AN ANALOG OF THE THETA-SHEAF ON THE 3-FOLD METAPLECTIC COVER OF G_2?

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0.0.1. We try to geometrize (in the global setting) the construction of an automorphic sheaf on the 3-fold metaplectic cover of Bun_G, where G is simple of type G_2, given in the local setting by Savin in [8].

0.0.2. Work over an algebraically closed field k of charcateristic $p > 0$. Let X be a smooth complete irreducible curve. Let Ω be the canonical line bundle on X. Pick a square root $\Omega^{\frac{1}{2}}$ of Ω.

0.0.3. Let G be a split group of type G_2 over k. Pick a maximal torus and Borel $T \subset B \subset G$. Let $\check{\Lambda}$ be the weights lattice of T, Λ the coweights lattice. Let $\check{\alpha}, \check{\beta}$ be simple roots of (T, B), where $\check{\alpha}$ is short. Recall that the positive roots are

$$\check{\alpha}, \check{\beta}, \check{\alpha} + \check{\beta}, \check{\alpha} + 2\check{\beta}, \check{\alpha} + 3\check{\beta}, 2\check{\alpha} + 3\check{\beta}$$

Let P_β be the parabolic of G generated by B and the root space corresponding to $-\check{\beta}$. Let $U_\beta \subset P_\beta$ be the unipotent radical. Recall that U_β has a 1-dimensional center $Z(U_\beta)$ corresponding to the maximal root $2\check{\alpha} + 3\check{\beta}$, and $U_\beta/Z(U_\beta)$ is abelian of dimension 4. Let M_β be the Levi of P_β.

0.0.4. Let $\check{\omega}_\alpha, \check{\omega}_\beta$ be the fundamental weight corresponding to the coroots α, β. Let V be the irreducible M_β-module with h.w. $\check{\omega}_\beta$. The map so obtained $M_\beta \to \mathrm{GL}(V)$ is an isomorphism, we identify M_β with $\mathrm{GL}(V)$ this way. Then the $\mathrm{GL}(V)$-representations $Z(U_\beta)$ and $\det V$ are identified. Besides, $U_\beta/Z(U_\beta)$ identifies naturally with $\mathcal{V} := (\mathrm{Sym}^3 V) \otimes \det V^*$.

Lemma 0.0.5. *There is a natural* $\mathrm{GL}(V)$*-equivariant symplectic form* $\omega : \wedge^2 \mathcal{V} \to \det V$.

Proof. We have $V \widetilde{\to} V^* \otimes \det V$. This gives $\mathrm{Sym}^3(V^* \otimes \det V) \widetilde{\to} (\mathrm{Sym}^3 V^*) \otimes (\det V)^3$. The obtained isomorphism $\mathcal{V} \widetilde{\to} \mathcal{V}^* \otimes \det V$ is the desired symplectic form. $\square$

Let $H(\mathcal{V}) = \mathcal{V} \oplus \det V$ be the Heisenberg group with the product

$$(v_1, z_1)(v_2, z_2) = (v_1 + v_2, z_1 + z_2 + \frac{1}{2}\omega(v_1, v_2))$$

In ([2], Section 3.2) it is shown that $P_\beta \widetilde{\to} M_\beta \ltimes H(\mathcal{V})$.

So, Bun_{P_β} classifies $V \in \mathrm{Bun}_2$ and a torsor under $H(\mathcal{V})$, where now $\mathcal{V} = (\mathrm{Sym}^3 V) \otimes \det V^*$ is a vector bundle on X with the above symplectic form, and $H(\mathcal{V})$ is the corresponding Heisenberg group scheme on X.

0.0.6. Recall the extended theta-sheaf from ([4], Section 2.2). Namely, for $n > 0$ we have the stacks $\mathrm{Bun}_{\mathbb{G}_n}, \widetilde{\mathrm{Bun}}_{\mathbb{G}_n}$ and the perverse sheaf Aut^e_ψ on $\widetilde{\mathrm{Bun}}_{\mathbb{G}_n}$ defined in *loc.cit.* It is understood that we fixed a nontrivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell$ and work with $\bar{\mathbb{Q}}_\ell$-sheaves.

Here $\mathrm{Bun}_{\mathbb{G}_n}$ classifies $M_1 \in \mathrm{Bun}_{2n+2}$ with a symplectic form $\wedge^2 M_1 \to \Omega$ and a section $v : \Omega \hookrightarrow M_1$, which is a subbundle.

