1. Comments to Deformation of local systems and Eisestein series [3]

1.0.1. Let L be a Lie algebra, $A = \mathrm{Sym}(L^*[-1])$ be a (super)-commutative k-algebra, let $Y = \operatorname{Spec} A$, let M be a A-module. Then we get a natural augmentation $A \to k$. According to introduction, L acts on M. Why it should also act on $k \overset{L}{\otimes}_A M$?

1.0.2. For a Lie algebra L they call $C_\bullet(L) = \mathrm{Sym}(L[1])$ the standard complex of L. In which sense the universal enveloping algebra $U(L)$ is Koszul dual of $C_\bullet(L)$? What is $U(L)$ in the case when L is a DG-Lie algebra?

1.0.3. We assume throughout this text that we work with $\bar{\mathbb{Q}}_\ell$-sheaves.

Case of $G = \mathrm{GL}_2$. The exact sequence (2.1) in Sect. 2.4 roughly answers the question how the perverse sheaf $(i_0^{d_1,d_2})_! \mathrm{IC}$ on $\overline{\mathrm{Bun}}_B^{d_1,d_2}$ decomposes into irreducibles in the abelian category of perverse sheaves on $\overline{\mathrm{Bun}}_B^{d_1,d_2}$. For any $d \geq 0$, the $*$-restriction is

$$(i_d^{d_1,d_2})^* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \widetilde{\to} \bar{\mathbb{Q}}_\ell[2m] \boxtimes \mathrm{IC}$$

for $i_d^{d_1,d_2} : X^{(d)} \times \mathrm{Bun}_B^{d_1+d,d_2-d} \hookrightarrow \overline{\mathrm{Bun}}_B^{d_1,d_2}$.

Let $\theta \in \Lambda^{pos}$. Write $\overline{\mathrm{Bun}}_{B,\geq\theta}$ for the image of the finite map $X^\theta \times \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B$. Write α for the unique simple coroot of G.

Consider the general situation of a stack Y with a stratification $Y_0, Y_1, Y_2, \ldots$ such that $\bar{Y}_m = \cup_{k\geq m} Y_k$. Assume F is a perverse sheaf on Y, let $j_m : Y_m \hookrightarrow Y$ be the inclusion. Assume $j_m^* F$ is placed in perverse degree $-m$. Assume the inclusion $Y_m \hookrightarrow \bar{Y}_m$ is affine, so $F_m := j_{m!} j_m^* F[-m]$ is perverse on Y for any $m \geq 0$. The substack

$$Y_{\leq m} = \underset{k\leq m}{\cup} Y_k \subset Y$$

is assumed open. Now apply the idea from ([7], 1.2.2.3). For $k \leq m$ let $Y_{[k,m]} = \cup_{k\leq i\leq m} Y_i$, this is a closed substack in $Y_{\leq m}$. In the stable category of sheaves on Y we get a filtered object, hence a $\mathbb{Z}$-complex on Y in the sense of ([7], Def. 1.2.2.2). Write $F\mid_{Y_{[m-1,m]}}$ for the $*$-restriction. We get an exact triangle $(j_{m-1})_! j_{m-1}^* F \to F\mid_{Y_{[m-1,m]}} \to (j_m)_! j_m^* F$ on $Y_{[m-1,m]}$, and extend it by zero to get an exact triangle on Y. It gives a morphism $F_m \to F_{m-1}$. As Lurie explains in ([7], 1.2.2.3), these morphisms form complex

$$\ldots \to F_2 \to F_1 \to F_0$$

(that is, the square of the differential is zero). The claim is that the complex

$$\ldots \to F_2 \to F_1 \to F_0 \to F$$

is then exact (in the abelian category of perverse sheaves).

1.0.4. About Koszul complex in Th.2.3. they use the following notion of a Koszul complex. Let R be a commutative ring, V an R-module, $\delta : V \to R$ a R-linear map. Then we have the Koszul complex $\ldots \wedge^2 V \stackrel{d_2}{\to} \wedge^1 V \stackrel{d_1}{\to} R$, where $d_1 = \delta$, and the map $d_r : \wedge^r V \to \wedge^{r-1} V$ for $r \geq 2$ is the composition

$$\wedge^r V \to \wedge^{r-1} V \otimes V \stackrel{\mathrm{id}\otimes\delta}{\to} \wedge^{r-1} V$$

Here to be precise I think $\wedge^r V \to \wedge^{r-1} V \otimes V$ sends $e_1 \wedge \ldots \wedge e_r$ to

$$\sum_{i=1}^{r} (-1)^{i+1} (e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_r) \otimes e_i$$

It is understood here that $\wedge^r V$ is interpreted as $\wedge^{r,!} V \subset V^{\otimes r}$ in the sense of ([4], Section 5.1). This is found in (Stack project, Definition 15.26.1).

They apply this for the symmetric algebra $R = \operatorname{Sym} W$ over k and $V = W \otimes_k R$ with the natural map $\delta : V \to R$ given by the product. Now if M is an R-module, this Koszul complex for M becomes $\ldots \to (\wedge^2 W) \otimes_k M \to (\wedge^1 W) \otimes_k M \stackrel{\partial_1}{\to} M$, where ∂_1 is a part of the action map $R \otimes_k M \to M$. In their case moreover R is $\mathbb{Z}_+$-graded, M is graded and the action preserves the gradings. They write in Th. 2.3 only the corresponding complex in the degree (d_1, d_2).

In Th. 2.3 they use that the $\bar{\mathbb{Q}}_\ell$-algebra $R := \operatorname{Sym} W$ is regular, so the Koszul complex for this algebra gives a resolution of $\bar{\mathbb{Q}}_\ell$ by free R-modules.

The Koszul resolution of $\operatorname{Eis}_!(E_{\check{T}})$ is $\ldots (\wedge^2 W) \otimes R \otimes_R \operatorname{Eis}_!(E_{\check{T}}) \to (\wedge^1 W) \otimes R \otimes_R \operatorname{Eis}_!(E_{\check{T}}) \stackrel{\partial_1}{\to} \operatorname{Eis}_!(E_{\check{T}})$.

1.0.5. The diagram at the end of Sect. 2.4 says the following, here $G = \mathrm{GL}_2$. Write for brevity $\overline{\operatorname{Bun}}^{\lambda}_{B,\le d}$ for the open substack $\cup_{i=0}^{d} X^{(i)} \times \operatorname{Bun}_B^{\lambda + i\alpha}$, here α is the simple coroot. Similarly, we have $\overline{\operatorname{Bun}}^{\lambda}_{B,[r,d]}$ for $r \le d$, this is $\cup_{i=r}^{d} X^{(i)} \times \operatorname{Bun}_B^{\lambda+i\alpha}$. The claim is that the transition map in the complex ((2.1), p. 1802) described at the end of ([**?**], Sect. 2.4) is also the transition map described in my Section 1.0.3. To see this, consider the diagram

$$\begin{array}{ccc} X^{(d-1)} \times X \times \operatorname{Bun}_B^{\lambda+d\alpha} & \stackrel{\mathrm{id} \times i_1}{\to} & X^{(d-1)} \times \overline{\operatorname{Bun}}^{\lambda+(d-1)\alpha}_{B,\le 1} \\ \downarrow sum \times \mathrm{id} & & \downarrow \beta \\ X^{(d)} \times \operatorname{Bun}_B^{\lambda+d} & \stackrel{i}{\to} & \overline{\operatorname{Bun}}^{\lambda}_{B,[d-1,d]} \end{array}$$

We get

$$i_! \operatorname{IC} \mapsto i_!(sum \times \mathrm{id})_! \operatorname{IC} \widetilde{\to} \beta_!(\mathrm{id} \times i_1)_! \operatorname{IC}$$

Let $j : \operatorname{Bun}_B^{\lambda+(d-1)\alpha} \hookrightarrow \overline{\operatorname{Bun}}^{\lambda+(d-1)\alpha}_{B,\le 1}$ be the open immersion. Since $(i_1)_! \operatorname{IC} \to j_! \operatorname{IC}$ naturally, composing we get the map

$$i_! \operatorname{IC} \mapsto i_!(sum \times \mathrm{id})_! \operatorname{IC} \widetilde{\to} \beta_!(\mathrm{id} \times i_1)_! \operatorname{IC} \to \beta_!(\operatorname{IC} \boxtimes j_! \operatorname{IC})$$

This map (extended by zero to $\overline{\operatorname{Bun}}^{\lambda}_B$) gives the transition map in the complex ((2.1), p. 1802). We get an exact triangle on $X^{(d-1)} \times \overline{\operatorname{Bun}}^{\lambda+(d-1)\alpha}_{B,\le 1}$

$$\operatorname{IC} \boxtimes i_{1!} \operatorname{IC} \to \operatorname{IC} \boxtimes j_! \operatorname{IC} \to \operatorname{IC}$$

it yields an exact triangle

$$\beta_!(\operatorname{IC} \boxtimes i_{1!} \operatorname{IC}) \to \beta_!(\operatorname{IC} \boxtimes j_! \operatorname{IC}) \to \beta_! \operatorname{IC} \tag{1}$$

on $\overline{\operatorname{Bun}}^{\lambda}_{B,[d-1,d]}$. Now $\beta_! \operatorname{IC}$ contains $\bar{\mathbb{Q}}_\ell[d + \dim \operatorname{Bun}_B^{\lambda+d\alpha}]$ as a direct summand. Let $i_{d-1} : X^{(d-1)} \times \operatorname{Bun}_B^{\lambda+(d-1)\alpha} \hookrightarrow \overline{\operatorname{Bun}}^{\lambda}_{B,[d-1,d]}$ be the open stratum. Then the exact triangle

$$i_!\bar{\mathbb{Q}}_\ell[-1] \to (i_{d-1})_!\bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_\ell$$

(appropriately shifted) is now contained in the triangle (1) as a direct summand, I think. Is this correct? This is why the two definitions of the transition maps match.

