $\mathbb{U}\to V$ such that $\mathbb{U}(1)\to V(1)$ is an isomorphism. This is naturally a monoidal category.

Consider the Koszul duality for the monoidal category $\mathrm{Vect}_r^{\Sigma_{\leq m}}$, namely the adjoint pair

$$Bar^{enh} : AssAlg^{aug}(\mathrm{Vect}_r^{\Sigma_{\leq m}}) \leftrightarrows CoassCoalg^{aug}(\mathrm{Vect}_r^{\Sigma_{\leq m}}) : coBar^{enh}$$

These functors equivalences.

This is a particular case of the following more general claim. Let $C \in \mathrm{DGCat}_{cont}$, whose image in $1-\mathcal{C}\mathrm{at}$ is equipped with a structure of an object of $Alg(1-\mathcal{C}\mathrm{at})$. My understanding is that then Vect^{Σ} acts on C as above by the $\star$-action. Let now $C' \subset C$ be a full subcategory, which is closed under the multiplication and is pronilpotent. Let C_0 be the essential image of $C' \to C, x \mapsto 1+x$. So, $C_0 \subset C$ is a full subcategory, which is a monoidal subcategory of C. Then $AssAlg^{aug}(C_0)$ identifies with $AssAlg^{aug}-mod(C')$ in the notations of ([14], ch. IV.2, 1.1.5). Bisedes, $coAsscpAlg^{aug}(C_0)$ identifies with $coAsscpAlg^{aug}-mod(C')$, and ([9], 4.1.2) gives an equivalence

$$AssAlg^{aug}(C_0) \widetilde{\to} coAsscpAlg^{aug}(C_0)$$

This is how ([14], ch. IV.2, 2.3.1) establishes the equivalence

$$Bar^{enh} : Operads \leftrightarrows coOperads : coBar^{enh}$$

Here $Operads = AssAlg^{aug}(\mathrm{Vect}_r^{\Sigma})$ identifies with $AssAlg^{aug}-mod(\mathrm{Vect}_{>}^{\Sigma})$, and

$$coOperads = coAsscoAlg^{aug}(\mathrm{Vect}_r^{\Sigma})$$

identifies with $coAsscoAlg^{aug}-mod(\mathrm{Vect}_{>}^{\Sigma})$, so we may apply ([9], 4.1.2).

9.6.17. Let Q be a reduced cooperad and $P = Q^{\vee}$ be the corresponding operad, so we have the natural map $P \to \lim\limits_{[n]\in\boldsymbol{\Delta}^{\leq m}} Q^n := Tot^{\leq m}(Q^{\bullet})$. Then its restriction to $\Sigma^{\leq m}$ is an isomorphism, right? This was used for the proof of ([9], 4.1.6(b)) I think.

9.6.18. For ([9], 4.1.10). Their map (4.5), that is, for a cooperad $\mathcal{P}$ and $B \in \mathcal{P}-coalg_{dp}^{nil}(C)$, the map $Cobar_{\mathcal{P}}(B) = \lim_{[n]\in\boldsymbol{\Delta}} \mathcal{P}^n \circ B \to oblv_{\mathcal{P}}(B)$ is the projection from $\lim\limits_{[n]\in\boldsymbol{\Delta}} \mathcal{P}^n \circ B$ to the 0-th term of this cosimplicial diagram.

9.6.19. For ([14], ch. IV.2, 2.7.7). We do not need the presentability of $Q-Coalg(O)$ to conclude that $triv_Q : O \to Q-Coalg(O)$ has a right adjoint by ([27], Remark 5.5.2.10).

9.6.20. For ([14], ch. IV.2, 3.2.1). For any $O \in Alg(1-\mathcal{C}\mathrm{at})$ admitting limits and colimits they consider $Bar^{\bullet} : AssocAlg^{aug}(O) \to O^{\boldsymbol{\Delta}^{op}}$, it sends A to the functor $[n] \mapsto A^{\otimes n}$, and the same formula for $Bar^{\bullet}_{1//1}$.

For ([14], ch. IV.2, 3.2.2). Assume O is symmetric monoidal. Then they equip $O^{\boldsymbol{\Delta}^{op}}$ with the pointwise symmetric monoidal structure: for functors $f, g : \boldsymbol{\Delta}^{op} \to O$ the tensor product is $f \otimes g$ given by $[n] \mapsto f(n) \otimes g(n)$. The functors $Bar_{1//1}$ and Bar inherit the symmetric monoidal structure, because $\boldsymbol{\Delta}^{op}$ is sifted.

Indeed, the functor $\mathrm{colim} : O^{\boldsymbol{\Delta}^{op}} \to O$, $f \mapsto \mathrm{colim}\, f$ is symmetric monoidal.

For ([14], ch. IV.2, 3.2.3). They assume there that O is stable, more precisely $O \in Alg(\mathrm{DGCat}_{cont})$, so $triv_{Assoc^{aug}} : O_{1//1} \to AssAlg^{aug}(O)$ makes sense. Here we may identify $O_{1//1} \widetilde{\to} O$ via the map sending $1 \to z \to 1$ to $Fib(z \to 1)$.

9.6.21. For ([14], ch. IV.2, 3.2.3). Let $O \in Alg(\mathrm{DGCat}_{cont})$. They claim that $Bar_{1//1}: AssocAlg^{aug}(O) \to O_{1//1}$ is left adjoint to the composition

$$O_{1//1} \stackrel{\Omega}{\to} O_{1//1} \stackrel{triv_{Assoc^{aug}}}{\to} AssocAlg^{aug}(O)$$

Formal proof is not clear, here is the idea:

Let $Free_*: O_{1//1} \to AssocAlg^{aug}(O)$ be the left adjoint to $\mathrm{oblv}: AssocAlg^{aug}(O) \to O_{1//1}$. By ([28], 5.2.2.13), we see that the composition

$$O_{1//1} \stackrel{Free_*}{\to} AssocAlg^{aug}(O) \stackrel{Bar_{1//1}}{\to} O_{1//1}$$

is left adjoint to the composition

$$O_{1//1} \stackrel{\Omega}{\to} O_{1//1} \stackrel{triv_{Assoc^{aug}}}{\to} AssocAlg^{aug}(O) \stackrel{\mathrm{oblv}}{\to} O_{1//1}$$

The functor $Bar_{1//1}$ preserves sifted colimits, and each $B \in AssocAlg^{aug}(O)$ can be written by ([28], 4.7.3.14) as $\mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}} Free_*(x_n)$ for some $x_n \in O_{1//1}$. Then $Bar_{1//1}(B) \widetilde{\to} \mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{op}}(1 \sqcup_{x_n} 1)$, and for $z \in O_{1//1}$ we get

$$\mathrm{Map}_{O_{1//1}}(Bar_{1//1}(B), z) \widetilde{\to} \lim_{n\in\boldsymbol{\Delta}} \mathrm{Map}_{O_{1//1}}(1 \sqcup_{x_n} 1, z) \widetilde{\to} \lim_{n\in\boldsymbol{\Delta}} \mathrm{Map}_{O_{1//1}}(x_n, \Omega(z)) \widetilde{\to}$$
$$\lim_{n\in\boldsymbol{\Delta}} \mathrm{Map}_{AssocAlg^{aug}(O)}(Free_*(x_n), triv(\Omega(z))) \widetilde{\to} \mathrm{Map}_{AssocAlg^{aug}(O)}(B, triv(\Omega(z)))$$

Recall that $triv_{Assoc^{aug}}$ sends $1 \oplus b = B \in O_{1//1}$ to $1 \oplus b$, where the product on it is such that $b \times b \stackrel{m}{\to} b$ vanishes.

9.6.22. For ([14], ch. IV.2, 4.1.1). Let $O \in CAlg(\mathrm{DGCat}_{cont})$. They refer to ([28], 3.2.4.7) in the 2nd paragraph claiming that the maps (3.1) are isomorphisms (in those maps the coporducts are understood in the symmetric monoidal category $CAlg(O)$).

The category $CocomCoalg(O)$ is not known to be presentable, as far as I understand, though admits colimits, and $\mathrm{oblv}: CocomCoalg(O) \to O$ detects colimits. The existence of its right adjoint in general is not clear, I think, this is why they use instead the functor $Sym: O \to CocomCoalg(O)$ from ([14], ch. IV.2, 4.2.1).

9.6.23. For ([14], ch. IV.2, 4.1.2). There is the Lie operad $\mathcal{L}\mathrm{ie}$ descibed in my file ([38], 1.1.1). They take Lie equal to the augmentation of $\mathcal{L}\mathrm{ie}$, that is $\mathcal{L}\mathrm{ie} \oplus \mathbb{U} = \mathrm{Lie}$ in Vect^{Σ}. In ([14], ch. IV.2, 2.3.3) they have $\mathrm{Lie}^{\vee} \widetilde{\to} Cocom^{aug}[1]$ and $\mathrm{Lie}[-1] \widetilde{\to} (Cocom^{aug})^{\vee}$, there Lie is augmented!

The cooperads $Cocom^{aug}$ is defined in ([14], IV.2, 2.1.3) as $(Com^{aug})^*$. In other words, for any $I \in \Sigma$, $Cocom^{aug}(I) = e$, here e is the field of coefficients. Recall also that $Coassoc^{aug} \in \mathrm{Vect}^{\Sigma}$ is also defined there as the functor $I \mapsto (k^{ord(I)})^*$, here $ord(I)$ is the set of linear orders on I. As a mere functor of $I \in \Sigma$, it identifies with $Assoc^{aug}$.

Note that the functor $free_{\mathrm{Lie}}: O \to LieAlg(O)$ sends x to $x \oplus \wedge^2 x \oplus \ldots$, where $\wedge^2 x \widetilde{\to} \underset{B(S_2)}{\mathrm{colim}}\ \mathrm{Lie}_2 \otimes x^{\otimes 2}$.

10. For [14], ch. 2, Basics of derived algebraic geometry

10.0.1. The full subcategory $\mathrm{Vect}^{fd} \subset \mathrm{Vect}$ defined in (ch. 1, 10.1.3) is stable under finite colimits by (HTT, 5.3.4.15). Using the dualization equivalence $(\mathrm{Vect}^{ft})^{op} \widetilde{\to} \mathrm{Vect}^{ft}$ we see that for any small $K \in 1-\mathcal{C}at$ a diagram $K^{\triangleleft} \to \mathrm{Vect}^{fd}$ is a limit diagram iff $(K^{op})^{\triangleright} \to \mathrm{Vect}^{fd}$ is a colimit diagram. Note that $\mathrm{Vect}^{fd} \subset \mathrm{Vect}$ is a stable subcategory.

10.0.2. The t-structure on Vect is accessible in the sense of (HA, 1.4.4.12), that is, $\mathrm{Vect}^{\le 0}$ is presentable (by HA, 1.3.5.21). For any $n \in \mathbb{Z}$, $\mathrm{Vect}^{\ge -n}$ is accessible by (HA, 1.4.4.13). Applying (HTT, 5.4.6.6), we see that for $a \le b$, $\mathrm{Vect}^{[a,b]}$ is accessible. It is also cocomplete, hence presentable.

10.0.3. The category $\mathrm{Vect}^{\le 0}$ admits all finite limits, see my Section 4.0.7. The full subcategory $\mathrm{Vect}^{\le 0} \subset \mathrm{Vect}$ is stable under small colimits and preserved under the tensor product in Vect. The tensor product $\mathrm{Vect} \times \mathrm{Vect} \to \mathrm{Vect}$ is exact in each variable (the t-structure is compatible with the symmetric monoidal structure in the sense of HA, 2.2.1.3). So, by (HA, 2.2.1.3), $\mathrm{Vect}^{\le 0}$ inherits a symmetric monoidal structure, and the inclusion $\mathrm{Vect}^{\le 0} \to \mathrm{Vect}$ is symmetric monoidal. Its right adjoint is $\tau^{\le 0} : \mathrm{Vect} \to \mathrm{Vect}^{\le 0}$ has a right-lax nonunital monoidal structure by (HA, 2.2.1.3).

For 1.2.1. Let $\tau^{\ge -n} : \mathrm{Vect}^{\le 0} \to \mathrm{Vect}^{\ge -n, \le 0}$ be the left adjoint to the inclusion $\mathrm{Vect}^{\ge -n, \le 0} \hookrightarrow \mathrm{Vect}^{\le 0}$. Then according to (HA, 2.2.1.7), $\tau^{\ge -n}$ is compatible with the symmetric monoidal structure on $\mathrm{Vect}^{\le 0}$ in the sense of (HA, 2.2.1.6), this is also proved in (HA, 2.2.1.8). So, by (HA, 2.2.1.9), we get a symmetric monoidal structure on $\mathrm{Vect}^{\ge -n, \le 0}$ such that $\tau^{\ge -n} : \mathrm{Vect}^{\le 0} \to \mathrm{Vect}^{\ge -n, \le 0}$ is symmetric monoidal, and the inclusion $\mathrm{Vect}^{\ge -n, \le 0} \hookrightarrow \mathrm{Vect}^{\le 0}$ is right-lax nonunital monoidal functor (see also HA, Example 2.2.1.10). This means, in particular, that for $K, M \in \mathrm{Vect}^{\ge -n, \le 0}$ there is a natural map $K \otimes M \to \tau^{\ge -n}(K \otimes M)$.

The fact that

$$\tau^{\ge -n} : CAlg(\mathrm{Vect}^{\le 0}) \to CAlg(\mathrm{Vect}^{\ge -n, \le 0})$$

is left adjoint to the full embedding $CAlg(\mathrm{Vect}^{\ge -n, \le 0}) \hookrightarrow CAlg(\mathrm{Vect}^{\le 0})$ follows from my Section 3.0.20.

In 1.2.5 the functor $S \mapsto {}^{\le n}S$ is not fully faithful (this is a misprint in the published version).

For 1.3.2: the functor $LKE : {}^{\le n}\mathrm{PreStk} \to \mathrm{PreStk}$ is fully faithful because of the following general remark. Let $\mathcal{C} \in 1-\mathcal{C}at$, $\mathcal{C}^0 \subset \mathcal{C}$ be a full subcategory, $\mathcal{D}$ admits colimits. Then the left Kan extension $L : \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful. Indeed, if R is its right adjoint then $RL \widetilde{\to} \mathrm{id}$.

For 1.3.7: The right adjoint to the restriction functor $\mathrm{PreStk} \to {}^{\le n}\mathrm{PreStk}$ exists because of (HTT, 5.2.6.6).

Since $\tau^{\ge -n} : \mathrm{Vect}^{\le 0} \to \mathrm{Vect}^{\ge -n, \le 0}$ is symmetric monoidal, it preserves relative tensor products: if $B \leftarrow A \to C$ is a diagram in $CAlg(\mathrm{Vect}^{\le 0})$ then

$$\tau^{\ge -n}(B \otimes_A C) \widetilde{\to} (\tau^{\ge -n}B) \otimes_{\tau^{\ge -n}A} \tau^{\ge -n}C,$$

where in the RHS the tensor product is taken in the symmetric monoidal category $\mathrm{Vect}^{\ge -n, \le 0}$.

The category Sch^{aff} admits colimits, and ${}^{\le n}\mathrm{Sch}^{aff} \subset \mathrm{Sch}^{aff}$ is stable under colimits. Since the tensor product in $\mathrm{Vect}^{\le 0}$ preserves colimits separately in each variable, and $\mathrm{Vect}^{\le 0}$ is presentable, the tensor product in $\mathrm{Vect}^{\le 0}$ is compatible with colimits in the sense of (HA, 3.1.1.18). So, by (HA, 3.2.3.3), $CAlg(\mathrm{Vect}^{\le 0})$ admits small colimits, Sch^{aff} admits small limits.

10.0.4. (For 1.3.3). Let $\bar{L} : \mathcal{A} \to \mathcal{B}$ be a left adjoint functor to $\bar{R} : \mathcal{B} \to \mathcal{A}$, here $\mathcal{A}, \mathcal{B} \in 1-\mathcal{C}at$ are small. Let $R : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{A})$ be the functor of composing with $\bar{L} : \mathcal{A}^{op} \to \mathcal{B}^{op}$. Then R has a left adjoint $L : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ given by the LKE along $\bar{L} : \mathcal{A}^{op} \to \mathcal{B}^{op}$. For the Yoneda embeddings $j : \mathcal{A} \to \mathcal{P}(\mathcal{A}), j : \mathcal{B} \to \mathcal{P}(\mathcal{B})$ and $a \in \mathcal{A}$ one has canonically $L(j(a)) \widetilde{\to} j(\bar{L}(a))$. Indeed, for $F \in \mathcal{P}(\mathcal{B})$ one has

$$\mathrm{Map}_{\mathcal{P}(\mathcal{B})}(j(\bar{L}(a)), F) \widetilde{\to} F(\bar{L}(a)) \widetilde{\to} (RF)(a) \widetilde{\to} \mathrm{Map}_{\mathcal{P}(\mathcal{A})}(j(a), RF) \widetilde{\to} \mathrm{Map}_{\mathcal{P}(\mathcal{B})}(L(j(a)), F)$$

This was used in 1.3.3 for $\bar{L} : \mathcal{A} = {}^{\le n}\mathrm{Sch}^{aff} \hookrightarrow \mathcal{B} = \mathrm{Sch}^{aff}$ the natural inclusion. This gives that if $S \in \mathrm{Sch}^{aff}$ then $\tau^{\le n} S$ as a scheme represents the prestack $\tau^{\le n} S$.

In the above general setting given $b \in \mathcal{B}$ the natural map $LRj(b) \to j(b)$ is an isomorphism in $\mathcal{P}(\mathcal{B})$ iff $\bar{L}\bar{R}(b) \to b$ is an isomorphism in $\mathcal{B}$. Indeed, $LRj(b) \widetilde{\to} Lj(\bar{R}b) \widetilde{\to} j\bar{L}\bar{R}(b)$.

10.0.5. For 1.4.1. For $S \in \mathrm{Sch}^{aff}$ we have canonical maps $\tau^{\le n}(S) \to \tau^{\le n+1}(S) \to S$ for any $n \ge 0$. This gives the definition of convergence. We have $\mathrm{Vect}^{\le 0} \widetilde{\to} \lim_n \mathrm{Vect}^{\ge -n, \le 0}$. Using Lemma 2.2.68, this implies

$$CAlg(\mathrm{Vect}^{\le 0}) \widetilde{\to} \lim_{n \ge 0} CAlg(\mathrm{Vect}^{\ge -n, \le 0})$$

So, for $A, B \in CAlg(\mathrm{Vect}^{\le 0})$ we then get

$$\mathrm{Map}_{CAlg(\mathrm{Vect}^{\le 0})}(A, B) \widetilde{\to} \lim_{n \ge 0} \mathrm{Map}_{CAlg(\mathrm{Vect}^{\ge -n, \le 0})}(\tau^{\ge -n}(A), \tau^{\ge -n}(B))$$

Since $\mathrm{Map}_{CAlg(\mathrm{Vect}^{\le 0})}(A, \tau^{\ge -n}(B)) \widetilde{\to} \mathrm{Map}_{CAlg(\mathrm{Vect}^{\ge -n, \le 0})}(\tau^{\ge -n}(A), \tau^{\ge -n}(B))$, we get

$$\mathrm{Map}_{CAlg(\mathrm{Vect}^{\le 0})}(A, B) \widetilde{\to} \lim_{n \ge 0} \mathrm{Map}_{CAlg(\mathrm{Vect}^{\le 0})}(A, \tau^{\ge -n}(B))$$

This means precisely that $\mathrm{Spec}\, A$ represents a convergent prestack.

10.0.6. For the proof of ([14], ch. 2, Prop. 1.4.7). Given a map $S' \to S$ in Sch^{aff} with $S' \in {}^{<\infty}\mathrm{Sch}^{aff}$, consider the category classifying $n \ge 0$ and a map $S' \to \tau^{\le n}(S)$ in Sch^{aff}/S. If $S' \in {}^{\le m}\mathrm{Sch}^{aff}$ then for any $n \ge m$ this category has exactly one object corresponding to n. It follows that this category is filtered, hence also contractible by (HTT, 5.3.1.20). Now, according to their definition, this means that $\mathbb{Z}^{\ge 0} \to ({}^{<\infty}\mathrm{Sch}^{aff})_{/S}$ is cofinal.

Remark: if $\mathcal{C} \in 1-\mathcal{C}at$ admits an initial object and push-out squares then it admits finite colimits, hence is filtered (HTT, 4.4.2.4).

10.0.7. For 1.4.8: Let $F \in \mathrm{PreStk}, F^0 : ({}^{<\infty}\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$ be its restriction under $({}^{<\infty}\mathrm{Sch}^{aff})^{op} \to (\mathrm{Sch}^{aff})^{op}$. Write $RKE(F^0) : (\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$ for the RKE of F^0 under $({}^{<\infty}\mathrm{Sch}^{aff})^{op} \to (\mathrm{Sch}^{aff})^{op}$. There is a natural map $F \to RKE(F^0)$ functorial in F. The functor $\mathrm{PreStk} \to {}^{conv}\mathrm{PreStk}, F \mapsto RKE(F^0)$ is the left adjoint to the fully faithful embedding ${}^{conv}\mathrm{PreStk} \hookrightarrow \mathrm{PreStk}$.

For 1.4.9. For $Y \in \mathrm{PreStk}$ we have a natural map $\mathrm{colim}_n \tau^{\leq n} Y \to Y$. If $S \in {}^{<\infty}\mathrm{Sch}^{aff}$ then evaluating this map on S we get an isomorphism

$$\mathrm{Map}(S, \mathrm{colim}_n \tau^{\leq n} Y) \widetilde{\to} Y(S)$$

Now if $Y_1 \in {}^{conv}\mathrm{PreStk}$ then $\mathrm{Map}(Y, Y_1) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}(({}^{<\infty}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc})}(Y^0, Y_1^0)$, where Y^0, Y_1^0 are the restrictions of Y, Y_1. So, $\mathrm{Map}(\mathrm{colim}_n \tau^{\leq n} Y, Y_1) \to \mathrm{Map}(Y, Y_1)$ is an isomorphism.

10.0.8. For 1.5.7: ${}^{\leq n}\mathrm{Sch}^{aff}_{ft}$ is clearly closed under retracts, so (HTT, 5.4.2.4) applies and gives $({}^{\leq n}\mathrm{Sch}^{aff}_{ft})^{op} \widetilde{\to} (({}^{\leq n}\mathrm{Sch}^{aff})^{op})^c$.

10.0.9. For 1.6.8. We want to check that the full embedding $LKE : {}^{\leq n}\mathrm{PreStk}_{lft} \to {}^{\leq n}\mathrm{PreStk}$ commutes with finite limits. The category $({}^{\leq n}\mathrm{Sch}^{aff}_{ft})^{op}$ admits finite colimits. Applying (HTT, 5.3.4.7), we see that the inclusion ${}^{\leq n}\mathrm{PreStk}_{lft} \subset {}^{\leq n}\mathrm{PreStk}$ is stable under finite limits. We are done.

10.0.10. For 1.7.3. Let $0 \leq n < m$. If $Y \in {}^{\leq n}\mathrm{PreStk}_{lft}$, consider its LKE Y' under the full embedding $({}^{\leq n}\mathrm{Sch}^{aff})^{op} \subset ({}^{\leq m}\mathrm{Sch}^{aff})^{op}$. Then $Y' \in {}^{\leq m}\mathrm{PreStk}_{lft}$.

10.0.11. For the proof of Prop. 1.7.6. For $n \geq 0$ the diagram commutes

$$\begin{array}{ccc}
\mathrm{Fun}(({}^{<\infty}\mathrm{Sch}^{aff}_{ft})^{op}, Spc) & \stackrel{LKE}{\to} & \mathrm{Fun}(({}^{<\infty}\mathrm{Sch}^{aff})^{op}, Spc) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(({}^{\leq n}\mathrm{Sch}^{aff}_{ft})^{op}, Spc) & \stackrel{LKE}{\to} & \mathrm{Fun}(({}^{\leq n}\mathrm{Sch}^{aff})^{op}, Spc),
\end{array}$$

where the vertical arrows are the natural restrictions. Indeed, let $S \in {}^{\leq n}\mathrm{Sch}^{aff}$ and $Y \in \mathrm{Fun}(({}^{<\infty}\mathrm{Sch}^{aff}_{ft})^{op}, Spc)$. The value of $LKE(Y)$ on S is $\underset{S \to S'}{\mathrm{colim}}\, Y(S')$, where $S' \in {}^{<\infty}\mathrm{Sch}^{aff}_{ft}$ and the map $S \stackrel{\alpha}{\to} S'$ is in ${}^{<\infty}\mathrm{Sch}^{aff}$. However,

$$\mathrm{Map}_{\mathrm{Sch}^{aff}}(S, S') \widetilde{\to} \mathrm{Map}_{{}^{\leq n}\mathrm{Sch}^{aff}}(S, {}^{\leq n}S')$$

by adjointness. So, α factors as $S \to {}^{\leq n}S' \to S'$. So, this is also the colimit over the full subcategory given by the condition that $S' \in {}^{\leq n}\mathrm{Sch}^{aff}_{ft}$. Thus, the above square commutes.

10.0.12. For 1.8.2. The embedding $\mathrm{Spc}_{\leq k} \subset \mathrm{Spc}$ admits an accessible left adjoint by (HTT, 5.5.6.18). If $S \in \mathrm{Spc}$ then $S \to \lim_{n \geq 0} P_{\leq n}(S)$ is an equvalence in Spc, because in Spc the Postnikov towers are convergent by (HTT, 7.2.1.10, 7.2.1.9) and (HTT, 5.5.6.25).

The category $\mathrm{Vect}^{\geq -n, \leq 0}$ is equivalent to a $n+1$-category in the terminology of Lurie (see Sect.4.0.26). By the way, $CAlg(\mathrm{Vect}^{\geq -n, \leq 0}) \widetilde{\to} \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Vect}^{\geq -n, \leq 0})$ for $k > n+1$ by (HA, 5.1.1.7).

10.0.13. For 1.8.9. The embedding ${}^{\le n}\operatorname{PreStk}_{\le k}\subset{}^{\le n}\operatorname{PreStk}$ admits a left adjoint by my Lemma 2.2.56.

10.0.14. Lm 2.1.3 is immediate from the first of the three equvalent properties in ([14], ch. 2, 2.1.1).

If $\alpha: S'\to S$ in Sch^{aff} is flat then $\tau^{cl}(S)\times_S S'\widetilde{\to}\tau^{cl}(S')$ by property iii) in 2.1.1). If $n\ge 0$ then $\tau^{\le n}S\times_S S'\widetilde{\to}\tau^{\le n}S'$. Indeed, the tensor product is exact in each variable. Let $A\to A'$ in $CAlg(\mathrm{Vect}^{\le 0})$ correspond to α. The fibre sequence $(\tau^{<-n}A)\otimes_A A'\to A'\to(\tau^{\ge -n}A)\otimes_A A'$ shows that $(\tau^{<-n}A)\otimes_A A'$ is in degrees $<-n$, $(\tau^{\ge -n}A)\otimes_A A'$ is in degrees $\ge -n$. If $M\in\mathrm{Vect}$ is an A-module then $(\tau^{<-n}M)\otimes A$ lives in degrees $<-n$. Applying $\tau^{<-n}$ to the composition $(\tau^{<-n}M)\otimes A\to M\otimes A\to M$, one gets the multiplication $(\tau^{<-n}M)\otimes A\to\tau^{<-n}M$, which equips it with a A-module structure.

Note also that for any $A\in CAlg(\mathrm{Vect}^{\le 0})$ the map $A\to\tau^{\ge -n}A$ is a map in $CAlg(\mathrm{Vect}^{\le 0})$, so can also be seen as a map in A-mod.

If $A\to A'$ in $CAlg(\mathrm{Vect}^{\le 0})$ is not flat then the base change $\tau^{\ge -n}A\otimes_A A'$ is not necessarily in $\mathrm{Vect}^{[-n,0]}$. For example, A could be classical, and A' placed in many degrees below zero. For example, consider a diagram of algebras $B\leftarrow A\to A'$ in $CAlg(\mathrm{Vect}^{\heartsuit})$ then $B\otimes_A A'$ could be placed in many degrees ≤ 0.

So, the inclusion ${}^{\le n}\mathrm{Sch}^{aff}\hookrightarrow\mathrm{Sch}^{aff}$ does not preserve finite limits.

10.0.15. For 2.1.4. For the definition of Zariski map. To be precise, a map $S'\to S$ in Sch^{aff} is Zariski if f is flat, and there is a disjoint union ${}^{cl}S'=\sqcup_i T_i$ in the category ${}^{cl}\mathrm{Sch}^{aff}$ such that each map $T_i\to{}^{cl}S$ in ${}^{cl}\mathrm{Sch}^{aff}$ is an open immersion. (The above coproduct can not be understood in classical prestacks, as to would not be an affine scheme). In this situation let $T^i\to S$ be an affine open embedding corresponding to $T_i\to{}^{cl}S$ in the sense of (ch. I.2, Lm. 2.1.5). Then $\alpha:\sqcup_i T^i\to T$ is an isomorphism in Sch^{aff}, where the coproduct is understood in Sch^{aff}. Indeed, $\sqcup_i T^i\to S$ is etale, so to check that α is an isomorphism it suffices to check that ${}^{cl}\alpha:{}^{cl}\sqcup_i T^i\to{}^{cl}T$ is an isomorphism. This is true because $\tau^{\ge 0}: CAlg(\mathrm{Vect}^{\le 0})\to CAlg(\mathrm{Vect}^{\heartsuit})$ preserves limits by my Section 9.1.

To summarize, a map $S'\to S$ in Sch^{aff} is affine Zariski iff there is an isomorphism $S'\widetilde{\to}\sqcup_i S'_i$ in Sch^{aff}, here $S'_i\in\mathrm{Sch}^{aff}$ and the coporduct is understood in Sch^{aff}, such that each map $S'_i\to S$ is an open embedding.

If $\operatorname{Spec}B\to\operatorname{Spec}A$ is a flat map in Sch^{aff}, and $\operatorname{Spec}A'\to\operatorname{Spec}A$ is any map then $B\otimes_A A'$ is flat over A'. This follows from (ch. I.2, property (3) in 2.1.1).

10.0.16. In (ch. 2, 2.2.1) there is a non usual definition of the Cech nerve. Let $\mathcal{F}$in be the category of finite sets. We have a functor $\Delta\to\mathcal{F}\mathrm{in}$, $[n]\mapsto[n]^{\mathrm{Spc}}$, where each $[n]$ is viewed as a category $0\to 1\to\ldots\to n$. Their functor $\mathcal{F}\mathrm{in}^{op}\to\mathcal{C}$ is given by $I\mapsto c^I$. For $\alpha: J\to I$ a map in $\mathcal{F}$in the induced map $c^I\to c^J$ is such that the composition $c^I\to c^J\stackrel{j}{\to}c$, where j is j-th projection, equals i-th projection, where $\alpha(j)=i$. This is the usual exponent, that is, we use the fact that $\mathcal{C}$ is cotensored over $\mathcal{F}$in.

Remark 10.0.17. *Given* $Y\stackrel{\beta}{\to}X\stackrel{\alpha}{\leftarrow}S$ *in* PreStk *with* $S\in\mathrm{Sch}^{aff}$ *the fibre of* $Y(S)\to X(S)$ *over* α *is* $\mathrm{Map}_{\mathrm{PreStk}_{/X}}(S,Y)$. *Besides,* β *lies in* $\tau_{\le -1}(\mathrm{PreStk}_{/X})$ *iff for any*

$S\in \mathrm{Sch}^{aff}$ the induced map $Y(S)\to X(S)$ is a full subspace. Since each $S\in \mathrm{Sch}^{aff}$ is a stack (in etale topology), we get an analog for stacks: let $Y\stackrel{\beta}{\to} X\stackrel{\alpha}{\leftarrow} S$ in Stk *with $S\in \mathrm{Sch}^{aff}$. The fibre of $Y(S)\to X(S)$ over α is $\mathrm{Map}_{\mathrm{Stk}/X}(S,Y)$. We have $\beta\in\tau_{\le -1}(\mathrm{Stk}_{/X})$ iff for any $S\in \mathrm{Sch}^{aff}$, $\beta: Y(S)\to X(S)$ is a full subspace.*

Proof. If $\mathcal{Z}\in$ PreStk then $\mathcal{Z}\widetilde{\to}\operatorname{colim}_{S\to\mathcal{Z}} S$, the colimit taken over $\mathrm{Sch}^{aff}\times_{\mathrm{PreStk}}\mathrm{PreStk}_{/\mathcal{Z}}$ in PreStk. Let $\beta: Y\to X$ be a map in PreStk such that for any $S\in\mathrm{Sch}^{aff}$, $Y(S)\to X(S)$ is a full subspace. Then $\mathrm{Map}(\mathcal{Z},Y)\widetilde{\to}\lim_{S\to\mathcal{Z}}(S,Y)$, so passing to the limit we get a full subspace $Y(\mathcal{Z})\to X(\mathcal{Z})$. Similar argument works also for stack. □

10.0.18. For 2.3.1. The notion of an etale covering is given in 2.1.6. Now one should proceed as in (SGA4, Exp.7, Sect.1) to define the etale Grothendieck topology on Sch^{aff}. Namely, given $X\in\mathrm{Sch}^{aff}$, we should call a collection $(Y_i\to X)_{i\in I}$ of maps in Sch^{aff} a covering family if each $Y_i\to X$ is etale, and the union of the images of ${}^{cl}Y_i\to {}^{cl}X$ equals ${}^{cl}X$. A sieve on X is a covering sieve iff it contains a covering family of X (by SGA4, Exp. 2, Prop. 1.4).

Now the functor L : PreStk $\to$ Stk left adjoint to the inclusion has no simple description in general. The construction $F\mapsto F^+$ in (HTT, 6.2.2.9) even applied any finite number of times does not make from F a sheaf (see HTT, 6.5.3 for this)!

10.0.19. If $f: Z\to X$ is an etale surjection in PreStk then let $g: Z'\to X$ be the object $\tau_{\le -1}(f)$ in $\mathrm{PreStk}_{/X}$. Then g is an etale surjection.

10.0.20. *Remark.* Let $f: Y\to Y'$ be a morphism in Stk, $Z'\to Y'$ an etale surjection, let $Z=Z'\times_{Y'}Y$. Assume the map $Z\to Z'$ is an isomorphism. Then f is an isomorphism. Indeed, $Y'=L(\mid Z'^{\bullet}/Y'\mid_{\mathrm{PreStk}})$ and $Y\widetilde{\to}L(\mid Z^{\bullet}/Y\mid_{\mathrm{PreStk}})$. For any n, the natural map $Z^n/Y\to Z'^n/Y'$ is an isomorphism, our claim follows.

10.0.21. Let $\mathcal{Z}\to S$ be a map in Stk with $S\in\mathrm{Sch}^{aff}$. Consider the functor $\bar{\mathcal{Z}}$: $((\mathrm{Sch}^{aff})_{/S})^{op}\to\mathrm{Spc}$, $T\mapsto\mathcal{Z}(T)\times_{S(T)} *$, where the map $T\to S$ is the structure one. Then $\bar{\mathcal{Z}}$ satisfies the descent for an etale cover $S_1\to S_2$ in $(\mathrm{Sch}^{aff})_{/S}$. Indeed, limits commute with limits.

Remark 10.0.22. *Let $f:\mathcal{Z}\to\mathcal{X}$ be a map in* Stk*, which is an etale surjection. Assume for any $S\in\mathrm{Sch}^{aff}$, $\mathcal{Z}(S)\to\mathcal{X}(S)$ is a full subspace. In other words, $f\in\tau_{\le -1}(\mathrm{Stk}_{/\mathcal{X}})$. Then f is an isomorphism.*

Proof. Let $a: S\to\mathcal{X}$ in PreStk, where $S\in\mathrm{Sch}^{aff}$. Let $\mathcal{Z}_a\in\mathrm{Spc}$ be the fibre of $\mathcal{Z}(S)\to\mathcal{X}(S)$ over a. We have to show it is nonempty. There is an etale cover $S'\to S$ in Sch^{aff} such that the composition $S'\to S\to\mathcal{X}$ factors through $S'\to\mathcal{Z}$. This implies that $\mathcal{Z}_a$ is nonempty as follows.

