Assumptions on the sheaf theory for the 2nd joint paper with Dennis[1]

0.0.1. k is algebraically closed of any characteristic, e is an algebraically closed field of characteristic zero. The notation DGCat stands for the category denoted DGCat_{cont} in [7].

We are given a right-lax symmetric monoidal functor

$$(\mathrm{Sch}^{aff}_{ft})^{op} \to \mathrm{DGCat}, S \mapsto Shv(S), \text{ and } (S_1 \xrightarrow{f} S_2) \text{ goes to } f^! : Shv(S_2) \to Shv(S_1)$$

Its right Kan extension under $(\mathrm{Sch}^{aff}_{ft})^{op} \subset (\mathrm{PreStk}_{lft})^{op}$ defines a functor

$$Shv : (\mathrm{PreStk}_{lft})^{op} \to \mathrm{DGCat}$$

It is assumed that the latter functor satisfies the etale descent for etale covers in PreStk_{lft}.

0.0.2. Probably the functor Shv should be defined in a larger category than PreStk_{lft}? For example, there should be $Shv(\mathrm{Hecke}^{loc}_{G,\mathrm{Ran}})$, though the latter is not locally of finite type. Indeed, for a closed $\mathfrak{L}^+(G)_{\mathrm{Ran}}$-equivariant subscheme $Y \subset \mathrm{Gr}_{G,\mathrm{Ran}}$ we may define $Shv((\mathfrak{L}^+(G)_{\mathrm{Ran}})\backslash Y)$ and pass to the colimit (or limit). Similarly, do we need $Shv(\mathrm{Hecke}^{loc}_{G,x})$?

Also, we need to make sense of invariants under $(\mathfrak{L}(N)_x, \chi_N)$, and $\mathfrak{L}(N)$ is not locally of finite type. At least, give a reference to Appendix C of [10].

0.0.3. For a map $f : Y_1 \to Y_2$ in PreStk_{lft} the left adjoint $f_!$ to $f^!$ is only partially defined in general (everywhere defined in the constructible context). If f is schematic open embedding, $f_* : Shv(Y_1) \to Shv(Y_2)$ is defined as the right adjoint to $f^! = f^*$. Moreover f_* satisfies the base change with respect to $g^!$ for $g : Y_2' \to Y_2$.

When we say f is ind-schematic, this means that f is ind-schematic of ind-finite type, as Shv was only defined for PreStk_{lft}. For f ind-schematic we have the functor $f_* : Shv(Y_1) \to Shv(Y_2)$. What is its definition? It has a partially defined left adjoint f^*. Is f^* always defined in the constructible context? For this f has to be of finite type, I think. For example, for $p : Y \to k$, where Y is an ind-scheme of ind-finite type the functor $p_* : Shv(Y) \to \mathrm{Vect}$ does not admit a left adjoint unless Y is a scheme of finite type (see [25], 1.2.7).

For f ind-schematic, f_* satisfies the base change formula with respect to $g^!$, where $g : Y_2' \to Y_2$. If f is ind-proper then $f_* = f_!$. My understanding is that this holds more generally for f pseudo-proper.

If f is etale then $f^! = f^*$ is the left adjoint of f_*.

The functor f_* should be defined more generally under the assumption that after a base change $S \to Y_2$ with $S \in \mathrm{Sch}^{aff}_{ft}$, $S \times_{Y_2} Y_1$ is an ind-algebraic stack. In this case f_* should also satisfy the base change formula with respect to $g^!$.

For example, the following is crucial: the category $Shv(B_{et}(e^{*,tors}))$ is monoidal for the convolution monoidal structure. For Y a prestack this is used to define a twist of $Shv(Y)$ by a $e^{*,tors}$-gerbe over Y.

[1]Date: 6 May, 2026

0.0.4. For a scheme of finite type S we have $Shv(S)^{constr} \subset Shv(S)$ is the full category of bounded complexes with constructible cohomology sheaves, then $Shv(S)^{constr} = Shv(S)^c$, and it coincides with $D^b(\mathrm{Perv}(S))$ in the constructible context. The subcategory $Shv(S)^{constr} \subset Shv(S)$ is closed under $\otimes, \otimes^!$. For Y an ind-scheme we have an equivalence $\mathbb{D} : (Shv(Y)^c)^{op} \tilde{\to} Shv(Y)^c$. Its definition is given in ([8], 7.1.3) for any $Y \in \mathrm{PreStk}_{lft}$ such that the diagonal $Y \to Y \times Y$ is pseudo-proper.

For an Artin stack Y locally of finite type with an affine diagonal we should define $Shv(Y)^{constr} \subset Shv(Y)$ as the full subcategory of objects that !-pull back to an object of $Shv(S)^c$ for any $S \to Y$, where $S \in \mathrm{Sch}^{aff}_{ft}$. Then by ([1], Appendix C),

$$\mathbb{D} : (Shv(Y)^{constr})^{op} \tilde{\to} Shv(Y)^{constr} \tag{1}$$

is an equivalence. Indeed, we have $Shv(Y) \tilde{\to} \lim_{S\to Y} Shv(S)$ taken over the category opposite to the one classifying smooth maps $S \to Y$ with $S \in \mathrm{Sch}^{aff}$. For $a : S \to Y$ smooth, we may use $a^!$ or a^* to test compactness, they differ by a shift. Then $Shv(Y)^{constr} \tilde{\to} \lim_{S\to Y} Shv(S)^c$ in $\mathrm{DGCat}^{non-cocompl}$. Recall that $\mathrm{DGCat}^{non-cocompl}$ admits limits. This gives $(Shv(Y)^{constr})^{op} \tilde{\to} \lim_{S\to Y} (Shv(S)^c)^{op}$ in $\mathrm{DGCat}^{non-cocompl}$. So, the Verdier duality for schemes of finity type gives the equivalence (1).