1.0.6. A possible construction of the sheaf $A_{X^{(n)}}$ from ([3], p. 1805, Sect. 3.1) in a special case. Let A be a $\bar{\mathbb{Q}}_\ell$-local system on X, which is a sheaf of coalgebras on X. Consider the sheaf $A^{(n)}$ on $X^{(n)}$.

Lemma 1.0.7. *There is a unique subsheaf $F_n \subset A^{(n)}$ such that for $D = \sum d_k x_k$ one gets $(F_n)_D = \otimes_k A_{x_k} \subset \otimes_k \operatorname{Sym}^{d_k}(A_{x_k})$.*

Proof. One may check that all the cospecialization maps preserve the fibres of F_n. □

I wonder if F_n coincides with $A_{X^{(n)}}$ in this case.

1.0.8. The construction of $A_{X^{(n)}}$ from ([3], p. 1805, Sect. 3.1). The standard complex of a DG-Lie algebra L is $\operatorname{Sym}(L[1])$, they view it as a cocommutative DG-coalgebra. This is because the costandard complex $\operatorname{Sym} L^*[-1]$ of a DG-Lie algebra L is a commutative DG-algebra. Notation: $C_\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})$ is the standard complex of the sheaf of Lie algebras $\check{\mathfrak{n}}_{X,E_{\check{T}}}$ on X, this is a sheaf of cocommutative DG-coalgebras on X. Better to say, this is a cocommutative DG-coalgebra in the category of local systems on X.

In ([8], Section 1.3.53) we associated to a commutative DG-algebra $E^{\cdot}$ in the category of local systems on X a complex of factorization algebras $\mathbb{D}(B_{X^I} \otimes \eta_I)$, $I \in \mathcal{S}$, here B_{X^I} is a complex of perverse sheaves on X^I. The $*$-fibre of $\mathbb{D}(B_{X^I} \otimes \eta_I)$ at $mx \in X^I$ is the complex $E_x^{\cdot *}$. Take the direct image of $\mathbb{D}(B_{X^I} \otimes \eta_I)$ under $X^I \to X^{(m)}$, where $m = \mid I \mid$, and take $\operatorname{Aut}(I)$-invariants. The complex that we obtain on $X^{(n)}$ is denoted by $A_{X^{(n)}}$ in ([3], p. 1805) for the cocommutative DG-coalgebra $A = E^{\cdot *} = \mathcal{H}om(E^{\cdot}, \bar{\mathbb{Q}}_\ell)$.

Consider the standard complex $C_\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})$ as a cocommuative DG-coalgebra in the category of local systems on X. It is actually a $\check{\Lambda}^{pos}$-graded cocommuative DG-coalgebra in the category of local systems on X. Indeed, if $\check{\alpha}, \check{\beta}$ are positive coroots of G then $[\check{\mathfrak{n}}_{\check{\alpha}}, \check{\mathfrak{n}}_{\check{\beta}}] \subset \check{\mathfrak{n}}_{\check{\alpha}+\check{\beta}}$. Therefore, the differential on $\operatorname{Sym}(\check{\mathfrak{n}}[1])$ preserves the $\check{\Lambda}^{pos}$-grading.

So, $C^*_\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}}) = \mathcal{H}om(C_\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}}), \bar{\mathbb{Q}}_\ell)$ is the costandard complex, a $-\check{\Lambda}^{pos}$-graded commutative DG-algebra in the category of local systems on X. In ([8], Section 1.3.53) we associated to such an object a complex B in $\mathcal{F}\mathrm{A}(X)_{-\check{\Lambda}^{pos}}$. Let $\check{\lambda} \in \check{\Lambda}^{pos} - 0$, write $\check{\lambda} = \sum_j n_j \check{\alpha}_j$, where $\check{\alpha}_j$ are simple coroots of G. Consider $(I, \check{\lambda}_I) \in \mathcal{S}_{-\check{\Lambda}^{pos}}$ such that $\check{\lambda}_I : I \to -\check{\Lambda}^{pos}$ takes values only in minus simple coroots, and each $-\check{\alpha}_j$ appear with multiplicity n_j. Recall that $B_{X^{\check{\lambda}}} = sum_!(B_{X^I} \otimes \eta_I)^{\operatorname{Aut}(I,\check{\lambda})}$ for the natural map $sum : X^I \to X^{\check{\lambda}}$. Then

$$\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}} = \mathbb{D}B_{X^{\check{\lambda}}}$$

For $\check{\lambda}x \in X^{\check{\lambda}}$ the $*$-fibre of $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ at this point is $\wedge^\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}_x$. Here the superscript $\check{\lambda}$ denotes the corresponding component of $\check{\Lambda}^{pos}$-grading, and $\wedge^i\check{\mathfrak{n}}$ is placed in cohomological degree $-i$. Since $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ factorize, this gives a description of the $*$-fibre at any point of $X^{\check{\lambda}}$.

For example, if $\check{\lambda}=\check{\alpha}$ a simple coroot then $X^{\check{\alpha}}=X$, and $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\alpha}}=\check{\mathfrak{n}}^{\check{\alpha}}_{X,E_{\check{T}}}[1]$ is a smooth perverse sheaf on X.

If $\check{\mathfrak{n}}$ is a 1-dimensional abelian Lie algebra, and $\check{\alpha}$ is the unique simple coroot, for $\check{\lambda}=n\check{\alpha}$ we get $\Upsilon(\check{\mathfrak{n}}_X)^{\check{\lambda}}\widetilde{\to}(\wedge^{(n)}\check{\mathfrak{n}}_X)[n]$. Here for a local system F on X we write $\wedge^{(n)}F$ for the n-th exteriour power of F.

Note that for a rank one local system F on X, $\wedge^{(n)}F$ is the extension by zero from the open subscheme $U\hookrightarrow X^{(n)}$, the complement to all the diagonals, and for $D=\sum_{i=1}^n x_i$ with x_i pairwise distinct the $*$-fibre of $\wedge^{(n)}F$ at D is isomorphic to $\otimes_i F_{x_i}$.

Remark 1.0.9. *If E, E' are commutative DG-algebras then $E\otimes E'$ is a commutative DG-algebra naturally. If L_i is a Lie algebra then for standard complexes we have an isomorphism of complexes* $\mathrm{Sym}(L_1[1])\otimes\mathrm{Sym}(L_2[1])\widetilde{\to}\mathrm{Sym}((L_1\oplus L_2)[1])$ *of vector spaces. Moreover, this is an isomorphism of cocommutative DG-coalgebras.*

In ([8], Section 1.3.55) we described the object of $\mathcal{F}\mathrm{A}(X)_{-\Lambda^{pos}}$ associated to a tensor product of two $-\Lambda^{pos}$-graded commutative DG-algebras in local systems on X.

Note that $\mathfrak{n}=\oplus_{\check{\alpha}\in\check{\Delta}^+}\check{\mathfrak{n}}_{\check{\alpha}}$ is the sum of the coroot subspaces, here $\check{\Delta}^+$ is the set of all positive coroots. If $\check{\mathfrak{n}}$ is a commutative Lie algebra, we get an isomorphism

$$C^*_\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})=\underset{\check{\alpha}\in\check{\Delta}^+}{\otimes}C^*_\bullet(\check{\mathfrak{n}}_{\check{\alpha},E_{\check{T}}})$$

of DG-algebras. In this case we get

$$\Upsilon(\check{\mathfrak{n}}_X)^{\check{\lambda}}=\underset{\sum\limits_{i\in\check{\Delta}^+}n_i\check{\alpha}_i=\check{\lambda}}{\oplus}\;(\underset{i\in\check{\Delta}^+}{\star}\Upsilon(\check{\mathfrak{n}}_{\check{\alpha}_i})^{n_i\check{\alpha}_i})$$

The sum is taken over all decompositions of $\check{\lambda}$ as indicated. Since each $\Upsilon(\check{\mathfrak{n}}_{\check{\alpha}_i})^{n_i\check{\alpha}_i}$ is perverse, and $\star$ is exact, the above sum is also perverse.

This gives the formula for the associated graded of $\Upsilon(\check{\mathfrak{n}}_X)^{\check{\lambda}}$ on p. 1806.

1.0.10. The map $i_{\check{\lambda}}: X^{\check{\lambda}}\times\mathrm{Bun}_B\to\overline{\mathrm{Bun}}_B$ is a locally closed immersion, so $i^!F$ is placed in perverse degrees ≥ 0 for a perverse sheaf F on $\overline{\mathrm{Bun}}_B$ (used in Th. 4.2).

1.0.11. **Question**. Can we localize the scheme of moduli of local systems in general, in the same way as for $E_{\check{T}}$ regular the collection $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}$ for $\check{\lambda}\in\check{\Lambda}^{pos}$ gives $\mathcal{O}_{Def_{\check{B}}(E_{\check{T}})}$? Do we have a $\check{\Lambda}^{pos}$-factorization algebra for any $E_{\check{T}}$?