Consider the functor $Z_a: (\mathrm{Sch}^{aff}/S)^{op}\to\mathrm{Spc}$ sending $T\stackrel{b}{\to}S$ to $\mathcal{Z}(T)\times_{\mathcal{X}(T)}\{ab\}$. This is a subfunctor of the functor $*: (\mathrm{Sch}^{aff}/S)^{op}\to\mathrm{Spc}$ given by $T\mapsto *$. Then Z_a is a stack in etale topology on Sch^{aff}/S. Indeed, for any $S\in\mathrm{Sch}^{aff}$, $S\in$ Stk. So, $\mathcal{Z}\times_{\mathcal{X}}S\in$ Stk, and our claim follows from Section 10.0.21. Now $Z_a(S')$ is nonempty, so $Z_a(S'^n/S)\widetilde{\to} *$ for any $n\ge 0$, and $\mathrm{Tot}(Z_a(S'^{\cdot}/S))\widetilde{\to} *$. So, $Z_a(S)\widetilde{\to} *$. □

For 2.3.9. Let $\mathcal{X} \in \mathrm{PreStk}$ then $\mathcal{X} \to L(\mathcal{X})$ is an etale surjection. The proof is probably complicated using a transfinite induction, because $L(\mathcal{X})$ can not be obtained via a finite number of operations $\mathcal{X} \mapsto \mathcal{X}^+$ of (HTT, 6.2.2.12). If it is obtained via a finite number of such steps then we can proceed as follows. Given $y : S \to L(\mathcal{X})$ with $S \in \mathrm{Sch}^{aff}$, from (HTT, 6.2.2.12) we know that there is a covering sieve $\mathcal{C}^{(0)}_{/S} \subset \mathrm{Sch}^{aff}/S$ on S such that y comes from $\lim_{(S' \to S) \in \mathcal{C}^{(0)}_{/S}} \mathcal{X}(S')$. Assume that $S' \to S$ is an etale covering lying in $\mathcal{C}^{(0)}_{/S}$. Let $y' \in \mathcal{X}(S')$ be the image of y under the projection $\lim_{(T \to S) \in \mathcal{C}^{(0)}_{/S}} \mathcal{X}(T) \to \mathcal{X}(S')$. Then the image of y in $L(\mathcal{X})(S')$ identifies with the image y' under $\mathcal{X}(S') \to L(\mathcal{X})(S')$. We are done.

For 2.3.8. Let $f : Y_1 \to Y_2$ be an etale surjection in PreStk. Then the map $L(f) : L(Y_1) \to L(Y_2)$ satisfies $\tau_{\le -1}(L(f)) = \mathrm{id} : L(Y_2) \to L(Y_2)$. Indeed, let $Z \subset L(Y_2)$ be a map in Stk lying in $\tau_{\le -1}(\mathrm{Stk}_{/L(Y_2)})$ such that $L(f)$ factors through $Z \subset L(Y_2)$. We must show $Z = L(Y_2)$. By (ch 2, 2.3.9), $Y_2 \to L(Y_2)$ is an etale surjection. This gives immediately that $Z \to L(Y_2)$ is an etale surjection. Our claim follows now from my Remark 10.0.22. To finish the proof, use (HTT, 6.2.3.5).

10.0.23. For Corollary 2.4.4. It may be strenthened as follows: if $f : Y_1 \to Y_2$ is an open embedding in PreStk then $LY_1 \to LY_2$ is also an open embedding (same proof).

10.0.24. In 2.5.2 there is a misprint in the definition of the functor (2.1), $\mathcal{Y} \mapsto \mathcal{Y}^+$. Namely, this should be the construction (HTT, 6.2.2.12).

For 2.5.9: Let $Y' : ({}^{<\infty}\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$, let $Y'_n : ({}^{\le n}\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$ be its restriction for any n. Assume for each $n \ge 0$, $Y'_n \in {}^{\le n}\mathrm{Stk}$. Then $Y := RKE(Y) : \mathrm{Sch}^{aff} \to \mathrm{Spc}$ lies in Stk. Indeed, for $S \in \mathrm{Sch}^{aff}$ we have $Y(S) \widetilde{\to} \lim_{n \ge 0} Y'_n({}^{\le n}S) = \lim_{n \ge 0} RKE(Y'_n)(S)$. So, $Y \widetilde{\to} \lim_{n \ge 0} RKE(Y'_n)$ in PreStk. However, each $RKE(Y'_n) \in \mathrm{Stk}$ by (ch. 2, 2.5.6). Besides, $\mathrm{Stk} \subset \mathrm{PreStk}$ is stable under all limits by my Remark 4.0.45.

10.0.25. For 2.6.1. To see that ${}^L LKE_{\le n \mathrm{Sch}^{aff} \hookrightarrow \mathrm{Sch}^{aff}}$ is fully faithful use (ch. 2, 2.5.7): if $Y \in {}^{\le n}\mathrm{Stk}$ then let $LKE(Y)$ be its LKE under $({}^{\le}\mathrm{Sch}^{aff})^{op} \hookrightarrow (\mathrm{Sch}^{aff})^{op}$. Then

$$^{\le n}(L(LKE(Y))) \widetilde{\to} ({}^{\le n}L)({}^{\le n}(LKE(Y))) \widetilde{\to} ({}^{\le n}L)(Y) \widetilde{\to} Y$$

We are done.

Formula in 2.6.2 is correct, because $\tau^{\le n} : \mathrm{PreStk} \to \mathrm{PreStk}$ is the composition $\mathrm{PreStk} \stackrel{Y \mapsto {}^{\le n}Y}{\to} {}^{\le n}\mathrm{PreStk} \stackrel{LKE}{\to} \mathrm{PreStk}$. Once again, ${}^L\tau^{\le n} : \mathrm{Stk} \to \mathrm{Stk}$ is the functor

$$\mathrm{Stk} \stackrel{Y \mapsto {}^{\le n}Y}{\to} {}^{\le n}\mathrm{Stk} \hookrightarrow {}^{\le n}\mathrm{PreStk} \stackrel{LKE}{\to} \mathrm{PreStk} \stackrel{L}{\to} \mathrm{Stk},$$

where the second arrow is a full subcategory.

10.0.26. For 2.7.1. For $n \ge 0$ realize ${}^{\le n}\mathrm{PreStk}_{lft}$ as $\mathrm{Fun}(({}^{\le n}\mathrm{Sch}^{aff}_{ft})^{op}, \mathrm{Spc})$. Then

$$^{\le n}\mathrm{NearStk}_{lft} \subset \mathrm{Fun}(({}^{\le n}\mathrm{Sch}^{aff}_{ft})^{op}, \mathrm{Spc})$$

is the full subcategory of objects satisfying the descent condition for etale coverings. The corresponding localization functor ${}^{\le n}L_{ft} : {}^{\le n}\mathrm{PreStk}_{lft} \to {}^{\le n}\mathrm{PreStk}_{lft}$ is left exact, so sends k-truncated objects to k-truncated objects.

10.0.27. For (ch. 2, Lm. 2.7.4). Let $S \in {}^{\le n}\mathrm{Sch}^{aff}$ and $S' \to S$ be an etale cover in ${}^{\le n}\mathrm{Sch}^{aff}$, let $\mathcal{Y} \in {}^{\le n}\mathrm{PreStk}_{lft}$ viewed as a functor $({}^{\le n}\mathrm{Sch}^{aff}_{ft})^{op} \to \mathrm{Spc}$. Let $\bar{Y}$ be the RKE of Y under $({}^{\le n}\mathrm{Sch}^{aff}_{ft})^{op} \subset ({}^{\le n}\mathrm{Sch}^{aff})^{op}$. The notation S'^n/S they use means $S' \times_S \ldots \times_S S'$, where S' appear n times, and their S_0^n is not a power of S_0, but an element of ${}^{\le n}\mathrm{Sch}^{aff}_{ft}$. We get a diagram in Spc indexed by $\boldsymbol{\Delta}$ sending $n-1$ to

$$\lim_{S_0^n \to (S'^n/S),\, S_0^n \in {}^{\le n}\mathrm{Sch}^{aff}_{ft}} Y(S_0^n)$$

We have to show that the limit of this diagram is $\lim\limits_{S_0 \to S,\, S_0 \in {}^{\le n}\mathrm{Sch}^{aff}_{ft}} Y(S_0)$.

There is another misprint in the proof: the functor $({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{/S} \to ({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{(S'^n/S)}$, $S_0 \mapsto S_0^n := S_0 \times_S (S'^n/S)$ is cofinal (not the opposite one). They actually need the following:

Lemma 10.0.28. *Given any map $Z \to S$ in ${}^{\le n}\mathrm{Sch}^{aff}$ the map $({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{/S} \to ({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{/Z}$ given by $(S' \to S) \mapsto (S' \times_S Z \to Z)$ is cofinal. (The fact that $Z \to S$ is etale is not needed).*

Proof. Given $(Z' \to Z) \in ({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{/Z}$ let h be the composition $Z' \to Z \to S$. The category

$$\mathcal{X} := ({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{/S} \times_{({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{/Z}} (({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{/Z})_{Z'/}$$

identifies with the category, whose objects are diagrams $Z' \xrightarrow{a} S' \xrightarrow{b} S$ in ${}^{\le n}\mathrm{Sch}^{aff}$ such that $ba = h$ and $S' \in {}^{\le n}\mathrm{Sch}^{aff}_{ft}$. A morphism in this category is a map $S' \to S''$ such that the diagram commutes

$$\begin{array}{ccccc} Z' & \to & S' & \to & S \\ & \searrow & \downarrow & \nearrow & \\ & & S'' & & \end{array}$$

This category is clearly nonempty. It does not admit fibred coproducts in general. Namely, ${}^{\le n}\mathrm{Sch}^{aff}$ has coproducts, this is easy. However, the inclusion ${}^{\le n}\mathrm{Sch}^{aff}_{ft} \subset {}^{\le n}\mathrm{Sch}^{aff}$ is not stable under coproducts! This happens already for $n = 0$.

The problem for $n = 0$ reduces to the following one: given a diagram $A'_2 \overset{\alpha_2}{\leftarrow} A' \overset{\alpha_1}{\to} A'_1$ of finitely generated k-algebras, is it true that $A'_1 \times_{A'} A'_2$ is a finitely generated k-algebra? The answer is no. A more precise claim holds: if α_1, α_2 are injections then $A'_1 \cap A'_2$ is not always finitely generated. An example is found in [3] and also at mathoverflow discussion [40].

However, the category $\mathcal{X}$ is contractible by my Lemma 2.2.105, as it has an initial object. □

10.0.29. For (ch. 2, 2.7.5)(b) follows from (a). Indeed, for $Y \in {}^{\le n}\mathrm{PreStk}$ the map $Y \to ({}^{\le n}L)(Y)$ is an etale equivalence in ${}^{\le n}\mathrm{PreStk}$, so its restriction to ${}^{\le n}\mathrm{Sch}^{aff}_{ft}$ is also an etale equivalence. Besides, $(({}^{\le n}L)(Y)) \mid_{{}^{\le n}\mathrm{Sch}^{aff}_{ft}}$ lies in ${}^{\le n}\mathrm{NearStk}_{lft}$.

10.0.30. For 2.7.7: for $\mathcal{Y} \in {}^{\le n}\mathrm{PreStk}_{lft}$ the notion of 'being k-truncated' is unambiguous. We may think this this means that the functor $\mathcal{Y} : {}^{\le n}\mathrm{Sch}^{aff} \to \mathrm{Spc}$ takes values in $\tau_{\le k}\mathrm{Spc}$, or equivalently, that its restriction $\mathcal{Y}' : {}^{\le n}\mathrm{Sch}^{aff}_{ft} \to \mathrm{Spc}$ factors through $\tau_{\le k}\mathrm{Spc}$.

For 2.7.8: the intersection ${}^{\le n}\mathrm{Stk} \cap {}^{\le n}\mathrm{PreStk}_{lft}$ is taken inside $\mathrm{Fun}(({}^{\le n}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc})$.

10.0.31. For 2.8.1. For $S \in {}^{\le n}\mathrm{Sch}^{aff}$ the category $(({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{S/})^{op}$ is filtered. Indeed, the category $({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{S/}$ has fibred products: if $S_1 \to \bar{S} \leftarrow S_2$ is a diagram in $({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{S/}$ then the fibred product $S_1 \times_{\bar{S}} S_2$ in ${}^{\le n}\mathrm{Sch}^{aff}$ will lie in ${}^{\le n}\mathrm{Sch}^{aff}_{ft}$. Then $S_1 \times_{\bar{S}} S_2$ will be their fibred product in $({}^{\le n}\mathrm{Sch}^{aff}_{ft})_{S/}$ also.

Recall that ${}^{\le n}\mathrm{Sch}^{aff}$ has all limits and colimits. By (HTT, 5.3.4.15), the inclusion $({}^{\le n}\mathrm{Sch}^{aff}_{ft})^{op} \subset ({}^{\le n}\mathrm{Sch}^{aff})^{op}$ is stable under all finite colimits. In particular, ${}^{\le n}\mathrm{Sch}^{aff}_{ft}$ admits all finite limits. See 1.5.3-1.5.4 also.

For ([14], ch. 2, Lm. 2.8.2). In the definition of f_{ft} no condition is imposed on morphisms between diagrams. I think one has to impose the condition that a map in f_{ft} is a morphism from $f' : S_1' \to S_2'$ to $f'' : S_1'' \to S_2''$ such that the square is cartesian

$$\begin{array}{ccc} S_1' & \to & S_1'' \\ \downarrow f' & & \downarrow f'' \\ S_2' & \to & S_2'' \end{array}$$

Then indeed the category f_{ft} has fibred products, hence $(f_{ft})^{op}$ is filtered.

For the first part of the lemma: given an object $(S_1 \xrightarrow{b} S_1')$ in ${}^{\le n}\mathrm{Sch}^{aff}_{ft})_{S_1/}$ there is indeed always an object given by $(S_1'' \to S_2'')$ of f_{ft} such that b factors as $S_1 \to S_1'' \to S_1'$. For $n = 0$ this follows from the next claim: given an etale morphism $A \to B$ of commutative k-algebras, there is an isomorphism $B \widetilde{\to} A[x_1, \ldots, x_d]/(f_1, \ldots, f_d)$ such that $\det(\frac{\partial f_i}{\partial x_j})$ is invertible in B. Indeed, given a finitely generated k-subalgebra in B, it is generated by some finite collection $p_i \in A[x_1, \ldots, x_d]/(f_1, \ldots, f_d)$. Consider then the k-subalgebra A' of A generated by all the coefficients of polynomials f_j and of coefficients of polynomials p_i. Then we may form $A'[x_1, \ldots, x_d]/(f_1, \ldots, f_d)$, and $B' \to B$ factors through it.

At the end of the proof a general claim is used, which I documented as Lemma 2.5.26.

10.0.32. For 2.7.10: If $Y \in {}^{\le n}\mathrm{PreStk}_{lft}$ is k-truncated for some k, the map $Y \to {}^{\le n}L_{ft}(Y)$ is an etale equivalence in ${}^{\le n}\mathrm{PreStk}_{lft}$. By 2.7.3, $LKE(Y) \to LKE({}^{\le n}L_{ft}(Y))$ is an etale equivalence in ${}^{\le n}\mathrm{PreStk}$, where LKE is taken with respect to $({}^{\le n}\mathrm{Sch}^{aff}_{ft})^{op} \to ({}^{\le n}\mathrm{Sch}^{aff})^{op}$. Now $LKE({}^{\le n}L_{ft}(Y)) \in {}^{\le n}\mathrm{Stk}_{lft}$ by 2.7.7, and 2.7.10 follows.

10.0.33. Since $\mathrm{Vect}^{\leq 0}$ admits limits, $CAlg(\mathrm{Vect}^{\leq 0})$ also admits limits by (HA, 3.2.2.5), and the forgetful functor $CAlg(\mathrm{Vect}^{\leq 0}) \to \mathrm{Vect}^{\leq 0}$ commutes with limits. Given a diagram $A_1 \xrightarrow{f_1} A \xleftarrow{f_2} A_2$ in $\mathrm{Vect}^{\leq 0}$, its limit in Vect is the fibre $Fib \to A_1 \oplus A_2 \xrightarrow{f_1,-f_2} A$, it lies in $\mathrm{Vect}^{\leq 1}$. In turn, $\tau^{\leq 0}(Fib)$ is the limit of this diagram in $\mathrm{Vect}^{\leq 0}$.

The category $\mathrm{Vect}^{[-n,0]}$ admits limits. If $f : K \to \mathrm{Vect}^{[-n,0]}$ is a diagram, take first its limit in $\mathrm{Vect}^{\geq -n}$ and the apply the limit-preserving functor $\tau^{\leq 0}$. This gives $\lim f$. This implies by (HA, 3.2.2.5) that $CAlg(\mathrm{Vect}^{[-n,0]})$ admits limits, and the forgetful functor $CAlg(\mathrm{Vect}^{[-n,0]}) \to \mathrm{Vect}^{[-n,0]}$ preserves limits. So, ${}^{\leq n}\mathrm{Sch}^{aff}$ admits colimits.

Recall that $({}^{\leq n}\mathrm{Sch}^{aff}_{ft})^{op} = (({}^{\leq n}\mathrm{Sch}^{aff})^{op})^c$, so the inclusion ${}^{\leq n}\mathrm{Sch}^{aff}_{ft} \subset {}^{\leq n}\mathrm{Sch}^{aff}$ is stable under all finite limits, which exist in ${}^{\leq n}\mathrm{Sch}^{aff}$ by (HTT, 5.3.4.15). All finite limits indeed exist, because $({}^{\leq n}\mathrm{Sch}^{aff})^{op}$ is presentable, see below.

Is the full subcategory ${}^{\leq n}\mathrm{Sch}^{aff}_{ft}$ stable under finite colimits in ${}^{\leq n}\mathrm{Sch}^{aff}$? No, this is not true already for $n = 0$. The example is given in [3], [40]. Namely, if A is a finitely generated k-algebra, $A_i \subset A$ are f.gen. k-subalgebras then $A_1 \times_A A_2$ may be not finitely generated.

The category $\mathrm{Vect}^{[-n,0]}$ is presentable, see Section 10.0.2. If $f : K \to \mathrm{Vect}^{[-n,0]}$ is a small diagram, let $\bar{f} : K^{\triangleright} \to \mathrm{Vect}^{\leq 0}$ be the colimit of f in $\mathrm{Vect}^{\leq 0}$ then $\tau_{\geq -n}\bar{f}$ is a colimit diagram in $\mathrm{Vect}^{[-n,0]}$.

Lemma 10.0.34. *The tensor product in the symmetric monoidal ∞-category* $\mathrm{Vect}^{[-n,0]}$ *preserves small colimits separately in each variable. So, the tensor product in* $\mathrm{Vect}^{[-n,0]}$ *is compatible with small colimits in the sense of (HA, 3.1.1.18).*

Proof. Given $z_i \in \mathrm{Vect}^{[-n,0]}$, write for clarity $z_1 \bar{\otimes} z_2$ for the tensor product in $\mathrm{Vect}^{[-n,0]}$. Let $I \in 1-\mathfrak{C}\mathrm{at}$ be small, $f : I \to \mathrm{Vect}^{[-n,0]}$ be a diagram, $i \mapsto x_i$, and $y \in \mathrm{Vect}^{[-n,0]}$. Let x be the colimit of f in $\mathrm{Vect}^{\leq 0}$, so $\bar{x} := \tau^{\geq -n}x$ is the colimit of f. In Vect the tensor product preserves colimits separately in each variable, so $x \otimes y \widetilde{\to} \mathrm{colim}_i(x_i \otimes y)$, here on the right the colimit is taken in $\mathrm{Vect}^{\leq 0}$. Applying $\tau^{\geq -n}$, we get

$$\tau^{\geq -n}(x \otimes y) \widetilde{\to} \tau^{\geq -n}(\mathrm{colim}_i(x_i \otimes y)) = \mathrm{colim}\, g,$$

where $g : I \to \mathrm{Vect}^{[-n,0]}$, $g(i) = x_i \bar{\otimes} y$. The fibre sequence $(\tau^{<-n}x)\otimes \to x \otimes y \to \bar{x} \otimes y$ in Vect yields an isomorphism $\tau^{\geq -n}(x \otimes y) \widetilde{\to} \tau^{\geq -n}(\bar{x} \otimes y) \widetilde{\to} \bar{x} \bar{\otimes} y$. □

From this we derive using (HA, 3.2.3.5) that both $CAlg(\mathrm{Vect}^{[-n,0]}), CAlg(\mathrm{Vect}^{\leq 0})$ are presentable, so ${}^{\leq n}\mathrm{Sch}^{aff}, \mathrm{Sch}^{aff}$ admit all limits and colimits.

10.0.35. For 2.9.4: to prove 2.9.4, one proves first the following.

Lemma 10.0.36. *The functor* $\mathrm{PreStk}_{laft} \to \mathrm{PreStk}$ *of RKE along* $({}^{<\infty}Sch^{aff}_{ft})^{op} \hookrightarrow (\mathrm{Sch}^{aff})^{op}$ *sends* $\mathrm{NearStk}_{laft}$ *to* Stk.

Proof. Let $S \in \mathrm{Sch}^{aff}$, $Y : ({}^{<\infty}Sch^{aff}_{ft})^{op} \to \mathrm{Spc}$ be a functor, $\bar{Y} : (\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$ be its RKE. One has $\bar{Y}(S) \widetilde{\to} \lim_{S_0 \to S} Y(S_0)$, where the limit is taken over $(S_0 \to S) \in (({}^{<\infty}\mathrm{Sch}^{aff}_{ft})_{/S})^{op}$. Let now $S' \to S$ be an etale cover in Sch^{aff}. We must show that

$\bar{Y}(S) \to \mathrm{Tot}(\bar{Y}(S'^{\cdot}/S))$ is an isomorphism. Write

$$\bar{Y}(S'^n/S) \widetilde{\to} \lim_{(S_0^n \to S'^n/S) \in ((^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/(S'^n/S)})^{op}} Y(S_0^n)$$

By Lemma 10.0.37, the above identifies with

$$\lim_{(S_0 \to S) \in ((^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/S})^{op}} Y(S_0^n),$$

where $S_0^n = S_0 \times_S (S'^n/S) = (S_0')^n/S_0$. Permuting the two limits, we get

$$\mathrm{Tot}(\bar{Y}(S'^{\cdot}/S)) \widetilde{\to} \lim_{(S_0 \to S) \in ((^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/S})^{op}} \mathrm{Tot}(Y((S_0')^n/S_0)) \widetilde{\to} \bar{Y}(S)$$

□

Lemma 10.0.37. *Let $S' \to S$ be a map in Sch^{aff}. Assume that S' is "of finite type" over S. The map $(^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/S} \to (^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/S'}$, $S_0 \mapsto S_0 \times_S S'$ is cofinal.*

Proof. (analogous to Lemma 10.0.28). The assumption S' is "of finite type" over S is needed to conclude that $S_0 \mapsto S_0 \times_S S'$ lies in $^{<\infty}\mathrm{Sch}_{ft}^{aff}$. I don't see a good choice of such definition, but if $S' \to S$ is etale then S' should be of finite type over S. If $S' \to S$ is flat then S' is of finite type over S simply means that $^{cl}S'$ is of finite type over ^{cl}S.

Given $(Z' \to S') \in (^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/S'}$, the category

$$\mathcal{X} := ((^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/S}) \times_{(^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/S'}} ((^{<\infty}\mathrm{Sch}_{ft}^{aff})_{/S'})_{Z'/}$$

is the category whose objects are diagrams $Z' \to S_0 \to S$, whose composition is $h : Z' \to S' \to S$. The category $\mathcal{X}$ has an initial object, hence it is contractible. □

10.0.38. For 2.9.5: let $\mathcal{Y} \in NearStk_{laft}$ viewed as prestack via

$$NearStk_{laft} \subset \mathrm{PreStk}_{laft} \subset \mathrm{PreStk}$$

For any $n \geq 0$, the restriction to $\mathcal{Y}$ to $^{\leq n}\mathrm{Sch}_{ft}^{aff}$ lies in $^{\leq n}NearStk_{lft}$. Assume $\mathcal{Y} \mid_{(^{\leq n}\mathrm{Sch}^{aff})^{op}}$ is k_n-truncated for some $k_n \in \mathbb{Z}$. Then by (ch. I.2, 2.7.7), $^{\leq n}\mathcal{Y}$ lies in $^{\leq n}$ Stk. So, by (ch. I.2, 2.5.9), $\mathcal{Y} \in \mathrm{Stk}$.

10.0.39. For 3.1. Let $f : X \to Y$ be a flat morphism of prestacks, assume $Y \in {}^{\leq n}\mathrm{PreStk}$ for some $n \geq 0$. Then $X \in {}^{\leq n}\mathrm{PreStk}$ also. Indeed, viewing Y as an object of $\mathrm{Fun}((^{\leq n}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc})$, we have $Y \widetilde{\to} \mathrm{colim}_{S \to Y}\, j(S)$, the colimit taken in $\mathrm{Fun}((^{\leq n}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc})$ over the category $(^{\leq n}\mathrm{Sch}^{aff})_{/Y}$. The functor say L of LKE along $(^{\leq n}\mathrm{Sch}^{aff})^{op} \to (\mathrm{Sch}^{aff})^{op}$ preserves colimits. So, in PreStk we get

$$LY \widetilde{\to} \mathrm{colim}_{S \to Y}\, j(\bar{L}(S)),$$

where $\bar{L} : {}^{\leq n}\mathrm{Sch}^{aff} \to \mathrm{Sch}^{aff}$ is the inclusion. We used here the fact that $L(j(S)) \widetilde{\to} j(\bar{L}(S))$ from my Section 10.0.4. To simplify, we write $LY \widetilde{\to} \mathrm{colim}_{S \to Y}\, S$ for the colimit in PreStk. Since the colimits in PreStk are universal, we get

$$X \times_Y (\mathrm{colim}_{S \to Y}\, S) \widetilde{\to} \mathrm{colim}_{S \to Y}(X \times_Y S)$$

However, $X \times_Y S \in {}^{\leq n}\mathrm{Sch}^{aff}$, so lies in ${}^{\leq n}\,\mathrm{PreStk}$ viewed as a full subcategory of PreStk. Since ${}^{\leq n}\,\mathrm{PreStk} \subset \mathrm{PreStk}$ is closed under colimits, our claim follows.

Remark 10.0.40. *Let $f : Y \to Z$ be a flat morphism in* PreStk*. Then for $n \geq 0$ one has $Y \times_Z \tau^{\leq n} Z \widetilde{\to} \tau^{\leq n}(Y)$.*

Proof. By the above section, $Y \times_Z \tau^{\leq n} Z \in {}^{\leq n}\,\mathrm{PreStk}$. So, it suffices to show that restricting both side to the category $({}^{\leq n}\mathrm{Sch}^{aff})^{op}$ we get the same functor. This is true because this restriction preserves limits. □

10.0.41. Question: let $Y \in \mathrm{PreStk}$. Is it true that the functor $X \mapsto {}^{cl}X$ induces an equivalence of the full subcategory of $\mathrm{PreStk}_{/Y}$ spanned by $f : X \to Y$ with f etale and the full subcategory of ${}^{cl}\,\mathrm{PreStk}_{/{}^{cl}Y}$ spanned by $\tilde{f} : \tilde{X} \to {}^{cl}Y$ with $\tilde{f}$ etale? A similar claim for affine schemes is Lemma 2.1.5.

10.0.42. For 3.1.4. If $Z \in \mathrm{Sch}$ then ${}^{cl}Z$ is a classical scheme. Indeed, let $f_i : S_i \to Z$ be a Zariski atlas. Then ${}^{cl}S_i \to {}^{cl}Z$ is open immersion, and the union of ${}^{cl}S_i$ is ${}^{cl}Z$.

Besides, ${}^{cl}Z$ is separated.

10.0.43. For 3.1.8. Here $T \in \mathrm{Sch}^{aff}$. If $Z_i \to T$ are etale morphisms of classical schemes and $h : Z_1 \to Z_2$ is a map over T then h is etale. So, if ${}^{cl}T \to {}^{cl}S'$ is a lift of ${}^{cl}T \to {}^{cl}S$ then the induced map ${}^{cl}T \to {}^{cl}S' \times_{{}^{cl}S} {}^{cl}T$ is etale, hence comes from the desired etale map $T \to S' \times_S T$.

Case (b) is similar: since $Z \to Z'$ is an affine Zariski map, $Z' \times_Z T \to T$ is also an affine Zariski map, and $Z' \times_Z T \in \mathrm{Sch}^{aff}$.

10.0.44. For 3.2.4. A general remark: let $T^i \in \mathrm{Sch}^{aff}, S \in \mathrm{Sch}^{aff}$. Consider a map $S \to \sqcup_i T^i$ in PreStk, where the coproduct is understood in PreStk. Then it factors through $S \to T^i$ for some i by (HTT, 5.3.4.17), as $T \in (\mathrm{PreStk})^c$. The category Sch^{aff} has an initial object, the empty affine scheme. It corresponds to the object zero vector space $\{0\} \in CAlg(\mathrm{Vect}^{\leq 0})$. Now for any j the natural map $T^j \to \sqcup_i T^i$ is an affine open embedding of prestacks. Indeed, for any $S \to \sqcup_i T^i$ with $S \in \mathrm{Sch}^{aff}$, its base change by S is either $\emptyset \to S$ or $\mathrm{id} : S \to S$.

By $\mid S^\bullet \mid$ they mean the geometric realization in PreStk. The coproducts $\sqcup_i S^0_i, \sqcup_j S^1_j$ are understood in PreStk. The use the following:

Remark 10.0.45. *Let $E^\cdot$ be a simplicial object in* Spc*, $E = \mid E \mid$ in* Spc*, $e \in E$. Then there is $e_0 \in E^0$ whose image in E is isomorphic to e.*

Proof. $\pi_0 : \mathrm{Spc} \to \mathrm{Sets}$ preserves colimits. Any element of $\mathrm{colim}_{[n] \in \boldsymbol{\Delta}^{op}} \pi_0(E^n)$ is represented by some element of $\pi_0(E^0)$. □

In their proof $T \in \mathrm{Sch}^{aff}$ is any. Then any element of $\mathrm{Map}(T, \mid S^\cdot \mid)$ comes from a map $T \to S^0$. Further they use the fact that any groupoid in PreStk is effective, as it is an ∞-topos. So, the precise assumption in this proposition says that for any i, $\mathrm{pr}_0 : S^1 \times_{\mathrm{pr}_1, S^0} S^0_i \to S^0$ is an open embedding, where $\mathrm{pr}_0, \mathrm{pr}_1 : S^1 \to S_0$ are the two maps in our groupoid.

In version May 4, 2020 of their ch. I.2 there is a mistake in the formulation of Pp. 3.2.4, the assumption has to be corrected as above.

10.0.46. Let $Z \in \text{PreStk}$. We have the functor sending $U \to Z$ to ${}^{cl}U \to {}^{cl}Z$ from the full subcategory of $\text{PreStk}_{/Z}$ spanned by the open immersions to the full subcategory of ${}^{cl}\,\text{PreStk}_{/{}^{cl}Z}$ spanned by open immersions. Is it an equivalence? Not sure, as we can not really glue open pieces of Z. But maybe this is true if $Z \in \text{Stk}$. For scheme the answer is given in their Cor. 3.2.6.

10.0.47. For 3.2.6. The functor $\tau^{\geq 0} : CAlg(\text{Vect}^{\leq 0}) \to CAlg(\text{Vect}^{\heartsuit})$ preserves limits by my Section 9.1. So, for $Y_i \in \text{Sch}^{aff}$ we get ${}^{cl}(\sqcup_i Y_i) \widetilde{\to} \sqcup_i {}^{cl}Y_i$, where we understand $\sqcup$ as coproduct in Sch^{aff} and in ${}^{cl}\text{Sch}^{aff}$ respectively.

The property of a morphism $Z' \to Z$ in Sch^{aff} to be affine Zariski is local in Zariski topology of the target. This is mentioned in their (ch. I.2, Section 2.1.7).

If $f : Y \to Z$ is an affine Zariski morphism in Sch then ${}^{cl}Y \to {}^{cl}Z$ is also affine Zariski morphism. Conversely, assume $h : \tilde{Y} \to {}^{cl}Z$ is an affine Zariski morphism of classical schemes. Pick Zariski atlas $Z_i \to Z$ of Z, $i \in I$. Let $\tilde{Y}_i = \tilde{Y} \times_{{}^{cl}Z} {}^{cl}Z_i$, the preimage in usual classical schemes. Then $\tilde{Y}_i \to {}^{cl}Z_i$ is an affine Zarizki map. So, by their Lemma 2.1.5, it lifts to an affine Zariski map $Y_i \to Z_i$ in Sch^{aff}, and $\tilde{Y}_i = {}^{cl}Y_i$. How to create the corresponding groupoid object $S^\cdot$ to produce Y via Prop. 3.2.4? One has to take $S^0 = \sqcup_i Y_i$. "On the classical level", we should take the Cech nerve of $\sqcup_i {}^{cl}Y_i \to {}^{cl}Y$. It indeed lifts to a groupoid object with values in affine scheme, because of Lemma 2.1.5: at each step we lift an open embedding. For example, if $I = \{1, 2\}$, we have $S^0 = Y_1 \sqcup Y_2$, and we need to define open embeddings $Y_{12} \hookrightarrow Y_i$ for $i = 1, 2$ in particular. They come from the open embeddings on the level of classical affine schemes ${}^{cl}Y_1 \times_{{}^{cl}Y} {}^{cl}Y_2 \hookrightarrow {}^{cl}Y_i$. Then $Y = L(\mid S^\cdot \mid)$. Do we have ${}^{cl}Y = \tilde{Y}$?

By their Cor. 2.5.7, ${}^{\leq 0}L({}^{\leq 0}(\mid S^\cdot \mid)) \widetilde{\to} {}^{\leq 0}Y$, and the functor

$$\text{PreStk} \to \text{Fun}(({}^{cl}\text{Sch}^{aff})^{op}, \text{Spc}),\ U \mapsto {}^{\leq 0}U$$

preserves colimits, so ${}^{\leq 0}(\mid S^\cdot \mid)$ is the geometric realization of the Cech nerve of $\sqcup_i {}^{cl}Y_i \to {}^{cl}Y$. So, ${}^{\leq 0}L({}^{\leq 0}(\mid S^\cdot \mid))$ identifies with ${}^{cl}Y$ by their Lemma 3.1.6. So, the assumptions of Prop. 3.2.4 are verified.

10.0.48. By definition, ${}^{\leq n}\text{Sch}$ is the following full subcategory of ${}^{\leq n}\,\text{Stk}$. First, for $F, F' \in {}^{\leq n}\,\text{PreStk}$ and a map $\alpha : F \to F'$ say that α is affine schematic if for any $S \in {}^{\leq n}\text{Sch}^{aff}$ and $S \to F$, $F \times_{F'} S$ is represented by some object of ${}^{\leq n}\text{Sch}^{aff}$. Similarly, for an affine schematic map $F \to F'$ as above, one defines a notion of being flat, etale, ppf, open immersion, Zariski. Viewing ${}^{\leq n}\,\text{Stk} \subset \text{Fun}(({}^{\leq n}\text{Sch}^{aff})^{op}, \text{Spc})$ as a full subcategory, say that $Z \in {}^{\leq n}\,\text{Stk}$ lies in ${}^{\leq n}\text{Sch}$ if the diagonal map $Z \to Z \times Z$ is affine schematic, and for any $T \in {}^{\leq n}\text{Sch}^{aff}{}_{/Z\times Z}$ the induces map ${}^{cl}(T \times_{Z\times Z} Z) \to {}^{cl}T$ is a closed immersion.

Besides, it is required that there is a collection of $S_i \in {}^{\leq n}\text{Sch}^{aff}$ and maps $f_i : S_i \to Z$ such that

- each f_i (which is affine schematic by the above) is an open embeddings;
- for any $T \in {}^{\leq n}\text{Sch}^{aff}_{/Z}$, the images of ${}^{cl}(T \times_Z S_i) \to {}^{cl}T$ cover ${}^{cl}T$.

So, this definition "does not know" about the existence of bigger categories than ${}^{\leq n}\text{Sch}$ as Sch^{aff}.

Remark. if $Y \to Y'$ is an affine schematic morphism in PreStk (resp., ppf, smooth, etale, open embedding, Zariski) then ${}^{\le n}Y \to {}^{\le n}Y'$ is affine schematic in ${}^{\le n}\operatorname{PreStk}$ (resp., ppf, smooth, etale, open embedding, Zariski).