For $F_i \in Shv(Y)$ write $\mathcal{H}om_{Shv}(F_1, F_2) \in \mathrm{Vect}$ for the relative inner hom for the Vect-action on $Shv(Y)$. For ind-schemes or Artin stacks $\mathbb{D}$ satisfies the formula

$$\mathcal{H}om_{Shv}(\mathbb{D}(F_1), F_2) \tilde{\to} \mathrm{R}\Gamma(Y, F_1 \otimes^! F_2)$$

for $F_1 \in Shv(Y)^{constr}$. This property charaterizes $\mathbb{D}(F_1)$ uniquely. For example see ([1], F.2.5, F.1.3, F.4).

Say that $Y \in \mathrm{PreStk}_{lft}$ is a pseudo-scheme if it is of the form $Y \tilde{\to} \mathrm{colim}_{i\in I} Y_i$, where I is small, $Y_i \in \mathrm{Sch}_{ft}$, and for $i \to j$ in I the map $Y_i \to Y_j$ is proper. For a pseudo-scheme Y, $Shv(Y)$ is canonially self-dual (use [7], I.1, 6.3.4).

0.0.5. For $Y \in \mathrm{PreStk}_{lft}$ and $F_i \in Shv(Y)$ write $\mathcal{H}om^!(F_1, F_2)$ for the relative inner hom in $Shv(Y)$ for the !-pointwise monoidal structure. For Y smooth of dimension n we get $\mathrm{R}\Gamma\, \mathcal{H}om^!(F_1, F_2)[-2n] \tilde{\to} \mathcal{H}om_{Shv}(F_1, F_2)$.

In which generality the category $Shv(Y)$ admits a symmetric monoidal structure given by $(F_1, F_2) \mapsto F_1 \otimes F_2 = d^*(F_1 \boxtimes F_2)$ for the diagonal $d : Y \to Y \times Y$? This should be always the case in the constructible context, and we reserve the notation $\otimes$ for this tensor product structure on $Shv(Y)$.

If the monoidal structure on $Shv(Y)$ given by $\otimes$ exists, we reserve the notation $\mathcal{H}om(F_1, F_2)$ for the inner hom for $Shv(Y)$ for this monoidal structure.

Lemma 0.0.6. *In the constructible context the Verdier duality for a scheme Y of finite type (or an Artin stack locally of finite type with an affine diagonal) satisfies a stronger property: for $F_1 \in Shv(Y)^{constr}, F_2 \in Shv(Y)$,*

$$\mathcal{H}om(\mathbb{D}(F_1), F_2) \tilde{\to} F_1 \otimes^! F_2$$

in $Shv(Y)$.

Proof. For a map $f : S \to Y$ with $S \in \mathrm{Sch}^{aff}$ let us construct an isomorphism $f^! \mathcal{H}om(\mathbb{D}(F_1), F_2) \tilde{\to} f^!(F_1 \otimes^! F_2)$ in a way compatible with compositions $S' \to S$ for

$S' \in Sch^{aff}$. We have

$$f^!\mathcal{H}om(\mathbb{D}(F_1), F_2) \widetilde{\to} \mathcal{H}om(\mathbb{D}(f^!F_1), f^!F_2) \widetilde{\to} (f^!F_1) \otimes^! (f^!F_2) \widetilde{\to} f^!(F_1 \otimes^! F_2)$$

as desired. □

For $F \in Shv(Y)^c$ the functor $Shv(Y) \to Shv(Y), G \mapsto \mathcal{H}om(F, G)$ preserves filtered colimits.

If we assume that there is an adjoint pair $p^* : \mathrm{Vect} \leftrightarrows Shv(Y) : p_*$ for $p : Y \to \operatorname{Spec} k$ then given $F_i \in Shv(Y)$ we get

$$\mathcal{H}om_{Shv}(F_1, F_2) \widetilde{\to} \mathrm{R}\Gamma\, \mathcal{H}om(F_1, F_2)$$

0.0.7. For which maps $f : Y \to \operatorname{Spec} k$ the functor f^* is defined on e? If defined, it gives the constant sheaf on Y. This happens at least for algebraic stack locally of finite type (with an affine diagonal).

Let now Y be a scheme of finite type or an algebraic stack locally of finite type (with an affine diagonal). Assume we are in the constructible context and $F \in Shv(Y)^{constr}$. Then the functor $Shv(Y) \to Shv(Y), K \mapsto K \otimes^! F$ admits a continuous left adjoint given by $K \mapsto K \otimes (\mathbb{D}F)$. Indeed, for $L \in Shv(Y)$ we get

$$\mathcal{H}om(L, K \otimes^! F) \widetilde{\to} \mathcal{H}om(L, \mathcal{H}om(\mathbb{D}F, K)) \widetilde{\to} \mathcal{H}om(L \otimes (\mathbb{D}F), K)$$

Recall that here $\mathcal{H}om$ denotes the inner hom in $(Shv(Y), \otimes)$.

Claim Let $X, Y \in Sch_{ft}$. Note that the exteriour product $h : Shv(X) \otimes Shv(Y) \to Shv(X \times Y)$ is a map of $Shv(X) \otimes Shv(Y)$-modules, where the action of $L \in Shv(X)$ (resp., of $L' \in Shv(Y)$) on $Shv(X \times Y)$ sends K to $(L \boxtimes \omega) \otimes^! K$ (resp., to $(\omega \boxtimes L') \otimes^! K$). So, its right adjoint h^R is a right-lax morphism of $Shv(X) \otimes Shv(Y)$-modules. In fact, this right-lax structure is strict.

