COMMENTS: SHEAVES OF CATEGORIES AND NOTION OF 1-AFFINENESS

0.0.1. For 1.1.5. The fact that $ShvCat(Y)$ admits small colimits is proved as in ([1], around Lm 2.2.67).

0.0.2. For 1.2.2. This follows from ([1], 2.2.41).

0.0.3. For 1.2.3. Let $\mathcal{Z} \to Y$ be a morphism in PreStk. He means $\Gamma(\mathcal{Z}, \cdot) : ShvCat(Y) \to \mathrm{DGCat}_{cont}$ is right-lax symmetric monoidal. Indeed, let $F, F' \in ShvCat(Y)$

$$\Gamma(\mathcal{Z}, F \otimes F') \widetilde{\to} \lim_{S \to \mathcal{Z}} \Gamma(S, F \otimes F') \widetilde{\to} \lim_{S \to \mathcal{Z}} \Gamma(S, F) \otimes_{\mathrm{QCoh}(S)} \Gamma(S, F')$$

The limit over $((\mathrm{Sch}^{aff})_{/Z})^{op}$. For each $S \to Z$ we have the projection

$$\Gamma(\mathcal{Z}, F) \otimes \Gamma(\mathcal{Z}, F') \to \Gamma(S, F) \otimes \Gamma(S, F') \to \Gamma(S, F) \otimes_{QCoh(S)} \Gamma(S, F')$$

They are compatible with morphisms in $(\mathrm{Sch}^{aff})_{/\mathcal{Z}}$, so yield a morphism

$$\Gamma(\mathcal{Z}, F) \otimes \Gamma(\mathcal{Z}, F') \to \Gamma(\mathcal{Z}, F \otimes F')$$

For $F' = \mathrm{QCoh}_{/Y}$ we get a map $\Gamma(\mathcal{Z}, F) \otimes \Gamma(\mathcal{Z}, \mathrm{QCoh}_{/Y}) \to \Gamma(\mathcal{Z}, F \otimes (\mathrm{QCoh}_{/Y}))$. Since $F \otimes (\mathrm{QCoh}_{/Y}) \widetilde{\to} F$, we get the $\mathrm{QCoh}(\mathcal{Z})$-module structure on $\Gamma(\mathcal{Z}, F)$.

The above map for F, F' factors to give a map

$$\Gamma(\mathcal{Z}, F) \otimes_{\mathrm{QCoh}(\mathcal{Z})} \Gamma(\mathcal{Z}, F') \to \Gamma(\mathcal{Z}, F \otimes F')$$

of $\mathrm{QCoh}(\mathcal{Z})$-modules.

0.0.4. For 1.3.1. Let us justify that $\mathrm{Loc}_{\mathcal{Y}}$ is a left adjoint to $\Gamma^{enh}_{\mathcal{Y}}$ indeed. First, given $\mathcal{C} \in ShvCat(\mathcal{Y})$, we may view for any $S \in (\mathrm{Sch}^{aff})_{/\mathcal{Y}}$, $\Gamma(S, \mathcal{C})$ as a $\mathrm{QCoh}(\mathcal{Y})$-module. Moreover, for $f : S \to S'$ the restriction functor $\mathcal{C}(S') \to \mathcal{C}(S)$ is a morphism of $\mathrm{QCoh}(\mathcal{Y})$-modules. So, in the definition of global sections, we may directly define $\Gamma(\mathcal{Y}, \mathcal{C}) = \lim_{S \to \mathcal{Y}} \mathcal{C}(S)$ over the category $((\mathrm{Sch}^{aff})_{/\mathcal{Y}})^{op}$, the limit being taken in $\mathrm{QCoh}(\mathcal{Y})$-modules. The projection $\mathrm{QCoh}(\mathcal{Y}) - mod \to \mathrm{DGCat}_{cont}$ preserves limits.

Recall that

$$ShvCat(\mathcal{Y}) \widetilde{\to} \lim_{S \to \mathcal{Y}} \mathrm{QCoh}(S) - mod$$

in $1 - \mathcal{C}\mathrm{at}$, the limit over $((\mathrm{Sch}^{aff})_{/\mathcal{Y}})^{op}$. Now given $M \in \mathrm{QCoh}(\mathcal{Y}) - mod$, using the description of the Map spaces in the limit of categories from ([1], Cor. 2.5.3), we get

$$\mathrm{Map}_{ShvCat(\mathcal{Y})}(\mathrm{Loc}_{\mathcal{Y}}(M), \mathcal{C}) \widetilde{\to} \lim_{S \to \mathcal{Y}} \mathrm{Map}_{\mathrm{QCoh}(S)-mod}(M \otimes_{\mathrm{QCoh}(\mathcal{Y})} \mathrm{QCoh}(S), \mathcal{C}(S)) \widetilde{\to}$$
$$\lim_{S \to \mathcal{Y}} \mathrm{Map}_{\mathrm{QCoh}(\mathcal{Y})-mod}(M, \mathcal{C}(S)) \widetilde{\to} \mathrm{Map}_{\mathrm{QCoh}(\mathcal{Y})-mod}(M, \mathcal{C}(\mathcal{Y}))$$

We use here the fact that the projection $\mathrm{QCoh}(\mathcal{Y}) - mod \to \mathrm{DGCat}_{cont}$ preserves limits.

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0.0.5. For 1.3.5. Let $A \in CAlg(\mathrm{DGCat}_{cont})$, by $A-mod$ he denotes the category of A-modules in DGCat_{cont}. If $C \in A-mod$ is dualizable as an object of the symmetric monoidal ∞-category $A-mod$ then the functor $A-mod \to A-mod$, $M \mapsto M \otimes_A C$ preserves small limits. This follows from ([1], Lemma 3.1.2), which is a strengthened version of ([4], ch. 1, Lm 4.1.6 a)).

0.0.6. For Lm 1.4.6. My understanding is that the diagram $\alpha \mapsto O_\alpha$ is a functor $I \to 1-\mathcal{C}\mathrm{at}^{SymMon}$, the notation $1-\mathcal{C}\mathrm{at}^{SymMon}$ is from ([4], ch. 1, 3.3.1). Another version is $I \to \mathrm{DGCat}^{SymMon}_{cont}$.

0.0.7. For 1.4.8(b). If $\mathrm{QCoh}(Y)$ is rigid then for any $S \to Y$ with $S \in \mathrm{Sch}^{aff}$, $\mathrm{QCoh}(S)$ is dualizable as an object of $\mathrm{QCoh}(Y)-mod$ by ([4], ch. I.1, 9.4.4). So, the functor $\mathrm{QCoh}(Y)-mod \to \mathrm{QCoh}(S)-mod$, $C \mapsto \mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(Y)} C$ commutes with limits.

0.0.8. For 1.5.5. (a) follows from an analog of ([4], ch. I.2, 2.3.8) for flat topology instead of etale one. Indeed, the functor $(\mathrm{PreStk}/Y)^{op} \to \mathrm{DGCat}_{cont}$, $Z \mapsto \Gamma(Z, C)$ preserves limits.

0.0.9. For 2.2.1. Given a lax prestack Y, he defined $Shv^!(Y) \in 1-\mathcal{C}\mathrm{at}$. As in [2], one shows that the functor $(\mathrm{PreStk}^{lax})^{op} \to 1-Cat^{St,cocmpl}_{cont}$, $Y \mapsto Shv^!(Y)$ transforms lax colimits in PreStk^{lax} (relative over Sch^{op}) into oplax limits.