0.0.7. Recall the definition of octonions from [9]. Let $D = Mat_2(k)$ and $C = D \oplus D$. Define the product in C by the formula from [9]. Namely, the conjugation $x \mapsto \bar{x}$ on D sends

$$x = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \mapsto \bar{x} = \begin{pmatrix} \xi_{22} & -\xi_{12} \\ -\xi_{21} & \xi_{11} \end{pmatrix}$$

The product in C is given by

$$(x, y)(u, v) = (xu + \bar{v}y, vx + y\bar{u})$$

for $x, y, u, v \in D$. The norm $N : C \to k$ is given by $N((x, y)) = \det x - \det y$ for $(x, y) \in D \oplus D = C$. We assume $G = \mathrm{Aut}(C)$.

0.0.8. Set $H = \mathbb{SO}(C_0, N)$. Then $G \subset H$ is a closed group subscheme, we may also view $G \subset \bar{H} := \mathbb{SO}(C, N)$.

The stack Bun_H classifies $W \in \mathrm{Bun}_7$ with a non-degenerate symmetric form $\mathrm{Sym}^2 W \to \mathcal{O}$ and a compatible trivialization $\det W \widetilde{\to} \mathcal{O}$.

Write $\mathrm{Bun}_{\mathbb{O}_7,\Omega}$ the stack classifying $W \in \mathrm{Bun}_7$ with a non)degenerate symmetric form $\mathrm{Sym}^2 W \to \Omega$. We have a canonical Pfaffian line bundle Pf on $\mathrm{Bun}_{\mathbb{O}_7,\Omega}$ defined in ([1], Section 4.2.1). Pick $i \in k$ with $i^2 = -1$. This choice yields an isomorphism $Pf(W)^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, W)$ for $W \in \mathrm{Bun}_{\mathbb{O}_7,\Omega}$ as in *loc.cit.*.

We have the map $\mathrm{Bun}_H \to \mathrm{Bun}_{\mathbb{O}_7,\Omega}$, $W \mapsto W \otimes \Omega^{\frac{1}{2}}$. By abuse of notations, the restriction of Pf to Bun_H is still denoted Pf.

Recall that Pf is a generator of $\mathrm{Pic}(\mathrm{Bun}_H) \widetilde{\to} \mathbb{Z}$. Write Pf_G for its restriction to Bun_G. Recall that Pf_G is a generator of $\mathrm{Pic}(\mathrm{Bun}_G)$ by [5].

0.0.9. Write $\widetilde{\mathrm{Bun}}_G$ for the gerbe of degree 3 roots of Pf_G over Bun_G. We look for a conjectural construction of an automorphic sheaf K on $\widetilde{\mathrm{Bun}}_G$ in the sense of the metaplectic geometric Langlands.

0.0.10. Let $R \subset \bar{H}$ be the parabolic preserving the isotropic subspace $L(0, 1_D) \subset C$, where

$$L = \{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\} \subset D$$

Let $U(R) \subset R$ be its unipotent radical.

Think of a point of $\mathrm{Bun}_{\bar{H}}$ as $\bar{W} \in \mathrm{Bun}_8$ with a nondegenerate symmetric form $\mathrm{Sym}^2 \bar{W} \to \mathcal{O}$ and a compatible isomorphism $\det \bar{W} \widetilde{\to} \mathcal{O}$.

Now Bun_R classifies a point of $\mathrm{Bun}_{\bar{H}}$ as above together with an isotropic subbundle $V \subset \bar{W}$ with $\mathrm{rk}(V) = 2$. Write $V^\perp \subset \bar{W}$ for its orthogonal complement in $\bar{W}$. Then $\bar{W}/V^\perp \widetilde{\to} V^*$, and we have a nondegenerate form $\mathrm{Sym}^2(V^\perp/V) \to \mathcal{O}$ with a compatible isomorphism $\det(V^\perp/V) \widetilde{\to} \mathcal{O}$.

For such point of Bun_R, $V\otimes(V^\perp/V)$ is equipped with a symplectic form $\omega_R : \wedge^2(V\otimes(V^\perp/V))\to(\det V)$. So, Bun_R classifies: $V\in\mathrm{Bun}_2$, $V^\perp/V\in\mathrm{Bun}_{\mathbb{SO}_4}$ and a torsor under the Heisenberg group scheme

$$H(V\otimes(V^\perp/V),\omega_R)=(V\otimes(V^\perp/V))\oplus\det V$$

on X.