Proof. Let $K \in Shv(X \times Y)$ and $F \in Shv(X)$. We must show that the natural map $(F \boxtimes \omega) \otimes h^R(K) \to h^R((F \boxtimes \omega) \otimes^! K)$ is an isomorphism in $Shv(X) \otimes Shv(Y)$. We may and do assume $F \in Shv(X)^c$. It is understood that $Shv(Y), Shv(X)$ is equipped with the $\otimes^!$-symmetric monoidal structures, so $Shv(X) \otimes Shv(Y)$ is also symmetric monoidal. By the above, the functor $Shv(Y) \otimes Shv(Y) \to Shv(Y) \otimes Shv(Y)$, $S \mapsto (F \boxtimes \omega) \otimes S$ admits a continuous left adjoint sending $K_1 \boxtimes K_2$ to $(K_1 \otimes \mathbb{D}(F)) \boxtimes K_2$ for $K_i \in Shv(Y)$.

Now for $K_1 \in Shv(X), K_2 \in Shv(Y)$ we get

$$\mathrm{Map}_{Shv(X)\otimes Shv(Y)}(K_1 \boxtimes K_2, h^R((F \boxtimes \omega) \otimes^! K) \widetilde{\to} \mathrm{Map}_{Shv(X \times Y)}(K_1 \boxtimes K_2, (F \boxtimes \omega) \otimes^! K)$$
$$\widetilde{\to} \mathrm{Map}_{Shv(X \times Y)}((K_1 \otimes (\mathbb{D}F)) \boxtimes K_2, K) \widetilde{\to} \mathrm{Map}_{Shv(X)\otimes Shv(Y)}((K_1 \otimes (\mathbb{D}F)) \boxtimes K_2, h^R(K))$$
$$\widetilde{\to} \mathrm{Map}_{Shv(X)\otimes Shv(Y)}(K_1 \boxtimes K_2, (F \boxtimes \omega) \otimes h^R(K))$$

Let us underline that in the above formulas $(F \boxtimes \omega) \otimes h^R(K)$ denotes the tensor product in the symmetric monoidal category $Shv(X) \otimes Shv(Y)$. □

Recall also that h^R coincides with $h^\vee$ with respect to the Verdier self-dualities, see ([22], Sect. 1.0.1).

0.0.8. If $i : Z' \to Z$ is a closed immersion and $F \in Shv(Z)$ satisfies $i^! F = 0$ then F is in the essential image of $j_* : Shv(Z - Z') \to Shv(Z)$. Here $j : Z - Z' \to Z$. For $F \in Shv(Z)$ one as a fibre sequence

$$i_! i^! F \to F \to j_* j^! F$$

in $Shv(Z)$. In particular, if $M \in Shv(Z)$ satisfies $j^* M = 0$ then M is in the essential image of $i_!$.

0.0.9. Let S be an ind-scheme of ind-finite type. In the constructible context, the tensor product $\otimes^! : Shv(S) \otimes Shv(S) \to Shv(S)$ admits a continuous right adjoint. Indeed, for $F_i \in Shv(S)^c$ it suffices to show that $F_1 \otimes^! F_2$ is compact. For this, it suffices to show that $\mathbb{D}(\vartriangle^* (\mathbb{D}F_1 \boxtimes \mathbb{D}F_2))$ is compact, and in turn that $\vartriangle^* (\mathbb{D}F_1 \boxtimes \mathbb{D}F_2))$ is compact. This is true, because for $\vartriangle: S \to S \times S$, $\vartriangle^*$ has a continuous right adjoint $\vartriangle_*$.

This is not the case for $\mathcal{D}$-modules, as far as I understand, because $\vartriangle^*$ is not always defined.

0.0.10. What are the t-structures on $Shv(Y)$ and under which assumptions and how they are defined? Perverse one, usual one?

For $Y \in \mathrm{Sch}_{ft}$ there is a t-structure on $Shv(Y)$ that we think of as being perverse. It is important that this t-structure is accessible. It is also compatible with filtered colimits (this reduces to the fact that the t-structure on Vect is compatible with filtered colimits).

The t-structure on $Shv(Y)$ for Y an ind-scheme is defined as follows. If $Y = \operatorname{colim}_{i \in I} Y_i$ with I filtered and $Y_i \in \mathrm{Sch}_{ft}$ then $Shv(Y)^{\le 0} \subset Shv(Y)$ should be the smallest full subcategory containing $Shv(Y_i)^{\le 0}$ for any i, closed under extensions and closed under small colimits. By (HA, 1.4.4.11), $Shv(Y)^{\le 0}$ is then presentable and defines an accessible t-structure on Y. We use here the fact that $Shv(Y_i)$ is generated by a small set of objects.

Note that for an ind-scheme Y of ind-finite type $F \in Shv(Y)$ lies in $Shv(Y)^{\ge 0}$ iff for any closed subscheme $i : Y' \subset Y$ one has $i^! F \in Shv(Y')^{\ge 0}$. This implies that the t-structure on $Shv(Y)$ is compatible with filtered colimits.

You should also explain what is assumed about right or left completeness of the t-structure on $Shv(S)$ for $S \in \mathrm{Sch}_{ft}$. Apparently, you assume it is right complete, as you want to use maps like $\mathrm{D}^+(Shv(Y)^\heartsuit) \to Shv(Y)$?