Moreover, the functor $\operatorname{Sch} \to {}^{\le n}\operatorname{PreStk}$, $Y \mapsto {}^{\le n}Y$ sends Sch to ${}^{\le n}\operatorname{Sch}$. If $S_i \to Z$ is a Zariski cover of Z then ${}^{\le n}S_i \to {}^{\le n}Z$ is a Zarizki cover of ${}^{\le n}Z$.

10.0.49. *Definition.* Let $f : Y \to Y'$ be a morphism in ${}^{\le n}\operatorname{PreStk}$. Say that it is schematic (resp., schematic flat, schematic ppf, schematic smooth, schematic etale, schematic Zariski) if for any $S \in {}^{\le n}\operatorname{Sch}^{aff}$ and a map $S \to Y'$ the base change $S \times_{Y'} Y \in {}^{\le n}\operatorname{PreStk}$ lies in ${}^{\le n}\operatorname{Sch}$ (resp., and for any Zarizki cover $T_i \to S \times_{Y'} Y$ the map $T_i \to S$ has the corresponding property).

Remark. If $f : Y \to Y'$ is a schematic morphism in PreStk (resp., schematic flat, schematic ppf, schematic smooth, schematic etale, schematic Zariski) then ${}^{\le n}f : {}^{\le n}Y \to {}^{\le n}Y'$ is a schematic morphism in ${}^{\le n}\operatorname{PreStk}$ (resp., schematic flat, schematic ppf, schematic smooth, schematic etale, schematic Zariski).

10.0.50. For 3.3.3. If $Z \in {}^{cl}\operatorname{Sch}$ and $T \in {}^{cl}\operatorname{Sch}^{aff}$ then $Z(T) = \operatorname{Map}_{\operatorname{PreStk}}(T, Z)$ is a set, so is 0-truncated in Spc. Now if $Z \in {}^{\le n}\operatorname{Sch}$ they claim that Z is n-truncated as an object of ${}^{\le n}\operatorname{PreStk}$ actually (a misprint in the formulation).

It suffices to show that for $T \in {}^{\le n}\operatorname{Sch}^{aff}$, $\operatorname{Map}(T, Z)$ is n-truncated in Spc. We have a map $\alpha : \operatorname{Map}(T, Z) \to \operatorname{Map}({}^{cl}T, Z)$, and $\operatorname{Map}({}^{cl}T, Z) \,\tilde{\to}\, \operatorname{Map}_{{}^{cl}\operatorname{PreStk}}({}^{cl}T, {}^{cl}Z)$ is 0-truncated. so, it suffices to show that each fibre of α is n-truncated indeed.

10.0.51. For 3.3.5. Let $Z \in {}^{\le n}\operatorname{Sch}$. Write for brevity $F(Z) = {}^{L}LKE(Z)$, where the LKE is along $({}^{\le n}\operatorname{Sch}^{aff})^{op} \hookrightarrow (\operatorname{Sch}^{aff})^{op}$. We have to show that $F(Z) \in \operatorname{Sch}$.

Let $Z' = \sqcup_i S_i \to Z$ be a Zariski atlas with $S_i \in {}^{\le n}\operatorname{Sch}$. Consider the Cech nerve $Z'^{\bullet}/Z$. For each $n \ge 0$, $Z'^{n}/Z \in {}^{\le n}\operatorname{Sch}^{aff}$. An analog of (ch. I.2, Lemma 3.1.6) holds for the category ${}^{\le n}\operatorname{Sch}$ and gives $Z \,\tilde{\to}\, ({}^{\le n}L)(\mid Z'^{\bullet}/Z \mid_{\le n\operatorname{PreStk}})$ in ${}^{\le n}\operatorname{Stk}$. The functor $F : {}^{\le n}\operatorname{PreStk} \to \operatorname{PreStk}$ sends etale equivalences to isomorphisms. We get $F(Z) \,\tilde{\to}\, L(\mid F(Z'^{\bullet}/Z) \mid_{\operatorname{PreStk}})$. Since $Z'^{n}/Z \in {}^{\le n}\operatorname{Sch}$, $F(Z'^{n}/Z) \,\tilde{\to}\, Z'^{n}/Z$ for each n. From Cor. 2.5.7 we get ${}^{\le n}(F(Z)) \,\tilde{\to}\, Z$ naturally, in particular, ${}^{cl}F(Z) \,\tilde{\to}\, {}^{cl}Z$ is a classical scheme. The assumptions of (ch. I.2, Prop. 3.2.4) are verified, so $F(Z) \in \operatorname{Sch}$, and $Z' \to F(Z)$ is a Zariski atlas of Z.

10.0.52. For 3.3.6. We get a full embedding ${}^{\le n}\operatorname{Sch} \subset \operatorname{Sch}$. In (ch. I.2, 2.6.2) they defined a full embedding ${}^{\le n}\operatorname{Stk} \subset \operatorname{Stk}$. Prop. 3.3.5 allows to see ${}^{\le n}\operatorname{Sch}$ as a full subcategory of Sch, namely the image of the composition ${}^{\le n}\operatorname{Sch} \subset {}^{\le n}\operatorname{Stk} \subset \operatorname{Stk}$.

10.0.53. For 3.3.8. In the proof the following should be added. The colimit $\operatorname{colim}_{a \in A} S_a$ is understood in ${}^{\le n}\operatorname{PreStk}$ first, that is, in $\operatorname{Fun}(({}^{\le n}\operatorname{Sch}^{aff})^{op}, \operatorname{Spc})$. This is actually $V^{\cdot}/{}^{\le n}Z$, where $V = \sqcup S_i$, and $S_i \to Z$ is a Zariski atlas with $S_i \in {}^{\le n}\operatorname{Sch}^{aff}$. So, each $V^m \in {}^{\le n}\operatorname{Sch}^{aff}$. Since $Z \in \operatorname{Stk}$, ${}^{\le n}Z \in {}^{\le n}\operatorname{Stk}$. Further, we need an analog of (ch. I.2, Lm. 2.3.8) for ${}^{\le n}\operatorname{Sch}^{aff}$. Namely, the natural map

$$\mid V^{\cdot}/{}^{\le n}Z \mid_{\le n\operatorname{PreStk}} \to {}^{\le n}Z$$

becomes an isomorphism after applying ${}^{\le n}L$, that is, is an etale equivalence. Apply (ch. I.2, 2.5.5(b)) to see that

$$LKE(\mid V^{\cdot}/^{\le n}Z \mid_{\le n \, \mathrm{PreStk}}) \to LKE({}^{\le n}Z) \tag{18}$$

is an etale equivalence, here LKE is with respect to $({}^{\le n}\mathrm{Sch}^{aff})^{op} \to (\mathrm{Sch}^{aff})^{op}$. The left hand side in (18) identifies with

$$\mid V^{\cdot}/^{\le n}Z \mid_{\mathrm{PreStk}}$$

Thus, applying L to (18) we get

$$Z \widetilde{\to} L(\mid V^{\cdot}/^{\le n}Z \mid_{\mathrm{PreStk}}) \widetilde{\to} L(LKE({}^{\le n}Z)),$$

the first isomorphism being given by (ch. I.2, 2.3.8), as $Z \in \mathrm{Stk}$. So, $Z \in {}^{\le n}\mathrm{Stk}$ indeed.

10.0.54. For (ch. I.2, 3.4.4). Given $S \in \mathrm{Sch}^{aff}$, $\mathrm{colim}_{n\ge 0}\, \tau^{\le n}S \widetilde{\to} S$ in Sch^{aff}, however not in PreStk, precisely because there are non-convergent prestacks. The Yoneda $\mathrm{Sch}^{aff} \hookrightarrow \mathrm{PreStk}$ does not preserve colimits.

The maps $S_{i,n} \to {}^{\le n}Z$ are obtained from (ch. I.2, Corollary 3.2.6). Besides, $S_{i,n} \in {}^{\le n}\mathrm{Sch}^{aff}$ by Prop. 3.3.8. Why $S_i \to Z$ is an affine open embedding? This follows from Lemma 10.0.56 below.

Remark 10.0.55. *Let $Y \in {}^{conv}\mathrm{PreStk}$, assume for any $n \ge 0$, ${}^{\le n}Y \in {}^{\le n}\mathrm{Sch}^{aff}$. Then $Y \in \mathrm{Sch}^{aff}$.*

Proof. The compatible system $\tau^{\le 0}Y \to \tau^{\le 1}Y \to \dots$ of affine schemes has a colimit in Sch^{aff}, let $S \in \mathrm{Sch}^{aff}$ be this colimit. Then S is convergent. For $T \in \mathrm{Sch}^{aff}$ we get

$$\mathrm{Map}(\tau^{\le n}T, S) \widetilde{\to} \mathrm{Map}(\tau^{\le n}T, \tau^{\le n}S) \widetilde{\to} \mathrm{Map}_{\le n\, \mathrm{PreStk}}({}^{\le n}T, {}^{\le n}Y) \widetilde{\to} \mathrm{Map}(\tau^{\le n}T, Y)$$

Passing to the limit over n, we get $\mathrm{Map}(T, S) \widetilde{\to} \mathrm{Map}(T, Y)$. □

Lemma 10.0.56. *Let $f : Z' \to Z$ be a map in ${}^{conv}\mathrm{PreStk}$. Assume that for any n the induced map ${}^{\le n}Z' \to {}^{\le n}Z$ in $\mathrm{Fun}(({}^{\le n}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc})$ is affine schematic (resp., flat, etale, smooth, open embedding). Then f is affine schematic (resp., flat, etale, smooth, open embedding).*

Proof. Let $T \in \mathrm{Sch}^{aff}$, $T' = T \times_Z Z'$ in PreStk. Then T' is convergent, since ${}^{\mathrm{conv}}\mathrm{PreStk} \subset \mathrm{PreStk}$ is closed under small limits (being a localization). By Remark 10.0.55, it suffices to show that ${}^{\le n}T' = {}^{\le n}T \times_{\le n Z} {}^{\le n}Z'$ lies in ${}^{\le n}\mathrm{Sch}^{aff}$. This follows from the assumption on f.

Assume in addition that each map ${}^{\le n}Z' \to {}^{\le n}Z$ is flat. Then ${}^{\le n}T' \to {}^{\le n}T$ is flat for any n. So, T' is flat over T by (ch. I.2, 2.1.3).

The arguments for etale, smooth, open embedding are clear now by definition. □

10.0.57. For 3.5.1. Their definition of ${}^{\le n}\mathrm{Sch}_{lft}$ is badly given. Recall that ${}^{\le n}\mathrm{Stk}_{lft} \subset \mathrm{Fun}(({}^{\le n}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc})$ was defined in (ch. I.2, 2.7.8) as ${}^{\le n}\mathrm{Stk} \cap {}^{\le n}\mathrm{PreStk}_{lft}$. They first view ${}^{\le n}\mathrm{Stk}_{lft}$ as a full subcategory of Stk vie the inclusions ${}^{\le n}\mathrm{Stk}_{lft} \subset {}^{\le n}\mathrm{Stk} \subset \mathrm{Stk}$, where the second inclusion is that of (ch. I.2, 2.6.2). So, the intersection ${}^{\le n}\mathrm{Sch}_{lft} = \mathrm{Sch} \cap {}^{\le n}\mathrm{Stk}_{lft}$ is taken inside Stk.

If we view ${}^{\leq n}\mathrm{Sch}$ as a full subcategory in $\mathrm{Fun}(({}^{\leq n}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc})$ then its full subcategory ${}^{\leq n}\mathrm{Sch}_{lft} \subset {}^{\leq n}\mathrm{Sch}$ consists of $Z \in {}^{\leq n}\mathrm{Sch}$ such that $Z : ({}^{\leq n}\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$, Z is a LKE under $({}^{\leq n}\mathrm{Sch}^{aff}_{ft})^{op} \subset ({}^{\leq n}\mathrm{Sch}^{aff})^{op}$ of its own resriction.

The category Stk_{laft} was defined in (ch. I.2, 2.9.6), the intersection $\mathrm{Sch}_{laft} = \mathrm{Sch} \cap \mathrm{Stk}_{laft}$ takes place in Stk.

10.0.58. For the proof of 3.5.3. Let $Z : ({}^{\leq n}\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$ lie in ${}^{\leq n}\mathrm{Sch}$. Assume it has a Zariski atlas consisting of elements $S_i \in {}^{\leq n}\mathrm{Sch}^{aff}_{ft}$. Then in their formula $Z \widetilde{\to} ({}^{\leq n}L)(\mathrm{colim}_{a\in A} S_a)$ it is understood that

$$Z \widetilde{\to} ({}^{\leq n}L)(|\ (\sqcup_i S_i)^{\cdot}/Z\ |_{\leq n\ \mathrm{PreStk}})$$

Since all $S_a \in {}^{\leq n}\mathrm{PreStk}_{lft}$, $\mathrm{colim}_{a\in A} S_a \in {}^{\leq n}\mathrm{PreStk}_{lft}$, because ${}^{\leq n}\mathrm{PreStk}_{lft}$ admits all small colimits. Si, indeed by their Cor. 2.7.10 we see that Z is the LKE from $({}^{\leq n}\mathrm{Sch}^{aff}_{ft})^{op}$.

The next part of the prove uses the following.

Lemma 10.0.59. *Let $A \in 1-\mathcal{C}\mathrm{at}$ be filtered, $A \times [1] \to \mathrm{Spc}$ be the functor sending $a \in A$ to a monomorphism $X_a \subset Y_a$ in* Spc. *Then $X = \mathrm{colim}_a X_a \to \mathrm{colim}_a Y_a = Y$ is a monomorphism, that is, (-1)-truncated.*

Proof. By assumption, $X_a \to X_a \times_{Y_a} X_a$ is an isomorphism for any a (cf. HTT, 5.5.6.15). By (HTT, 5.3.3.3), $\mathrm{colim}_a(X_a \times_{Y_a} X_a) \widetilde{\to} X \times_Y X$, so the map $X \to X \times_Y X$ is an isomorphism, so $X \to Y$ is a monomorphism. □

Lemma 10.0.60. *Let $A \to B$ be a map in $CAlg(\mathrm{Vect}^{[-n,0]})$ with $\mathrm{Spec}\, A \in {}^{\leq n}\mathrm{Sch}^{aff}$ of finite type such that $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is an open embedding. Then B is of finite type.*

Proof. For $n = 0$ we know this. Now for any $i < 0$, $H^{-i}(B) \widetilde{\to} H^0(B) \otimes_{\mathrm{H}^0(A)} \mathrm{H}^{-i}(A)$. Since $H^{-i}(A)$ is a finite type $\mathrm{H}^0(A)$-module, the same holds for B. □

10.0.61. For the proof of 3.6.2. They use the following general observation.

Lemma 10.0.62. *If $Z' \to Z$ is a map in* PreStk *such that for any $S \in \mathrm{Sch}^{aff}$ and $S \to Z$, $S \times_Z Z'$ is a stack, then Z' itself is a stack.*

Proof. Let $T \to S$ be an étale cover with $S, T \in \mathrm{Sch}^{aff}$. We must show that the map $\beta : Z'(S) \to Tot(Z'(T^{\cdot}/S))$ is an isomorphism. This is a map over $Z(S)$. Pick a morphism $\alpha \in Z(S)$. It suffices to show that the fibre of β over α is an isomorphism. Let $S' = Z' \times_Z S$, this is a stack. The fibre of $Z'(S)$ over α is $S'(S) \times_{S(S)} \{\mathrm{id}\}$. The fibre of $Tot(Z'(T^{\cdot}/S))$ over α is

$$Tot(Z'(T^{\cdot}/S) \times_{Z(T^{\cdot}/S)} \{\alpha\}) \widetilde{\to} Tot(S'(T^{\cdot}/S) \times_{S(T^{\cdot}/S)} *)$$

Since $\mathrm{Tot}(S'(T^{\cdot}/S)) \widetilde{\to} S'(S)$ and $Tot(S(T^{\cdot}/S)) \widetilde{\to} S(S)$, the latter space identifies with $S'(S) \times_{S(S)} \{\mathrm{id}\}$. □

One more thing: for a morphism of classical schemes $Z' \to Z$ the property of Z' being separeted is local in Zariski topology of Z: if there is a Zariski atlas $S_i \to Z$ such that $Z' \times_Z S_i$ is separated then Z' is separated. This is used in the proof of Prop. 3.6.2: namely we apply Prop. 3.2.2 to get it.

10.0.63. For 3.6.4. Let for example $f : Y_1 \to Y_2$ in ${}^{conv}\mathrm{PreStk}$ be such that for any $S \in {}^{<\infty}\mathrm{Sch}^{aff}$ and $S \to Y_2$, $S \times_{Y_2} Y_1$ is a scheme. Then f is schematic. Indeed, for any $S \to Y_2$ with $S \in\in \mathrm{Sch}^{aff}$ we get $S \times_{Y_2} Y_1$ is convergent, as ${}^{conv}\mathrm{PreStk}$ is closed under limits in PreStk. Besides, for $n \geq 0$, ${}^{\leq n}(S \times_{Y_2} Y_1) \widetilde{\to} {}^{\leq n}({}^{\leq n}S \times_{Y_2} Y_1)$. Since ${}^{\leq n}S \times_{Y_2} Y_1 \in \mathrm{Sch}$ by assumption, we are done by my Section 10.0.48.

10.0.64. For 4.1.1. The category $CAlg(\mathrm{Vect}^{\leq 0})$ admits all small limits, and the projection $CAlg(\mathrm{Vect}^{\leq 0}) \to \mathrm{Vect}^{\leq 0}$ preserves limits by (HA, 3.2.2.4). We used the fact that $\mathrm{Vect}^{\leq 0}$ is presentable. So, Sch^{aff} admits all small colimits. So, given $S_i \in \mathrm{Sch}^{aff}$ with $S_i = \mathrm{Spec}\, A_i$, one may consider the coproduct $\sqcup_i S_i = \mathrm{Spec}(\prod A_i)$ in Sch^{aff}. In their formula $L(\sqcup_i S_i)$ one can not remove L, because the coproduct $\sqcup_i S_i$ is understood in PreStk, and the inclusion $\mathrm{Sch}^{aff} \hookrightarrow \mathrm{PreStk}$ does not preserve colimits.

It is understood that $\emptyset \to T$ is flat (an open embedding) for any $T \in \mathrm{Sch}^{aff}$. So, $0 \in CAlg(\mathrm{Vect}^{\heartsuit})$ is a flat A-module for any $A \in CAlg(\mathrm{Vect}^{\heartsuit})$. For $S_i \in \mathrm{Sch}^{aff}$, $S_i \to \sqcup_j S_j$ is an affine schematic, open embedding, so $S_i \to L(\sqcup_j S_j)$ is also affine schematic open embedding by (ch. I.2, 2.4.6).

To check that $L(\sqcup_j S_j) \in \mathrm{Sch}$ let us show that for $T \in \mathrm{Sch}^{aff}$ with $T \overset{\alpha}{\to} L(\sqcup_j S_j)$ the images of ${}^{cl}(T_i)$ cover ${}^{cl}T$. Here $T_i = S_i \times_{L(\sqcup_j S_j)} T$. Using (ch. I.2, 2.3.10), pick an etale cover $T' \overset{\beta}{\to} T$ with $T' \in \mathrm{Sch}^{aff}$ such that $\alpha\beta$ factors as $T' \to S_j \to L(\sqcup_j S_j)$ for some j. This shows that we may assume $T = S_j$ and α is the canonical map $S_j \to L(\sqcup_j S_j)$. In this case $S_i \times_{L(\sqcup_j S_j)} S_j \widetilde{\to} S_i$ for $i = j$ and empty otherwise.

10.0.65. The notion of 0-*representable morphism* in PreStk from (ch. I.2, 4.1.2) is as follows. A map $f : X \to Y$ of prestacks is 0-representable iff for any $S \to Y$ with $S \in \mathrm{Sch}^{aff}$, $S \times_Y X$ is in Stk^{0-Artn}, that is, of the form $L(\sqcup_{i\in I} S_i)$. Here $S_i \in \mathrm{Sch}^{aff}$, the coproduct is taken in PreStk, and L is the sheafification. Moreover, the 0-representable morphism f is flat (resp., ppf, smooth, etale, surjective) if the morphism $S \times_Y X \to S$ of schemes is flat (resp., ppf, smooth, etale, surjective). Surjectivity of a morphism $Z \to Z'$ in Sch means, I think, that ${}^{cl}Z \to {}^{cl}Z'$ is surjective.

10.0.66. Let $Y_1, Y_2 \in \mathrm{Stk}$, $Z \in \mathrm{Sch}^{aff}$. Assume given a map $a : Z \to L(Y_1 \sqcup Y_2)$. Then there is an etale cover $Z' \to Z$ with $Z' \in \mathrm{Sch}^{aff}$ such that the restriction to Z' factors through $Y_i \to L(Y_1 \sqcup Y_2)$ for some i. Is it true that a factors through $Z \to Y_1 \to L(Y_1 \sqcup Y_2)$?

To set up induction in the definition of Arthin stacks in (ch. I.2, 4.1), let us show that the composition of 0-representable morphisms is 0-representable. Let $Y \overset{\alpha}{\to} L(\sqcup_i S_i) \to S$ be diagram, where $S_i \in \mathrm{Sch}^{aff}$, and α is 0-representable (in particular, $Y \in \mathrm{Sch}$). By assumption, for each i there is an isomorphism $Y \times_{L(\sqcup_i S_i)} S_i \widetilde{\to} L(\sqcup_j S_{ij})$ for some $S_{ij} \in \mathrm{Sch}^{aff}$. Recall that $L(\sqcup_j S_{ij}) \in \mathrm{Sch}$, and the collection $S_{ij} \to L(\sqcup_j S_{ij})$ is itz Zarizki cover. Then the collection $S_{ij} \to Y$ is an affine open embedding, this is a Zariski cover. Besides, for two different pairs (i,j), (i',j') one has $S_{ij} \times_Y S_{i'j'} = \emptyset$. This should imply $Y \widetilde{\to} L(\sqcup_{i,j} S_{ij})$.

More generally, if $Z \in \mathrm{Sch}$, $S_i \to Z$, $i \in I$ is a Zariski cover, assume that for $i \neq j$, $S_i \times_Z S_j = \emptyset$. Then I think $Z \widetilde{\to} L(\sqcup_i S_i)$.

10.0.67. For (ch. I.2, 4.2.2 a)). My understanding here is as follows. Let $f : Y_1 \to Y_2$ in PreStk be k-representable. Let $S \to Y_1 \times_{Y_2} Y_1$ be a map with $S \in \mathrm{Sch}^{aff}$. We get the map $S \to Y_2$, and may base change f by this map, so we may assume $Y_2 = S$. Then the datum of $S \to Y_1 \times_S Y_1$ yields in particular a map $a : S \to Y_1 \times Y_1$. Let $Z = S \times_{Y_1 \times Y_1} Y_1$. My understanding is that $Z \widetilde{\to} S \times_{Y_1 \times_{Y_2} Y_1} Y_1$. Since now Y_1 is a k-Artin stack, $Y_1 \to Y_1 \times Y_1$ is $(k-1)$-representable, so Z is a $(k-1)$-Artin stack.

10.0.68. Proof of (ch. I.2, 4.2.4) contains in particular the following claim: if Y_i are k-Artin stacks for $k \geq 0$ then $L(\sqcup_i Y_i)$ is also a k-Artin stack. Proof not clear.

(ch. I.2, Lm. 4.3.2) gives in particular: if $S' \to S$ is a morphism in Sch^{aff}, which is smooth and surjective then there is an etale cover $T \to S$ such that $T \to S$ factors through S'.

(ch. I.2, 4.3.3) follows from 4.3.2 and 2.3.8.

Lemma 10.0.69. *Let* $Y \in \mathrm{Stk}$*,* $S_i \in \mathrm{Sch}^{aff}$*. Assume given a map* $Y \to L(\sqcup_i S_i)$*, where the coproduct is in* PreStk*. Let* $Y_i = Y \times_{L(\sqcup_i S_i)} S_i$*. Then the natural map* $L(\sqcup_i Y_i) \to Y$ *is an isomorphism.*

Proof. Let $Y' = Y \times_{L(\sqcup_i S_i)} \sqcup_i S_i$. Then $Y' \times_{\sqcup_i S_i} S_i \widetilde{\to} Y_i$, hence $Y' \widetilde{\to} \sqcup_i Y_i$. Since L preserves fibres products, applying L to the cartesian square $\sqcup_i Y_i \widetilde{\to} Y \times_{L(\sqcup_i S_i)} \sqcup_i S_i$, we get that $Y \widetilde{\to} L(\sqcup_i Y_i)$. □

10.0.70. For (ch. I.2, 4.3.4). It uses without a reference a result that I documented as my Lemma 7.3.3.

10.0.71. *Structure of comodule categories.* Recall that $coAlg(\mathrm{Vect}) = (Alg(\mathrm{Vect}^{op}))^{op}$. By ([28], 3.2.2.4), $coAlg(\mathrm{Vect})$ admits colimits, and the projection $coAlg(\mathrm{Vect}) \to \mathrm{Vect}$ reflects colimits.

Assume $A_i \in coAlg(\mathrm{Vect})$, let $A = \oplus_{i \in I} A_i$ in $coAlg(\mathrm{Vect})$ or Vect. We have the natural functor

$$\underset{i \in I}{\oplus} A_i - comod(\mathrm{Vect}) \to A - comod(\mathrm{Vect})$$

coming from the system of functors $A_i - comod \to A - comod$, $M \mapsto M$. Dima claims it is an equivalence (in which generality, for A_i classical?). Recall also that coproducts in DGCat_{cont} coincide avec products, so this is also the product.

For example, if T is a torus over k then $k[T]$ as a coalgebra is $\oplus_{\check{\lambda} \in \check{\Lambda}} kt^{\check{\lambda}}$. Here $\check{\Lambda}$ is the weight lattice of T, and $kt^{\check{\lambda}}$ is the 1-dimensional unital coalgebra (dual of the 1-dimensional k-algebra k). The coproduct is $t^{\check{\lambda}} \to t^{\check{\lambda}} \otimes t^{\check{\lambda}}$, and the counit is $t^{\check{\lambda}} \to 1$.

10.1. **Quasi-coherent sheaves.** For (ch. 3, 1.1.3). The functor $\mathrm{QCoh}^*_{\mathrm{PreStk}} : \mathrm{PreStk}^{op} \to \mathrm{DGCat}_{cont}$ preserves small limits.

For (ch. 3, Lm 1.2.2). If $Y \in {}^{\leq n}\mathrm{PreStk}$ then $Y \widetilde{\to} \mathrm{colim}_{S \to Y} S$ in PreStk, where the colimit is over the category $({}^{\leq n}\mathrm{Sch}^{aff})_{/Y}$, see my Section 10.0.39. This implies the lemma.

For Lm. 1.2.4 same argument: let $Y_0 : ({}^{\leq n}\mathrm{Sch}^{aff}_{ft})^{op} \to \mathrm{Spc}$ be a functor, $Y : (\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$ its LKE along $({}^{\leq n}\mathrm{Sch}^{aff}_{ft})^{op} \to (\mathrm{Sch}^{aff})^{op}$, so Y is n-coconnective locally of finite type. Then $Y \widetilde{\to} \operatorname{colim}_{S\to Y} S$ in PreStk, the colimit is taken over the category $({}^{\leq n}\mathrm{Sch}^{aff}_{ft})_{/Y}$. This implies the lemma.

10.1.1. For (ch. 3, 1.3.4): they claimed that $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}at$ preserves limits. The projection $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ preserves limits and colimits. The inclusion $1-\mathcal{C}at^{St,cocmpl}_{cont} \subset \mathcal{P}r^L$ is stable under all limits (see my Section 4.0.31). The functor $\mathcal{P}r^L \to 1-\mathcal{C}at$ preserves all small limits (HTT, 5.5.3.13). Our claim follows.

Cor. 1.3.5 uses the fact that $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}at$ is conservative. To explain this property, note that $\mathrm{DGCat}_{cont} \to \mathrm{DGCat}^{non-cocompl}$ is conservative, since it is 1-replete. Further, the inclusion $\mathrm{DGCat}^{non-cocompl} \subset \mathrm{Vect}^{f.d.}-mod(1-\mathcal{C}at)$ is conservative. The projection $\mathrm{Vect}^{f.d.}-mod(1-\mathcal{C}at) \to 1-\mathcal{C}at$ is conservative by my Section 3.0.53.

Claim: The forgetful functor $\mathrm{DGCat}_{cont} \to \mathcal{P}r^L$ preserves all limits and colimits.

Proof. We have seen above that $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}at^{St,cocmpl}_{cont}$ preserves limits and colimits. By Sect. Â 4.0.31, the forgetful functor $1-\mathcal{C}at^{St,cocmpl}_{cont} \to \mathcal{P}r^L$ preserves all limits and colimits. □

10.1.2. By (ch. 1, 8.5.10), the functor $\mathrm{Alg}(\mathrm{Vect})^{op} \to \mathrm{Vect}-mod^{St,cocmpl}_{cont} = \mathrm{DGCat}_{cont}$, $\mathcal{A} \mapsto \mathcal{A}-mod$ is symmetric monoidal. By (HA, 2.2.1.1) the inclusion $\mathrm{Vect}^{\leq 0} \to \mathrm{Vect}$ is a symmetric monoidal functor, so $\mathrm{Alg}(\mathrm{Vect}^{\leq 0}) \to \mathrm{Alg}(\mathrm{Vect})$ is also symmetric monoidal. Passing to opposite, $\mathrm{Alg}(\mathrm{Vect}^{\leq 0})^{op} \to \mathrm{Alg}(\mathrm{Vect})^{op}$ is also symmetric monoidal. The composition becomes the functor $\mathrm{Sch}^{aff} \to \mathrm{DGCat}_{cont}$, $S \mapsto \mathrm{QCoh}(S)$, it is the functor defined also in (ch. 3, 1.1.1, formula (1.1)). In (ch. 3, 1.1.1) it is not mentioned that it is symmetric monoidal. My understanding is that the functor $\mathrm{QCoh}^*_{\mathrm{Sch}^{aff}} : (\mathrm{Sch}^{aff})^{op} \to \mathrm{DGCat}_{cont}$, $S \mapsto \mathrm{QCoh}(S)$ (obtained by passing to left adjoints) is also symmetric monoidal.

We have $CAlg((\mathrm{Sch}^{aff})^{op}) \widetilde{\to} (\mathrm{Sch}^{aff})^{op}$. So, the symmetric monoidal structure on the above functor $\mathrm{QCoh}^*_{\mathrm{Sch}^{aff}}$ yields a functor $(\mathrm{Sch}^{aff})^{op} \to CAlg(\mathrm{DGCat}_{cont})$. That is, for $S \in \mathrm{Sch}^{aff}$, $\mathrm{QCoh}(S)$ is symmetric monoidal stable category, and for $\alpha : S \to S'$ the functor $\alpha^* : \mathrm{QCoh}(S') \to \mathrm{QCoh}(S)$ is symmetric monoidal, see [11].

10.1.3. For Corollary 1.3.7, proof: Let $\mathcal{C} \in 1-\mathcal{C}at$. We can consider the category $\mathrm{PreStk}_{\mathcal{C}} = \mathrm{Fun}((\mathrm{Sch}^{aff})^{op}, \mathcal{C})$. Then for the flat (resp, etale etc.) topology we can consider the full subcategory $\mathrm{Stk}_{\mathcal{C}} \subset \mathrm{PreStk}_{\mathcal{C}}$ of functors satisfying the corresponding descent. Let now $f : Y_1 \to Y_2$ be a map in PreStk, which is a flat equivalence. Assume $Z : \mathrm{PreStk}^{op} \to \mathcal{C}$ is the RKE if its restriction under $(\mathrm{Sch}^{aff})^{op} \hookrightarrow \mathrm{PreStk}^{op}$, assume this restriction of Z lies in $\mathrm{Stk}_{\mathcal{C}}$. Let us show that the natural map $f^* : Z(Y_2) \to Z(Y_1)$ is an equivalence in $\mathcal{C}$. Let $c \in \mathcal{C}$. It suffices to show that the map $\mathrm{Map}_{\mathcal{C}}(c, Z(Y_2)) \to \mathrm{Map}_{\mathcal{C}}(c, Z(Y_1))$ is an isomorphism in Spc. Write $\bar{Z} : \mathrm{PreStk}^{op} \to \mathrm{Spc}$ for the functor $Y \mapsto \mathrm{Map}_{\mathcal{C}}(c, Z(Y))$. Let $Z' : (\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$ be the restriction of $\bar{Z}$. Then $\bar{Z}$ is the RKE of Z', and Z' is a stack for the flat topology. So, the map $\mathrm{Map}_{\mathrm{PreStk}}(Y_2, Z') \to \mathrm{Map}_{\mathrm{PreStk}}(Y_1, Z')$ is an isomorphism in Spc. Since $Y_2 \widetilde{\to} \operatorname{colim}_{S \xrightarrow{y} Y_2} S$ in PreStk, the colimit over the category $(\mathrm{Sch}^{aff})_{/Y}$, we get

$$\mathrm{Map}_{\mathrm{PreStk}}(Y_2, Z') \widetilde{\to} \lim_{(S,y)\in((\mathrm{Sch}^{aff})_{/Y})^{op}} \mathrm{Map}_{\mathrm{PreStk}}(S, Z') \widetilde{\to} \lim_{(S,y)\in((\mathrm{Sch}^{aff})_{/Y})^{op}} Z'(S) \widetilde{\to} \bar{Z}(Y_2)$$

So, $\bar{Z}(Y_2) \to \bar{Z}(Y_1)$ is an isomorphism.

10.1.4. For (ch. 3, 1.5). Recall that if $A \in \mathrm{Alg}(\mathrm{Vect})$ then $A - mod = A - mod(\mathrm{Vect})$ is stable presentable (see my Section 4.0.32). Consider the forgetful functor $oblv : A-mod \to \mathrm{Vect}$, it is a right adjoint (and preserves both limits and colimits). Applying (HTT, 5.5.4.17) we see that $\{M \in A - mod \mid oblv(M) \in \mathrm{Vect}^{\geq 1}\}$ is strongly reflective subcategory. Consider on the other hand $\mathcal{C} = \{M \in A - mod \mid oblv(M) \in \mathrm{Vect}^{\leq 0}\}$. It is closed under extensions, because $\mathrm{Vect}^{\leq 0}$ is closed under extensions. It is also closed under small colimits, because $\mathrm{Vect}^{\leq 0} \subset \mathrm{Vect}$ is closed under colimits. We have $\mathcal{C} \widetilde{\to} A - mod \times_{\mathrm{Vect}} \mathrm{Vect}^{\leq 0}$. Since the forgetful functor $\mathcal{P}r^L \to 1 - \mathcal{C}at$ preserves limits, we see that $\mathcal{C}$ is presentable (this is also [14], ch. 1, 2.5.2). Applying (HA, 1.4.4.11) we see that there is a t-structure on $A - mod$ such that $A - mod^{\leq 0} = \mathcal{C}$. This t-structure is accessible in the sense of (HA, 1.4.4.12). This is the t-structure defined in Remark 9.3.12.

In particular, if $A \in CAlg(\mathrm{Vect}^{\leq 0})$, on $A - mod$ we get a t-structure. If $A \to B$ is a map in $CAlg(\mathrm{Vect}^{\leq 0})$ then the functor $A - mod \to B - mod$, $M \mapsto B \otimes_A M$ is right t-exact in the sense of (HA, 1.3.3.1).

If $Y \in \mathrm{PreStk}$ they define $\mathrm{QCoh}(Y)^{\leq 0} = \lim_{S\to Y} \mathrm{QCoh}(S)^{\leq 0}$, the limit in $\mathcal{P}r^L$. So, $\mathrm{QCoh}(Y)^{\leq 0}$ is presentable by construction. It is also closed under colimits and extensions, because the functors $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(S)$ preserve colimits and are exact. So, (HA, 1.4.4.11) shows that this defines a t-structure on $\mathrm{QCoh}(Y)$.

If $S \in \mathrm{Sch}^{aff}$ then for $f : S \to \mathrm{Spec}\, k$ the map $f_* : \mathrm{QCoh}(S) \to \mathrm{Vect}$ is t-exact.

