1. Comments to: D. Gaitsgory, The local and global versions of the Whittaker category, version Oct. 9, 2020

1.0.1. For 1.4.5. If $S \in \mathrm{Sch}_{ft}, Y \in \mathrm{PreStk}_{lft}$ then the functor of exterior product $Shv(S) \otimes Shv(Y) \to Shv(S \times Y)$ is fully fauthful for all the 4 sheaf theories, and its right adjoint is continuous ([21], 0.0.28). For $T \in \mathrm{Sch}_{ft}$ let $h_T : Shv(S) \otimes Shv(T) \to Shv(S \times T)$ be the exteriour product functor, then its right adjoint h_T^R identifies with the dual $h^\vee$.

I see this as follows. The composition $Shv(S) \otimes Shv(T) \stackrel{dual \otimes dual}{\to} Shv(S)^\vee \otimes Shv(T)^\vee = \mathrm{Fun}_{k,cont}(Shv(S) \otimes Shv(T), \mathrm{Vect}) \widetilde{\to} \mathrm{Fun}_{k,bi-ex}(Shv(S)^c \times Shv(T)^c, \mathrm{Vect})$ sends L to the functor $(K_S, K_T) \mapsto \mathcal{H}om_{Shv(S) \otimes Shv(T)}(\mathbb{D}(K_S) \otimes \mathbb{D}(K_T), L) \in \mathrm{Vect}$. Here $\mathcal{H}om$ denotes the relative inner hom for the Vect-action on $Shv(S) \otimes Shv(T)$. Indeed, it suffices to check this for L of the form $L_S \boxtimes L_T$ with $L_S \in Shv(S), L_T \in Shv(T)$, where this follows from ([5], ch. I.1, 10.5.8). Here $\mathbb{D} : (Shv(S)^c)^{op} \widetilde{\to} Shv(S)^c$ denotes the Verdier duality and similarlty for T. Now the claim follows from the fact that for $K \in Shv(S)^c, K' \in Shv(T)^c$, $\mathbb{D}(K \boxtimes K') \widetilde{\to} \mathbb{D}(K) \boxtimes \mathbb{D}(K')$.

Dualizing the monoidal structure on $Shv(H)$, one gets a coalgebra structure on $Shv(H)$, where the coproduct $Shv(H) \to Shv(H) \otimes Shv(H)$ is the composition $Shv(H) \stackrel{m^!}{\to} Shv(H \times H) \stackrel{h^R}{\to} Shv(H) \otimes Shv(H)$, here h^R is right adjoint to the exteriour product functor $h : Shv(H) \otimes Shv(H) \to Shv(H)$. In fact, $(Shv(H)*) - mod \widetilde{\to} (Shv(H), m^!) - comod$, see ([1], B.1.2). So, $Shv(Y)$ becomes a $(Shv(H), m^!)$-comodule category. The corresponding cosimplicial category $[n] \mapsto Shv(H)^{\otimes n} \otimes Shv(Y)$ is obtained from the usual cosimplicial category $[n] \mapsto \mathrm{Fun}(Shv(H)^{\otimes n}, Shv(Y))$, whose limit is

$$\mathrm{Fun}_{Shv(H)}(\mathrm{Vect}, Shv(Y)),$$

via the equivalences $\mathrm{Fun}(Shv(H)^{\otimes n}, Shv(Y)) \widetilde{\to} Shv(H)^{\otimes n} \otimes Shv(Y)$ given by the canonical self-duality on $Shv(H)$ (coming from the Verdier duality). For any $T' \to Y/H$, with $T' \in \mathrm{Sch}_{ft}^{aff}$, let $T = Y \times_{Y/H} T'$. Then we get a compatible action of H on T.

Then the collection of functors $h_T^R : Shv(H^n \times T) \to Shv(H)^{\otimes n} \otimes Shv(T)$ yields for any $(T' \to Y/H) \in (\mathrm{Sch}_{ft}^{aff})_{/(Y/H)}$ a morphism of cosimplicial categories

$$\begin{array}{ccccc} Shv(T) & \rightrightarrows & Shv(H \times T) & \substack{\to\\ \to\\ \to} \dots \\ \downarrow \mathrm{id} & & \downarrow h_T^R & \\ Shv(T) & \rightrightarrows & Shv(H) \otimes Shv(T) & \substack{\to\\ \to\\ \to} \dots \end{array}$$

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(because it is a morphism after dualization). Then passing to the limit over $T' \in (\mathrm{Sch}^{aff}_{ft})^{op}_{/(Y/H)}$, one get the desired morphism of cosimplicial categories

$$\begin{array}{ccccc} Shv(Y) & \rightrightarrows & Shv(H\times Y) & \substack{\to\\ \to\\ \to} \dots \\ \downarrow & & \downarrow h^R_Y & \\ Shv(Y) & \rightrightarrows & Shv(H)\otimes Shv(Y) & \substack{\to\\ \to\\ \to} \dots \end{array}$$

Indeed, we have seen in ([21], 0.0.28) that h^R_Y is the limit of h^R_T. Every functor $Shv(H^n\times Y)\to Shv(H)^{\otimes n}\otimes Shv(Y)$ restricted to the full subcategory $Shv(H)^{\otimes n}\otimes Shv(Y)$ becomes the identity functor.

Any object of $\mathrm{Tot}\, Shv(H^\bullet\times Y)$ is given by the collection of objects $\omega_H^{\boxtimes n}\boxtimes F\in Shv(H^\bullet\times Y)$ for some $F\in Shv(Y)$ with the corresponding isomorphisms for any map $[n]\to[m]$ in $\boldsymbol{\Delta}$. For each n we have actually

$$\omega_H^{\boxtimes n}\boxtimes F\in Shv(H)^{\otimes n}\otimes Shv(Y)\subset Shv(H^n\times Y)$$

For two such objects $(\omega_H^{\boxtimes n}\boxtimes F),(\omega_H^{\boxtimes n}\boxtimes F')\in \mathrm{Tot}\, Shv(H^\bullet\times Y)$ the mapping space between them in $\mathrm{Tot}\, Shv(H^\bullet\times Y)$ is

$$\mathrm{Tot}_{[n]\in\boldsymbol{\Delta}}\mathrm{Map}_{Shv(H^\bullet\times Y)}(\omega_H^{\boxtimes n}\boxtimes F,\omega_H^{\boxtimes n}\boxtimes F')\,\widetilde{\to}\,\mathrm{Tot}_{[n]\in\boldsymbol{\Delta}}\mathrm{Map}_{Shv(H)^{\otimes n}\otimes Shv(Y)}(\omega_H^{\boxtimes n}\boxtimes F,\omega_H^{\boxtimes n}\boxtimes F')$$

because $[n]$-th terms lie in the corresponding full subcategories. The latter identifies with the mapping space in $\mathrm{Tot}_{[n]\in\boldsymbol{\Delta}}Shv(H)^{\otimes n}\otimes Shv(Y)$. So, $Shv(Y/H)\to Shv(Y)^H$ is fully faithful.

There is no obvious reason for the essentially surjectivity. We don't have a map of cosimplicial categories in the opposite direction, so to get the essential surjectivity one shows that the cosimplicial category $\mathrm{Fun}(Shv(H)^{\otimes\bullet},Shv(Y))$ satisfies the comonadic Beck-Chevalley conditions.

1.1. **For Appendix B.**

1.1.1. For B.1.3. Av^H_* is defined by applying $\mathrm{Fun}_{Shv(H)}(\cdot,C)$ to the functor $p^*:\mathrm{Vect}\to Shv(H)$ in $Shv(H)-mod$. Here $p:H\to\mathrm{Spec}\,k$. For $F\in Shv(H)$ we get $e_Y*F\,\widetilde{\to}\, e_Y\otimes C(F)$, where $C(F)=p_*F\in\mathrm{Vect}$.

Taking $C=Shv(H)$ with the $Shv(H)$-action by left translations, we get

$$\mathrm{Fun}_{Shv(H)}(\mathrm{Vect},Shv(H))\,\widetilde{\to}\,\mathrm{Fun}_{Shv(H)}(Shv(H),\mathrm{Vect}),$$

where now the first argument $Shv(H)$ is equipped with the $Shv(H)$-action by right translations (by Verdier duality). This gives $\mathrm{Fun}_{Shv(H)}(Shv(H),\mathrm{Vect})\,\widetilde{\to}\,\mathrm{Vect}$.

