

**COMMENTS: SHEAVES OF CATEGORIES AND NOTION OF
1-AFFINENESS**

0.0.1. For 1.1.5. The fact that $ShvCat(Y)$ admits small colimits is proved as in ([1], around Lm 2.2.67).

0.0.2. For 1.2.2. This follows from ([1], 2.2.41).

0.0.3. For 1.2.3. Let $\mathcal{Z} \rightarrow Y$ be a morphism in $PreStk$. He means $\Gamma(\mathcal{Z}, \cdot) : ShvCat(Y) \rightarrow DGCat_{cont}$ is right-lax symmetric monoidal. Indeed, let $F, F' \in ShvCat(Y)$

$$\Gamma(\mathcal{Z}, F \otimes F') \xrightarrow{\sim} \lim_{S \rightarrow \mathcal{Z}} \Gamma(S, F \otimes F') \xrightarrow{\sim} \lim_{S \rightarrow \mathcal{Z}} \Gamma(S, F) \otimes_{QCoh(S)} \Gamma(S, F')$$

The limit over $((Sch^{aff})_{/Z})^{op}$. For each $S \rightarrow Z$ we have the projection

$$\Gamma(\mathcal{Z}, F) \otimes \Gamma(\mathcal{Z}, F') \rightarrow \Gamma(S, F) \otimes \Gamma(S, F') \rightarrow \Gamma(S, F) \otimes_{QCoh(S)} \Gamma(S, F')$$

They are compatible with morphisms in $(Sch^{aff})_{/Z}$, so yield a morphism

$$\Gamma(\mathcal{Z}, F) \otimes \Gamma(\mathcal{Z}, F') \rightarrow \Gamma(\mathcal{Z}, F \otimes F')$$

For $F' = QCoh_{/Y}$ we get a map $\Gamma(\mathcal{Z}, F) \otimes \Gamma(\mathcal{Z}, QCoh_{/Y}) \rightarrow \Gamma(\mathcal{Z}, F \otimes (QCoh_{/Y}))$. Since $F \otimes (QCoh_{/Y}) \xrightarrow{\sim} F$, we get the $QCoh(\mathcal{Z})$ -module structure on $\Gamma(\mathcal{Z}, F)$.

The above map for F, F' factors to give a map

$$\Gamma(\mathcal{Z}, F) \otimes_{QCoh(\mathcal{Z})} \Gamma(\mathcal{Z}, F') \rightarrow \Gamma(\mathcal{Z}, F \otimes F')$$

of $QCoh(\mathcal{Z})$ -modules.

0.0.4. For 1.3.1. Let us justify that $Loc_{\mathcal{Y}}$ is a left adjoint to $\Gamma_{\mathcal{Y}}^{enh}$ indeed. First, given $\mathcal{C} \in ShvCat(\mathcal{Y})$, we may view for any $S \in (Sch^{aff})_{/Y}$, $\Gamma(S, \mathcal{C})$ as a $QCoh(\mathcal{Y})$ -module. Moreover, for $f : S \rightarrow S'$ the restriction functor $\mathcal{C}(S') \rightarrow \mathcal{C}(S)$ is a morphism of $QCoh(\mathcal{Y})$ -modules. So, in the definition of global sections, we may directly define $\Gamma(\mathcal{Y}, \mathcal{C}) = \lim_{S \rightarrow \mathcal{Y}} \mathcal{C}(S)$ over the category $((Sch^{aff})_{/Y})^{op}$, the limit being taken in $QCoh(\mathcal{Y})$ -modules. The projection $QCoh(\mathcal{Y}) - mod \rightarrow DGCat_{cont}$ preserves limits.

Recall that

$$ShvCat(\mathcal{Y}) \xrightarrow{\sim} \lim_{S \rightarrow \mathcal{Y}} QCoh(S) - mod$$

in $1 - Cat$, the limit over $((Sch^{aff})_{/Y})^{op}$. Now given $M \in QCoh(\mathcal{Y}) - mod$, using the description of the Map spaces in the limit of categories from ([1], Cor. 2.5.3), we get

$$\begin{aligned} \text{Map}_{ShvCat(\mathcal{Y})}(Loc_{\mathcal{Y}}(M), \mathcal{C}) &\xrightarrow{\sim} \lim_{S \rightarrow \mathcal{Y}} \text{Map}_{QCoh(S) - mod}(M \otimes_{QCoh(\mathcal{Y})} QCoh(S), \mathcal{C}(S)) \xrightarrow{\sim} \\ &\lim_{S \rightarrow \mathcal{Y}} \text{Map}_{QCoh(\mathcal{Y}) - mod}(M, \mathcal{C}(S)) \xrightarrow{\sim} \text{Map}_{QCoh(\mathcal{Y}) - mod}(M, \mathcal{C}(\mathcal{Y})) \end{aligned}$$

We use here the fact that the projection $QCoh(\mathcal{Y}) - mod \rightarrow DGCat_{cont}$ preserves limits.

0.0.5. For 1.3.5. Let $A \in \mathcal{CAlg}(\mathrm{DGCat}_{cont})$, by $A - mod$ he denotes the category of A -modules in DGCat_{cont} . If $C \in A - mod$ is dualizable as an object of the symmetric monoidal ∞ -category $A - mod$ then the functor $A - mod \rightarrow A - mod$, $M \mapsto M \otimes_A C$ preserves small limits. This follows from ([1], Lemma 3.1.2), which is a strengthened version of ([3], ch. 1, Lm 4.1.6 a)).