0.0.10. For 3.1.3. Let us show that $cores_f : ShvCat(\mathcal{Y}_2) \to ShvCat(\mathcal{Y}_1)$ is left adjoint to $coind_f$. Given $\mathcal{C}_i \in ShvCat(\mathcal{Y}_i)$, we get

$$\mathrm{Map}_{ShvCat(\mathcal{Y}_2)}(\mathcal{C}_2, coind_f(\mathcal{C}_1)) \widetilde{\to} \lim_{S\to Y_2} \mathrm{Map}_{\mathrm{QCoh}(S)-mod}(\mathcal{C}_2(S), \mathcal{C}_1(S \times_{\mathcal{Y}_2} \mathcal{Y}_1)) \widetilde{\to}$$
$$\lim_{S\to Y_2,\, T\to S\times_{\mathcal{Y}_2}\mathcal{Y}_1} \mathrm{Map}_{\mathrm{QCoh}(S)-mod}(\mathcal{C}_2(S), \mathcal{C}_1(T)) \widetilde{\to} \lim_{T\to \mathcal{Y}_1} \mathrm{Map}_{\mathrm{QCoh}(T)-mod}(\mathcal{C}_2(T), \mathcal{C}_1(T))$$
$$\widetilde{\to} \mathrm{Map}_{ShvCat(\mathcal{Y}_1)}(cores_f(\mathcal{C}_2), \mathcal{C}_1)$$

The first limit is over $((\mathrm{Sch}^{aff})_{/\mathcal{Y}_2})^{op}$. The second limit is over pairs of maps $S \to \mathcal{Y}_2, T \to S \times_{\mathcal{Y}_2} \mathcal{Y}_1$ with $S, T \in \mathrm{Sch}^{aff}$. Inside we have the full subcategory given by the property that the composition $T \to S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \to S$ is an isomorphism. The corresponding inclusion is cofinal, so the second limit rewrites as the third one.

0.0.11. If $S \to T \leftarrow \mathcal{Y}$ is a diagram in PreStk with $S, T \in \mathrm{Sch}^{aff}$ then

$$\mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(T)} \mathrm{QCoh}(\mathcal{Y}) \widetilde{\to} \mathrm{QCoh}(S \times_T \mathcal{Y})$$

Indeed, the functor Γ^{enh}_T is an equivalence, apply their Prop. 3.1.9 now. More generally, one has

Lemma 0.0.12. *Let $Y_1 \xrightarrow{f} Y_2 \xleftarrow{h} Z$ be a diagram in* PreStk *with Y_i 1-affine. Then one has canonically*

$$\mathrm{QCoh}(Y_1) \otimes_{\mathrm{QCoh}(Y_2)} \mathrm{QCoh}(Z) \widetilde{\to} \mathrm{QCoh}(Y_1 \times_{Y_2} Z)$$

Proof. Consider the sheaf $\mathrm{QCoh}_{/Z}$ on Z, it sends $S \to Z$ to $\mathrm{QCoh}(S)$, let $\mathcal{C} = coind_h(\mathrm{QCoh}_{/Z})$. Apply ([5], Lemma 3.2.4) to $\mathcal{C}$. We get $\Gamma^{enh}(Y_2, \mathcal{C}) \widetilde{\to} \mathrm{QCoh}(Z)$ and $\Gamma^{enh}(Y_1, cores_f(\mathcal{C})) \widetilde{\to} \lim_{S\to Y_1} \mathcal{C}(S) \widetilde{\to} \lim_{S\to Y_1} \mathrm{QCoh}(S \times_{Y_2} Z) \widetilde{\to} \mathrm{QCoh}(Y_1 \times_{Y_2} Z)$. Here the limit is over $(\mathrm{Sch}^{aff}_{/Y_1})^{op}$. We used the fact that the colimits in PreStk are universal. $\square$

0.0.13. For 3.2.8. Let us check that given $Y, Z \in \mathrm{PreStk}$ 1-affine the product $Y \times Z$ is 1-affine. We want to apply ([5], Pp. 3.2.6) to the projection $f : Y \times Z \to Z$.

By my Lemma 0.0.12, for any $S \to Z$ with $S \in \mathrm{Sch}^{aff}$, we get $\mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(Z)} \mathrm{QCoh}(Y \times Z) \widetilde{\to} \mathrm{QCoh}(Y \times S)$. Now ([5], Pp. 3.2.6) reduces the claim to the particular case when $Z = T \in \mathrm{Sch}^{aff}$.

So, we assume $Z = T$. Let us check that Loc is fully faithful for $T \times Y$. We check that the natural map $C \to \Gamma^{enh}_{T\times Y}(\mathrm{Loc}_{T\times Y} C)$ is an isomorphism for $C \in \mathrm{QCoh}(T \times Y) - module$. Use ([1], 9.2.10), which is the base change property for products of affine schemes. We get $\Gamma(T \times Y, \mathrm{Loc}(C)) \widetilde{\to} \lim_{S'\to T\times Y} C \otimes_{\mathrm{QCoh}(T\times Y)} \mathrm{QCoh}(S')$, the limit over $(\mathrm{Sch}^{aff}_{/T\times Y})^{op}$. We may rewrite it as

$$\lim_{S'\to Y} C \otimes_{\mathrm{QCoh}(T\times Y)} \mathrm{QCoh}(T \times S')$$

over $(\mathrm{Sch}^{aff}_{/Y})^{op}$. Here

$$C \otimes_{\mathrm{QCoh}(T\times Y)} \mathrm{QCoh}(T \times S') \widetilde{\to} C \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(S')$$

as $\mathrm{QCoh}(S')$-modules, because $\mathrm{QCoh}(T \times Y) \widetilde{\to} \mathrm{QCoh}(T) \otimes \mathrm{QCoh}(Y)$. So, our limit identifies with $\Gamma_Y(\mathrm{Loc}_Y C) \widetilde{\to} C$. So, $\mathrm{Loc}_{T\times Y}$ is fully faithful.

To show that $\Gamma^{enh}_{T\times Y}$ is fully faithful, use the projection $q : T \times Y \to Y$. The assumptions of his Prop. 3.2.6 are satisfied for q. Namely, given $S' \in \mathrm{Sch}^{aff}$ with $S' \to Y$, one has $\mathrm{QCoh}(S') \otimes \mathrm{QCoh}(Y)\, \mathrm{QCoh}(T \times Y) \widetilde{\to} \mathrm{QCoh}(S') \otimes \mathrm{QCoh}(T) \widetilde{\to} \mathrm{QCoh}(S' \times T)$. Besides, the base change of q by $S' \to Y$ with S' an affine scheme gives a 1-affine prestack $S' \times T$. Since Γ^{enh}_Y is fully fauthful, by Pp. 3.2.6(b), $\Gamma^{enh}_{T\times Y}$ is also fully faithful.

