1. Comments to: D. Gaitsgory, The local and global versions of the Whittaker category, version Oct. 9, 2020

1.0.1. For 1.4.5. If $S \in \mathrm{Sch}_{ft}, Y \in \mathrm{PreStk}_{lft}$ then the functor of exterior product $Shv(S)\otimes Shv(Y)\to Shv(S\times Y)$ is fully fauthful for all the 4 sheaf theories, and its right adjoint is continuous ([23], 0.0.31). For $T\in \mathrm{Sch}_{ft}$ let $h_T: Shv(S)\otimes Shv(T)\to Shv(S\times T)$ be the exteriour product functor, then its right adjoint h_T^R identifies with the dual $h^\vee$.

I see this as follows. The composition $Shv(S)\otimes Shv(T)\stackrel{dual\otimes dual}{\to} Shv(S)^\vee\otimes Shv(T)^\vee = \mathrm{Fun}_{k,cont}(Shv(S)\otimes Shv(T),\mathrm{Vect})\widetilde{\to}\mathrm{Fun}_{k,bi-ex}(Shv(S)^c\times Shv(T)^c,\mathrm{Vect})$ sends L to the functor $(K_S,K_T)\mapsto \mathcal{H}om_{Shv(S)\otimes Shv(T)}(\mathbb{D}(K_S)\otimes\mathbb{D}(K_T),L)\in\mathrm{Vect}$. Here $\mathcal{H}om$ denotes the relative inner hom for the Vect-action on $Shv(S)\otimes Shv(T)$. Indeed, it suffices to check this for L of the form $L_S\boxtimes L_T$ with $L_S\in Shv(S), L_T\in Shv(T)$, where this follows from ([6], ch. I.1, 10.5.8). Here $\mathbb{D}: (Shv(S)^c)^{op}\widetilde{\to} Shv(S)^c$ denotes the Verdier duality and similarlty for T. Now the claim follows from the fact that for $K\in Shv(S)^c, K'\in Shv(T)^c$, $\mathbb{D}(K\boxtimes K')\widetilde{\to}\mathbb{D}(K)\boxtimes\mathbb{D}(K')$.

Dualizing the monoidal structure on $Shv(H)$, one gets a coalgebra structure on $Shv(H)$, where the coproduct $Shv(H)\to Shv(H)\otimes Shv(H)$ is the composition $Shv(H)\stackrel{m^!}{\to} Shv(H\times H)\stackrel{h^R}{\to} Shv(H)\otimes Shv(H)$, here h^R is right adjoint to the exteriour product functor $h: Shv(H)\otimes Shv(H)\to Shv(H)$. In fact, $(Shv(H)*)-mod\widetilde{\to}(Shv(H),m^!)-comod$, see ([1], B.1.2). So, $Shv(Y)$ becomes a $(Shv(H),m^!)$-comodule category. The corresponding cosimplicial category $[n]\mapsto Shv(H)^{\otimes n}\otimes Shv(Y)$ is obtained from the usual cosimplicial category $[n]\mapsto \mathrm{Fun}(Shv(H)^{\otimes n}, Shv(Y))$, whose limit is

$$\mathrm{Fun}_{Shv(H)}(\mathrm{Vect}, Shv(Y)),$$

via the equivalences $\mathrm{Fun}(Shv(H)^{\otimes n}, Shv(Y))\widetilde{\to} Shv(H)^{\otimes n}\otimes Shv(Y)$ given by the canonical self-duality on $Shv(H)$ (coming from the Verdier duality). For any $T'\to Y/H$, with $T'\in\mathrm{Sch}_{ft}^{aff}$, let $T=Y\times_{Y/H}T'$. Then we get a compatible action of H on T.

Then the collection of functors $h_T^R: Shv(H^n\times T)\to Shv(H)^{\otimes n}\otimes Shv(T)$ yields for any $(T'\to Y/H)\in(\mathrm{Sch}_{ft}^{aff})_{/(Y/H)}$ a morphism of cosimplicial categories

$$\begin{array}{ccccc} Shv(T) & \rightrightarrows & Shv(H\times T) & \substack{\to\\ \to\\ \to} & \dots \\ \downarrow \mathrm{id} & & \downarrow h_T^R & & \\ Shv(T) & \rightrightarrows & Shv(H)\otimes Shv(T) & \substack{\to\\ \to\\ \to} & \dots \end{array}$$

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(because it is a morphism after dualization). Then passing to the limit over $T' \in (\mathrm{Sch}^{aff}_{ft})^{op}_{/(Y/H)}$, one get the desired morphism of cosimplicial categories

$$\begin{array}{ccccc} Shv(Y) & \rightrightarrows & Shv(H\times Y) & \substack{\to\\ \to\\ \to} \dots \\ \downarrow & & \downarrow h^R_Y & \\ Shv(Y) & \rightrightarrows & Shv(H)\otimes Shv(Y) & \substack{\to\\ \to\\ \to} \dots \end{array}$$

Indeed, we have seen in ([23], 0.0.28) that h^R_Y is the limit of h^R_T. Every functor $Shv(H^n\times Y)\to Shv(H)^{\otimes n}\otimes Shv(Y)$ restricted to the full subcategory $Shv(H)^{\otimes n}\otimes Shv(Y)$ becomes the identity functor.

Any object of $\mathrm{Tot}\, Shv(H^{\bullet}\times Y)$ is given by the collection of objects $\omega_H^{\boxtimes n}\boxtimes F\in Shv(H^{\bullet}\times Y)$ for some $F\in Shv(Y)$ with the corresponding isomorphisms for any map $[n]\to[m]$ in $\boldsymbol{\Delta}$. For each n we have actually

$$\omega_H^{\boxtimes n}\boxtimes F\in Shv(H)^{\otimes n}\otimes Shv(Y)\subset Shv(H^n\times Y)$$

For two such objects $(\omega_H^{\boxtimes n}\boxtimes F),(\omega_H^{\boxtimes n}\boxtimes F')\in \mathrm{Tot}\, Shv(H^{\bullet}\times Y)$ the mapping space between them in $\mathrm{Tot}\, Shv(H^{\bullet}\times Y)$ is

$$\mathrm{Tot}_{[n]\in\boldsymbol{\Delta}}\mathrm{Map}_{Shv(H^{\bullet}\times Y)}(\omega_H^{\boxtimes n}\boxtimes F,\omega_H^{\boxtimes n}\boxtimes F')\widetilde{\to}\,\mathrm{Tot}_{[n]\in\boldsymbol{\Delta}}\mathrm{Map}_{Shv(H)^{\otimes n}\otimes Shv(Y)}(\omega_H^{\boxtimes n}\boxtimes F,\omega_H^{\boxtimes n}\boxtimes F')$$

because $[n]$-th terms lie in the corresponding full subcategories. The latter identifies with the mapping space in $\mathrm{Tot}_{[n]\in\boldsymbol{\Delta}}Shv(H)^{\otimes n}\otimes Shv(Y)$. So, $Shv(Y/H)\to Shv(Y)^H$ is fully faithful.

There is no obvious reason for the essentially surjectivity. We don't have a map of cosimplicial categories in the opposite direction, so to get the essential surjectivity one shows that the cosimplicial category $\mathrm{Fun}(Shv(H)^{\otimes\bullet},Shv(Y))$ satisfies the comonadic Beck-Chevalley conditions.

1.1. **For Appendix B.**

1.1.1. For B.1.3. Av^H_* is defined by applying $\mathrm{Fun}_{Shv(H)}(\cdot,C)$ to the functor $p^*:\mathrm{Vect}\to Shv(H)$ in $Shv(H)-mod$. Here $p:H\to\mathrm{Spec}\, k$. For $F\in Shv(H)$ we get $e_Y * F\widetilde{\to} e_Y\otimes C(F)$, where $C(F)=p_*F\in\mathrm{Vect}$.

Taking $C=Shv(H)$ with the $Shv(H)$-action by left translations, we get

$$\mathrm{Fun}_{Shv(H)}(\mathrm{Vect},Shv(H))\widetilde{\to}\mathrm{Fun}_{Shv(H)}(Shv(H),\mathrm{Vect}),$$

where now the first argument $Shv(H)$ is equipped with the $Shv(H)$-action by right translations (by Verdier duality). This gives $\mathrm{Fun}_{Shv(H)}(Shv(H),\mathrm{Vect})\widetilde{\to}\mathrm{Vect}$.

1.1.2. For B.2.1. Let Y be a scheme (of finite type). We assume that for $p:Y\to\mathrm{Spec}\, k$, $p^*e=e_Y$ exists. The category $Shv(Y)^0$ is compactly generated by e_Y. The fact that $Shv(Y)^0\to C(Y)-mod$ is an equivalence follows from ([22], Lemma 9.2.18). The product map on $C(H)$ is $\mathrm{R}\Gamma(Y\times Y,e)\to\mathrm{R}\Gamma(Y,e)$ is given by $\vartriangle^*(e_Y\boxtimes e_Y)\widetilde{\to} e_Y$, which gives a map $e_Y\boxtimes e_Y\to\vartriangle_* e_Y$, and passing to $\mathrm{R}\Gamma$, the product map. Since Y is a commutative coalgebra in Sch, $C(Y)=p_*(e_Y)$ is a commutative algebra in Vect.

We used the Kunneth formula: for smooth schemes of finite type Y_i,

$$\mathrm{R}\Gamma(Y_1 \times Y_2, e) \widetilde{\to} \mathrm{R}\Gamma(Y_1, e) \otimes \mathrm{R}\Gamma(Y_2, e)$$

So, the functor $(Sch_{smooth,ft})^{op} \to \mathrm{Vect}$, $Y \mapsto \mathrm{R}\Gamma(Y, e)$ is symmetric monoidal.

1.1.3. For B.2.2. The coalgebra structure on $C(H) = \mathrm{R}\Gamma(H, e_Y)$ is as follows. Note that e_H is a coalgebra in $Shv(H)$ for the covolution monoidal structure naturally. The coproduct is given by $e_H \to m_* m^* e_H$ for $m : H \times H \to H$. The functor $p_* : Shv(H) \to \mathrm{Vect}$ for $p : H \to \operatorname{Spec} k$ is monoidal, so $p_* e_H = C(H)$ is a coalgebra in Vect. The compatibility of algebra and coalgebra structure comes from the diagram

$$\begin{array}{ccc} H \times H & \stackrel{\triangle \times \triangle}{\to} & H \times H \times H \times H \\ \downarrow m & & \downarrow m_{13} \times m_{24} \\ H & \stackrel{\triangle}{\to} & H \times H \end{array}$$

1.1.4. For B.2.2. Let us check that the lax monoidal functor $R : Shv(H) \to Shv(H)^0$, the right adjoint to the inclusion, is actually monoidal.

For $F_i \in Shv(H)$ we have a natural map $R(F_1) * R(F_2) \to R(F_1 * F_2)$. To check that this is an isomorphism, it suffices to show that $\mathcal{H}om_{Shv(H)}(e_H, R(F_1) * R(F_2)) \to \mathcal{H}om_{Shv(H)}(e_H, R(F_1 * F_2))$ is an isomorphism. Here $\mathcal{H}om_{\mathcal{C}}(c_1, c_2)$ denotes the inner hom in $\mathcal{C} \in \mathrm{DGCat}_{cont}$ with respect to the Vect-action. Indeed, e_H generates $Shv(H)^0$. This is the map $C(R(F_1)) \otimes C(R(F_2)) \to C(F_1 * F_2)$. However, $C(R(F_i)) \widetilde{\to} C(F_i)$ naturally.

1.1.5. For B.2.3. The full subcategory $Shv(H)^0 \subset Shv(H)$ is preserved by both left and right $Shv(H)$-actions, because for $F \in Shv(H), F * e_H \widetilde{\to} e_H * F \widetilde{\to} e_H \otimes C(F)$.