1.1.2. For B.2.1. Let Y be a scheme (of finite type). We assume that for $p:Y\to\mathrm{Spec}\,k$, $p^*e=e_Y$ exists. The category $Shv(Y)^0$ is compactly generated by e_Y. The fact that $Shv(Y)^0\to C(Y)-mod$ is an equivalence follows from ([20], Lemma 9.2.18). The product map on $C(H)$ is $\mathrm{R}\Gamma(Y\times Y,e)\to\mathrm{R}\Gamma(Y,e)$ is given by $\vartriangle^*\,(e_Y\boxtimes e_Y)\,\widetilde{\to}\, e_Y$, which gives a map $e_Y\boxtimes e_Y\to\vartriangle_* e_Y$, and passing to $\mathrm{R}\Gamma$, the product map. Since Y is a commutative coalgebra in Sch, $C(Y)=p_*(e_Y)$ is a commutative algebra in Vect.

We used the Kunneth formula: for smooth schemes of finite type Y_i,

$$\mathrm{R}\Gamma(Y_1 \times Y_2, e) \widetilde{\to} \mathrm{R}\Gamma(Y_1, e) \otimes \mathrm{R}\Gamma(Y_2, e)$$

So, the functor $(Sch_{smooth,ft})^{op} \to \mathrm{Vect}$, $Y \mapsto \mathrm{R}\Gamma(Y, e)$ is symmetric monoidal.

1.1.3. For B.2.2. The coalgebra structure on $C(H) = \mathrm{R}\Gamma(H, e_Y)$ is as follows. Note that e_H is a coalgebra in $Shv(H)$ for the covolution monoidal structure naturally. The coproduct is given by $e_H \to m_* m^* e_H$ for $m : H \times H \to H$. The functor $p_* : Shv(H) \to \mathrm{Vect}$ for $p : H \to \operatorname{Spec} k$ is monoidal, so $p_* e_H = C(H)$ is a coalgebra in Vect. The compatibility of algebra and coalgebra structure comes from the diagram

$$\begin{array}{ccc} H \times H & \stackrel{\triangle\times\triangle}{\to} & H \times H \times H \times H \\ \downarrow m & & \downarrow m_{13}\times m_{24} \\ H & \stackrel{\triangle}{\to} & H \times H \end{array}$$

1.1.4. For B.2.2. Let us check that the lax monoidal functor $R : Shv(H) \to Shv(H)^0$, the right adjoint to the inclusion, is actually monoidal.

For $F_i \in Shv(H)$ we have a natural map $R(F_1) * R(F_2) \to R(F_1 * F_2)$. To check that this is an isomorphism, it suffices to show that $\mathcal{H}om_{Shv(H)}(e_H, R(F_1) * R(F_2)) \to \mathcal{H}om_{Shv(H)}(e_H, R(F_1 * F_2))$ is an isomorphism. Here $\mathcal{H}om_{\mathcal{C}}(c_1, c_2)$ denotes the inner hom in $\mathcal{C} \in \mathrm{DGCat}_{cont}$ with respect to the Vect-action. Indeed, e_H generates $Shv(H)^0$. This is the map $C(R(F_1)) \otimes C(R(F_2)) \to C(F_1 * F_2)$. However, $C(R(F_i)) \widetilde{\to} C(F_i)$ naturally.

1.1.5. For B.2.3. The full subcategory $Shv(H)^0 \subset Shv(H)$ is preserves by both left and right $Shv(H)$-actions, because for $F \in Shv(H), F * e_H \widetilde{\to} e_H * F \widetilde{\to} e_H \otimes C(F)$.

The functor $R : Shv(H) \to Shv(H)^0$ commutes with left (and right) $Shv(H)$-action by translations. Indeed, for $F, F_1 \in Shv(H)$ we have a natural map $R(F_1) \to F_1$, which gives a morphism $R(F_1 * F) \widetilde{\to} R(F_1) * R(F) \to F_1 * R(F)$. To check that it is an isomorphism, it suffices to apply R, because R is monoidal.

The adjoint pair $i : Shv(H)^0 \leftrightarrows Shv(H) : R$ takes place in $Shv(H) - mod$.

1.1.6. For B.2.3. The fact that $Shv(H)^0$ is dualizable as a left (or right) $Shv(H)$-module follows from ([20], Remark 3.1.4).

In B.2.4 misprint: the right adjoint to the restriction $Shv(H)^0 - mod \to Shv(H) - mod$ along $R : Shv(H) \to Shv(H)^0$ is given by $M \mapsto (Shv(H)^0)^\vee \otimes_{Shv(H)} M = \mathrm{Fun}_{Shv(H)}(Shv(H)^0, M)$ by ([20], 3.2.1). Here the dual is calculated in the category $Shv(H) - mod$. The self-duality of $Shv(H)^0$ is ([20], Remark 3.1.4).

1.1.7. For B.3.1. The existence of Ξ follows from (HTT, 5.3.5.10). The fact that Ξ is fully faithful is (HTT, 5.3.5.11(1)).

1.1.8. For B.3.2. Recall that $C(H) - mod \widetilde{\to} Shv(H)^0$, and this is an equivalence of monoidal categories. In the adjoint pair $\Xi : C(H) - mod \rightleftarrows C(H) - mod^{ren} : \Psi$ both Ψ and Ξ are monoidal functors. For Ξ this follows from $e_H * e_H \widetilde{\to} e_H \otimes C(H)$ in $Shv(H)^0$.

The fact that $(C(H) - mod) - mod \to (C(H) - mod^{ren}) - mod$ is fully faithful comes from ([20], 9.2.32), namely we use the adjoint pair $\Xi : C(H) - mod \rightleftarrows C(H) - mod^{ren} : \Psi$, where both functors are monoidal, and Ξ is fully-faithful.

1.1.9. For B.3.3. The category $C(H)-mod^{ren}$ is rigid, because each $F\in C(H)-mod^{fin-dim}$ admits a left and right dual: this is the dual in Vect with the induced $C(H)$-action?

Note that $e\in C(H)-mod$ is the unit of the symmetric monoidal category $C(H)-mod^{ren}$. It is compact in $C(H)-mod^{ren}$, because any object lying in $C(H)-mod^{fin.dim}$ is compact by construction.

1.1.10. For B.4.1. The equivalence $Shv(B(H))\widetilde{\to}\mathrm{Fun}_{Shv(H)}(\mathrm{Vect},\mathrm{Vect})$ used there without a reference is explained in [21].

The fact that the functor $H-mod\to \mathrm{DGCat}_{cont}$, $C\mapsto C^H=\mathrm{Fun}_{Shv(H)}(\mathrm{Vect},C)$ upgrades to a functor $H-mod\to \mathrm{Fun}_{Shv(H)}(\mathrm{Vect},\mathrm{Vect})-mod$ is a residual 2-categorical phenimenon. Namely, $H-mod$ is a 2-category naturally ([5], ch. I.1, 8.3). The mapping category $\mathbf{Map}(?,\bullet)$ is naturally a module over the endomorphisms category $\mathbf{Map}(?,?)$.

Recall that the composition $Shv(H)\overset{R}{\to} Shv(H)^0\overset{\mathrm{R}\Gamma}{\to}\mathrm{Vect}$ is p_* for $p:H\to\mathrm{Spec}\,k$, see Sect. 1.1.4.

For any $C^r\in Shv(H)^0-mod^r, C\in Shv(H)^0-mod$ viewing them as $Shv(H)$-modules via $R: Shv(H)\to Shv(H)^0$, we get by ([20], 9.2.32)

$$C^r\otimes_{Shv(H)^0} C\widetilde{\to} C^r\otimes_{Shv(H)} C$$

The claim from B.4.1 follows now from the equivalence

$$\mathrm{Vect}\otimes_{Shv(H)^0} Shv(H)^0\otimes_{Shv(H)} C\widetilde{\to} Vect\otimes_{Shv(H)} C$$

together with Th. B.1.2.

1.1.11. For B.4.4. The symmetric monoidal structure on $Shv(B(H))$ given by $\otimes^!$ coincides with the monoidal structure on $\mathrm{Fun}_{(Shv(H),\star)}(\mathrm{Vect},\mathrm{Vect})$ given by the composition.

In the formula (B.8) we may view Vect as a right module over $(Shv(B(H)),\otimes^!)$, where $K\in Shv(B(H))$ acts on $V\in\mathrm{Vect}$ as $(p^!K)\otimes V$. Here $p:\mathrm{Spec}\,k\to B(H)$. We also view Vect there as a left module over $(Shv(H),\star)$, so this is a bimodule, an object of $(Shv(H),\star)-mod-Shv(B(H))$. Then we apply (HA, 4.6.2.1(2)) showing that the functor (B.8) is a left adjoint, and its right adjoint is given by tensoring by $\mathrm{Vect}\in Shv(B(H))-mod-(Shv(H),\star)$.