0.0.14. For 3.3. Evident remark: let $f : Y_1 \to Y_2$ be a morphism in PreStk, $\mathcal{C}_1 \in ShvCat(Y_1)$, let $\bar{f} : Z_1 \to Z_2$ be obtained from f by the base change $Z_2 \to Y_2$. Then

$$(coind_f(\mathcal{C}_1)) \mid_{Z_2} \widetilde{\to} coind_{\bar{f}}(\mathcal{C}_1 \mid_{Z_1})$$

0.0.15. For 3.2.6, the proof of (b). Assume $coind_f$ is conservative. Then given $C_1 \in ShvCat(Y_1)$, we have to show that $\mathrm{Loc}_f(\Gamma_{Y_1}(C_1)) \to C_1$ is an isomorphism. We check that $coind_f \,\mathrm{Loc}_f(\Gamma_{Y_1}(C_1)) \to coind_f C_1$ is an isomorphism in $ShvCat(Y_2)$. By (a)i), $coind_f \,\mathrm{Loc}_f(\Gamma_{Y_1}(C_1))\, \mathrm{Loc}_{Y_2}(res_f \Gamma_{Y_1}(C_1))$. Since $\Gamma_{Y_2}(coind_f C_1) \widetilde{\to} res_f \Gamma_{Y_1}(C_1)$, our claim follows from the fact that $\mathrm{Loc}_{Y_2} \Gamma_{Y_2}(C) \to C$ is an isomorphism for $C = coind_f(C_1)$.

In the proof he means that if $T \to Y_1$ is a map with $T \in \mathrm{Sch}^{aff}$ then $\Gamma(T \times_{Y_2} Y_1, \mathcal{C}'_i) \widetilde{\to} \Gamma(T, coind_f(\mathcal{C}'_i)$, so does not depend on i, the corresponding map is an isomorphism. So, indeed, (3.11) is an isomorphism, and we may evaluate at $T \to T \times_{Y_2} Y_1$. We are done.

0.0.16. If $S \in \mathrm{Sch}^{aff}$ then S_{dR} is 1-affine.

0.1. In Appendix C, Def. C.1.2 Dennis means by the last face map $\partial_i : [i] \to [i+1]$ the injective map in $\boldsymbol{\Delta}$ whose image does not contain $i+1$. I think for $\alpha : [j] \to [i]$ in $\boldsymbol{\Delta}$ he denotes by $\alpha + 1 : [j+1] \to [i+1]$ the map given by α on $\{0, \dots, j\}$ and sending $j+1$ to $i+1$.

By definition, a simplicial category $C^\bullet$ satisfies the co-monadic Beck-Chevalley conditions if for every i and the last face $\partial_i : [i] \to [i+1]$ the functor $T^{\partial_i} : C^{i+1} \to C^i$ has a right adjoint $(T^{\partial_i})^R$, and for every $\alpha : [j] \to [i]$ in $\boldsymbol{\Delta}$ the diagram

$$\begin{array}{ccc} C^i & \stackrel{(T^{\partial_i})^R}{\to} & C^{i+1} \\ \downarrow T^\alpha & & \downarrow T^{\alpha+1} \\ C^j & \stackrel{(T^{\partial_j})^R}{\to} & C^{j+1} \end{array}$$

commutes. The version of Lm. C.1.6 for co-monadic Beck-Chevalley conditions:

Lemma 0.1.1. *Let $C^\bullet$ be a simplicial category. Assume for any $\alpha : [j] \to [i]$ in $\boldsymbol{\Delta}$, $T^\alpha : C^i \to C^j$ admits a left adjoint $(T^\alpha)^L$, and for any i and the last face map $\partial_i : [i] \to [i+1]$ in $\boldsymbol{\Delta}$, T^{∂_i} has a right adjoint $(T^{\partial_i})^R$. Let $C^{\bullet,L}$ be the co-simplicial category obtained from $C^\bullet$ by passing to left adjoints. Then $C^{\bullet,L}$ satisfies the co-monadic Beck-Chevalley conditions iff $C^\bullet$ satisfies the co-monadic Beck-Chevalley conditions.* □

To derive his Cor. C.2.3 from Lemma C.2.2, one applies (HA, 4.7.5.2), this is the *canonical reference* to Beck-Chevalley conditions!

0.1.2. *Example of application of Cor. C.2.3.* Let $S \in \mathrm{Sch}^{aff}$ be an affine derived scheme, $q : S \to *$. Assume S is a group object in Sch^{aff}, let $m : S \times S \to S$ be the product map. Equip $\mathrm{QCoh}(S)$ with the convolution monoidal structure. Then the functor $q_* : \mathrm{QCoh}(S) \to \mathrm{Vect}$ is monoidal. So, Vect is a bimodule over $\mathrm{QCoh}(S)$. Then $\mathrm{Vect} \otimes_{\mathrm{QCoh}(S)} \mathrm{Vect}$ can be calculated using his Cor. C.2.3. Namely, q_*, m_* admit left adjoints, and $q^* : \mathrm{Vect} \to \mathrm{QCoh}(S)$ is a map of left $\mathrm{QCoh}(S)$-module categories. Indeed, this follows from the fact that the map $\mathrm{pr}_2 \times m : S \times S \to S \times S$ is an isomorphism. The assumptions of Lemma C.2.2(ii) are verified, so $\mathrm{Vect} \otimes_{\mathrm{QCoh}(S)} \mathrm{Vect} \widetilde{\to} \mathcal{A} - mod$, where $\mathcal{A}$ is a monad on Vect given as $q_*(q_*)^R$ for the diagram

$$\mathrm{Vect} \stackrel{(q_*)^R}{\to} \mathrm{QCoh}(S) \stackrel{q}{\to} \mathrm{Vect},$$

here $(q_*)^R$ is the right adjoint to q_*. The functor $\mathcal{A} : \mathrm{Vect} \to \mathrm{Vect}$ sends V to $\underline{\mathrm{Hom}}(q_*\mathcal{O}, V)$. For example, if $q_*\mathcal{O}$ is dualizable then $\mathcal{A}$ becomes the tensoring by $(q_*\mathcal{O})^\vee$.

0.1.3. For Appendix D.1.2. If $\mathbf{O}$ is a rigid monoidal DG-category then the map $\phi : \mathbf{O} \widetilde{\to} \mathbf{O}^\vee$ is defined as in ([1], 6.1.5 "For 9.2.1"), that is, $\phi(x)$ is the functor $\mathbf{O} \to \mathrm{Vect}$, $b \mapsto u^R(b \otimes x)$. Here $u^R : \mathbf{O} \to \mathrm{Vect}$ is the right adjoint to the unit map $u : \mathrm{Vect} \to \mathbf{O}$. The map ϕ is an isomorphism of left $\mathbf{O}$-modules.

If $a \in \mathcal{O}^c$, $\mathbf{O}$ is rigid and compactly generated then the left dual $a^{\vee,L}$ is obtained as follows. For $b, c \in \mathbf{O}$ write $\mathcal{M}aps_{k,\mathbf{O}}(b, c) \in \mathrm{Vect}$ for the relative inner hom (as in [4], 10.3.7). Then $a^{\vee,L}\widetilde{\to}(\mathrm{id}\otimes f)m^R(1)$, where $m^R:\mathbf{O}\to\mathbf{O}\otimes\mathbf{O}$ is the right adjoint to m, and $f:\mathbf{O}\to\mathrm{Vect}$, $f(b)\widetilde{\to}\mathcal{M}aps_{k,\mathbf{O}}(a,b)$.

0.1.4. For D.3. By a comonoidal DG-category he means a monoid in $(\mathrm{DGCat}_{cont})^{op}$.

0.2. For Appendix E.

0.2.1. Let O be a symmetric monoidal ∞-category, A be a bialgebra in O (recall here [1],3.2.4). It is not clear if the antipode $S:A\to A$ is assumed to be an isomorphism, maybe this is not needed.