1.0.12. If L is a Lie algebra then $U(L)$ is a cocommutative coalgebra. Now $U(\check{\mathfrak{n}}_{X,E_{\check{T}}})$ is a sheaf of $\check{\Lambda}^{pos}$-graded cocommutative coalgebras on X. Taking the graded dual $\oplus_{\check{\lambda}\in\check{\Lambda}^{pos}}(U(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}})^*$, we get a sheaf of $\check{\Lambda}^{neg}$-graded commutative algebras on X. We view it as $\check{\Lambda}^{neg}$-graded commutative algebra in local systems on X. The construction of ([8], 1.3.53) attaches to it an object B of $\mathcal{F}\mathrm{A}(X)_{\check{\Lambda}^{neg}}$, hence for any $\check{\lambda}\in\check{\Lambda}^{pos}$ a complex $B_{X^{\check{\lambda}}}$ on $X^{\check{\lambda}}$. Then $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ is defined as $\mathbb{D}B_{X^{\check{\lambda}}}$. The $*$-fibre of $\mathbb{D}B_{X^{\check{\lambda}}}$ at $\check{\lambda}x$ equals $U(\check{\mathfrak{n}}_{E_{\check{T}},x})^{\check{\lambda}}$, it is placed in usual degree zero. So, $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ is a constructible sheaf. By construction, we have for $\check{\lambda}_i\in\check{\Lambda}^{pos}$ a natural map

$$\mathfrak{U}(\check{\mathfrak{n}})^{\check{\lambda}_1+\check{\lambda}_2}\to\mathfrak{U}(\check{\mathfrak{n}})^{\check{\lambda}_1}\star\mathfrak{U}(\check{\mathfrak{n}})^{\check{\lambda}_2}$$

For example, if $\check{\lambda} = n\check{\alpha}$, where $\check{\alpha}$ is a simple coroot, then $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}} = (E_{\check{T}}^{\check{\alpha}})^{(n)}$ on $X^{(n)}$.

The !-restriction of $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ to $X^{\check{\lambda}} \times \mathrm{Bun}_B$ is described in ([1], Cor. 4.7). It is of the form $\mathcal{M}^{\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B}$. Here

$$\mathcal{M}^{\check{\lambda}} = \underset{\mathfrak{B}(\check{\lambda})}{\oplus} i_{\mathfrak{B}(\check{\lambda})*}\bar{\mathbb{Q}}_\ell \tag{2}$$

They claim in Prop. 4.4 that $\mathcal{M}^{\check{\lambda}} \widetilde{\to} \mathfrak{U}(\check{\mathfrak{n}})^{\check{\lambda}}$, I don't see why this is true.

More precisely, for a partition $\mathfrak{B}(\check{\lambda})$ given by $\check{\lambda} = \sum_j n_j \check{\lambda}_j$, where $\check{\lambda}_j$ are not necessarily simple positive coroots of G, on $X^{\mathfrak{B}(\check{\lambda})} = \prod_j X^{(n_j)}$ we get $\boxtimes_j (\check{\mathfrak{n}}^{\check{\lambda}_j})^{(n_j)}$ instead of $\bar{\mathbb{Q}}_\ell$ in (2). With this correction the isomorphism $\mathcal{M}^{\check{\lambda}} \widetilde{\to} \mathfrak{U}(\check{\mathfrak{n}})^{\check{\lambda}}$ should be canonical.

Note that $U(\check{\mathfrak{n}})$ has a filtration $0 = F_0 \subset F_1 \subset \dots$ with $F_n/F_{n-1} = \mathrm{Sym}^n \check{\mathfrak{n}}$. This filtration is compatible with the coalgebra structure on $U(\check{\mathfrak{n}})$. Namely, $\triangle\ (F_n) \subset \sum_{i_1+i_2=n} F_{i_1} \otimes F_{i_n} \subset U(\check{\mathfrak{n}}) \otimes U(\check{\mathfrak{n}})$. For any $\check{\lambda} \in \check{\Lambda}$ this filtration induces one on $U(\check{\mathfrak{n}})^{\check{\lambda}}$.

EXAMPLE: take $\check{\lambda} = \check{\alpha}_1 + \check{\alpha}_2$, where $\check{\alpha}_i$ are simple coroots of G and assume $\check{\lambda}$ is a positive coroot. We have $U^{\check{\lambda}} = \mathfrak{n}_{\check{\lambda}} + (\mathfrak{n}_{\check{\alpha}_1} \otimes \mathfrak{n}_{\check{\alpha}_2})$. Let $I = \{1,2\}$, and $\check{\lambda}_I : I \to \check{\Lambda}^{pos}_*$ take values $\check{\alpha}_i$. The corresponding complex B_{X^I} on X^I is as follows. The corresponding part of the Chevalley complex on X^2 is $j_*j^*(\mathfrak{n}^*_{\check{\alpha}_1} \boxtimes \mathfrak{n}^*_{\check{\alpha}_2}) \to \triangle_*\ (U^{\check{\lambda}})^*$ placed in degrees $-2,-1$. It is canonically isomorphic to a direct sum

$$(\mathfrak{n}^*_{\check{\alpha}_1} \boxtimes \mathfrak{n}^*_{\check{\alpha}_2}[2]) \oplus \triangle_*\ \mathfrak{n}^*_{\check{\lambda}}[1]$$

So, $B_{X^{\check{\lambda}}} = (\mathfrak{n}^*_{\check{\alpha}_1} \boxtimes \mathfrak{n}^*_{\check{\alpha}_2}[2]) \oplus \triangle_*\ \mathfrak{n}^*_{\check{\lambda}}[1]$ as a constant complex on $X^2 = X^{\check{\lambda}}$, and

$$\mathbb{D}B_{X^{\check{\lambda}}} \widetilde{\to} (\mathfrak{n}_{\check{\alpha}_1} \boxtimes \mathfrak{n}_{\check{\alpha}_2}) \oplus \triangle_*\ \mathfrak{n}_{\check{\lambda}}$$

canonically.

1.0.13. For Cor. 4.5. Recall that for $F \in \mathrm{D}^b(S)$ its image in the Grothendieck group of perverse sheaves on S is given by $\sum_i (-1)^i [\mathrm{H}^i(F)]$, where $\mathrm{H}^i(F)$ denotes the i-th perverse cohomology sheaf. In the proof of Cor. 4.5 the first step is to note that

$$[\mathcal{T} \star \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}] = \sum_{\check{\lambda}' \in \check{\Lambda}^{pos}} [\mathcal{T} \star \mathbb{D}\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}'} \star j_!\, \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}+\check{\lambda}'}}]$$

They then use the fact that $[\mathbb{D}\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}}] = [\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}}]$ is the Grothendieck group of $X^{\check{\lambda}}$, this is evident for $\mathcal{M}^{\check{\lambda}}$. This is unfortunate use, as this means that we identify $\check{\mathfrak{n}}_{\check{\alpha}}$ with $\check{\mathfrak{n}}^*_{\check{\alpha}}$ for each positive coroot $\check{\alpha}$, which we tried not to do before! In their Section 6.4, which they reffer to, a correction formula is proved.

1.0.14. For the proof of Cor. 4.6. For any $\check{\lambda} \in \check{\Lambda}^{pos}$, $\check{\mu} \in \check{\Lambda}$ we have $\overline{\mathrm{Bun}}_B^{\check{\mu},\le\check{\lambda}}$ defined in their proof of Cor. 4.6. By Cor. 4.5, in the Groth. group of perverse sheaves on $\overline{\mathrm{Bun}}_B^{\check{\mu},\le\check{\lambda}}$ we have

$$[j_!\, \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}] = \sum_{0 \le \check{\lambda}' \le \check{\lambda}} [\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}'} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}'}}]$$

In this group the LHS is a sum of irreducible perverse sheaves with some nonnegative coefficients, and the RHS is also a sum of some irreducible perverse sheaves with some nonnegative coefficients. So, this equality says that there are filtrations on perverse sheaves $j_! \mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}$ and on $\underset{0\le\check\lambda'\le\check\lambda}{\oplus} \Omega(\check{\mathfrak n}_X)^{-\check\lambda'} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda'}}$ such that their *gr* are isomorphic! We want to show that

$$[h^0 i^!_{\check\lambda} j_! \mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}] = [\Omega(\check{\mathfrak n}_X)^{-\check\lambda} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}}]$$

in the Grothendieck group of $X^{\check\lambda}\times \mathrm{Bun}_B^{\check\mu}$ (misprint in their formulation of Cor.4.6).

For any $0\le\check\lambda'<\check\lambda$ and any irreducible subquotient $\mathcal T$ of $\Omega(\check{\mathfrak n}_X)^{-\check\lambda'}$ the complex

$$i^!_{\check\lambda}(\mathcal T\star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda'}})$$

is placed in perverse degrees >0. Indeed, $\mathcal T\star\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda'}}$ is the intermediate extension from $X^{\check\lambda'}\times\mathrm{Bun}_B^{\check\mu+\check\lambda'}$. So, because of the factorization property, it suffices to show that for any $0\le\check\lambda'<\check\lambda$ and any irreducible subquotient $\mathcal T$ of $\Omega(\check{\mathfrak n}_X)^{-\check\lambda'}$, any exact sequence of perverse sheaves on $\overline{\mathrm{Bun}}_B^{\check\mu,\le\check\lambda}$

$$0\to\mathcal T\star\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda'}}\to\mathcal T'\to i_{\check\lambda !}(\triangle_!\,\mathrm{IC}_X\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\lambda}})\to 0 \tag{3}$$

splits. Here $\triangle\colon X\to X^{\check\lambda}$ is the diagonal. The cartesian square

$$\begin{array}{ccc} X^{\check\lambda'}\times\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda'} & \to & \overline{\mathrm{Bun}}_B^{\check\mu,\le\check\lambda}\\ \uparrow \mathrm{id}\times i_{\check\lambda-\check\lambda'} & & \uparrow i_{\check\lambda}\\ X^{\check\lambda'}\times X^{\check\lambda-\check\lambda'}\times\mathrm{Bun}_B^{\check\mu+\check\lambda} & \to & X^{\check\lambda}\times\mathrm{Bun}_B^{\check\mu+\check\lambda}\end{array}$$

gives $i^!_{\check\lambda}(\mathcal T\star\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda'}})\widetilde{\to}(\mathcal T\star\mathfrak U(\check{\mathfrak n}_X)^{\check\lambda-\check\lambda'})\boxtimes\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}}$. To show that (3) is trivial it suffices to check that $\triangle^!\,(\mathcal T\star\mathfrak U(\check{\mathfrak n}_X)^{\check\lambda-\check\lambda'})$ is placed in perverse degrees ≥ 2. The latter property is easy to see, because $\triangle^!\,\mathfrak U(\check{\mathfrak n}_X)^{\check\lambda-\check\lambda'}$ is placed in perverse degrees ≥ 1.