The corresponding duality datum is given by the functor

$$Shv(B(H))\to\mathrm{Vect}\otimes_{(Shv(H),\star)}\mathrm{Vect}$$

in $Shv(B(H))-mod-Shv(B(H))$. The latter functor is an equivalence, so that the unit of the adjunction is an isomorphism, hence (B.8) is fully faithful.

1.1.12. For B.4.7. By $C.(H)$ he means homology, that is, $C(H)^\vee$, the dual of $C(H)$ in Vect. Since $C(H)\in\mathrm{Vect}^c$, this dual exists. The functor $p^!: Shv(B(H))\to\mathrm{Vect}$ for $p:\mathrm{Spec}\,k\to B(H)$ is comonadic, and the corresponding comonad is $C(H)$. So, $Shv(B(H)\widetilde{\to} C(H)-comod\widetilde{\to} C(H)^\vee-mod$.

My understanding is that the coalgebra $C(H)$ is non-commutative, so the algebra $C(H)^\vee$ is also non-commutative. However, $C(H)^\vee$ is a cocommutative coalgebra. Note that $e\in C(H)-comod$ corresponds to $e_{B(H)}\in Shv(B(H))$.

Let us try a simplier thing. We consider $(Shv(B(H)), \otimes^!)$ as a monoidal category, and want to calculate $\mathrm{Vect}\otimes_{(Shv(B(H)),\otimes^!)}\mathrm{Vect}$. It is the colimit of $Bar^\bullet(\mathrm{Vect}, Shv(B(H)), \mathrm{Vect})$, that is of

$$Shv(B(H))^{\otimes 2} \substack{\to\\ \to\\ \to} Shv(B(H)) \rightrightarrows \mathrm{Vect}$$

Does it satisfy the monadic Beck-Chevalley condition ([6], Def. C.1.5)? Convent that for $n\ge 0$ the functor corresponding to the last face map $\partial_n : [n]\to[n+1]$ is $Shv(B(H))^{\otimes n+1}\to Shv(B(H))^{\otimes n}$ sending $F_1\otimes\ldots\otimes F_{n+1}$ to $F_1\otimes\ldots\otimes F_n\otimes p^!F_{n+1}$.

The action functor $p^! : Shv(B(H))\to \mathrm{Vect}$ admits contunuous right adjoint $p_*[-2m]$ with $m=\dim H$. We have

$$Shv(B(H))\otimes Shv(B(H))\widetilde{\to}(C(H)^\vee-mod)\otimes(C(H)^\vee-mod)\widetilde{\to}C(H)^\vee\otimes C(H)^\vee-mod$$

The monoidal operation $\otimes^!$ on $Shv(B(H))$ becomes the restriction functor $C(H)^\vee\otimes C(H)^\vee-mod\to C(H)^\vee-mod$ via $C(H)^\vee\to C(H)^\vee\otimes C(H)^\vee$. Does it have a continuous right adjoint? Not clear for me.

Here is a proof of Sam:

1) Let Y be a a scheme of finite type with H-action. Write $p: Y\to Y/H$ for the quotient in the sense of prestacks. Let $Shv(Y)^{H-mod}\subset Shv(Y)$ be the full subcategory generated by the image of $p^*: Shv(Y/H)\to Shv(Y)$. We show that $p_*: Shv(Y)^{H-mod}\to Shv(Y/H)$ is monadic. Indeed, the latter functor is continuous, it has the left adjoint $p^*: Shv(Y/H)\to Shv(Y)^{H-mod}$, and this left adjoint generates $Shv(Y)^{H-mod}$. So, $p_*: Shv(Y)^{H-mod}\to Shv(Y/H)$ is conservative, hence monadic. The corresponding monad on $Shv(Y/H)$ is $F\mapsto F\otimes p_*(e)$.

2) Special case $Y=\operatorname{Spec} k$. In this case $p:\operatorname{Spec} k\to B(H)$, let $A=p_*(e)\in Alg(Shv(B(H)),\otimes)$. We get $\mathrm{Vect}\widetilde{\to} A-mod(Shv(B(H),\otimes)$.

3) Let $D\in(Shv(B(H)),\otimes)-mod$. Then by ([5], ch. I.1, 8.5.7),

$$Vect\otimes_{(Shv(B(H)),\otimes)} D\widetilde{\to} A-mod(D)$$

Applying this for $D=\mathrm{Vect}$ with an action of $(Shv(B(H)),\otimes)$ given by p^* for $p:\operatorname{Spec} k\to B(H)$, we get $Vect\otimes_{(Shv(B(H)),\otimes)}\mathrm{Vect}\widetilde{\to}A-mod(\mathrm{Vect})$. The corresponding monad of Vect becomes $C(H)$ as desired.

Consider the functor $\tau:(Shv(B(H)),\otimes)\to(Shv(B(H)),\otimes^!)$ sending K to $K[-2n]$ with $n=\dim H$. This is a monoidal equivalence. Indeed, the diagonal map $\vartriangle: B(H)\to B(H)\times B(H)$ is smooth of relative dimension $n=\dim H$, hence $\vartriangle^!=\vartriangle^*[2n]$. Under this equivalence for $p:\operatorname{Spec} k\to B(H)$ the monoidal functor $p^*:(Shv(B(H)),\otimes)\to\mathrm{Vect}$ identifies with $p^!:(Shv(B(H)),\otimes^!)\to\mathrm{Vect}$. This finishes the proof of Th. B.4.3.

1.2. **For Appendix C.** For C.1.1. In the definition of placidity, the transition maps $Z_\alpha\to Z_\beta$ are assumed smooth surjective and affine. Note that a placid scheme is quasi-compact and quasi-separated.

1.2.1. For C.1.4. If Y is an ind-scheme of ind-finite type then Y is a placid ind-scheme.

1.2.2. For C.1.6 line 1: for the map $f_{\alpha,\beta} : \mathcal{Y}_\alpha \to \mathcal{Y}_\beta$ we must assume that for any $S \in \mathrm{Sch}_{ft}$ and a map $S \to \mathcal{Y}_\beta$ the base change $S \times_{\mathcal{Y}_\beta} \mathcal{Y}_\alpha \to S$ is smooth, affine and surjective morphism, and moreover $S \times_{\mathcal{Y}_\beta} \mathcal{Y}_\alpha$ is of finite type. So, it is better to say that $f_{\alpha,\beta}$ is smooth, surjective, affine and quasi-compact.

The fact that $Z_i \to Z_j$ is a placid closed embedding is obtained as follows. For $i \to j$, $Y_i \times_{Y_j} Z_j \widetilde{\to} Z_i$. Indeed, $Y_i \times_{Y_j} \lim_\alpha Y_{j,\alpha} \widetilde{\to} \lim_\alpha Y_{i,\alpha}$. Here $Y_{i,\alpha} = Y_i \times_{\mathcal{Y}_{\alpha_0}} \mathcal{Y}_\alpha$.

Recall that the category Sch^{aff} of classical affine schemes admits all limits, and $\mathrm{Sch}^{aff} \subset \mathrm{PreStk}$ is closed under limits by HTT. Assume I is filtered, $I^{op} \to \mathrm{Sch}_{ft}$, $i \mapsto Z_i$ is a diagram such that the transition maps $Z_i \to Z_j$ are smooth, surjective and affine. Let $Z = \lim_{i \in I^{op}} Z_i$ is PreStk. We claim that Z is a scheme. Indeed, we may assume I has an initial object i_0. For any open affine subscheme $S \subset Z_{i_0}$, $S \times_{Z_{i_0}} Z_i$ is affine, hence $\lim_{i \in I^{op}} S \times_{Z_{i_0}} Z_i \widetilde{\to} S \times_{Z_{i_0}} Z$ is an affine scheme. Thus, Z admits an open covering by affine schemes, hence $Z \in \mathrm{Sch}$.

1.2.3. Let Z be a placid scheme written as $Z = \lim_{i \in I^{op}} Z_i$ is PreStk, where the transition maps $Z_i \to Z_j$ are smooth, surjective and affine with $Z_i \in \mathrm{Sch}_{ft}$. Let $S \in \mathrm{Sch}_{ft}$ and $f : Z \to S$ be a morphism of schemes. Then there is i such that f factors as $Z \to Z_i \to S$. Indeed, first it suffices to assume S affine. Suppose for each affine open subscheme $U \subset S$, $U \times_S Z \to S$ factors through $U \times_S Z_i$ for some i. Since S is quasi-compact, for a finite subset $I' \subset I$ we pick an element j for which there is a map $i \to j$ in I for any $i \in I'$. Then $Z \to S$ factors through Z_j.