0.0.6. For Lm 1.4.6. My understanding is that the diagram $\alpha \mapsto O_\alpha$ is a functor $I \rightarrow 1 - \mathcal{C}at^{SymMon}$, the notation $1 - \mathcal{C}at^{SymMon}$ is from ([3], ch. 1, 3.3.1). Another version is $I \rightarrow \mathrm{DGCat}_{cont}^{SymMon}$.

0.0.7. For 1.4.8(b). If $\mathrm{QCoh}(Y)$ is rigid then for any $S \rightarrow Y$ with $S \in \mathrm{Sch}^{aff}$, $\mathrm{QCoh}(S)$ is dualizable as an object of $\mathrm{QCoh}(Y) - mod$ by ([3], ch. I.1, 9.4.4). So, the functor $\mathrm{QCoh}(Y) - mod \rightarrow \mathrm{QCoh}(S) - mod$, $C \mapsto \mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(Y)} C$ commutes with limits.

0.0.8. For 1.5.5. (a) follows from an analog of ([3], ch. I.2, 2.3.8) for flat topology instead of etale one. Indeed, the functor $(\mathrm{PreStk}/Y)^{op} \rightarrow \mathrm{DGCat}_{cont}$, $Z \mapsto \Gamma(Z, C)$ preserves limits.

0.0.9. For 2.2.1. Given a lax prestack Y , he defined $Shv^!(Y) \in 1 - \mathcal{C}at$. As in [2], one shows that the functor $(\mathrm{PreStk}^{lax})^{op} \rightarrow 1 - \mathcal{C}at_{cont}^{St, cocmpl}$, $Y \mapsto Shv^!(Y)$ transforms lax colimits in PreStk^{lax} (relative over Sch^{op}) into oplax limits.

0.0.10. For 3.1.3. Let us show that $cores_f : ShvCat(\mathcal{Y}_2) \rightarrow ShvCat(\mathcal{Y}_1)$ is left adjoint to $coind_f$. Given $\mathcal{C}_i \in ShvCat(\mathcal{Y}_i)$, we get

$$\begin{aligned} \mathrm{Map}_{ShvCat(\mathcal{Y}_2)}(\mathcal{C}_2, coind_f(\mathcal{C}_1)) &\xrightarrow{\sim} \lim_{S \rightarrow \mathcal{Y}_2} \mathrm{Map}_{\mathrm{QCoh}(S) - mod}(\mathcal{C}_2(S), \mathcal{C}_1(S \times_{\mathcal{Y}_2} \mathcal{Y}_1)) \xrightarrow{\sim} \\ \lim_{S \rightarrow \mathcal{Y}_2, T \rightarrow S \times_{\mathcal{Y}_2} \mathcal{Y}_1} \mathrm{Map}_{\mathrm{QCoh}(S) - mod}(\mathcal{C}_2(S), \mathcal{C}_1(T)) &\xrightarrow{\sim} \lim_{T \rightarrow \mathcal{Y}_1} \mathrm{Map}_{\mathrm{QCoh}(T) - mod}(\mathcal{C}_2(T), \mathcal{C}_1(T)) \\ &\xrightarrow{\sim} \mathrm{Map}_{ShvCat(\mathcal{Y}_1)}(cores_f(\mathcal{C}_2), \mathcal{C}_1) \end{aligned}$$

The first limit is over $((\mathrm{Sch}^{aff})_{/\mathcal{Y}_2})^{op}$. The second limit is over pairs of maps $S \rightarrow \mathcal{Y}_2, T \rightarrow S \times_{\mathcal{Y}_2} \mathcal{Y}_1$ with $S, T \in \mathrm{Sch}^{aff}$. Inside we have the full subcategory given by the property that the composition $T \rightarrow S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \rightarrow S$ is an isomorphism. The corresponding inclusion is cofinal, so the second limit rewrites as the third one.

0.0.11. If $S \rightarrow T \leftarrow \mathcal{Y}$ is a diagram in PreStk with $S, T \in \mathrm{Sch}^{aff}$ then

$$\mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(T)} \mathrm{QCoh}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{QCoh}(S \times_T \mathcal{Y})$$

Indeed, the functor Γ_T^{enh} is an equivalence, apply their Prop. 3.1.9 now. More generally, one has

Lemma 0.0.12. *Let $Y_1 \xrightarrow{f} Y_2 \xleftarrow{h} Z$ be a diagram in PreStk with Y_i 1-affine. Then one has canonically*

$$\mathrm{QCoh}(Y_1) \otimes_{\mathrm{QCoh}(Y_2)} \mathrm{QCoh}(Z) \xrightarrow{\sim} \mathrm{QCoh}(Y_1 \times_{Y_2} Z)$$

Proof. Consider the sheaf QCoh/Z on Z , it sends $S \rightarrow Z$ to $\mathrm{QCoh}(S)$, let $\mathcal{C} = \mathrm{coind}_h(\mathrm{QCoh}/Z)$. Apply ([4], Lemma 3.2.4) to \mathcal{C} . We get $\Gamma^{\mathrm{enh}}(Y_2, \mathcal{C}) \xrightarrow{\sim} \mathrm{QCoh}(Z)$ and $\Gamma^{\mathrm{enh}}(Y_1, \mathrm{cores}_f(\mathcal{C})) \xrightarrow{\sim} \lim_{S \rightarrow Y_1} \mathcal{C}(S) \xrightarrow{\sim} \lim_{S \rightarrow Y_1} \mathrm{QCoh}(S \times_{Y_2} Z) \xrightarrow{\sim} \mathrm{QCoh}(Y_1 \times_{Y_2} Z)$. Here the limit is over $(\mathrm{Sch}/Y_1)^{\mathrm{aff}, \mathrm{op}}$. We used the fact that the colimits in PreStk are universal. \square