The functor $R : Shv(H) \to Shv(H)^0$ commutes with left (and right) $Shv(H)$-action by translations. Indeed, for $F, F_1 \in Shv(H)$ we have a natural map $R(F_1) \to F_1$, which gives a morphism $R(F_1 * F) \widetilde{\to} R(F_1) * R(F) \to F_1 * R(F)$. To check that it is an isomorphism, it suffices to apply R, because R is monoidal.

The adjoint pair $i : Shv(H)^0 \leftrightarrows Shv(H) : R$ takes place in $Shv(H) - mod$.

1.1.6. For B.2.3. The fact that $Shv(H)^0$ is dualizable as a left (or right) $Shv(H)$-module follows from ([22], Remark 3.1.4).

In B.2.4 misprint: the right adjoint to the restriction $Shv(H)^0 - mod \to Shv(H) - mod$ along $R : Shv(H) \to Shv(H)^0$ is given by $M \mapsto (Shv(H)^0)^\vee \otimes_{Shv(H)} M = \mathrm{Fun}_{Shv(H)}(Shv(H)^0, M)$ by ([22], 3.2.1). Here the dual is calculated in the category $Shv(H) - mod$. The self-duality of $Shv(H)^0$ is ([22], Remark 3.1.4).

1.1.7. For B.3.1. The existence of Ξ follows from (HTT, 5.3.5.10). The situation here is similar to ([6], ch. II.1, Lm. 1.1.7). Namely, consider the smallest stable and Vect^{fd}-submodule subcategory $\mathcal{C} \subset C^\cdot(Y) - mod$ containing $C^\cdot(Y)$. Then $\mathrm{Ind}(\mathcal{C}) \widetilde{\to} C^\cdot(Y) - mod$, and $\mathcal{C} \in \mathrm{DGCat}^{non-cocmpl}$. The inclusion $\mathcal{C} \subset C^\cdot(Y) - mod^{fd}$ gives the induced functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(C^\cdot(Y) - mod^{fd}) = C^\cdot(Y) - mod^{ren}$.

The fact that Ξ is fully faithful is (HTT, 5.3.5.11(1)).

1.1.8. For B.3.2. Recall that $C(H)-mod\widetilde{\to} Shv(H)^0$, and this is an equivalence of monoidal categories. In the adjoint pair $\Xi: C(H)-mod \rightleftarrows C(H)-mod^{ren}:\Psi$ both Ψ and Ξ are monoidal functors. For Ξ this follows from $e_H * e_H\widetilde{\to} e_H\otimes C(H)$ in $Shv(H)^0$.

The fact that $(C(H)-mod)-mod\to (C(H)-mod^{ren})-mod$ is fully faithful comes from ([22], 9.2.32), namely we use the adjoint pair $\Xi: C(H)-mod \rightleftarrows C(H)-mod^{ren}:\Psi$, where both functors are monoidal, and Ξ is fully-faithful.

1.1.9. For B.3.3. The category $C(H)-mod^{ren}$ is rigid, because each $F\in C(H)-mod^{fin-dim}$ admits a left and right dual: this is the dual in Vect with the induced $C(H)$-action?

Note that $e\in C(H)-mod$ is the unit of the symmetric monoidal category $C(H)-mod^{ren}$. It is compact in $C(H)-mod^{ren}$, because any object lying in $C(H)-mod^{fin.dim}$ is compact by construction.

1.1.10. For B.4.1. The equivalence $Shv(B(H))\widetilde{\to}\mathrm{Fun}_{Shv(H)}(\mathrm{Vect},\mathrm{Vect})$ used there without a reference is explained in [23].

The fact that the functor $H-mod\to \mathrm{DGCat}_{cont}$, $C\mapsto C^H=\mathrm{Fun}_{Shv(H)}(\mathrm{Vect}, C)$ upgrades to a functor $H-mod\to \mathrm{Fun}_{Shv(H)}(\mathrm{Vect},\mathrm{Vect})-mod$ is a residual 2-categorical phenimenon. Namely, $H-mod$ is a 2-category naturally ([6], ch. I.1, 8.3). The mapping category $\mathbf{Map}(?,\bullet)$ is naturally a module over the endomorphisms category $\mathbf{Map}(?,?)$.

Recall that the composition $Shv(H)\stackrel{R}{\to} Shv(H)^0\stackrel{\mathrm{R}\Gamma}{\to}\mathrm{Vect}$ is p_* for $p: H\to\operatorname{Spec} k$, see Sect. 1.1.4.

For any $C^r\in Shv(H)^0-mod^r, C\in Shv(H)^0-mod$ viewing them as $Shv(H)$-modules via $R: Shv(H)\to Shv(H)^0$, we get by ([22], 9.2.32)

$$C^r\otimes_{Shv(H)^0} C\widetilde{\to} C^r\otimes_{Shv(H)} C$$

The claim from B.4.1 follows now from the equivalence

$$\mathrm{Vect}\otimes_{Shv(H)^0} Shv(H)^0\otimes_{Shv(H)} C\widetilde{\to} Vect\otimes_{Shv(H)} C$$

together with Th. B.1.2.

1.1.11. For B.4.4. The symmetric monoidal structure on $Shv(B(H))$ given by $\otimes^!$ coincides with the monoidal structure on $\mathrm{Fun}_{(Shv(H),\star)}(\mathrm{Vect},\mathrm{Vect})$ given by the composition.

In the formula (B.8) we may view Vect as a right module over $(Shv(B(H)),\otimes^!)$, where $K\in Shv(B(H))$ acts on $V\in\mathrm{Vect}$ as $(p^!K)\otimes V$. Here $p:\operatorname{Spec} k\to B(H)$. We also view Vect there as a left module over $(Shv(H),\star)$, so this is a bimodule, an object of $(Shv(H),\star)-mod-Shv(B(H))$. Then we apply (HA, 4.6.2.1(2)) showing that the functor (B.8) is a left adjoint, and its right adjoint is given by tensoring by $\mathrm{Vect}\in Shv(B(H))-mod-(Shv(H),\star)$.

The corresponding duality datum is given by the functor

$$Shv(B(H))\to\mathrm{Vect}\otimes_{(Shv(H),\star)}\mathrm{Vect}$$

in $Shv(B(H))-mod-Shv(B(H))$. The latter functor is an equivalence, so that the unit of the adjunction is an isomorphism, hence (B.8) is fully faithful.

1.1.12. For B.4.7. By $C_\cdot(H)$ he means homology, that is, $C(H)^\vee$, the dual of $C(H)$ in Vect. Since $C(H) \in \mathrm{Vect}^c$, this dual exists. The functor $p^! : Shv(B(H)) \to \mathrm{Vect}$ for $p : \operatorname{Spec} k \to B(H)$ is comonadic, and the corresponding comonad is $C(H)$. So, $Shv(B(H) \widetilde{\to} C(H) - comod \widetilde{\to} C(H)^\vee - mod$.

My understanding is that the coalgebra $C(H)$ is non-commutative, so the algebra $C(H)^\vee$ is also non-commutative. However, $C(H)^\vee$ is a cocommutative coalgebra. Note that $e \in C(H) - comod$ corresponds to $e_{B(H)} \in Shv(B(H))$.

Let us try a simplier thing. We consider $(Shv(B(H)), \otimes^!)$ as a monoidal category, and want to calculate $\mathrm{Vect} \otimes_{(Shv(B(H)),\otimes^!)} \mathrm{Vect}$. It is the colimit of $Bar^\bullet(\mathrm{Vect}, Shv(B(H)), \mathrm{Vect})$, that is of

$$Shv(B(H))^{\otimes 2} \overset{\to}{\underset{\to}{\to}} Shv(B(H)) \rightrightarrows \mathrm{Vect}$$

Does it satisfy the monadic Beck-Chevalley condition ([7], Def. C.1.5)? Convent that for $n \geq 0$ the functor corresponding to the last face map $\partial_n : [n] \to [n+1]$ is $Shv(B(H))^{\otimes n+1} \to Shv(B(H))^{\otimes n}$ sending $F_1 \otimes \ldots \otimes F_{n+1}$ to $F_1 \otimes \ldots \otimes F_n \otimes p^! F_{n+1}$.

The action functor $p^! : Shv(B(H)) \to \mathrm{Vect}$ admits contunuous right adjoint $p_*[-2m]$ with $m = \dim H$. We have

$$Shv(B(H)) \otimes Shv(B(H)) \widetilde{\to} (C(H)^\vee - mod) \otimes (C(H)^\vee - mod) \widetilde{\to} C(H)^\vee \otimes C(H)^\vee - mod$$

The monoidal operation $\otimes^!$ on $Shv(B(H))$ becomes the restriction functor $C(H)^\vee \otimes C(H)^\vee - mod \to C(H)^\vee - mod$ via $C(H)^\vee \to C(H)^\vee \otimes C(H)^\vee$. Does it have a continuous right adjoint? Not clear for me.

Here is a proof of Sam:

1) Let Y be a a scheme of finite type with H-action. Write $p : Y \to Y/H$ for the quotient in the sense of prestacks. Let $Shv(Y)^{H-mod} \subset Shv(Y)$ be the full subcategory generated by the image of $p^* : Shv(Y/H) \to Shv(Y)$. We show that $p_* : Shv(Y)^{H-mod} \to Shv(Y/H)$ is monadic. Indeed, the latter functor is continuous, it has the left adjoint $p^* : Shv(Y/H) \to Shv(Y)^{H-mod}$, and this left adjoint generates $Shv(Y)^{H-mod}$. So, $p_* : Shv(Y)^{H-mod} \to Shv(Y/H)$ is conservative, hence monadic. The corresponding monad on $Shv(Y/H)$ is $F \mapsto F \otimes p_*(e)$.

2) Special case $Y = \operatorname{Spec} k$. In this case $p : \operatorname{Spec} k \to B(H)$, let $A = p_*(e) \in Alg(Shv(B(H)), \otimes)$. We get $\mathrm{Vect} \widetilde{\to} A - mod(Shv(B(H), \otimes)$.

3) Let $D \in (Shv(B(H)), \otimes) - mod$. Then by ([6], ch. I.1, 8.5.7),

$$Vect \otimes_{(Shv(B(H)),\otimes)} D \widetilde{\to} A - mod(D)$$

Applying this for $D = \mathrm{Vect}$ with an action of $(Shv(B(H)), \otimes)$ given by p^* for $p : \operatorname{Spec} k \to B(H)$, we get $Vect \otimes_{(Shv(B(H)),\otimes)} \mathrm{Vect} \widetilde{\to} A - mod(\mathrm{Vect})$. The corresponding monad of Vect becomes $C(H)$ as desired.

Consider the functor $\tau : (Shv(B(H)), \otimes) \to (Shv(B(H)), \otimes^!)$ sending K to $K[-2n]$ with $n = \dim H$. This is a monoidal equivalence. Indeed, the diagonal map $\vartriangle : B(H) \to B(H) \times B(H)$ is smooth of relative dimension $n = \dim H$, hence $\vartriangle^! = \vartriangle^* [2n]$. Under this equivalence for $p : \operatorname{Spec} k \to B(H)$ the monoidal functor $p^* : (Shv(B(H)), \otimes) \to \mathrm{Vect}$ identifies with $p^! : (Shv(B(H)), \otimes^!) \to \mathrm{Vect}$. This finishes the proof of Th. B.4.3.

1.2. **For Appendix C.** For C.1.1. In the definition of placidity, the transition maps $Z_\alpha \to Z_\beta$ are assumed smooth surjective and affine. Note that a placid scheme is quasi-compact and quasi-separated.