Now assume S affine. We may assume I has an initial object i_0. For any open affine subscheme $V \subset Z_{i_0}$, $V \times_{Z_{i_0}} Z$ is affine, hence the map $V \times_{Z_{i_0}} Z \to S$ factors through $V \times_{Z_{i_0}} Z_i$ for some i. Since Z_{i_0} is quasi-compact, we may pick j large enough in I such that now $V \times_{Z_{i_0}} Z \to S$ factors through $V \times_{Z_{i_0}} Z_j$ for any V in some open covering of Z_{i_0}, and we are done.

Indeed, if there are two morphisms $f, f' : Z_j \to S$ such that the compositions $Z \to Z_j \to S$ coincide then $f = f'$. Indeed, this can checked locally in Zariski topology of Z_{i_0}.

1.2.4. For C.2.4. There $f : \mathcal{Z} \to \mathcal{Z}'$ is any morphism in PreStk, and $\mathcal{Z}, \mathcal{Z}'$ are placid schemes. By the previous subsection, $f_* : Shv(\mathcal{Z}) \to Shv(\mathcal{Z}')$ is well-defined.

Is it true that for a placid scheme $\mathcal{Z}$ written as $Z = \lim \mathcal{Z}_i$, any object $F \in Shv(\mathcal{Z})$ is of the form f^*K for some $i \in I$ and $f : Z \to Z_i$, $K \in Shv(\mathcal{Z}_i)$? Probably not, because $\mathrm{DGCat}_{cont} \to 1 - \mathfrak{Cat}$ does not preserve filtered colimits ([20], 9.2.29).

1.2.5. For a placid closed embedding $f : \mathcal{Z}' \to \mathcal{Z}$ of placid schemes, $f_* : Shv(\mathcal{Z}') \to Shv(\mathcal{Z})$, which was initially defined in C.2.4 admits a different interpretation. Namely, assume $\mathcal{Z} = \lim\limits_{\alpha \in I^{op}} \mathcal{Z}_\alpha$, and for a closed embedding $\mathcal{Z}'_{\alpha_0} \hookrightarrow \mathcal{Z}_{\alpha_0}$ we have $\mathcal{Z}' = \mathcal{Z}'_{\alpha_0} \times_{\mathcal{Z}_{\alpha_0}} \mathcal{Z}$.

Then we may assume α_0 is an initial object of I. Then for any $\alpha \in I$ set $\mathcal{Z}'_\alpha = \mathcal{Z}'_{\alpha_0} \times_{\mathcal{Z}_{\alpha_0}} \mathcal{Z}_\alpha$, so that $\mathcal{Z}' = \lim \mathcal{Z}'_\alpha$, and $Shv(\mathcal{Z}') \widetilde{\to} \operatorname*{colim}\limits_{\alpha \in I^{op}} Shv(\mathcal{Z}'_\alpha)$. For each α consider the closed embedding $i_\alpha : \mathcal{Z}'_\alpha \hookrightarrow \mathcal{Z}_\alpha$. The functors $(i_\alpha)_*$ are compatible with the corresponding colimit systems, and yield a functor $\operatorname{colim} Shv(\mathcal{Z}'_\alpha) \to \operatorname{colim} Shv(\mathcal{Z}_\alpha)$, which is f_*.

In addition, consider for $\alpha \to \beta$ in I the diagram

$$\begin{array}{ccc} \mathcal{Z}'_\beta & \stackrel{i_\beta}{\to} & \mathcal{Z}_\beta \\ \downarrow \pi'_{\alpha,\beta} & & \downarrow \pi_{\alpha,\beta} \\ \mathcal{Z}'_\alpha & \stackrel{i_\alpha}{\to} & \mathcal{Z}_\alpha \end{array}$$

We get $i^!_\alpha(\pi_{\alpha,\beta})_* \widetilde{\to} (\pi'_{\alpha,\beta})_* i^!_\beta$. So, the functors $i^!_\alpha$ form a morphism of the corresponding inverse systems, yielding a functor $f^! : Shv(\mathcal{Z}) \to Shv(\mathcal{Z}')$. It is the right adjoint to f_* by ([5], ch. I.1, 2.6.4).

Proposition 1.2.6. *Let Z be a quasi-compact quasi-separated scheme, Y a placid ind-scheme written as $Y \widetilde{\to} \operatorname{colim}_{i\in I} Y_i$ with I filtered, Y_i placid scheme, and $Y_i \to Y_j$ a placid closed immersion for $i \to j$ in I. Let $f : Z \to Y$ be a morphism in* PreStk *then there is $i \in I$ such that it factors through $Z \to Y_i \to Y$.*

Proof. Step 1. We reduce this to the case of affine Z. Assume this known for Z affine. Since Z is quasi-compact, we pick an affine cover $Z = \cup_\beta S_\beta$. Each map $S_\beta \to Y$ factors through some Y_i, we may assume this is the same i for all β. Since Z is quasi-separated, our claim follows from the next lemma.

Lemma 1.2.7. *Let S be an affine scheme, $\gamma, \gamma' : S \to Y_i$ two maps such that the compositions $S \to Y_i \to Y$ are the same. Then there is $i \to j$ in I such that the compositions $S \to Y_i \to Y_j$ are the same.*

Proof. *Sets* $\subset$ Spc is closed under filtered colimits, so $Y(S)$ is a set. Filtered colimits in *Sets* are obtained as a quotient by the equvalence relation of $\sqcup_i Y_i(S)$ by $(i, \gamma) \sim (i', \gamma')$ iff there is a diagram $i \to j \leftarrow i'$ such that the images of γ and γ' in $Y_j(S)$ are equal. □

Step 2. Assume Z affine. Then any object of $Y(Z)$ comes from some element of $Y_i(Z)$ by ([20], 13.1.14). □

For example, if Z is a placid scheme then Z is quasi-compact and quasi-separated, so satisfies the assumptions of Proposition 1.2.6. Another example is when Y is a ind-scheme of ind-finite type.

1.2.8. For C.2.7. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between placid ind-schemes. Write $\mathcal{Y}_2 = \operatorname{colim}_{i\in I} \mathcal{Y}^i_2$ and $\mathcal{Y}_1 = \operatorname{colim}_{j\in J} \mathcal{Y}^j_1$ with I, J filtered.

The composition $\mathcal{Y}^j_1 \to \mathcal{Y}_1 \to \mathcal{Y}_2$ factors through $f_{ji} : \mathcal{Y}^j_1 \to \mathcal{Y}^i_2$ for some i by the above, so we get a morphism $(f_{ji})_* : Shv(\mathcal{Y}^j_1) \to Shv(\mathcal{Y}^i_2)$. Composing with $(\gamma_i)_* : Shv(\mathcal{Y}^i_2) \to Shv(\mathcal{Y}_2)$ for $\gamma_i : \mathcal{Y}^i_2 \to \mathcal{Y}_2$, we get a morphism $Shv(\mathcal{Y}^j_1) \to Shv(\mathcal{Y}_2)$. This is a compatible system, hence induces a functor $Shv(\mathcal{Y}_1) \widetilde{\to} \operatorname{colim}_{j\in J} Shv(\mathcal{Y}^j_1) \to Shv(\mathcal{Y}_2)$.

1.2.9. For C.2.8. Fully-fathfulness follows from ([20], 2.2.17). Now write $Y_j = \lim_{i\in I^{op}_j} Y^i_j$, where Y^i_j is a scheme of finite type, and the transition maps are smooth affine surjective. Then $Shv(Y_1) \otimes Shv(Y_1)$ is compactly generated by objects of the form $F_1 \otimes F_2$ with $F_1 \in Shv(Y^i_1)^c, i \in I_1$, $F_2 \in Shv(Y^j_2)^c, j \in I_2$ for some i, j. Since then $F_1 \boxtimes F_2 \in Shv(Y^i_1 \times Y^j_2)$ is compact, we see that the functor $Shv(Y_1) \otimes Shv(Y_2) \to Shv(Y_1 \times Y_2)$ has a continuous right adjoint.

1.2.10. For C.2.9. The functors $(f_{\alpha,\beta})_*$ exist, because $f_{\alpha,\beta}$ is schematic quasi-compact, and $\mathcal{Y}_\alpha, \mathcal{Y}_\beta \in \mathrm{PreStk}_{lft}$. The functor $f^*_{\alpha,\beta}$ exists, because $f_{\alpha,\beta}$ is smooth.

In the situation of Section C.1.6, let $\mathcal{Y}_{\alpha,i} = Y_i \times_{\mathcal{Y}_{\alpha_0}} \mathcal{Y}_\alpha$. The maps $\mathcal{Z}_i = \lim_\beta \mathcal{Y}_{\beta,i} \to \mathcal{Y}_{\alpha,i} \to \mathcal{Y}_\alpha$ form a compatible system over $i \in I$, hence yield a morphism $f_\alpha : \mathcal{Y} = \mathrm{colim}_i \mathcal{Z}_i \to \mathcal{Y}_\alpha$. Since f_α is a map between placid ind-schemes, $(f_\alpha)_*$ exists.