For an algebraic stack with an affine diagonal Y we define the perverse t-structure on $Shv(Y)$ by

$$Shv(Y)^{\le 0} \widetilde{\to} \lim_{S \overset{\alpha}{\to} Y} Shv(S)^{\le -\operatorname{dim.rel}(\alpha)},$$

where the limit is over the category whose objects are smooth maps $\alpha : S \to Y$ with $S \in \mathrm{Sch}_{ft}$, and morphisms from (S, α) to (S', α') is a smooth map $S \to S'$ compatible with α, α'. The transition functors here are the !-pullbacks. This defines an accessible t-structure by ([16], 1.4.4.11) or better by ([7], ch. I.3, Lemma 1.5.8). We have $Shv(Y)^{>0} \widetilde{\to} \lim_{S \overset{\alpha}{\to} Y} Shv(S)^{> -\operatorname{dim.rel}(\alpha)}$ taken over the same category with the transition functors being !-pullbacks. This t-structure is compatible with filtered colimits and both left and right complete by *loc.cit.*

Claim. If Y is an algebraic stack with an affine diagonal then in the constructible context the t-structure on $Shv(Y)$ is right complete.

Proof. The t-structure on $Shv(Y)$ is accessible, so by ([21], 4.0.10) it suffices to show that for $L \in Shv(Y)$ the natural map $\mathrm{colim}_n \tau^{\leq n} L \to L$ is an isomorphism. This property is local in Zariski topology, so it suffices to show this is an isomorphism over any open quasi-compact substack $U \subset Y$.

For each U the category $Shv(U)$ is right complete. Indeed, we have an adjoint pair $ren_U : Shv(U) \leftrightarrows Shv(U)^{ren} : un - ren_U$ in DGCat_{cont} as in ([1], F.5.3) with ren_U fully faithful. The t-structure on $Shv(U)^{ren}$ is right complete by ([21], 9.3.18). The t-structure on $Shv(U)$ is accessible, so by ([21], 4.0.10) it suffices to show that for $K \in Shv(U)$ the natural map $\mathrm{colim}_n \tau^{\leq n} K \to K$ is an isomorphism in $Shv(U)$. To see this, let $K' = ren_U(K)$. Then the natural map $\mathrm{colim}_n \tau^{\leq n} K' \to K'$ is an isomorphism in $Shv(U)^{ren}$. Since $un - ren_U$ is t-exact, $K \widetilde{\to} un - ren_U(K')$ identifies with

$$\mathrm{colim}_n\, un - ren_U(\tau^{\leq n} K') \widetilde{\to} \mathrm{colim}_n\, \tau^{\leq n}(un - ren_U(K')) \widetilde{\to} \mathrm{colim}_n\, \tau^{\leq n}(K)$$

We are done. □

0.0.11. For $S \in \mathrm{Sch}_{ft}$ mention that $Shv(S)$ is assumed compactly generated. So, for an ind-scheme of ind-finite type Y, $Shv(Y)$ is also compactly generated. Moreover, the Verdier duality provides an equivalence $Shv(Y)^\vee \widetilde{\to} Shv(Y)$, which is an isomorphism of $Shv(Y)$-modules. The corresponding map $Shv(Y) \otimes Shv(Y) \to \mathrm{Vect}$ sends (F_1, F_2) to $\mathrm{R}\Gamma(Y, F_1 \otimes^! F_2)$.

If now $f : Y_1 \to Y_2$ is a morphism of ind-schemes of ind-finite type then the dual to $f^! : Shv(Y_2) \to Shv(Y_1)$ identifies with $f_* : Shv(Y_1) \to Shv(Y_2)$.

If moreover, we are in the **constructible context**, since $(f_!, f^!)$ is an adjoint pair, its dual $((f^!)^\vee, (f_!)^\vee)$ is also an adjoint pair. So, the dual to $f_! : Shv(Y_1) \to Shv(Y_2)$ is the right adjoint to $f_* : Shv(Y_1) \to Shv(Y_2)$.

Assume $f : Y_1 \to Y_2$ schematic of finite type. In the conctructible context, f_* has a left adjoint f^*, hence $(f^!, (f^*)^\vee)$ is an adjoint pair, so $f^!$ has a continuous right adjoint.

Example: let T be a split torus. Then e on $B(T)$ is not compact in the constructible context, that is $\mathrm{R}\Gamma : Shv(B(T)) \to \mathrm{Vect}$ is not continuous. So, this functor can not be the dual of $f^!$ for $f : B(T) \to \mathrm{Spec}\, k$.

There is a projection formula for maps $f : Y \to Y'$, where Y is a quasi-compact classical algebraic stack with affine diagonal and Verdier compatible, it is formulated in ([2], B). This f_* satisfies the projection formula (even if not continuous).

0.0.12. Consider the 1-full subcategory $\mathrm{PreStk}_{ind-sch} \subset \mathrm{PreStk}_{lft}$, where we restrict 1-morphisms to be ind-schematic. Then we have a well-defined functor

$$Shv_{\mathrm{PreStk}_{ind-sch}} : \mathrm{PreStk}_{ind-sch} \to \mathrm{DGCat}_{cont}$$

sending Y to $Shv(Y)$ and a morphism $f : Y \to Y'$ to $f_* : Shv(Y) \to Shv(Y')$. Moreover, this functor is right-lax symmetric monoidal, so sends algebras to algebras. So, if G is an algebra in $\mathrm{PreStk}_{ind-sch}$, $(Shv(G), \star)$ will become a monoidal DG-category with the monoidal convolution structure.

So, we may talk about strong actions of $Shv(G)$ on some $C \in$ DGCat, this is an object of $(Shv(G), \star) - mod(\mathrm{DGCat})$.