Another proof in the simplest case $\check\lambda=\check\alpha$ a simple coroot: we have an exact sequence of perverse sheaves

$$0\to\mathbb D\mathfrak U(\check{\mathfrak n})^{\check\alpha}\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\alpha}}[-1]\to j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}\to\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}\to 0$$

on $\overline{\mathrm{Bun}}_B^{\check\mu,\le\check\alpha}$. So, $h^0 i^!_{\check\alpha} j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}\widetilde{\to}\Omega^{-\check\alpha}\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\alpha}}$ over $X^{\check\alpha}\times\mathrm{Bun}_B^{\check\mu+\check\alpha}$. Here $\Omega^{-\check\alpha}=\check{\mathfrak n}^*_{\check\alpha}[1]$ on $X=X^{\check\alpha}$.

Remark: an analog of Cor. 4.6 holds for Zastava spaces.

1.0.15. In the first displayed formula on p. 1813 the parentheses are not correct, this is a misprint. It should be $\underset{i\in I}{\boxtimes}(\wedge^{(n_i)}\check{\mathfrak n}^*_{\check\alpha_i,X})[n_i]$ over $\overset{\circ}{X}{}^{\check\lambda}$.

Lm.4.8: if $C^\bullet(\check{\mathfrak n})^{-\check\lambda}$ is the $-\check\lambda$-component of the costandard complex then for the diagonal $\triangle\colon X\to X^{\check\lambda}$ one has $\triangle^!\,\Omega(\check{\mathfrak n}_X)^{-\check\lambda}\widetilde{\to}C^\bullet(\check{\mathfrak n})^{-\check\lambda}[2]$ on X. So, indeed the proof of Lm.4.8 reduces to the fact that $C^\bullet(\check{\mathfrak n})^{-\check\lambda}$ has no cohomologies in degrees ≤ 1. This is true, because the image of $[.,.]:\wedge^2\check{\mathfrak{n}}\to\check{\mathfrak{n}}$ is the direct sum $\oplus_{\check{\alpha}}\check{\mathfrak{n}}_{\check{\alpha}}$ over all nonsimple positive coroots $\check{\alpha}$ of G. So, $(\check{\mathfrak{n}}^*)^{-\check{\lambda}}=0$ for $\check{\lambda}\in\check{\Lambda}^{pos}$, which is not a simple coroot.

1.0.16. Recall that we know a filtration on $\Upsilon^{\check{\lambda}}$, hence also on $\Omega(\check{\mathfrak{n}})^{\check{\lambda}}$ given in their Sect. 3.3. I wonder if each associated graded perverse sheaf

$$\underset{k}{\star}(\wedge^{(n_k)}E_{\check{T}}^{\check{\alpha}})[j] \tag{4}$$

for this filtration is irreducible.

For Sect. 4.10. If $\check{\lambda}$ is not a positive coroot then each perverse sheaf (4) is the intermediate extension from the image of $(\prod_k \overset{\circ}{X}{}^{n_k\check{\alpha}_k})_{disj}\to X^{\check{\lambda}}$. So, $\Omega(\check{\mathfrak{n}})^{-\check{\lambda}}$ has no irreducible subquotients, which are extensions by zero from the diagonal $\triangle\colon X\hookrightarrow X^{\check{\lambda}}$.

Explanation for *Case 2*. If $\check{\lambda}=\check{\alpha}$ is a positive coroot then $\triangle_!\,(\check{\mathfrak{n}}^{\check{\alpha}})^*[1]$ is a unique irreducible subquotient (actually, quotient) of $\Omega(\check{\mathfrak{n}})^{-\check{\lambda}}$, which is the extension by zero under $\triangle$. Moreover, by the induction hypothesis, $\Omega(\check{\mathfrak{n}})^{-\check{\lambda}}\to h^0 i^!_{\check{\lambda}} j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}$ is an isomorphism over $X^{\check{\lambda}}-X$. So, if $\Omega(\check{\mathfrak{n}})^{-\check{\lambda}}\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}\to\mathcal{T}$ is not surjective then $\mathcal{T}$ is given by the exact sequence

$$0\to\Omega(\check{\mathfrak{n}})^{-\check{\lambda}}\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}\to\mathcal{T}\to\triangle_!\,\mathrm{IC}_X\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \tag{5}$$

If Y,Z are schemes, $F_i\in\mathrm{D}(Y), G_i\in\mathrm{D}(Z)$ then

$$\mathrm{R}\mathcal{H}om(F_1\boxtimes G_1,F_2\boxtimes G_2)\widetilde{\to}\mathrm{R}\mathcal{H}om(F_1,F_2)\boxtimes\mathrm{R}\mathcal{H}om(G_1,G_2)$$

Since $\mathrm{Hom}_{X^{\check{\lambda}}}(\triangle_!\,\mathrm{IC}_X,\Omega(\check{\mathfrak{n}})^{-\check{\lambda}})=0$, to show that the above sequence splits it suffices to prove their Lemma 4.11. Since the sequence (5) splits, we see that $\triangle_!\,\mathrm{IC}_X\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}$ would be a subsheaf of $h^0 i^!_{\check{\lambda}} j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}$. This contradicts Pp. 4.9.

For the proof of their Lemma 4.11 recall that $\triangle^!\,\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}}\widetilde{\to}C^\bullet(\check{\mathfrak{n}})^{-\check{\lambda}}[2]$ on X. We must show that $\mathrm{H}^1(X,C^\bullet(\check{\mathfrak{n}})^{-\check{\lambda}}[1])=0$. We may assume $\check{\lambda}=\check{\alpha}$ is a positive coroot of G, which is not simple. Then the complex on X that we have to integrate is $h^2(C^\bullet(\check{\mathfrak{n}}))^{-\check{\lambda}}$. That is, we must show that $\mathrm{H}^0(X,h^2(C^\bullet(\check{\mathfrak{n}}))^{-\check{\lambda}})=0$, which means simply that $h^2(C^\bullet(\check{\mathfrak{n}}))^{-\check{\lambda}}=0$. By definition, $h^2(C^\bullet(\check{\mathfrak{n}}))=\mathrm{H}^2(\check{\mathfrak{n}},\bar{\mathbb{Q}}_\ell)$.

1.0.17. For 5.4. If $h^0 i^!_{\check{\lambda}} j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}$ has a subsheaf, which is the extension by zero from $\triangle\,(X)\times\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$ then $\check{\lambda}$ is a positive coroot of G, and this subsheaf is $\triangle_!\,\mathrm{IC}_X\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}$.

Let $\check{\lambda}=\check{\alpha}$ be a positive coroot of G, not a simple coroot. Then there are positive coroots $\check{\beta},\check{\gamma}$ of G such that $\check{\lambda}=\check{\beta}+\check{\gamma}$, and $0\neq[x_{\check{\beta}},x_{\check{\gamma}}]\in\check{\mathfrak{n}}_{\check{\gamma}}$ for the corresponding nonzero root vectors. Their perverse sheaf $\mathcal{F}_1'$ is indeed a quotient of

$$\bar{H}:=h^0 i^!_{\check{\lambda}} j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}/(\triangle_!\,\mathrm{IC}_X\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}})$$

and not just a subquotient, because over $X^{\check{\lambda}}-X$ the sheaf $\Omega(\check{\mathfrak{n}})^{-\check{\lambda}}$ has no perverse subsheaves supported on the diagonal divisor.

Their $\mathcal{F}'$ on p. 1816 is defined as the perverse sheaf on $\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}$, the quotient of $j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}$ by H. Here $H=\mathrm{Ker}(\bar{H}\to\mathcal{F}_1')$. So, $\mathcal{F}_1'\subset\mathcal{F}'$ is a perverse subsheaf.

To see which irreducible subquotients $\mathcal{E}$ of $j_! \mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}} \mid_{\overline{\mathrm{Bun}}_B^{\check\mu, \le\check\alpha}}$ could admit a nontrivial extension $0 \to \mathcal{F}'_1 \to ? \to \mathcal{E} \to 0$, let $0 \le \check\tau < \check\alpha$. We have a cartesian square

$$\begin{array}{ccc} X^{\check\tau} \times \overline{\mathrm{Bun}}_B^{\check\mu+\check\tau} & \to & \overline{\mathrm{Bun}}_B^{\check\mu, \le\check\alpha} \\ \uparrow & & \uparrow \\ X^{\check\tau} \times X^{\check\alpha-\check\tau} \times \mathrm{Bun}_B^{\check\mu+\check\alpha} & \to & X^{\check\alpha} \times \overline{\mathrm{Bun}}_B^{\check\mu+\check\alpha} \end{array}$$

Recall that $i^*_{\check\alpha-\check\tau} \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\tau}} \widetilde{\to} \mathbb{D}\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\alpha-\check\tau} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\alpha}}$ by Prop. 4.4. If $\mathcal{E}'$ is an irreducible perverse sheaf on $X^{\check\tau}$ appearing as a subquotient of $\Omega(\check{\mathfrak{n}})^{-\check\tau}$, then this could happen if

$$\mathrm{Ext}^1_{X^{\check\alpha}}(\mathcal{E}' \star \mathbb{D}\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\alpha-\check\tau}, \mathcal{P}) \ne 0,$$

where $\mathcal{P}$ is the direct image of $\mathrm{IC}_X \boxtimes \mathrm{IC}_X$ under $\xi : X^2 \to X^{\check\alpha}$, $(x, y) \mapsto \check\beta x + \check\gamma y$. Note that $\check\beta \ne \check\gamma$, so ξ is a closed immersion.