We have a morphism

$$\operatorname*{colim}_{i\in I} \lim_{\alpha\in \mathcal{J}^{op}} \mathcal{Y}_{\alpha,i} \to \lim_{\alpha\in\mathcal{J}^{op}} \operatorname*{colim}_{i\in I} \mathcal{Y}_{\alpha,i},$$

that is, the map $\mathcal{Y} \to \lim_{\alpha\in\mathcal{J}^{op}} \mathcal{Y}_\alpha$. Is it an isomorphism in PreStk?

By definition, $Shv(\mathcal{Y}) \widetilde{\to} \mathrm{colim}_{i\in I} \mathrm{colim}_\alpha Shv(\mathcal{Y}_{\alpha,i})$, where we use for the transition map $\tau_{\alpha,\beta} : \mathcal{Y}_{\alpha,i} \to \mathcal{Y}_{\beta,i}$ the functor $\tau^*_{\alpha,\beta}$. So,

$$Shv(\mathcal{Y}) \widetilde{\to} \mathrm{colim}_\alpha \mathrm{colim}_{i\in I} Shv(\mathcal{Y}_{\alpha,i}) \widetilde{\to} \mathrm{colim}_\alpha Shv(\mathcal{Y}_\alpha)$$

Here for $\alpha \to \beta$ and $f_{\alpha,\beta} : \mathcal{Y}_\alpha \to \mathcal{Y}_\beta$ we are using $f^*_{\alpha,\beta} : Shv(\mathcal{Y}_\beta) \to Shv(\mathcal{Y}_\alpha)$. The functor $f^*_{\alpha,\beta}$ has a continuous right adjoint $(f_{\alpha,\beta})_*$, because $f_{\alpha,\beta}$ is schematic quasi-compact. So, we may pass to right adjoints and get $Shv(\mathcal{Y}) \widetilde{\to} \lim_\alpha Shv(\mathcal{Y}_\alpha)$.

1.2.11. Let Z be a placid scheme written as $Z = \lim_{i\in I^{op}} Z_i$. For $i \to j$ in I let $f_{ij} : Z_j \to Z_i$ be the corresponding morphism, it is smooth of relative dimension d_{ij}, affine, surjective. Since $Shv(Z) \widetilde{\to} \mathrm{colim}_i Shv(Z_i)$ via the maps f^*_{ij}, $Shv(Z)$ is compactly generated, hence dualizable. By ([5], ch. I.1, 6.3.4), by applying the dualization functor to the functor

$$I \to \mathrm{DGCat}_{cont}, i \mapsto Shv(Z_i),\ (i \to j) \mapsto f^*_{ij},$$

we get a functor $I^{op} \to \mathrm{DGCat}_{cont}, i \mapsto Shv(Z_i), (i \to j) \mapsto (f_{ij})_*[-2d_{ij}]$. Moreover,

$$Shv(Z)^\vee \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)$$

with respect to the transition maps $(f_{ij})_*[-2d_{ij}]$. Consider for $i \in I$ the isomorphism $Shv(Z_i) \overset{\otimes e[2d_i]}{\to} Shv(Z_i)$ with $d_i = \dim Z_i$. So, $d_{ij} = d_j - d_i$. The diagram commutes

$$\begin{array}{ccc} Shv(Z_j) & \overset{\otimes e[2d_j]}{\to} & Shv(Z_j) \\ \downarrow (f_{ij})_* & & \downarrow (f_{ij})_*[-2d_{ij}] \\ Shv(Z_i) & \overset{\otimes e[2d_i]}{\to} & Shv(Z_i) \end{array}$$

Passing to the limit over I^{op}, we obtain an equivalence $Shv(Z) \widetilde{\to} Shv(Z)^\vee$. So, for each placid scheme Z, $Shv(Z)$ is canonically self-dual.

However, if Z is a scheme of finite type, this self-duality is not the same as the usual one for a finite type scheme. Everything can be here over a base scheme $S \in Sch_{ft}$, which is not necessary of pure dimension. Then d_i is such that $Z_i \to S$ is smooth of relative dimension d_i.

1.2.12. Let now Z, Z' be placid schemes and $i : Z' \to Z$ a placid closed immersion. What is the dual of the adjoint pair $i_* : Shv(Z') \to Shv(Z) : i^!$?

We explain the dual of i_*. If $Z = \lim_{i\in I^{op}} Z_i$ and, assume for simplicity I has an initial object i_0 such that $Z' = Z'_{i_0} \times_{Z_{i_0}} Z$. So, $Z' = \lim_{i\in I^{op}} Z'_i$ with $Z'_i = Z_i \times_{Z_{i_0}} Z'_{i_0}$. For $i \to j$ in I let $f_{ij} : Z_j \to Z_i$ be the corresponding transition map. For the closed embeddings $i_i : Z'_i \to Z_i$ writing $Shv(Z) = \lim_{i\in I^{op}} Shv(Z_i)$ for $(f_{ij})_* : Shv(Z_j) \to Shv(Z_i)$ and similarly for $Shv(Z')$, the dual functor is given by the collection of functors $i_i^![2d_i - 2d'_i] : Shv(Z_i) \to Shv(Z'_i)$, here $d_i = \dim Z_i, d'_i = \dim Z'_i$ as locally constant functions, they form a morphism of the corresponding inverse systems. The number $d_i - d'_i$ does depend on i, and can be denoted $\operatorname{codim}_Z(Z') = d_i - d'_i$. So, the dual of $i_* : Shv(Z') \to Shv(Z)$ is $i^![2\operatorname{codim}_Z(Z')]$.

1.3. For C.3.1. Here K_n is the kernel of $G(\mathcal{O}) \to G(\mathcal{O}/t^n)$. Note for $n < m$ then $K_m \subset K_n$ is a subgroup, hence a map $f_{n,m} : \mathfrak{L}(G)/K_m \to \mathfrak{L}(G)/K_n$. The functor $f^*_{n,m} : Shv(\mathfrak{L}(G)/K_n) \to Shv(\mathfrak{L}(G)/K_m)$ exists, see my previous section (and [9], C.2.9), and $Shv(\mathfrak{L}(G)) = \operatorname{colim}_n Shv(\mathfrak{L}(G)/K_n)$, where the transition functors are $f^*_{n,m}$.

If e_{K_n} denotes the sheaf e extended by zero under $K_n/K_n \hookrightarrow \mathfrak{L}(G)/K_n$. Let $f_n : \mathfrak{L}(G) \to \mathfrak{L}(G)/K_n$ be the natural map. Then $(f_n)_*\delta_1 \widetilde{\to} e_{K_n}$. By ([20], 9.2.6), the natural map $\operatorname{colim}_n (f_n)^*(f_n)_*\delta_1 \to \delta_1$ is an isomorphism.

1.3.1. For 1.5.3. The formula for the "new action" is wrong. Here is the correct formula.

If H is a smooth group scheme of finite type, L is a local system on H equipped with associative isomorphism $m^*L \widetilde{\to} L \boxtimes L$ and a compatible trivialization $i^*L \widetilde{\to} e$ for $i : \operatorname{Spec} k \to H$ then $f : (Shv(H), \star) \to (Shv(H), \star)$, $F \mapsto F \otimes L^{-1}$ is a monoidal functor. Indeed,

$$f(F_1 * F_2) \widetilde{\to} (F_1 * F_2) \otimes L^{-1} \widetilde{\to} (F_1 \otimes L^{-1}) * (F_2 \otimes L^{-1}) = f(F_1) * f(F_2)$$

Now given $C \in (Shv(H), *) - mod$, we twist the action by L as follows. The object $C_L \in H - mod$ is defined as $C \in \mathrm{DGCat}_{cont}$ with the new action given by $Shv(H) \otimes C \stackrel{f\otimes \mathrm{id}}{\to} Shv(H) \otimes C \stackrel{act}{\to} C$.