0.1. We may and do assume $P_\beta=R\cap G$ inside $\bar H$. Now a point of Bun_{P_β} gives rise to $\bar W\in\mathrm{Bun}_{\bar H}$ with an isotropic subbundle $V\subset\bar W$ and an isomorphism

$$\mathcal{E}nd(V)\widetilde{\to}V^\perp/V$$

compatible with trivializations of the determinants and with the symmetric forms on both sides.

Remark 0.1.1. *In general, let $M\in\mathrm{Bun}_{2n}$ with a given symplectic form $\wedge^2M\to\mathcal{A}$ for some $\mathcal{A}\in\mathrm{Bun}_1$. Then we let $H(M)=M\oplus\mathcal{A}$ denote the corresponding Heisenberg group scheme on X. The stack $\mathrm{Bun}_{H(M)}$ is as follows. It classifies $M_1\in\mathrm{Bun}_{2n+2}$ with a symplectic form $\wedge^2M_1\to\mathcal{A}$ and an inclusion $\mathcal{A}\hookrightarrow M_1$, whose image is a subbundle. Let $M'\subset M_1$ be the orthogonal complement to $\mathcal{A}$, so we get a symplectic form $\wedge^2(M'/\mathcal{A})\to\mathcal{A}$. We require in addition a given isomorphism $M'/\mathcal{A}\widetilde{\to}M$ compatible with symplectic forms.*

0.1.2. The above remark gives the following description of Bun_{P_β}. This is the stack classifying $V\in\mathrm{Bun}_2$ and $M_1\in\mathrm{Bun}_6$ with a symplectic form $\wedge^2M_1\to\det V$ and a subbundle $\det V\subset M_1$. Denote by M' the orthogonal complement of $\det V$ in M_1. Then we are given in addition an isomorphism $M'/\det V\widetilde{\to}(\mathrm{Sym}^3V)\otimes\det V^*$ compatible with symplectic forms on both sides.

0.1.3. The latter description allows to descibe the map $\mathrm{Bun}_{P_\beta}\to\mathrm{Bun}_R$ explicitly as follows.

Given $V\in\mathrm{Bun}_2$, we have canonically

$$V\otimes\mathcal{E}nd_0(V)\widetilde{\to}\mathcal{V}\oplus V, \tag{1}$$

where $\mathcal{V}$ is that of Section 0.0.4. Here $\mathcal{E}nd_0(V)=(\det V)^*\otimes\mathrm{Sym}^2V$ is the sheaf of traceless endomorphisms of V. Moreover, (1) is compatible with symplectic forms (with values in $\det V$) on both sides. So,

$$\tau:H(\mathcal{V})\hookrightarrow H(V\otimes\mathcal{E}nd(V))$$

is naturally a subgroup.

The map $\mathrm{Bun}_{P_\beta}\to\mathrm{Bun}_R$ sends a pair $(V\in\mathrm{Bun}_2,\mathcal{F})$, where $\mathcal{F}$ is a $H(\mathcal{V})$-torsor on X to the triple $(V,V^\perp/V:=\mathcal{E}nd(V),\mathcal{F}')$, where $\mathcal{E}nd(V)\in\mathrm{Bun}_{\mathbb{SO}_4}$ naturally, and $\mathcal{F}'$ is the torsor under $H(V\otimes(V^\perp/V))$ obtained from $\mathcal{F}$ via the extension of scalars by τ.

0.1.4. Let $\mathrm{Bun}_{P_\beta,\Omega}$ be the stack classifying a point of Bun_{P_β} given as above together with an isomorphism $\det V \widetilde{\to} \Omega$. Consider the natural map

$$\nu : \mathrm{Bun}_{P_\beta,\Omega} \to \mathrm{Bun}_G \tag{2}$$

It lifts to a morphism $\tilde{\nu} : \mathrm{Bun}_{P_\beta,\Omega} \to \widetilde{\mathrm{Bun}}_G$.

Then $\tilde{\nu}^* K[\dim.\mathrm{rel}(\tilde{\nu})]$ is easily expressed explicitly using Aut^e_ψ.

For this purpose, first define $f : \mathrm{Bun}_{P_\beta,\Omega} \to \mathrm{Bun}_{\mathbb{G}_2}$ as follows. In terms of the description from Section 0.1.2, a point of $\mathrm{Bun}_{P_\beta,\Omega}$ is a collection

$$(V, M_1, \wedge^2 M_1 \to \Omega, \Omega \hookrightarrow M_1)$$

and an isomorphism $M'/\Omega \widetilde{\to} (Sym^3 V) \otimes \Omega^{-1}$ compatible with the symplectic forms on both sides. The map f sends this collection to

$$(M_1, \wedge^2 M_1 \to \Omega, \Omega \hookrightarrow M_1) \in \mathrm{Bun}_{\mathbb{G}_2}$$

Let $\mathrm{Bun}_{M_\beta,\Omega}$ be the stack classifying $V \in \mathrm{Bun}_2$ with an isomorphism $\det V \widetilde{\to} \Omega$.