If A is an augmented coalgebra in O, that is, we are given a map of coalgebras $1\to A$ then the cosimplicial object co-$\mathrm{Bar}^\bullet(A)$ is nothing but the simplicial object $Bar^\bullet(A)$ in O^{op}, the latter would calculate the tensor product $1\otimes_A 1$ in O^{op}.

Let now A be a bialgebra in O. The unit $1\stackrel{u}{\to}A$ defines the agmentation of the coalgebra A. In E.3.1 he then considers the object of $\mathrm{Fun}(\boldsymbol{\Delta},1-\mathcal{C}\mathrm{at})$ denoted

$$\text{co-Bar}^\bullet(A)-mod,$$

a co-simplicial category sending $[n]\in\boldsymbol{\Delta}$ to $A^{\otimes n}-mod$. If $\alpha:[n]\to[m]$ is a map in $\boldsymbol{\Delta}$ then the corresponding functor $A^{\otimes n}-mod\to A^{\otimes m}-mod$ is $M\mapsto A^{\otimes m}\otimes_{A^{\otimes n}}M$, where we used the map $\bar{\alpha}:A^{\otimes n}\to A^{\otimes m}$ from the diagram co-$\mathrm{Bar}^\bullet(A)$. Then $\mathrm{Tot}(\text{co-Bar}^\bullet(A)-mod)$ is calculated in $1-\mathcal{C}\mathrm{at}$.

0.2.2. He defines $coAlg+comod(O)$ as the category of augmented coalgebra plus a comodule over it in O. If $(A_i,M_i)\in coAlg{+}comod(O)$ then a map $(A_1,M_1)\to(A_2,M_2)$ in $coAlg+comod(O)$ is essentially a pair of maps $A_1\to A_2$ in $coAlg^{aug}(O)$ and a map $M_1\to M_2$ in $A_2-comod(O)$, where on M_1 the A_2-comodule structure is obtained via "restriction of scalars".

Given $(A,M)\in coAlg+comod(O)$ by co-$\mathrm{Bar}^\bullet(A,M)\in\mathrm{Fun}(\boldsymbol{\Delta},O)$ he means the cosimplicial object, which in O^{op} would calculate the tensor product $1\otimes_A M$. So, it sends $[n]\in\boldsymbol{\Delta}$ to $M\otimes A^{\otimes n}$, and we could consider $\lim\limits_{[n]\in\boldsymbol{\Delta}} M\otimes A^{\otimes n}$ in O. The functor

$$\mathcal{F}:coAlg+comod(O)\to\mathrm{Fun}(\boldsymbol{\Delta},O),\ (A,M)\mapsto\text{co-Bar}^\bullet(A,M)$$

is symmetric monoidal.

0.2.3. Let $A\in O$ be a bialgebra. Then $(A,1)\in coAlg+comod(O)$ is indeed an algebra in this category. The map $(A,1)\otimes(A,1)\to(A,1)$ consists of the product map $A\otimes A\stackrel{m}{\to}A$, which is a map in $coAlg^{aug}(O)$, and the map $1\stackrel{\mathrm{id}}{\to}1$, which is a map in $A-comod(O)$, where on the source the A-comodule structure comes from the composition $1\stackrel{u\otimes u}{\to}A\otimes A\stackrel{m}{\to}A$, the latter composition is identified with the unit u.

The unit of $(A,1)$ is the map $(1,1)\to(A,1)$, where $1\stackrel{u}{\to}A$, and $\mathrm{id}:1\to 1$ is a map in $A-comod(O)$.

Now given $M\in A-comod(O)$. Then $(A,M)\in coAlg+comod(O)$ upgrades to an object of $(A,1)-mod(coAlg+comod(O))$. The action map $(A,1)\otimes(A,M)\to(A,M)$ is the map $(A\otimes A,M)\to(A,M)$ in $coAlg+comod(O)$, where $A\otimes A\to A$ is the multiplication m, and $M \to M$ is the identity. The fact that $id : M \to M$ is indeed a morphism in $A - comod(O)$ comes from the commutativity of the diagram

$$\begin{array}{ccc} M & \stackrel{coact}{\to} & M \otimes A \\ \downarrow 1\otimes coact & & \nearrow \mathrm{id}_M \otimes m \\ A \otimes M \otimes A & & \end{array}$$

(it says that the multiplication by 1 is the identity).

Since $\mathcal{F}$ is symmetric monoidal, it yileds a functor

$$(A,1) - mod(coAlg + comod(O)) \to \mathcal{F}(A,1) - mod(\mathrm{Fun}(\boldsymbol{\Delta}, O))$$

Now $\mathcal{F}(A,1) = \text{co-Bar}^\bullet(A)$. Composing, we get the desired functor

$$A - comod(O) \to \mathcal{F}(A,1) - mod(\mathrm{Fun}(\boldsymbol{\Delta}, O))$$

This makes sense for any bialgebra A.

My underdstanding is that his functor (E.4) makes sense for any bialgebra, not necessarily a Hopf algebra. Is the existence of the antipode used to show that the co-simplicial category $\text{co-Bar}^\bullet(A) - mod$ satisfies the comonadic Beck-Chevalley conditions?

1. Appendix

1.0.1. Let $S \in \mathrm{Sch}^{aff}$, $p : S \to *$. Then the functor $p^* : ShvCat(*) = \mathrm{DGCat}_{cont} \to QCoh(S) - mod$ has a left adjoint $p_!$ sending $C \in QCoh(S) - mod$ to $C \in \mathrm{DGCat}_{cont}$, where we forget the $QCoh(S)$-module structure. Indeed, $\mathrm{QCoh}(S)$ is rigid, so we apply ([1], 3.2). Recall that for $A = \mathrm{QCoh}(S)$ the dualization map $\phi_A : A \to A^\vee$ is an isomorphism of left A-modules by ([4], ch. I.1, 9.2.3).

Let $f : S' \to S$ be a map in Sch^{aff} giving $A = \mathrm{QCoh}(S) \to \mathrm{QCoh}(S') = B$. Recall that B is dualizable and self-dual in $A - mod(\mathrm{DGCat}_{cont})$. So, the restriction of scalars functor $B - mod \to A - mod$ is left adjoint to $f^* : A - mod \to B - mod, D \mapsto B \otimes_A D$ by ([1], 3.2). Let us write $f_! : B - mod \to A - mod$ for the functor of restriction of scalars. For $C \in A - mod$ we get a natural map $f_! f^* C \to C$ in $A - mod$.

Let $Y \in \mathrm{PreStk}$, $p : Y \to *$. Does the functor $p^* : ShvCat(*) \to ShvCat(Y)$ has a left adjoint? For $C \in ShvCat(Y)$ consider the functor $\mathrm{Sch}^{aff}_{/Y} \to \mathrm{DGCat}_{cont}$ sending S to $\Gamma(S, Y)$, and a morphism $f : S' \to S$ in $\mathrm{Sch}^{aff}_{/Y}$ to the above map $\Gamma(S', C) \to \Gamma(S, C)$. Passing to the colimit, we get a functor $p_! : ShvCat(Y) \to \mathrm{DGCat}_{cont}$ sending C to $\underset{S \in \mathrm{Sch}^{aff}_{/Y}}{\mathrm{colim}}\ \Gamma(S, C)$.