For the trivial action of H on Vect we get Vect_L. This is Vect equipped with the action of $Shv(H)$, where $F \in Shv(H)$ sends F to $C^\cdot(H, F \otimes L^{-1})$. We have the adjoint pair $g^L : \mathrm{Vect}_L \leftrightarrows Shv(H) : g$, where $g(F) = C^\cdot(H, F \otimes L^{-1})$, and $(g^L)(V) = V \otimes L$. We view it as an adjoint pair in $(Shv(H), *) - mod$, where the $(Shv(H), *)$-action on Vect_L is given by g. Recall that g is monoidal. For $C \in H - mod$ applying the functor $\mathrm{Fun}_{Shv(H)}(\bullet, C)$ we get for $C^{H,L} := \mathrm{Fun}_{Shv(H)}(\mathrm{Vect}_L, C)$ the adjoint pair

$$\mathrm{oblv}_{H,L} : C^{H,L} \leftrightarrows C : Av_*^{H,L}$$

The composition $\mathrm{oblv}_{H,L} \circ Av_*^{H,L} : C \to C$ is $c \mapsto (e_H \otimes L) * c$. Note that $C^{H,L} \widetilde{\to} (C_{L^{-1}})^H$.

Remark 1.3.2. *If we are not in the constructible context, it is better not to use the $\otimes$, but only $\otimes^!$. For this we should start with an object $\mathcal{L} \in Shv(H)$ invertible for the $\otimes^!$-monoidal structure and satisfying $m^!\mathcal{L} \widetilde{\to} \mathcal{L} \boxtimes \mathcal{L}$ associatively and $i^!\mathcal{L} \widetilde{\to} e$ for $i : Speck \stackrel{1}{\to} H$. Then the functor $(Shv(H), *) \to (Shv(H), *)$, $F \mapsto F \otimes^! \mathcal{L}$ is a monoidal equivalence. If moreover we are in the constructible context then for a multiplicative local system L in the usual sense, $\mathcal{L} := L \otimes \omega_H$ satisfies the above properties.*

Lemma 1.3.3. *Let H be a unipotent group scheme of finite type (or pro-unipotent), $C \in H - mod$ and L is a character local system on H, so for $m : H \times H \to H$, $m^*L \widetilde{\to} L \boxtimes L$. In this case the functor $\mathrm{oblv}_{H,L} : C^{H,L} \to C$ is fully faithful.*

Proof. In the notations of the previous section, for $g^L : \mathrm{Vect}_L \leftrightarrows Shv(H) : g$ the unit of the adjunction $\mathrm{id} \to gg^L$ is $C^{\cdot}(H) \widetilde{\to} e$. So, g^L is fully faithful. This gives $Av_*^{H,L} \circ \mathrm{oblv}_{H,L} \widetilde{\to} \mathrm{id}$. □

If U is a pro-unipotent group scheme then $C^{\cdot}(U) \widetilde{\to} e$. Indeed, e comes as the object $(e_{U_i}) \in \lim Shv(U_i)$ for $U = \lim U_i$, where e_{U_i} is the constant sheaf on U_i.

1.3.4. Let H, G be placid group ind-schemes and $f : H \to G$ a homomorphism of groups in PreStk. Then the functor $f_* : (Shv(H), *) \to (Shv(G), *)$ is monoidal. So, if $C, D \in G - mod$, we get the morphism of inverse systems

$$\begin{array}{ccccc} \mathrm{Fun}(D, C) & \rightrightarrows & \mathrm{Fun}(Shv(G) \otimes D, C) & \substack{\to\\ \to\\ \to} & \mathrm{Fun}(Shv(G)^{\otimes 2} \otimes D, C) \dots \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Fun}(D, C) & \rightrightarrows & \mathrm{Fun}(Shv(H) \otimes D, C) & \substack{\to\\ \to\\ \to} & \mathrm{Fun}(Shv(H)^{\otimes 2} \otimes D, C) \dots \end{array}$$

given by composing with the maps $(f_*)^{\otimes n} : Shv(H)^{\otimes n} \to Shv(G)^{\otimes n}$. This yields a morphism between the totalizations

$$\mathrm{Fun}_{Shv(G)}(D, C) \to \mathrm{Fun}_{Shv(H)}(D, C)$$

Assume now $G = \mathrm{colim}_{i \in I} G_i$ in PreStk, where I is filtered, each G_i is a placid scheme, a group scheme, and for $i \to j$ in I the map $i_{ij} : G_i \to G_j$ is a placid closed embedding, and a homomorphism of group schemes. Then the natural map

$$\mathrm{Fun}_{Shv(G)}(D, C) \to \lim_{i \in I^{op}} \mathrm{Fun}_{Shv(G_i)}(D, C)$$

is an equivalence. Indeed, I is sifted, so $\mathrm{colim}_{i \in I} Shv(G_i)^{\otimes n} \widetilde{\to} Shv(G)^{\otimes n}$, where we use $(i_{ij})_* : Shv(G_i) \to Shv(G_j)$ as the transition maps. We get

$$\mathrm{Fun}_{Shv(G)}(D, C) \widetilde{\to} \lim_{[n] \in \boldsymbol{\Delta}} \mathrm{Fun}(Shv(G)^{\otimes n} \otimes D, C) \widetilde{\to} \lim_{[n] \in \boldsymbol{\Delta}} \mathrm{Fun}(\mathrm{colim}_i\, Shv(G_i)^{\otimes n} \otimes D, C)$$
$$\widetilde{\to} \lim_{i \in I^{op}} \lim_{[n] \in \boldsymbol{\Delta}} \mathrm{Fun}(Shv(G_i)^{\otimes n} \otimes D, C) \widetilde{\to} \lim_{i \in I^{op}} \mathrm{Fun}_{Shv(G_i)}(D, C)$$

This implies the formula (2.1) in 2.1.2. More generally, if $C \in G - mod$ then $C^G \widetilde{\to} \lim_{i \in I^{op}} C^{G_i}$.

1.3.5. Let $G = \lim_{i \in I^{op}} G_i$, where I is filtered, G_i is a smooth group scheme of finite type, for $i \to j$ the map $\pi_{ji} : G_j \to G_i$ is a homomorphism of group schemes, which is smooth, affine, surjective. So, G is a placid scheme. For $i \to j$ in I the functor $(\pi_{ij})_* : Shv(G_j) \to Shv(G_i)$ is monoidal, and $Shv(G) \widetilde{\to} \lim_{i \in I^{op}} Shv(G_i)$ in $Alg(\mathrm{DGCat}_{cont})$.

1.3.6. Let G be a group scheme, which is a placid scheme, $C \in G - mod$. Consider the cosimplicial category defining C^G:

$$\mathrm{Fun}(\mathrm{Vect}, C) \rightrightarrows \mathrm{Fun}(Shv(G), C) \substack{\to\\ \to\\ \to} \mathrm{Fun}(Shv(G)^{\otimes 2}, C) \dots$$

Let us show that it satisfies the comonadic Beck-Chevalley conditions.

The functor corresponding to the last face map $\partial_n : [n] \to [n+1]$ (its image avoids $n+1$) is the following functor F_n. We consider $Shv(G)^{\otimes n+1} \to Shv(G)^{\otimes n}$, $\mathrm{id} \otimes \mathrm{R}\Gamma$, and compose it with $\mathrm{Fun}(\cdot, C)$. For $p : G \to \operatorname{Spec} k$ the functor p_* has a left adjoint p^*. Let T_n be the functor obtained from $Shv(G)^{\otimes n} \to Shv(G)^{\otimes n+1}$, $\mathrm{id} \otimes p^*$ by composing with $\mathrm{Fun}(\cdot, C)$. Then T_n is the right adjoint to F_n. Let now $\alpha : [m] \to [n]$ be a map in $\boldsymbol{\Delta}$. Consider the corresponding diagram

$$\begin{array}{ccc} \mathrm{Fun}(Shv(G)^{\otimes n}, C) & \stackrel{T_n}{\leftarrow} & \mathrm{Fun}(Shv(G)^{\otimes n+1}, C) \\ \uparrow F_\alpha & & \uparrow F_{\alpha+1} \\ \mathrm{Fun}(Shv(G)^{\otimes m}, C) & \stackrel{T_m}{\leftarrow} & \mathrm{Fun}(Shv(G)^{\otimes m+1}, C) \end{array}$$

We show that it commutes. It suffices to prove this for α injective, becase of the following. Let $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ be the full subcategory with the same class of object, where we keep only injective maps. Then $\boldsymbol{\Delta}_s \to \boldsymbol{\Delta}$ is cofinal by [15]. If $\alpha : [m] \to [n]$ is injective, and $0, n$ are in the image then the desired commutativity follows from the commutativity of

$$\begin{array}{ccc} Shv(G)^{\otimes n} & \stackrel{p^* \otimes \mathrm{id}}{\to} & Shv(G)^{\otimes n+1} \\ \downarrow (m_\alpha)_* & & \downarrow (m_{\alpha+1})_* \\ Shv(G)^{\otimes m} & \stackrel{p^* \otimes \mathrm{id}}{\to} & Shv(G)^{\otimes m+1}, \end{array}$$

where $(m_\alpha)_*$ is the product along α in the monoidal category $Shv(G)$.