0.0.13. For 3.2.8. Let us check that given $Y, Z \in \mathrm{PreStk}$ 1-affine the product $Y \times Z$ is 1-affine. We want to apply ([4], Pp. 3.2.6) to the projection $f : Y \times Z \rightarrow Z$.

By my Lemma 0.0.12, for any $S \rightarrow Z$ with $S \in \mathrm{Sch}^{\mathrm{aff}}$, we get $\mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(Z)} \mathrm{QCoh}(Y \times Z) \xrightarrow{\sim} \mathrm{QCoh}(Y \times S)$. Now ([4], Pp. 3.2.6) reduces the claim to the particular case when $Z = T \in \mathrm{Sch}^{\mathrm{aff}}$.

So, we assume $Z = T$. Let us check that Loc is fully faithful for $T \times Y$. We check that the natural map $C \rightarrow \Gamma_{T \times Y}^{\mathrm{enh}}(\mathrm{Loc}_{T \times Y} C)$ is an isomorphism for $C \in \mathrm{QCoh}(T \times Y)$ – *module*. Use ([1], 9.2.10), which is the base change property for products of affine schemes. We get $\Gamma(T \times Y, \mathrm{Loc}(C)) \xrightarrow{\sim} \lim_{S' \rightarrow T \times Y} C \otimes_{\mathrm{QCoh}(T \times Y)} \mathrm{QCoh}(S')$, the limit over $(\mathrm{Sch}/T \times Y)^{\mathrm{aff}, \mathrm{op}}$. We may rewrite it as

$$\lim_{S' \rightarrow Y} C \otimes_{\mathrm{QCoh}(T \times Y)} \mathrm{QCoh}(T \times S')$$

over $(\mathrm{Sch}/Y)^{\mathrm{aff}, \mathrm{op}}$. Here

$$C \otimes_{\mathrm{QCoh}(T \times Y)} \mathrm{QCoh}(T \times S') \xrightarrow{\sim} C \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(S')$$

as $\mathrm{QCoh}(S')$ -modules, because $\mathrm{QCoh}(T \times Y) \xrightarrow{\sim} \mathrm{QCoh}(T) \otimes \mathrm{QCoh}(Y)$. So, our limit identifies with $\Gamma_Y(\mathrm{Loc}_Y C) \xrightarrow{\sim} C$. So, $\mathrm{Loc}_{T \times Y}$ is fully faithful.

To show that $\Gamma_{T \times Y}^{\mathrm{enh}}$ is fully faithful, use the projection $q : T \times Y \rightarrow Y$. The assumptions of his Prop. 3.2.6 are satisfied for q . Namely, given $S' \in \mathrm{Sch}^{\mathrm{aff}}$ with $S' \rightarrow Y$, one has $\mathrm{QCoh}(S') \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(T \times Y) \xrightarrow{\sim} \mathrm{QCoh}(S') \otimes_{\mathrm{QCoh}(T)} \mathrm{QCoh}(Y) \xrightarrow{\sim} \mathrm{QCoh}(S' \times T)$. Besides, the base change of q by $S' \rightarrow Y$ with S' an affine scheme gives a 1-affine prestack $S' \times T$. Since Γ_Y^{enh} is fully faithful, by Pp. 3.2.6(b), $\Gamma_{T \times Y}^{\mathrm{enh}}$ is also fully faithful.

0.0.14. For 3.3. Evident remark: let $f : Y_1 \rightarrow Y_2$ be a morphism in PreStk , $\mathcal{C}_1 \in \mathrm{ShvCat}(Y_1)$, let $\bar{f} : Z_1 \rightarrow Z_2$ be obtained from f by the base change $Z_2 \rightarrow Y_2$. Then

$$(\mathrm{coind}_f(\mathcal{C}_1))|_{Z_2} \xrightarrow{\sim} \mathrm{coind}_{\bar{f}}(\mathcal{C}_1|_{Z_1})$$

0.0.15. For 3.2.6, the proof of (b). Assume coind_f is conservative. Then given $C_1 \in \mathrm{ShvCat}(Y_1)$, we have to show that $\mathrm{Loc}_f(\Gamma_{Y_1}(C_1)) \rightarrow C_1$ is an isomorphism. We check that $\mathrm{coind}_f \mathrm{Loc}_f(\Gamma_{Y_1}(C_1)) \rightarrow \mathrm{coind}_f C_1$ is an isomorphism in $\mathrm{ShvCat}(Y_2)$. By (a)i), $\mathrm{coind}_f \mathrm{Loc}_f(\Gamma_{Y_1}(C_1)) \mathrm{Loc}_{Y_2}(\mathrm{res}_f \Gamma_{Y_1}(C_1))$. Since $\Gamma_{Y_2}(\mathrm{coind}_f C_1) \xrightarrow{\sim} \mathrm{res}_f \Gamma_{Y_1}(C_1)$, our claim follows from the fact that $\mathrm{Loc}_{Y_2} \Gamma_{Y_2}(C) \rightarrow C$ is an isomorphism for $C = \mathrm{coind}_f(C_1)$.