Lemma 1.0.2. *The functor $p_! : ShvCat(Y) \to \mathrm{DGCat}_{cont}$ is left adjoint to $p^* : ShvCat(*) \to ShvCat(Y)$.*

Proof. For $C \in ShvCat(Y), D \in \mathrm{DGCat}_{cont}$ we have

$$\mathrm{Map}_{ShvCat(Y)}(C, p^* D) \widetilde{\to} \lim_{S \in (\mathrm{Sch}^{aff}_{/Y})^{op}} \mathrm{Map}_{ShvCat(S)}(C\mid_S, D\mid_S) \widetilde{\to}$$

$$\lim_{S \in (\mathrm{Sch}^{aff}_{/Y})^{op}} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(\Gamma(S, C), D)) \widetilde{\to} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(p_! C, D)$$

□

1.0.3. Let now $f: Y \to Z$ be a map in PreStk and $C \in ShvCat(Y)$. If $Z \in \mathrm{Sch}^{aff}$ then for any $S \in \mathrm{Sch}^{aff}_{/Y}$, $\Gamma(S, C) \in \mathrm{QCoh}(Z) - mod$. If $f: S' \to S$ is a map in $\mathrm{Sch}^{aff}_{/Y}$ then the above map $\Gamma(S', C) \to \Gamma(S, C)$ is a morphism in $\mathrm{QCoh}(Z) - mod$. Define $f_! : ShvCat(Y) \to ShvCat(Z)$ by

$$C \mapsto (\underset{S \in \mathrm{Sch}^{aff}_{/Y}}{\mathrm{colim}} \Gamma(S, C)) \in \mathrm{QCoh}(Z) - mod$$

As in Lemma 1.0.2 we see that $f_!$ is left adjoint to f^*.

Let now $g: Z' \to Z$ be a morphism in Sch^{aff} and $f': Y' \to Z'$ be obtained from f by the base change $g: Z' \to Z$. We want to check that $f'_!(C \mid_{Y'}) \widetilde{\to} g^* f_! C$ naturally. We get

$$(1) \qquad \Gamma(Z', g^* f_! C) \widetilde{\to} \underset{S \in \mathrm{Sch}^{aff}_{/Y}}{\mathrm{colim}} \Gamma(S, C) \otimes_{\mathrm{QCoh}(Z)} \mathrm{QCoh}(Z') \widetilde{\to} \underset{S \in \mathrm{Sch}^{aff}_{/Y}}{\mathrm{colim}} \Gamma(S \times_Z Z', C)$$

Lemma 1.0.4. *Let $f: Y \to Z$ be a morphism in* PreStk *with $Z \in \mathrm{Sch}^{aff}$, let $Z' \to Z$ be a map in Sch^{aff}, and $Y' = Y \times_Z Z'$. Then the functor $\mathrm{Sch}^{aff}_{/Y} \to \mathrm{Sch}^{aff}_{/Y'}$, $S \mapsto S \times_Z Z'$ is cofinal.*

Proof. Let $U \overset{h}{\to} Y'$ be an object of $\mathrm{Sch}^{aff}_{/Y'}$. The category $\mathrm{Sch}^{aff}_{/Y} \times_{\mathrm{Sch}^{aff}_{/Y'}} (\mathrm{Sch}^{aff}_{/Y'})_{h/}$ is contractible, because it has an initial object. Indeed, it identifies with the category of diagrams

$$\begin{array}{ccc} U & \overset{\nu}{\to} & S \\ \downarrow & & \downarrow \\ Y' & \to & Y \end{array}$$

with $S \in \mathrm{Sch}^{aff}$. The initial object is the one for which ν is an isomorphism. □

By the above lemma, (1) identifies with $\underset{S' \in \mathrm{Sch}^{aff}_{/Y'}}{\mathrm{colim}} \Gamma(S', C) \widetilde{\to} f'_!(C \mid_{Y'})$.

Corollary 1.0.5. *Let $f: Y \to Z$ be a morphism in* PreStk*. Then the functor $f^*: ShvCat(Z) \to ShvCat(Y)$ admits a left adjoint $f_!$ compatible with arbitrary base change. The functor $f_!$ is given by the following formula. Let $C \in ShvCat(Y)$. If $(Z' \overset{g}{\to} Z) \in \mathrm{Sch}^{aff}_{/Z}$ then $g^* f_! C \widetilde{\to} (f')_! g'^* C$ canonically, where $g': Y' \to Y$ and $f': Y' \to Z'$ are obtained by the base changes as above.*

1.0.6. Let us study the same property for lax prestacks in the sense of [7]. I am using freely the notations of [7, 2]. Let $Y \in \mathrm{PreStk}^{lax}$, $p: Y \to *$. Recall that by formula (4) from [2],

$$ShvCat^{naive}_{/Y} \widetilde{\to} \lim_{(S \to S') \in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Fun}(Y(S'), ShvCat(S))$$

For $C \in ShvCat^{naive}_{/Y}, D \in \mathrm{DGCat}_{cont}$ we get

$$(2) \quad \mathrm{Map}_{ShvCat^{naive}_{/Y}}(C, p^* D) \widetilde{\to} \lim_{(S \overset{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Map}_{\mathrm{Fun}(Y(S'), ShvCat(S))}(C_\alpha, r_\alpha(D))$$

Here $r_\alpha : \mathrm{DGCat}_{cont} \to \mathrm{Fun}(Y(S'), ShvCat(S))$ is the restriction functor. Here C_α is the image of C in $\mathrm{Fun}(Y(S'), ShvCat(S))$.

For each $\alpha \in Tw(\mathrm{Sch}^{aff})$ as above, r_α has a left adjoint sending C_α to $\Gamma(S, \operatorname{colim} C_\alpha)$. Assume given a map $\alpha \to \beta$ in $Tw(\mathrm{Sch}^{aff})$, where $(T \stackrel{\beta}{\to} T') \in Tw(\mathrm{Sch}^{aff})$ let $q : S \to T$ be the corresponding morphism. Consider the corresponding diagram

$$\begin{array}{ccc} \mathrm{DGCat}_{cont} & \stackrel{r_\alpha}{\to} & \mathrm{Fun}(Y(S'), ShvCat(S)) \\ & \searrow r_\beta & \uparrow t \\ & & \mathrm{Fun}(Y(T'), ShvCat(T)), \end{array}$$

where t is the transition functor in our diagram indexed by $Tw(\mathrm{Sch}^{aff})^{op}$. The functor t admits a left adjoint t^L given as the composition

$$\mathrm{Fun}(Y(S'), ShvCat(S)) \stackrel{\bar{q}}{\to} \mathrm{Fun}(Y(S'), ShvCat(T)) \stackrel{LKE}{\to} \mathrm{Fun}(Y(T'), ShvCat(T)),$$

here we denoted by $\bar{q}$ the functor of composition with $q_! : ShvCat(S) \to ShvCat(T)$.

Consider the functor $\Gamma_c^{naive} : ShvCat^{naive}_{/Y} \to \mathrm{DGCat}_{cont}$ given by

$$C \mapsto \Gamma_c^{naive}(Y, C) := \operatorname*{colim}_{(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})} \Gamma(S, \operatorname{colim} C_\alpha) \tag{3}$$

In the above formula $\Gamma(S, \operatorname{colim} C_\alpha) \tilde{\to} \operatorname{colim}_{Y(S')} \Gamma(S, C_\alpha)$ naturally. For a morphism $\alpha \to \beta$ in $Tw(\mathrm{Sch}^{aff})$ as above the corresponding transition map in this system is the natural map $r_\alpha^L C_\alpha \to r_\beta^L C_\beta$ coming from the isomorphism $tC_\beta \tilde{\to} C_\alpha$ by adjointness.