Lemma 0.1.5. *For $V \in \mathrm{Bun}_{M_\beta,\Omega}$, there is a canonical isomorphism*

$$\det \mathrm{R}\Gamma(X, (\mathrm{Sym}^3 V) \otimes \Omega^{-1}) \widetilde{\to} \det \mathrm{R}\Gamma(X, V)^{10}$$

Proof. Let $\mathcal{L}$ be the line bundle on $\mathrm{Bun}_{M_\beta,\Omega}$ with fibre $\det \mathrm{R}\Gamma(X, V)$ at V. Recall that $\mathcal{L}$ is a generator of $\mathrm{Pic}(\mathrm{Bun}_{M_\beta,\Omega})$. So, there is m such that $\mathcal{L}^m$ is isomorphic to the line bundle with fibre $\det \mathrm{R}\Gamma(X, (\mathrm{Sym}^3 V) \otimes \Omega^{-1})$.

We check that $m = 10$ and moreover this isomorphism is canonically normalized. For this purpose, consider the map $h : \mathrm{Bun}_1 \to \mathrm{Bun}_{M_\beta,\Omega}$ sending $\mathcal{A}$ to $\mathcal{A} \oplus \mathcal{A}^* \otimes \Omega$. Note that $h^*\mathcal{L}$ is the line bundle with fibre $\det \mathrm{R}\Gamma(X, \mathcal{A})^2$ at $\mathcal{A} \in \mathrm{Bun}_1$. So, it suffices to get the desired canonical isomorphism after restriction under h.

This is easily reduced to the lemma below. □

Lemma 0.1.6. *The line bundle on* Bun_1 *with fibre*

$$\det \mathrm{R}\Gamma(X, \mathcal{A}^3 \otimes \Omega^{-1}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{A})^{-9}$$

at $\mathcal{A} \in \mathrm{Bun}_1$ is canonically constant on the whole of Bun_1.

Proof. left to a reader. □

0.1.7. Now we extend the map f to a morphism $\tilde{f} : \mathrm{Bun}_{P_\beta,\Omega} \to \widetilde{\mathrm{Bun}}_{\mathbb{G}_2}$. It sends a point of $\mathrm{Bun}_{P_\beta,\Omega}$ to its image by f together with

$$\mathcal{B} = \det \mathrm{R}\Gamma(X, V)^5$$

equipped with an isomorphism $\det \mathrm{R}\Gamma(X, (\mathrm{Sym}^3 V) \otimes \Omega^{-1}) \widetilde{\to} \mathcal{B}^2$ given by Lemma 0.1.5.

I first thought that we could propose the following.

Wrong conjecture 0.1.8. *There is a genuine automorphic sheaf K on $\widetilde{\mathrm{Bun}}_G$ equipped with an isomorphism*

$$\tilde{f}^* \mathrm{Aut}^e_\psi[\dim.\mathrm{rel}(\tilde{f})] \widetilde{\to} \nu^* K[\dim.\mathrm{rel}(\nu)]$$

and satisfying the Hecke property on $\widetilde{\mathrm{Bun}}_G$. It should geometrize the representation constructed by Savin in [8].

It is wrong already for $X=\mathbb{P}^1$. Namely, the sheaf K proposed in the conjecture would depend on ψ. However, it should not depend on ψ. The conclusion is that one should really first induce along $[M_\beta, M_\beta]U_\beta\hookrightarrow P_\beta$, and then hope for a descent along $\operatorname{Bun}_{P_\beta}\to \widetilde{\operatorname{Bun}}_G$. We will correct the conjecture below.

0.1.9. What is the smoothness locus ${}^0\operatorname{Bun}_{P_\beta,\Omega}$ of (2)?

Let $T_\beta\subset M_\beta$ be the maximal torus contained in T. We may pick a T_β-torsor $\mathcal{F}_0$ with the following properties. Let M^0_β be the twist of $[M_\beta, M_\beta]$ by $\mathcal{F}_0$ via the adjoint action of T_β. So, M^0_β is a group scheme on X. Then $\operatorname{Bun}_{M^0_\beta}$ is the stack classifying $V\in\operatorname{Bun}_2$ with an isomorphism $\det V\widetilde{\to}\Omega$. Let P^0_β be the twist of $[M_\beta,M_\beta]U_\beta$ by $\mathcal{F}_0$ on X. Then $\operatorname{Bun}_{P^0_\beta}\widetilde{\to}\operatorname{Bun}_{P_\beta,\Omega}$ naturally.