Let us show that a placid scheme Z is quasi-separated. Write $Z \widetilde{\to} \lim Z_\alpha$ as in the definition of placidity. We may assume α_0 is an initial object of the corresponding filtered category I. Let U_j, $j \in J$ is a finite open covering of Z_{α_0} by affine schemes. Write $Z^j = Z \times_{Z_{\alpha_0}} U_j$, this is an affine scheme. Then for each pair $(j, j') \in J \times J$ it suffices to show that the preimage $\mathcal{V}$ of $Z^j \times Z^{j'}$ under $Z \to Z \times Z$ is quasi-compact. Pick a finite open covering by affine schemes V_k (with $k \in S$, where S is a finite set) of $U_j \cap U_{j'}$. Let $\mathcal{V}_k$ be the base change of $\mathcal{V}$ under $V_k \to U_j \cap U_j := (U_i \times U_j) \times_{Z_{\alpha_0} \times Z_{\alpha_0}} Z_{\alpha_0}$. Then $\mathcal{V} = \cup_{k \in S} \mathcal{V}_k$, so it suffices to show that each $\mathcal{V}_k$ is quasi-compact. However, $\mathcal{V}_k$ is a placid scheme over V_k, hence $\mathcal{V}_k$ is quasi-compact.

1.2.1. For C.1.4. If Y is an ind-scheme of ind-finite type then Y is a placid ind-scheme.

1.2.2. For C.1.6 line 1: for the map $f_{\alpha,\beta} : \mathcal{Y}_\alpha \to \mathcal{Y}_\beta$ we must assume that for any $S \in \mathrm{Sch}_{ft}$ and a map $S \to \mathcal{Y}_\beta$ the base change $S \times_{\mathcal{Y}_\beta} \mathcal{Y}_\alpha \to S$ is smooth, affine and surjective morphism, and moreover $S \times_{\mathcal{Y}_\beta} \mathcal{Y}_\alpha$ is of finite type. So, it is better to say that $f_{\alpha,\beta}$ is smooth, surjective, affine and quasi-compact.

The fact that $Z_i \to Z_j$ is a placid closed embedding is obtained as follows. For $i \to j$, $Y_i \times_{Y_j} Z_j \widetilde{\to} Z_i$. Indeed, $Y_i \times_{Y_j} \lim_\alpha Y_{j,\alpha} \widetilde{\to} \lim_\alpha Y_{i,\alpha}$. Here $Y_{i,\alpha} = Y_i \times_{\mathcal{Y}_{\alpha_0}} \mathcal{Y}_\alpha$.

Recall that the category Sch^{aff} of classical affine schemes admits all limits, and $\mathrm{Sch}^{aff} \subset \mathrm{PreStk}$ is closed under limits by HTT. Assume I is filtered, $I^{op} \to \mathrm{Sch}_{ft}$, $i \mapsto Z_i$ is a diagram such that the transition maps $Z_i \to Z_j$ are smooth, surjective and affine. Let $Z = \lim_{i \in I^{op}} Z_i$ is PreStk. We claim that Z is a scheme. Indeed, we may assume I has an initial object i_0. For any open affine subscheme $S \subset Z_{i_0}$, $S \times_{Z_{i_0}} Z_i$ is affine, hence $\lim_{i \in I^{op}} S \times_{Z_{i_0}} Z_i \widetilde{\to} S \times_{Z_{i_0}} Z$ is an affine scheme. Thus, Z admits an open covering by affine schemes, hence $Z \in \mathrm{Sch}$.

1.2.3. Let Z be a placid scheme written as $Z = \lim_{i \in I^{op}} Z_i$ is PreStk, where the transition maps $Z_i \to Z_j$ are smooth, surjective and affine with $Z_i \in \mathrm{Sch}_{ft}$. Let $S \in \mathrm{Sch}_{ft}$ and $f : Z \to S$ be a morphism of schemes. Then there is i such that f factors as $Z \to Z_i \to S$. Indeed, first it suffices to assume S affine. Suppose for each affine open subscheme $U \subset S$, $U \times_S Z \to S$ factors through $U \times_S Z_i$ for some i. Since S is quasi-compact, for a finite subset $I' \subset I$ we pick an element j for which there is a map $i \to j$ in I for any $i \in I'$. Then $Z \to S$ factors through Z_j.

Now assume S affine. We may assume I has an initial object i_0. For any open affine subscheme $V \subset Z_{i_0}$, $V \times_{Z_{i_0}} Z$ is affine, hence the map $V \times_{Z_{i_0}} Z \to S$ factors through $V \times_{Z_{i_0}} Z_i$ for some i. Since Z_{i_0} is quasi-compact, we may pick j large enough in I such that now $V \times_{Z_{i_0}} Z \to S$ factors through $V \times_{Z_{i_0}} Z_j$ for any V in some open covering of Z_{i_0}, and we are done.

Indeed, if there are two morphisms $f, f' : Z_j \to S$ such that the compositions $Z \to Z_j \to S$ coincide then $f = f'$. Indeed, this can checked locally in Zariski topology of Z_{i_0}.

1.2.4. For C.2.4. There $f : \mathcal{Z} \to \mathcal{Z}'$ is any morphism in PreStk, and $\mathcal{Z}, \mathcal{Z}'$ are placid schemes. By the previous subsection, $f_* : Shv(\mathcal{Z}) \to Shv(\mathcal{Z}')$ is well-defined.

Is it true that for a placid scheme $\mathcal{Z}$ written as $Z = \lim \mathcal{Z}_i$, any object $F \in Shv(\mathcal{Z})$ is of the form f^*K for some $i \in I$ and $f : Z \to Z_i$, $K \in Shv(\mathcal{Z}_i)$? Probably not, because $\mathrm{DGCat}_{cont} \to 1-\mathcal{C}\mathrm{at}$ does not preserve filtered colimits ([22], 9.2.29).

1.2.5. For a placid closed embedding $f : \mathcal{Z}' \to \mathcal{Z}$ of placid schemes, $f_* : Shv(\mathcal{Z}') \to Shv(\mathcal{Z})$, which was initially defined in C.2.4 admits a different interpretation. Namely, assume $\mathcal{Z} = \lim_{\alpha \in I^{op}} \mathcal{Z}_\alpha$, and for a closed embedding $\mathcal{Z}'_{\alpha_0} \hookrightarrow \mathcal{Z}_{\alpha_0}$ we have $\mathcal{Z}' = \mathcal{Z}'_{\alpha_0} \times_{\mathcal{Z}_{\alpha_0}} \mathcal{Z}$.

Then we may assume α_0 is an initial object of I. Then for any $\alpha \in I$ set $\mathcal{Z}'_\alpha = \mathcal{Z}'_{\alpha_0} \times_{\mathcal{Z}_{\alpha_0}} \mathcal{Z}_\alpha$, so that $\mathcal{Z}' = \lim \mathcal{Z}'_\alpha$, and $Shv(\mathcal{Z}') \widetilde{\to} \operatorname{colim}_{\alpha \in I^{op}} Shv(\mathcal{Z}'_\alpha)$. For each α consider the closed embedding $i_\alpha : \mathcal{Z}'_\alpha \hookrightarrow \mathcal{Z}_\alpha$. The functors $(i_\alpha)_*$ are compatible with the corresponding colimit systems, and yield a functor $\operatorname{colim} Shv(\mathcal{Z}'_\alpha) \to \operatorname{colim} Shv(\mathcal{Z}_\alpha)$, which is f_*.

In addition, consider for $\alpha \to \beta$ in I the diagram

$$\begin{array}{ccc} \mathcal{Z}'_\beta & \stackrel{i_\beta}{\to} & \mathcal{Z}_\beta \\ \downarrow \pi'_{\alpha,\beta} & & \downarrow \pi_{\alpha,\beta} \\ \mathcal{Z}'_\alpha & \stackrel{i_\alpha}{\to} & \mathcal{Z}_\alpha \end{array}$$

We get $i^!_\alpha (\pi_{\alpha,\beta})_* \widetilde{\to} (\pi'_{\alpha,\beta})_* i^!_\beta$. So, the functors $i^!_\alpha$ form a morphism of the corresponding inverse systems, yielding a functor $f^! : Shv(\mathcal{Z}) \to Shv(\mathcal{Z}')$. It is the right adjoint to f_* by ([6], ch. I.1, 2.6.4).

Proposition 1.2.6. *Let Z be a quasi-compact quasi-separated scheme, Y a placid ind-scheme written as $Y \widetilde{\to} \operatorname{colim}_{i \in I} Y_i$ with I filtered, Y_i placid scheme, and $Y_i \to Y_j$ a placid closed immersion for $i \to j$ in I. Let $f : Z \to Y$ be a morphism in* PreStk *then there is $i \in I$ such that it factors through $Z \to Y_i \to Y$.*

Proof. Step 1. We reduce this to the case of affine Z. Assume this known for Z affine. Since Z is quasi-compact, we pick an affine cover $Z = \cup_\beta S_\beta$. Each map $S_\beta \to Y$ factors through some Y_i, we may assume this is the same i for all β. Since Z is quasi-separated, our claim follows from the next lemma.

Lemma 1.2.7. *Let S be an affine scheme, $\gamma, \gamma' : S \to Y_i$ two maps such that the compositions $S \to Y_i \to Y$ are the same. Then there is $i \to j$ in I such that the compositions $S \to Y_i \to Y_j$ are the same.*

Proof. *Sets* $\subset$ Spc is closed under filtered colimits, so $Y(S)$ is a set. Filtered colimits in *Sets* are obtained as a quotient by the equvalence relation of $\sqcup_i Y_i(S)$ by $(i, \gamma) \sim (i', \gamma')$ iff there is a diagram $i \to j \leftarrow i'$ such that the images of γ and γ' in $Y_j(S)$ are equal. □

Step 2. Assume Z affine. Then any object of $Y(Z)$ comes from some element of $Y_i(Z)$ by ([22], 13.1.14). □

For example, if Z is a placid scheme then Z is quasi-compact and quasi-separated, so satisfies the assumptions of Proposition 1.2.6. Another example is when Y is a ind-scheme of ind-finite type.

1.2.8. For C.2.7. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between placid ind-schemes. Write $\mathcal{Y}_2 = \mathrm{colim}_{i\in I} \mathcal{Y}_2^i$ and $\mathcal{Y}_1 = \mathrm{colim}_{j\in J} \mathcal{Y}_1^j$ with I, J filtered.

The composition $\mathcal{Y}_1^j \to \mathcal{Y}_1 \to \mathcal{Y}_2$ factors through $f_{ji} : \mathcal{Y}_1^j \to \mathcal{Y}_2^i$ for some i by the above, so we get a morphism $(f_{ji})_* : Shv(\mathcal{Y}_1^j) \to Shv(\mathcal{Y}_2^i)$. Composing with $(\gamma_i)_* : Shv(\mathcal{Y}_2^i) \to Shv(\mathcal{Y}_2)$ for $\gamma_i : \mathcal{Y}_2^i \to \mathcal{Y}_2$, we get a morphism $Shv(\mathcal{Y}_1^j) \to Shv(\mathcal{Y}_2)$. This is a compatible system, hence induces a functor $Shv(\mathcal{Y}_1) \widetilde{\to} \mathrm{colim}_{j\in J} Shv(\mathcal{Y}_1^j) \to Shv(\mathcal{Y}_2)$.

1.2.9. For C.2.8. Fully-fathfulness follows from ([22], 2.2.17). Now write $Y_j = \lim\limits_{i\in I_j^{op}} Y_j^i$, where Y_j^i is a scheme of finite type, and the transition maps are smooth affine surjective. Then $Shv(Y_1) \otimes Shv(Y_2)$ is compactly generated by objects of the form $F_1 \otimes F_2$ with $F_1 \in Shv(Y_1^i)^c, i \in I_1$, $F_2 \in Shv(Y_2^j)^c, j \in I_2$ for some i, j. Here we used ([2], Cor. 1.9.4). Since then $F_1 \boxtimes F_2 \in Shv(Y_1^i \times Y_2^j)$ is compact, we see that the functor $Shv(Y_1) \otimes Shv(Y_2) \to Shv(Y_1 \times Y_2)$ has a continuous right adjoint.