We claim that Γ_c^{naive} is left adjoint to p^*. Indeed, the mapping space (2) identifies with

$$\lim_{(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(\Gamma(S, \operatorname{colim} C_\alpha), D) \tilde{\to} \mathrm{Map}_{\mathrm{DGCat}_{cont}}(\Gamma_c^{naive}(Y, C), D)$$

What is $\Gamma_c^{naive}(Y, \mathrm{QCoh}_Y)$? For $C = \mathrm{QCoh}_Y$ we get for any $\alpha \in Tw(\mathrm{Sch}^{aff})$ as above $C_\alpha : Y(S') \to ShvCat(S)$ is the constant functor with value QCoh_S. We then get $\operatorname{colim} C_\alpha \tilde{\to} \mathrm{Fun}(Y(S')^{inv}, \mathrm{QCoh}(S))$. Here for $\mathcal{E} \in 1-\mathcal{C}\mathrm{at}$ we denote by $\mathcal{E}^{inv}$ the space obtained by inverting all the arrows in $\mathcal{E}$. Indeed, the projection $\mathrm{QCoh}(S)-mod \to \mathrm{DGCat}_{cont}$ preserves colimits. The colimit of a constant functor $\mathcal{E} \to \mathrm{DGCat}_{cont}$ with value D is (by passing to the right adjoint) the limit of the constant functor $\mathcal{E}^{op} \to \mathrm{DGCat}_{cont}$ with value D. The latter is $\mathrm{Fun}((\mathcal{E}^{op})^{inv}, D)$, because the projection $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}\mathrm{at}$ preserves limits (by [1], 10.1.1). Finally, $\mathcal{E}^{inv} \tilde{\to} (\mathcal{E}^{op})^{inv}$ for any $\mathcal{E} \in 1-\mathcal{C}\mathrm{at}$ naturally. So,

$$\Gamma_c^{naive}(Y, \mathrm{QCoh}_Y) \tilde{\to} \operatorname*{colim}_{(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})} \mathrm{Fun}(Y(S')^{inv}, \mathrm{QCoh}(S)),$$

the colimit calculated in DGCat_{cont}. The latter colimit identifies (by passing to right adjoints) with

$$\lim_{(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Fun}(Y(S')^{inv}, \mathrm{QCoh}(S)) \tilde{\to} \mathrm{QCoh}(Y^{inv})$$

The latter formula for $\mathrm{QCoh}(Y^{inv})$ was established in ([2], 0.0.7 and 0.2.14). We have proved actually that $\Gamma_c^{naive}(Y, \mathrm{QCoh}_Y) \tilde{\to} \Gamma^{naive}(Y, \mathrm{QCoh}_Y) \tilde{\to} \mathrm{QCoh}(Y^{inv})$. That is, we may remove "compact support" here.

Since $p^* : ShvCat(*) \to ShvCat(Y)$ is symmetric monoidal, Γ_c^{naive} is left-lax symmetric monoidal (by [4], ch. I.1, 3.2.4).

1.0.7. We claim that (3) can be given also by a simplier formula

$$C \mapsto \underset{S\in \mathrm{Sch}^{aff}}{\mathrm{colim}}\ \underset{y\in Y(S)}{\mathrm{colim}}\ \Gamma(S, y^*C)$$

More precisely, we consider the cartesian fibration $coGroth(Y) \to \mathrm{Sch}^{aff}$ corresponding to $Y : (\mathrm{Sch}^{aff})^{op} \to 1-\mathcal{C}\mathrm{at}$, and the above colimit is over $coGroth(Y)$. Write also $Groth(Y) \to (\mathrm{Sch}^{aff})^{op}$ for the cocartesian fibration corresponding to Y.

As we have seen already in ([2], 0.2.15), we have the natural functor, say $\xi : Tw(\mathrm{Sch}^{aff}) \times_{(\mathrm{Sch}^{aff})^{op}} Groth(Y) \to coGroth(Y)$ sending $(S \stackrel{\alpha}{\to} S', y' \in Y(S'))$ to $(S, y \in Y(S))$, where y is the composition $S \to S' \stackrel{y'}{\to} Y$. This functor has the following property established in *loc.cit.* For any $\mathcal{D} \in 1-\mathcal{C}\mathrm{at}$ admitting small colimits and any $h : coGroth(Y) \to \mathcal{D}$, the canonical map $LKE_\xi(h\circ\xi) \to h$ is an isomorphism. Here LKE_ξ denotes the LKE along ξ.

1.0.8. Let now $f : Y \to Z$ be a morphism in PreStk^{lax}. We want to find a left adjoint $f_!$ to $f^* : ShvCat^{naive}_{/Z} \to ShvCat^{naive}_{/Y}$. First, if $Z \in \mathrm{Sch}^{aff}$ then we still can define $f_!$ by (3), because the corresponding diagram indexed by $Tw(\mathrm{Sch}^{aff})$ takes values in $\mathrm{QCoh}(Z)-mod$ actually.

1.0.9. Assume now $Z \in \mathrm{PreStk}^{lax}$ and $D \in ShvCat^{naive}_{/Z}$, $C \in ShvCat^{naive}_{/Y}$. For $(S \stackrel{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})$, f yields a functor $Y(S') \to Z(S')$, hence an adjoint pair

$$LKE_\alpha : \mathrm{Fun}(Y(S'), ShvCat(S)) \leftrightarrows \mathrm{Fun}(Z(S'), ShvCat(S)) : res_\alpha,$$

where res_α is the restriction. We get

$$\mathrm{Map}_{ShvCat^{naive}_{/Y}}(C, f^*D) \widetilde{\to} \lim_{(S\stackrel{\alpha}{\to}S')\in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Map}_{\mathrm{Fun}(Y(S'),ShvCat(S))}(C_\alpha, (f^*D)_\alpha)$$

Here $(f^*D)_\alpha : Y(S') \to ShvCat(S)$ is the functor $res_\alpha(D_\alpha)$. So, the above limit identifies with

$$\lim_{(S\stackrel{\alpha}{\to}S')\in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Map}_{\mathrm{Fun}(Z(S'),ShvCat(S))}(LKE_\alpha C_\alpha, D_\alpha)$$

One wants to define $f_! : ShvCat^{naive}_{/Y} \to ShvCat^{naive}_{/Y}$ by requiring that it sends C to the projective system in $\lim_{(S\stackrel{\alpha}{\to}S')\in Tw(\mathrm{Sch}^{aff})^{op}} \mathrm{Fun}(Z(S'), ShvCat(S))$, whose value at α is

$$LKE_\alpha(C_\alpha)$$

Is this indeed a projective system? Given a map $\alpha \to \beta$ in $Tw(\mathrm{Sch}^{aff})$ as above, let $t_Z : \mathrm{Fun}(Z(T'), ShvCat(T)) \to \mathrm{Fun}(Z(S'), ShvCat(S))$ be the corresponding transition functor. We have a natural map $LKE_\alpha C_\alpha \to t_T(LKE_\beta C_\beta)$, but it is not clear if it is an equivalence.