We could also consider the twist G^0 of G by $\mathcal{F}_0$ via the adjoint action of T_β, but we would get anyway $\operatorname{Bun}_{G^0}\widetilde{\to}\operatorname{Bun}_G$ naturally. Let $P^d_\beta=[M_\beta,M_\beta]U_\beta$. Let $\mathfrak{p}_\beta,\mathfrak{p}^d,\mathfrak{g}$ be the Lie algebras of P_β, P^d_β, G.

The above gives the following. The map ν is smooth at a P_β-torsor $\mathcal{F}\in\operatorname{Bun}_{P_\beta,\Omega}$ iff $\mathrm{H}^1(X,(\mathfrak{g}/\mathfrak{p}^b_\beta)_{\mathcal{F}})=0$.

In terms of the description of Section 0.1.2 for a point $\mathcal{F}\in\operatorname{Bun}_{P_\beta,\Omega}$ we get $(\mathfrak{g}/\mathfrak{p}^b_\beta)^*_{\mathcal{F}}\widetilde{\to} M_1$. So, the smoothness locus ${}^0\operatorname{Bun}_{P_\beta,\Omega}$ of ν is given precisely by the property that $\mathrm{H}^0(X, M_1\otimes\Omega)=0$.

In particular, for this smoothness locus to be nonempty, we need $\mathrm{H}^0(X,\Omega^2)=0$. This happens only for $g=0$, where g is the genus of X. Indeed, $\chi(\Omega^2)=3g-3$. If $\mathrm{H}^0(X,\Omega^2)=0$ then $\chi(\Omega^2)\le 0$, so $g\le 1$. For $g=1$ we have $\Omega\widetilde{\to}\mathcal{O}$ and $\mathrm{H}^0(X,\Omega^2)\ne 0$.

For $g=0$ the stack ${}^0\operatorname{Bun}_{P_\beta,\Omega}$ is non empty. Indeed, let ${}^0\operatorname{Bun}_{M_\beta,\Omega}\subset\operatorname{Bun}_{M_\beta,\Omega}$ be the open substack classifying V isomorphic to $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Then the preimage of ${}^0\operatorname{Bun}_{M_\beta,\Omega}$ under the projection $\operatorname{Bun}_{P_\beta,\Omega}\to\operatorname{Bun}_{M_\beta,\Omega}$ is contained in ${}^0\operatorname{Bun}_{P_\beta,\Omega}$.

0.2. It seems there is no natural section of the restriction of the gerbe $\widetilde{\operatorname{Bun}}_G\to\operatorname{Bun}_G$ under $\operatorname{Bun}_{P_\beta}\to\operatorname{Bun}_G$. Even over $\operatorname{Bun}_{M_\beta}$ there seems no natural section.

Indeed, for $V\in\operatorname{Bun}_{M_\beta}$ the question is, in particular, if

$$\mathcal{F}(V):=\det\mathrm{R}\Gamma(X,V\otimes\Omega^{\frac12})\otimes\det\mathrm{R}\Gamma(X,\mathcal{E}nd(V)\otimes\Omega^{\frac12})\otimes\det\mathrm{R}\Gamma(X,V^*\otimes\Omega^{\frac12})$$

is naturally a 6-th power of some line bundle. We have by ([6], Lemma 1)

$$\det\mathrm{R}\Gamma(X,\mathcal{E}nd(V)\otimes\Omega^{\frac12})\widetilde{\to}\frac{\det\mathrm{R}\Gamma(X,V\otimes\Omega^{\frac12})^2\otimes\det\mathrm{R}\Gamma(X,V^*)^2}{\det\mathrm{R}\Gamma(X,\det V^*)^2\otimes\det\mathrm{R}\Gamma(X,\mathcal{O})^2}$$

and

$$\det\mathrm{R}\Gamma(X,V^*)\widetilde{\to}\frac{\det\mathrm{R}\Gamma(X,V\otimes\Omega^{\frac12})\otimes\det\mathrm{R}\Gamma(X,\Omega^{\frac{-1}{2}})\otimes\det\mathrm{R}\Gamma(X,\Omega^{\frac12}\otimes\det V)}{\det\mathrm{R}\Gamma(X,\det V)\otimes\det\mathrm{R}\Gamma(X,\mathcal{O})}$$