If $\alpha : [n-1] \to [n]$ is the last face map then $\alpha + 1 : [n] \to [n+1]$ avoids n. The functor $F_{\alpha+1}$ is the composition with $Shv(G)^{\otimes n+1} \to Shv(G)^{\otimes n}$, $K_1 \otimes \ldots \otimes K_{n+1} \mapsto K_1 \otimes \ldots \otimes K_{n-1} \otimes K_n * K_{n+1}$. In this case the desired commutativity follows from $K * e_G \widetilde{\to} \mathrm{R}\Gamma(G, K) \otimes e_G$.

If $\alpha : [n-1] \to [n]$ is injective and avoids 0 then F_α sends f to the functor

$$K_1 \otimes \ldots \otimes K_n \mapsto K_1 * f(K_2 \otimes \ldots \otimes K_n)$$

and the commutativity is tautological. So, it always hold.

By ([6], Lemma C.1.9), the functor $\mathrm{oblv}_G : C^G \to C$ is comonadic, and the corresponding monad on C is $C \to C, c \mapsto e_G * c$. We conclude that $C^G \widetilde{\to} e_G - comod(C)$.

Does the same holds if $G \in Grp(\mathrm{PreStk})$ is only a placid ind-scheme? I and Sam think no, namely if G is not a group scheme then the comonadic Beck-Chevalley conditions do not hold for the above cosimplicial diagram, because the left adjoint to $\mathrm{R}\Gamma : Shv(G) \to \mathrm{Vect}$ does not exist.

Lemma 1.3.7. *Let be given an exact sequence $1 \to U \to G \stackrel{q}{\to} H \to 1$ of group schemes, where G, H, U are placid schemes. Assume U is pro-unipotent. Let $C \in H-mod$, which we view by restriction as an object of $G - mod$. Then $C^G \widetilde{\to} C^H$ canonically.*

Proof. We apply Section 1.3.6 and get $C^H \widetilde{\to} e_H - comod(C)$ where $e_H \in Shv(H)$ is the constant sheaf, and similarly, $C^G \widetilde{\to} e_G - comod(C)$. We have a morphism of monad in one direction, and we check that the corresponding monads in $\mathrm{Fun}(C, C)$ are the same. This follows from $q_* e_G \widetilde{\to} e_H$. □

1.3.8. Let $G \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, let G be a placid ind-scheme, $C \in G - mod$. The functor $\mathrm{oblv}_G : C^G \to C$ is comonadic. Indeed, It has a right adjoint oblv_G^R, because $ev^0 : C^G \to C$ in the corresponding cosimplicial category is continuous, hence admits a maybe discontinuous right adjoint. By (HA, 4.7.5.1), oblv_G is comonadic, so $C^G \widetilde{\to} E - comod(C)$, where $E = \mathrm{oblv}_G \circ \mathrm{oblv}_G^R$. In general, oblv_G^R is not continuous.

1.3.9. Let $N\in Grp(\mathrm{PreStk})$ be a placid ind-scheme. Assume $N=\mathrm{colim}_{i\in I} N_i$, where each N_i is a placid scheme, and a group scheme. Assume I filtered. Assume for $i\to j$ the map $N_i\to N_j$ is a placid closed immersion and a homomorphism of group schemes. Let $C\in N-mod$. Recall that, by Sect. 1.3.4, $C^N\widetilde{\to}\lim_{i\in I^{op}} C^{N_i}$.

Assume each N_i is pro-unipotent. Then $C^{N_i}\subset C$ is a full subcategory. Then $C^N\widetilde{\to}\cap_i C^{N_i}$ as full subcategories in C by ([20], 2.7.7), because the forgetful functor $\mathrm{DGCat}_{cont}\to 1-\mathcal{C}at$ preserves limits.

1.3.10. Let Z be a placid scheme written as $\lim_{i\in I^{op}} Z_i$, where Z_i is a smooth scheme of finite type over a base scheme $S\in \mathrm{Sch}_{ft}$, and for $i\to j$ the map $Z_j\to Z_i$ is smooth, affine and surjective. Write $p: Z\to \operatorname{Spec} S$ for the structure map. Under the self-duality of $Shv(Z)$ from Section 1.2.11, the dual to $p_*: Shv(Z)\to Shv(S)$ identifies with $p^*: Shv(S)\to Shv(Z)$. So, p_* identifies with $(p^*)^\vee$.

Let U be a pro-unipotent group scheme, $C\in U-mod$. Then we have the adjoint pair $p_*: Shv(U)\leftrightarrows \mathrm{Vect}: (p_*)^R$ in the constructible context. However, in general $(p^*)^R$ is not a strict morphism of $Shv(U)$-modules. In general for $C\in Shv(U)-mod$, the left adjoint to $\mathrm{oblv}_U: C^U\to C\,\mathrm{oblv}_U$ does not exist in the constructible context, see ([22], after 1.12.3).

1.3.11. Let G be a placid group scheme, $G=\lim_{i\in I^{op}} G_i$, where G_i is smooth group scheme of finite type, for $i\to j$ in I the map $f_{ij}: G_j\to G_i$ is smooth, affine and surjective homomorphism, and I is filtered category. Assume I has an initial object i_0. Let $f_{i_0}: G\to G_{i_0}$ be the projection, L_0 a character local system on G_{i_0}. We assume it is invertible in $(Shv(G_{i_0}),\otimes)$. Let $L=f_{i_0}^*L_0$ be a character local system on G.

Consider the action of G on itself by left translations.

Claim. The continuous functor $h:\mathrm{Vect}\to Shv(G)^{G,L}$ sending e to L is an equivalence.

Proof. Write $Shv(G)\widetilde{\to}\lim_{i\in I^{op}} Shv(G_i)$, where we use $(f_{ij})_*$ as the transition functors. For any i, G acts on G_i, and f_{ij} are G-equivariant, so $(f_{ij})_*$ is a map in $G-mod$. Thus, $Shv(G)^{G,L}\widetilde{\to}\lim_{i\in I^{op}} Shv(G_i)^{G,L}$. By Lemma 1.3.7, $Shv(G_i)^{G,L}\widetilde{\to} Shv(G_i)^{G_i,L_i}$, where $L_i=f_{i_0 i}^*L_0$. Thus, it suffices to establish the equivalence $Shv(G_i)^{G_i,L_i}\widetilde{\to}\mathrm{Vect}$. So, we are reduced to the case, where G is a smooth group scheme of finite type. We assume this.

By Section 1.3.6, $Shv(G)^{G,L}\widetilde{\to} L-comod(Shv(G))$. Here we use the comonad $K\mapsto L*K$. Note that L is a coalgebra in $Shv(G)$, hence $L\in L-comod(Shv(G))$. This justifies the definition of $h:\mathrm{Vect}\to L-comod(Shv(G))$.

Let $\bar h:\mathrm{Vect}\to Shv(G)$ be the continuous functor sending e to L. Its right adjoint $\bar h^R$ is given by $(\bar h^R)(F)=C^\cdot(L^{-1}\otimes K)$. Since the composition $\mathrm{Vect}\stackrel{\bar h}{\to} Shv(G)\stackrel{\delta^!_1}{\to}\mathrm{Vect}$ is the identity up to a shift, $\bar h$ is conservative.

Does the functor $\bar h$ preserve totalizations? In the constructibel context $\bar h$ has a left adjoint given by $F\mapsto p_!(F\otimes L^{-1})[2d]$ for $p: G\to \operatorname{Spec} k$, so the answer is yes. But what about $\mathcal{D}$-modules? Here is another argument.