To get a rigorous definition of $f_!$ it is better to proceed as follows. Assume given a map $f : Y \to Z$ in PreStk^{lax} with $Z \in \mathrm{Sch}^{aff}$. Let $g : Z' \to Z$ be a morphism in Sch^{aff} and $f' : Y' \to Z'$ be obtained from f by the base change, same for $g' : Y' \to Y$. It suffices to prove the base change property for the functor $f_! : ShvCat^{naive}_{/Y} \to ShvCat(Z)$.

Namely, we have a natural map $f'_!(g'^*C) \to g^* f_! C$, and we want to show it is an isomorphism. The above morphism is

$$\underset{S' \in \mathrm{Sch}^{aff}, y \in Y'(S')}{\mathrm{colim}} \Gamma(S', y^*(g')^*C) \to \mathrm{QCoh}(Z') \otimes_{\mathrm{QCoh}(Z)} f_! C \widetilde{\to} \underset{S \in \mathrm{Sch}^{aff}, y \in Y(S)}{\mathrm{colim}} \Gamma(S \times_Z Z', y^*C)$$

Lemma 1.0.10. *Let $f : Y \to Z$ be a morphism in* PreStk^{lax} *with $Z \in \mathrm{Sch}^{aff}$. Let $g : Z' \to Z$ be a morphism in Sch^{aff} and $g' : Y' \to Y$ be obtained from g by the base change f. Then $R : coGroth(Y) \to coGroth(Y')$, $(S \overset{y}{\to} Y) \mapsto (S \times_Z Z' \to Y')$ is cofinal. Here $coGroth(Y) \to \mathrm{Sch}^{aff}$ is the cartesian fibration corresponding to Y.*

Proof. Consider the functor $L : coGroth(Y') \to coGroth(Y)$ obtained by funtoriality of *coGroth* from $g' : Y' \to Y$. We show that L is left adjoint to R. This suffices by ([4], ch. I.1, 2.2.3).

Indeed, let $(S' \overset{y'}{\to} Y') \in coGroth(Y'), (T \overset{t}{\to} Y) \in coGroth(Y)$. The mapping space in $coGroth(Y)$ from $(S' \overset{y'}{\to} Y' \to Y)$ to $(T \to Y)$ consists of pairs: a map $S' \to T$ in Sch^{aff} such that the diagram commutes

$$\begin{array}{ccc} S' & \to & T \\ \downarrow & & \downarrow \\ Z' & \to & Z \end{array}$$

and a 1-morphism ϵ from $(S' \to Y' \to Y)$ to $(S' \to T \to Y)$ in $Y(S)$.

Set $(T \times_Z Z' \overset{t'}{\to} Y') = R(T \overset{t}{\to} Y)$. The mapping space in $coGroth(Y')$ from (S', y') to $(T \times_Z Z' \overset{t'}{\to} Y')$ consists of pairs: a morphism $S' \to T \times_Z Z'$ over Z' and a 1-morphism ϵ from $(S' \to Y' \to Y)$ to $(S' \to T \to Y)$ in $Y(S)$ as above. Thes spaces are identified. □

Since we have now the base change property, we derive the following.

Corollary 1.0.11. *Let $Y \overset{f}{\to} Z \overset{g}{\leftarrow} Z'$ be a diagram in* PreStk^{lax} *with $g : Z' \to Z$ a morphism in* PreStk. *Then the functor $f^* : ShvCat^{naive}_{/Z} \to ShvCat^{naive}_{/Y}$ admits a left adjoint $f_!$ compatible with the base change by g. That is, the natural map $f'_!(g')^* \to g^* f_!$ for $C \in ShvCat^{naive}_{/Y}$ is an isomorphism in $ShvCat^{naive}_{/Z'}$.*

As in ([7], A.10), for a diagram $X \overset{z}{\to} Z \overset{f}{\leftarrow} Y$ in PreStk^{lax} denote by $Y_{z/}$ the lax prestack sending $S \in \mathrm{Sch}^{aff}$ to the category of triples: $S \overset{x}{\to} X, S \overset{y}{\to} Y$ and a morphism $zx \to fy$ in $Z(S)$. This definition is transitive in the following sense: if $X \in \mathrm{Sch}^{aff}$ and $X' \overset{\alpha}{\to} X$ is a morphism in Sch^{aff} then $X' \times_X Y_{z/} \widetilde{\to} Y_{(z\alpha)/}$ canonically.

So, for each $S \overset{z}{\to} Z$ with $S \in \mathrm{Sch}^{aff}$ denote by $f_z : Y_{z/} \to S$ the projection. Let $C \in ShvCat^{naive}_{/Y}$. Then the sheaves of categories $(f_z)_!(C \mid_{Y_{z/}})$ form a sheaf of categories on Z, so a well-defined object $D \in ShvCat^{naive}_{/Z}$ by Corollary 1.0.11. More precisely, for each $(S \overset{\alpha}{\to} S') \in Tw(\mathrm{Sch}^{aff})$ we get $D_\alpha : Z(S') \to ShvCat(S)$ sending a given map $S' \overset{z'}{\to} Z$ to $(f_{z'\alpha})_!(C \mid_{Y_{z'\alpha/}})$. I think the functor so obtained $f_! : ShvCat^{naive}_{/Y} \to ShvCat^{naive}_{/Z}$ is left adjoint to f^*, but I have not checked this.

2. Sheaves on categories for other sheaf theories

2.0.1. Let $Shv : (\mathrm{Sch}^{aff}_{ft})^{op} \to \mathrm{DGCat}_{cont}$ be one of our 4 sheaf theories from [6]. We consider the functor $(\mathrm{Sch}^{aff}_{ft})^{op} \to 1-\mathfrak{Cat}$, $S \mapsto Shv(S)-mod(\mathrm{DGCat}_{cont})$. It sends $S' \to S$ to the functor $Shv(S)-mod \to Shv(S')-mod$, $M \mapsto M \otimes_{Shv(S)} Shv(S')$.

We right-Kan-extend it along $(\mathrm{Sch}^{aff}_{ft})^{op} \subset (\mathrm{PreStk}_{lft})^{op}$ to a functor $(\mathrm{PreStk}_{lft})^{op} \to 1-\mathfrak{Cat}$, $Y \mapsto ShvCat(Y)$. For a morphism $f : Y' \to Y$ in PreStk_{lft} this gives a restriction $ShvCat(Y) \to ShvCat(Y')$, which I sometimes denote by f^* (this notation was used by Dennis and Sam for quasi-coherent case), or maybe by $C \mid_{Y'}$ for $C \in ShvCat(Y)$. Note that $ShvCat : (\mathrm{PreStk}_{lft})^{op} \to 1-\mathfrak{Cat}$ preserves limits.

For $Y \in \mathrm{PreStk}_{lft}$, $ShvCat(Y)$ admits colimits. For $\mathcal{D}$-modules it also admits limits, because for $T \to S$ in Sch_{ft}, $Shv(T)$ is dualzable as a $Shv(S)$-module category in this case. In the constructible context this is not clear.

The global sections are defined in the usual way. This gives a functor $\Gamma^{enh}_Y : ShvCat(Y) \to Shv(Y)-mod$. It has a left adjoint $\mathrm{Loc}_Y : Shv(Y)-mod \to ShvCat(Y)$ sending C to the sheaf of categories whose sections over $S \to Y$ are $\Gamma(S, \mathrm{Loc}_Y(C)) = C \otimes_{Shv(Y)} Shv(S)$.