First, they claim that for $\check\tau = 0$ this is impossible. Indeed, for a partition $\mathfrak{B}(\check\alpha)$ we have the map $i_{\mathfrak{B}(\check\alpha)} : X^{\mathfrak{B}(\check\alpha)} \to X^{\check\alpha}$ and the direct summand $i_{\mathfrak{B}(\check\alpha)!}\bar{\mathbb{Q}}_\ell[2 \mid \mathfrak{B}(\check\alpha) \mid]$ in $\mathbb{D}\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\alpha}$. The complex $\mathrm{R}\mathcal{H}om(i_{\mathfrak{B}(\check\alpha)!}\bar{\mathbb{Q}}_\ell[2 \mid \mathfrak{B}(\check\alpha) \mid], \xi_! \mathrm{IC})$ is placed in usual degrees $\ge 2 \mid \mathfrak{B}(\check\alpha) \mid -2$. So, we could possible have $\mathrm{Ext}^1 \ne 0$ only for the trivial partition $\mathfrak{B}(\check\alpha) = \check\alpha$. In this case we get $\mathrm{RHom}_{X^2}(\triangle_! \bar{\mathbb{Q}}_\ell[2], \mathrm{IC}) \widetilde{\to} \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell[-2])$, it has no Ext^1 either.

Let now $\check\tau > 0$. Assume that $\mathcal{E}'$ comes from the decimpsotion $\mathfrak{B}(\check\tau) = \sum_k n_k\check\alpha_k$, where $\check\alpha_k$ are positive coroots of G, so $\mid \mathfrak{B}(\check\tau) \mid = \sum_k n_k$. Then

$$\mathrm{R}\mathcal{H}om(\mathcal{E}' \boxtimes i_{\mathfrak{B}(\check\alpha-\check\tau)!}\bar{\mathbb{Q}}_\ell[2 \mid \mathfrak{B}(\check\alpha - \check\tau) \mid], \mathcal{P})$$

is placed in usual degrees $\ge \mathfrak{B}(\check\alpha - \check\tau) + \mid \mathfrak{B}(\check\tau) \mid -2$. So, Ext^1 could be nonzero only if both $\check\alpha - \check\tau, \check\tau$ are positive coroots of G and $\mathfrak{B}(\check\alpha - \check\tau) = \check\alpha - \check\tau$, $\mathfrak{B}(\check\tau) = \check\tau$. Moreover, we must have $\check\tau = \check\beta$ or $\check\tau = \check\gamma$. In both cases the corresponding space Ext^1 is 1-dimensional.

1.0.18. Recall that $\mathfrak{U}(\check{\mathfrak{n}})^{\check\lambda} = \oplus_{\mathfrak{B}(\check\lambda)} i_{\mathfrak{B}(\check\lambda)!}\bar{\mathbb{Q}}_\ell$, but this isomorphism is not canonical!!! For $\check\lambda_i \in \check\Lambda^{pos}$ two partitions $\mathfrak{B}(\check\lambda_i)$ for $i = 1, 2$ give rise to their sum $\mathfrak{B}(\check\lambda_1) + \mathfrak{B}(\check\lambda_2)$, which is a partition $\mathfrak{B}(\check\lambda_1 + \check\lambda_2)$ of $\check\lambda_1 + \check\lambda_2$ and a diagram

$$\begin{array}{ccc} X^{\check\lambda_1} \times X^{\check\lambda_2} & \to & X^{\check\lambda_1+\check\lambda_2} \\ \uparrow & & \uparrow \\ X^{\mathfrak{B}(\check\lambda_1)} \times X^{\mathfrak{B}(\check\lambda_2)} & \to & X^{\mathfrak{B}(\check\lambda_1)+\mathfrak{B}(\check\lambda_2)} \end{array}$$

This gives a map $\mathcal{M}^{\check\lambda_1} \star \mathcal{M}^{\check\lambda_2} \to \mathcal{M}^{\check\lambda_1+\check\lambda_2}$, but it is different from the map denoted (5.5) in their Sect. 5.5! Indeed, the map $\mathfrak{U}(\check{\mathfrak{n}})^{\check\lambda_1} \star \mathfrak{U}(\check{\mathfrak{n}})^{\check\lambda_2} \to \mathfrak{U}(\check{\mathfrak{n}})^{\check\lambda_1+\check\lambda_2}$ is not commutative, as the product in $U(\check{\mathfrak{n}})$ is not commutative. The map denoted (5.5) in their paper is constructed as in ([8], Section 1.3.58).

I have given a direct proof of ([3], Theorem 5.6) in ([9], Proposition 1.1.1).

1.0.19. If $\mathcal{G}$ is a sheaf on a scheme Y with a filtration $\mathcal{G}^1 \subset \mathcal{G}^2 \subset \mathcal{G}^3 = \mathcal{G}$ with $\mathcal{G}_i = \mathcal{G}^i/\mathcal{G}^{i-1}$, we get natural maps $\mathcal{G}_2 \to \mathcal{G}_1[1]$, $\mathcal{G}_3 \to \mathcal{G}_2[1]$ on X. Their composition is a map $\mathcal{G}_3 \to \mathcal{G}_1[2]$, it vanishes (by [7], Remark 1.2.2.3). This is used in their Section 5.7. There is a sign issue, I think, because the natural isomorphism $\mathbb{C}_X^{\check\beta}[1] \star \mathbb{C}_X^{\check\gamma}[1] \widetilde{\to} \mathbb{C}_X^{\check\gamma}[1] \star \mathbb{C}_X^{\check\beta}[1]$ contains the sign, which appears when we permute $\check\beta$ and $\check\gamma$.

1.0.20. For 6.2. The differential on $\mathfrak{U}(\check{\mathfrak{n}})^{\bullet,\check{\lambda}}$ is explained, I think, by ([5], Proposition 4.4.1). This is the $\check{\lambda}$-component of the tensor algebra of Υ in the tensor category $\mathrm{D}(\sqcup_{\check{\lambda}\in\check{\Lambda}^{pos}} X^{\check{\lambda}})$. Namely, the fact that $d^2 = 0$ comes from the associativity of the product on $\Omega(\check{\mathfrak{n}})$.

Namely, let $E = \mathrm{Sym}(\check{\mathfrak{n}}^*[-1])$ be the costandard complex of $\check{\mathfrak{n}}$. This is a finite-dimensional algebra, so we get the complex in the category of complexes (double complex)

$$E^* \xrightarrow{d} E^*\otimes E^* \xrightarrow{d} E^*\otimes E^*\otimes E^* \xrightarrow{d} \dots, \tag{6}$$

here $(E^*)^{\otimes m}$ is placed in 'horizontal' degree m, the horizontal differential d is an odd derivation of degree 1. Here $d: E^*\to E^*\otimes E^*$ is the dual of the product map $E\otimes E\to E$ (giving the structure of a DGA on E), and it extends uniquely to an odd derivation d of degree 1. The 'vertical differential' is the usual differential on $(E^*)^{\otimes m}$, we have one because E^* is the standard complex of $\check{\mathfrak{n}}$. For example, $d: E^*\otimes E^*\to E^*\otimes E^*\otimes E^*$ is given by $d(v_1\otimes v_2) = (dv_1)\otimes v_2 - v_1\otimes(dv_2)$.

Recall further that E^* is $\check{\Lambda}^{pos}$-graded and

$$((E^*)^{\otimes m})^{\check{\lambda}} = \mathop{\oplus}_{\check{\lambda}_i\in\check{\Lambda}^{pos}, \sum\check{\lambda}_i=\check{\lambda}} (E^*)^{\check{\lambda}_1}\otimes\dots\otimes(E^*)^{\check{\lambda}_m}$$

We first pass to the $\check{\lambda}$-component in the above complex. If we further keep only the summands with $\check{\lambda}_i\neq 0$, we get a quotient complex. The reason is that in the dual complex $E^{-\check{\lambda}}\leftarrow(E\otimes E)^{-\check{\lambda}}\leftarrow\dots$ we have a subcomplex whose m-th term is

$$\mathop{\oplus}_{\check{\lambda}_i\neq 0, \sum\check{\lambda}_i=\check{\lambda}} E^{-\check{\lambda}_1}\otimes\dots\otimes E^{-\check{\lambda}_m}$$

Finally, we got a double complex, let us denote it

$$(E^*)^{\check{\lambda}}_{\neq 0} \xrightarrow{d} (E^*\otimes E^*)^{\check{\lambda}}_{\neq 0} \xrightarrow{d} (E^*\otimes E^*\otimes E^*)^{\check{\lambda}}_{\neq 0} \xrightarrow{d} \dots, \tag{7}$$

The 0-th term of the total complex of this double complex is

$$\mathop{\oplus}_{m\geq 0}\ \mathop{\oplus}_{\check{\lambda}_i\neq 0, \sum_{i=1}^m\check{\lambda}_i=\check{\lambda}} \check{\mathfrak{n}}^{\check{\lambda}_1}\otimes\dots\otimes\check{\mathfrak{n}}^{\check{\lambda}_m} \tag{8}$$

Indeed, it is like this

$$\begin{array}{ccccc}
\check{\mathfrak{n}} & \to & 0 & & \\
\uparrow & & \uparrow & & \\
\wedge^2\check{\mathfrak{n}} & \to & \check{\mathfrak{n}}\otimes\check{\mathfrak{n}} & \to & 0 \\
 & & & & \uparrow \\
 & & & & \check{\mathfrak{n}}\otimes\check{\mathfrak{n}}\otimes\check{\mathfrak{n}}
\end{array}$$

(where we further take $\check{\lambda}$-component and pass to the part $\neq 0$). Now (8) maps naturally to $U(\check{\mathfrak{n}})^{\check{\lambda}}$ surjectively, and this yields a quasi-isomorphism between the total complex of (7) and $U(\check{\mathfrak{n}})^{\check{\lambda}}$.