In the proof he means that if $T \rightarrow Y_1$ is a map with $T \in \mathrm{Sch}^{\mathrm{aff}}$ then $\Gamma(T \times_{Y_2} Y_1, \mathcal{C}'_i) \xrightarrow{\sim} \Gamma(T, \mathrm{coind}_f(\mathcal{C}'_i))$, so does not depend on i , the corresponding map is an isomorphism. So, indeed, (3.11) is an isomorphism, and we may evaluate at $T \rightarrow T \times_{Y_2} Y_1$. We are done.

0.0.16. If $S \in \text{Sch}^{aff}$ then S_{dR} is 1-affine.

0.1. In Appendix C, Def. C.1.2 Dennis means by the last face map $\partial_i : [i] \rightarrow [i+1]$ the injective map in $\mathbf{\Delta}$ whose image does not contain $i+1$.

0.1.1. For Appendix D.1.2. If \mathbf{O} is a rigid monoidal DG -category then the map $\phi : \mathbf{O} \xrightarrow{\sim} \mathbf{O}^\vee$ is defined as in ([1], 6.1.5 "For 9.2.1"), that is, $\phi(x)$ is the functor $\mathbf{O} \rightarrow \text{Vect}$, $b \mapsto u^R(b \otimes x)$. Here $u^R : \mathbf{O} \rightarrow \text{Vect}$ is the right adjoint to the unit map $u : \text{Vect} \rightarrow \mathbf{O}$. The map ϕ is an isomorphism of left \mathbf{O} -modules.

If $a \in \mathbf{O}^c$, \mathbf{O} is rigid and compactly generated then the left dual $a^{\vee,L}$ is obtained as follows. For $b, c \in \mathbf{O}$ write $\text{Maps}_{k, \mathbf{O}}(b, c) \in \text{Vect}$ for the relative inner hom (as in [3], 10.3.7). Then $a^{\vee,L} \xrightarrow{\sim} (\text{id} \otimes f)m^R(1)$, where $m^R : \mathbf{O} \rightarrow \mathbf{O} \otimes \mathbf{O}$ is the right adjoint to m , and $f : \mathbf{O} \rightarrow \text{Vect}$, $f(b) \xrightarrow{\sim} \text{Maps}_{k, \mathbf{O}}(a, b)$.

0.1.2. For D.3. By a comonoidal DG -category he means a monoid in $(\text{DGCat}_{cont})^{op}$.

1. APPENDIX

1.0.1. Let $S \in \text{Sch}^{aff}$, $p : S \rightarrow *$. Then the functor $p^* : \text{ShvCat}(*) = \text{DGCat}_{cont} \rightarrow \text{QCoh}(S) - \text{mod}$ has a left adjoint $p_!$ sending $C \in \text{QCoh}(S) - \text{mod}$ to $C \in \text{DGCat}_{cont}$, where we forget the $\text{QCoh}(S)$ -module structure. Indeed, $\text{QCoh}(S)$ is rigid, so we apply ([1], 3.2). Recall that for $A = \text{QCoh}(S)$ the dualization map $\phi_A : A \rightarrow A^\vee$ is an isomorphism of left A -modules by ([3], ch. I.1, 9.2.3).

Let $f : S' \rightarrow S$ be a map in Sch^{aff} giving $A = \text{QCoh}(S) \rightarrow \text{QCoh}(S') = B$. Recall that B is dualizable and self-dual in $A - \text{mod}(\text{DGCat}_{cont})$. So, the restriction of scalars functor $B - \text{mod} \rightarrow A - \text{mod}$ is left adjoint to $f^* : A - \text{mod} \rightarrow B - \text{mod}$, $D \mapsto B \otimes_A D$ by ([1], 3.2). Let us write $f_! : B - \text{mod} \rightarrow A - \text{mod}$ for the functor of restriction of scalars. For $C \in A - \text{mod}$ we get a natural map $f_! f^* C \rightarrow C$ in $A - \text{mod}$.

Let $Y \in \text{PreStk}$, $p : Y \rightarrow *$. Does the functor $p^* : \text{ShvCat}(*) \rightarrow \text{ShvCat}(Y)$ has a left adjoint? For $C \in \text{ShvCat}(Y)$ consider the functor $\text{Sch}_{/Y}^{aff} \rightarrow \text{DGCat}_{cont}$ sending S to $\Gamma(S, Y)$, and a morphism $f : S' \rightarrow S$ in $\text{Sch}_{/Y}^{aff}$ to the above map $\Gamma(S', C) \rightarrow \Gamma(S, C)$. Passing to the colimit, we get a functor $p_! : \text{ShvCat}(Y) \rightarrow \text{DGCat}_{cont}$ sending C to $\text{colim}_{S \in \text{Sch}_{/Y}^{aff}} \Gamma(S, C)$.