So,

$$\mathcal{F}(V)\widetilde{\to}\det\mathrm{R}\Gamma(X,V\otimes\Omega^{\frac12})^6\otimes\frac{\det\mathrm{R}\Gamma(X,\Omega^{-\frac12})^2\otimes\det\mathrm{R}\Gamma(X,\Omega^{\frac12}\otimes\det V)^2}{\det\mathrm{R}\Gamma(X,\det V)^2\otimes\det\mathrm{R}\Gamma(X,\mathcal{O})^4\otimes\det\mathrm{R}\Gamma(X,\det V^*)^2}$$

Lemma 0.2.1. *For $\mathcal{A} \in \mathrm{Bun}_1$ one has canonically*
$$\frac{\det \mathrm{R}\Gamma(X, \mathcal{A} \otimes \Omega^{\frac{1}{2}}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{A}^* \otimes \Omega^{\frac{1}{2}})}{\det \mathrm{R}\Gamma(X, \mathcal{A}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{A}^*)} \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}})^2}{\det \mathrm{R}\Gamma(X, \mathcal{O})^2}$$

Proof. For $\mathcal{A}, \mathcal{B}\, \mathrm{Bun}_1$ set
$$\mathcal{K}(\mathcal{A}, \mathcal{B}) = \frac{\det \mathrm{R}\Gamma(X, \mathcal{A} \otimes \mathcal{B}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})}{\det \mathrm{R}\Gamma(X, \mathcal{A}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{B})}$$
We calculate
$$\mathcal{K}(\mathcal{A}, \mathcal{A}^*) \otimes \mathcal{K}(\mathcal{A} \otimes \Omega^{\frac{1}{2}}, \mathcal{A}^* \otimes \Omega^{\frac{1}{2}})^{-1}$$
Recall that $\mathcal{K}(\mathcal{A}, \mathcal{B})$ is canonically bilinear and symmetric by ([7], Section 4.2.1 - 4.2.2). This gives $\mathcal{K}(\mathcal{A} \otimes \Omega^{\frac{1}{2}}, \mathcal{A}^* \otimes \Omega^{\frac{1}{2}}) \widetilde{\to} \mathcal{K}(\mathcal{A}, \mathcal{A}^*) \otimes \mathcal{K}(\Omega^{\frac{1}{2}}, \Omega^{\frac{1}{2}})$. □

Lemma 0.2.1 gives for $V \in \mathrm{Bun}_{M_\beta}$
$$\mathcal{F}(V) \widetilde{\to} \det \mathrm{R}\Gamma(X, V \otimes \Omega^{\frac{1}{2}})^6 \otimes \frac{\det \mathrm{R}\Gamma(X, \Omega^{\frac{-1}{2}})^2 \otimes \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}})^4}{\det \mathrm{R}\Gamma(X, \mathcal{O})^8 \otimes \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}} \otimes \det V)^2}$$
Applying Lemma 0.2.1 for $\mathcal{A} = \Omega^{-\frac{1}{2}}$ we get
$$\det \mathrm{R}\Gamma(X, \Omega^{-\frac{1}{2}}) \otimes \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}})^3 \widetilde{\to} \det \mathrm{R}\Gamma(X, \mathcal{O})^4$$
So, the above expression simplifies: for $V \in \mathrm{Bun}_{M_\beta}$ we get canonically
$$\mathcal{F}(V) \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, V \otimes \Omega^{\frac{1}{2}})^6}{\det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}})^2 \otimes \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}} \otimes \det V)^2} \tag{3}$$

So, the question roughly becomes the following: for $V \in \mathrm{Bun}_{M_\beta}$ does
$$\det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}} \otimes \det V)$$
have a natural 3rd root? There seems no such natural 3rd root.

0.2.2. Let us look for a natural cover of M_β over which the gerbe $\widetilde{\mathrm{Bun}}_G \to \mathrm{Bun}_G$ trivializes.

This cover should be of the form Bun_{M_β} again for a suitable homomorphism $M_\beta \to M_\beta$ of the form $x \mapsto x(\det x)^r$ for some r. This is the covering $\mathrm{GL}_2 \to \mathrm{GL}_2 / \mu_{2r+1}$.

Let us try $r = 1$. We arrive at the following.