10.1.5. For (ch. I.3, 1.5.8). Let $I \to \mathrm{DGCat}_{cont}$ be a diagram, $i \mapsto C_i$, for $i \to j$ in I let $F_{ij} : C_i \to C_j$ be the transition functor. Let $C = \mathrm{colim}\, C_i$ in DGCat_{cont}. Recall that also $C \widetilde{\to} \lim_{i\in I^{op}} C_i$ in DGCat. Let $ev_i : C \to C_i$ be the projection functor. Let $C^{\geq 0} = \{c \in C \mid \text{for each } i \in I, ev_i(c) \in C_i^{\geq 0}\}$. For a map $i \to j$ in I let $F_{ij}^R : C_j \to C_i$ be the right adjoint to F_{ij}. Then F_{ij}^R is left t-exact.

Indeed, given $x \in C_j^{\geq 0}$ let $y \in C_i^{<0}$. Then $\pi_0 \mathrm{Map}_{C_i}(y, F_{ij}^R(x)) \widetilde{\to} \pi_0 \mathrm{Map}_{C_j}(F_{ij}(y), x) \widetilde{\to} 0$, because $F_{ij}(y) \in C_j^{<0}$. So, $F_{ij}^R(x) \in C_i^{\geq 0}$.

Thus, we obtain $\lim_{i\in I^{op}} C_i^{\geq 0} \subset C$ a full subcategory by my Lemma 2.2.17. In fact $C^{\geq 0} = \lim_{i\in I^{op}} C_i^{\geq 0}$. For each i we have an adjoint pair $\tau^{\geq 0} : C_i \leftrightarrows C_i^{\geq 0} : G_i$ with G_i being the inclusion.

It is not clear why $C^{\geq 0}$ is a localization of C.

Assume for a moment that the t-structure on each C_i is accessible. Then $C_i^{\geq 0}$ is presentable, and we may rewrite $C^{\geq 0}$ as $\mathrm{colim}_{i\in I}\, C_i^{\geq 0}$ in $\mathcal{P}r^L$. Passing to the colimit in DGCat_{cont} in the system of maps $\tau^{\geq 0} : C_i \to C_i^{\geq 0}$ indexed by I, we get a functor $g : C \to C^{\geq 0}$. Is g left adjoint to the inclusion $i : C^{\geq 0} \hookrightarrow C$?

Passing to the colimit in the system $C_i^{\geq 0} \hookrightarrow C_i \stackrel{\tau^{\geq 0}}{\to} C_i^{\geq 0}$, we see that the composition $C^{\geq 0} \to C \to C^{\geq 0}$ is isomorphic to the identity. In $\mathrm{Fun}(C_i, C_i)$ we have a morphism $\mathrm{id} \to i_i \circ \tau^{\geq 0}$, where $i_i : C_i^{\geq 0} \hookrightarrow C_i$ is the inclusion. I think we can pass to the colimit here over $i \in I$ and get a morphism $\mathrm{id} \to i \circ g$, where $i : C^{\geq 0} \hookrightarrow C$ is the inclusion. I think that i is right adjoint to g indeed.

For a map $i \to j$ in I, $F_{ij} : C_i^{\leq 0} \to C_j^{\leq 0}$ preserves colimits, hence admits a right adjoint $G_{ij} : C_j^{\leq 0} \to C_i^{\leq 0}$. In fact, $G_{ij} = \tau^{\leq 0} F_{ij}^R$. The map $\tau^{\leq 0} : C_i \to C_i^{\leq 0}$ becomes a morphism of projective systems indexed by I^{op}, where we use functors $F_{ij}^R : C_j \to C_i$ and $G_{ij} : C_j^{\leq 0} \to C_i^{\leq 0}$ as transition functors. Passing to the limit, we get a functor $\alpha : C \to \lim_{i \in I^{op}} C_i^{\leq 0}$, where the limit can be seen as a limit in $\mathcal{P}\mathrm{r}^R$. We get $\lim_{i \in I^{op}} C_i^{\leq 0} \widetilde{\to} \mathrm{colim}_{i \in I} C_i^{\leq 0}$. The compatible system of functors $C_i^{\leq 0} \hookrightarrow C_i$ yields by passing to the colimit over I a functor $\mathrm{colim}_{i \in I} C_i^{\leq 0} \to C$. Is it left adjoint to α?

Note that $C_i^{<0} \widetilde{\to} C_i \times_{C_i^{\geq 0}} 0$, the fibred product taken in $\mathcal{P}\mathrm{r}^L$. (Here the functor $0 \to C_i^{\geq 0}$ sends the unique object of 0 to the zero object of $C_i^{\geq 0}$). This is an inductive system in $\mathcal{P}\mathrm{r}^L$ indexed by I, and

$$\mathrm{colim}_{i \in I}(C_i \times_{C_i^{\geq 0}} 0) \widetilde{\to} \mathrm{colim}_{i \in I} C_i^{<0}$$

in $\mathcal{P}\mathrm{r}^L$ rewrites as $\lim_{i \in I^{op}}(C_i \times_{C_i^{\geq 0}} 0)$ in $\mathcal{P}\mathrm{r}^R$ with respect to the right adjoint system of transition functors.

The claim is wrong as stated. For example, assume $I = [1]$, and the diagram $I \to \mathrm{DGCat}_{cont}$ is $F : C_0 \to C_1$ then $\mathrm{colim}\, C_i \widetilde{\to} C_1$, and the t-structure we get is the old one on C_1. However, $F^R : C_1 \to C_0$ is not t-exact in general.

Remark 10.1.6. *Let $f : C' \to C$ be a map in* DGCat_{cont} *and $f^R : C' \to C$ its right adjoint. Assume C, C' are equipped with t-structures. Then f is right t-exact iff f^R is left t-exact.*

Proof. Argument as in my Section 10.1.5. □

This was a misprint in their (ch. I.3, 1.5.8), one should consider $C = \lim_i C_i$ for $I \to \mathrm{DGCat}_{cont}$. Assume for simplicity all the t-structures on C_i accessible. Then $\{c \in C \mid ev_i(c) \in C_i^{\leq 0} \text{ for all } i\}$ is presentable (as a limit of presentable categories $\lim_i C_i^{\leq 0}$ in $\mathcal{P}\mathrm{r}^L$), stable under colimits and extensions in C, hence defines a t-structure on C (HA, 1.4.4.11) with $C^{\leq 0} = \{c \in C \mid ev_i(c) \in C_i^{\leq 0} \text{ for all } i\}$. Moreover, $C^{>0} = \{c \in C \mid \text{for any } i, ev_i(c) \in C_i^{>0}\}$.

If for each i the t-structure on C_i is compatible with filtered colimits then $C^{>0} \subset C$ is stable under filtered colimits, so the t-structure on C is also compatible with filtered colimits.

If $S \in \mathrm{Sch}^{aff}$ then the t-structure on $QCoh(S)$ is compatible with filtered colimits.

The characterization of right completeness of the t-structures given in (ch. I.3, 1.5.7) use the fact that the t-structures on C_i are accessible and are given in my Section 4.0.10.

For (d): assume each C_i is left complete. Let us show that C is also left complete. For each $i \in I$, $C_i \widetilde{\to} \lim_{n \in \mathbb{Z}^{op}} C_i^{\geq -n}$. Besides, $C^{\geq 0} \widetilde{\to} \lim_{i \in I} C_i^{\geq 0}$, so $C^{\geq -n} \widetilde{\to} \lim_{i \in I} C_i^{\geq -n}$

for any $n \in \mathbb{Z}$. Thus,

$$\lim_{n\in\mathbb{Z}^{op}} C^{\geq -n} \widetilde{\to} \lim_{i\in I} \lim_{n\in\mathbb{Z}^{op}} C_i^{\geq -n} \widetilde{\to} \lim_{i\in I} C_i \widetilde{\to} C$$

as required.

10.2. The following definitions are from ([14], vol 2, ch. 1, 8.1): for a classical prestack Y one has ${}^{red}Y$, its restriction to the category $({}^{red}\mathrm{Sch}^{aff})^{op}$, one has the notion of a closed embedding of classical prestacks.

10.2.1. For (ch. I.3, 2.1.3). If $S_i \in \mathrm{Sch}^{aff}$, I is small then then $S := \sqcup_{i\in I} S_i$ in Sch^{aff} is not the same as $\sqcup_{i\in I} S_i$ in PreStk. The latter is not an affine scheme in general. We have a natural map $\sqcup_{i\in S} S_i \to S$ in PreStk, which is not an isomorphism in general. Indeed, $\mathrm{Sch}^{aff} \to \mathrm{PreStk}$ does not preserve colimits. For example, if $I = \mathbb{N}$ and $S_i = *$.

Already if $S_1, S_2 \in \mathrm{Sch}^{aff}$ are nonempty then the coproduct $S_1 \sqcup S_2$ in PreStk is never the coproduct of S_1 and S_2 in Sch^{aff}. Let S be the coproduct of S_1 and S_2 in Sch^{aff}. Then $L(S_1 \sqcup S_2) \widetilde{\to} S$. This follows from (ch. I.2, 3.1.6). Indeed, the simplicial object $(S_1 \sqcup S_2)^\bullet/S$ is constant with value $S_1 \sqcup S_2$, where the coproduct is in PreStk. However, if I is infinite, let $S_i \in \mathrm{Sch}^{aff}$ and S be the coproduct of S_i in Sch^{aff}. Then $\sqcup_i S_i \to S$ is usually not a Zarizki cover, here $\sqcup_i S_i$ is calculated in PreStk.

The coproducts in Sch^{aff} do not commute with finite products. Indeed, let S be the coproduct of $*$ in Sch^{aff} indexed by $I = \mathbb{N}$. The natural map $k[t] \otimes_k (\prod_{i\in\mathbb{N}} k) \to \prod_{i\in\mathbb{N}} k[t_i]$ is not an isomorphism, because it is not surjective. If we take a point $(p_i) \in \prod_{i\in\mathbb{N}} k[t_i]$ such that the $\deg p_i$ are not bounded for $i \in \mathbb{N}$ then this point is not in the image.

Example of a noncontinuous functor, which is a map in DGCat but not in DGCat_{cont} is as follows. Consider the projection $\pi : X = \sqcup_{i\in\mathbb{N}} * \to *$, here the coproduct is taken in PreStk. We get $\mathrm{QCoh}(X) = \prod_{i\in\mathbb{N}} \mathrm{Vect}$, and $\pi_* : \prod_{i\in\mathbb{N}} \mathrm{Vect} \to \mathrm{Vect}$ sends (K_n) to $\prod_{n\in\mathbb{N}} K_n$, where the latter product is taken in Vect. This functor is exact, but not continuous. Indeed, if $\{e_i, i \in \mathbb{N}\}$ is a base of the vector space M and M_n is vector space with the base $\{e_1, \ldots, e_n\}$ then $\mathrm{colim}_{n\in\mathbb{N}} M_n = M$ in Vect. However, the natural map $\mathrm{colim}_{i\in\mathbb{N}}(\prod_{n\in\mathbb{N}} M_i) \to \prod_{n\in\mathbb{N}} M$ is not an isomorphism of vector spaces, here the colimit and product are calculated in Vect. The above map is not surjective.

10.2.2. *Remark.* Let $Z \in \mathrm{Sch}$ and $U_i \to Z$ is a Zariski cover for $i = 1, 2$. Let $U_{12} = U_1 \times_Z U_2$. We claim that the natural map $f : L(U_1 \sqcup_{U_{12}} U_2) \to Z$ is an isomorphism, where the coproduct $U := U_1 \sqcup_{U_{12}} U_2$ is calculated in PreStk. Indeed, $U \times_Z U_i \widetilde{\to} U_i$, so the base change of f by $a : U_1 \sqcup U_2 \to Z$ becomes an isomorphism. Since a is an etale surjection, our claim follows from my Section 10.0.20.

In particular, this gives an eqiuvalence $\mathrm{QCoh}(Z) \widetilde{\to} \mathrm{QCoh}(U_1) \times_{\mathrm{QCoh}(U_{12})} \mathrm{QCoh}(U_2)$ in DGCat_{cont}, because QCoh^* preserves limits (we use ch. I.3, 1.3.4).

10.2.3. For (ch. I.3, 2.2.2). A schematic morphism $f : X \to Y$ in PreStk is quasi-compact if for $S \to Y$ with $S \in \mathrm{Sch}^{aff}$, $S \times_Y X$ is a quasi-compact derived scheme.

In the proof it is the functor $\mathrm{Sch}_{qc}/Y_2 \to \mathrm{Sch}_{qc}/Y_1$, which is cofinal, not its opposite. We want to show that for the diagram $W' = Z' \times_Z W$ we have the base change for $f : W \to Z$, where W, Z, Z' are quasi-compact schemes. It suffices to assume Z

affine. Indeed, if we know this for Z affine, we may pass to the limit I think over $(S \to Z) \in (\mathrm{Sch}^{aff}_{/Z})^{op}$ in the system of the corresponding base change isomorphisms for

$$\begin{array}{ccc} W'_S & \to & W_S \\ \downarrow & & \downarrow \\ Z'_S & \to & S \end{array}$$

Here the subscript S denote the base change under $S \to Z$. The next step is to do the same for maps $S \to Z'$ and reduce to the case when Z' is also affine.

Remark: let $W \in \mathrm{Sch}$, $U_i \to W$ for $i = 1, 2$ a Zariski atlas. Then for $K \in \mathrm{QCoh}(W)$ let $K_i = K \mid_{U_i}$, $K_{12} = K \mid_{U_{12}}$ with $U_{12} = U_1 \times_W U_2$. Let $j_i : U_i \to W$ and $j_{12} : U_{12} \to W$ be the open immersions. Then $K \to (j_1)_*F_1 \oplus (j_2)_*F_2 \to (j_{12})_*F_{12}$ is a fibre sequence in $\mathrm{QCoh}(W)$. Indeed, let $U = U_1 \sqcup U_2$, the coproduct in PreStk. Recall that $\mathrm{QCoh}(W) \widetilde{\to} Tot(\mathrm{QCoh}(U^\bullet/W))$, where $U^\bullet/W$ is the Cech nerve of $U \to W$. So, it suffices to show that for any $n \geq 0$, the image of our triangle in $\mathrm{QCoh}(U^n/W)$ is a fibre sequence. This follows from the fact that this is true for $\mathrm{QCoh}(U_i)$ and $\mathrm{QCoh}(U_{12})$, as U^n/W is a coproduct of such.

10.2.4. If $j : U \hookrightarrow X$ is a schematic quasi-compact morphism in PreStk, which is a monomorphism then $\mathrm{id} \widetilde{\to} j^*j_*$ by (ch. I.3, 2.2.2).

10.2.5. If $f : X \to Y$ is a schematic quasi-compact morphism in PreStk then from (ch. I.3, 3.2.3) one derives the following. Let $H \in \mathrm{QCoh}(X), F \in \mathrm{QCoh}(Y)$ then $(\mathrm{id} \times f)_*(F \boxtimes H) \widetilde{\to} F \boxtimes (f_*H)$. This combines the base change isomorphism of f by $\mathrm{pr}_1 : Y \times Y \to Y$ and the projection formula.

10.2.6. For 2.2.4. Recall the 1-subcategory $\mathrm{PreStk}_{sch,qc}$, where one restricts 1-morphisms to be schematic and quasi-compact. My understanding is that the functor denoted $\mathrm{QCoh}_{\mathrm{PreStk}_{sch,qc-qs}} : \mathrm{PreStk}_{sch,qc} \to \mathrm{DGCat}_{cont}$ in (ch. 3, 2.2.4) is right-lax symmetric monoidal. This functor sends X to $\mathrm{QCoh}(X)$, and a schematic quasi-compact morphism $f : X \to Y$ to $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$.

Now if $G \in \mathrm{PreStk}$ has a structure of an algebra in $\mathrm{PreStk}_{sch,qc}$ then $\mathrm{QCoh}(G)$ gets a structure of an algebra in DGCat_{cont} given by convolution. Namely, for the product map $m : G \times G \to G$ the product on $\mathrm{QCoh}(G)$ is given as the composition $\mathrm{QCoh}(G) \otimes \mathrm{QCoh}(G) \to \mathrm{QCoh}(G \times G) \stackrel{m_*}{\to} \mathrm{QCoh}(G)$.

This is offen applied for $G \in \mathrm{Sch}_{qc}$, which is an algebra in Sch_{qc}.

If $i : \mathrm{Spec}\, k \to G$ is the unit then $i_*\mathcal{O}$ is the unit of $\mathrm{QCoh}(G)$ for the convolution monoidal structure.

10.2.7. For (ch. I.3, 2.3.2). Let $S_i \in \mathrm{Sch}^{aff}$ and $Y = L(\sqcup_i S_i)$, recall that $Y \in \mathrm{Sch}$. Then Y is quasi-compact iff I is finite (that is, S_i is empty for i not in this finite subset). Indeed, assume $T \in \mathrm{Sch}^{aff}$ and $T \to Y$ a smooth atlas. For each i, $S_i \to Y$ is an affine open embedding, hence $S_i \times_Y T \to T$ is an affine open embedding, and $S_i \to Y$ for $i \in I$ form a Zariski atlas of Y, so $S_i \times_Y T$ form a Zariski atlas of T. However, $\mathrm{Spec}\, T$ is quasi-compact, so already for a finite subset $I' \subset I$, $S_i \times_Y T$ cover T, so $L(\sqcup_{i \in I'} S_i) = Y$.

Let Y_1 be an Artin stack, and $Z \to Y_1$ be a smooth atlas with $Z \in \mathrm{Sch}$. For $i \geq 0$ let Z^i/Y_1 be the corresponding element of the Cech nerve of $Z \to Y_1$. Let $f^i : Z^i/Y_1 \to Y_1$

be the projection. Consider the category $\mathrm{QCoh}(Y_1)\widetilde{\to}Tot(\mathrm{QCoh}(Z^\bullet/Y_1))$. Applying my Section 2.7.5 for $\mathcal{F}\in\mathrm{QCoh}(Y_1)$ we get

$$\mathcal{F}\widetilde{\to}Tot((f^\bullet)_*(f^\bullet)^*\mathcal{F})$$

This was used in the proof.

10.2.8. For 2.4.3. If $A\in CAlg(\mathrm{Vect}^{\le 0})$ then $(A-mod)^\heartsuit\widetilde{\to}(\mathrm{H}^0(A)-mod)^\heartsuit$ is a Grothendieck abelian category by (HA, 1.3.5.23). Indeed, $A-mod$ is presentable stable, and the t-structure is compatible with filtered colimits.

For the proof of 2.4.3 by $\mathrm{Hom}_{\mathrm{QCoh}(Y)}(F_1,F_1)$ they mean $\pi_0\,\mathrm{Map}_{\mathrm{QCoh}(Y)}(F_1,F_2)$. If $f:S\to Y$ is flat, Y is a 1-Artin stack quasi-compact and quasi-separated, then for $\mathcal{F}\in\mathrm{QCoh}(Y)^\heartsuit,\mathcal{F}_S\in\mathrm{QCoh}(S)^\heartsuit$ if $a:f^*\mathcal{F}\to\mathcal{F}_S$ is injective then $\mathcal{F}\to f_*\mathcal{F}_S$ is also injective. Indeed, if the latter factors as $\mathcal{F}\to\mathcal{F}/\mathcal{F}_0\to f_*\mathcal{F}_S$ for some subsheaf $\mathcal{F}_0$ then a factors as $f^*\mathcal{F}\to f^*(\mathcal{F}/\mathcal{F}_0)\to\mathcal{F}_S$.

For Lm. 2.4.5: I think this lemma is probably wrong as stated. It should be essentially a reformulation of (HA, 1.3.3.7). There is an assumption missing: c_0 is injective. With this assumption added lemma becomes true, and follows from (HA, 1.3.3.7). Namely, given $c,c'\in C^\heartsuit$ with c injective we claim that $\mathrm{Ext}^i(c',c)=0$ for $i>0$. Indeed, pick an injection $c\stackrel{\alpha}{\to}c_0$ in $C^\heartsuit$ such that c_0 is injective, and $\mathrm{Ext}^i(c',c_0)=0$ for $i>0$. Then there is $\beta:c_0\to c$ such that $\beta\alpha=\mathrm{id}$. So, $\mathrm{id}:\mathrm{Ext}^i(c',c)\to\mathrm{Ext}^i(c',c)$ factors as $\mathrm{Ext}^i(c',c)\to\mathrm{Ext}^i(c',c_0)\to\mathrm{Ext}^i(c',c)$, and $\mathrm{Ext}^i(c',c)=0$.

By $\mathrm{D}(C^\heartsuit)$ then mean the version of the derived category defined as in (HA, 1.3.2) but for an abelian category $C^\heartsuit$ having enough injective objects. The condition that the t-structure is compatible with filtered colimits is only needed to assure that $C^\heartsuit$ has enough injective objects. Then we have indeed a functor $\mathrm{D}(C^\heartsuit)^+\to C$, and we want to check it is fully faithful using (HA, 1.3.3.7). However, their condition does not seem to garantee the assumptions of (HA, 1.3.3.7). With the above correction their Lemma 2.4.5 becomes true.

Note that if in addition the t-structure on C is accessible then $C^\heartsuit$ is a Grothendieck abelian category by (HA, 1.3.5.23).

10.2.9. In 3.2.6 for an abelian variety A he means by $\Gamma(A,\mathcal{O}_A)$ the complex $\pi_*\mathcal{O}_A$ for $\pi:A\to *$. In classical terms this is $\mathrm{R}\Gamma$. He means that $\pi_*\mathcal{O}_A$ is known to be isomorphic to $\mathrm{Sym}(\mathrm{H}^1(X,\mathcal{O})[-1])$.

As in (ch. I.3, 3.2.5), $\pi_*\mathcal{O}_A$ is an algebra in Vect. Since A is a group, $\pi_*\mathcal{O}_A$ also gets a structure of a coalgebra in Vect. Namely, the product $m:A\times A\to A$ yields $\pi_*\mathcal{O}_A\to(\pi_*\mathcal{O}_A)\otimes(\pi_*\mathcal{O}_A)$ in Vect, hence the dual $(\pi_*\mathcal{O}_A)^\vee$ is naturally an algebra in Vect. Let $q:*\to B(A)$ be the natural map. The idea is to check that $q^*:\mathrm{QCoh}(B(A))\to\mathrm{Vect}$ is comonadic.

First, A acts trivially on $*$. This yields an action of $\mathrm{QCoh}(A)$ with the convolution monoidal structure (defined in the previous subsection) on Vect. The action map is $\pi_*:\mathrm{QCoh}(A)=\mathrm{QCoh}(A)\otimes\mathrm{Vect}\to\mathrm{Vect}$. Here by tensor product we mean the tensor product in DGCat_{cont}, that is, over Vect. Now $\mathcal{O}_A\in\mathrm{QCoh}(A)$ is naturally a coalgebra in $\mathrm{QCoh}(A)$ with the convolution monoidal structure. Let $i:\mathrm{Spec}\,k\to A$ be the unit then $i_*\mathcal{O}$ is the unit of the convolution monoidal structure on $\mathrm{QCoh}(A)$. The counit map for $\mathcal{O}_A$ is the natural map $\mathcal{O}_A\to i_*\mathcal{O}$. We also have the natural map $\mathcal{O}_A\to m_*\mathcal{O}$.

What is the corresponding comonad on Vect? For any $S \in \mathrm{Sch}^{aff}$ one gets $* \times_{B(A)} S \widetilde{\to} S \times A$. So, $q : * \to B(A)$ is schematic quasi-compact, so the right adjoint $q_* : \mathrm{Vect} \to \mathrm{QCoh}(B(A))$ is continuous, and we get the comonad $\mathcal{A} = q^*q_* : \mathrm{Vect} \to \mathrm{Vect}$. It is given by $M \mapsto M \otimes \pi_*\mathcal{O}_A$ by (ch. I.3, 2.2.2), as q is schematic quasi-compact. We get the functor $(q^*)^{enh} : \mathrm{QCoh}(A) \to \mathcal{A} - comod(\mathrm{Vect})$. The functor q^* is conservative, it remains to show it preserves totalizations. This is not evident, and it is better to apply here ([15], Lemma C.1.9).

Namely, consider the cosimplicial category $[\mathrm{Vect} \rightrightarrows \mathrm{QCoh}(A) \overset{\to}{\rightrightarrows} \mathrm{QCoh}(A^2)\ldots]$, where the functors are the pull-backs (coming from the group object $\boldsymbol{\Delta}^{op} \to \mathrm{PreStk}$ given by A). It suffices to check that this cosimplicial category satisfies the comonadic Beck-Chevalley condition ([15], Def. C.1.3). Namely, for any map $\alpha : [j] \to [i]$ in $\boldsymbol{\Delta}$ we get the cartesian square of schemes

$$\begin{array}{ccc} A^{i-1} & \overset{p_i}{\leftarrow} & A^i \\ \downarrow q_i & & \downarrow q_j \\ A^{j-1} & \overset{p_j}{\leftarrow} & A^j, \end{array}$$

where p_i is the projection on first $i-1$ factors. The base change for this diagram $q_i^*(p_j)_* \widetilde{\to} (p_i)_* q_j^*$ garantees the condition ([15], Def. C.1.2). So, q^* is comonadic.

My understanding is that for $\mathcal{A} = \pi_*\mathcal{O}_A$ the product map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a morphism of coalgebras in Vect, so the dual map $\tau : \mathcal{A}^\vee \to \mathcal{A}^\vee \otimes \mathcal{A}^\vee$ is a morphism of algebras. This provides a symmetric monoidal structure on $\mathcal{A}^\vee - mod(\mathrm{Vect})$ such that the forgetful functor $\mathcal{A}^\vee - mod \to \mathrm{Vect}$ is symmetric monoidal. I think this symmetric monoidal structure on $\mathcal{A}^\vee - mod$ corresponds to the pointwise symmetric monoidal structure on $\mathrm{QCoh}(B(A))$ (the one existing on $\mathrm{QCoh}(Y)$ for any prestack Y).

Since A is a commutative group, $\mathcal{A}$ is a cocommutative coalgebra, so $\mathcal{A}^\vee$ is a commutative algebra in $\mathrm{Vect}^{\leq 0}$, so we get $S := \mathrm{Spec}\,\mathcal{A}^\vee \in \mathrm{Sch}^{aff}$. The morphism τ gives a map $h : S \times S \to S$. Since $\mathcal{A}$ is a commutative algebra in Vect, $\mathcal{A}^\vee$ is a cocommutative coalgebra in $\mathrm{Vect}^{\leq 0}$, so S is a commutative monoid in Sch^{aff}.

Let $f : S \to \mathrm{Spec}\, k$ be the natural map. Consider the tensor product $\mathrm{Vect} \otimes_{\mathcal{A}^\vee - mod} \mathrm{Vect}$, here both functors $\mathcal{A}^\vee - mod \to \mathrm{Vect}$ are f_*, and the symmetric monoidal structure on $\mathcal{A}^\vee - mod$ is given by the above morphism of algebras $\tau : \mathcal{A}^\vee \to \mathcal{A}^\vee \otimes \mathcal{A}^\vee$. Are the monadic Beck-Chevalley conditions satisfied for the simplicial category $Bar^\bullet(\mathrm{Vect}, \mathrm{QCoh}(S), \mathrm{Vect})$, namely those of ([15], Def. C.1.2)? Note that $h_* : \mathrm{QCoh}(S) \otimes \mathrm{QCoh}(S) \to \mathrm{QCoh}(S)$ and $f_* : \mathrm{QCoh}(S) \to \mathrm{Vect}$ have left adjoints h^*, f^*.

In fact $\mathcal{A}^\vee \widetilde{\to} \mathrm{Sym}(V[1]) \widetilde{\to} k \otimes_{\mathrm{Sym}\, V} k$, where $V = \mathrm{H}^1(A, \mathcal{O})^*$. So, $S \widetilde{\to} * \times_V *$, where $* \to V$ is given by zero. This is a group in Sch^{aff} naturally. This implies the Beck-Chevalley conditions for the simplicial category $Bar^\bullet(\mathrm{Vect}, \mathrm{QCoh}(S), \mathrm{Vect})$ by ([15], C.2.2), because the square is cartesian

$$\begin{array}{ccc} S \times S & \overset{h}{\to} & S \\ \downarrow \mathrm{pr}_2 & & \downarrow \\ S & \to & * \end{array}$$

So, by ([15], Cor. C.2.3) the right adjoint to the natural functor $\mathrm{Vect} \to \mathrm{Vect} \otimes_{\mathrm{QCoh}(S)} \mathrm{Vect}$ is monadic, and the corresponding monad is given by $\mathcal{A} \in Alg(Vect)$. So, $\mathcal{A} - mod \tilde{\to} \mathrm{Vect} \otimes_{\mathrm{QCoh}(S)} \mathrm{Vect}$.

However, ([14], ch. I.3, 3.2.6) claims another answer for $\mathrm{Vect} \otimes_{\mathrm{QCoh}(S)} \mathrm{Vect}$.

10.2.10. Question: given a coalgebra A in Vect, consider $oblv : A - comod \to \mathrm{Vect}$. Assume A compact, hence dualizable. We have $A - comod \tilde{\to} A^\vee - mod$ (see my Section 3.2.1). Do we get the structure of a $A - comod$-module on Vect? How to calculate $\mathrm{Vect} \otimes_{A-comod} \mathrm{Vect}$?

10.2.11. For (ch. I.3, 3.4.2). In the proof when one shows (ii) implies (iii) we use the following diagram

$$\begin{array}{ccc} Y & \stackrel{\triangle}{\to} & Y \times Y \\ \downarrow \triangle & & \downarrow \triangle \times \mathrm{id} \\ Y \times Y & \stackrel{\mathrm{id} \times \triangle}{\to} & Y \times Y \times Y, \end{array}$$

with $\triangle$ being the diagonal map. Since the maps are schematic quasi-compact, base change holds for this diagram: for $M, F \in \mathrm{QCoh}(Y)$ we get $(\mathrm{id} \times \triangle)^*((\triangle_* M) \boxtimes F) \tilde{\to} \triangle_* (M \otimes F)$, which means that $\triangle_*$: $\mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y)$ is compatible with the right action of $\mathrm{QCoh}(Y)$. Similarly, $(\triangle \times \mathrm{id})^*(M \boxtimes (\triangle_* F)) \tilde{\to} \triangle_* (M \otimes F)$, so $\triangle_*$ is also compatible with the left action of $\mathrm{QCoh}(Y)$.

10.2.12. Let G be a finite group. Let $B = \mathrm{H}^0(G, \mathcal{O})$, this is an algebra and a coalgebra in Vect (placed in degree zero). The category $\mathrm{QCoh}(B(G))$ is described as in Section 10.2.9, namely, $\mathrm{QCoh}(B(G)) \tilde{\to} B - comod(\mathrm{Vect})$. By my Section 3.2.1, $B - comod(\mathrm{Vect}) \tilde{\to} B^\vee - mod(\mathrm{Vect})$. What is the structure of the algebra $B^\vee$? (It is non commutative for G nonabelian).

If G is abelian, write $G^\vee$ for the group of characters of G. Then we have a canonical isomorphism $B^\vee \tilde{\to} H^0(G^\vee, \mathcal{O})$ of commutative algebras in Vect (we assume $char(k) = 0$). Namely, for a character $\chi : G \to k^*$ write $f_\chi \in H^0(G^\vee, \mathcal{O})$ for the chacateristic function of χ. The set $\{\chi\}_{\chi \in G^\vee}$ forms a base in B, write $\{\epsilon_\chi\}_{\chi \in G^\vee}$ for the dual base in $B^\vee$. The above isomorphism sends ϵ_χ to f_χ. For G abelian we obtain an equivalence $QCoh(B(G)) \tilde{\to} \mathrm{QCoh}(\check{G})$. The latter identifies with the symmetric monoidal category $\prod_{\chi \in G^\vee} \mathrm{Vect}$, the product taken in DGCat_{cont}.

I think that for $n \geq 1$, writing $\sqcup_{i=1}^n *$ for the coproduct in PreStk, $L(\sqcup_{i=1}^n *)$ is the corresponding coproduct in Sch^{aff}. Then this would follow from the fact that the functor $\mathrm{QCoh}^* : \mathrm{PreStk}^{op} \to \mathrm{DGCat}_{cont}$ preserves limits.

10.2.13. Since PreStk admits small colimits, its tensored over Spc. Namely, for $Y \in \mathrm{PreStk}, X \in \mathrm{Spc}$, $Y \otimes X$ is the prestack sending $S \in \mathrm{Sch}^{aff}$ to $Y(S) \times X$. We can also write $Y \times X$ instead of $Y \otimes X$, the product with the constant prestack. We have $\mathrm{QCoh}(Y \times X) \tilde{\to} \lim_{x \in X} \mathrm{QCoh}(Y)$, the limit taken in DGCat_{cont} or in $1 - \mathcal{C}at$. So, $\mathrm{QCoh}(Y \times X) \tilde{\to} \mathrm{Fun}(X, \mathrm{QCoh}(Y))$.

10.2.14. Let $Y \in \mathrm{Fun}(\mathrm{Sch}^{aff}, Sets)$, let $Y' : \mathrm{Sch}^{aff} \to \mathrm{Spc}$ be the corresponding prestack. The inclusion $Sets \subset \mathrm{Spc}$ preserves limits. So, to check that $Y' \in \mathrm{Stk}$ in the etale topology, we have to calculate a totalization in Sets. Namely, $Y' \in \mathrm{Stk}$ if for any etale covering $f : S \to T$ in Sch^{aff} the natural map $Y(T) \to \mathrm{Tot}(Y(S^\bullet/T))$ should be an isomorphism, where the totalization is calculated in $Sets$. By Lemma2.5.24 of this file, $\mathrm{Tot}(Y(S^\bullet/T)) \widetilde{\to} \lim_{\bullet \in \boldsymbol{\Delta}^{\leq 1}} Y(S^\bullet/T)$. So, this condition is a finite limit.

Let now $I \in 1-\mathcal{C}at$ be filtered, $Z : I \to \mathrm{Fun}(\mathrm{Sch}^{aff}, Sets)$ be a functor taking values in Stk. Then $Y := \mathrm{colim}_I Z$ (taken in PreStk) also takes values in $Sets$, because $\tau_{\leq m} \mathrm{Spc} \subset \mathrm{Spc}$ is stable under filtered colimits for any m. We claim that $Y \in \mathrm{Stk}$. Indeed, let $f : S \to T$ be an etale cover in Sch^{aff}. We have

$$\lim_{\bullet \in \boldsymbol{\Delta}^{\leq 1}} Y((S^\bullet/T) \widetilde{\to} \lim_{\bullet \in \boldsymbol{\Delta}^{\leq 1}} \mathrm{colim}_{i \in I} Z_i(S^\bullet/T) \widetilde{\to} \mathrm{colim}_{i \in I} \lim_{\bullet \in \boldsymbol{\Delta}^{\leq 1}} Z_i(S^\bullet/T)$$

by (HTT, 5.3.3.3). Since $Z_i \in \mathrm{Stk}$ for any $i \in I$, we get $Y(T) \widetilde{\to} \lim_{\bullet \in \boldsymbol{\Delta}^{\leq 1}} Y((S^\bullet/T)$.

This is especially used for ind-schemes, see ([17], Sect. 2.1).

10.2.15. For (ch. I.3, 3.6.10). Let $S = \mathrm{Spec}\, A \in \mathrm{Sch}^{aff}$, $i_n : {}^{\leq n}S \to S$ the natural map. First, the functor $\mathrm{QCoh}(S) \to \lim_n \mathrm{QCoh}({}^{\leq n}S)$, $M \mapsto (i_n^* M)_{n \geq 0}$ has a right adjoint sending a compatible family $(\mathcal{F}_n)$ to $\lim_n (i_n)_* \mathcal{F}_n$. In particular, the same holds with $\mathrm{QCoh}(S)$ replaced by $\mathrm{QCoh}(S)^-$. We still have to explain that $\lim_n (i_n)_* \mathcal{F}_n$ taken in $\mathrm{QCoh}(S)$ is upper-bounded.

Let now $M \in \mathrm{QCoh}(S)^-$. Then the natural map $M \to \lim_{n \geq 0} M \otimes_A \tau^{\geq -n} A$ in $\mathrm{QCoh}(S)$ is an isomorphism? If M is in degrees $\leq N$ then $M \otimes_A \tau^{<-n} A$ is in degrees $< N - n$. It suffices to show that $\lim_n (M \otimes_A \tau^{-n} A) = 0$ in Vect. This is a particular case of the following.