Lemma 1.0.2. *The functor $p_! : \text{ShvCat}(Y) \rightarrow \text{DGCat}_{cont}$ is left adjoint to $p^* : \text{ShvCat}(*) \rightarrow \text{ShvCat}(Y)$.*

Proof. For $C \in \text{ShvCat}(Y)$, $D \in \text{DGCat}_{cont}$ we have

$$\begin{aligned} \text{Map}_{\text{ShvCat}(Y)}(C, p^* D) &\xrightarrow{\sim} \lim_{S \in (\text{Sch}_{/Y}^{aff})^{op}} \text{Map}_{\text{ShvCat}(S)}(C|_S, D|_S) \xrightarrow{\sim} \\ &\lim_{S \in (\text{Sch}_{/Y}^{aff})^{op}} \text{Map}_{\text{DGCat}_{cont}}(\Gamma(S, C), D) \xrightarrow{\sim} \text{Map}_{\text{DGCat}_{cont}}(p_! C, D) \end{aligned}$$

□

1.0.3. Let now $f : Y \rightarrow Z$ be a map in PreStk and $C \in \text{ShvCat}(Y)$. If $Z \in \text{Sch}^{aff}$ then for any $S \in \text{Sch}_{/Y}^{aff}$, $\Gamma(S, C) \in \text{QCoh}(Z) - \text{mod}$. If $f : S' \rightarrow S$ is a map in $\text{Sch}_{/Y}^{aff}$ then the above map $\Gamma(S', C) \rightarrow \Gamma(S, C)$ is a morphism in $\text{QCoh}(Z) - \text{mod}$. Define $f_! : \text{ShvCat}(Y) \rightarrow \text{ShvCat}(Z)$ by

$$C \mapsto \left(\text{colim}_{S \in \text{Sch}_{/Y}^{aff}} \Gamma(S, C) \right) \in \text{QCoh}(Z) - \text{mod}$$

As in Lemma 1.0.2 we see that $f_!$ is left adjoint to f^* .

Let now $g : Z' \rightarrow Z$ be a morphism in Sch^{aff} and $f' : Y' \rightarrow Z'$ be obtained from f by the base change $g : Z' \rightarrow Z$. We want to check that $f'_!(C|_{Y'}) \xrightarrow{\sim} g^* f_! C$ naturally. We get

$$(1) \quad \Gamma(Z', g^* f_! C) \xrightarrow{\sim} \text{colim}_{S \in \text{Sch}_{/Y}^{aff}} \Gamma(S, C) \otimes_{\text{QCoh}(Z)} \text{QCoh}(Z') \xrightarrow{\sim} \text{colim}_{S \in \text{Sch}_{/Y}^{aff}} \Gamma(S \times_Z Z', C)$$

Lemma 1.0.4. *Let $f : Y \rightarrow Z$ be a morphism in PreStk with $Z \in \text{Sch}^{aff}$, let $Z' \rightarrow Z$ be a map in Sch^{aff} , and $Y' = Y \times_Z Z'$. Then the functor $\text{Sch}_{/Y}^{aff} \rightarrow \text{Sch}_{/Y'}^{aff}$, $S \mapsto S \times_Z Z'$ is cofinal.*

Proof. Let $U \xrightarrow{h} Y'$ be an object of $\text{Sch}_{/Y'}^{aff}$. The category $\text{Sch}_{/Y}^{aff} \times_{\text{Sch}_{/Y'}^{aff}} (\text{Sch}_{/Y'}^{aff})_{h/}$ is contractible, because it has an initial object. Indeed, it identifies with the category of diagrams

$$\begin{array}{ccc} U & \xrightarrow{\nu} & S \\ \downarrow & & \downarrow \\ Y' & \rightarrow & Y \end{array}$$

with $S \in \text{Sch}^{aff}$. The initial object is the one for which ν is an isomorphism. \square

By the above lemma, (1) identifies with $\text{colim}_{S' \in \text{Sch}_{/Y'}^{aff}} \Gamma(S', C) \xrightarrow{\sim} f'_!(C|_{Y'})$.

Corollary 1.0.5. *Let $f : Y \rightarrow Z$ be a morphism in PreStk . Then the functor $f^* : \text{ShvCat}(Z) \rightarrow \text{ShvCat}(Y)$ admits a left adjoint $f_!$ compatible with arbitrary base change. The functor $f_!$ is given by the following formula. Let $C \in \text{ShvCat}(Y)$. If $(Z' \xrightarrow{g} Z) \in \text{Sch}_{/Z}^{aff}$ then $g^* f_! C \xrightarrow{\sim} (f')_! g^* C$ canonically, where $g' : Y' \rightarrow Y$ and $f' : Y' \rightarrow Z'$ are obtained by the base changes as above.*

1.0.6. Let us study the same property for lax prestacks in the sense of [5]. I am using freely the notations of [5, 2]. Let $Y \in \text{PreStk}^{lax}$, $p : Y \rightarrow *$. Recall that by formula (4) from [2],

$$\text{ShvCat}_{/Y}^{naive} \xrightarrow{\sim} \lim_{(S \rightarrow S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S'), \text{ShvCat}(S))$$

For $C \in \text{ShvCat}_{/Y}^{naive}$, $D \in \text{DGCat}_{cont}$ we get

$$(2) \quad \text{Map}_{\text{ShvCat}_{/Y}^{naive}}(C, p^* D) \xrightarrow{\sim} \lim_{(S \xrightarrow{\alpha} S') \in \text{Tw}(\text{Sch}^{aff})^{op}} \text{Map}_{\text{Fun}(Y(S'), \text{ShvCat}(S))}(C_\alpha, r_\alpha(D))$$

Here $r_\alpha : \text{DGCat}_{cont} \rightarrow \text{Fun}(Y(S'), \text{ShvCat}(S))$ is the restriction functor. Here C_α is the image of C in $\text{Fun}(Y(S'), \text{ShvCat}(S))$.