Both functors $\Gamma(Y, \cdot) : ShvCat(Y) \to \mathrm{DGCat}_{cont}$ and $\Gamma^{enh}(Y, \cdot) : ShvCat(Y) \to Shv(Y)-mod$ are right-lax symmetric monoidal. Indeed, for $F, F' \in ShvCat(Y)$,

$$\Gamma(Y, F \otimes F') \widetilde{\to} \lim_{S \to Y} \Gamma(S, F \otimes F') \widetilde{\to} \lim_{S \to Y} \Gamma(S, F) \otimes_{Shv(S)} \Gamma(S, F')$$

For each $S \to Y$ in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$ we have the pojection

$$\Gamma(Y, F) \otimes \Gamma(Y, F') \to \Gamma(S, F) \otimes \Gamma(S, F') \to \Gamma(S, F) \otimes_{Shv(S)} \Gamma(S, F')$$

they are compatible with the transition maps, so yield the desired morphism $\Gamma(Y, F) \otimes \Gamma(Y, F') \to \Gamma(Y, F \otimes F')$. It factors through $\Gamma(Y, F) \otimes_{Shv(Y)} \Gamma(Y, F') \to \Gamma(Y, F \otimes F')$.

2.0.2. If $C \in ShvCat(Y)$ consider the functor $\Gamma(\bullet, C) : ((\mathrm{PreStk}_{lft})_{/Y})^{op} \to \mathrm{DGCat}_{cont}$, recall that it is the right Kan extension of its restriction to $((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op}$. Then $\Gamma(\bullet, C)$ preserves limits.

For $f : Z \to Y$ a map in PreStk_{lft} this gives a morphism $\Gamma(Y, C) \to \Gamma(Z, f^*C)$ in DGCat_{cont}.

If Y, Z are preudo-indschemes, and f is pseudo-indproper in the sense of ([7], 7.15.1) then the above restriction functor $\Gamma(Y, C) \to \Gamma(Z, f^*C)$ admits a left adjoint $f_{*,C} : \Gamma(Z, f^*C) \to \Gamma(Y, C)$, see ([3], 3.6.8).

2.0.3. By definition, $Y \in \mathrm{PreStk}_{lft}$ is 1-affine if Loc_Y is an equivalence. Lin Chen have proved that each ind-scheme of ind-finite type is 1-affine. He proved also that for $S, T \in \mathrm{Sch}_{ft}$ and a Zariski cover $T \to S$ the base change functor $Shv(S)-mod \to Shv(T)-mod$ is conservative.

Let $Y \in \mathrm{PreStk}_{lft}$, $C \in ShvCat(Y)$. Then the functor $(\mathrm{Sch}_{ft}/Y)^{op} \to \mathrm{DGCat}_{cont}$, $(S \to Y) \mapsto \Gamma(S, C)$ satisfies Zarizki and proper descent, hence h-descent. Recall that $Shv : (\mathrm{PreStk}_{lft})^{op} \to \mathrm{DGCat}_{cont}$ satisfies the proper and Zarizki descent, etale descent also.

2.0.4. if $C \in Shv(Y)-mod$ is dualizable in $Shv(Y)-mod$ then the natural map $C \to \Gamma(Y, \mathrm{Loc}_Y(C))$ is an isomorphism, as in ([5], 1.3.5).

2.0.5. Recall that $ShvCat(Y)$ is symmetric monoidal. Namely, the limit

$$ShvCat(Y) \widetilde{\to} \lim_{(S\to Y)\in((\mathrm{Sch}^{aff}_{ft})/Y)^{op}} Shv(S)-mod$$

can be understood in $CAlg(1-\mathrm{\mathcal{C}at})$.

For $Y, Z \in \mathrm{PreStk}_{lft}$ we have the functor $ShvCat(Y) \times ShvCat(Z) \to ShvCat(Y \times Z)$ sending (C, D) to $C \boxtimes D := \mathrm{pr}_1^* C \otimes \mathrm{pr}_2^* D$.

Let $C \in ShvCat(Y), D \in ShvCat(Z)$. We have a natural functor

$$\Gamma(Y, C) \otimes \Gamma(Z, D) \to \Gamma(Y \times Z, C \boxtimes D) \tag{4}$$

Indeed, $Y \widetilde{\to} \underset{S\to Y}{\mathrm{colim}}\, S$ in PreStk_{lft}, the colimit taken over $(\mathrm{Sch}^{aff}_{ft})_{/Y}$. So,

$$Y \times Z \widetilde{\to} \underset{(S\to Y, T\to Z)}{\mathrm{colim}} S \times T$$

in PreStk_{lft}, the colimit taken over $(\mathrm{Sch}^{aff}_{ft})_{/Y} \times (\mathrm{Sch}^{aff}_{ft})_{/Z}$. So,

$$\Gamma(Y \times Z, C \boxtimes D) \widetilde{\to} \lim_{(S\to Y, T\to Z)} \Gamma(S \times T, C \boxtimes D)$$

Now $\Gamma(S \times T, C \boxtimes D) \widetilde{\to} (\Gamma(S, C) \otimes \Gamma(T, D)) \otimes_{Shv(S)\otimes Shv(T)} Shv(S \times T)$, and we have a natural map

$$\Gamma(Y, C) \otimes \Gamma(Z, D) \to \Gamma(S, C) \times \Gamma(T, D) \to \Gamma(S \times T, C \boxtimes D)$$

These maps are compatible with the transition maps, so define the morphism (4).

2.0.6. For a morphism $f : Z \to Y$ in PreStk_{lft} we have the functor $cores_f : ShvCat(Y) \to ShvCat(Z)$ tautologically. It is not clear if it has a right adjoint in general.

For $Y = \mathrm{Spec}\, k$ this right adjoint exists, this is $ShvCat(Z) \to \mathrm{DGCat}_{cont}$, $C \mapsto \Gamma(Z, C)$. More generally, if $Y \in \mathrm{PreStk}_{lft}$ is 1-affine then this right adjoint $coind_f$: $ShvCat(Z) \to Shv(Y)-mod$ exists. Indeed, $cores_f$ factors as $Shv(Y)-mod \to Shv(Z)-mod \overset{\mathrm{Loc}_Z}{\to} ShvCat(Z)$, where the first functor sends C to $Shv(Z) \otimes_{Shv(Y)} C$. So its right adjoint is the composition $ShvCat(Z) \overset{\Gamma^{enh}(Z,\bullet)}{\to} Shv(Z)-mod \to Shv(Y)-mod$, where the second functor is the restriction along $Shv(Y) \overset{f^!}{\to} Shv(Z)$.

Assume Y is 1-affine. The base change in general does not hold. Namely, if say $S \in \mathrm{Sch}_{ft}$ and $S \to Y$ is given then we have a natural map $\Gamma(Z, C) \otimes_{Shv(Y)} Shv(S) \to \Gamma(Z \times_Y S, C)$. This is not an isomorphism in general. For example, consider the simplest sheaf of categories $Shv \in ShvCat(Z)$ and $Y = \mathrm{Spec}\, k$. Then the above map becomes $Shv(Z) \otimes Shv(S) \to Shv(Z \times S)$, it is not an isomorphism in the constructible setting.

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