1.2.10. For C.2.9. The functors $(f_{\alpha,\beta})_*$ exist, because $f_{\alpha,\beta}$ is schematic quasi-compact, and $\mathcal{Y}_\alpha, \mathcal{Y}_\beta \in \mathrm{PreStk}_{lft}$. The functor $f^*_{\alpha,\beta}$ exists, because $f_{\alpha,\beta}$ is smooth.

In the situation of Section C.1.6, let $\mathcal{Y}_{\alpha,i} = Y_i \times_{\mathcal{Y}_{\alpha_0}} \mathcal{Y}_\alpha$. The maps $\mathcal{Z}_i = \lim_\beta \mathcal{Y}_{\beta,i} \to \mathcal{Y}_{\alpha,i} \to \mathcal{Y}_\alpha$ form a compatible system over $i \in I$, hence yield a morphism $f_\alpha : \mathcal{Y} = \mathrm{colim}_i \mathcal{Z}_i \to \mathcal{Y}_\alpha$. Since f_α is a map between placid ind-schemes, $(f_\alpha)_*$ exists.

We have a morphism

$$\mathop{\mathrm{colim}}_{i\in I} \lim_{\alpha\in \mathcal{J}^{op}} \mathcal{Y}_{\alpha,i} \to \lim_{\alpha\in \mathcal{J}^{op}} \mathop{\mathrm{colim}}_{i\in I} \mathcal{Y}_{\alpha,i},$$

that is, the map $\mathcal{Y} \to \lim_{\alpha\in\mathcal{J}^{op}} \mathcal{Y}_\alpha$. Is it an isomorphism in PreStk? Yes, see ([23], 0.0.36).

By definition, $Shv(\mathcal{Y}) \widetilde{\to} \mathrm{colim}_{i\in I} \mathrm{colim}_\alpha Shv(\mathcal{Y}_{\alpha,i})$, where we use for the transition map $\tau_{\alpha,\beta} : \mathcal{Y}_{\alpha,i} \to \mathcal{Y}_{\beta,i}$ the functor $\tau^*_{\alpha,\beta}$. So,

$$Shv(\mathcal{Y}) \widetilde{\to} \mathrm{colim}_\alpha \mathrm{colim}_{i\in I} Shv(\mathcal{Y}_{\alpha,i}) \widetilde{\to} \mathrm{colim}_\alpha Shv(\mathcal{Y}_\alpha)$$

Here for $\alpha \to \beta$ and $f_{\alpha,\beta} : \mathcal{Y}_\alpha \to \mathcal{Y}_\beta$ we are using $f^*_{\alpha,\beta} : Shv(\mathcal{Y}_\beta) \to Shv(\mathcal{Y}_\alpha)$. The functor $f^*_{\alpha,\beta}$ has a continuous right adjoint $(f_{\alpha,\beta})_*$, because $f_{\alpha,\beta}$ is schematic quasi-compact. So, we may pass to right adjoints and get $Shv(\mathcal{Y}) \widetilde{\to} \lim_\alpha Shv(\mathcal{Y}_\alpha)$.

1.2.11. Let Z be a placid scheme written as $Z = \lim_{i\in I^{op}} Z_i$. For $i \to j$ in I let $f_{ij} : Z_j \to Z_i$ be the corresponding morphism, it is smooth of relative dimension d_{ij}, affine, surjective. Since $Shv(Z) \widetilde{\to} \mathrm{colim}_i Shv(Z_i)$ via the maps f^*_{ij}, $Shv(Z)$ is compactly generated, hence dualizable. By ([6], ch. I.1, 6.3.4), by applying the dualization functor to the functor

$$I \to \mathrm{DGCat}_{cont}, i \mapsto Shv(Z_i),\ (i \to j) \mapsto f^*_{ij},$$

we get a functor $I^{op} \to \mathrm{DGCat}_{cont}, i \mapsto Shv(Z_i)$, $(i \to j) \mapsto (f_{ij})_*[-2d_{ij}]$. Moreover,

$$Shv(Z)^\vee \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)$$

with respect to the transition maps $(f_{ij})_*[-2d_{ij}]$. Consider for $i \in I$ the isomorphism $Shv(Z_i) \stackrel{\otimes e[2d_i]}{\to} Shv(Z_i)$ with $d_i = \dim Z_i$. So, $d_{ij} = d_j - d_i$. The diagram commutes

$$\begin{array}{ccc} Shv(Z_j) & \stackrel{\otimes e[2d_j]}{\to} & Shv(Z_j) \\ \downarrow (f_{ij})_* & & \downarrow (f_{ij})_*[-2d_{ij}] \\ Shv(Z_i) & \stackrel{\otimes e[2d_i]}{\to} & Shv(Z_i) \end{array}$$

Passing to the limit over I^{op}, we obtain an equivalence $Shv(Z) \widetilde{\to} Shv(Z)^\vee$. So, for each placid scheme Z, $Shv(Z)$ is canonically self-dual.

However, if Z is a scheme of finite type, this self-duality is not the same as the usual one for a finite type scheme. Everything can be here over a base scheme $S \in Sch_{ft}$, which is not necessary of pure dimension. Then d_i is such that $Z_i \to S$ is smooth of relative dimension d_i. (Probably, smoothness of Z_i over S is not needed here).

If $f_i : Z \to Z_i$ is the projection then the dual of the functor $f_i^* : Shv(Z_i) \to Shv(Z)$ identifies with $(f_i)_* : Shv(Z) \to Shv(Z_i)$.

1.2.12. Let now Z, Z' be placid schemes and $i : Z' \to Z$ a placid closed immersion. What is the dual of the adjoint pair $i_* : Shv(Z') \to Shv(Z) : i^!$?

We explain the dual of i_*. If $Z = \lim_{i \in I^{op}} Z_i$ and, assume for simplicity I has an initial object i_0 such that $Z' = Z'_{i_0} \times_{Z_{i_0}} Z$. So, $Z' = \lim_{i \in I^{op}} Z'_i$ with $Z'_i = Z_i \times_{Z_{i_0}} Z'_{i_0}$. For $i \to j$ in I let $f_{ij} : Z_j \to Z_i$ be the corresponding transition map. For the closed embeddings $i_i : Z'_i \to Z_i$ writing $Shv(Z) = \lim_{i \in I^{op}} Shv(Z_i)$ for $(f_{ij})_* : Shv(Z_j) \to Shv(Z_i)$ and similarly for $Shv(Z')$, the dual functor is given by the collection of functors $i_i^![2d_i - 2d'_i] : Shv(Z_i) \to Shv(Z'_i)$, here $d_i = \dim Z_i, d'_i = \dim Z'_i$ as locally constant functions, they form a morphism of the corresponding inverse systems. The number $d_i - d'_i$ does depend on i, and can be denoted $\operatorname{codim}_Z(Z') = d_i - d'_i$. So, the dual of $i_* : Shv(Z') \to Shv(Z)$ is $i^![2 \operatorname{codim}_Z(Z')]$.

1.3. For C.3.1. Here K_n is the kernel of $G(\mathcal{O}) \to G(\mathcal{O}/t^n)$. Note for $n < m$ then $K_m \subset K_n$ is a subgroup, hence a map $f_{n,m} : \mathfrak{L}(G)/K_m \to \mathfrak{L}(G)/K_n$. The functor $f_{n,m}^* : Shv(\mathfrak{L}(G)/K_n) \to Shv(\mathfrak{L}(G)/K_m)$ exists, see my previous section (and [10], C.2.9), and $Shv(\mathfrak{L}(G)) = \operatorname{colim}_n Shv(\mathfrak{L}(G)/K_n)$, where the transition functors are $f_{n,m}^*$.

If e_{K_n} denotes the sheaf e extended by zero under $K_n/K_n \hookrightarrow \mathfrak{L}(G)/K_n$. Let $f_n : \mathfrak{L}(G) \to \mathfrak{L}(G)/K_n$ be the natural map. Then $(f_n)_*\delta_1 \widetilde{\to} e_{K_n}$. By ([22], 9.2.6), the natural map $\operatorname{colim}_n (f_n)^*(f_n)_*\delta_1 \to \delta_1$ is an isomorphism.

For D.1.2. We show that the natural map $\operatorname{colim}_n C^{K_n} \to C$ is an isomorphism, where the colimit is calculated in DGCat_{cont}. For each n, K_n is a placid group scheme, so the adjoint pair $p_n^* : \mathrm{Vect} \leftrightarrows Shv(K_n) : (p_n)_*$ for $p_n : K_n \to \operatorname{Spec} k$ gives an adjoint pair $\mathrm{oblv} : C^{K_n} \leftrightarrows C : \mathrm{Av}_*^{K_n}$ in DGCat_{cont}. Here for $n \geq 1$, oblv is fully faithful, because K_n is prounipotent. For $n \in \mathbb{N}$, $K_{n+1} \subset K_n$, so we have the oblivion functor $C^{K_n} \to C^{K_{n+1}} \to C$, this shows that $C^{K_n} \to C^{K_{n+1}}$ is fully faithful. Now by ([4], Lm. 1.3.6),

$$C^{K_m} \to \operatorname*{colim}_{n \in \mathbb{N}^{op}} C^{K_n} := D$$

is fully faithful for $m \geq 1$, where the colimit is calculated in DGCat_{cont}. Moreover, the inclusion $C^{K_n} \to C^{K_{n+1}}$ has a continuous right adjoint given by $\mathrm{Av}_*^{K_n}$ restricted to $C^{K_{n+1}}$. So, by passing to right adjoints $D \widetilde{\to} \lim_n C^{K_n}$. Since for each $n \geq 1$, $C^{K_n} \hookrightarrow C$ is fully faithful, we get the fully faithfulness of $D \to C$ by passing to the limit. There remains to show that $D \to C$ is essentially surjective. Given $c \in C$, we write $\delta_1 \widetilde{\to} \operatorname{colim}_n e_{K_n}$, so $\delta_1 * c \widetilde{\to} \operatorname{colim}_n e_{K_n} * c$. Here $e_{K_n} * c \in C^{K_n}$ and $e_{K_n} * c \widetilde{\to} \mathrm{Av}_*^{K_n}(c)$. The projective system $(\mathrm{Av}_*^{K_n}(c))_{n\in\mathbb{N}}$ defines an object of D which can be seen as $\operatorname{colim}_n ins_n \mathrm{Av}_*^{K_n}(c) \widetilde{\to} c$, where $ins_n : C^{K_n} \hookrightarrow C$ is the natural inclusion. Thus, the canonical map $D \to C$ is an isomorphism.

In fact, $G(F) \widetilde{\to} \lim_{n\mathbb{N}^{op}} G(F)/K_n$ as prestacks, here $F = k((t))$, this follows from ([15], 4.4.2).

1.3.1. For 1.5.3. If H is a smooth group scheme of finite type, L is a local system on H equipped with associative isomorphism $m^*L \widetilde{\to} L \boxtimes L$ and a compatible trivialization $i^*L \widetilde{\to} e$ for $i : \operatorname{Spec} k \to H$ then $f : (Shv(H), \star) \to (Shv(H), \star)$, $F \mapsto F \otimes L^{-1}$ is a monoidal functor. Indeed,

$$f(F_1 * F_2) \widetilde{\to} (F_1 * F_2) \otimes L^{-1} \widetilde{\to} (F_1 \otimes L^{-1}) * (F_2 \otimes L^{-1}) = f(F_1) * f(F_2)$$

Now given $C \in (Shv(H), *) - mod$, we twist the action by L as follows. The object $C_L \in H - mod$ is defined as $C \in \mathrm{DGCat}_{cont}$ with the new action given by $Shv(H) \otimes C \stackrel{f\otimes \mathrm{id}}{\to} Shv(H) \otimes C \stackrel{act}{\to} C$.