0.0.13. If G is an ind-scheme of ind-finite type, assume $m : G \times G \to G$ ind-proper. Then $(Shv(G), \star)$ is rigid for any sheaf theory. My understanding is that there is no hope for it to be rigid without the ind-properness assumption.

0.0.14. If G is a group ind-scheme of ind-finite type then $(Shv(G), m_*)$ is monoidal (convolution monoidal structure).

The functor $Shv(G) \otimes Shv(G) \to Shv(G \times G)$ sends a compact object $F_1 \otimes F_2$ to a compact object $F_1 \boxtimes F_2$. [2] So, this functor admits a continuous right adjoint. In the contstructible context the functor $m_* : Shv(G \times G) \to Shv(G)$ admits a continuous right adjoint. Besides, the dual to m_* is the functor $m^!$. Thus, passing to the dual in $(Shv(G), m_*)$, in the constructible context we get a coalgebra $(Shv(G), m^!)$ in DGCat_{cont}. Recall that $(Shv(G), m_*) - mod \widetilde{\to} (Shv(G), m^!) - comod$ (cf. [21]).

For any ind-scheme of ind-finite type Y, Y is a cocommutative coalgebra in PreStk_{lft} via the maps $Y \to Y \times Y$ and $Y \to \operatorname{Spec} k$, hence a commutative algebra in $(\mathrm{PreStk}_{lft})^{op}$. Applying the right-lax monoidal functor Shv, we get on $Shv(Y)$ a commutative algebra structure in $CAlg(\mathrm{DGCat}_{cont})$. The product is $Shv(Y) \otimes Shv(Y) \to Shv(Y \times Y) \stackrel{\vartriangle^!}{\to} Shv(Y)$. We denote this algebra $(Shv(Y), \vartriangle^!)$. It makes sense for any sheaf theory. Applying the duality, we get a coalgebra structure on $Shv(Y)$, which we denote $(Shv(Y), \vartriangle_*)$ following [3]. Recall that this duality exchanges the functors $\vartriangle_*$ and $\vartriangle^!$.

Sam says $(Shv(G), \vartriangle^!, m^!)$ is probably not a Hopf algebra in the constructible context (only for $\mathcal{D}$-modules). Similarly for $(Shv(G), m_*, \vartriangle_*)$. For $\mathcal{D}$-modules this was explained in [3]. Though $(Shv(G), m_*) - mod$ is a symmetric monoidal category for $\mathcal{D}$-modules, this does not seem to be the case in the constructible context.

Sam's idea: if this was the case, consider the diagonal action of $(Shv(G), m_*)$ on $Shv(G) \otimes Shv(G)$. It is given by a map of algebras $h^R \circ \Delta_* : Shv(G) \to Shv(G) \otimes Shv(G)$, which is the coproduct. Here $h : Shv(G) \otimes Shv(G) \hookrightarrow Shv(G \times G)$ is the exteriour product, and h^R is its right adjoint. Besides, $\Delta_* : Shv(G) \to Shv(G \times G)$ is a morphism in $Alg(\mathrm{DGCat}_{cont})$. Is it true that h^R or h then becomes a morphism in $Shv(G) - mod$? Then we could consider the map between the invariants, hopefully to get a contradiction. We have in mind that $\Delta_* \omega_G$ is invariant under the diagonal action, but does not lie in the essential image of h, here $\Delta : G \to G \times G$ is the diagonal. Not clear.

0.0.15. If $Y \in \mathrm{PreStk}_{lft}$ is equipped with a G-action then the action map $a : G \times Y \to Y$ is ind-schematic (isomorphic to the projection $Y \times G \to Y$). So, $(Shv(G), \star)$ acts on $Shv(Y)$ on the left via $F \in Shv(G), K \in Shv(Y) \mapsto a_*(F \boxtimes K)$. If $f : Y_1 \to Y_2$ is an ind-schematic morphism in PreStk_{lft} commuting with G-actions then $f_* : Shv(Y_1) \to Shv(Y_2)$ is a map of $(Shv(G), \star)$-modules. Besides, $f^!$ is a map of $(Shv(G), \star)$-modules. Consider the prestack quotient $Y/G \in \mathrm{PreStk}_{lft}$. The map $f : Y \to Y/G$ commutes

[2] Is it true for any sheaf theory? In ([10], 1.2.5(b)) you mentioned this only for two sheaf theories, but not for constructible sheaves in the classical topology. I imagine this is a misprint there! You actually claim this for any placid ind-schemes Y_1, Y_2 in ([10], C.2.8), so I assume this is true for any sheaf theory.

with G-actions, where G acts trivially on Y/G. So, $f^! : Shv(Y/G) \to Shv(Y)$ is a map of $(Shv(G), \star)$-modules. Thus, by ([25], 1.10.10) it induces a functor

$$Shv(Y/G) \to \mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, Shv(Y)) \tag{2}$$

Is it an equivalence?

0.0.16. In general the answer is not clear. Assume G smooth of finite type. Then this is an equivalence, as Lin Chen shows (there is a different proof in ([10], 1.4.5)). Here is his argument.

One shows that $Shv(Y/G) \widetilde{\to} e-comod(Shv(Y))$ by verifying the comonadic Beck-Chevalley conditions. Here e is the constant sheaf on G, it is a coalgebra in $(Shv(G), \star)$, and we consider the corresponding category of comodules with the convolution action of $Shv(G)$ on $Shv(Y)$. The forgetful functor $e-comod(Shv(Y)) \to Shv(Y)$ is $f^!$ for $f : Y \to Y/G$. The self-functor underlying the comonad is $p_*a^* : Shv(Y) \to Shv(Y)$. It also identifies with a_*p^*, here $a : G \times Y \to Y$ is the action map, $p : G \times Y \to Y$ is the projection.