We have $Shv(G)^{G,L}\widetilde{\to}\mathrm{Fun}_{Shv(G)}(\mathrm{Vect}, Shv(G)_{L^{-1}})$, where $Shv(G)_{L^{-1}}$ is the category $Shv(G)$ equipped with the action of $K\in Shv(G)$ as $F\mapsto (K\otimes L^{-1})*F$. Consider the equivalence $f: Shv(G)\to Shv(G)_{L^{-1}}$, $F\mapsto F\otimes L^{-1}$. For $K,F\in Shv(G)$ we have

$(K\otimes L^{-1})*(F\otimes L^{-1})\widetilde{\to}(K*F)\otimes L^{-1}$. Thus, f is an isomorphism in $G-mod$, where $K\in Shv(G)$ acts on $F\in Shv(G)$ by $K*F$. It induces an equivalence

$$\mathrm{Vect}_{Shv(G)}(\mathrm{Vect}, Shv(G))\widetilde{\to}\mathrm{Vect}_{Shv(G)}(\mathrm{Vect}, Shv(G)_{L^{-1}})=Shv(G)^{G,L}$$

Since we know that $Shv(G)^G\widetilde{\to}\mathrm{Vect}$, our claim holds for any sheaf theory. □

1.3.12. Let $G=\mathrm{colim}_{i\in I}G_i$ in PreStk, where I is a filtered small category, each G_i is a placid scheme, a group scheme, and for $i\to j$ in I the map $i_{ij}:G_i\to G_j$ is a homomorphism of group schemes and a placid closed embedding. Recall that $Shv(G)\widetilde{\to}\mathrm{colim}_{i\in I}Shv(G_i)$. Let $M\in G-mod^r, C\in G-mod$. Then one has

$$\mathrm{colim}_{i\in I}M\otimes_{Shv(G_i)}C\widetilde{\to}M\otimes_{Shv(G)}C$$

Indeed, I is sifted, so $\mathrm{colim}_{i\in I}Shv(G_i)^{\otimes n}\widetilde{\to}Shv(G)^{\otimes n}$ as in Section 1.3.4 of this file. So,

$$M\otimes_{Shv(G)}C\widetilde{\to}\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}M\otimes Shv(G)^{\otimes n}C\widetilde{\to}$$
$$\underset{i\in I}{\mathrm{colim}}\underset{[n]\in\boldsymbol{\Delta}^{op}}{\mathrm{colim}}M\otimes Shv(G_i)^{\otimes n}C\widetilde{\to}\underset{i\in I}{\mathrm{colim}}M\otimes_{Shv(G_i)}C$$

In particular, $C_G\widetilde{\to}\mathrm{colim}_{i\in I}C_{G_i}$ in DGCat_{cont}, the transition maps are $\mathrm{Av}_{G_i,G_j,*}: C_{G_i}\to C_{G_j}$ for $i\to j$ in I.

1.3.13. Let $f:H\to G$ be a map in $\mathcal{G}$rp(PreStk), where H,G are placid ind-schemes. Recall that $f_*:(Shv(H),*)\to(Shv(G),*)$ is monoidal. Let $D\in G-mod^r, C\in G-mod$ then we have a natural functor $D\otimes_{Shv(H)}C\to D\otimes_{Shv(G)}C$. Indeed, this holds for any morphism $A\to B$ in $Alg(\mathrm{DGCat}_{cont}$ and $D\in B-mod^r, C\in B-mod$.

In particular, we have a natural functor $\mathrm{Av}_{H,G,*}:C_H\to C_G$.

1.3.14. Let $p:G\to\mathrm{Spec}\,k$ be a placid scheme, and a group scheme. Let $C\in G-mod$. Viewing $p^*:\mathrm{Vect}\leftrightarrows Shv(G):p_*$ as an adjoint pair in $G-mod^r$ and applying $\bullet\otimes_{Shv(G)}C$, we get an adjoint pair $\mathrm{oblv}_G:C_G\leftrightarrows C:\mathrm{Av}_{G,*}$ in DGCat_{cont}.

1.3.15. Let Z be a placid ind-scheme. Is $Shv(Z)$ canonically self-dual?

Write $Z=\mathrm{colim}_{i\in I}Z_i$ with Z_i a placid scheme, I small filtered, and for $i\to j$ the map $f_{ij}:Z_i\to Z_j$ is a placid closed immersion. We have $Shv(Z)=\mathrm{colim}_{i\in I}Shv(Z_i)$ with respect to the transition functors $(f_{ij})_*$.

Consider the functor $I\to\mathrm{DGCat}_{cont}$, $i\mapsto Shv(Z_i)$, $(i\to j)\mapsto(f_{ij})_*$. By ([5], ch. I.1, 6.3.4), the colimit of this functor $\mathrm{colim}_{i\in I}Shv(Z_i)=Shv(Z)$ is dualizable, and $Shv(Z)^\vee\widetilde{\to}\lim_{i\in I^{op}}Shv(Z_i)^\vee$, the limit of the dual functor.

Recall for each i the canonical self-duality on $Shv(Z_i)$ introduced in Sect. 1.2.11 of this file. It allows to rewrite $Shv(Z)^\vee\widetilde{\to}\lim_{i\in I^{op}}Shv(Z_i)$, where the transition functors for $i\to j$ in I is $(f_{ij})^![2\,\mathrm{codim}_{Z_j}(Z_i)]$ in the notations of Section 1.2.12.

Pick an element $i_0\in I$. Consider for $i\to j$ in I a commutative diagram

$$\begin{array}{ccc} Shv(Z_j) & \overset{\otimes e[-2\,\mathrm{codim}_{Z_j}(Z_{i_0})]}{\to} & Shv(Z_j)\\ \downarrow f_{ij}^! & & \downarrow f_{ij}^![2\,\mathrm{codim}_{Z_j}(Z_i)]\\ Shv(Z_i) & \overset{\otimes e[-2\,\mathrm{codim}_{Z_i}(Z_{i_0})]}{\to} & Shv(Z_i)\end{array}$$

Indeed, we have $\mathrm{codim}_{Z_j}(Z_i) + \mathrm{codim}_{Z_i}(Z_{i_0}) = \mathrm{codim}_{Z_j}(Z_{i_0})$. Passing to the limit over I^{op}, this provides an equivalence $Shv(Z)^\vee \widetilde{\to} Shv(Z)$.

This duality maybe depend on a choice of an element $i_0 \in I$.

1.3.16. For 3.4.6. For $\mu \in \Lambda^+$ let $S^\mu \subset \mathrm{Gr}_{G,x}$ be the $\mathfrak{L}(N)_x$-orbit through t^μ. What is $\mathrm{R}\Gamma(S^\mu, \omega_{S^\mu})$? Let $N_\mu \subset \mathfrak{L}(N)_x$ be the stabilizor of $t^\mu \in \mathrm{Gr}_{G,x}$. We may pick a sequence $N_\mu \subset N^k \subset \mathfrak{L}(N)_x$ such that $N^k \subset N^{k+1}$ is closed, and N^k/N_μ is isomorphic to an affine space. Recall that $\omega_{S^\mu} \widetilde{\to} \mathrm{colim}_k\, \omega_{N^k/N_\mu}$ in $Shv(S^\mu)$ (by [20], 9.2.6). So, $\mathrm{R}\Gamma(S^\mu, \omega_{S^\mu}) \widetilde{\to} \mathrm{colim}_k\, \mathrm{R}\Gamma(N^k/N_\mu, \omega_{N^k/N_\mu}) \widetilde{\to} e$. Indeed, $\omega_{N^k/N_\mu} \widetilde{\to} e[2\dim(N^k/N_\mu]$.

Question: what is $\mathrm{R}\Gamma(\mathrm{Gr}_G^\gamma, \omega_{\mathrm{Gr}_G})$ for $\gamma \in \pi_1(G)$?

1.3.17. For 5.2.7. The following is used. Let $x, y \in X$ be distinct. Then $N_{X-\{x,y\}}$ acts transitively on $\mathfrak{L}_y(N)/\mathfrak{L}_y^+(N)$. Indeed, $X - x$ is affine. Let be given an N-torsor $\mathcal{F}_N$ on $X - x$ with a trivialization over $X - \{x, y\}$. This gives an element of $\mathfrak{L}_y(N)/\mathfrak{L}_y^+(N)$. Pick a global trivialization over $X - x$ of this N-torsor. This gives the claim.

In 5.2.7 (ii) he means that $\mathcal{F}$ is $(N_{X-\{x,y\}}), -\chi_y)$-equivariant under the action on the factor $\mathfrak{L}_y(N)/\mathfrak{L}_y^+(N)$ by right translations (if one uses the left translations, it would be χ_y instead).

1.3.18. For 5.2.7, a question: Let N be a group ind-scheme of the form $\mathrm{colim}_{i\in I} N_i$ with N_i pro-unipotent, for $i \to j$ in I, $N_i \to N_j$ is a placid closed immersion, and I is small filtered category. Let 0 be the initial object of I. Let S be an ind-scheme of ind-finite type. Assume given an action of N on S, and consider $(N/N_0) \times S$.

How to better identify $Shv((N/N_0) \times S)^{N\times N}$ with $Shv(S)^N$? Dennis proposed the claim that the !-restriction along $S \hookrightarrow (N/N_0) \times S$ gives an equivalence

$$Shv((N/N_0) \times S)^{N\times N} \to Shv(S)^N$$

Is this correct? Consider the projection $\mathrm{pr} : (N/N_0) \times S \to S$. Is it true that $\mathrm{pr}^!$ identifies $Shv(S)^N$ with $Shv((N/N_0) \times S)^{N\times N}$?

1.3.19.

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