For the trivial action of H on Vect we get Vect_L. This is Vect equipped with the action of $Shv(H)$, where $F \in Shv(H)$ sends F to $C^{\cdot}(H, F \otimes L^{-1})$. We have the adjoint pair $g^L : \mathrm{Vect}_L \leftrightarrows Shv(H) : g$, where $g(F) = C^{\cdot}(H, F \otimes L^{-1})$, and $(g^L)(V) = V \otimes L$. We view it as an adjoint pair in $(Shv(H), *) - mod$, where the $(Shv(H), *)$-action on Vect_L is given by g, and $Shv(H)$ acts on itself by left translations. Recall that g is monoidal. For $C \in H - mod$ applying the functor $\mathrm{Fun}_{Shv(H)}(\bullet, C)$ we get for

$$C^{H,L} := \mathrm{Fun}_{Shv(H)}(\mathrm{Vect}_L, C)$$

the adjoint pair

$$\mathrm{oblv}_{H,L} : C^{H,L} \leftrightarrows C : Av_*^{H,L}$$

The composition $\mathrm{oblv}_{H,L} \circ Av_*^{H,L} : C \to C$ is $c \mapsto (e_H \otimes L) * c$. Note that $C^{H,L} \widetilde{\to} (C_{L^{-1}})^H$.

By Section 1.3.12, we get $C^{H,L} \widetilde{\to} L - comod(C)$.

Example: for $C = Shv(H)$ note that $L \in L - comod(Shv(H))$ naturally, as $L \in coAlg(Shv(H), *)$. So, this is a good definition of twisted invariants.

Remark: let $Y \in \mathrm{PreStk}_{lft}$ with a H-action. Let $a : H \times Y \to Y$ be the action map. Let H act on $H \times Y$ via its action on H by left translations. Then a is H-equivariant, so $a^! : L - comod(Shv(Y)) \to L - comod(Shv(H \times Y))$. We have naturally $L^{-1} \boxtimes \omega \in L^{-1} - comod(Shv(H \times Y)$. Let us show that for $K \in L - comod(Shv(Y))$, $(\mathrm{pr}_1^! L^{-1}) \otimes^! a^! K$ becomes an objects of $e_H - comod(Shv(H \times Y))$, so is of the form $\omega \boxtimes \mathcal{F}$ for some $\mathcal{F} \in Shv(Y)$.

By Section 1.3.3 below, $(L^{-1} \boxtimes \omega) \boxtimes a^! K \in Shv(H \times Y \times H \times Y)^{H \times H, L^{-1} \boxtimes L}$.

Restricting the action of $H\times H$ to the diagonal, the map $d: Y\to Y\times Y$ becomes H-equivariant. By the functoriality of Section 1.3.2,

$$(L^{-1}\boxtimes\omega)\boxtimes a^!K\in Shv(H\times Y\times H\times Y)^H$$

Further taking $d^!$, we get $(\mathrm{pr}_1^! L^{-1})\otimes^! a^!K\in Shv(H\times Y)^H$, where H acts via the left transations on H. We are done.

1.3.2. Let $f: G\to H$ be a homomorphism of placid group schemes, L a character local system on H. Since $f_*: Shv(G)\to Shv(H)$ is monoidal, f^* is left-lax monoidal, so yields a functor $f^*: coAlg(H)\to coAlg(G)$. Thus, f^*L is a coalgebra. This structure also comes from the fact that f^*L is a character local system on G. Since f_* is monoidal, it yields a functor $f_*: coAlg(Shv(G))\to coAlg(Shv(H))$, and we have an adjoint pair $f^*: coAlg(Shv(H))\leftrightarrows coAlg(Shv(G)): f_*$ by ([25], Remark 1.2.9). So, the map $L\to f_*f^*L$ is a morphism of coalgebras, hence yields the forgetful functor for any $C\in Shv(H)-mod$,

$$L-comod(C)\to f_*f^*L-comod(C)$$

The target here can be seen as $f^*L-comod(C)$, where $f^*L\in coAlg(Shv(G))$ with the action of $Shv(G)$ on C obtained by restriction via f_*. We constructed the functor $\mathrm{oblv}: C^{H,L}\to C^{G,f^*L}$.

1.3.3. Let G_i be a placid group scheme acting on $Y_i\in \mathrm{PreStk}_{lft}$ for $i=1,2$, let L_i be a character local system on G_i. Then $G_1\times G_2$ acts on $Y_1\times Y_2$, and $L_1\boxtimes L_2$ is a character local system on $G_1\times G_2$. The exteriour product functor $Shv(Y_1)\otimes Shv(Y_2)\to Shv(Y_1\times Y_2)$ is $Shv(G_1)\otimes Shv(G_2)$-linear, where it acts on $Shv(Y_1\times Y_2)$ via the exteriour product map $Shv(G_1)\otimes Shv(G_2)\to Shv(G_1\times G_2)$. This used ([23], 0.0.52).

Now by ([22], 9.2.61), there is a natural functor

$$\mathrm{Fun}_{Shv(G_1)}(\mathrm{Vect}_{L_1}, Shv(Y_1))\otimes \mathrm{Fun}_{Shv(G_2)}(\mathrm{Vect}_{L_2}, Shv(Y_2))\to \\ \mathrm{Fun}_{Shv(G_1)\otimes Shv(G_2)}(\mathrm{Vect}_{L_1}\otimes \mathrm{Vect}_{L_1}, Shv(Y_1)\otimes Shv(Y_2))$$

By ([22], 9.2.61), the evaluation functor

$$\mathrm{Fun}_{Shv(G_1)\otimes Shv(G_2)}(\mathrm{Vect}_{L_1}\otimes \mathrm{Vect}_{L_1}, Shv(Y_1)\otimes Shv(Y_2))\stackrel{\gamma}{\to} Shv(Y_1)\otimes Shv(Y_2)$$

is comonadic. Let us calculate the corresponding comonad.

As in Section 1.3.1, we have the adjoint pairs $g_i^L: \mathrm{Vect}_{L_i}\leftrightarrows Shv(G_i): g_i$ in $Shv(G_i)-mod$. Tensoring, we get an adjoint pair $\mathrm{Vect}_{L_1}\otimes\mathrm{Vect}_{L_2}\leftrightarrows Shv(G_1)\otimes Shv(G_2)$ in $Shv(G_1)\otimes Shv(G_2)-mod$. Applying $\mathrm{Fun}_{Shv(G_1)\otimes Shv(G_2)}(\bullet, Shv(Y_1)\otimes Shv(Y_2))$, this gives the adjoint pair

$$\gamma: \mathrm{Fun}_{Shv(G_1)\otimes Shv(G_2)}(\mathrm{Vect}_{L_1}\otimes \mathrm{Vect}_{L_1}, Shv(Y_1)\otimes Shv(Y_2))\leftrightarrows Shv(Y_1)\otimes Shv(Y_2):\gamma^R$$

This shows that $\gamma(\gamma^R)$ is the functor of action by $L_1\boxtimes L_2$. Thus,

$$\mathrm{Fun}_{Shv(G_1)\otimes Shv(G_2)}(\mathrm{Vect}_{L_1}\otimes \mathrm{Vect}_{L_1}, Shv(Y_1)\otimes Shv(Y_2))\widetilde{\to} \\ (L_1\boxtimes L_2)-comod(Shv(Y_1)\otimes Shv(Y_2))$$

Composing with the exteriour product $Shv(Y_1)\otimes Shv(Y_2) \to Shv(Y_1\times Y_2)$, we obtain the morphism

$$(L_1 \boxtimes L_2) - comod(Shv(Y_1) \otimes Shv(Y_2)) \to (L_1 \boxtimes L_2) - comod(Shv(Y_1 \times Y_2))$$

Combining, we get finally the functor

$$Shv(Y_1)^{G_1,L_1} \otimes Shv(Y_2)^{G_2,L_2} \to Shv(Y_1 \times Y_2)^{G_1\times G_2, L_1\boxtimes L_2}$$

Remark 1.3.4. *If we are not in the constructible context, it is better not to use the* $\otimes$*, but only* $\otimes^!$*. For this we should start with an object* $\mathcal{L} \in Shv(H)$ *invertible for the* $\otimes^!$*-monoidal structure and satisfying* $m^!\mathcal{L} \widetilde{\to} \mathcal{L} \boxtimes \mathcal{L}$ *associatively and* $i^!\mathcal{L} \widetilde{\to} e$ *for* $i : Speck \xrightarrow{1} H$*. Then the functor* $(Shv(H), *) \to (Shv(H), *)$*,* $F \mapsto F \otimes^! \mathcal{L}$ *is a monoidal equivalence. If moreover we are in the constructible context then for a multiplicative local system* L *in the usual sense,* $\mathcal{L} := L \otimes \omega_H$ *satisfies the above properties.*

1.3.5. Let $G \widetilde{\to} \operatorname{colim}_{i\in I} G_i$ in PreStk, where G_i is a prosmooth placid group scheme, I is small filtered, for $i \to j$ in I, $G_i \to G_j$ is a placid closed immersion. Assume $0 \in I$ is initial. Let H be a group scheme of finite type, L be a character local system on H, $\alpha : G \to H$ be a homomorphism. Let Y be an ind-scheme of ind-finite type with a G_0-action, $K \in Shv(Y)^{G_0,L}$. We want to define $(\alpha^!L)\widetilde{\boxtimes} K \in Shv(G \times^{G_0} Y)$. Here $\alpha^!L$ makes sense only after some additional choice, see below.

Write $Y \widetilde{\to} \operatorname{colim}_{j\in J} Y_j$, where J is small filetred, Y_j is a G_0-invariant scheme of finite type, and for $j \to j' \in J$, $Y_j \to Y_{j'}$ is a closed immersion. It suffices to define our object in the case when $Y \in \mathrm{Sch}_{ft}$ in a way compatible with !-pullbacks under the closed immersions $Y_j \to Y_{j'}$. So, we simply assume $Y \in \mathrm{Sch}_{ft}$.

The definition is not canonical and depends on an additional choice. Namely, pick a normal prounipotent subgroup $N \subset G_0$ of finite codimension such that $\alpha : G \to H$ factors through $\bar{\alpha} : G/N \to H$. Let us also assume G_0-action on Y factors through G_0/N. Our constructon depends on a choice of N.

By definition, $Shv(Y)^{G_0,L} \widetilde{\to} Shv(Y)^{G_0/N,L}$. Let $(g_1, g_2) \in (G_0/N) \times (G_0/N)$ act on $(G/N) \times Y$ sending (g, y) to (gg_1^{-1}, g_2y). For this action

$$(\bar{\alpha}^!L) \boxtimes K \in Shv(G/N \times Y)^{G_0/N\times G_0/N, L^{-1}\boxtimes L}$$

So, for the diagonal action of G_0/N, $\bar{\alpha}^!L\boxtimes K \in Shv(G/N\times Y)^{G_0/N}$. Let $q : G/N\times Y \to G/N \times^{G_0/N} Y$ be the stack quotient map. Denote by $\alpha^!L\widetilde{\boxtimes}K \in Shv(G/N \times^{G_0/N} Y)$ the object such that $q^!(\bar{\alpha}^!L\widetilde{\boxtimes}K) \widetilde{\to} \mathrm{oblv}(\bar{\alpha}^!L\boxtimes K)$. Now $G/N \times^{G_0/N} Y \widetilde{\to} G \times^{G_0} Y$, and $(\alpha^!L)\widetilde{\boxtimes}K \in Shv(G \times^{G_0} Y)$ as desired.