Since $Shv(G)$ is self-dual, $Shv(Y)^G$ identifies with the limit of

$$Shv(Y) \rightrightarrows Shv(G) \otimes Shv(Y) \overset{\to}{\underset{\to}{\to}} Shv(G)^{\otimes 2} \otimes Shv(Y) \dots$$

(For $\mathcal{D}$-modules, since $Shv(G)^{\otimes n} \otimes Shv(Y) \widetilde{\to} Shv(G^n \times Y)$, this finishes the proof). Assume now we are in the constructible context.

The above cosimplicial diagram is also

$$Shv(Y) \rightrightarrows \mathrm{Fun}(Shv(G), Shv(Y)) \overset{\to}{\underset{\to}{\to}} \mathrm{Fun}(Shv(G)^{\otimes 2}, Shv(Y)) \dots$$

The functors $Shv(Y) \rightrightarrows \mathrm{Fun}(Shv(G), Shv(Y))$ are: F goes to $(K \mapsto K * F)$, and F goes to $(K \mapsto \mathrm{R}\Gamma(G, K) \otimes F)$. The second functor identifies via the Verdier duality with $Shv(Y) \to Shv(G) \otimes Shv(Y)$, $F \mapsto \omega_G \otimes F$. Its right adjoint is $p_*[-2n] \otimes \mathrm{id} : Shv(G) \otimes Shv(Y) \to Shv(Y)$ for $p : G \to \operatorname{Spec} k$, where $n = \dim G$.

The comonadic Beck-Chevalley condition for the above cosimplicial diagram holds, it is mentioned in [10], 1.4.6 without a proof. We also check this in bigger generality in Section 0.0.23 of this file.

The corresponding comonad on $Shv(Y)$ is $Shv(Y) \to \mathrm{Fun}(Shv(G), Shv(Y)) \overset{T^0}{\to} Shv(Y)$, where the first functor sends F to $(K \mapsto K * F)$. Thus, this comonad sends F to $e * F$. We see that both comonads are the same.

0.0.17. Let G be a smooth group scheme of finite type, $Y \in \mathrm{PreStk}_{lft}$. The equivalence $Shv(B(G)) \widetilde{\to} \mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, \mathrm{Vect})$ given by (2) transforms the symmetric monoidal structure on $Shv(B(G))$ given by $\otimes^!$ to the composition monoidal structure on $\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, \mathrm{Vect})$.

The projection $q : Y/G \to B(G)$ yields an action of $(Shv(B(G)), \otimes^!)$ on $Shv(Y/G)$. Namely, $K \in Shv(B(G))$ acts on $M \in Shv(Y/G)$ as $(q^!K)\otimes^! M$. Similarly, the monoidal category $\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, \mathrm{Vect})$ acts on $\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, Shv(Y))$ by composition on the left. The equivalence (2) is compatible with these actions via the above monoidal equivalence

$$Shv(B(G)) \widetilde{\to} \mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, \mathrm{Vect})$$

0.0.18. We need the following claim: for $Y \in \mathrm{Sch}_{ft}$, its cohomology $C(Y)$ is bounded, and the dimension of each H^i is finite. It was used in ([10], B.3.1) to show that for a smooth group scheme of finite type H and $C \in Shv(H) - mod$, $C_H \to C^H$ is an equivalence. In the constructible context this is automatic, because $p_* : Shv(Y) \to \mathrm{Vect}$ for $p : Y \to \mathrm{Spec}\, k$ admits a continuous right adjoint, and the constant sheaf e_Y is compact, so $p_*(e_Y)$ is also compact.

So, a suitable finiteness assumption on the functor p_* should be formulated which holds for any sheaf theory. How it is formulated?

0.0.19. Consider a cartesian square

$$\begin{array}{ccc} X & \stackrel{f_X}{\to} & X' \\ \downarrow g & & \downarrow g' \\ Y & \stackrel{f_Y}{\to} & Y', \end{array} \tag{3}$$

in PreStk, where all objects are placid ind-schemes. For which morphisms g' we have the functors $(g')^!, (g')^*$? When do we have the base change with respect to $(f_Y)_*$?

Lemma 0.0.20. *let $Y' \in Sch_{ft}$ and Y, X' be placid schemes over Y', recall then X is also a placid scheme. Assume $Y \widetilde{\to} \lim_{i\in I^{op}} Y_i$, where I is filtered, $f_{Y,i} : Y_i \to Y'$ is smooth, $Y_i \in Sch_{ft}$, and for $i \to j$ in I, $Y_j \to Y_i$ is smooth affine surjective morphism in Sch_{ft}. Then one has $f_Y^* g'_* \widetilde{\to} g_* f_X^*$.*

Proof. 1) Assume first $g' : X' \to Y'$ a morphism in Sch_{ft}. Set $X_i = Y_i \times_{Y'} X'$ for $i \in I$, so $X \widetilde{\to} \lim_{i\in I^{op}} X_i$. For each i we get a cartesian square

$$\begin{array}{ccc} X_i & \stackrel{f_{X,i}}{\to} & X' \\ \downarrow g_i & & \downarrow g' \\ Y_i & \stackrel{f_{Y,i}}{\to} & Y', \end{array}$$

So, $f_{Y,i}^* g'_* \widetilde{\to} (g_i)_* f_{X,i}^*$ naturally. So, $(g_i)_*$ form a morphism of the corresponding colmit systems giving $g_* : Shv(X) \widetilde{\to} \mathrm{colim}_{i\in I} Shv(X_i) \to \mathrm{colim}_{i\in I} Shv(Y_i) \widetilde{\to} Shv(Y)$. The claim follows.