For each $\alpha \in Tw(\text{Sch}^{aff})$ as above, r_α has a left adjoint sending C_α to $\Gamma(S, \text{colim } C_\alpha)$. Assume given a map $\alpha \rightarrow \beta$ in $Tw(\text{Sch}^{aff})$, where $(T \xrightarrow{\beta} T') \in Tw(\text{Sch}^{aff})$ let $q : S \rightarrow T$ be the corresponding morphism. Consider the corresponding diagram

$$\begin{array}{ccc} \text{DGCat}_{cont} & \xrightarrow{r_\alpha} & \text{Fun}(Y(S'), \text{ShvCat}(S)) \\ & \searrow r_\beta & \uparrow t \\ & & \text{Fun}(Y(T'), \text{ShvCat}(T)), \end{array}$$

where t is the transition functor in our diagram indexed by $Tw(\text{Sch}^{aff})^{op}$. The functor t admits a left adjoint t^L given as the composition

$$\text{Fun}(Y(S'), \text{ShvCat}(S)) \xrightarrow{\bar{q}} \text{Fun}(Y(S'), \text{ShvCat}(T)) \xrightarrow{LKE} \text{Fun}(Y(T'), \text{ShvCat}(T)),$$

here we denoted by \bar{q} the functor of composition with $q_! : \text{ShvCat}(S) \rightarrow \text{ShvCat}(T)$.

Consider the functor $\Gamma_c^{naive} : \text{ShvCat}_{/Y}^{naive} \rightarrow \text{DGCat}_{cont}$ given by

$$(3) \quad C \mapsto \Gamma_c^{naive}(Y, C) := \underset{(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})}{\text{colim}} \Gamma(S, \text{colim } C_\alpha)$$

In the above formula $\Gamma(S, \text{colim } C_\alpha) \xrightarrow{\sim} \text{colim}_{Y(S')} \Gamma(S, C_\alpha)$ naturally. For a morphism $\alpha \rightarrow \beta$ in $Tw(\text{Sch}^{aff})$ as above the corresponding transition map in this system is the natural map $r_\alpha^L C_\alpha \rightarrow r_\beta^L C_\beta$ coming from the isomorphism $tC_\beta \xrightarrow{\sim} C_\alpha$ by adjointness.

We claim that Γ_c^{naive} is left adjoint to p^* . Indeed, the mapping space (2) identifies with

$$\lim_{(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})^{op}} \text{Map}_{\text{DGCat}_{cont}}(\Gamma(S, \text{colim } C_\alpha), D) \xrightarrow{\sim} \text{Map}_{\text{DGCat}_{cont}}(\Gamma_c^{naive}(Y, C), D)$$

What is $\Gamma_c^{naive}(Y, \text{QCoh}_Y)$? For $C = \text{QCoh}_Y$ we get for any $\alpha \in Tw(\text{Sch}^{aff})$ as above $C_\alpha : Y(S') \rightarrow \text{ShvCat}(S)$ is the constant functor with value QCoh_S . We then get $\text{colim } C_\alpha \xrightarrow{\sim} \text{Fun}(Y(S')^{inv}, \text{QCoh}(S))$. Here for $\mathcal{E} \in 1 - \text{Cat}$ we denote by \mathcal{E}^{inv} the space obtained by inverting all the arrows in \mathcal{E} . Indeed, the projection $\text{QCoh}(S) - \text{mod} \rightarrow \text{DGCat}_{cont}$ preserves colimits. The colimit of a constant functor $\mathcal{E} \rightarrow \text{DGCat}_{cont}$ with value D is (by passing to the right adjoint) the limit of the constant functor $\mathcal{E}^{op} \rightarrow \text{DGCat}_{cont}$ with value D . The latter is $\text{Fun}((\mathcal{E}^{op})^{inv}, D)$, because the projection $\text{DGCat}_{cont} \rightarrow 1 - \text{Cat}$ preserves limits (by [1], 10.1.1). Finally, $\mathcal{E}^{inv} \xrightarrow{\sim} (\mathcal{E}^{op})^{inv}$ for any $\mathcal{E} \in 1 - \text{Cat}$ naturally. So,

$$\Gamma_c^{naive}(Y, \text{QCoh}_Y) \xrightarrow{\sim} \underset{(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})}{\text{colim}} \text{Fun}(Y(S')^{inv}, \text{QCoh}(S)),$$

the colimit calculated in DGCat_{cont} . The latter colimit identifies (by passing to right adjoints) with

$$\lim_{(S \xrightarrow{\alpha} S') \in Tw(\text{Sch}^{aff})^{op}} \text{Fun}(Y(S')^{inv}, \text{QCoh}(S)) \xrightarrow{\sim} \text{QCoh}(Y^{inv})$$

The latter formula for $\text{QCoh}(Y^{inv})$ was established in ([2], 0.0.7 and 0.2.14). We have proved actually that $\Gamma_c^{naive}(Y, \text{QCoh}_Y) \xrightarrow{\sim} \Gamma^{naive}(Y, \text{QCoh}_Y) \xrightarrow{\sim} \text{QCoh}(Y^{inv})$. That is, we may remove "compact support" here.