Dependence on N is as follows. Let $N' \subset N$ be another normal prounipotent group subscheme of G_0. Then let $\epsilon : G/N' \to G/N$ be the natural map and $\bar{\alpha}' = \bar{\alpha}\circ\epsilon$. By definition, we identify the two categories via the equivalence $\epsilon^! : Shv(G/N)\widetilde{\to}Shv(G/N')$, and similarly for $(\epsilon\times\mathrm{id})^! : Shv(G/N\times Y)\widetilde{\to}Shv(G/N'\times Y)$, so the result is independent of N in this sense.

1.3.6. Assume for this subsection we are in the constructible context. Let G be a group scheme of finite type, $Y \in \mathrm{PreStk}_{lft}$ with a G-action. Let L be a character local system on Y. Recall that $Shv(Y)^{G,L} \widetilde{\to} L - comod(Shv(Y))$. Let $\mathfrak{a}^R : Shv(Y) \to Shv(Y)$ be the comonad $K \mapsto L * K$. Then $\mathfrak{a}^R$ admits a left adjoint $\mathfrak{a} : Shv(Y) \to Shv(Y)$, which is a monad, and

$$\mathfrak{a} - mod(Shv(Y)) \widetilde{\to} \mathfrak{a}^R - comod(Shv(Y))$$

canonically, and this isomorphisms respect the oblivion functors to $Shv(Y)$. In particular $\mathrm{oblv} : Shv(Y)^{G,L} \to Shv(Y)$ has a left adjoint $\mathrm{Av}_!^{G,L} : Shv(Y) \to Shv(Y)^{G,L}$.

Let us generalize this claim as follows. Let $f : H \to G$ be a homomorphism of group schemes of finite type. Then $\mathrm{oblv} : Shv(Y)^{G,L} \to Shv(Y)^{H,L}$ has a left adjoint.

Proof. Our f^*L is a character local system on H, write $\mathfrak{b}^R : Shv(Y) \to Shv(Y)$ for the comonad $K \mapsto f^*L * K$. As in Section 1.3.2, we have a morphism of comonads $\mathfrak{a}^R \to \mathfrak{b}^R$ on $Shv(Y)$. Let $\mathfrak{b} : Shv(Y) \to Shv(Y)$ be the left adjoint to $\mathfrak{b}^R$. Then we get by functoriality a morphism of monads $\mathfrak{b} \to \mathfrak{a}$ on $Shv(Y)$. Now oblv becomes the forgetful functor $\mathfrak{a} - mod(Shv(Y)) \to \mathfrak{b} - mod(Shv(Y))$. Its left adjoint is given by the induction $\mathfrak{b} - mod(Shv(Y)) \to \mathfrak{a} - mod(Shv(Y))$, $K \mapsto \mathfrak{a} \otimes_{\mathfrak{b}} K$. The relative tensor product here is taken i, the sense of [17] for the action of $\mathrm{Fun}_{e,cont}(Shv(Y), Shv(Y))$ on $Shv(Y)$. □

Lemma 1.3.7. *Let H be a unipotent group scheme of finite type (or pro-unipotent), $C \in H - mod$ and L is a character local system on H, so for $m : H \times H \to H$, $m^*L \widetilde{\to} L \boxtimes L$. In this case the functor $\mathrm{oblv}_{H,L} : C^{H,L} \to C$ is fully faithful. Its essential image is the image of the functor $C \to C, c \mapsto (e_H \otimes L) * c$.*

Proof. In the notations of the previous section, for $g^L : \mathrm{Vect}_L \leftrightarrows Shv(H) : g$ the unit of the adjunction $\mathrm{id} \to gg^L$ is $C^\cdot(H) \widetilde{\to} e$. So, g^L is fully faithful. This gives $Av_*^{H,L} \circ \mathrm{oblv}_{H,L} \widetilde{\to} \mathrm{id}$. □

If U is a pro-unipotent group scheme then $C^\cdot(U) \widetilde{\to} e$. Indeed, e comes as the object $(e_{U_i}) \in \lim Shv(U_i)$ for $U = \lim U_i$, where e_{U_i} is the constant sheaf on U_i.

For a pro-unipotent group scheme $p : U \to \mathrm{Spec}\, k$ consider the adjoint pair $p^* : \mathrm{Vect} \leftrightarrows Shv(U) : p_*$ in $Shv(U) - mod^r$. For $C \in Shv(U) - mod$ applying $\cdot \otimes_{Shv(U)} C$, one gets the adjoint pair $\mathrm{pr}^L : C_U \leftrightarrows C : \mathrm{pr}$, where pr^L is fully faithful, because $\mathrm{id} \widetilde{\to} p_*p^*$. So,

1.3.8. Let H, G be placid group ind-schemes and $f : H \to G$ a homomorphism of groups in PreStk. Then the functor $f_* : (Shv(H), *) \to (Shv(G), *)$ is monoidal. So, if $C, D \in G - mod$, we get the morphism of inverse systems

$$\begin{array}{ccccc}
\mathrm{Fun}(D, C) & \rightrightarrows & \mathrm{Fun}(Shv(G) \otimes D, C) & \substack{\to\\\to\\\to} & \mathrm{Fun}(Shv(G)^{\otimes 2} \otimes D, C) \ldots \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fun}(D, C) & \rightrightarrows & \mathrm{Fun}(Shv(H) \otimes D, C) & \substack{\to\\\to\\\to} & \mathrm{Fun}(Shv(H)^{\otimes 2} \otimes D, C) \ldots
\end{array}$$

given by composing with the maps $(f_*)^{\otimes n} : Shv(H)^{\otimes n} \to Shv(G)^{\otimes n}$. This yields a morphism between the totalizations

$$\mathrm{Fun}_{Shv(G)}(D, C) \to \mathrm{Fun}_{Shv(H)}(D, C)$$

Assume now $G = \operatorname{colim}_{i\in I} G_i$ in PreStk, where I is filtered, each G_i is a placid scheme, a group scheme, and for $i \to j$ in I the map $i_{ij} : G_i \to G_j$ is a placid closed embedding, and a homomorphism of group schemes. Then the natural map

$$\mathrm{Fun}_{Shv(G)}(D, C) \to \lim_{i\in I^{op}} \mathrm{Fun}_{Shv(G_i)}(D, C)$$

is an equivalence. Indeed, I is sifted, so $\operatorname{colim}_{i\in I} Shv(G_i)^{\otimes n} \widetilde{\to} Shv(G)^{\otimes n}$, where we use $(i_{ij})_* : Shv(G_i) \to Shv(G_j)$ as the transition maps. We get

$$\mathrm{Fun}_{Shv(G)}(D, C) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}(Shv(G)^{\otimes n}\otimes D, C) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}(\operatorname{colim}_i Shv(G_i)^{\otimes n}\otimes D, C)$$
$$\widetilde{\to} \lim_{i\in I^{op}} \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Fun}(Shv(G_i)^{\otimes n} \otimes D, C) \widetilde{\to} \lim_{i\in I^{op}} \mathrm{Fun}_{Shv(G_i)}(D, C)$$

This implies the formula (2.1) in 2.1.2. More generally, if $C \in G - mod$ then $C^G \widetilde{\to} \lim_{i\in I^{op}} C^{G_i}$.

1.3.9. Let $G = \lim_{i\in I^{op}} G_i$, where I is filtered, G_i is a smooth group scheme of finite type, for $i \to j$ the map $\pi_{ji} : G_j \to G_i$ is a homomorphism of group schemes, which is smooth, affine, surjective. So, G is a placid scheme. For $i \to j$ in I the functor $(\pi_{ij})_* : Shv(G_j) \to Shv(G_i)$ is monoidal, and $Shv(G) \widetilde{\to} \lim_{i\in I^{op}} Shv(G_i)$ in $Alg(\mathrm{DGCat}_{cont})$.

1.3.10. Let G be a group scheme, which is a pro-smooth placid scheme, $C \in G - mod$. Consider the cosimplicial category defining C^G:

$$\mathrm{Fun}(\mathrm{Vect}, C) \rightrightarrows \mathrm{Fun}(Shv(G), C) \substack{\to\\ \to\\ \to} \mathrm{Fun}(Shv(G)^{\otimes 2}, C) \dots$$

Let us show that it satisfies the comonadic Beck-Chevalley conditions.

The functor corresponding to the last face map $\partial_n : [n] \to [n+1]$ (its image avoids $n+1$) is the following functor F_n. We consider $Shv(G)^{\otimes n+1} \to Shv(G)^{\otimes n}$, $\mathrm{id} \otimes \mathrm{R}\Gamma$, and compose it with $\mathrm{Fun}(\cdot, C)$. For $p : G \to \operatorname{Spec} k$ the functor p_* has a left adjoint p^*. Let T_n be the functor obtained from $Shv(G)^{\otimes n} \to Shv(G)^{\otimes n+1}$, $\mathrm{id} \otimes p^*$ by composing with $\mathrm{Fun}(\cdot, C)$. Then T_n is the right adjoint to F_n. Let now $\alpha : [m] \to [n]$ be a map in $\boldsymbol{\Delta}$. Consider the corresponding diagram

$$\begin{array}{ccc} \mathrm{Fun}(Shv(G)^{\otimes n}, C) & \stackrel{T_n}{\leftarrow} & \mathrm{Fun}(Shv(G)^{\otimes n+1}, C) \\ \uparrow F_\alpha & & \uparrow F_{\alpha+1} \\ \mathrm{Fun}(Shv(G)^{\otimes m}, C) & \stackrel{T_m}{\leftarrow} & \mathrm{Fun}(Shv(G)^{\otimes m+1}, C) \end{array}$$

We show that it commutes. It suffices to prove this for α injective, becase of the following. Let $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ be the full subcategory with the same class of object, where we keep only injective maps. Then $\boldsymbol{\Delta}_s \to \boldsymbol{\Delta}$ is cofinal by [17]. If $\alpha : [m] \to [n]$ is injective, and $0, n$ are in the image then the desired commutativity follows from the commutativity of

$$\begin{array}{ccc} Shv(G)^{\otimes n} & \stackrel{p^*\otimes\mathrm{id}}{\to} & Shv(G)^{\otimes n+1} \\ \downarrow (m_\alpha)_* & & \downarrow (m_{\alpha+1})_* \\ Shv(G)^{\otimes m} & \stackrel{p^*\otimes\mathrm{id}}{\to} & Shv(G)^{\otimes m+1}, \end{array}$$

where $(m_\alpha)_*$ is the product along α in the monoidal category $Shv(G)$.

If $\alpha : [n-1] \to [n]$ is the last face map then $\alpha + 1 : [n] \to [n+1]$ avoids n. The functor $F_{\alpha+1}$ is the composition with $Shv(G)^{\otimes n+1} \to Shv(G)^{\otimes n}$, $K_1 \otimes \ldots \otimes K_{n+1} \mapsto K_1 \otimes \ldots \otimes K_{n-1} \otimes K_n * K_{n+1}$. In this case the desired commutativity follows from $K * e_G \widetilde{\to} \mathrm{R}\Gamma(G, K) \otimes e_G$.

If $\alpha : [n-1] \to [n]$ is injective and avoids 0 then F_α sends f to the functor

$$K_1 \otimes \ldots \otimes K_n \mapsto K_1 * f(K_2 \otimes \ldots \otimes K_n)$$

and the commutativity is tautological. So, it always hold.

By ([7], Lemma C.1.9), the functor $\mathrm{oblv}_G : C^G \to C$ is comonadic, and the corresponding comonad on C is $C \to C, c \mapsto e_G * c$. We conclude that $C^G \widetilde{\to} e_G - comod(C)$.