Lemma 10.2.16. *Let* $\mathbb{Z}_{<0} \to \mathrm{Vect}$ *be a diagram* $\ldots \to M_{n-1} \overset{f_{n-1}}{\to} M_n \overset{f_n}{\to} \ldots$*. Assume* $\alpha(n) \in \mathbb{Z}$*,* $M_n \in \mathrm{Vect}^{\leq \alpha(n)}$ *and* $\lim \alpha(n) = -\infty$ *as* n *goes to* $-\infty$*. Then* $\lim_n M_n = 0$ *in* Vect*.*

Proof. Let $M = \lim M_n$ in Vect. We have the fibre sequence $M \to \prod_n M_n \overset{b}{\to} \prod_n M_n$, where b is the product of maps $b_n : \prod_n M_n \to M_n$. Here b_n is the composition $\prod_n M_n \overset{\mathrm{pr}}{\to} M_{n-1} \times M_n \overset{f_{n-1} - \mathrm{id}}{\to} M_n$. Recall that $\mathrm{H}^i : \mathrm{Vect} \to \mathrm{Vect}^{\heartsuit}$ commutes with products. For $i \in \mathbb{Z}$ we get a part of the long exact sequence

$$\mathrm{H}^i(M) \to \prod_n \mathrm{H}^i(M_n) \overset{\mathrm{H}^i(b)}{\to} \prod_n \mathrm{H}^i(M_n) \to \mathrm{H}^{i+1}(M)$$

Note that the products in the above are actually finite. The kernel of $\mathrm{H}^i(b)$ in the category of abelian groups is $\lim_n \mathrm{H}^i(M_n) = 0$. So, $H^{i+1}(M)$ is the cokernel of $\mathrm{H}^i(b)$ in $\mathrm{Vect}^{\heartsuit}$. We claim that $\mathrm{H}^i(b)$ is surjective in $\mathrm{Vect}^{\heartsuit}$. Indeed, it suffices to show that for any $N < 0$ the map $\prod_{n=N-1}^0 \mathrm{H}^i(M_n) \to \prod_{n=N}^0 \mathrm{H}^i(M_n)$ is surjective. This is easy. □

By this lemma, $\mathrm{QCoh}(S)^- \to \lim_n \mathrm{QCoh}({}^{\leq n}S)^-$ is fully faithful. It remain to show it is essentially surjective. This follows from Corollary 10.2.18 below. Namely, given $(\mathcal{F}_n) \in \lim_n \mathrm{QCoh}({}^{\leq n}S)$, for any N, the sequence $\tau^{\geq -N}(i_n)_* \mathcal{F}_n$ stabilizes, so $\tau^{\geq -N}(\lim_n (i_n)_* \mathcal{F}_n$ identifies with $\tau^{\geq -N}(i_m)_* \mathcal{F}_m$ for m large enough.

In turn, if $L \in A-mod^-$ then to calculate $\tau^{\geq N}(L \otimes_A \tau^{\geq -n} A)$ we may replace L by $\tau^{\geq c} L$ for c small enough. Yuchen Fu also suggests here a reference to ([19], Prop. 4.6).

10.2.17. The following is due to Dima. Let $S = \operatorname{Spec} A \in \mathrm{Sch}^{aff}$, $\ldots \stackrel{f}{\to} M_{-1} \stackrel{f}{\to} M_0$ is a sequence in $A-mod$ indexed by $\mathbb{Z}_{<0}$. Let $M = \lim M_i$. Let $N \in \mathbb{Z}$ and $M' = \lim_n \tau^{\geq N} M_n$ taken in Vect. Then $M \to M'$ induces an isomorphism $\tau^{>N} M \widetilde{\to} \tau^{>N} M'$.

Indeed, we have a fibre sequence in Vect

$$M \to \prod_n M_n \to \prod_n M_n$$

Here each projection $\prod_n M_n \to M_m$ is the composition $\prod_n M_n \to M_{m-1} \times M_m \stackrel{f-\mathrm{id}}{\to} M_m$. Now apply Section 4.0.15 to the latter fibre sequence.

Corollary 10.2.18. *Let $S = \operatorname{Spec} A \in \mathrm{Sch}^{aff}$. Let $\ldots M_{n-1} \to M_n \to \ldots \to M_0$ be a diagram in $(A-mod)$ indexed by $\mathbb{Z}_{<0}$. Let $M = \lim_n M_n$ in $A-mod$. Assume that for any $N \geq 0$ the diagram $\ldots \to \tau^{\geq -N} M_{n-1} \to \tau^{\geq -N} M_n \to \ldots \to \tau^{\geq -N} M_0$ stabilizes. Let $\bar{M}_N$ be the limit in* Vect *of the latter diagram, that is, $\bar{M}_N = \tau^{\geq -N} M_n$ for n small enough. We have a natural map*

$$\tau^{\geq -N-1} M_\bullet \to \tau^{\geq -N} M_\bullet$$

Passing to the limit over $n \in \mathbb{Z}_{<0}$, we get a map $\bar{M}_{N+1} \to \bar{M}_N$. Then $\tau^{\geq -N} \bar{M}_{N+1} \widetilde{\to} \bar{M}_N$ canonically. So, $\bar{M}_\bullet$ is an object of $\lim_N \mathrm{Vect}^{\geq -N} \widetilde{\to} \mathrm{Vect}$. We denote the corresponding object of Vect *by $\bar{M} = \lim_N \bar{M}_N$. The natural maps $M \to \bar{M}_N$ are compatible with transition maps $\bar{M}_{N+1} \to \bar{M}_N$, hence a map $M \to \bar{M}$ in* Vect*. The latter map is an isomorphism.*

10.2.19. For (ch. I.3, 3.7). If $Y \in \mathrm{Stk}$ is perfect then $\mathcal{O}_Y$ is compact in $\mathrm{Ind}(\mathrm{QCoh}(Y)^{perf})$, because $\mathcal{O}_Y \in \mathrm{QCoh}(Y)^{perf}$ and (HTT, 5.3.5.5), the compact generation of $\mathrm{QCoh}(Y)$ also follows from (HTT, 5.3.5.5).

10.2.20. Let $f : X \to Y$ be a morphism in Sch with X, Y quasi-compact. Then for $F \in \mathrm{QCoh}^-$, $f_* F \in \mathrm{QCoh}^-$. Indeed, we may assume Y affine. Then make induction on the number of affine open subschemes U_i which cover X. If X is affine then $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is exact, so this is true. Induction step: assume $X = U_1 \cup U_2$ be open covering with U_2 affine. Assume one may cover U_1 by n open affine subschemes, then one may cover $U_{12} = U_1 \cap U_2$ also by n open affine subschemes. For $F \in \mathrm{QCoh}(X)$ we get an exact triangle $F \to (j_1)_* F_1 \oplus (j_2)_* F_2 \to (j_{12})_* F_{12}$ as in my Section 10.2.3. We know that $f_*(j_i)_* F_i, f_*(j_{12})_* F_{12} \in \mathrm{QCoh}(Y)^-$, so the same holds for $f_* F$, and we actually get an estimation of the cohomological amplitude of f_*. In particular, $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is right t-exact up to a finite shift.

10.2.21. Let M be a connected reductive group, recall that $\mathrm{QCoh}(B(M))$ is rigid. For $q : \operatorname{Spec} k \to B(M)$ consider the dual pair $q^* : \mathrm{QCoh}(B(M)) \leftrightarrows \mathrm{Vect} : q_*$ in DGCat_{cont}. Being rigid, Vect and $\mathrm{QCoh}(B(M))$ are equipped with canonical self-dualities. Under this duality the dual of $q^* : \mathrm{QCoh}(B(M)) \to \mathrm{Vect}$ is q_*.

Proof This is the projection formula for q_*. Namely, for $V \in \mathrm{Vect}, F \in \mathrm{QCoh}(B(M))$ one has $\mathcal{H}om_{\mathrm{QCoh}(B(M))}(e, F \otimes q_* V) \widetilde{\to} \mathcal{H}om_{\mathrm{Vect}}(e, V \otimes q^* F)$.

10.3. **Ind-coherent sheaves.** For (ch. II.1, 1.1). Recall that Sch_{aft} is the category of quasi-compact schemes such that for $n \geq 0$, ${}^{\leq n}Z$ is locally if finite type as a prestack (so, ${}^{\leq n}Z$ admits a Zariski atlas consisting of affine schemes in ${}^{\leq n}\mathrm{Sch}^{aff}_{ft}$ by [14], I.2, 3.5.3).

If $X \in \mathrm{Sch}_{aft}$ then for $F \in \mathrm{QCoh}(X)^{\heartsuit}$ the property of being coherent means that for any open affine subscheme $S = \operatorname{Spec} A \subset X$, $F\mid_S$ is a $\mathrm{H}^0(A)$-module of finite type.

If $X \in \mathrm{Sch}_{aft}$ then $\mathrm{Coh}(X)$ admits finite colimits, so $\operatorname{Ind}\mathrm{Coh}(X)$ is presentable.

In Lm. 1.1.3 an assumption is missing: X is quasi-compact.

10.3.1. In (HA, 7.2.2.10) there is the following definition useful for derived algebraic geometry. Let $S = \operatorname{Spec} A \in \mathrm{Sch}^{aff}$, $M \in A-mod$. Then M is a *flat A-module* iff $\mathrm{H}^0(M)$ is a flat $\mathrm{H}^0(A)$-module in the usual sense, and for any $n \in \mathbb{Z}$ the natural map $\mathrm{H}^n(A) \otimes_{\mathrm{H}^0(A)} \mathrm{H}^0(M) \to \mathrm{H}^n(M)$ is an isomorphism. (In particular, $M \in A-mod^{\leq 0}$).

Let $S = \operatorname{Spec} A \in \mathrm{Sch}^{aff}$, $M \in A-mod^{\leq 0}$. In (HA, 7.2.2.4) Lurie defines the notion for M to be projective. It has several equivalent reformulations in (HA, 7.2.2.6). A free A-module is an A-module of the form $\oplus_{i\in I} A$, where I is a set. Then by (HA, 7.2.2.7), M is projective iff M is a direct summand of a free A-module in $A - mod^{\leq 0}$. Since $\mathrm{H}^n : \mathrm{Vect} \to \mathrm{Vect}^{\heartsuit}$ commutes with direct sums, we see that a projective A-module is flat (HA, 7.2.2.14).

10.3.2. In (HA, 7.2.4.1) Lurie introduces a notion of a perfect A-module different from the one in [14]. Namely, let $S = \operatorname{Spec} A \in \mathrm{Sch}^{aff}$. He defines $(A-mod)^{perf}$ as the smallest stable subcategory of $A-mod$ containing A and closed under retracts. Then (HA, 7.2.4.2) shows that $(A-mod)^c = (A-mod)^{perf}$, so this definition coincides with that of ([14], ch. I.2, 3.6.1). In addition, $\operatorname{Ind}((A-mod)^c) \widetilde{\to} A-mod$ by (HA, 7.2.4.2).

(HA, 7.2.4.5) says in partocular: let $S = \operatorname{Spec} A \in \mathrm{Sch}^{aff}$, $M \in (A-mod)^{perf}$. Then M is upper-bounded, and for any $n \in \mathbb{Z}$, $\mathrm{H}^n(M)$ is finitely-presented as $\mathrm{H}^0(A)$-module.

This implies that if $S \in {}^{\leq n}\mathrm{Sch}^{aff}_{ft}$ for some n then $(A-mod)^{perf} \subset Coh(S)$. Indeed, by Lurie's definition, a perfect A-module M is a retract of some $\mathcal{M} \in A-mod$, where $\mathcal{M}$ is a finite extension of objects of the form $A[i]$, $i \in \mathbb{Z}$. Since A is bounded, M is also bounded. By the above, $\mathrm{H}^i(M)$ is a finitely generated $\mathrm{H}^0(A)$-module.

This implies the fact used in ([14], ch. II.1, 1.1.7): let $X \in \mathrm{Sch}_{aft}$ be eventually coconnective then $\mathrm{QCoh}(X)^{perf} \subset Coh(X)$ (recall that X is quasi-compact).

in Lm. 1.1.7 the categories $\mathrm{QCoh}(X)^{perf}, \mathrm{Coh}(X)$ admit finite colimits, and the inclusion $\mathrm{QCoh}(X)^{perf} \subset \mathrm{Coh}(X)$ preserves finite colimits, so is right exact. Thus, by (HTT, 5.3.5.13), the induced functor $Ind(\mathrm{QCoh}(X)^{perf}) \to \operatorname{Ind}(\mathrm{Coh}(X))$ admits a right adjoint given by sending $f : \mathrm{Coh}(X)^{op} \to \mathrm{Spc}$ to the composition $(\mathrm{QCoh}(X)^{perf})^{op} \to \mathrm{Coh}(X)^{op} \xrightarrow{f} \mathrm{Spc}$. The latter functor coincides with the ind-extension of the inclusion $\mathrm{Coh}(X) \subset \operatorname{Ind}(\mathrm{QCoh}(X)^{perf})$.

Remark: let C be small admitting finite colimits, $C \subset C' \overset{\alpha}{\hookrightarrow} \operatorname{Ind}(C)$ be a full subcategories. Then the inclusion $C \subset C'$ preserves finite colimits by (HTT, 5.3.5.14), and the ind-extension $\operatorname{Ind}(C) \to \operatorname{Ind}(C')$ of the inclusion $C \subset C'$ admits a right adjoint $g : \operatorname{Ind}(C') \to \operatorname{Ind}(C)$. Moreover, the composition $C' \to \operatorname{Ind}(C') \xrightarrow{g} \operatorname{Ind}(C)$ identifies with α by (HTT, 5.3.5.13).

Remark: for any $Y \in$ PreStk, $QCoh(Y)^{perf}$ admits finite colimits, to see this use (ch. I.3, 3.6.4) and the fact that for $Y \in Sch^{aff}$, $\mathrm{QCoh}(Y)^{perf} = \mathrm{QCoh}(Y)^c$.

10.3.3. Proof of (ch. II.1, Lm. 1.2.4), here $C_0 \in \mathrm{DGCat}^{non-cocmpl}$. Let $C = \mathrm{Ind}(C_0)$. Recall that C is stable by (HA, 1.1.3.6), presentable, because C_0 admits finite colimits. It also get a Vect action by passing to colimits, so $C \in \mathrm{DGCat}_{cont}$. The continuous extension $\mathrm{Ind}(C_0^{\le 0}) \to C$ of the inclusion $C_0^{\le 0} \to C$ is fully faithful by (HTT, 5.3.5.11). Since $C_0^{\le 0} \subset C_0$ is closed under finite colimits, $\mathrm{Ind}(C_0^{\le 0})$ is presentable by (HTT, 5.5.1.1). To see that there is a unique t-structure on C with $C^{\le 0} = \mathrm{Ind}(C_0^{\le 0})$ it remains to show, by (HA, 1.4.4.11), that $C^{\le 0} \subset C$ is closed under extensions and colimits. The inclusions $C_0^{\le 0} \subset C_0 \subset \mathrm{Ind}(C_0)$ preserve finite colimits (by HTT, 5.3.5.14). So, by (HTT, 5.5.1.9), $C^{\le 0} \subset C$ preserves colimits. However, it is not evident that $C^{\le 0} \subset C$ is stable under extensions.

We argue instead as follows. Set $C^{>0} = \mathrm{Ind}(C_0^{>0})$. Note that $C_0^{>0}$ admits finite colimits, because C_0 does and $\tau^{>0} : C \to C^{>0}$ preserves colimits. So, $C^{>0}$ is presentable. We have $C^{\le 0}[1] \subset C^{\le 0}$. Indeed, if $x \in C^{\le 0}$ is written as $\mathrm{colim}_{i \in I}\, x_i$ with I filtered, $x_i \in C_0^{\le 0}$ then $\mathrm{colim}_i\, x_i[1] \widetilde{\to} x[1] \in C^{\le 0}$ also. If $y \in C^{>0}$ write $y \widetilde{\to} \mathrm{colim}_{i \in I}\, y_i$ in $\mathcal{P}(C_0)$ with $y_i \in C_0^{>0}$. The natural map $\mathrm{colim}_i(y_i[-1]) \to \Omega y$ in $\mathcal{P}(C_0)$ is an isomorphism by (HTT, 5.3.3.3). Note that $\mathrm{Ind}(C_0^{>0}) \subset \mathrm{Ind}(C_0) \subset \mathcal{P}(C_0)$ are full subcategories. So, $(C^{>0})[-1] \subset C^{>0}$. For $x \in C$ write $x \widetilde{\to} \mathrm{colim}_{i \in I}\, x_i$ in C with $x_i \in C_0$ and I filtered. For each i, we have the fibre sequence $\tau^{\le 0} x_i \to x_i \to \tau^{>0} x_i$ in C_0. Passing to the colimit over $i \in I$, we get a fibre sequence $y \to x \to z$ with $y \in C^{\le 0}$, $z \in C^{>0}$.

Let now $y \in C^{\le 0}, z \in C^{>0}$. Let us show that $\mathrm{Map}_C(y, z) \widetilde{\to} *$. Write $y \widetilde{\to} \mathrm{colim}_{i \in I}\, y_i$ with $y_i \in C_0^{\le 0}$, we see that we may and do assume $y \in C_0^{\le 0}$. Write $z \widetilde{\to} \mathrm{colim}_{j \in J}\, z_j$ in C with $z_j \in C_0^{>0}$ and J filtered. Since $y \in C^c$ by (HTT, 5.3.5.5), we get

$$\mathrm{Map}(y, z) \widetilde{\to} \mathrm{colim}_j\, \mathrm{Map}_{C_0}(y, z_j) \widetilde{\to} *,$$

because $\mathrm{Map}_{\mathcal{C}_0}(y, z_j) \widetilde{\to} *$. By (HA, 1.2.1.1), this is a t-structure on C. The inclusion $\mathrm{Ind}(C_0^{>0}) \to \mathrm{Ind}(C_0)$ preserves filtered colimits by construction. Lm. 1.2.4 is proved.

10.3.4. Proof of (ch. II.1, Lm. 1.2.5) is given in ([13], Pp. 1.2.4).

10.3.5. For (ch. II.1, 1.2.10). Let $X \in \mathrm{Sch}_{aft}$. First, X is a perfect prestack, so $\mathrm{Ind}(\mathrm{QCoh}(X)^{perf}) \widetilde{\to} \mathrm{QCoh}(X)$ by (ch. I.3, 3.7.4).

The $\mathrm{QCoh}(X)^{perf}$-action on $\mathrm{QCoh}(X)$ preserves the full subcategory $\mathrm{Coh}(X)$. Since X is quasi-compact, the question is local so we may assume $X = \mathrm{Spec}\, A$, in which case $\mathrm{QCoh}(X)^{perf}$ is given by (HA, 7.2.4.1). First, for any $n \in \mathbb{Z}$, the action of $\mathcal{O}_X[n]$ preserves it. If now $F \in \mathrm{QCoh}(X)^{perf}$ is a finite extension of objects of the form $\mathcal{O}_X[n]$ then the same by induction. Finally, if F' is a direct summand of F then it remains to prove the following. Let $\mathcal{F} \in \mathrm{Coh}(X)$ be written as $\mathcal{F}_1 \oplus \mathcal{F}_2 = \mathcal{F}$ in $\mathrm{QCoh}(X)$. Then $\mathcal{F}_1 \in \mathrm{Coh}(X)$. Indeed, if $M \in (A-mod)^{\heartsuit}$ is a finite type module over $A = \mathrm{H}^0(X)$ then a direct summand of M is also a finite A-module, because A is a finite type k-algebra.

10.3.6. For (ch. II.1, 2.1.2), they use (ch. I.3, 2.3.2) in the proof.

For 2.3.1: If $f : X \to Y$ is a map in Sch_{aft} then $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is a morphism of $\mathrm{QCoh}(Y)$-modules by (ch. I.1, 9.3.6).

10.3.7. For 2.2.4. For the definition of DGCat^t_{cont}. For an object $C \in \mathrm{DGCat}^t_{cont}$ when they say that C is "compactly generated by objects from C^+", it is meant that there is a full subcategory $C_0 \subset C^c \cap C^+$ such that C_0 generates C in the sense of (ch. I.1, 5.4.1).

10.3.8. For 3.1.2: if $f : X \to Y$ is a map in Sch_{aft} let $M \in \mathrm{QCoh}(Y)^-$ with coherent cohomologies. Then $f^*M \in \mathrm{QCoh}(X)^-$ also has coherent cohomologies. Indeed, since f^* sends $\mathrm{QCoh}(Y)^{\le m}$ to $\mathrm{QCoh}(X)^{\le m}$, we may assume M bounded from below. Hence, it suffices to treat the case when $M \in \mathrm{Coh}(Y)^{\heartsuit}$. The problem is local, assume X, Y affine, let $A \to B$ be the corresponding map in $CAlg(\mathrm{Vect}^{\le 0})$. Then M is a finite type $\mathrm{H}^0(A)$-module. As in mentionned in (HA, after 7.2.4.7), there is a resolution $\ldots \to P_{-1} \to P_0 \to M \to 0$, where each P_i is a free A-module of finite rank. So, $M \otimes_A B$ will be represented by a complex $\ldots P_{-1} \otimes_A B \to P_0 \otimes_A B$ consisting of free B-modules of finite type. Thus, each cohomology of $M \otimes_A B$ will be a finite type $\mathrm{H}^0(B)$-module.

10.3.9. For 3.1.4: the class of eventually coconnective morphsms in Sch_{aft} is stable under base change. In [13] this is 3.6.3, 3.6.8.

10.3.10. For 3.3.1. Let $f : X \to Y$ be eventually coconnective map in Sch_{aft}. By (3.5) they mean the functor sending $G \in \mathrm{QCoh}(X), F \in \mathrm{IndCoh}(Y)$ to $G \otimes f^{\mathrm{IndCoh},*}F$.

10.3.11. For 3.3.2, it appears as ([13], 4.4.2). First, $\mathrm{QCoh}(X) \otimes \mathrm{IndCoh}(Y)$ is compactly generated by objects of the form $F \boxtimes G$ with $F \in \mathrm{QCoh}(X)^{perf}, G \in \mathrm{Coh}(Y)$ by (ch. I.1, 7.4.2). We used the fact that $\mathrm{QCoh}(X)$ is perfect (ch. I.3, 3.7.4). Further, the right adjoint to $\mathrm{QCoh}(X) \otimes \mathrm{IndCoh}(Y) \to \mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{IndCoh}(Y)$ is continuous by (ch. I.1, 8.7.2), because $\mathrm{QCoh}(Y)$ is rigid. So, for $F \in \mathrm{QCoh}(X)^{perf}, G \in \mathrm{Coh}(Y)$, $F \otimes G$ is compact in $\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{IndCoh}(Y)$. by (ch. I.1, 8.7.2). Such compact objects generate the latter category by (ch. I.1, 8.2.6 and 8.7.4). Then continue as in ([13], 4.4.2).

10.3.12. For 3.3.4. They apply their general remark about D_i and F_i to the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{IndCoh}(Y) & \stackrel{T}{\to} & \mathrm{IndCoh}(X) \\ & \nwarrow f^*\otimes\mathrm{id} & \uparrow f^{\mathrm{IndCoh},*} \\ & & \mathrm{IndCoh}(Y) \end{array}$$

with T being (3.5). Clearly, $f^* \otimes \mathrm{id}$ here has a right adjoint $f_* \otimes \mathrm{id}$, where $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is the direct image for QCoh.

10.3.13. For 3.3.9. The fact that $f_*(\mathcal{E}_X) \in \mathrm{QCoh}(Y)$ is upper-bounded follows from my Section 10.2.20. Their $f_*(\mathcal{E}_X) \in \mathrm{QCoh}(Y)^b$ is of bounded tor dimension in the sense that there is N such that for any $M \in \mathrm{QCoh}(Y)^{\heartsuit}$, $M \otimes f_*(\mathcal{E}_X) \in \mathrm{QCoh}(Y)$ is placed in degrees $\geq N$. This follows from $M \otimes f_*(\mathcal{E}_X) \widetilde{\to} f_*(M \otimes \mathcal{E}_X)$, as we may assume $\mathcal{E}_X$ is bounded, finite extension of objects of the form $\mathcal{O}_X[i]$ (maybe after Zariski localization).

10.3.14. For 4.1.1. If $X \in \mathrm{Sch}_{aft}$, and $X^0 \subset X$ is an open embedding then $X^0 \in \mathrm{Sch}_{aft}$. To see that X^0 is quasi-compact, we may assume X classical, and moreover affine. If the complement to X^0 is given by the ideal $(f_1, \dots, f_n) \subset A$ with $\operatorname{Spec} A = X$ then $\cup_i \operatorname{Spec} A_{f_i} = X^0$ is a finite covering by affine open subschemes.

10.3.15. For 4.2.2. If $X \in \mathrm{Sch}$, $U_i \subset X$ open affine subschemes with $U_1 \cup U_2 = X$, for $F \in \mathrm{QCoh}(X)$ one has the fibre sequence described in Remark of my Section 10.2.3, they used this at the end of the proof.

10.3.16. For 4.2.4: $\mathrm{IndCoh}(X)^{\geq 0} \subset \mathrm{IndCoh}(X)$ is closed under limits, and f_*^{IndCoh} is left t-exact.

10.3.17. For 5.1.2. If $f : X \to Y$ is a closed immersion in Sch_{aft} and $Y \in \mathrm{Sch}^{aff}$ then X is also affine, because ${}^{cl}X$ is affine (by ch. I.2, 3.2.7).

10.3.18. For 5.1.8. If $f : X \to Y$ is a morphism in Sch_{aft} then $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ is right t-exact up to a finite shift. Indeed, this is true for $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$, see my Section 10.2.20. Recall that $\mathrm{IndCoh}(X)^{\leq n} = \{F \in \mathrm{IndCoh}(X) \mid \Psi_X(F) \in \mathrm{QCoh}(X)^{\leq n}\}$. Using (ch. II.1, 2.1.2), we are done.

If f_*^{IndCoh} sends $\mathrm{IndCoh}(X)^{\leq 0}$ to $\mathrm{IndCoh}(Y)^{\leq n}$ for any m then $f^!$ sends $\mathrm{IndCoh}(Y)^{\geq 0}$ to $\mathrm{IndCoh}(X)^{\geq -n}$.

10.3.19. For 5.1.10. Let $Y = \operatorname{Spec} k[t]/(t^2)$. Then Y is not smooth, but is eventually coconnective. So, $\Psi_Y : \mathrm{IndCoh}(Y) \to \mathrm{QCoh}(Y)$ is not an equivalence, and$\mathrm{QCoh}(Y)$ is a colocalization of $\mathrm{IndCoh}(Y)$. So, there is $F \in \mathrm{IndCoh}(Y)$ with $\Psi_Y(F) \widetilde{\to} 0$. Note that $\mathcal{O}_Y$ does not lie in $\mathrm{QCoh}(Y)^{perf}$.

10.3.20. If $X \in \mathrm{Sch}_{aft}$ then $\mathrm{IndCoh}(X)^{\heartsuit} \widetilde{\to} \mathrm{Ind}(\mathrm{Coh}(X)^{\heartsuit})$ by (HTT, 5.3.5.6), we may also see it as a localization of $\mathrm{IndCoh}(X)^{\leq 0}$.

For the proof of 6.1.3. The formula $\mathrm{IndCoh}(X)^{\heartsuit}_Z = \mathrm{Coh}(X)^{\heartsuit}_Z$ that they wrote is wrong. I think they meant $\mathrm{IndCoh}(X)^{\heartsuit}_Z = \mathrm{Ind}(\mathrm{Coh}(X)^{\heartsuit}_Z)$. Indeed, if $F \in \mathrm{IndCoh}(X)^{\heartsuit}_Z$, write $F \widetilde{\to} \operatorname{colim}_{i\in I} F_i$ in $\mathrm{IndCoh}(X)$ with $F_i \in \mathrm{Coh}(X)^{\heartsuit}$. We want to show that we may assume that $j^{\mathrm{IndCoh},*} F_i = 0$ for all i. Let $K = \mathrm{Ker}(\mathcal{O}_X \to j_*\mathcal{O}_U)$. Then we get $F \widetilde{\to} \operatorname{colim}_i(\mathcal{K} \otimes F_i) \widetilde{\to} \operatorname{colim}_i \tau^{\geq 0}(\mathcal{K} \otimes F_i)$ in $\mathrm{IndCoh}(X)$. Moreover, each $\mathcal{K} \otimes F_i \in \mathrm{Coh}(X)^{\leq 0}$, hence $\tau^{\geq 0}(\mathcal{K} \otimes F_i) \in \mathrm{Coh}(X)^{\heartsuit}$. Since $j^{\mathrm{IndCoh},*}(\mathcal{K} \otimes F_i) = 0$, we are done.

10.3.21. For 6.1.5. I think that for $X \in Sch_{aft}$ and the closed immersion $i : {}^{cl}X \to X$ the composition $\mathrm{IndCoh}({}^{cl}X) \xrightarrow{i_*} \mathrm{IndCoh}(X) \xrightarrow{i^!} \mathrm{IndCoh}({}^{cl}X)$ is not the identity.

Consider the example $X = \operatorname{Spec} A$, where $A = k[\eta]$ with $\deg \eta = -2$. Then ${}^{cl}X = \operatorname{Spec} k$, what is $i^! k$? We have a fibre sequence $A[2] \to A \to k$ in $\mathrm{QCoh}(X)$. For $n \leq 0$ it yields a fibre sequence

$$\mathrm{Map}_{A-mod}(k, k[n]) \to \mathrm{Map}_{\mathrm{Vect}}(k, k[n]) \to \mathrm{Map}_{\mathrm{Vect}}(k[2], k[n])$$

in Spc. For $n = 0, 1$ this gives $\mathrm{Map}_{A-mod}(k, k[n]) \widetilde{\to} Dold - Kan(k[n])$. For $n \geq 2$ we get a fibre sequence $\mathrm{Map}_{A-mod}(k, k[n]) \to Dold - Kan(k[n]) \to Dold - Kan(k[n-2])$

in Spc. In particular, for $n\ge 2$ the natural map

$$\mathrm{Map}_{\mathrm{IndCoh}(^{cl}X)}(k,k[n])\widetilde{\to}\mathrm{Map}_{\mathrm{Coh}(^{cl}X)}(k,k[n])\to\mathrm{Map}_{\mathrm{Coh}(X)}(i_*k,i_*k[n])\widetilde{\to}$$
$$\mathrm{Map}_{\mathrm{IndCoh}(^{cl}X)}(k,i^!i_*^{\mathrm{IndCoh}}(k[n]))$$

is not an isomorphism in general. It also shows that the assumption $n\ge k$ in (ch. II.1, Lm. 6.4.4) is necessary.

10.3.22. For 6.2.2. One is tempted to consider a general situation here. Let $C_1\overset{i_!}{\to} C\overset{j_*}{\leftarrow} C_0$ be full embeddings in DGCat_{cont}. Assume we have adjoint pairs of functors $j^*: C\leftrightarrows C_0: j_*$ and $i_!: C_1\leftrightarrows C: i^!$ in DGCat_{cont}, and $C_1=\{c\in C\mid j^*c=0\}$. Assume moreover that $i^!j_*=0$. Then for any $F\in C$ we get a fibre sequence

$$i_!i^!F\to F\to j_*j^*F$$

in C. Indeed, let K be the fibre of $F\to j_*j^*F$. Then $K\in C_1$ and $i^!K\to i^!F$ is an isomorphism, so $K\widetilde{\to} i_!i^!F$.

This is what happens for $C=\mathrm{IndCoh}(X)$, with a closed embedding $i:Z\to X$ and $C_1=\mathrm{IndCoh}(X)_Z$, $C_0=\mathrm{IndCoh}(U)$ for the complement $j:U\to X$ of Z in X.

For 6.2.4. Let $f:X\to Y$ be a proper surjective morphism of classical (quasi-compact separated) schemes, where Y is smooth. To see that the essential image f_* of $\mathrm{QCoh}(X)$ generates $\mathrm{QCoh}(Y)$ Dima suggests to show that $f_*\,\mathrm{QCoh}(X)^{perf}$ generates $\mathrm{QCoh}(Y)$ under direct summands and cones.

10.3.23. For 6.3.4. In the proof the isomorphism in the first displayed diagram is that of (ch. I.1, 7.4.2).

For 6.4.3. Let $\mathcal{F}_1,\mathcal{F}_2\in \mathrm{Coh}(^{cl}X)^{\heartsuit}$. Let $\bar{i}_n: {}^{cl}X\to {}^{\le n}X$ be the natural closed immersion. By $\mathrm{colim}_n\,\mathrm{Map}_{\mathrm{Coh}(^{\le n}X)}(\mathcal{F}_1,\mathcal{F}_2[k])$ then mean

$$\mathrm{colim}_n\,\mathrm{Map}_{\mathrm{Coh}(^{\le n}X)}((\bar{i}_n)_*\mathcal{F}_1,(\bar{i}_n)_*\mathcal{F}_2[k])$$

taken in Spc. This sequence stabilizes by their Lemma 6.4.4 with value

$$\mathrm{Map}_{\mathrm{Coh}(X)}((i_0)_*\mathcal{F}_1,(i_0)_*\mathcal{F}_2[k])$$

They implicitly use (HA, 1.1.4.6) saying that $1-\mathcal{C}\mathrm{at}^{St}$ admits filtered colimits, and the inclusion $1-\mathcal{C}\mathrm{at}^{St}\to 1-\mathcal{C}\mathrm{at}$ preserves filtered colimits. The description of the mapping spaces in filtered colimits in $1-\mathcal{C}\mathrm{at}$ given in [46] is also used. They also implicitly use (ch. I.1, 7.2.7), which is written with a mistake, see my comments about this in Section 4.2.8.

For 6.4.4. For $\mathcal{F}_1\in\mathrm{QCoh}(^{\le n}X)^{\le 0}$, $(i_n)_*\mathcal{F}_1$ is a direct summand in $(i_n)_*i_n^*(i_n)_*\mathcal{F}_1$ in $\mathrm{QCoh}(X)$ by my Section 9.2.26.

10.3.24. Assume we work with the classical prestacks (no derived algebraic geometry). Let $Y_i\in\mathrm{PreStk}$ for $i=1,2$. Recall that the inner hom $\underline{\mathrm{Hom}}(Y_1,Y_2)$ in PreStk is given by $\mathrm{Map}(S,\underline{\mathrm{Hom}}(Y_1,Y_2))\widetilde{\to}\mathrm{Map}(S\times Y_1,Y_2)$ for $S\in\mathrm{Sch}^{aff}$. If $Y_2\in\mathrm{PreStk}_{lft}$ and $Y_1\in\mathrm{Sch}^{aff}$ then $\underline{\mathrm{Hom}}(Y_1,Y_2)\in\mathrm{PreStk}_{lft}$ also.

Indeed, let $S\in \mathrm{Sch}^{aff}$ be written as $S\widetilde{\to}\lim_{i\in I^{op}} S_i$ with $S_i\in \mathrm{Sch}^{aff}$ and I filtered. Then $S\times Y_1\widetilde{\to}\lim_{i\in I^{op}} S_i\times Y_1$ in Sch^{aff}, so

$$\mathrm{Map}(S,\underline{\mathrm{Hom}}(Y_1,Y_2))\widetilde{\to}\operatorname{colim}_{i\in I}\mathrm{Map}(S_i\times Y_1,Y_2)\widetilde{\to}\operatorname{colim}_{i\in I}\mathrm{Map}(S_i,\underline{\mathrm{Hom}}(Y_1,Y_2)).$$

11. Correspondences

I am using the version on Dennis' homepage with old numbering of sections.

11.1. For ([14], ch. V.1, 1.1.2). We explain the second claim. After the base change by $c_3\to c_3'$, we get the diagram

$$\begin{array}{ccccc} c & \stackrel{v}{\to} & c_1'' & \to & c_1\\ \downarrow u & & \downarrow u'' & & \downarrow b\\ c_2'' & \stackrel{v''}{\to} & \bar{c}' & \to & \bar{c}_1'\\ \downarrow & & \downarrow & & \downarrow\\ c_2 & \stackrel{a}{\to} & \bar{c}_2' & \to & c_3\end{array}$$

with $c=c_1\times_{c_3}c_2$, and it suffices to show that $v''u$ is in adm. Here all the four small squares are cartesian, and $a,b\in adm$. So, $u,v''\in adm$, hence $v''u\in adm$ as required.