Since $p^* : \text{ShvCat}(*) \rightarrow \text{ShvCat}(Y)$ is symmetric monoidal, Γ_c^{naive} is left-lax symmetric monoidal (by [3], ch. I.1, 3.2.4).

1.0.7. We claim that (3) can be given also by a simpler formula

$$C \mapsto \operatorname{colim}_{S \in \operatorname{Sch}^{aff}} \operatorname{colim}_{y \in Y(S)} \Gamma(S, y^* C)$$

More precisely, we consider the cartesian fibration $coGroth(Y) \rightarrow \operatorname{Sch}^{aff}$ corresponding to $Y : (\operatorname{Sch}^{aff})^{op} \rightarrow 1 - \mathcal{C}at$, and the above colimit is over $coGroth(Y)$. Write also $Groth(Y) \rightarrow (\operatorname{Sch}^{aff})^{op}$ for the cocartesian fibration corresponding to Y .

As we have seen already in ([2], 0.2.15), we have the natural functor, say $\xi : Tw(\operatorname{Sch}^{aff}) \times_{(\operatorname{Sch}^{aff})^{op}} Groth(Y) \rightarrow coGroth(Y)$ sending $(S \xrightarrow{\alpha} S', y' \in Y(S'))$ to $(S, y \in Y(S))$, where y is the composition $S \rightarrow S' \xrightarrow{y'} Y$. This functor has the following property established in *loc.cit.* For any $\mathcal{D} \in 1 - \mathcal{C}at$ admitting small colimits and any $h : coGroth(Y) \rightarrow \mathcal{D}$, the canonical map $LKE_\xi(h \circ \xi) \rightarrow h$ is an isomorphism. Here LKE_ξ denotes the LKE along ξ .

1.0.8. Let now $f : Y \rightarrow Z$ be a morphism in $\operatorname{PreStk}^{lax}$. We want to find a left adjoint $f_!$ to $f^* : ShvCat_{/Z}^{naive} \rightarrow ShvCat_{/Y}^{naive}$. First, if $Z \in \operatorname{Sch}^{aff}$ then we still can define $f_!$ by (3), because the corresponding diagram indexed by $Tw(\operatorname{Sch}^{aff})$ takes values in $\operatorname{QCoh}(Z) - mod$ actually.

1.0.9. Assume now $Z \in \operatorname{PreStk}^{lax}$ and $D \in ShvCat_{/Z}^{naive}$, $C \in ShvCat_{/Y}^{naive}$. For $(S \xrightarrow{\alpha} S') \in Tw(\operatorname{Sch}^{aff})$, f yields a functor $Y(S') \rightarrow Z(S')$, hence an adjoint pair

$$LKE_\alpha : \operatorname{Fun}(Y(S'), ShvCat(S)) \rightleftarrows \operatorname{Fun}(Z(S'), ShvCat(S)) : res_\alpha,$$

where res_α is the restriction. We get

$$\operatorname{Map}_{ShvCat_{/Y}^{naive}}(C, f^* D) \xrightarrow{\sim} \lim_{(S \xrightarrow{\alpha} S') \in Tw(\operatorname{Sch}^{aff})^{op}} \operatorname{Map}_{\operatorname{Fun}(Y(S'), ShvCat(S))}(C_\alpha, (f^* D)_\alpha)$$

Here $(f^* D)_\alpha : Y(S') \rightarrow ShvCat(S)$ is the functor $res_\alpha(D_\alpha)$. So, the above limit identifies with

$$\lim_{(S \xrightarrow{\alpha} S') \in Tw(\operatorname{Sch}^{aff})^{op}} \operatorname{Map}_{\operatorname{Fun}(Z(S'), ShvCat(S))}(LKE_\alpha C_\alpha, D_\alpha)$$

One wants to define $f_! : ShvCat_{/Y}^{naive} \rightarrow ShvCat_{/Z}^{naive}$ by requiring that it sends C to the projective system in $\lim_{(S \xrightarrow{\alpha} S') \in Tw(\operatorname{Sch}^{aff})^{op}} \operatorname{Fun}(Z(S'), ShvCat(S))$, whose value at α

is

$$LKE_\alpha(C_\alpha)$$

Is this indeed a projective system? Given a map $\alpha \rightarrow \beta$ in $Tw(\operatorname{Sch}^{aff})$ as above, let $t_Z : \operatorname{Fun}(Z(T'), ShvCat(T)) \rightarrow \operatorname{Fun}(Z(S'), ShvCat(S))$ be the corresponding transition functor. We have a natural map $LKE_\alpha C_\alpha \rightarrow t_T(LKE_\beta C_\beta)$, but it is not clear if it is an equivalence.

To get a rigorous definition of $f_!$ it is better to proceed as follows. Assume given a map $f : Y \rightarrow Z$ in $\operatorname{PreStk}^{lax}$ with $Z \in \operatorname{Sch}^{aff}$. Let $g : Z' \rightarrow Z$ be a morphism in Sch^{aff} and $f' : Y' \rightarrow Z'$ be obtained from f by the base change, same for $g' : Y' \rightarrow Y$. It suffices to prove the base change property for the functor $f_! : ShvCat_{/Y}^{naive} \rightarrow ShvCat(Z)$.