Does the same holds if $G \in Grp(\mathrm{PreStk})$ is only a placid ind-scheme? I and Sam think no, namely if G is not a group scheme then the comonadic Beck-Chevalley conditions do not hold for the above cosimplicial diagram, because the left adjoint to $\mathrm{R}\Gamma : Shv(G) \to \mathrm{Vect}$ does not exist.

Lemma 1.3.11. *Let be given an exact sequence* $1 \to U \to G \stackrel{q}{\to} H \to 1$ *of group schemes, where* G, H, U *are placid schemes. Assume* U *is pro-unipotent. Let* $C \in H - mod$*, which we view by restriction as an object of* $G - mod$*. Then* $C^G \widetilde{\to} C^H$ *canonically.*

Proof. We apply Section 1.3.10 and get $C^H \widetilde{\to} e_H - comod(C)$ where $e_H \in Shv(H)$ is the constant sheaf, and similarly, $C^G \widetilde{\to} e_G - comod(C)$. We have a morphism of comonads in one direction, and we check that the corresponding monads in $\mathrm{Fun}(C, C)$ are the same. This follows from $q_* e_G \widetilde{\to} e_H$. □

1.3.12. Let $G \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, let G be a placid ind-scheme, $C \in G - mod$. The functor $\mathrm{oblv}_G : C^G \to C$ is comonadic. Indeed, It has a right adjoint oblv_G^R, because $ev^0 : C^G \to C$ in the corresponding cosimplicial category is continuous, hence admits a maybe discontinuous right adjoint. By (HA, 4.7.5.1), oblv_G is comonadic, so $C^G \widetilde{\to} E - comod(C)$, where $E = \mathrm{oblv}_G \circ \mathrm{oblv}_G^R$. In general, oblv_G^R is not continuous.

1.3.13. Let $N \in Grp(\mathrm{PreStk})$ be a placid ind-scheme. Assume $N = \mathrm{colim}_{i \in I} N_i$, where each N_i is a placid scheme, and a group scheme. Assume I filtered. Assume for $i \to j$ the map $N_i \to N_j$ is a placid closed immersion and a homomorphism of group schemes. Let $C \in N - mod$. Recall that, by Sect. 1.3.8, $C^N \widetilde{\to} \lim_{i \in I^{op}} C^{N_i}$.

Assume each N_i is pro-unipotent. Then $C^{N_i} \subset C$ is a full subcategory. Then $C^N \widetilde{\to} \cap_i C^{N_i}$ as full subcategories in C by ([22], 2.7.7), because the forgetful functor $\mathrm{DGCat}_{cont} \to 1 - \mathcal{C}\mathrm{at}$ preserves limits.

1.3.14. Let Z be a placid scheme written as $\lim_{i \in I^{op}} Z_i$, where Z_i is a smooth scheme of finite type over a base scheme $S \in \mathrm{Sch}_{ft}$, and for $i \to j$ the map $Z_j \to Z_i$ is smooth, affine and surjective. Write $p : Z \to \mathrm{Spec}\, S$ for the structure map. Under the self-duality of $Shv(Z)$ from Section 1.2.11, the dual to $p_* : Shv(Z) \to Shv(S)$ identifies with $p^* : Shv(S) \to Shv(Z)$. So, p_* identifies with $(p^*)^\vee$.

Let U be a pro-unipotent group scheme, $C \in U - mod$. Then we have the adjoint pair $p_* : Shv(U) \leftrightarrows \mathrm{Vect} : (p_*)^R$ in the constructible context. However, in general $(p^*)^R$ is not a strict morphism of $Shv(U)$-modules. In general for $C \in Shv(U) - mod$, the left adjoint to $\mathrm{oblv}_U : C^U \to C$ does not exist in the constructible context, see ([24], after 1.12.3).

1.3.15. Let G be a placid group scheme, $G = \lim_{i\in I^{op}} G_i$, where G_i is smooth group scheme of finite type, for $i \to j$ in I the map $f_{ij} : G_j \to G_i$ is smooth, affine and surjective homomorphism, and I is filtered category. Assume I has an initial object i_0. Let $f_{i_0} : G \to G_{i_0}$ be the projection, L_0 a character local system on G_{i_0}. We assume it is invertible in $(Shv(G_{i_0}), \otimes)$. Let $L = f_{i_0}^* L_0$ be a character local system on G.

Consider the action of G on itself by left translations.

Claim. The continuous functor $h : \mathrm{Vect} \to Shv(G)^{G,L}$ sending e to L is an equivalence.

Proof. Write $Shv(G) \widetilde{\to} \lim_{i\in I^{op}} Shv(G_i)$, where we use $(f_{ij})_*$ as the transition functors. For any i, G acts on G_i, and f_{ij} are G-equivariant, so $(f_{ij})_*$ is a map in $G-mod$. Thus, $Shv(G)^{G,L} \widetilde{\to} \lim_{i\in I^{op}} Shv(G_i)^{G,L}$. By Lemma 1.3.11, $Shv(G_i)^{G,L} \widetilde{\to} Shv(G_i)^{G_i,L_i}$, where $L_i = f_{i_0 i}^* L_0$. Thus, it suffices to establish the equivalence $Shv(G_i)^{G_i,L_i} \widetilde{\to} \mathrm{Vect}$. So, we are reduced to the case, where G is a smooth group scheme of finite type. We assume this.

By Section 1.3.10, $Shv(G)^{G,L} \widetilde{\to} L - comod(Shv(G))$. Here we use the comonad $K \mapsto L * K$. Note that L is a coalgebra in $Shv(G)$, hence $L \in L - comod(Shv(G))$. This justifies the definition of $h : \mathrm{Vect} \to L - comod(Shv(G))$.

Let $\bar{h} : \mathrm{Vect} \to Shv(G)$ be the continuous functor sending e to L. Its right adjoint $\bar{h}^R$ is given by $(\bar{h}^R)(F) = C^{\cdot}(L^{-1} \otimes K)$. Since the composition $\mathrm{Vect} \xrightarrow{\bar{h}} Shv(G) \xrightarrow{\delta^!_1} \mathrm{Vect}$ is the identity up to a shift, $\bar{h}$ is conservative.

Does the functor $\bar{h}$ preserve totalizations? In the constructibel context $\bar{h}$ has a left adjoint given by $F \mapsto p_!(F \otimes L^{-1})[2d]$ for $p : G \to \mathrm{Spec}\, k$, so the answer is yes. But what about $\mathcal{D}$-modules? Here is another argument.

We have $Shv(G)^{G,L} \widetilde{\to} \mathrm{Fun}_{Shv(G)}(\mathrm{Vect}, Shv(G)_{L^{-1}})$, where $Shv(G)_{L^{-1}}$ is the category $Shv(G)$ equipped with the action of $K \in Shv(G)$ as $F \mapsto (K \otimes L^{-1}) * F$. Consider the equivalence $f : Shv(G) \to Shv(G)_{L^{-1}}$, $F \mapsto F \otimes L^{-1}$. For $K, F \in Shv(G)$ we have $(K \otimes L^{-1}) * (F \otimes L^{-1}) \widetilde{\to} (K * F) \otimes L^{-1}$. Thus, f is an isomorphism in $G - mod$, where $K \in Shv(G)$ acts on $F \in Shv(G)$ by $K * F$. It induces an equivalence

$$\mathrm{Vect}_{Shv(G)}(\mathrm{Vect}, Shv(G)) \widetilde{\to} \mathrm{Vect}_{Shv(G)}(\mathrm{Vect}, Shv(G)_{L^{-1}}) = Shv(G)^{G,L}$$

Since we know that $Shv(G)^G \widetilde{\to} \mathrm{Vect}$, our claim holds for any sheaf theory. □

1.3.16. Let $G = \mathrm{colim}_{i\in I} G_i$ in PreStk, where I is a filtered small category, each G_i is a placid scheme, a group scheme, and for $i \to j$ in I the map $i_{ij} : G_i \to G_j$ is a homomorphism of group schemes and a placid closed embedding. Recall that $Shv(G) \widetilde{\to} \mathrm{colim}_{i\in I} Shv(G_i)$. Let $M \in G - mod^r, C \in G - mod$. Then one has

$$\mathrm{colim}_{i\in I} M \otimes_{Shv(G_i)} C \widetilde{\to} M \otimes_{Shv(G)} C$$

Indeed, I is sifted, so $\operatorname{colim}_{i\in I} Shv(G_i)^{\otimes n} \widetilde{\to} Shv(G)^{\otimes n}$ as in Section 1.3.8 of this file. So,

$$M \otimes_{Shv(G)} C \widetilde{\to} \operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{op}} M \otimes Shv(G)^{\otimes n} C \widetilde{\to}$$
$$\operatorname*{colim}_{i\in I} \operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{op}} M \otimes Shv(G_i)^{\otimes n} C \widetilde{\to} \operatorname*{colim}_{i\in I} M \otimes_{Shv(G_i)} C$$

In particular, $C_G \widetilde{\to} \operatorname{colim}_{i\in I} C_{G_i}$ in DGCat_{cont}, the transition maps are $\mathrm{Av}_{G_i,G_j,*} : C_{G_i} \to C_{G_j}$ for $i \to j$ in I.

1.3.17. Let $f : H \to G$ be a map in $\mathcal{G}\mathrm{rp}(\mathrm{PreStk})$, where H, G are placid ind-schemes. Recall that $f_* : (Shv(H), *) \to (Shv(G), *)$ is monoidal. Let $D \in G - mod^r, C \in G - mod$ then we have a natural functor $D \otimes_{Shv(H)} C \to D \otimes_{Shv(G)} C$. Indeed, this holds for any morphism $A \to B$ in $Alg(\mathrm{DGCat}_{cont}$ and $D \in B - mod^r, C \in B - mod$.

In particular, we have a natural functor $\mathrm{Av}_{H,G,*} : C_H \to C_G$. The composition $C \overset{\mathrm{Av}_{H,*}}{\to} C_H \overset{\mathrm{Av}_{H,G,*}}{\to} C_G$ is $\mathrm{Av}_{G,*}$.

1.3.18. Let $p : G \to \operatorname{Spec} k$ be a pro-smooth placid scheme, and a group scheme. Let $C \in G - mod$. Viewing $p^* : \mathrm{Vect} \leftrightarrows Shv(G) : p_*$ as an adjoint pair in $G - mod^r$ and applying $\bullet \otimes_{Shv(G)} C$, we get an adjoint pair $\mathrm{oblv}_G : C_G \leftrightarrows C : \mathrm{Av}_{G,*}$ in DGCat_{cont}.

1.3.19. Let Z be a placid ind-scheme. Is $Shv(Z)$ canonically self-dual?

Write $Z = \operatorname{colim}_{i\in I} Z_i$ with Z_i a placid scheme, I small filtered, and for $i \to j$ the map $f_{ij} : Z_i \to Z_j$ is a placid closed immersion. We have $Shv(Z) = \operatorname{colim}_{i\in I} Shv(Z_i)$ with respect to the transition functors $(f_{ij})_*$.

Consider the functor $I \to \mathrm{DGCat}_{cont}$, $i \mapsto Shv(Z_i)$, $(i \to j) \mapsto (f_{ij})_*$. By ([6], ch. I.1, 6.3.4), the colimit of this functor $\operatorname{colim}_{i\in I} Shv(Z_i) = Shv(Z)$ is dualizable, and $Shv(Z)^\vee \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)^\vee$, the limit of the dual functor.

Recall for each i the canonical self-duality on $Shv(Z_i)$ introduced in Sect. 1.2.11 of this file. It allows to rewrite $Shv(Z)^\vee \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)$, where the transition functors for $i \to j$ in I is $(f_{ij})^![2\operatorname{codim}_{Z_j}(Z_i)]$ in the notations of Section 1.2.12.