Lemma 0.2.3. *For $V' \in \mathrm{Bun}_2$ let $V = V' \otimes \det V'$. Then there is a canonical isomorphism*
$$\mathcal{F}(V) \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, V \otimes \Omega^{\frac{1}{2}})^6 \otimes \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}})^{14}}{\det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}} \otimes \det V')^{18}}$$

Proof. **Step 1**. For for $\mathcal{A} \in \mathrm{Bun}_1$ we have canonically
$$\det \mathrm{R}\Gamma(X, \mathcal{A}^3 \otimes \Omega^{\frac{1}{2}}) \widetilde{\to} \det \mathrm{R}\Gamma(X, \mathcal{A} \otimes \Omega^{\frac{1}{2}})^9 \otimes \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}})^{-8}$$
Indeed, set $\mathcal{B} = \mathcal{A} \otimes \Omega^{\frac{1}{2}}$ and apply Lemma 0.1.6 to $\mathcal{B}$. This gives
$$\det \mathrm{R}\Gamma(X, \mathcal{A}^3 \otimes \Omega^{\frac{1}{2}}) \widetilde{\to} \det \mathrm{R}\Gamma(X, \mathcal{B}^3 \otimes \Omega^{-1}) \widetilde{\to} \det \mathrm{R}\Gamma(X, \mathcal{B})^9 \otimes \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}})^{-8}$$

Step 2. Let $V' \in \mathrm{Bun}_2$ and $V = V' \otimes \det V'$ then $\det V \widetilde{\to} (\det V')^3$. Applying Step 1 for $\mathcal{A} = \det V'$ we get

$$\det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}} \otimes \det V) \widetilde{\to} \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}} \otimes \det V')^9 \otimes \det \mathrm{R}\Gamma(X, \Omega^{\frac{1}{2}})^{-8}$$

Our claim follows now from (3). □

Consider the map $Cov : \mathrm{Bun}_2 \to \mathrm{Bun}_{M_\beta}$, $V' \mapsto V' \otimes \det V'$. Write $\mathrm{Bun}_2 \times_{\mathrm{Bun}_{M_\beta}} \mathrm{Bun}_{P_\beta}$ for the base change, where we used the map Cov. Let

$$\nu_{Cov} : \mathrm{Bun}_2 \times_{\mathrm{Bun}_{M_\beta}} \mathrm{Bun}_{P_\beta} \to \mathrm{Bun}_G$$

be the composition

$$\mathrm{Bun}_2 \times_{\mathrm{Bun}_{M_\beta}} \mathrm{Bun}_{P_\beta} \stackrel{\mathrm{pr}}{\to} \mathrm{Bun}_{P_\beta} \stackrel{\nu}{\to} \mathrm{Bun}_G$$

From Lemma 0.2.3 we see that ν_{Cov} lifts to a morphism

$$\tilde{\nu}_{Cov} : \mathrm{Bun}_2 \times_{\mathrm{Bun}_{M_\beta}} \mathrm{Bun}_{P_\beta} \to \widetilde{\mathrm{Bun}}_G$$

0.2.4. To get a conjectural construction of K on $\widetilde{\mathrm{Bun}}_G$, we could look for an explicit formula for the complex $\tilde{\nu}^*_{Cov} K$ inspired by the formula for the R-model of the minimal sheaf for $\mathbb{SO}_{2n}$ from [4]. However, we propose something else.

Idea: consider the metapelctic dual H of G equipped with its metapelctic data as in [3]. Restricting our metaplectic data from G to M_β we get some metaplectic data for M_β, hence its metaplectic dual group say $M(H)_\beta$, it will be a Levi of H. Then induction along $[M_\beta, M_\beta]U_\beta \to \widetilde{P}_\beta$ should mean some global version of averaging under the Hecke action of $M(H)_\beta$.

Somethig similar happened for the R-model in [4].

Let $\Lambda_H \subset \Lambda$ be the sublattice attached to our metaplectic data as in [3].

0.2.5. Concretely, we can try the following. Let $\overline{\mathcal{H}}_{M_\beta}$ be the stack classifying $V \in \mathrm{Bun}_{M_\beta}$, $V_1 \in \mathrm{Bun}_{M_\beta,\Omega}$ a divisor $D \in X^{(d)}$, an isomorphism $\gamma : V \widetilde{\to} V_1 \mid_{X-D}$. Let now $\mathcal{H}^0_{M_\beta} \subset \overline{\mathcal{H}}_{M_\beta}$ be the substack given by the properties: $D \in X^{(d),rss}$, the complement of all diagonals in $X^{(d)}$, and for each $x \in D$, $V_1(\mathcal{O}_x)$ is in relative position $(-2,-1)$ with respect to $V(\mathcal{O}_x)$. Here $\mathcal{O}_x$ is the completion of the local ring of X at x. Let $\mathcal{H}_{M_\beta} \subset \overline{\mathcal{H}}_{M_\beta}$ be the closure of $\mathcal{H}^0_{M_\beta}$ in $\overline{\mathcal{H}}_{M_\beta}$.