2) Let now $g' : X' \to Y'$ be any placid scheme over Y'. Write $X' \widetilde{\to} \lim_{j\in J^{op}} X'_j$ with $X'_j \in Sch_{ft}$, J filtered, and for $j \to j'$ in J the map $X'_{j'} \to X'_j$ is smooth affine and surjective. Set $X_j = Y \times_{Y'} X'_j$ for $j \in J$. Then X_j is a placid scheme, and we get the diagram

$$\begin{array}{ccc} X_j & \stackrel{f_{X,j}}{\to} & X'_j \\ \downarrow g_j & & \downarrow g'_j \\ Y & \stackrel{f_Y}{\to} & Y', \end{array}$$

for $j \in J$. Note that $Shv(X) \widetilde{\to} \lim_{j\in J^{op}} Shv(X_j)$ with respect to the $*$-direct image transition functors. By 1), for each $j \in J$,

$$f_Y^*(g'_j)_* \widetilde{\to} (g_j)_* f_{X,j}^* \tag{4}$$

naturally. The functors $f_{X,j}^*$ are compatible with the corresponding inverse systems and give in the limit over J^{op} the functor f_X^*. Pick any $j \in J$. Then g' is the composition $X' \stackrel{ev'_j}{\to} X'_j \stackrel{g'_j}{\to} Y'$. Since $(f_{X,j}^*)(ev'_j)_* \widetilde{\to} (ev_j)_* f_X^*$ our claim follows from (4). □

If $S \in \mathrm{Sch}_{ft}$ and $Z \to S$ is a placid S-scheme, let $i : S' \to S$ be a map in Sch_{ft} and $h : Z' \to Z$ be obtained by base change. Then $h^! : Shv(Z) \to Shv(Z')$ is defined: write $Z \widetilde{\to} \lim_{i\in I^{op}} Z_i$, where I is small filtered, $Z_i \in \mathrm{Sch}_{ft}/S$, and for $i \to j$ in I the map $Z_j \to Z_i$ in Sch_{ft}/S is smooth affine surjective. Then let $Z'_i = Z_i \times_S S'$, let $h_i : Z'_i \to Z_i$ be the corresponding map. The functors $h_i^!$ are compatible with $*$-pushforwards in the diagrams $Shv(Z) \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)$, $Shv(Z') \widetilde{\to} \lim_{i\in I^{op}} Shv(Z'_i)$. In the limit they yield the functor $h^!$. For the projections $p_i : Z \to Z_i$, $p'_i : Z' \to Z'_i$ we get $h_i^!(p_i)_* \widetilde{\to} (p'_i)_* h^!$ and $h^! p_i^* \widetilde{\to} (p'_i)^* h_i^!$ canonically.

Lemma 0.0.21. *Let $S \in \mathrm{Sch}_{ft}$, assume given a cartesian square in $\mathrm{PreStk}_{/S}$*

$$\begin{array}{ccc} Y & \stackrel{g}{\leftarrow} & Y' \\ \downarrow f & & \downarrow f' \\ Z & \stackrel{h}{\leftarrow} & Z' \end{array}$$

Assume I is a filtered category, and we are given a morphism $f_i : Y_i \to Z_i$ in $(\mathrm{Sch}_{ft})_{/S}$ functorial in $i \in I^{op}$, where f_i is smooth. We assume for $i \to j$ in I the transition maps $Y_j \to Y_i$, $Z_j \to Z_i$ are smooth affine surjective. We assume that $f : Y \to Z$ is obtained from f_i by passing to the limit over I^{op}. We assume $i : S' \to S$ is a map in Sch_{ft}, and $f' : Y' \to Z'$ is obtained from f by the base change $i : S' \to S$. Then $g^! f^* \widetilde{\to} (f')^* h^!$ naturally. We do not assume here that the squares

$$\begin{array}{ccc} Y_j & \to & Y_i \\ \downarrow f_j & & \downarrow f_i \\ Z_j & \to & Z_i \end{array}$$

are cartesian.

Proof. By definition, $f^* : Shv(Z) \to Shv(Y)$ is obtained by passing to the colimit over I in $f_i^* : Shv(Z_i) \to Shv(Y_i)$. Note that Y, Y', Z, Z' are placid S-schemes. Note that $h^!$ is obtained by passing to the colimit over I in $h_i^! : Shv(Z_i) \to Shv(Z'_i)$, and similarly for $g^!$. Recall that $Shv(Z) \widetilde{\to} \operatorname{colim}_{i\in I} Shv(Z_i)$.

For $i \in I$ and $K \in Shv(Z_i)$ we have $g_i^! f_i^* K \widetilde{\to} (f'_i)^* h_i^! K$ canonically. Passing to the colimit over I, one gets the desired claim. □

0.0.22. Let Z be a placid scheme written as $Z = \lim_{i\in I^{op}} Z_i$. For $i \to j$ in I let $f_{ij} : Z_j \to Z_i$ be the corresponding morphism, it is smooth of relative dimension d_{ij}, affine, surjective. Since $Shv(Z) \widetilde{\to} \operatorname{colim}_i Shv(Z_i)$ via the maps f_{ij}^*, $Shv(Z)$ is compactly generated, hence dualizable. By ([7], ch. I.1, 6.3.4), by applying the dualization functor to the functor

$$I \to \mathrm{DGCat}_{cont}, i \mapsto Shv(Z_i),\ (i \to j) \mapsto f_{ij}^*,$$

we get a functor $I^{op} \to \mathrm{DGCat}_{cont}, i \mapsto Shv(Z_i)$, $(i \to j) \mapsto (f_{ij})_*[-2d_{ij}]$. Moreover,

$$Shv(Z)^\vee \widetilde{\to} \lim_{i \in I^{op}} Shv(Z_i)$$

with respect to the transition maps $(f_{ij})_*[-2d_{ij}]$. Consider for $i \in I$ the isomorphism $Shv(Z_i) \stackrel{\otimes e[2d_i]}{\to} Shv(Z_i)$ with $d_i = \dim Z_i$. So, $d_{ij} = d_j - d_i$. The diagram commutes

$$\begin{array}{ccc} Shv(Z_j) & \stackrel{\otimes e[2d_j]}{\to} & Shv(Z_j) \\ \downarrow (f_{ij})_* & & \downarrow (f_{ij})_*[-2d_{ij}] \\ Shv(Z_i) & \stackrel{\otimes e[2d_i]}{\to} & Shv(Z_i) \end{array}$$