Namely, we have a natural map $f'_!(g'^*C) \rightarrow g^*f_!C$, and we want to show it is an isomorphism. The above morphism is

$$\operatorname{colim}_{S' \in \operatorname{Sch}^{aff}, y \in Y'(S')} \Gamma(S', y^*(g')^*C) \rightarrow \operatorname{QCoh}(Z') \otimes_{\operatorname{QCoh}(Z)} f_!C \xrightarrow{\sim} \operatorname{colim}_{S \in \operatorname{Sch}^{aff}, y \in Y(S)} \Gamma(S \times_Z Z', y^*C)$$

Lemma 1.0.10. *Let $f : Y \rightarrow Z$ be a morphism in $\operatorname{PreStk}^{lax}$ with $Z \in \operatorname{Sch}^{aff}$. Let $g : Z' \rightarrow Z$ be a morphism in Sch^{aff} and $g' : Y' \rightarrow Y$ be obtained from g by the base change f . Then $R : \operatorname{coGroth}(Y) \rightarrow \operatorname{coGroth}(Y')$, $(S \xrightarrow{y} Y) \mapsto (S \times_Z Z' \rightarrow Y')$ is cofinal. Here $\operatorname{coGroth}(Y) \rightarrow \operatorname{Sch}^{aff}$ is the cartesian fibration corresponding to Y .*

Proof. Consider the functor $L : \operatorname{coGroth}(Y') \rightarrow \operatorname{coGroth}(Y)$ obtained by functoriality of $\operatorname{coGroth}$ from $g' : Y' \rightarrow Y$. We show that L is left adjoint to R . This suffices by ([3], ch. I.1, 2.2.3).

Indeed, let $(S' \xrightarrow{y'} Y') \in \operatorname{coGroth}(Y')$, $(T \xrightarrow{t} Y) \in \operatorname{coGroth}(Y)$. The mapping space in $\operatorname{coGroth}(Y)$ from $(S' \xrightarrow{y'} Y' \rightarrow Y)$ to $(T \rightarrow Y)$ consists of pairs: a map $S' \rightarrow T$ in Sch^{aff} such that the diagram commutes

$$\begin{array}{ccc} S' & \rightarrow & T \\ \downarrow & & \downarrow \\ Z' & \rightarrow & Z \end{array}$$

and a 1-morphism ϵ from $(S' \rightarrow Y' \rightarrow Y)$ to $(S' \rightarrow T \rightarrow Y)$ in $Y(S)$.

Set $(T \times_Z Z' \xrightarrow{t'} Y') = R(T \xrightarrow{t} Y)$. The mapping space in $\operatorname{coGroth}(Y')$ from (S', y') to $(T \times_Z Z' \xrightarrow{t'} Y')$ consists of pairs: a morphism $S' \rightarrow T \times_Z Z'$ over Z' and a 1-morphism ϵ from $(S' \rightarrow Y' \rightarrow Y)$ to $(S' \rightarrow T \rightarrow Y)$ in $Y(S)$ as above. These spaces are identified. \square

Since we have now the base change property, we derive the following.

Corollary 1.0.11. *Let $Y \xrightarrow{f} Z \xleftarrow{g} Z'$ be a diagram in $\operatorname{PreStk}^{lax}$ with $g : Z' \rightarrow Z$ a morphism in PreStk . Then the functor $f^* : \operatorname{ShvCat}_{/Z}^{naive} \rightarrow \operatorname{ShvCat}_{/Y}^{naive}$ admits a left adjoint $f_!$ compatible with the base change by g . That is, the natural map $f'_!(g')^* \rightarrow g^*f_!$ for $C \in \operatorname{ShvCat}_{/Y}^{naive}$ is an isomorphism in $\operatorname{ShvCat}_{/Z'}^{naive}$.*

As in ([5], A.10), for a diagram $X \xrightarrow{z} Z \xleftarrow{f} Y$ in $\operatorname{PreStk}^{lax}$ denote by $Y_{z/}$ the lax prestack sending $S \in \operatorname{Sch}^{aff}$ to the category of triples: $S \xrightarrow{x} X, S \xrightarrow{y} Y$ and a morphism $zx \rightarrow fy$ in $Z(S)$. This definition is transitive in the following sense: if $X \in \operatorname{Sch}^{aff}$ and $X' \xrightarrow{\alpha} X$ is a morphism in Sch^{aff} then $X' \times_X Y_{z/} \xrightarrow{\sim} Y_{(z\alpha)/}$ canonically.

So, for each $S \xrightarrow{z} Z$ with $S \in \operatorname{Sch}^{aff}$ denote by $f_z : Y_{z/} \rightarrow S$ the projection. Let $C \in \operatorname{ShvCat}_{/Y}^{naive}$. Then the sheaves of categories $(f_z)_!(C|_{Y_{z/}})$ form a sheaf of categories on Z , so a well-defined object $D \in \operatorname{ShvCat}_{/Z}^{naive}$ by Corollary 1.0.11. More precisely, for each $(S \xrightarrow{\alpha} S') \in \operatorname{Tw}(\operatorname{Sch}^{aff})$ we get $D_\alpha : Z(S') \rightarrow \operatorname{ShvCat}(S)$ sending a given map $S' \xrightarrow{z'} Z$ to $(f_{z'\alpha})!(C|_{Y_{z'\alpha/}})$. I think the functor so obtained $f_! : \operatorname{ShvCat}_{/Y}^{naive} \rightarrow \operatorname{ShvCat}_{/Z}^{naive}$ is left adjoint to f^* , but I have not checked this.

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