11.1.1. For 1.1.1. The assumptions imply that all the isomorphisms are in adm, because for $\alpha: x\widetilde{\to}y$ in C, $\alpha^{-1}:y\to x$ is the base change of id by α.

12. Appendices to Lurie, HTT

12.0.1. About model categories. Explanation of the proof of ([27], A.2.3.1). Recall that cofibration (resp., trivial cofibrations) are preserved by push-outs. So, $B\sqcup B\to C(A)\sqcup_{A\sqcup A}(B\sqcup B)$ is a cofibration. The object $C(A)\sqcup_A B$ that he considers is defined by the diagram $C(A)\leftarrow A\sqcup A\stackrel{l}{\leftarrow} A\stackrel{i}{\to} B$, here l means the left map. First, pick a morphism $h': C(A)\to X$ such that the composition $A\sqcup A\to C(A)\stackrel{h'}{\to} X$ is $(g'i,f)$. This gives the unique map $h_0: C(A)\sqcup_A B\to X$ given as (h',g'). The map $\kappa: C(A)\sqcup_A B\to C(B)$ that he uses is defined as the composition

$$C(A)\sqcup_A B\stackrel{a}{\to} C(A)\sqcup_{A\sqcup A}(B\sqcup B)\to C(B),$$

where a comes from the left map $l: B\to B\sqcup B$. The map a is a cofibration, it is obtained as a push-out of the cofibration $C(A)\to C(A)\sqcup_A B$. So, κ is a cofibration.

Let us show that κ is a w.eq. It suffices to show that κ composed with $C(B)\to B$ is a week equivalence. For this in turn it suffices to show that the map $B\to C(A)\sqcup_A B$ is a w.eq. But the latter map is the push-out of the trivial cofibration $A\to C(A)$ (the left one) via the map $i: A\to B$. The map $A\to C(A)$ is a cofibration, as it is the composition of cofibrations $A\to A\sqcup A\to C(A)$; it is also a weak equivalence, because the composition $A\to C(A)\to A$ is the identity.

Now we get $h: C(B)\to X$ extending h_0, because the trivial cofibration κ has the left lifting property with respect to the fibration $X\to *$.

12.0.2. In a model category a composition of cofibrations (resp., of fibrations) is a cofibration (resp., fibration).

13. Appendice: $(\infty, 2)$-categories

13.1. By ([28], 4.2.1.35), one may define an $(\infty, 2)$-category as a $(\infty, 1)$-category enriched over $1 - \mathcal{C}at$. Here $1 - \mathcal{C}at$ is viewed as monoidal $(\infty, 1)$-category with its cartesian symmetric monoidal structure. Here the notion of being enriched is that of ([28], 4.2.1.28).

A different approach via complete Segal spaces is taken in ([14], A.1).

For example, if E_1 is a group object in Spc then this is a Segal space, and $\pi_0(E_1)$ is a group in *Sets*. Then E_1 is a complete Segal space iff $E_1 = *$.

13.1.1. There is a notion of a *strict 2-category.* Namely, look at the usual category $\mathcal{C}at$, whose objects are categories, and whose morphisms are functors. This is a monoidal category with respect to the cartesian product. Now a strict 2-category is a $\mathcal{C}at$-enriched category in the sense of (HTT, A.1.4). Each strict 2-category should give rise to an object of $2 - \mathcal{C}at$. The axioms of a strict 2-category given nlab are also clear.

The basic example of a strict 2-category is say $\underline{\mathcal{C}at}$. Its objects are usual categories. For $A, B \in \underline{\mathcal{C}at}$, $\mathbf{Map}_{\underline{\mathcal{C}at}}(A, B)$ is the usual category of functors $\mathrm{Fun}(A, B)$, and the composition is the composition of functors. So, the 2-morphisms are natural transformations of functors. We keep as in [14] the symbol **Map** to denote for any $(\infty, 2)$-category $\mathcal{C}$ the $(\infty, 1)$-category $\mathbf{Map}_{\mathcal{C}}(c_1, c_2)$ defined in (ch. 10, 2.2.7).

Another example, if A is an abelian group, $B^2(A) \in$ Spc can be seen as a strict 2-category: one object $*$; one 1-morphism $\mathrm{id} : * \to *$, and 2-morphisms $\mathrm{id} \to \mathrm{id}$ are A. Both vertical and horizontal compositions are given by the product in A.

If $X \in \tau_{\le 2}\,\mathrm{Spc}$ then for any $x, x' \in X$, $\mathrm{Map}_X(x, x') \in \tau_{\le 1}\,\mathrm{Spc}$ is a usual groupoid. However, we can not think of this X as a strict 2-category, because for $x, x', x'' \in X$ the composition $\mathrm{Map}_X(x', x'') \times \mathrm{Map}_X(x, x') \to \mathrm{Map}_X(x, x'')$ is associative only up to coherent homotopy!

13.1.2. There is a mistake in (Ch. 10, Sect. 1.2.2), it is already corrected in the version of May 4, 2020. Namely, let $E : \boldsymbol{\Delta}^{op} \to \mathrm{Spc}$ be a functor such that for any n, m, $E_{n+m} \widetilde{\to} E_n \times_{E_0} E_m$. The map $a : \pi_0(E_1 \times_{E_0} E_1) \to \pi_0(E_1) \times_{\pi_0(E_0)} \pi_0(E_1)$ is a surjection. But in general the product map $m : \pi_0(E_1 \times_{E_0} E_1) \to \pi_0(E_1)$ does not factor through a.

For example, suppose E comes from the Cech nerve corresponding to a map $B(H) \to B(G)$, where $H \to G$ is a homomorphism of descrete groups. Then $\pi_0(E_1)$ is the set of diagonal G-orbits on $G/H \times G/H$, $\pi_0(E_2)$ is the set of diagonal G-orbits on $G/H \times G/H \times G/H$. Say the product map $E_1 \times_{E_0} E_1 \to E_1$ is given by the projection on $(1, 3)$-factors. In this case we don't have in general a map m as above.

For example, take $G = S_n$ and $H = S_{n-1}$. Then $G/H = \{1, ..., n\}$. If $n \ge 3$ then m does not exist.

Let *comp* : $E_2 \to E_1$ be the map corresponding to $[1] \stackrel{0,2}{\to} [2]$. The good definition: an object $\beta \in E_1$ lies in the full subspace E_1^{invert} if there is $(\beta, \gamma) \in E_1 \times_{E_0} E_1 \widetilde{\to} E_2$ such that $comp(\beta, \gamma)$ lies in the essential image of $\delta : E_0 \to E_1$, and there is $(\gamma', \beta) \in E_1 \times_{E_0} E_1$ such that $comp(\gamma', \beta)$ lies in the essential image of $\delta : E_0 \to E_1$. The latter maps corresponds to $[1] \to [0]$. This is essentially the definition from ([32], 1.1.6).

13.1.3. The functor $Seq_\bullet : 1-\mathcal{C}\mathrm{at} \to \mathrm{Funct}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ from ([14], A.1, 1.3.1) is rigotously defined as follows. It is easier to define a functor $1-\mathcal{C}\mathrm{at} \times \boldsymbol{\Delta}^{op} \to Spc$, it sends $(\mathcal{C}, [n])$ to $\mathrm{Funct}([n], \mathcal{C})^{\mathrm{Spc}} = \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], \mathcal{C})$. It makes sense because there is a functor $1-\mathcal{C}\mathrm{at} \times 1-\mathcal{C}\mathrm{at}^{op} \to 1-\mathcal{C}\mathrm{at}$, $(\mathcal{C}, \mathcal{D}) \mapsto \mathrm{Funct}(\mathcal{D}, \mathcal{C})$.

The existence of the left adjoint to $Seq_\bullet$ follows from my Lemma 2.2.40.

13.1.4. For (A.1 , Sect. 1.4.5): let $f : C \to D$ be a map in $1-\mathcal{C}\mathrm{at}$ such that $\mathrm{Seq}_1(C) \to \mathrm{Seq}_1(D)$ is fully faithful. Then $Seq_0(C) \to \mathrm{Seq}_0(D)$ is also fully faithful, as these are retracts of Seq_1 (cf. my Section 2.2.17). So, inside $\mathrm{Seq}_1(D)$ we get the full subcategories $\mathrm{Seq}_1(C)$ and $\mathrm{Seq}_1(D) \times_{\mathrm{Seq}_0(D)\times \mathrm{Seq}_0(D)} \mathrm{Seq}_0(C) \times \mathrm{Seq}_0(C)$. Thus, $C \to D$ is 1-fully faithful by (A.1, 1.4.3) and $C^{\mathrm{Spc}} \to D^{\mathrm{Spc}}$ is fully faithful. So, f is 1-replete.

13.1.5. For ([14], ch. 12, Sect. 2.1). If $E_\bullet \in \mathrm{Funct}(\boldsymbol{\Delta}, 1-\mathcal{C}\mathrm{at})$ satisfies Conditions 0,1 then $(E_\bullet)^{\mathrm{Spc}} \in \mathrm{Funct}(\boldsymbol{\Delta}, \mathrm{Spc})$ is the functor sending $[n]$ to $E_1^{\mathrm{Spc}} \times_{E_0} \ldots \times E_1^{\mathrm{Spc}}$. Indeed, $1-\mathcal{C}\mathrm{at} \to \mathrm{Spc}$, $(\mathcal{X} \mapsto \mathcal{X}^{\mathrm{Spc}})$ preserves limits.

13.1.6. For ([14], ch. 12, Sect. 2.2.5). The functor $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}^{ordn}$ sending $\mathcal{C}$ to its homotopy category $\mathcal{C}^{ordn}$ does not commutes with fibred products. For example, if A is an abelian group in $Sets$ then the diagram is cartesian

$$\begin{array}{ccc} B^2(A) & \to & B^2(A) \times B^2(A) \\ \uparrow & & \uparrow \\ B(A) & \to & * \end{array}$$

Indeed, for any $\mathcal{A} \in 1-\mathcal{C}\mathrm{at}$ with finite limits and final object $* \in \mathcal{A}$, for $x \in \mathcal{A}$ we get $x \times_{x\times x} * \times * \widetilde{\to} * \times_x *$ in $\mathcal{A}$. Besides, $\Omega B^2(A) \widetilde{\to} B(A)$. Applying $ordn$ to the above square, we get $B(A)^{ordn} = B(A)$, it is different from

$$* = (B^2(A))^{ordn} \times_{(B^2(A)\times B^2(A))^{ordn}} \quad *$$

For any spaces S_i, the natural map $\pi_0(S_1 \times_{S_2} S_3) \to \pi_0(S_1) \times_{\pi_0(S_2)} \pi_0(S_3)$ is surjective. For any diagram $\mathcal{A} \to \mathcal{B} \leftarrow \mathcal{C}$ in $1-\mathcal{C}\mathrm{at}$ the functor $(\mathcal{A} \times_{\mathcal{B}} \mathcal{C})^{ordn} \to \mathcal{A}^{ordn} \times_{\mathcal{B}^{ordn}} \mathcal{C}^{ordn}$ is essentially surjective.

The construction of left adjoint functor to $2-\mathcal{C}\mathrm{at}^{2-ordn} \hookrightarrow 2-\mathcal{C}\mathrm{at}$. Given $E \in 2-\mathcal{C}\mathrm{at}$ the corresponding object E^{2-ordn} should be an element of $\mathrm{Funct}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}\mathrm{at}^{ordn})$ given by

$$[n] \mapsto E_1^{1-ordn} \times_{E_0^{1-ordn}} \ldots \times E_1^{1-ordn}$$

Let $E \in 2-\mathcal{C}\mathrm{at}$. They claim that the natural functor $E_n^{ordn} \to E_1^{ordn} \times_{E_0^{ordn}} \ldots \times E_1^{ordn}$ is an equivalence. It is clearly essentially surjective. For $n = 2$ let us try to check this is fully faithful. Let $r, l : E_1 \to E_0$ be the end and the source of the map. Any object of E_2 is isomorphic to an object of the form $(x_1, x_2, r(x_1) \stackrel{\mathrm{id}}{\to} l(x_2))$. Indeed, if $(x_1, x_2, \alpha : r(x_1) \widetilde{\to} l(x_2)) \in E_1$ then replace x_2 by $d(\alpha) \circ x_2$, where $d : E_0 \to E_1$ is the corresponding map. Let $x, y \in E_2$, whose images under the two maps $r, l : E_1 \to E_0$ are the same, x_0, y_0 respectively. So, $x = (x_1, x_2, \mathrm{id} : r(x_1) \to l(x_2) = x_0)$ and similarly for y. Recall that $\mathrm{Map}_{E_2}(x, y) \widetilde{\to} \mathrm{Map}_{E_1}(x_1, y_1) \times_{\mathrm{Map}_{E_0}(x_0,y_0)} \mathrm{Map}_{E_1}(x_2, y_2)$. We want to show that the induces map

$$\pi_0 \mathrm{Map}_{E_2}(x, y) \to \pi_0 \mathrm{Map}_{E_1}(x_1, y_1) \times_{\pi_0 \mathrm{Map}_{E_0}(x_0,y_0)} \pi_0 \mathrm{Map}_{E_1}(x_2, y_2) \tag{19}$$

is bijective, only the injectivity is nontrivial. For any diagram of spaces $S_1 \to S_2 \leftarrow S_3$ and a point $(s_1, s_3) \in S_1 \times S_3$ whose image in $S_2 \times S_2$ is (s_2, s_2), one has a cartesian square

$$\begin{array}{ccc} \Omega(S_2, s_2) & \widetilde{\to} & \{s_1, s_3\} \\ \downarrow & & \downarrow \\ S_1 \times_{S_2} S_3 & \to & S_1 \times S_3 \end{array}$$

and triviality of the fibre of $\pi_0(S_1 \times_{S_2} S_3) \to \pi_0(S_1) \times \pi_0(S_3)$ over (s_1, s_3) is equivalent to the surjectivity of the map $\pi_1(S_1, s_1) \times \pi_1(S_3, s_3) \to \pi_1(S_2, s_2)$.

Since E_0 is a space, arguing as above we may assume in addition that $x_0 = y_0$, and we are analyzing the fibre of (19) over (α_1, α_2), where the images of α_i in $\pi_0 \operatorname{Map}_{E_0}(x_0, x_0)$ are the identities. Then we have to show that

$$\pi_1(\operatorname{Map}_{E_1}(x_1, y_1), \alpha_1) \times \pi_1(\operatorname{Map}_{E_1}(x_2, y_2), \alpha_2) \to \pi_1(\operatorname{Map}_{E_0}(x_0, y_0), \mathrm{id}) \tag{20}$$

is surjective. Why this is so?

Remark 13.1.7. *1) Let $X \stackrel{a}{\to} Y \leftarrow Z$ be a diagram in* Spc*, assume for any $x \in X$, $\operatorname{Map}_X(x, x) \to \operatorname{Map}_Y(a(x), a(x))$ is essentially surjective, that is, $\pi_1(x, X) \to \pi_1(a(x), Y)$ is surjective. Then $\pi_0(X \times_Y Z) \to \pi_0(X) \times_{\pi_0(Y)} \pi_0(Z)$ is an isomorphism. 2) Let $X \stackrel{a}{\to} Y \leftarrow Z$ be a diagram in $1-\mathcal{C}\mathrm{at}$. Assume for any map $\beta : x_1 \to x_2$ in X the natural map $\pi_1(\beta, \operatorname{Map}_X(x_1, x_2)) \to \pi_1(a(\beta), \operatorname{Map}_Y(\bar{x}_1, \bar{x}_2))$ is surjective. Then the natural functor $\alpha : (X \times_Y Z)^{ordn} \to X^{ordn} \times_{Y^{ordn}} Z^{ordn}$ is an equivalence.*

Proof. 1) is easy. 2) Our α is always essentially surjective. Let us check that it is fully faithful. Let $(x_1, z_1), (x_2, z_2) \in X \times_Y Z$, write $\bar{x}_1, \bar{x}_2$ for the images of x_1, x_2 in Y. Recall that

$$\operatorname{Map}_{X \times_Y Z}((x_1, z_1), (x_2, z_2)) \widetilde{\to} \operatorname{Map}_X(x_1, x_2) \times_{\operatorname{Map}_Y(\bar{x}_1, \bar{x}_2)} \operatorname{Map}_Z(z_1, z_2)$$

We have to show that the natural map

$$\pi_0(\operatorname{Map}_X(x_1, x_2) \times_{\operatorname{Map}_Y(\bar{x}_1, \bar{x}_2)} \operatorname{Map}_Z(z_1, z_2)) \to \pi_0 \operatorname{Map}_X(x_1, x_2) \times_{\pi_0 \operatorname{Map}_Y(\bar{x}_1, \bar{x}_2)} \pi_0 \operatorname{Map}_Z(z_1, z_2)$$

is a bijection. To this end, it suffices to show that for any $\beta \in \operatorname{Map}_X(x_1, x_2)$ the natural map $\Omega(\beta, \operatorname{Map}_X(x_1, x_2)) \to \Omega(a(\beta), \operatorname{Map}_Y(\bar{x}_1, \bar{x}_2))$ is essentially surjective, that is, $\pi_1(\beta, \operatorname{Map}_X(x_1, x_2)) \to \pi_1(a(\beta), \operatorname{Map}_Y(\bar{x}_1, \bar{x}_2))$ is surjective. □

13.1.8. For ch. 10. Let $\mathbb{S} \in 2-\mathcal{C}\mathrm{at}$ and $E = \operatorname{Seq}_\bullet(\mathbb{S})$. Recall their notation for $c, c' \in E_0$, $\mathbf{Map}_{\mathbb{S}}(c, c') = E_1 \times_{E_0 \times E_0} \{c, c'\}$. If we think of $\mathbb{S}$ as enriched over $1-\mathcal{C}\mathrm{at}$ then this is the corresponding mapping category. For $c, c', c'' \in E_0$ the composition $\mathbf{Map}_{\mathbb{S}}(c', c'') \times \mathbf{Map}_{\mathbb{S}}(c, c') \to \operatorname{Map}_{\mathbb{S}}(c, c''')$ is defined as

$$\begin{aligned} \mathbf{Map}_{\mathbb{S}}(c', c'') \times \mathbf{Map}_{\mathbb{S}}(c, c') \widetilde{\to} (E_1 \times_{E_0} E_1) \times_{E_0 \times E_0 \times E_0} \{c, c', c''\} \stackrel{m}{\to} \\ E_1 \times_{E_0 \times E_0} \{c, c'''\} \widetilde{\to} \mathbf{Map}_{\mathbb{S}}(c, c''') \end{aligned} \tag{21}$$

where m is the product map.

Let now $a, b \in E_0, f : a \to b$ be a map in E_1. For strict 2-categories there is a notion of left whiskering, see (nlab, strict 2-categories). Its analog in this setting is as follows. Let $d : E_0 \to E_1$ be the map corresponding to $[1] \to [0]$, we think of it as sending b to the identity map $\mathrm{id} : b \to b$ in $\mathbb{S}$. The map d yields the natural map

$\mathrm{Map}_{E_0}(b,b)\widetilde{\to}\Omega(b,E_0)\to *\times_{E_1} *\to \mathrm{Map}_{\mathbf{Map}_{\mathbb{S}}(b,b)}(d(b),d(b)))$, where the maps $*\to E_1$ are both given by $d(b)$. Now (21) yields a morphism

$$\mathrm{Map}_{\mathbf{Map}_{\mathbb{S}}(b,b)}(d(b),d(b)))\times \mathrm{Map}_{\mathbf{Map}_{\mathbb{S}}(a,b)}(f,f)\to \mathrm{Map}_{\mathbf{Map}_{\mathbb{S}}(a,b)}(f,f),$$

because the composition $d(f)\circ f=f$. Now ristrict the above diagram to get a map $\mathrm{Map}_{E_0}(b,b)\times\{id_f\}\to \mathrm{Map}_{\mathbf{Map}_{\mathbb{S}}(a,b)}(f,f)$. Similarly, one gets a right whiskering.

More generally, we define the horizontal composition of 2-morphisms as follows. Given $a,b,c\in E_0$ and maps $f,g\in E_1$, which we visualize as maps $a\xrightarrow{f} b\xrightarrow{g} c$ in $\mathbb{S}$, the diagram (21) yields by passing to the mapping spaces at the pair (g,f)

$$\mathrm{Map}_{\mathbf{Map}_{\mathbb{S}}(b,c)}(g,g)\times \mathrm{Map}_{\mathbf{Map}_{\mathbb{S}}(a,b)}(f,f)\to \mathrm{Map}_{\mathbf{Map}_{\mathbb{S}}(a,c)}(gf,gf)$$

13.1.9. For (ch. 10, 2.2.5). Let $E_\bullet\in 2-\mathcal{C}\mathrm{at}$. If E_1 is ordinary then E_0 is also ordinary, because E_0 is a retract of E_1 (and $\tau_{\le 1}\mathrm{Spc}\subset\mathrm{Spc}$ is stable under retracts).

Let $E_\bullet\in 2-\mathcal{C}\mathrm{at}$. Consider $(E_\bullet)^{ordn}$, that is, the object of $\mathrm{Fun}(\boldsymbol{\Delta}^{op},1-\mathcal{C}\mathrm{at}_{ordn})$ sending $[n]$ to E_n^{ordn}. They claim that it is a category object, that is, lies in $\mathcal{C}\mathrm{at}(1-\mathcal{C}\mathrm{at}_{ordn})$. We check that the natural functor $E_2^{ordn}\to E_1^{ordn}\times_{E_0^{ordn}} E_1^{ordn}$ is an equivalence. For $n>2$ the argument should be similar.

By Remark 13.1.7, it suffices to show that for any map $\beta: x\to x'$ in E_1 the natural map $\Omega(\beta,\mathrm{Map}_{E_1}(x,x'))\to\Omega(r(\beta),\mathrm{Map}_{E_0}(r(x),r(x')))$ is essentially surjective. Here $r: E_1\to E_0$ is the projection on the target of a 1-morphism. Not clear, though I proposed some approach in an email to Sam (19/09/2018).

I would expect the following proof: define a functor $2-\mathcal{C}\mathrm{at}\to 2-\mathcal{C}\mathrm{at}^{2-ordn}$ sending $E_\bullet$ to the simplicial category $[n]\mapsto \bar{E}_n := E_1^{ordn}\times_{E_0^{ordn}}\dots\dots\times_{E_0^{ordn}} E_1^{ordn}$. One checks that the biggest simplicial subgroupoid is constant, so this is indeed an object of $2-\mathcal{C}\mathrm{at}^{2-ordn}$. Then given a map $E_\bullet\to G_\bullet$ in $2-\mathcal{C}\mathrm{at}$ with $G_\bullet\in 2-\mathcal{C}\mathrm{at}^{2-ordn}$, it factors naturally through a morphism $\bar{E}_n\to G_\bullet$ in $2-\mathcal{C}\mathrm{at}^{2-ordn}$. There remains to check the induced map

$$\mathrm{Map}_{2-\mathcal{C}\mathrm{at}^{2-ordn}}(\bar{E},G)\to \mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(E,G)$$

is an isomorphism. It is clearly essentially surjective. Why is it fully faithful?

13.1.10. Given $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$, let q be the composition $\boldsymbol{\Delta}\times_{1-\mathcal{C}\mathrm{at}}(1-\mathcal{C}\mathrm{at})/\mathcal{C}\to\boldsymbol{\Delta}\xrightarrow{h} 1-\mathcal{C}\mathrm{at}$, where h is the natural map. Is it true that $\underset{\boldsymbol{\Delta}\times_{1-\mathcal{C}\mathrm{at}}(1-\mathcal{C}\mathrm{at})/\mathcal{C}}{\mathrm{colim}} q\widetilde{\to}\mathcal{C}$?

Jacob says this is true, and is equivalent to the fact that $1-\mathcal{C}\mathrm{at}$ fully embeds into complete Segal spaces ([14], ch. A.1, 1.3.4). More precisely, consider the full embedding $a:\boldsymbol{\Delta}\hookrightarrow 1-\mathcal{C}\mathrm{at}$. Then $\mathrm{id}: 1-\mathcal{C}\mathrm{at}\to 1-\mathcal{C}\mathrm{at}$ is the left Kan extension of a along itself.

Let $\mathcal{X}$ be the usual category, whose objects are pair $[n]\in\boldsymbol{\Delta}, m\in[n]$. A map from $([n_1],m_1)$ to $([n_2],m_2)$ is a map $f:[n_2]\to[n_1]$ in $\boldsymbol{\Delta}$ such that $m_1\le f(m_2)$. Then $\mathcal{X}\to\boldsymbol{\Delta}^{op}, ([n],m)\mapsto[n]$ is a cartesian fibration corresponding to the natural functor $\boldsymbol{\Delta}\to 1-\mathcal{C}\mathrm{at}$.

13.1.11. *Definition of an $(\infty,2)$-category from* [32]. In ([32], 1.1.7) he means: for $K\in 1-\mathcal{C}\mathrm{at}$, $\boldsymbol{\Delta}_{/K}=\boldsymbol{\Delta}\times_{1-\mathcal{C}\mathrm{at}}1-\mathcal{C}\mathrm{at}_{/K}$. Let now $\mathcal{C}\in 1-\mathcal{C}\mathrm{at}$, $X:\boldsymbol{\Delta}^{op}\to\mathcal{C}$ in $1-\mathcal{C}\mathrm{at}$. For $K\in 1-\mathcal{C}\mathrm{at}$ his notation $X(K)$ means the value of the RKE of X along $\boldsymbol{\Delta}^{op}\to(1-\mathcal{C}\mathrm{at})^{op}$. He defines $\mathcal{C}\mathrm{at}(\mathcal{C})\subset\mathrm{Fun}(\boldsymbol{\Delta}^{op},\mathcal{C})$ as the full subcategory of 'category objects'. It is important that Grpd(Spc) $\subset$ $\mathcal{C}$at(Spc) admits a right adjoint that he denotes $X_\bullet \mapsto X_\bullet^\sim$ in ([32], 1.1.9). In Dennis notations we get $X_1^\sim = X_1^{invert}$. Lurie proves that for any $\mathcal{C} \in 1-\mathcal{C}at$ admitting finite limits the inclusion $\mathrm{Grpd}(\mathcal{C}) \hookrightarrow \mathcal{C}at(\mathcal{C})$ admits a right adjoint denoted $X_\bullet \mapsto X_\bullet^\sim$ ([32], 1.1.14). This right adjoint in general remains misterious for me.

Let $\mathcal{X} \subset \mathcal{Y}$ be a distributor ([32], 1.2.1). Then $\mathcal{X}$ is an accessible localization of $\mathcal{Y}$ (so, $\mathcal{X}$ is a strongly reflective subcategory of $\mathcal{Y}$). Consider the cartesian square in $1-\mathcal{C}at$

$$\begin{array}{ccc} SS_{\mathcal{X}\subset\mathcal{Y}} & \hookrightarrow & \mathcal{C}at(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathcal{X} & \hookrightarrow & \mathcal{Y} \end{array}$$

Since $\mathcal{C}at(\mathcal{Y}) \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{Y})$ is stable under limits, the above functor $\mathcal{C}at(\mathcal{Y}) \to \mathcal{Y}$ preserves limits and admits a left adjoint, say $f : \mathcal{Y} \to \mathcal{C}at(\mathcal{Y})$. Now (HTT, 5.5.4.17) shows that $SS_{\mathcal{X}\subset\mathcal{Y}}$ is a strongly reflective subcategory of $\mathcal{C}at(\mathcal{Y})$.

The adjoint pair $i : \mathcal{X} \leftrightarrows \mathcal{Y} : R'$, where i is the natural inclusion, yields an adjoint pair $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{X}) \leftrightarrows \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{Y})$ given by composing with i and R. These functors restrict to functors, say $L : \mathcal{C}at(\mathcal{X}) \to \mathcal{C}at(\mathcal{Y})$ and $R : \mathcal{C}at(\mathcal{Y}) \to \mathcal{C}at(\mathcal{X})$ which form an adjoint pair $L : \mathcal{C}at(\mathcal{X}) \leftrightarrows \mathcal{C}at(\mathcal{Y}) : R$.

13.1.12. In Lurie ([32], Def. 1.2.1, (4)) there is a misprint: his functor χ should preserve colimits, not limits!!!

About the definition of a complete Segal space from ([32], 1.2.10). For $\mathcal{C} \in 1-\mathcal{C}at$ let Grd($\mathcal{C}$) be the category of groupoids in $\mathcal{C}$. If $\mathcal{X} \subset \mathcal{Y}$ is a distributor then, in the notations of loc.cit., the right adjoint Gp to full embedding $\mathrm{Grd}(\mathcal{X}) \subset SS_{\mathcal{X}\subset\mathcal{Y}}$ sends $E \in \mathcal{C}at(\mathcal{Y})$ with $\mathcal{Y}_0 \in \mathcal{X}$ to $Gp(E) = R(E)^\sim$. Namely, we have the diagram $\mathrm{Grd}(\mathcal{X}) \subset \mathcal{C}at(\mathcal{X}) \subset SS_{\mathcal{X}\subset\mathcal{Y}}$. Here $R : \mathcal{C}at(\mathcal{Y}) \to \mathcal{C}at(\mathcal{X})$ is the right adjoint to the inclusion $\mathcal{C}at(\mathcal{X}) \hookrightarrow \mathcal{C}at(\mathcal{Y})$. Here $\mathcal{C}at(\mathcal{X}) \to \mathrm{Grd}(\mathcal{X}), E \mapsto E^\sim$ is the right adjoint to $\mathrm{Grd}(\mathcal{X}) \to \mathcal{C}at(\mathcal{X})$ defined in ([32], 1.1.14).

Finally, **main definition** is ([32], 1.2.10): let $\mathcal{X} \subset \mathcal{Y}$ be a distributor. A Segal space object $Y_\bullet \in SS_{\mathcal{X}\subset\mathcal{Y}}$ is *complete* if $Gp(Y_\bullet) \in \mathrm{Grd}(\mathcal{X})$ is constant. Let $CSS_{\mathcal{X}\subset\mathcal{Y}} \subset SS_{\mathcal{X}\subset\mathcal{Y}}$ for the full subcategory of complete Segal space objects.

Recall the adjoint pair $\mathrm{Grpd}(\mathcal{X}) \leftrightarrows \mathcal{X}$, $E_\bullet \mapsto \mid E_\bullet \mid$, $x \in \mathcal{X}$ goes to the constant groupoid with value x ([32], 1.1.4). Here $\mathcal{X}$ is an accessible localization of Grpd($\mathcal{X}$). Again from (HTT, 5.5.4.17) we see that $CSS_{\mathcal{X}\subset\mathcal{Y}}$ is an accessible localization of $SS_{\mathcal{X}\subset\mathcal{Y}}$. All of the inclusions

$$CSS_{\mathcal{X}\subset\mathcal{Y}} \subset SS_{\mathcal{X}\subset\mathcal{Y}} \subset \mathcal{C}at(\mathcal{Y}) \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{Y})$$

preserve limits. So, $CSS_{\mathcal{X}\subset\mathcal{Y}}$ is presentable by ([32], 1.2.11).

In ([14], ch. 10, Sect. 2.1.1) the Condition 2 is badly explained. It should be by definition replaced by Condition 2'.

13.1.13. The inclusions $\mathcal{C}at(\mathrm{Spc}) \hookrightarrow \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ and $Grd(\mathrm{Spc}) \hookrightarrow \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ preserve filtered colimits (by HTT, 5.3.3.3). The diagonal map $\mathrm{Spc} \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ preserves all colimits, so the inclusion $\mathrm{Spc} \to Grd(\mathrm{Spc})$ preserves filtered colimits. By [46], the inclusion $CSS \hookrightarrow \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ commutes with filtered colimits. Evaluating at [0], we derive that $1-\mathcal{C}\mathrm{at} \to \mathrm{Spc}$, $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{Spc}} = \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(*, \mathcal{C})$ commutes with filtered colimits. In other words, $*$ is a compact object of $1-\mathcal{C}\mathrm{at}$.

By ([32], Remark 1.2.11), CSS is an accessible localization of $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$. The localization functor $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc}) \to CSS$ preserves compact objects.

Corollary 13.1.14. *1) if $I \in 1-\mathcal{C}\mathrm{at}$ is filtered (small), $I \to 1-\mathcal{C}\mathrm{at}, i \mapsto C_i$, $C = \mathrm{colim}_{i\in I} C_i$ in $1-\mathcal{C}\mathrm{at}$ then any object of C comes from some object of C_i.*
2) The inclusion $\mathrm{Spc} \hookrightarrow 1-\mathcal{C}\mathrm{at}$ *preserves compact objects, because its right adjoint is continuous (HTT, 5.5.7.2).*

13.1.15. In ([14], ch. 10, 2.1.6) there is a mistake. The involution $\mathbb{S} \to \mathbb{S}^{2-op}$ on $2-\mathcal{C}\mathrm{at}$ is interwined under $Seq_\bullet$ with the involution of $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}\mathrm{at})$ coming from $1-\mathcal{C}\mathrm{at} \to 1-\mathcal{C}\mathrm{at}, C \mapsto C^{op}$.

For ([14], ch. 10, 2.2.5). The inclusion $1-\mathcal{C}\mathrm{at}^{ordn} \subset 1-\mathcal{C}\mathrm{at}$ preserves limits. For this reason the inclusion $1-\mathcal{C}\mathrm{at}^{2-ordn} \hookrightarrow 2-\mathcal{C}\mathrm{at}$ preserves limits. I don't know why this functor is accessible, but it should be, so it has a left adjoint.

13.1.16. In any $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}$ assume given a map $Y \to Z$ in $\mathcal{C}$, one gets an isomorphism $(Y \times Y) \times_{Z\times Z} Z \widetilde{\to} Y \times_Z Y$. If for example, we are given in addition a map $a : Y' \to Y$ then this gives a cartesian square

$$\begin{array}{ccc} Y' \times Y' & \stackrel{a\times a}{\to} & Y \times Y \\ \uparrow & & \uparrow \\ Y' \times_Z Y' & \to & Y \times_Z Y \end{array}$$

13.1.17. an application of universality of colimits. Let $\mathcal{C}$ be a presentable category, in which colimits are universal. Let $E : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ be a simplicial object, $x = \mid E \mid \in \mathcal{C}$ be its colimit. We may consider the augmented simplicial object $\bar{E} : \boldsymbol{\Delta}_+^{op} \to \mathcal{C}$, which is the corresponding colimit diagram. It may be seen as a map $\bar{E} : \boldsymbol{\Delta}^{op} \to \mathcal{C}/x$. Given a morphism $f : y \to x$ in $\mathcal{C}$, we get the simplicial object E' obtained as the composition $\boldsymbol{\Delta}^{op} \to \mathcal{C}/x \to \mathcal{C}/y$, where the second functor is the pull-back along f. Then $\mathrm{colim}\, E' \widetilde{\to} y$, that is, the identity map $y \to y$. This kind of ideas is used in ([32], 1.2.22).

13.1.18. Consider the subcategory $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ with all objects and morphisms which are injective maps $[n] \to [m]$ ([27], 6.5.3.6). Let $X \subset \boldsymbol{\Delta}_s$ be the subcategory with all the objects, and where maps from $[n]$ to $[m]$ are injective maps $h : [n] \to [m]$ such that $h(0) = 0$. Let $Y \subset \boldsymbol{\Delta}$ be the subcategory with all objects, and maps in Y are surjective maps $f : [n] \to [m]$. For $f : [n] \to [m]$ surjective in $\boldsymbol{\Delta}$ define the map $h : [m] \to [n]$ by the property that $h(i)$ is the smallest element of $f^{-1}(i)$. Then h is injective, and $h(0) = 0$. The above construction defines an equivalence $Y \to X^{op}$.

Remark: by (HTT, 6.5.3.7), $\boldsymbol{\Delta}_s^{op} \hookrightarrow \boldsymbol{\Delta}^{op}$ is cofinal. So, any totalization can be rewritten as the limit over $\boldsymbol{\Delta}_s$.

13.1.19. If $\mathcal{C} \to X$ is a map in $1-\mathcal{C}\mathrm{at}$, $X \in \mathrm{Spc}$, let $X' \to X$ be an effective epimorphism in Spc. If $X' \times_X \mathcal{C} \in \mathrm{Spc}$ then $\mathcal{C} \in \mathrm{Spc}$. This is explained in ([32], 1.3.1 and 1.2.22).