Write $\widetilde{\mathrm{Bun}}_{M_\beta}, \widetilde{\mathrm{Bun}}_{P_\beta}$ for the pullbacks of the gerbe $\widetilde{\mathrm{Bun}}_G \to G$ under the corresponding maps. Let $\widetilde{\mathcal{H}}_{M_\beta}$ be μ_3-gerbe on $\mathcal{H}_{M_\beta}$, namely the pullback of our gerbe from V. Then over the open part $\widetilde{\mathcal{H}}^0_{M_\beta} \subset \widetilde{\mathcal{H}}_{M_\beta}$ we have a natural twisted spherical sheaf, as $(-2,-1)$ lies in Λ_H, and the gerbe over $\mathrm{Bun}_{M_\beta,\Omega}$ is trivialized. Take its intermediate extension to $\widetilde{\mathcal{H}}_{M_\beta}$, the result is denoted $\mathcal{S}$.

0.2.6. Consider

$$\widetilde{\mathcal{H}}_{M_\beta} \times_{\widetilde{\mathrm{Bun}}_{M_\beta}} \widetilde{\mathrm{Bun}}_{P_\beta}, \tag{4}$$

where the underlying map $\mathcal{H}_{M_\beta} \to \mathrm{Bun}_{M_\beta}$ sends (V, V_1, D, γ) to V. My understanding is as follows. Consider a point of (4), its projection to Bun_{P_β} is given by $V \in \mathrm{Bun}_{M_\beta}$ and a $H(\mathcal{V})$-torsor $\mathcal{F}$, where $\mathcal{V} = (\mathrm{Sym}^3 V) \otimes \det V$.

Our definition of $\mathcal{H}_{M_\beta}$ should imply simply that for $\mathcal{V}_1 := (\mathrm{Sym}^3 V_1) \otimes \det V_1$ the isomorphism $\mathcal{V} \widetilde{\to} \mathcal{V}_1 \mid_{X-D}$ extends to an inclusion $\mathcal{V} \subset \mathcal{V}_1$ everywhere on X. Similarly, the isomorphism $\det V \widetilde{\to} \det V_1 \mid_{X-D}$ extends to an inclusion $\det V \subset \det V_1$ on X. So, we get a morphism of group schemes $H(\mathcal{V}) \subset H(\mathcal{V}_1)$ over X. Define $\mathcal{F}_1$ as the push-out of $\mathcal{F}$ via the latter morphism of group schemes. This defines a map

$$\eta : \widetilde{\mathcal{H}}_{M_\beta} \times_{\widetilde{\mathrm{Bun}}_{M_\beta}} \widetilde{\mathrm{Bun}}_{P_\beta} \to \mathrm{Bun}_{P_\beta, \Omega}$$

sending a point of the source to $(V_1, \mathcal{F}_1)$. We get a diagram

$$\widetilde{\mathrm{Bun}}_{P_\beta} \stackrel{\mathrm{pr}}{\leftarrow} \widetilde{\mathcal{H}}_{M_\beta} \times_{\widetilde{\mathrm{Bun}}_{M_\beta}} \widetilde{\mathrm{Bun}}_{P_\beta} \stackrel{\eta}{\to} \mathrm{Bun}_{P_\beta, \Omega}$$

We may form the tensor product

$$(\eta^* \tilde{f}^* \mathrm{Aut}^e_\psi) \otimes \mathcal{S},$$

where $\mathcal{S}$ is pulled back from $\widetilde{\mathcal{H}}_{M_\beta}$. Now we propose the following. Let $\tilde{\nu}_P : \widetilde{\mathrm{Bun}}_{P_\beta} \to \widetilde{\mathrm{Bun}}_G$ be the natural map.

Conjecture 0.2.7. *There is K on $\widetilde{\mathrm{Bun}}_G$ with an isomorphism*

$$\mathrm{pr}_!((\eta^* \tilde{f}^* \mathrm{Aut}^e_\psi) \otimes \mathcal{S}) \widetilde{\to} \tilde{\nu}_P^* K,$$

and K is a geometric analog of Savin's representation from [8].

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