Note that (7) is the dual of some version of a reduced bar complex from ([5], Section 4.5).

We have similar picture for the complex $\mathfrak{U}(\check{\mathfrak{n}})^{\bullet,\check{\lambda}}$. I think we may first define for $\check{\lambda}\neq 0$ the complex

$$\Upsilon^{\check{\lambda}}\to(\Upsilon\star\Upsilon)^{\check{\lambda}}\to(\Upsilon\star\Upsilon\star\Upsilon)^{\check{\lambda}}\to\dots$$

of perverse sheaves on $X^{\check{\lambda}}$ coming as above from the algebra structure on the costandard complex. Here

$$(\Upsilon^{\star m})^{\check{\lambda}}=\underset{\check{\lambda}_i\in\check{\Lambda}^{pos},\sum\check{\lambda}_i=\check{\lambda}}{\oplus}\Upsilon^{\check{\lambda}_1}\star\dots\star\Upsilon^{\check{\lambda}_m}$$

and further pass to the quotient complex with terms with $\check{\lambda}_i\neq 0$.

Probably, the situation here is as follows. Consider the (double) complex

$$\dots\to E\otimes E\otimes E\stackrel{d^*}{\to}E\otimes E\stackrel{d^*}{\to}E$$

dual to (6). Each map in this complex is a morphism of $\check{\Lambda}^{neg}$-graded DGA (in the category of local systems on X). So, applying our construction from ([8], Section 1.3.53) to this complex, we get a complex in $\mathcal{F}\mathrm{A}(X)_{\check{\Lambda}^{neg}}$. This gives the desired complex $\mathfrak{U}(\check{\mathfrak{n}})^{\bullet,\check{\lambda}}$ on $X^{\check{\lambda}}$.

So, the fact that the total complex of (7) is quasi-isomorphic to $U(\check{\mathfrak{n}})^{\check{\lambda}}$ gives an isomorphism $\mathfrak{U}(\check{\mathfrak{n}})^{\bullet,\check{\lambda}}\widetilde{\to}\mathfrak{U}(\check{\mathfrak{n}})^{\check{\lambda}}$ in $\mathrm{D}(X^{\check{\lambda}})$ simply by functoriality! Similarly for their isomorphism (6.3) on p. 1820.

1.0.21. For Sect. 6.4. For $\check{\lambda}=0$ both $\Upsilon^{\check{\lambda}}$ and $\mathfrak{U}(\check{\mathfrak{n}})^{\bullet,\check{\lambda}}$ are $\bar{\mathbb{Q}}_\ell$ on $\operatorname{Spec}k=X^0$. So, for any $\check{\lambda}\in\check{\Lambda}^{pos}$

$$Kosz^{\bullet,\check{\lambda},*}\;=\;[\Upsilon^{\check{\lambda}}\;\to\underset{\substack{\check{\lambda}_1\neq 0\\ \check{\lambda}_1+\check{\lambda}_2=\check{\lambda}}}{\oplus}\Upsilon^{\check{\lambda}_1}\star\Upsilon^{\check{\lambda}_2}\;\to\underset{\substack{\check{\lambda}_1\neq 0,\check{\lambda}_2\neq 0\\ \check{\lambda}_1+\check{\lambda}_2+\check{\lambda}_3=\check{\lambda}}}{\oplus}\Upsilon^{\check{\lambda}_1}\star\Upsilon^{\check{\lambda}_2}\star\Upsilon^{\check{\lambda}_3}\;\to\;\dots]$$

Here $\check{\lambda}_i\in\check{\Lambda}^{pos}$.

Their first idea is that to find a $\check{\Lambda}^{neg}$-graded resolution of $\bar{\mathbb{Q}}_\ell$ by free $R_{E_{\check{T}}}$-modules, we first find a free graded resolution of the graded $C^*_\bullet(\check{\mathfrak{n}})$-module $\bar{\mathbb{Q}}_\ell$, here $C^*_\bullet(\check{\mathfrak{n}})$ is the costandard complex of $\check{\mathfrak{n}}$ viewed as a $\check{\Lambda}^{neg}$-graded DGA. But the latter question is standard (at least for usual algebras, as opposed to DGA). Namely, there is the reduced bar resolution (or a bar resolution), see [6],[10]. (In [6] there is a mistake corrected in [10]).

They actually use the reduced bar resolution of $\bar{\mathbb{Q}}_\ell$ by graded $C^*_\bullet(\check{\mathfrak{n}})$-modules, I think. Recall that $E=\mathrm{Sym}(\check{\mathfrak{n}}^*[-1])$ is $\check{\Lambda}^{neg}$-graded. Set $E_+=\underset{0\neq\check{\lambda}\in\check{\Lambda}^{pos}}{\oplus}E^{-\check{\lambda}}$. Then the reduced bar resolution of $\bar{\mathbb{Q}}_\ell$ by right E-modules is

$$\dots\to E_+^{\otimes 2}\otimes E\to E_+\otimes E\to E\to\bar{\mathbb{Q}}_\ell \tag{9}$$

where we will further take $-\check{\lambda}$-component. This complex should correspond to their $Kosz^{\bullet,-\check{\lambda}}$.

The differential in (9) is given by

$$d(a_0\otimes\dots\otimes a_i)=a_0a_1\otimes a_2\otimes\dots\otimes a_i+\sum_{j=1}^{i-1}(-1)^j a_0\otimes\dots\otimes(a_ja_{j+1})\otimes\dots\otimes a_i$$

The homotopy $s : E_+^{\otimes i}\otimes E\to E_+^{\otimes i+1}\otimes E$ for (9) is given as follows. The map $s:\bar{\mathbb{Q}}_\ell\to E$ is the natural inclusion, and $s: E_+^{\otimes i+1}\otimes E\to E_+^{\otimes i+2}\otimes E$ is given by

$$s(a_0\otimes\ldots\otimes a_i)=a_0\otimes\ldots\otimes a_{i-1}\otimes(a_i-\epsilon(a_i))\otimes 1,$$

where ϵ is the counit.

I think we should think of the collection $Kosz^{\bullet,-\check\lambda}$ as a right factorization module over the factorization algebra $\Omega(\check{\mathfrak{n}})^{-\check\lambda}$.

1.0.22. For $0\neq\check\lambda\in\check\Lambda^{pos}$ the acyclicity of $Kosz^{\bullet,-\check\lambda}$ implies the fact used in the proof of Cor. 4.5 (and referred to in their Sect. 6.4).

If $0\neq\check\lambda\in\check\Lambda^{pos}$ then $\mathfrak{U}(\check{\mathfrak{n}})^{0,\check\lambda}=0$. For $\check\lambda=0$ we get $\mathfrak{U}(\check{\mathfrak{n}})^{0,\check\lambda}=\bar{\mathbb{Q}}_\ell$ on $\operatorname{Spec}k=X^{\check\lambda}$. This is not precised in the paper.

In Sect. 6.4 they define the complex $K(E_{\check T})^{\bullet,-\check\lambda}$. It is placed in cohomological degrees ≥ 0, and the differential on it is obtained from the diffetential of $Kosz(E_{\check T})^{\bullet,-\check\lambda}$ by applying $\mathrm{R}\Gamma(X^{\check\lambda},?)$ term-wise.

1.0.23. In $Kosz^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu}}=\underset{\check\lambda'\in\check\Lambda^{pos}}{\oplus}\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda',*}\star j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\lambda'}}$ the interaction between various components of the $\check\Lambda^{pos}$-grading comes from the action map given by their formula (4.1) in Th. 4.2. This is a complex of perverse sheaves $[\ldots\to Kosz^{-2}\to Kosz^{-1}\to Kosz^0]$ on $\overline{\mathrm{Bun}}_B^{\check\mu}$, and we have a map of perverse sheaves $Kosz^0\to\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}$ on $\overline{\mathrm{Bun}}_B^{\check\mu}$. Theorem 6.6 claims that it induces a quasi-isomorphism.

This is the usual formula for the Koszul complex, I think.