If $\mathcal{X} \subset \mathcal{Y}$ is a distributor, and $E \in \mathrm{Grpd}(\mathcal{Y}/X)$ with $X \in \mathcal{X}$. Let $X' \to X$ be an effective epimorphism in $\mathcal{X}$. To check that the groupoid object E is constant, it suffices to check that for any $[n] \to [m]$ the induced map $E_m \times_X X' \to E_n \times_X X'$ is an equivalence. This is used in ([32], 1.3.1).

13.1.20. The ∞-category $1-\mathcal{C}at$ is presentable. This follows for example from ([32], 1.3.2) and the fact that $1-\mathcal{C}at$ can be realized as complete Segal spaces in Spc.

13.1.21. In ([32], proof of 1.2.4) there are several misprints: (a) is equivalent to $\bar{\mathcal{C}}^0 \subset \bar{\mathcal{C}}^1$, and (c) is equivalent to $\bar{\mathcal{C}}^0 = \bar{\mathcal{C}}^1$.

13.1.22. The colimits in $1-\mathcal{C}at$ are not universal. The example of Toen: consider $X = [1]$ with the map $f : X \to [2]$, $0 \mapsto 0, 1 \mapsto 2$. The colimit of $[1] \stackrel{0}{\leftarrow} [0] \stackrel{1}{\to} [1]$ in $1-\mathcal{C}at$ is $[2]$ according to our axioms. If the colimits were universal, the induced diagram

$$\begin{array}{ccc} X \times_{[2]} [1] & \to & X \\ \uparrow & & \uparrow \\ X \times_{[2]} [0] & \to & X \times_{[2]} [1] \end{array}$$

would be cocartesian. This is not the case, as $X \times_{[2]} [1] \widetilde{\to} [0]$ and $X \times_{[2]} [0] \widetilde{\to} \emptyset$. We use that the inclusion $1-\mathcal{C}at^{ordn} \hookrightarrow 1-\mathcal{C}at$ preserves limits. If $\mathcal{C}$ is the usual category with two objects $\{0,1\}$ and no nontrivial morphisms then this is the coproduct $[0] \sqcup [0]$ in $1-\mathcal{C}at$, which is different from $[1]$.

13.1.23. For ([14], Appendix A.1, 2.3.1). For $\mathbb{S} \in 2-\mathcal{C}at$ the map $\mathrm{Seq}_1(\mathbb{S}) \to \mathrm{Seq}_0(\mathbb{S}) \times \mathrm{Seq}_0(\mathbb{S})$ keeping the source and the target of an arrow is a cartesian and cocartesian fibration, as the base is a space. So, given a map $\mathbb{S} \to \mathbb{T}$ in $2-\mathcal{C}at$ the map

$$\mathrm{Seq}_1(\mathbb{S}) \to \mathrm{Seq}_1(\mathbb{T}) \times_{\mathrm{Seq}_0(\mathbb{T}) \times \mathrm{Seq}_0(\mathbb{T})} \mathrm{Seq}_0(\mathbb{S}) \times \mathrm{Seq}_0(\mathbb{S})$$

is a morphism of cartesian fibrations over $\mathrm{Seq}_0(\mathbb{S}) \times \mathrm{Seq}_0(\mathbb{S})$. By Lemma 2.2.100, this map is an isomorphism iff it becomes an isomorphism after passing to any fibre $* \to \mathrm{Seq}_0(\mathbb{S}) \times \mathrm{Seq}_0(\mathbb{S})$.

13.1.24. Let $f : X \to [n] \times [1]$ be a cartesian fibration, let X_0, X_1 be its fibres over $0, 1 \in [1]$. The base changed maps $f_0 : X_0 \to [n]$, $f_1 : X_1 \to [n]$ are cartesian fibrations, and $X \to [1]$ is cartesian. We used that the composition of cartesian fibrations is a cartesian fibration. The diagram commutes

$$\begin{array}{ccc} X_1 & \stackrel{h}{\to} & X_0 \\ \downarrow & \swarrow & \\ [n], & & \end{array}$$

where h is the functor obtained from the cartesian fibration $X \to [1]$ via strengthening. Indeed, let $x_1 \in X_1$ be over $(j,1) \in [n] \times [1]$. Then $(j,0) \to (j,1)$ in $[n] \times [1]$ is cartesian over $0 \to 1$ in $[1]$. Let $x_0 \to x_1$ in X be cartesian arrow over $(j,0) \to (j,1)$. Then $x_0 \to x_1$ is cartesian over $0 \to 1$ by ([27], 2.4.1.3). So, $h(x_1) \widetilde{\to} x_0$. Note also that $X_0 \to [n]$ factors through $X_0^{ordn} \to [n]$.

We claim that h sends a cartesian arrow over any map $i \to j$ in $[n]$ to a cartesian arrow. Indeed, this follows from ([27], 2.4.1.7). Another way to see this is to apply ([32], 1.4.14).

The map f can be seen as a morphism in $(Cart/[1])_{strict}$, namely, f sends an arrow of X cartesian over $0 \to 1$ in $[1]$ to an arrow in $[n] \times [1]$ cartesian over $0 \to 1$. So, f defines a map in $\mathrm{Funct}([1]^{op}, 1-\mathcal{C}\mathrm{at})$ from h to $\mathrm{id} : [n] \to [n]$.

13.1.25. For the swapping procedure ([14], Ch. 12, 2.1.1). Let $I, J \in 1-\mathcal{C}\mathrm{at}$, $f : C \to I \times J$ lie in $Cart - coCart_{I,J}$. For any $i \in I$ let $f_i : C_i \to J$ be the fibre of f. Then f_i is a cocartesian fibration. Indeed, given any arrow $a : j_1 \to j_2$ in J and $c_1 \in C$ over (i, j_1), let $\alpha : c_1 \to c_2$ be a cocartesian arrow in C over a. Then $f(\alpha)$ is a cocartesian arrow in $I \times J$ over a, so $f(\alpha)$ is isomorphic to $(i, j_1) \stackrel{a}{\to} (i, j_2)$. Thus, α can be seen as an arrow in C_i. By ([27], 2.4.1.3), α is f-cocartesian. So, by (HTT, 2.4.1.3(2)), f is cocartesian for the morphism $C_i \to J$. This shows that $Cart - coCart_{I,J}$ is a subcategory of $\mathrm{Funct}(I^{op}, coCart/J)$. ([14], Ch. 12, Pp. 2.1.3) seems very useful!

Comment for the proof of ([14], Ch. 12, Pp. 2.1.3): let $\mathcal{C}, I, J \in 1-\mathcal{C}\mathrm{at}$, assume given a map $I^{op} \to coCart/J$. Let $f : \mathcal{C} \to I \times J$ be the corresponding morphism in $(Cart/I)_{strict}$. How to prove that $\mathcal{C} \to J$ is a cocartesian fibration? We know that $\mathcal{C}_i \to J$ is a cocartesian fibration for any $i \in I$. Let $\alpha : j_1 \to j_2$ be a map in J, $c_1 \in \mathcal{C}$ over j_1. Write i for the image of c_1 in I. Let $\bar{\alpha} : c_1 \to c_2$ be a cocartesian arrow in $\mathcal{C}_i$ over α. We want to show that $\bar{\alpha}$ is a cocartesian arrow in $\mathcal{C}$ over α. Let $c_3 \in \mathcal{C}$ over $(i', j_3) \in I \times J$. We want to check that

$$\mathrm{Map}_{\mathcal{C}}(c_2, c_3) \to \mathrm{Map}_{\mathcal{C}}(c_1, c_3) \times_{\mathrm{Map}_J(j_1, j_3)} \mathrm{Map}_J(j_2, j_3) \tag{22}$$

is an isomorphism. Since $f(\bar{\alpha})$ is cocartesian over J, by ([27], 2.4.1.3), our claim is equivalent to the fact that $\bar{\alpha}$ if f-cocartesian. We have a projection $\mathrm{Map}_{\mathcal{C}}(c_1, c_3) \to \mathrm{Map}_I(i, i')$, and (22) is a map over $\mathrm{Map}_I(i, i')$. Pick a map $\beta : i \to i'$ in I, let $\bar{\beta} : \bar{c} \to c_3$ be a cartesian arrow in $\mathcal{C}$ over β. Passing to the fibres over β in (22), we get a map

$$\mathrm{Map}_{\mathcal{C}}(c_2, c_3)_\beta \to \mathrm{Map}_{\mathcal{C}}(c_1, c_3)_\beta \times_{\mathrm{Map}_J(j_1, j_3)} \mathrm{Map}_J(j_2, j_3)$$

It suffices to show that the latter map is an isomorphism. We have canonically $\mathrm{Map}_{\mathcal{C}}(c_2, c_3)_\beta \widetilde{\to} \mathrm{Map}_{\mathcal{C}_i}(c_2, \bar{c})$ and $\mathrm{Map}_{\mathcal{C}}(c_1, c_3)_\beta \widetilde{\to} \mathrm{Map}_{\mathcal{C}_i}(c_1, \bar{c})$. The corresponding map

$$\mathrm{Map}_{\mathcal{C}_i}(c_2, \bar{c}) \to \mathrm{Map}_{\mathcal{C}_i}(c_1, \bar{c}) \times_{\mathrm{Map}_J(j_1, j_3)} \mathrm{Map}_J(j_2, j_3)$$

is an isomorphism, because $\bar{\alpha}$ is cocartesian for $\mathcal{C}_i \to J$. We are done.

Corollary 13.1.26 (Nick, 14mars2018)**.** *Let $J \in 1-\mathcal{C}\mathrm{at}$, $G : C \to D \stackrel{q}{\to} J$ be a map in $coCart/J$, so $C, D \in coCart/J$. Assume for each $j \in J$ the functor $G_j : C_j \to D_j$ admits a left adjoint. Write $D^\vee \to J^{op}$ for the cartesian fibration corresponding to $q : J \to 1-\mathcal{C}\mathrm{at}$. Then there is a map $F : D^\vee \to C^\vee$ in $Cart/J^{op}$ such that for any $j \in J$, $F_j : D_j \to C_j$ is left adjoint to G_j.*

Proof. Let $[1]^{op} \to coCart/J$ be given by G. It is given by some $X \to [1] \times J$ in $Cart - coCart_{[1]\times J}$, so $X_1 = C, X_0 = D$. Let $\mathcal{F} : J \to Cart/[1]$ be the map obtained from G by the swapping procedure. Since each G_j admits a left adjoint, $\mathcal{F} : J \to biCart/[1]$. Composing $\mathcal{F}$ with $biCart/[1] \to coCart/[1]$, we get a morphism $J \to coCart/[1]$. The swapping procedure applied to the latter one gives a functor $[1] \to Cart/J^{op}$, which is a diagram $D^\vee \stackrel{F}{\to} C^\vee \to J^{op}$. $\square$

Proposition 13.1.27. *For $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}$ one has canonically $Cart/[1]^{op} \times_{1-\mathcal{C}\mathrm{at} \times 1-\mathcal{C}\mathrm{at}} \{\mathcal{C}, \mathcal{D}\} \widetilde{\to} \mathrm{Funct}(\mathcal{C}, \mathcal{D})$.*

Proof. (Nick in his email of Sept. 14, 2016). Set $M = Cart/[1]^{op} \times_{1-\mathcal{C}\mathrm{at} \times 1-\mathcal{C}\mathrm{at}} \{\mathcal{C}, \mathcal{D}\}$. By strengthening, $M^{\mathrm{Spc}} \widetilde{\to} \mathrm{Funct}(\mathcal{C}, \mathcal{D})^{\mathrm{Spc}}$. Now it suffices to establish an equivalence $\mathrm{Funct}([n], M)^{\mathrm{Spc}} \widetilde{\to} \mathrm{Funct}([n], \mathrm{Funct}(\mathcal{C}, \mathcal{D}))^{\mathrm{Spc}}$ natural in n. By ([14], ch. 12, 2.1.3) we have $\mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], Cart/[1]^{op}) \widetilde{\to} (Cart - coCart_{[1]^{op} \times [n]})^{\mathrm{Spc}}$. So, $\mathrm{Funct}([n], M)^{\mathrm{Spc}}$ identifies with the space of diagrams

$$\begin{array}{ccc} C & \stackrel{f}{\to} & [1]^{op} \times [n] \\ \downarrow & \swarrow & \\ [1]^{op} & & \end{array}$$

in $(Cart - coCart_{[1]^{op} \times [n]})^{\mathrm{Spc}}$ together with identifications of the fibres of the latter diagram over 0 with

$$\begin{array}{ccc} \mathcal{C} \times [n] & \stackrel{\mathrm{pr}}{\to} & [n] \\ \downarrow & \swarrow & \\ \{0\} & & \end{array}$$

and over 1 with

$$\begin{array}{ccc} \mathcal{D} \times [n] & \stackrel{\mathrm{pr}}{\to} & [n] \\ \downarrow & \swarrow & \\ \{1\} & & \end{array}$$

Since f sends $[1]^{op}$-cartesian arrows of $\mathcal{C}$ to $[1]^{op}$-cartesian arrows, this is the space $(\mathrm{Funct}_{[n]}(\mathcal{C} \times [n], \mathcal{D} \times [n]))^{\mathrm{Spc}} \widetilde{\to} \mathrm{Funct}(\mathcal{C} \times [n], \mathcal{D})^{\mathrm{Spc}} \widetilde{\to} \mathrm{Funct}([n], \mathrm{Funct}(\mathcal{C}, \mathcal{D}))^{\mathrm{Spc}}$. $\square$

Lemma 13.1.28. *For $n, m \geq 0$ the diagram is cartesian*

$$\begin{array}{ccc} Cart/[n] & \leftarrow & Cart/[n+m] \\ \downarrow & & \downarrow \\ Cart/[0] & \leftarrow & Cart/[m] \end{array} \tag{23}$$

Here the maps are given by the pull-back functors.

Proof. Since the coproduct $[n] \stackrel{n}{\leftarrow} [0] \stackrel{0}{\to} [m]$ is $[n+m]$ in $1-\mathcal{C}\mathrm{at}$, the diagram is cartesian in $1-\mathcal{C}\mathrm{at}$

$$\begin{array}{ccc} \mathrm{Funct}([n], 1-\mathcal{C}\mathrm{at}) & \leftarrow & \mathrm{Funct}([n+m], 1-\mathcal{C}\mathrm{at}) \\ \downarrow & & \downarrow \\ \mathrm{Funct}([0], 1-\mathcal{C}\mathrm{at}) & \leftarrow & \mathrm{Funct}([m], 1-\mathcal{C}\mathrm{at}) \end{array}$$

So, the diagram obtained from (23) by applying $\bullet \mapsto \bullet^{\mathrm{Spc}}$ is cartesian. It suffices to show now that the diagram of spaces obtained from (23) by applying $\bullet \mapsto \mathrm{Funct}([1], \bullet)^{\mathrm{Spc}}$ is also cartesian. By ([14], ch. 12, 2.1.3) we have to show that the diagram is cartesian

$$\begin{array}{ccc} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n]^{op}, coCart/[1]) & \leftarrow & \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n+m]^{op}, coCart/[1]) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([0]^{op}, coCart/[1]) & \leftarrow & \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([m]^{op}, coCart/[1]) \end{array}$$

This is clear. $\square$

13.1.29. For the definition of **1-Cat** from ([14], ch. A.1, 2.4.1). Recall that $Seq_n(\mathbf{1\text{-}Cat})$ denotes the 1-full subcategory of $Cart/[n]^{op}$, where we keep all objects and restrict 1-morphisms to functors that induce an equivalence over any $i \in [n]$. In particular, $Seq_0(\mathbf{1\text{-}Cat}) \widetilde{\to} (1-\mathcal{C}\mathrm{at})^{\mathrm{Spc}}$.

Lemma 13.1.30. *For $n, m \geq 0$ the Segal condition for Seq holds, that is, the natural map*

$$Seq_{n+m}(\mathbf{1\text{-}Cat}) \widetilde{\to} Seq_n(\mathbf{1\text{-}Cat}) \times_{Seq_0(\mathbf{1\text{-}Cat})} Seq_m(\mathbf{1\text{-}Cat})$$

is an equivalence.

Proof. We have seen already that (23) is cartesian. If $C_1 \stackrel{a}{\to} C_2 \stackrel{b}{\leftarrow} C_3$ is a diagram in $1-\mathcal{C}\mathrm{at}$, and $C_i' \subset C_i$ is a 1-full subcategory, where we keep all objects, assume that $a(C_1') \subset C_2', b(C_3') \subset C_2'$. Then we get the 1-full subcategory of $C_1' \times_{C_2'} C_3' \subset C_1 \times_{C_2} C_3$ by Remark 2.2.18. $\square$

13.1.31. If $\mathbb{S} \in 2-\mathcal{C}\mathrm{at}$ given by $E_\bullet \in \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}\mathrm{at})$ then $(E_n^{\mathrm{Spc}})^{op} \widetilde{\to} E_n^{\mathrm{Spc}}$ for any n, and as far as I understand, this identification can be made functorial in $[n] \in \boldsymbol{\Delta}$. So, $(\mathbb{S}^{2-op})^{1-\mathcal{C}\mathrm{at}} \widetilde{\to} \mathbb{S}^{1-\mathcal{C}\mathrm{at}}$. See (ch. 10, Remark 2.4.5).

Note that for $x, y \in \mathbb{S}$, $\mathbf{Map}_{\mathbb{S}^{2-op}}(x,y) \widetilde{\to} (\mathbf{Map}_{\mathbb{S}}(x,y))^{op}$. So, we should have

$$coCart_{/[1]} \times_{1-\mathcal{C}\mathrm{at} \times 1-\mathcal{C}\mathrm{at}} \{\mathcal{C}, \mathcal{D}\} \widetilde{\to} \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{op}$$

Does this explain the normalization of strengthening?

13.1.32. For (ch. 10, 2.5.4). If $\mathbb{S} \in 2-\mathcal{C}\mathrm{at}$, $\mathbb{S}^{\mathrm{Spc}}$ denotes $\mathrm{Seq}_0(\mathbb{S}) \in \mathrm{Spc}$. This is $(\mathbb{S}^{1-\mathcal{C}\mathrm{at}})^{\mathrm{Spc}}$.

For $\mathcal{C} \in 1-\mathcal{C}\mathrm{at}, \mathbb{S} \in 2-\mathcal{C}\mathrm{at}$ we get $\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(\mathcal{C}, \mathbb{S}) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(\mathcal{C}, \mathbb{S}^{1-\mathcal{C}\mathrm{at}}) \widetilde{\to} \mathrm{Fun}(\mathcal{C}, \mathbb{S}^{1-\mathcal{C}\mathrm{at}})^{\mathrm{Spc}}$. Now for $\mathbb{S}, \mathbb{T} \in 2-\mathcal{C}\mathrm{at}$ we get $* \times \mathbb{S} \widetilde{\to} \mathbb{S}$ in $2-\mathcal{C}\mathrm{at}$, so

$$\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(\mathbb{S}, \mathbb{T}) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}(*, \mathrm{Fun}(\mathbb{S}, \mathbb{T})^{1-\mathcal{C}\mathrm{at}}) \widetilde{\to} \mathrm{Fun}(\mathbb{S}, \mathbb{T})^{\mathrm{Spc}}$$

13.1.33. **Question** If $\mathbb{S} \in 2-\mathcal{C}\mathrm{at}$ and $E_\bullet = \mathrm{Seq}_\bullet(\mathbb{S}) \in \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ then it should be true that for $x, y \in \mathbb{S}$, $\mathrm{Map}_{\mathbb{S}^{ordn}}(x,y) \widetilde{\to} \mathrm{Map}_{\mathbb{S}}(x,y)^{ordn}$, why this is so? In other words, why the natural functor $(E_1 \times_{E_0 \times E_0} \{x,y\})^{ordn} \to E_1^{ordn} \times_{E_0^{ordn} \times E_0^{ordn}} \{x,y\}$ is an equivalence?

13.1.34. Let $X \to [1]^{op}$ be a Cartesian fibration such that X_0, X_1 are ordinary categories. Then X is ordinary. Indeed, for $x_0 \in X_0, x_1 \in X_1$, $\mathrm{Map}_X(x_0, x_1) = \emptyset$. Let $\bar{x}_1 \to x_0$ be a cartesian arrow in X over $1 \to 0$. Then $\mathrm{Map}_X(x_1, \bar{x}_1) \widetilde{\to} \mathrm{Map}_X(x_1, x_0)$. So, X is ordinary.

Recall that $1-\mathcal{C}\mathrm{at}_{ordn} \subset 1-\mathcal{C}\mathrm{at}$ is the full subcategory of ordinary categories. From Proposition 13.1.27 we see that if $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}_{ordn}$ then $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \in 1-\mathcal{C}\mathrm{at}_{ordn}$. Note also that $1-\mathcal{C}\mathrm{at}_{ordn} \subset 1-\mathcal{C}\mathrm{at}$ preserves limits. Now we may define the functor $E_\bullet : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ by requiring that E_n is the 1-full subcategory of $Cart_{/[n]^{op}}$ spanned by those maps $X \to [n]^{op}$ such that X is ordinary, and those morphisms which induce equivalences over each $i \in [n]$. This is a category object in $1-\mathcal{C}\mathrm{at}$, $E_0 \widetilde{\to} (1-\mathcal{C}\mathrm{at}_{ordn})^{\mathrm{Spc}}$ is a space, and $E_n^{\mathrm{Spc}} \widetilde{\to} \mathrm{Fun}([n], 1-\mathcal{C}\mathrm{at}_{ordn})^{\mathrm{Spc}}$. So, $(E_\bullet)^{1-\mathcal{C}\mathrm{at}}$ identifies with $1-\mathcal{C}\mathrm{at}_{ordn}$. Thus, we have lifted $1-\mathcal{C}\mathrm{at}_{ordn}$ to an $(\infty, 2)$-category such that for $\mathcal{C}, \mathcal{D} \in 1-\mathcal{C}\mathrm{at}_{ordn}$,

$\mathbf{Map}_E(\mathcal{C},\mathcal{D})\tilde{\to}\mathrm{Fun}(\mathcal{C},\mathcal{D})$. We have the natural map $E_\bullet \to \mathbf{1\text{-}Cat}$ in $2-\mathcal{C}at$. According to (ch. 10, 2.3.1), it is fully faithful functor of $(\infty, 2)$-categories.

13.1.35. *The* $(\infty,2)$-*category of modules.* Let A be a monoidal $(\infty,1)$-category. Consider $A-mod \in 1-\mathcal{C}at$ defined in (ch. 1, 3.4.4). Its objects are A-module categories M. We want to lift $A-mod$ to an $(\infty,2)$-category in a way analogous to (ch. 1, Sect. 8.3.1). So, we define a simplicial object $E_\bullet : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}at$ by E_n be the full subcategory of $A-mod \times_{1-\mathcal{C}at} \mathrm{Seq}_n(\mathbf{1\text{-}Cat})$ given by the following condition. First, the map $\mathrm{Seq}_n(\mathbf{1\text{-}Cat}) \to 1-\mathcal{C}at$ here sends $(X \to [n]^{op})$ to X, and $A-mod \to 1-\mathcal{C}at$ is the forgetfull functor. An object $((A, M) \in A-mod, M \to [n]^{op})$ lies in E_n if the diagram commutes

$$\begin{array}{ccc} A\times M & \stackrel{act}{\to} & M \\ & \searrow & \downarrow \\ & & [n]^{op}, \end{array}$$

and $act : A \times M \to M$ is a map in $(Cart_{/[n]^{op}})_{strict}$. Note that $A \times M \to [n]^{op}$ is a cartesian fibration.

Question. It is not clear even that $E_\bullet$ is indeed a functor. If $[m] \to [n]$ is a map in $\boldsymbol{\Delta}$, given $(A, M \to [n]^{op}) \in E_n$ why the natural map $A \times M_{[n]^{op}}[m]^{op}$ extends naturally to a structure of a A-module on $M_{[n]^{op}}[m]^{op}$? Why $E_\bullet$ is a category object?

We get $E_0 \tilde{\to} (A-mod)^{\mathrm{Spc}}$. Hopefully we have $E_n^{\mathrm{Spc}} \tilde{\to} \mathrm{Map}_{1-\mathcal{C}at}([n], A-mod)$, I can not check this. It is also not clear what is $E_1 \times_{E_0,E_0} \{M_0, M_1\}$, I expect it is naturally equivalent to $\mathrm{LinFun}_A(M_0, M_1)$ defined in my Section 3.0.49.

13.1.36. For (ch. 10, 2.3). If $F : \mathbb{S} \to \mathbb{T}$ be a functor between $(\infty,2)$-categories and $E_\bullet = \mathrm{Seq}_\bullet(\mathbb{S}) \in \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}at)$ then $E_1 \to E_0 \times E_0$ is a bicartesian fibration. So,

$$\mathrm{Seq}_1(\mathbb{S}) \to \mathrm{Seq}_1(\mathbb{T}) \times_{\mathrm{Seq}_0(\mathbb{T})\times \mathrm{Seq}_0(\mathbb{T})} (\mathrm{Seq}_0(\mathbb{S}) \times \mathrm{Seq}_0(\mathbb{S}))$$

is a map of bicartesian fibrations over $\mathrm{Seq}_0(\mathbb{S}) \times \mathrm{Seq}_0(\mathbb{S})$. Now by my Lemma 2.2.100, the above map is an isomorphism iff for any $s, s' \in \mathbb{S}$

$$\mathbf{Map}_{\mathbb{S}}(s, s') \to \mathbf{Map}_{\mathbb{T}}(F(s), F(s'))$$

is an equivalence.

13.1.37. Given $\mathbb{S}, \mathbb{T} \in 2-\mathcal{C}at$, what is $\mathrm{Fun}(\mathbb{S},\mathbb{T})^{1-\mathcal{C}at}$? This is not clear in general. If $\mathbb{S} \in 1-\mathcal{C}at$ then we get $\mathrm{Fun}(\mathbb{S},\mathbb{T})^{1-\mathcal{C}at} \tilde{\to} \mathrm{Fun}(\mathbb{S}, \mathbb{T}^{1-\mathcal{C}at})$.

Remark 13.1.38. *If* $I \to 2-\mathcal{C}at$ *is a morphism in* $1-\mathcal{C}at$, $i \mapsto \mathbb{T}_i$, *let* $\mathbb{T} = \lim_{i\in I} \mathbb{T}_i$ *and* $t, t' \in \mathbb{T}$. *Then* $\mathbf{Map}_{\mathbb{T}}(t, t') \tilde{\to} \lim_{i\in I} \mathbf{Map}_{\mathbb{T}_i}(t_i, t'_i)$ *in* $1-\mathcal{C}at$, *where* $t_i, t'_i \in \mathbb{T}_i$ *are the images of* t, t'. *Indeed, limits commute with limits.*

13.1.39. For ([14], A.1, 2.6.2). Let $\tilde{i} : \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}at) \to 1-\mathcal{C}at$ be the evaluation at $[0]$. Let $\tilde{g} : 1-\mathcal{C}at = \mathrm{Fun}(*, 1-\mathcal{C}at) \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}at)$ be the composition with $\boldsymbol{\Delta}^{op} \to *$. Define $\mathcal{X}$ as the preimage of $\mathrm{Spc} \subset 1-\mathcal{C}at$ under $\tilde{i}$. So, $\mathcal{X} \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}at)$ is a full subcategory. The adjoint pair $\tilde{g} : 1-\mathcal{C}at \leftrightarrows \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}at) : \tilde{i}$ restricts to an adjoint pair $g : \mathrm{Spc} \leftrightarrows \mathcal{X} : i$. So, the functor i is corepresentable by $g(*)$. Here $g(*)$ is the constant functor $\boldsymbol{\Delta}^{op} \to 1-\mathcal{C}at$ with value $*$. Now $2-\mathcal{C}at \subset \mathcal{X}$ is a full subcategory and $g(*) \in 2-\mathcal{C}\mathrm{at}$. So, the composition $2-\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{X} \xrightarrow{i} \mathrm{Spc}$ is also corepresentable by $g(*)$. This establishes the isomorphism $[m,0]^{\sim} \widetilde{\to} *$ from their Sect. 2.6.2.

By definition, for $E \in 2-\mathcal{C}\mathrm{at} \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}\mathrm{at})$,

$$\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}([m,n]^{\sim}, E) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([m], E_n)$$

Note that $g(*)$ is a final object of $2-\mathcal{C}\mathrm{at}$.

The inclusion $1-\mathcal{C}\mathrm{at} \to 2-\mathcal{C}\mathrm{at}$ admits a left adjoint also. Indeed, the inclusion $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc}) \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}\mathrm{at})$ is a right adjoint, hence preserves limits. The full embeddings $1-\mathcal{C}\mathrm{at} \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ and $2-\mathcal{C}\mathrm{at} \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}\mathrm{at})$ are stable under limits, so the full subcategory $1-\mathcal{C}\mathrm{at} \subset 2-\mathcal{C}\mathrm{at}$ is stable under limits.

Let us show that $[0,n]^{\sim} \widetilde{\to} [n]$ in $2-\mathcal{C}\mathrm{at}$. For $E \in 2-\mathcal{C}\mathrm{at}$ one has

$$\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}([n], E) \widetilde{\to} \mathrm{Map}_{1-\mathcal{C}\mathrm{at}}([n], E^{1-\mathcal{C}\mathrm{at}}) \widetilde{\to} E_n^{\mathrm{Spc}},$$

we are done.

13.1.40. For A.1, 2.6.3. Their explicit description of $[m,n]^{\sim}$ produces a strict 2-category, I think (see my Section 13.1.1). Do I understand correctly that each strict 2-category yields an object of $2-\mathcal{C}\mathrm{at}^{2-ordn}$?

13.1.41. For ([14], A.1, 3.1.3). If $\mathbb{S}, \mathbb{T} : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ are objects of $2-\mathcal{C}\mathrm{at}$ then a non-unital left-lax functor from $\mathbb{S}$ to $\mathbb{T}$ can be seen as also as a morphism $X_{\mathbb{S}} \to X_{\mathbb{T}}$ over $\boldsymbol{\Delta}$ sending a cartesian arrow over an inert map to a cartesian arrow. Here $X_{\mathbb{T}} \to \boldsymbol{\Delta}$ is the cartesian fibration attached to $\mathbb{T}$. The definition is compatible with a similar notion for monoidal ∞-categories.

For 3.1.6. If $\mathbb{S}, \mathbb{T} : \boldsymbol{\Delta}^{op} \to 1-\mathcal{C}\mathrm{at}$ are objects of $2-\mathcal{C}\mathrm{at}$, let $F : \mathbb{S} \dashrightarrow \mathbb{T}$ be a non-unital right-lax functor. Let $X_{\mathbb{S}} \to \boldsymbol{\Delta}^{op}$ be the corresponding cocartesian fibration, same for $\mathbb{T}$, so $F : X_{\mathbb{S}} \to X_{\mathbb{T}}$. Write $X_{\mathbb{S},n}$ for the fibre of $X_{\mathbb{S}}$ over $[n]$. We get $X_{\mathbb{S},1} \widetilde{\to} \mathbb{S}_1$. By the assumption, if $b : a_0 \to a_1$ is a map in $\mathbb{S}$, that is, $b \in \mathbb{S}_1$ with target $a_1 \in \mathbb{S}_0$ and source $a_0 \in \mathbb{S}_0$ then $F(b) : F(a_0) \to F(a_1)$ is an element of $\mathbb{T}_1$ with target $F(a_1)$ and source $F(a_0)$. Given a diagram $a_0 \xrightarrow{b} a_1 \xrightarrow{c} a_2$ in $\mathbb{S}$, let (b,c) be the corresponding element of $\mathbb{S}_2 \widetilde{\to} \mathbb{S}_1 \times_{\mathbb{S}_0} \mathbb{S}_1$. Let $\alpha : [1] \to [2]$ be the unique active map in $\boldsymbol{\Delta}$. Let $\bar{\alpha} : (b,c) \to c \circ b$ be a cocartesian arrow in $X_{\mathbb{S}}$ over α. Then $F(\bar{\alpha}) : (F(b), F(c)) \to F(c \circ b)$ is not necessarily cocartesian over $\boldsymbol{\Delta}$. However, there is a cocartesian arrow $\alpha' : ((F(b), F(c)) \to F(c) \circ F(b)$ in $X_{\mathbb{T}}$ over α. So, we get a morphism $F(c) \circ F(b) \to F(c \circ b)$ in $\mathbb{T}_1$. It is actually a morphism in $\mathbf{Map}_{\mathbb{T}}(F(a_0), F(a_2))$.

If F is moreover a right-lax functor (not just right-lax non-unital) then for any $s \in \mathbb{S}$, F sends $\mathrm{id} : s \to s$ to $F(\mathrm{id}) \widetilde{\to} \mathrm{id} : F(s) \to F(s)$. Besides, if as above $a_0 \xrightarrow{\mathrm{id}} a_0 \xrightarrow{c} a_1$ are maps in $\mathbb{S}$ then the above 2-morphism $F(c) \circ F(\mathrm{id}) \to F(c \circ \mathrm{id})$ in $\mathbb{T}$ is an isomorphism, because $F(\mathrm{id}) \widetilde{\to} \mathrm{id}$. In the above I think if $a_0 \xrightarrow{b} a_1 \xrightarrow{c} a_2$ is a diagram in $\mathbb{S}$ such that b or c is an isomorphism then the above 2-morphism $F(c) \circ F(b) \to F(c \circ b)$ in $\mathbb{T}$ is an isomorphism.

What a right-lax functor F does to 2-morphisms in $\mathbb{S}$? Assume given as above a diagram $a_0 \xrightarrow{b} a_1 \xrightarrow{c} a_2$, $b' : a_0 \to a_1$ in $\mathbb{S}$, and a morphism $\beta : b' \to b$ in $\mathbb{S}_1$. So, β is a 2-morphism in $\mathbb{S}$. Denote by $\bar{\beta} : c \circ b' \to c \circ b$ the 2-morphism in $\mathbb{S}$ given by composing with c. First, F produces a 2-moprhism $F(\beta) : F(b') \to F(b)$ in $\mathbb{T}$. That is, $F(\beta)$ is a

1-morphism in $\mathbb{T}_1$. Denote by $\overline{F(\beta)} : F(c) \circ F(b') \to F(c) \circ F(b)$ the 2-morphism in $\mathbb{T}$ obtained from $F(\beta)$ by composing with $F(c)$. Then the diagram commutes in $\mathbb{T}_1$

$$\begin{array}{ccc} F(c) \circ F(b') & \to & F(c \circ b') \\ \downarrow \overline{F(\beta)} & & \downarrow F(\bar{\beta}) \\ F(c) \circ F(b) & \to & F(c \circ b) \end{array}$$

This is a diagram of 2-morphisms in $\mathbb{T}$. (The situation here is similar to the case of right-lax functors between monoidal categories). This is the functoriality of the canonical 2-morphisms, which are the horizontal arrows in the above diagram.

Similarly for composing on the right of $F(b)$ instead of on the left.

13.1.42. For ([14], A.1, 3.2.1). There

$$\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(\mathbb{S}_1 \circledast \mathbb{S}_2 \ldots \circledast \mathbb{S}_n, \mathbb{T}) \subset \mathrm{Map}_{2-\mathcal{C}\mathrm{at}^{lax}}(\mathbb{S}_1 \times \ldots \times \mathbb{S}_n, \mathbb{T})$$

is a full subspace. Here $2-\mathcal{C}\mathrm{at}^{lax}$ means right-lax of course. The fact that their functor $2-\mathcal{C}\mathrm{at} \to \mathrm{Spc}, \mathbb{T} \mapsto \mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(\mathbb{S}_1 \circledast \mathbb{S}_2 \ldots \circledast \mathbb{S}_n, \mathbb{T})$ commutes with limits follows from my Remark 13.1.38.

If $T, X, S \in 2-\mathcal{C}\mathrm{at}$, there is a canonical map $(T \times X) \circledast S \to T \times (X \circledast S)$ in $2-\mathcal{C}\mathrm{at}$. It comes from the canonical functor $(T \times X) \circledast S \to T \times X \times S$, which yields a projection $(T \times X) \circledast S \to T$. The second projection $(T \times X) \circledast S \to X \circledast S$ comes from the morphism $T \times X \to X$ by applying the functor $\bullet \circledast S$. I think for $V \in 2-\mathcal{C}\mathrm{at}$ the corresponding map

$$\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(T \times (X \circledast S), V) \to \mathrm{Map}_{2-\mathcal{C}\mathrm{at}}((T \times X) \circledast S, V)$$

is a full subspace, I have not checked that.