Pick an element $i_0 \in I$. Consider for $i \to j$ in I a commutative diagram

$$\begin{array}{ccc} Shv(Z_j) & \overset{\otimes e[-2\operatorname{codim}_{Z_j}(Z_{i_0})]}{\to} & Shv(Z_j) \\ \downarrow f_{ij}^! & & \downarrow f_{ij}^![2\operatorname{codim}_{Z_j}(Z_i)] \\ Shv(Z_i) & \overset{\otimes e[-2\operatorname{codim}_{Z_i}(Z_{i_0})]}{\to} & Shv(Z_i) \end{array}$$

Indeed, we have $\operatorname{codim}_{Z_j}(Z_i) + \operatorname{codim}_{Z_i}(Z_{i_0}) = \operatorname{codim}_{Z_j}(Z_{i_0})$. Passing to the limit over I^{op}, this provides an equivalence $Shv(Z)^\vee \widetilde{\to} Shv(Z)$.

This duality maybe depend on a choice of an element $i_0 \in I$.

1.3.20. For 3.4.6. For $\mu \in \Lambda^+$ let $S^\mu \subset \mathrm{Gr}_{G,x}$ be the $\mathfrak{L}(N)_x$-orbit through t^μ. What is $\mathrm{R}\Gamma(S^\mu, \omega_{S^\mu})$? Let $N_\mu \subset \mathfrak{L}(N)_x$ be the stabilizor of $t^\mu \in \mathrm{Gr}_{G,x}$. We may pick a sequence $N_\mu \subset N^k \subset \mathfrak{L}(N)_x$ such that $N^k \subset N^{k+1}$ is closed, and N^k/N_μ is isomorphic to an affine space. Recall that $\omega_{S^\mu} \widetilde{\to} \operatorname{colim}_k \omega_{N^k/N_\mu}$ in $Shv(S^\mu)$ (by [22], 9.2.6). So, $\mathrm{R}\Gamma(S^\mu, \omega_{S^\mu}) \widetilde{\to} \operatorname{colim}_k \mathrm{R}\Gamma(N^k/N_\mu, \omega_{N^k/N_\mu}) \widetilde{\to} e$. Indeed, $\omega_{N^k/N_\mu} \widetilde{\to} e[2\dim(N^k/N_\mu]$.

Question: what is $\mathrm{R}\Gamma(\mathrm{Gr}_G^\gamma, \omega_{\mathrm{Gr}_G})$ for $\gamma \in \pi_1(G)$?

1.3.21. For 5.2.7. The following is used. Let $x, y \in X$ be distinct. Then $N_{X-\{x,y\}}$ acts transitively on $\mathfrak{L}_y(N)/\mathfrak{L}_y^+(N)$. Indeed, $X - x$ is affine. Let be given an N-torsor $\mathcal{F}_N$ on $X - x$ with a trivialization over $X - \{x, y\}$. This gives an element of $\mathfrak{L}_y(N)/\mathfrak{L}_y^+(N)$. Pick a global trivialization over $X - x$ of this N-torsor. This gives the claim.

In 5.2.7 (ii) he means that $\mathcal{F}$ is $(N_{X-\{x,y\}}), -\chi_y)$-equivariant under the action on the factor $\mathfrak{L}_y(N)/\mathfrak{L}_y^+(N)$ by right translations (if one uses the left translations, it would be χ_y instead).

1.3.22. For 5.2.7, a question: Let N be a group ind-scheme of the form $\operatorname{colim}_{i\in I} N_i$ with N_i pro-unipotent, for $i \to j$ in I, $N_i \to N_j$ is a placid closed immersion, and I is small filtered category. Let 0 be the initial object of I. Let S be an ind-scheme of ind-finite type. Assume given an action of N on S, and consider $(N/N_0) \times S$.

How to better identify $Shv((N/N_0) \times S)^{N\times N}$ with $Shv(S)^N$? Dennis proposed the claim that the !-restriction along $S \hookrightarrow (N/N_0) \times S$ gives an equivalence

$$Shv((N/N_0) \times S)^{N\times N} \to Shv(S)^N$$

Is this correct? Consider the projection $\mathrm{pr} : (N/N_0) \times S \to S$. Is it true that $\mathrm{pr}^!$ identifies $Shv(S)^N$ with $Shv((N/N_0) \times S)^{N\times N}$?

1.3.23. Let G be a placid ind-scheme, and $G \in \mathcal{G}\mathrm{rp}(\mathrm{PreStk})$. Let H be a group scheme of finite type over k, L be a character local system on H, $C \in (Shv(G), *) - mod$. Let $\alpha : G \to H$ be a morphism in $\mathcal{G}\mathrm{rp}(\mathrm{PreStk})$. The functor $\alpha^* : Shv(H) \to Shv(G)$ does not make sense in general. However, (G, α^*L)-invariants in C make sense as follows. We have a diagram, where all the functors are monoidal

$$Shv(G) \stackrel{\alpha_*}{\to} Shv(H) \stackrel{K\mapsto K\otimes L^{-1}}{\to} Shv(H) \stackrel{\mathrm{R}\Gamma(H,\cdot)}{\to} \mathrm{Vect}$$

Write Vect_L for Vect equipped with the new $(Shv(G), *)$-action given by the above composition. Then set

$$C^{G,\alpha^*L} = \mathrm{Fun}_{Shv(G)}(\mathrm{Vect}_L, C)$$

1.4. **Comments to arxiv version 6.**

1.4.1. In Thm. 6.4.8 the functor $\mathrm{unit}_I^! : \mathrm{Whit}(\mathcal{Y}_I) \to \mathrm{Whit}(X^I \times \mathcal{Y})$ is $Shv(X^I)$-linear, so its partially defined left adjoint has is a left-lax $Shv(X^I)$-linear. A more precise claim is that this lax structure is strict.

1.4.2. For 6.6.3. It is claimed at the end of Section 6.6.3 that it suffices to check that for every j, $\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I}$ is defined on the essential image of

$$(unit_I)_! \circ \mathrm{oblv}_{I^j,\chi_x} : Shv(X^I \times \mathcal{Y})^{I^j,\chi_x} \to Shv(\mathcal{Y}_I)$$

This has to be justified. I try to do it as follows. The point is that the group scheme $X^I \times \mathcal{L}_x(N)$ does not act on $\mathcal{Y}_I$. Instead, consider the map $\mathcal{H}_I^+(N)' \to X^I \times \mathfrak{L}_x(N)$ and recall that the oblivion functor

$$\mathrm{Whit}(X^I \times \mathcal{Y}) \widetilde{\to} Shv(X^I \times \mathcal{Y})^{\mathfrak{L}_x(N),\chi_x} \to Shv(X^I \times \mathcal{Y})^{\mathfrak{L}_I^+(N)',\chi_I}$$

is an equivalence. But the group scheme $\mathfrak{L}_I^+(N)'$ acts on $\mathcal{Y}_I$ over X^I. The map $\mathrm{unit}_I : X^I \times \mathcal{Y} \to \mathcal{Y}_I$ is $\mathfrak{L}_I^+(N)'$-equivariant.

Now for $j\geq 1$ the functors $\mathrm{Av}_!^{\mathfrak{L}_x(N),\chi_x}\mathrm{oblv}_{I^j,\chi_x}: Shv(X^I\times\mathcal{Y})^{I^j,\chi_x}\to \mathrm{Whit}(X^I\times\mathcal{Y})$ are well defined and their essential images generate $\mathrm{Whit}(X^I\times\mathcal{Y})$. Now for $K\in Shv(X^I\times\mathcal{Y})^{I^j,\chi_x}$,

$$\mathrm{Av}_!^{\mathfrak{L}_x(N),\chi_x}(K)\widetilde{\to}\mathrm{Av}_!^{\mathfrak{L}_I^+(N)',\chi_I}(K)\in\mathrm{Whit}(X^I\times\mathcal{Y})$$

Now at least in the constructible context we have

$$(unit_I)_!\,\mathrm{Av}_!^{\mathfrak{L}_I^+(N)',\chi_I}(K)\widetilde{\to}\mathrm{Av}_!^{\mathfrak{L}_I^+(N)',\chi_I}(unit_I)_!K$$

naturally. Further, $\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I}\mathrm{Av}_!^{\mathfrak{L}_I^+(N)',\chi_I}\widetilde{\to}\mathrm{Av}_!^{\mathfrak{L}_I(N),\chi_I}$.

1.4.3. Let G be a unipotent group scheme, $H\subset G$ a closed subgroup. Let $Y\in\mathrm{PreStk}_{lft}$ with a G-action. Let L be a character local system on G. Write $\mathrm{Av}_!^{G/H}$ for the partially defined left adjoint to $\mathrm{oblv}: Shv(Y)^{G,L}\to Shv(Y)^{H,L}$. (Recall that in the constructible context it is always defined by Section 1.3.6). Consider the map

$$a: G\times^H Y\to Y$$

coming from the action map. For $a^!: Shv(Y)\to Shv(G\times^H Y)$ we have the partially defined left adjoint $a_!: Shv(G\times^H Y)\to Shv(Y)$.

Let $\nu: G\times Y\to G\times^H Y$ be the projection. We convent that we identify

$$Shv(G\times^H Y)\widetilde{\to}Shv(G\times Y)^H$$

in such a way that $\mathrm{oblv}: Shv(G\times Y)^H\to Shv(G\times Y)$ corresponds to $\nu^!$. Then for $K\in Shv(Y)^{H,L}$ we define $L\widetilde{\boxtimes}K\in Shv(G\times^H Y)$ as the object $L\boxtimes K\in Shv(G\times Y)^H$ with respect to the action of $h\in H$ sending $(g,y)\in G\times Y$ to (gh^{-1},gy).

Claim: $\mathrm{Av}_!^{G/H}$ is defined on $K\in Shv(Y)^{H,L}$ iff the functor $a_!$ is defined on $L\widetilde{\boxtimes}K$.

Proof. Let $\mathrm{act}: G\times Y\to Y$ be the action map. Let $\mathrm{pr}_G: G\times Y\to G, \mathrm{pr}_Y: G\times Y\to Y$ be the projections. For $M\in Shv(Y)$, $K\in Shv(Y)^{H,L}$ one has

$$\mathcal{H}om(L\widetilde{\boxtimes}K, a^!M)\widetilde{\to}\mathcal{H}om_{Shv(G\times Y)}(L\boxtimes K,\mathrm{act}^! M)\widetilde{\to}$$
$$\mathcal{H}om(\mathrm{pr}_Y^* K,(\mathrm{pr}_G^!\mathbb{D}L)\otimes^!\mathrm{act}^! M)\widetilde{\to}\mathcal{H}om(K,(\mathrm{pr}_Y)_*((\mathrm{pr}_G^!\mathbb{D}L)\otimes^!\mathrm{act}^! M))$$

It is easy to see that $(\mathrm{pr}_Y)_*((\mathrm{pr}_G^!\mathbb{D}L)\otimes^!\mathrm{act}^! M)\widetilde{\to}\mathrm{act}_*(L\boxtimes M)[2\dim G]\widetilde{\to}L*M[2\dim G]$. So,

$$\mathcal{H}om(L\widetilde{\boxtimes}K[2\dim G],a^!M)\widetilde{\to}\mathcal{H}om(K,L*M).$$

The partially defined functor $Shv(Y)^{H,L}\to Shv(Y)$, $K\mapsto a_!(L\widetilde{\boxtimes}K)[2\dim G]$ is the partially left adjoint to $Shv(Y)\to Shv(Y)^{H,L}$, $M\mapsto L*M$.

If now $M\in Shv(Y)^{G,L}$ then $L*M\widetilde{\to}M$ and we get

$$\mathcal{H}om(L\widetilde{\boxtimes}K[2\dim G],a^!M)\widetilde{\to}\mathcal{H}om(K,M)$$

Our claim follows. □

This kind of claim is used in A.1.1 of the paper.

1.4.4.

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