Let us analyze this complex in the case $G=\mathrm{GL}_2$. Then let $\check\alpha$ be the unique positive coroot of G. For $\check\lambda=d\check\alpha$ we have $\mathfrak{U}(\check{\mathfrak{n}})^{\check\lambda}$ is the constant sheaf on $X^{(d)}=X^{\check\lambda}$ with fibre $\check{\mathfrak{n}}^{\otimes d}$. We also have $\Upsilon(\check{\mathfrak{n}})^{\check\lambda}=(\wedge^{(d)}\check{\mathfrak{n}})[d]$ on $X^{(d)}$. For $d>0$ the complex of $\mathfrak{U}(\check{\mathfrak{n}})^{\bullet,\check\lambda}$ of perverse sheaves is

$$(\wedge^{(d)}\check{\mathfrak{n}})[d]\to\underset{\substack{d_i>0\\ d_1+d_2=d}}{\oplus}(\wedge^{(d_1)}\check{\mathfrak{n}}\star\wedge^{(d_2)}\check{\mathfrak{n}})[d]\to\ldots\to(\check{\mathfrak{n}}\star\ldots\star\check{\mathfrak{n}})[d],$$

it placed in degrees $1,2,\ldots,d$. We view it also as a double complex on $X^{(d)}$, placed in horizontal degrees $1,2,\ldots,n$, so each term in the above is a "vertical complex". Its total complex is quasi-isomorphic to $\mathfrak{U}(\check{\mathfrak{n}})^{\check\lambda}$. The term of the total complex corresponding to the usual degree 0 is then $\check{\mathfrak{n}}\star\ldots\star\check{\mathfrak{n}}$, it maps naturally to $\check{\mathfrak{n}}^{\otimes d}$ and induces the above quasi-isomorphism.

Consider the open substack $\overline{\mathrm{Bun}}_B^{\check\mu,\leq\alpha}$. Over this stack the complex $Kosz^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu}}$ becomes a direct sum of cohomologically shifted perverse sheaves

$$(\check{\mathfrak{n}}^*[2]\star j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\alpha}})\oplus j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}$$

placed in perverse degrees $-1,0$ (in general in perverse degrees ≤ 0). The differential augments the perverse degrees by 1, so it becomes the complex of perverse sheaves $(\check{\mathfrak{n}}^*[1]\star j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\alpha}})\to j_!\,\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}$ placed in horizontal degrees $-1,0$. In this case the differential is precisely the action map

$$\Omega(\check{\mathfrak{n}})^{-\check{\alpha}} \star \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\alpha}}} \to j_! \, \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}$$

We know that by Th. 4.2 the action map induces an isomorphism

$$\Omega(\check{\mathfrak{n}})^{-\check{\alpha}} \star \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\alpha}}} \widetilde{\to} h^0 i^!_{\check{\alpha}} j_! \, \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}$$

So, this complex over $\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \alpha}$ is indeed quasi-isomorphic to $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}$.

1.0.24. The proof of Th. 6.6 inspires the following.

Question 1. Consider the category $\mathcal{C}$ of collections $F^{\check{\lambda}} \in \overline{\mathrm{Bun}}_B^{-\check{\lambda}}$ indexed by $\check{\lambda} \in \check{\Lambda}^{pos}$ together with a factorization structure of the collection $\tilde{F}^{\check{\lambda}}$. Here $\tilde{F}^{\check{\lambda}} = f^*_{\check{\lambda}} F^{\check{\lambda}}[\dim.\mathrm{rel}\, f_{\check{\lambda}}]$, where $f_{\check{\lambda}} : \overline{\mathcal{Z}}^{\check{\lambda}} \to \overline{\mathrm{Bun}}_B^{-\check{\lambda}}$ is the projection. Recall that $\overline{\mathcal{Z}}^{\check{\lambda}} \subset \overline{\mathrm{Bun}}_B^{-\check{\lambda}} \times_{\mathrm{Bun}_G} \mathrm{Bun}_{N^-}$ is the open substack, the Zastava space. Is this is a reasonable category? What is its structure?

For example, the collection $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{-\check{\lambda}}}$, $\check{\lambda} \in \check{\Lambda}^{pos}$ admits a natural factorization structure in the above sense, so becomes an object of $\mathcal{C}$. Another example, the collection $j_! \, \mathrm{IC}_{\mathrm{Bun}_B^{-\check{\lambda}}}$, $\check{\lambda} \in \check{\Lambda}^{pos}$ admits a natural factorization structure in the above sense.

Their Theorem 6.6 gives an example of the collection $Kosz^{\bullet}_{\overline{\mathrm{Bun}}_B^{-\check{\lambda}}}$, $\check{\lambda} \in \check{\Lambda}^{pos}$, which is also naturally an object of $\mathcal{C}$.

1.0.25. **Question 2**. I wonder if the Koszul complex that we got for $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}$ is a particupar case of a general situation. Assume we have a stack Y with a stratification indexed by $\check{\Lambda}^{pos}$ such that if $\check{\nu} \leq \check{\lambda}$ then the stratum $Y^{\check{\lambda}}$ is in the closure of $Y^{\check{\nu}}$. We have the open substack $Y^{\leq \check{\lambda}}$ for any $\check{\lambda}$. Let $i_{\check{\lambda}} : Y^{\check{\lambda}} \hookrightarrow Y$ be the inclusion. Now if we have a complex K on Y, we can consider the graded object

$$\bigoplus_{\check{\lambda} \in \check{\Lambda}^{pos}} i_{\check{\lambda}!} i^*_{\check{\lambda}} K$$

How to put a differential on this complex such that the corresponding total complex would be quasi-isomorphic to K? In a special case considered in my Section 1.0.3 we answered this question. Of course, ([7], 1.2.2) is useful here.

1.0.26. The following lemma is implicit in the proof of Prop. 10.3, p. 1842.

Lemma 1.0.27. *Let Y be a stack with a stratification indexed by $\check{\Lambda}^{pos}$. Assume for $\check{\mu} \leq \check{\lambda}$, the stratum $Y^{\check{\lambda}}$ is in the closure of the stratum $Y^{\check{\mu}}$. For $\check{\lambda} \in \check{\Lambda}^{pos}$ we get the open substack $Y^{\leq \check{\lambda}} = \cup_{\check{\mu} \leq \check{\lambda}} Y^{\check{\mu}}$. Let $i_{\check{\lambda}} : Y^{\check{\lambda}} \to Y$ be the inclusion. Let K be a perverse sheaf on Y, write $K^{\check{\lambda}} = h^0 i^!_{\check{\lambda}} K$. Let $\bar{K}^{\check{\lambda}} = (i_{\check{\lambda}})_{!*} K^{\check{\lambda}}$. Assume K has no perverse quotient sheaves supported on the complement of Y^0. Assume that for any open substack ${}^0Y \subset Y$ of the form ${}^1Y = \underset{i \in I}{\cup} Y^{\check{\lambda}_i}$, where I is a finite set, we have*

$$[K] = \sum_{i \in I} [\bar{K}^{\check{\lambda}_i}] \tag{10}$$

in the Grothendieck group of ^1Y. *Then* K *admits a filtration with* $grK \widetilde{\to} \underset{\check{\lambda}\in\check{\Lambda}^{pos}}{\oplus} \bar{K}^{\check{\lambda}}$.

Proof. Let $^0Y \overset{j}{\hookrightarrow} {}^1Y \subset Y$ be open substacks consisting of some strata such that $^1Y - {}^0Y = Y^{\check{\lambda}}$ for some $\check{\lambda} \in \check{\Lambda}^{pos}$. Let $i : Y^{\check{\lambda}} \to {}^1Y$ be the inclusion. Assume by induction the above filtration is constructed over ^0Y. We have an exact sequence of perverse sheaves $0 \to i_*h^0(i^!K) \to K \to j_{!*}j^*K \to 0$ on ^1Y. The functor $j_{!*}$ is not exact in general. So, on ^1Y we get a filtration, where some additional summands in grK may appear. The formula (10) shows that no additional summands appear. We constructed the desired filtration on $K\mid_{^1Y}$. □

The above lemma is applied in Proposition 10.3, p. 1842 for the perverse sheaf $K = j_! \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}$. The equality in the Grothendieck group needed for the lemma is proved in their Cor. 4.5. So, $j_! \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}$ admits a filtration by perverse subsheaves with the associated graded pieces

$$(\bar{i}_{\check{\mu}'-\check{\mu}})_!(\Omega(\check{\mathfrak{n}})^{\check{\mu}-\check{\mu}'} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}'}})$$

This saves the proof of Th. 10.2.

1.0.28. For 10.9. There is a misprint in the def of $\mathrm{Eis}_*^{\check{\mu}}(E_{\check{T}})$.The correct definition is $\mathfrak{p}_*(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}} \otimes \mathfrak{q}^{\check{\mu}*}\mathcal{S}(E_{\check{T}}))$. They do not calculate $CT^{\check{\mu}}(\overline{\mathrm{Eis}}^{\check{\mu}}(E_{\check{T}})$. They only derive Prop. 10.8 from the calculation of $CT^{\check{\mu}}(\mathrm{Eis}_*^{\check{\mu}}(E_{\check{T}}))$.

So, in Sect. 10.9 the definition of $\mathcal{Z}_w^{\check{\mu},\check{\mu}'}$ is as follows. This is the stack classifying $\mathcal{F}_B \in \mathrm{Bun}_B^{\check{\mu}'}$ and a section $\sigma : X \to \mathcal{F}_B \times^B \overline{Fl}_w$ such that over the generic point of X, σ hits $\mathcal{F}_B \times^B Fl_w$. Moreover, the second B-structure on $\mathcal{F}_B \times_B G$ is required to be of degree $\check{\mu}$.