13.1.43. For 3.2.4. Given $\mathbb{S}_i \in 2-\mathcal{C}\mathrm{at}$, they claim an isomorphism $(\mathbb{S}_n \circledast \ldots \circledast \mathbb{S}_1)^{2-op} \widetilde{\to} \mathbb{S}_1^{2-op} \circledast \ldots \circledast \mathbb{S}_n^{2-op}$. This is equivalent to a different definition of the Gray product.

Namely, the strengthening for cartesian fibrations gives an embedding $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathcal{C}\mathrm{at}) \hookrightarrow Cart_{/\boldsymbol{\Delta}}$. Denote by $2-\mathcal{C}\mathrm{at}^{llax} \subset Cart_{/\boldsymbol{\Delta}}$ the 1-full subcategory, whose objects are the same as those of $2-\mathcal{C}\mathrm{at}$, and we restrict 1-morphisms to those which send cartesian arrows over idle maps to cartesian edges. We have a diagram

$$\begin{array}{ccc} 2-\mathcal{C}\mathrm{at}_{rlax} & \to & 2-\mathcal{C}\mathrm{at}_{llax} \\ \uparrow & & \uparrow \\ 2-\mathcal{C}\mathrm{at} & \overset{\mathbb{S} \mapsto \mathbb{S}^{2-op}}{\to} & 2-\mathcal{C}\mathrm{at}, \end{array} \tag{24}$$

where horizontal arrows are equivalences, and vertical arrows are canonical inclusions.

Then $\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(\mathbb{S}_1 \circledast \ldots \circledast \mathbb{S}_n, \mathbb{T})$ can be equivalently defined as the full subspace of $\mathrm{Map}_{2-\mathcal{C}\mathrm{at}^{llax}}(\mathbb{S}_1 \times \ldots \times \mathbb{S}_n, \mathbb{T})$ consisting of those $F \dashrightarrow: \mathbb{S}_1 \times \ldots \times \mathbb{S}_n \to \mathbb{T}$ such that

- for each i and $\hat{s}_i \in \prod_{j \neq i} \mathbb{S}_j$, the composite lax functor $\mathbb{S}_i \overset{\hat{s}_i}{\to} \prod_j \mathbb{S}_j \dashrightarrow \mathbb{T}$ is strict;
- for any morphism $f = (f_i) : (s_1, \ldots, s_n) \to (s'_1, \ldots, s'_n)$ in $\prod_j \mathbb{S}_j$ and $1 \leq k \leq n-1$ the 2-morphism in $\mathbb{T}$ corresponding to splitting f as a composition

$$(s_1, \ldots, s_k, s_{k+1}, \ldots, s_n) \overset{(\mathrm{id}, \ldots, \mathrm{id}, f_{k+1}, \ldots, f_n)}{\to} (s_1, \ldots, s_k, s'_{k+1}, \ldots, s'_n) \overset{(f_1, \ldots, f_k, \mathrm{id}, \ldots, \mathrm{id})}{\to} (s'_1, \ldots, s'_n)$$

is an isomorphism.

I think this is correct.

13.1.44. For 3.2.7. I think in the definition of $\mathrm{Funct}(\mathbb{S},\mathbb{T})_{right-lax}$ there is a misprint, in the RHS one should replace $2-\mathcal{C}\mathrm{at}_{right-lax}$ by $2-\mathcal{C}\mathrm{at}$.

If $\mathbb{S}\in 2-\mathcal{C}\mathrm{at}$ then $*\circledast\mathbb{S}\widetilde{\to}\mathbb{S}\widetilde{\to}\mathbb{S}\circledast *$ in $2-\mathcal{C}\mathrm{at}$. So, for $\mathbb{S},\mathbb{T}\in 2-\mathcal{C}\mathrm{at}$ one gets

$$(\mathrm{Fun}(\mathbb{S},\mathbb{T})_{rlax})^{\mathrm{Spc}}\widetilde{\to}\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(\mathbb{S},\mathbb{T})\widetilde{\to}\mathrm{Fun}(\mathbb{S},\mathbb{T})^{\mathrm{Spc}}$$

If α is a 1-morphism in $\mathrm{Fun}(\mathbb{S},\mathbb{T})_{rlax}$ written symbolically $\alpha: F_1 \dashrightarrow F_2$, this is a right-lax functor $[1]\times\mathbb{S}\to\mathbb{T}$ with some additional properties. For $s\in\mathbb{S}$ it gives an element $\alpha(s): F_1(s)\to F_2(s)$ in $\mathbf{Map}_{\mathbb{T}}(F_1(s),F_2(s))$ as the image under α of the arrow $(0,s)\to(1,s)$. Now given a 1-morphism $\phi: s_0\to s_1$ in $\mathbb{S}$, one gets the decompositions $(0,s_0)\to(1,s_0)\stackrel{\mathrm{id}\times\phi}{\to}(1,s_1)$ and $(0,s_0)\stackrel{\mathrm{id}\times\phi}{\to}(0,s_1)\to(1,s_1)$ in $[1]\times\mathbb{S}$. The 2-morphism in $\mathbb{T}$ corresponding to the first decomposition is an isomoprhism. Therefore, the 2-morphism in $\mathbb{T}$ corresponding to the second decomposition becomes a 2-morphism

$$\alpha(s_1)\circ F_1(\phi)\to F_2(\phi)\circ\alpha(s_0)$$

Definition of $\mathbb{S}_1\circledast\mathbb{S}_2$ is not symmetric under permuting $\mathbb{S}_i$. Probably, the monoidal structure on $2-\mathcal{C}\mathrm{at}$ given by the Gray product is not symmetric.

Given $\mathbb{S},\mathbb{T}\in 2-\mathcal{C}\mathrm{at}$, the object $\mathrm{Fun}(\mathbb{S},\mathbb{T})_{left-lax}\in 2-\mathcal{C}\mathrm{at}$ is defined by the isomorphism in Spc

$$\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(\mathbb{X},\mathrm{Fun}(\mathbb{S},\mathbb{T})_{left-lax})\widetilde{\to}\mathrm{Map}_{2-\mathcal{C}\mathrm{at}}(\mathbb{S}\circledast\mathbb{X},\mathbb{T})$$

functorial in $\mathbb{X}\in 2-\mathcal{C}\mathrm{at}$. For this definition if α is a 1-morphism in $\mathrm{Fun}(\mathbb{S},\mathbb{T})_{left-lax}$ written symbolically as $\alpha: F_1\dashrightarrow F_2$, for $s\in\mathbb{S}$ we get $\alpha(s): F_1(s)\to F_2(s)$ in $\mathbf{Map}_{\mathbb{T}}(F_1(s),F_2(s))$ as the image under the right-lax functor $\mathbb{S}\times[1]\to\mathbb{T}$ of the arrow $(s,0)\to(s,1)$. Further, if $\phi: s_0\to s_1$ is as above, we get instead a 2-morphism

$$F_2(\phi)\circ\alpha(s_0)\to\alpha(s_1)\circ F_1(\phi)$$

in $\mathbb{T}$. This is a correct definition.

Lemma 13.1.45. *1) If* $S\widetilde{\to}\mathrm{colim}_{i\in I} S_i$ *in* $2-\mathcal{C}\mathrm{at}$ *for a diagram* $I\to 2-\mathcal{C}\mathrm{at}$*,* $i\mapsto S_i$ *and* $T\in 2-\mathcal{C}\mathrm{at}$ *then*

$$\mathrm{Fun}(S,T)_{llax}\widetilde{\to}\lim_{i\in I^{op}}\mathrm{Fun}(S_i,T)_{llax}$$

2) If $S\widetilde{\to}\mathrm{colim}_{i\in I} S_i$ *in* $1-\mathcal{C}\mathrm{at}$ *for a diagram* $I\to 1-\mathcal{C}\mathrm{at}$*,* $i\mapsto S_i$*,* $T\in 2-\mathcal{C}\mathrm{at}$ *then the same holds.*

Proof. 1) Follows for the fact that $\bullet\circledast\bullet$ preserves colimits in each variable ([14], ch. A.1, Pp. 3.2.6).
2) $1-\mathcal{C}\mathrm{at}\to 2-\mathcal{C}\mathrm{at}$ preserves colimits. □

13.1.46. For ([14], A.1, 3.2.8). The inclusion $2-\mathcal{C}at \hookrightarrow 2-\mathcal{C}at_{rlax}$ preserves limits. This is why for $\mathbb{S}_i \in 2-\mathcal{C}at$, right-lax functors $\mathbb{S}_1 \times \ldots \times \mathbb{S}_{n_1} \to \mathbb{S}_1 \circledast \ldots \circledast \mathbb{S}_{n_1}$, $\mathbb{S}_{n_1+1} \times \ldots \times \mathbb{S}_{n_1+n_2} \to \mathbb{S}_{n_1+1} \circledast \ldots \circledast \mathbb{S}_{n_1+n_2}$ yield a functor $\mathbb{S}_1 \times \ldots \times \mathbb{S}_{n_1+n_2} \to (\mathbb{S}_1 \circledast \ldots \circledast \mathbb{S}_{n_1}) \times (\mathbb{S}_{n_1+1} \circledast \ldots \circledast \mathbb{S}_{n_1+n_2})$.

The dispalyed formula in just before Remark 3.4.2 is probably correct. Here $Sq^{\sim}_{m,n}(E) = \operatorname{Map}_{1-\mathcal{C}at}([m], E_n)$ for $E \in 2-\mathcal{C}at$. It says that

$$\operatorname{Map}_{2-\mathcal{C}at}([1,1], \mathbb{S}) \widetilde{\to} \mathbb{S}_2^{\mathrm{Spc}} \times \mathbb{S}_2^{\mathrm{Spc}} \times_{(\mathbb{S}_1^{\mathrm{Spc}} \times \mathbb{S}_1^{\mathrm{Spc}})} \operatorname{Map}_{1-\mathcal{C}at}([1], \mathbb{S}_1),$$

however, it is not really justified! From the above formula one gets a cocartesian square in $2-\mathcal{C}at$

$$\begin{array}{ccc} [0,2]^{\sim} \sqcup [0,2]^{\sim} & \to & [1,1] \\ \uparrow & & \uparrow \\ [0,1]^{\sim} \sqcup [0,1]^{\sim} & \to & [1,1]^{\sim}, \end{array}$$

where the left vertical arrow come from the active map $[1] \to [2]$, and the low horizontal arrow is taking the source and a target of an arrow in E_1 for $E \in 2-\mathcal{C}at$.

On the picture at the end of their Sect. 2.6.2 they mean n horizontal arrows and m vertical arrows (for each pair of consecutive points).

13.1.47. For ([14], A.1, 3.4.8), $\mathbb{S} \in 2-\mathcal{C}at$. The isomorphism

$$[m,1] \sqcup_{[m]\sqcup[m]} (* \sqcup *) \widetilde{\to} [m,1]^{\sim}$$

gives an isomorphism of spaces

$$\operatorname{Map}_{1-\mathcal{C}at}([m], \mathbb{S}_1) \widetilde{\to} \operatorname{Map}_{2-\mathcal{C}at}([m,1], \mathbb{S}) \times_{\mathbb{S}_m^{\mathrm{Spc}} \times \mathbb{S}_m^{\mathrm{Spc}}} (\mathbb{S}_0 \times \mathbb{S}_0)$$

Here $\operatorname{Map}_{2-\mathcal{C}at}([m,1], \mathbb{S}) \widetilde{\to} \operatorname{Map}_{2-\mathcal{C}at}([m], \operatorname{Fun}([1], \mathbb{S})_{rlax})$. Fix $s_0, s_1 \in \mathbb{S}_0$ and make the base change in the above displayed formula by the map $\{s_0, s_1\} \to \mathbb{S}_0 \times \mathbb{S}_0$ using

$$\mathbf{Map}_{\mathbb{S}}(s_0, s_1) \widetilde{\to} \mathbb{S}_1 \times_{\mathbb{S}_0 \times \mathbb{S}_0} \{s_0, s_1\}$$

One gets

$$\begin{aligned} \operatorname{Map}_{1-\mathcal{C}at}([m], \mathbf{Map}_{\mathbb{S}}(s_0, s_1)) \widetilde{\to} \operatorname{Map}_{2-\mathcal{C}at}([m], \operatorname{Fun}([1], \mathbb{S})_{rlax}) \times_{\mathbb{S}_m^{\mathrm{Spc}} \times \mathbb{S}_m^{\mathrm{Spc}}} \{s_0, s_1\} \widetilde{\to} \\ \operatorname{Map}_{2-\mathcal{C}at}([m], \operatorname{Fun}([1], \mathbb{S})_{rlax} \times_{\mathbb{S} \times \mathbb{S}} \{s_0, s_1\}), \end{aligned}$$

because $\operatorname{Fun}(*, \mathbb{S})_{rlax} \widetilde{\to} \mathbb{S}$ in $2-\mathcal{C}at$. Since this isomorphism is functorial in $[m] \in \boldsymbol{\Delta}^{op}$, this gives

$$\mathbf{Map}_{\mathbb{S}}(s_0, s_1) \widetilde{\to} (\operatorname{Fun}([1], \mathbb{S})_{rlax} \times_{\mathbb{S} \times \mathbb{S}} \{s_0, s_1\})^{1-\mathcal{C}at}$$

in $1-\mathcal{C}at$.

We check below that $\operatorname{Fun}([1], \mathbb{S})_{rlax} \times_{\mathbb{S} \times \mathbb{S}} \{s_0, s_1\}$ actually lies in $1-\mathcal{C}at$.

Remark 13.1.48. *Given $E \in 2-\mathcal{C}at$, one has $E \in 1-\mathcal{C}at$ iff the map $\operatorname{Map}_{2-\mathcal{C}at}([1,1]^{\sim}, E) \to \operatorname{Map}_{2-\mathcal{C}at}([0,1]^{\sim}, E)$ given by $[0] \xrightarrow{0} [1]$ (or by $[0] \xrightarrow{1} [1]$) is an isomorphism in* Spc.

Essentially, we are trying to check if for $\mathbb{S} \in 2-\mathcal{C}at$ the map $\operatorname{Map}_{2-\mathcal{C}at}([1,1,1], \mathbb{S}) \to \operatorname{Map}_{2-\mathcal{C}at}([0,1,1], \mathbb{S})$ is an equivalence, I think.

13.1.49. Let $E \in 2-\mathfrak{Cat}$. We want to understand the space $\mathrm{Map}_{2-\mathfrak{Cat}}([1,1]^{\sim} \circledast [1], E)$. The category $[1,1]^{\sim}$ has two objects $0,1$, 1-morphisms $\beta_1, \beta_2 : 0 \to 1$ and a 2-morphism $u : \beta_1 \to \beta_2$. Denote the unique morhism $0 \to 1$ in $[1]$ by α. So, the category $[1,1]^{\sim} \times [1]$ has objects (i,j) with $0 \leq i, j \leq 1$, 1-morphisms given by squares

$$\begin{array}{ccccccc} (0,0) & \overset{\mathrm{id},\alpha}{\to} & (0,1) & \qquad & (0,0) & \overset{\mathrm{id},\alpha}{\to} & (0,1) \\ \downarrow \beta_1,\mathrm{id} & & \downarrow \beta_1,\mathrm{id} & & \downarrow \beta_2,\mathrm{id} & & \downarrow \beta_2,\mathrm{id} \\ (1,0) & \overset{\mathrm{id},\alpha}{\to} & (1,1) & & (1,0) & \overset{\mathrm{id},\alpha}{\to} & (1,1) \end{array}$$

and a 2-morphisms

$$(u, \mathrm{id}_0) : (\beta_1, \mathrm{id}_0) \to (\beta_2, \mathrm{id}_0) \text{ and } (u, \mathrm{id}_1) : (\beta_1, \mathrm{id}_1) \to (\beta_2, \mathrm{id}_1)$$

A map $f \in \mathrm{Map}_{2-\mathfrak{Cat}}([1,1]^{\sim} \circledast [1], E)$ gives the corresponding diagrams

$$\begin{array}{ccccccc} e_{00} & \overset{\alpha_0}{\to} & e_{01} & \qquad & e_{00} & \overset{\alpha_0}{\to} & e_{01} \\ \downarrow b_1 & & \downarrow c_1 & & \downarrow b_2 & & \downarrow c_2 \\ e_{10} & \overset{\alpha_1}{\to} & e_{11} & & e_{10} & \overset{\alpha_1}{\to} & e_{11} \end{array} \tag{25}$$

which are not commutative, but equipped with 2-morphisms $v_1 : c_1\alpha_0 \to \alpha_1 b_1$ and $v_2 : c_2\alpha_0 \to \alpha_1 b_2$. In fact, v_1 is the natural 2-morphism $f(\beta_1, \mathrm{id}) \circ f(\mathrm{id}, \alpha) \to f(\beta_1, \alpha)$, and v_2 is the natural 2-morphism $f(\beta_2, \mathrm{id}) \circ f(\mathrm{id}, \alpha) \to f(\beta_2, \alpha)$. Besides, we get 2-morphisms $f(u, \mathrm{id}_1) : f(\beta_1, \mathrm{id}_1) \to f(\beta_2, \mathrm{id}_1)$ and $f(u, \mathrm{id}_\alpha) : f(\beta_1, \alpha) \to f(\beta_2, \alpha)$. The diagram of 2-morphisms commutes

$$\begin{array}{ccccccc} c_1\alpha_0 & \widetilde{\to} & f(\beta_1, \mathrm{id}) \circ f(\mathrm{id}, \alpha) & \to & f(\beta_1, \alpha) & \widetilde{\to} & \alpha_1 b_1 \\ & & \downarrow f(u, \mathrm{id}_1) & & \downarrow f(u, \mathrm{id}_\alpha) & & \downarrow \alpha_1 \circ f(u, \mathrm{id}_0) \\ c_2\alpha_0 & \widetilde{\to} & f(\beta_2, \mathrm{id}) \circ f(\mathrm{id}, \alpha) & \to & f(\beta_2, \alpha) & \widetilde{\to} & \alpha_1 b_2 \end{array}$$

See my Section 13.1.41.

To summarize, the space

$$\mathrm{Map}_{2-\mathfrak{Cat}}([1,1]^{\sim} \circledast [1], E) \widetilde{\to} \mathrm{Map}_{2-\mathfrak{Cat}}([1,1]^{\sim}, \mathrm{Fun}([1], E)_{rlax})$$

classifies pairs of noncommutative diagrams (25) in E together with 2-morphisms v_1, v_2 in E as above and 2-morphisms $\bar{b} : b_1 \to b_2$, $\bar{c} : c_1 \to c_2$ such that the diagram commutes

$$\begin{array}{ccc} c_1\alpha_0 & \overset{v_1}{\to} & \alpha_1 b_1 \\ \downarrow \bar{c} & & \downarrow \bar{b} \\ c_2\alpha_0 & \overset{v_2}{\to} & \alpha_1 b_2 \end{array} \tag{26}$$

Lemma 13.1.50. *If* $E \in 2-\mathfrak{Cat}$ *and* $e_0, e_1 \in E$ *then* $\mathrm{Fun}([1], E)_{rlax} \times_{E \times E} \{e_0, e_1\}$ *actually lies in* $1-\mathfrak{Cat}$.

Proof. We check that any 2-morphism in $\mathrm{Fun}([1], E)_{rlax} \times_{E \times E} \{e_0, e_1\}$ is actually an isomorphism. Let $\alpha_0, \alpha_1 : e_0 \to e_1$ be 1-morphisms in E, so $\alpha_0, \alpha_1 \in \mathrm{Fun}([1], E)_{rlax} \times_{E \times E} \{e_0, e_1\}$. Suppose we are given $v_1, v_2 : b_1 \to b_2$, which are 2-morphisms in E. So, v_i are 1-morphisms in $\mathrm{Fun}([1], E)_{rlax} \times_{E \times E} \{e_0, e_1\}$. My Section 13.1.49 gives a description of 2-morphisms in $\mathrm{Fun}([1], E)_{rlax}$, and hence also in $\mathrm{Fun}([1], E)_{rlax} \times_{E \times E} \{e_0, e_1\}$. It shows that a 2-morphism $v_1 \to v_2$ in $\mathrm{Fun}([1], E)_{rlax} \times_{E \times E} \{e_0, e_1\}$ is given by the diagram

(25), in which $e_{00} = e_{10} = e_0$, $e_{01} = e_{11} = e_1$, the maps b_i, c_i are the identities. Moreover, $\bar{c}, \bar{b}$ must be the identities, because we made the base change by $\{e_0, e_1\} \to E \times E$. The diagram (26) implies our claim. $\square$

13.1.51. For (A.1, 4.1.5). The involution $2-\mathfrak{Cat} \to 2-\mathfrak{Cat}$, $\mathbb{T} \mapsto \mathbb{T}^{2-op}$ preserves $1-\mathfrak{Cat}$, and induces on it a functor isomorphic to the identity functor.

For 4.1.6. The functor $\boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \overset{rev\times rev}{\to} \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \overset{[m],[n]\mapsto[m,n]}{\to} 2-\mathfrak{Cat}$ is isomorphic to the functor $\boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \overset{[m],[n]\mapsto[m,n]}{\to} 2-\mathfrak{Cat} \overset{1\&2-op}{\to} 2-\mathfrak{Cat}$.

13.1.52. For 4.2.1. For $[n] \in \boldsymbol{\Delta}^{op}$ the functor $2-\mathfrak{Cat} \to 1-\mathfrak{Cat}$, $\mathbb{S} \mapsto \mathrm{Seq}_n^{ext}(\mathbb{S})$ is the functor $\mathbb{S} \mapsto (\mathrm{Fun}([n], \mathbb{S})_{rlax})^{1-\mathfrak{Cat}}$. According to my Section 13.1.49, $\mathrm{Fun}([1], \mathbb{S})_{rlax}$ does not lie in $1-\mathfrak{Cat}$ in general.

13.1.53. For (A.1, 4.2.4). The natural transformation $\mathrm{Seq}_\bullet \to \mathrm{Seq}_\bullet^{ext}$ they mean comes from the transformation $Sq^{\sim}_{\bullet,\bullet} \to Sq_{\bullet,\bullet}$. It is given for $E \in 2-\mathfrak{Cat}$ by a morphism $E_n \to (\mathrm{Fun}([n], E)_{rlax})^{1-\mathfrak{Cat}}$ in $1-\mathfrak{Cat}$ functorial in E.

13.1.54. For (A.1, 4.3.7). It seems their definition of $\mathrm{Seq}_\bullet^{Pair}$ simply sends $(\mathbb{S}, C)$ to the functor $\boldsymbol{\Delta}^{op} \to 1-\mathfrak{Cat}$, $[n] \mapsto (\mathrm{Fun}([n], C)_{llax})^{1-\mathfrak{Cat}}$. Is this true?

My understanding is that $\mathrm{Seq}_n^{Pair}(\mathbb{S}, C)$ is the object of $1-\mathfrak{Cat}$ given by the complete Segal space $[m] \mapsto Sq_{m,n}^{Pair}(\mathbb{S}, C)$. So, $\mathrm{Seq}_n^{Pair}(\mathbb{S}, C)$ should be a subcategory of $(\mathrm{Fun}([n], \mathbb{S})_{rlax})^{1-\mathfrak{Cat}}$, where we keep all objects, and impose some conditions on morphisms. The natural transformation mentioned in (A.1, 4.4.2) are maps in $1-\mathfrak{Cat}$

$$S_n \leftarrow \mathrm{Seq}_n^{Pair}(\mathbb{S}, C) \to (\mathrm{Fun}([n], \mathbb{S})_{rlax})^{1-\mathfrak{Cat}}$$

13.1.55. For (A.1, 4.4.1). The functor $2-\mathfrak{Cat} \to 2-\mathfrak{Cat}$, $T \mapsto \mathrm{Fun}([n], T)_{rlax}$ preserves limits.

By definition, $\mathrm{Seq}^{ext} : 2-\mathfrak{Cat} \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathfrak{Cat})$ sends E to the functor $[n] \mapsto (\mathrm{Fun}([n], E)_{rlax})^{1-\mathfrak{Cat}}$. So, it preserves limits.

13.1.56. For (A.1, 4.4.1). Consider the adjoint pair $\mathfrak{L} : \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathfrak{Cat}) \rightleftarrows 2-\mathfrak{Cat} : \mathrm{Seq}_\bullet$. I usually denote $Seq_\bullet$ as the canonical inclusion $E \mapsto E$. Let $E \in \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ then for any $T \in 2-\mathfrak{Cat}$ we get

$$\mathrm{Map}_{2-\mathfrak{Cat}}(\mathfrak{L}(E), T) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathfrak{Cat})}(E, T) \widetilde{\to} \mathrm{Map}_{\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})}(E, T^{\mathrm{Spc}})$$

Here $T \mapsto T^{\mathrm{Spc}}$ is the right adjoint to the inclusion $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc}) \subset \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathfrak{Cat})$. In particular, the identity map $\mathfrak{L}(E) \to \mathfrak{L}(E)$ factors through $\mathfrak{L}(E)^{\mathrm{Spc}}$, so $\mathfrak{L}(E) \in 1-\mathfrak{Cat}$. So, the restriction of $\mathfrak{L}$ gives a functor $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc}) \to 1-\mathfrak{Cat}$ left adjoint to the inclusion $Seq_\bullet : 1-\mathfrak{Cat} \hookrightarrow \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$.

The composition

$$\mathrm{Fun}(\boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op}, \mathrm{Spc}) \overset{\mathfrak{L}^{\boldsymbol{\Delta}^{op}}}{\to} \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathfrak{Cat}) \overset{\mathfrak{L}^{ext}}{\to} 2-\mathfrak{Cat}$$

is $\mathfrak{L}^{Sq}$. Here $\mathfrak{L}^{\boldsymbol{\Delta}^{op}}$ sends a functor $f_{\bullet,\bullet}$ to the functor $[n] \mapsto \mathfrak{L}(f_{\bullet,n})$.

13.1.57. For (A.1, 5.1.2). If $E \in \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathrm{Spc})$ is a complete Segal space then the degeneracy map $E_0 \to E_1$ is a full subspace.

Given a double category, for $0 \le i, j \le 1$ consider the map $E_{1,1} \to E_{0,0}$ given by $[0] \times [0] \stackrel{i,j}{\to} [1] \times [1]$, this gives (i, j)-vertex of a diagram $a \in E_{1,1}$. Similarly, we get arrows $a_{00} \to a_{01}, a_{00} \to a_{10}$ and so on.

13.1.58. For (A.1, proof of Prop. 5.3.3). For $[m], [n] \in \boldsymbol{\Delta}$ one has a functorial isomorphism

$$\mathrm{Seq}_m \mathrm{Seq}_n^{ext}(\textbf{1-Cat}) = \mathrm{Map}_{1-\mathfrak{Cat}}([m], Cart_{/[n]^{op}}) \widetilde{\to} \mathrm{Map}_{1-\mathfrak{Cat}}([n], coCart_{/[m]})$$

given by the swapping procedure (ch. 12, 2.1.3). This is why the bisimplicial space $\mathrm{Seq}_\bullet \mathrm{Seq}_\bullet^{ext}(\textbf{1-Cat})$ is a complete Segal space along each row and column.

In their proof $Sq'_{\bullet,\bullet}(\textbf{1-Cat})$ is a double category. Indeed,

$$\mathrm{Map}_{1-\mathfrak{Cat}}([m], (Cart_{/[n]^{op}})_{strict}) \widetilde{\to} \mathrm{Map}_{1-\mathfrak{Cat}}([m] \times [n], 1-\mathfrak{Cat})$$

is symmetric with respect to permuting m and n.

For any $T \in 1-\mathfrak{Cat}$ viewed as a 2-category one has

$$\mathrm{Map}_{2-\mathfrak{Cat}}([m, n], T) \widetilde{\to} \mathrm{Map}_{1-\mathfrak{Cat}}([m] \times [n], T)$$

This implies $Sq'_{\bullet,\bullet}(\textbf{1-Cat}) = Sq_{\bullet,\bullet}(1-\mathfrak{Cat})$.

13.1.59. For (A.1, 6.1.1). We have $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \textbf{1-Cat})^{\mathrm{Spc}} \widetilde{\to} \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathfrak{Cat})^{\mathrm{Spc}}$ canonically. So, a full subcategory of $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathfrak{Cat})$ gives rise to a full subcategory of $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \textbf{1-Cat})$.

One has $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \textbf{1-Cat})^{1-\mathfrak{Cat}} \widetilde{\to} \mathrm{Fun}(\boldsymbol{\Delta}^{op}, 1-\mathfrak{Cat})$ canonically. Indeed, calculate $\mathrm{Map}_{2-\mathfrak{Cat}}([n], \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \textbf{1-Cat}))$ as a functor of $[n] \in \boldsymbol{\Delta}^{op}$.

Remark 13.1.60. *If $\mathbb{S} \in 2-\mathfrak{Cat}$ and $T \subset \mathbb{S}^{1-\mathfrak{Cat}}$ is a full subcategory let $\mathbb{T} \subset \mathbb{S}$ be the corresponding full subcategory. Then $\mathbb{T}^{1-\mathfrak{Cat}} \widetilde{\to} T$ naturally.*

This is why $\textbf{2-Cat}^{1-\mathfrak{Cat}} \widetilde{\to} 1-\mathfrak{Cat}$.

13.1.61. Given $T, S, V \in 2-\mathfrak{Cat}$, let us construct a canonical map in $2-\mathfrak{Cat}$

$$\mathrm{Fun}(S, \mathrm{Fun}(T, V))_{rlax} \to \mathrm{Fun}(T, \mathrm{Fun}(S, V)_{rlax})$$

Let $X \in 2-\mathfrak{Cat}$. In Section 13.1.42 we introduced a canonical map $(T \times X) \circledast S \to T \times (X \circledast S)$ in $2-\mathfrak{Cat}$. It yields a morphism functorial in $V \in 2-\mathfrak{Cat}$

$$\begin{aligned}(27) \quad \mathrm{Map}_{2-\mathfrak{Cat}}(X, \mathrm{Fun}(S, \mathrm{Fun}(T, V))_{rlax}) \widetilde{\to} \mathrm{Map}_{2-\mathfrak{Cat}}(T \times (X \circledast S), V) \to \\ \mathrm{Map}_{2-\mathfrak{Cat}}((T \times X) \circledast S, V) \widetilde{\to} \mathrm{Map}_{2-\mathfrak{Cat}}(X, \mathrm{Fun}(T, \mathrm{Fun}(S, V)_{rlax}))\end{aligned}$$

It is also functorial in X, hence the desired map. It seems (27) is a full subspace.

This is used in ([14], A.1, 6.1.3) I think. The other thing they use there is as follows. If $S \in 1-\mathfrak{Cat}, \mathbb{T} \in 2-\mathfrak{Cat}$ then $\mathrm{Fun}(S, \mathbb{T})^{1-\mathfrak{Cat}} \widetilde{\to} \mathrm{Fun}(S, \mathbb{T}^{1-\mathfrak{Cat}})$, where in the RHS the symbol Fun denotes the $(\infty, 1)$-category of functors between objects of $1-\mathfrak{Cat}$.

13.1.62. Sam says: a $(\infty,1)$-category enriched over $1-\mathcal{C}at$ is the same as a $(\infty,2)$-category. This a rigorous statement in the sense of ([32], Th. 0.0.3).

Example of an application: if A is a symmetric monoidal $(\infty,1)$-category admitting inner homs, and $A\to 1-\mathcal{C}at$ is right-lax symmetric monoidal functor then A becomes an object of $2-\mathcal{C}at$.

13.1.63. Nick says: let $C\in 1-\mathcal{C}at$, $I\to C^{1-\mathcal{C}at}$, $i\mapsto c_i$ be a map in $1-\mathcal{C}at$ with $c=\operatorname{colim}_i c_i$ in $C^{1-\mathcal{C}at}$. It is not true in general that for $d\in C$ the natural map

$$\mathbf{Map}_C(c,d)\to\lim_{i\in I^{op}}\mathbf{Map}_C(c_i,d)$$

is an isomorphism. For example, let $A\in\mathbb{E}_2(\mathrm{Spc})$ be a $\mathbb{E}_2$-algebra in Spc. There is the corresponding object of $2-\mathcal{C}at$, say C with one object $c\in C$, only one 1-morphism $\mathrm{id}:c\to c$, and with $\mathrm{Map}(\mathrm{id},\mathrm{id})=A$, the mapping space in $\mathbf{Map}_C(c,c)$. We may assume for example that the only invertible element in A is the identity. (For example, take A to be a commutative monoid in Spc with this property). Then $C^{1-\mathcal{C}at}=*$, and c is the initial object of $C^{1-\mathcal{C}at}$. However, $\mathbf{Map}_C(c,c)$ is not the final object of $1-\mathcal{C}at$.

Note that if for a diagram $I^{\triangleright}\to C^{1-\mathcal{C}at}$ given by $i\to c_i$ with the image of the final object of $I^{\triangleright}$ in C given by c we have $\mathbf{Map}_C(c,d)\to\lim_{i\in I^{op}}\mathbf{Map}_C(c_i,d)$ for any $d\in C$ then $I^{\triangleright}\to C^{1-\mathcal{C}at}$ is a colimit diagram in $C^{1-\mathcal{C}at}$.

Nick: a way to guarantee the above property is to ask that for each object $x\in C$, there exists an object $[1]\otimes x$ such that

$$\mathrm{Map}_{C^{1-\mathcal{C}at}}([1]\otimes x,y)\widetilde{\to}\mathrm{Map}_{1-\mathcal{C}at}([1],\mathbf{Map}_C(x,y))$$

in Spc and that the corresponding functor $C^{1-\mathcal{C}at}\to C^{1-\mathcal{C}at}$ given by $x\mapsto[1]\otimes x$ preserves colimits. This happens for example when C is tensored and cotensored over $1-\mathcal{C}at$ (cf. Def. 6.5 and 8.2 in [18]).

14. Conventions

14.1. For $\emptyset,\mathcal{C}\in 1-\mathcal{C}at$, we should have $\mathrm{Funct}(\emptyset,\mathcal{C})=*$. For $X\in\mathrm{Spc}$ we should have $\mathrm{Map}_{\mathrm{Spc}}(\emptyset,X)=*$.

If $X\in\mathrm{Spc}$ is not empty then $\mathrm{Map}_{\mathrm{Spc}}(X,\emptyset)=\emptyset$. This is used to show that a 0-connective space X is the same as a non empty space. Indeed, a 0-connective space is space X such that $X\to 1$ induces an isomorphism $\mathrm{Map}_{\mathrm{Spc}}(*,y)\to\mathrm{Map}_{\mathrm{Spc}}(X,y)$ for any $y\in\tau_{\le -1}\mathrm{Spc}$. Such y is empty or $*$.

14.1.1. In the book [14], the conventions about the t-structures on a stable category are different from those of [28]. Namely, [28] uses the homological indexing conventions, and Dennis uses the cohomological ones. So, if $(\mathcal{C}^{\le 0}\subset\mathcal{C})$ defines a t-structure on a stable category $\mathcal{C}$ in the sense of [14] then for example $\tau_{\le 0}:\mathcal{C}\to\mathcal{C}^{\le 0}$ is the right adjoint to the inclusion $\mathcal{C}^{\le 0}\to\mathcal{C}$, and each $x\in\mathcal{C}$ admits a fiber sequence $x'\to x\to x''$ with $x'\in\mathcal{C}^{\le 0}, x''\in\mathcal{C}^{\ge 1}$. The conventions of Lurie ([28], 1.2.1.1) are different.

For example, in ([14], ch. I.1, 6.2.8), Dennis defines the full subcategory $Sptr^{>0}\subset Sptr$ as the one spanned by objects K with $\Omega^\infty(K)\widetilde{\to}*$. In Lurie's notation this would be $Sptr^{<0}$.

Terminology for Dennis' conventions: $\mathcal{C}^{\leq 0}$ = connective objects, $\mathcal{C}^{\geq 0}$ = coconnective objects. Eventually coconnective objects are $\mathcal{C}^{+} = \cup_n \mathcal{C}^{\geq n}$. Now $\mathcal{C}^{-} = \cup_n \mathcal{C}^{\leq n}$.

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