Write $\mathcal{V}^\lambda$ for the Weyl module of G, $\mathcal{V}^{\lambda,\geq w}$ be the sum of all subspaces of T-weights $\geq w(\lambda)$. A point of $\mathcal{Z}_w^{\check{\mu},\check{\mu}'}$ is rewritten as $\mathcal{F}_B \in \mathrm{Bun}_B^{\check{\mu}'}$ and a collection of line subbundles $\mathcal{L}^\lambda \subset \mathcal{V}_{\mathcal{F}_B}^{\lambda,\geq w}$ for $\lambda \in \Lambda^+$ with $\deg \mathcal{L}^\lambda = \langle \lambda, \check{\mu}\rangle$ satisfying the Plucker relations such that for any $\lambda \in \Lambda^+$ the composition

$$\mathcal{L}^\lambda \to \mathcal{V}_{\mathcal{F}_B}^{\lambda,\geq w} \to (\mathcal{V}^{\lambda,\geq w}/\mathcal{V}^{\lambda,>w})_{\mathcal{F}_B}$$

is injective. So, for $\mathcal{F}_T = \mathcal{F}_B \times_B T$ we get $D \in X^{w(\check{\mu}')-\check{\mu}}$ and a B-torsor $\mathcal{F}'_B$ with $\mathcal{F}'_B \times_B G \widetilde{\to} \mathcal{F}_B \times_B G$ and $\mathcal{F}'_B \times_B T \widetilde{\to} \mathcal{F}_T(-D)$.

1.0.29. For the proof of their Prop. 10.10. Recall that $B(B) \times_{B(G)} B(B) \widetilde{\to} B\backslash G/B$ canonically, the orbit corresponding to $w \in W$ is BwB. Let $N_0 \subset N$ be the subgroup whose Lie algebra is the sum of $\mathfrak{n}_\alpha$ such that $\alpha \in \Delta^+$, $w\alpha \in \Delta^+$. So, N_0 is the stabilizor of $wB/B \in Fl$ in N. Let $N' \subset N$ be the subgroup whose Lie algebra is the sum of $\mathfrak{n}_\alpha$ such that $\alpha \in \Delta^+$, $w\alpha \notin \Delta^+$. Then N' acts simply transitively on $BwB/B = Fl_w$. The stack classifying $\mathcal{F}_B \in \mathrm{Bun}_B^{\check{\mu}'}$ and a global section $X \to \mathcal{F}_B \times_B Fl_w$ becomes the component $\mathrm{Bun}_{T\rtimes N_0}^{\check{\mu}'}$. Indeed, $B\backslash Fl_w$ is the classifying stack $B(T \rtimes N_0)$. This gives the proof of Prop. 10.10 in the case $w(\check{\mu}') = \check{\mu}$.

Consider the special case: for any $\alpha\in\Delta^+$ with $w\alpha\in\Delta^+$ assume $\langle\alpha,\check{\mu}'\rangle>2g-2$. This assumption garantees that $f:\mathrm{Bun}_{B'}^{\check{\mu}'}\to\mathrm{Bun}_B^{\check{\mu}'}$ is smooth (a generalized affine fibration). In this case to prove Pp. 10.10, we my replace $\mathcal{Z}_w^{\check{\mu},\check{\mu}'}$ by $\mathcal{Z}_w^{\check{\mu},\check{\mu}'}\times_{\mathrm{Bun}_B^{\check{\mu}'}}\mathrm{Bun}_{B'}^{\check{\mu}'}$, and it suffices to describe the direct image with compact supports for the composition

$$\mathcal{Z}_w^{\check{\mu},\check{\mu}'}\times_{\mathrm{Bun}_B^{\check{\mu}'}}\mathrm{Bun}_{B'}^{\check{\mu}'}\to\mathcal{Z}_w^{\check{\mu},\check{\mu}'}\to\mathrm{Bun}_T^{\check{\mu}}\times X^{w(\check{\mu}')-\check{\mu}}$$

The proof in general is reduced to this case, roughly, by some twist, I think.

1.0.30. Consider the situation as in the proof of Proposition 10.10, so we have N' and N_0 as above.

There is a great factorization principle in ([2], Section 2.16). It applies to the stack $N'\backslash\overline{Fl}_w$. That is, the complement of Fl_w in $\overline{Fl}_w$ is indeed given by a finite union of Cartier divisors. Recall that $[G,G]$ is assumed simply-connected. So, we may choose fundamental weights ω_i, i runs through the set of vertices of the Dynkin diagram $\mathcal{I}$. Then a point of $\overline{Fl}_w$ is completely defined by the lines $\mathcal{L}^{\omega_i}\subset V^{\omega_i,\geq w}$ for $i\in\mathcal{I}$. So, we get Cartier divisors $\mathcal{T}_i$ in $\overline{Fl}_w$ given by the property that the composition $\mathcal{L}^{\omega_i}\to V^{\omega_i,\geq w}\to V^{\omega_i,\geq w}/V^{\omega_i,>w}$ vanishes.

So, we are led to study the following "w-version of Zastava" for $w\in W$. Let $\check{\mu}\in\check{\Lambda}^{pos}$. Consider the scheme $\mathcal{W}_w^{\check{\mu}}$ classifying $(\mathcal{F}_{N'},D,\sigma)$, where $D\in X^{\check{\mu}}$, $\mathcal{F}_{N'}$ is a N'-torsor on X, and $\sigma:X\to\mathcal{F}_{N'}\times^{N'}\overline{Fl}_w$ is a global section such that over $X-D$ it hits to Fl_w, and for any $\lambda\in\Lambda^+$ the divisor of zeros of the composition

$$\mathcal{L}^\lambda\to\mathcal{V}^{\lambda,\geq w}_{\mathcal{F}_{N'}}\to(\mathcal{V}^{\lambda,\geq w}/\mathcal{V}^{\lambda,>w})_{\mathcal{F}_{N'}}=\mathcal{O}\tag{11}$$

is $\langle D,\lambda\rangle$. So, on $\mathcal{F}_{N'}\times_{N'}G$ we get a B-structure with the corresponding T-torsor being $\mathcal{O}(-D)$. According to the above factorization principle, $\mathcal{W}^{\check{\mu}}$ factorizes over $X^{\check{\mu}}$ as usual Zastava spaces. Let $\pi^{\check{\mu}}:\mathcal{W}^{\check{\mu}}\to X^{\check{\mu}}$ be the projection.

For w the longuest element of W the scheme $\mathcal{W}_w^{\check{\mu}}$ is the same as the open part Z_{max} of the usual Zastava space.

Question. What can we say about $\pi^{\check{\mu}}\bar{\mathbb{Q}}_\ell$? Can we describe $\mathrm{IC}_{\mathcal{W}^{\check{\mu}}}$ in a way analogous to the usual Zastava space from [1]?

We may realize $\mathcal{W}_w^{\check{\mu}}$ in local terms as follows. Now $\mathcal{W}_w^{\check{\mu}}$ is the scheme classifying $D\in X^{\check{\mu}}$, a N'-torsor over the formal neighbourhood $\bar{D}$ of D, its trivlization over the punched formal neighbourhood $\bar{D}^0$ of D such that for any $\lambda\in\Lambda^+$ the map (11) yields $\mathcal{L}^\lambda\widetilde{\to}\mathcal{O}(-\langle D,\lambda\rangle)$, and moreover $\mathcal{L}^\lambda\to\mathcal{V}^{\lambda,\geq w}_{\mathcal{F}_{N'}}$ is a subbundle over $\bar{D}$.

Let $T_{X^{\check{\mu}}}$ denote the group scheme over $X^{\check{\mu}}$ classifying $D\in X^{\check{\mu}}$ and a section of T over $\bar{D}$. The group scheme $T_{X^{\check{\mu}}}$ acts on $\mathcal{W}_w^{\check{\mu}}$ over $X^{\check{\mu}}$ via its action on the trivial T-torsor $T\mid_{\bar{D}}$. Let $\mathrm{Bun}_{T,X^{\check{\mu}}}$ be the stack classifying $\mathcal{F}_T\in\mathrm{Bun}_T, D\in X^{\check{\mu}}$ and a trivialization of $\mathcal{F}_T$ over $\bar{D}$.

Let $B'=T\ltimes N'$. We may consider a version $\mathcal{W}^{\check{\mu}}_{w,\mathrm{Bun}_T}$ of $\mathcal{W}_w^{\check{\mu}}$ with $\mathcal{F}_T^0$ replaced by a 'background' T-torsor $\mathcal{F}_T\in\mathrm{Bun}_T$. It classifies $\mathcal{F}_{B'}\in\mathrm{Bun}_{B'}$, $D\in X^{\check{\mu}}$, $\sigma:X\to\mathcal{F}_{B'}\times^{B'}\overline{Fl}_w$ such that over $X-D$ it hits Fl_w, and the maps

$$\mathcal{L}^\lambda\to\mathcal{V}^{\lambda,\geq w}_{\mathcal{F}_{B'}}\to(\mathcal{V}^{\lambda,\geq w}/\mathcal{V}^{\lambda,>w})_{\mathcal{F}_{B'}}=\mathcal{L}^{w(\lambda)}_{\mathcal{F}_T}$$

identify $\mathcal{L}^\lambda$ with $\mathcal{L}^{w(\lambda)}_{\mathcal{F}_T}(-\langle D, \lambda\rangle)$.

I think then $\mathcal{W}^{\check{\mu}}_{w,\mathrm{Bun}_T}$ is obtained from $\mathcal{W}^{\check{\mu}}_w$ by a twist with the $\mathrm{Bun}_T \times T_{X^{\check{\mu}}}$-torsor $\mathrm{Bun}_{T,X^{\check{\mu}}} \to \mathrm{Bun}_T \times X^{\check{\mu}}$ and the above action of $T_{X^{\check{\mu}}}$ on $\mathcal{W}^{\check{\mu}}_w$.

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