Passing to the limit over I^{op}, we obtain an equivalence $Shv(Z) \widetilde{\to} Shv(Z)^\vee$. So, a possibility is to mention that for each placid scheme Z, $Shv(Z)$ is canonically self-dual. However, this self-duality is not compatible with the one for finite type schemes, so maybe it is not needed.

Example: assume $0 \in I$ is an initial object, let $K_0 \in Shv(Z_0)$. For the projection $f_0 : Z \to Z_0$ the image of $f_0^* K_0$ in $Shv(Z)^\vee$ under this duality is the composition $Shv(Z) \stackrel{(f_0)_*}{\to} Shv(Z_0) \to \mathrm{Vect}$, where the second functor is $M \mapsto \mathrm{R}\Gamma(Z_0, K_0 \otimes^! M)[2d_0]$.

0.0.23. Let G be a group scheme, which is a placid scheme, $C \in G-mod$. Consider the cosimplicial category defining C^G:

$$\mathrm{Fun}(\mathrm{Vect}, C) \rightrightarrows \mathrm{Fun}(Shv(G), C) \substack{\to \\ \to \\ \to} \mathrm{Fun}(Shv(G)^{\otimes 2}, C) \ldots$$

Let us show that it satisfies the comonadic Beck-Chevalley conditions.

The functor corresponding to the last face map $\partial_n : [n] \to [n+1]$ (its image avoids $n+1$) is the following functor F_n. We consider $Shv(G)^{\otimes n+1} \to Shv(G)^{\otimes n}$, $\mathrm{id} \otimes \mathrm{R}\Gamma$, and compose it with $\mathrm{Fun}(\cdot, C)$. For $p : G \to \operatorname{Spec} k$ the functor p_* has a left adjoint p^*. Let T_n be the functor obtained from $\mathrm{id} \otimes p^* : Shv(G)^{\otimes n} \to Shv(G)^{\otimes n+1}$ by composing with $\mathrm{Fun}(\cdot, C)$. Then T_n is the right adjoint to F_n. Let now $\alpha : [m] \to [n]$ be a map in $\boldsymbol{\Delta}$. Consider the corresponding diagram

$$\begin{array}{ccc} \mathrm{Fun}(Shv(G)^{\otimes n}, C) & \stackrel{T_n}{\leftarrow} & \mathrm{Fun}(Shv(G)^{\otimes n+1}, C) \\ \uparrow F_\alpha & & \uparrow F_{\alpha+1} \\ \mathrm{Fun}(Shv(G)^{\otimes m}, C) & \stackrel{T_m}{\leftarrow} & \mathrm{Fun}(Shv(G)^{\otimes m+1}, C) \end{array}$$

We show that it commutes. It suffices to prove this for α injective, becase of the following. Let $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ be the full subcategory with the same class of object, where we keep only injective maps. Then $\boldsymbol{\Delta}_s^{op} \to \boldsymbol{\Delta}^{op}$ is cofinal by ([15], 6.5.3.7). If $\alpha : [m] \to [n]$ is injective, and $0, n$ are in the image then the desired commutativity follows from the commutativity of

$$\begin{array}{ccc} Shv(G)^{\otimes n} & \stackrel{\mathrm{id}\otimes p^*}{\to} & Shv(G)^{\otimes n+1} \\ \downarrow (m_\alpha)_* & & \downarrow (m_{\alpha+1})_* \\ Shv(G)^{\otimes m} & \stackrel{\mathrm{id}\otimes p^*}{\to} & Shv(G)^{\otimes m+1}, \end{array}$$

where $(m_\alpha)_*$ is the product along α in the monoidal category $Shv(G)$.

If $\alpha : [n-1] \to [n]$ is the last face map then $\alpha + 1 : [n] \to [n+1]$ avoids n. The functor $F_{\alpha+1}$ is the composition with $Shv(G)^{\otimes n+1} \to Shv(G)^{\otimes n}$, $K_1 \otimes \ldots \otimes K_{n+1} \mapsto K_1 \otimes \ldots \otimes K_{n-1} \otimes K_n * K_{n+1}$. In this case the desired commutativity follows from $K * e_G \widetilde{\to} \mathrm{R}\Gamma(G, K) \otimes e_G$.

If $\alpha : [n-1] \to [n]$ is injective and avoids 0 then F_α sends f to the functor

$$K_1 \otimes \ldots \otimes K_n \mapsto K_1 * f(K_2 \otimes \ldots \otimes K_n)$$

and the commutativity is tautological. So, it always hold.

By ([9], Lemma C.1.9), the functor $\mathrm{oblv}_G : C^G \to C$ is comonadic, and the corresponding comonad on C is $C \to C, c \mapsto e_G * c$.

0.0.24. Consider a placid scheme $Y = \lim_{i\in I^{op}} Y_i$, where I is filtered, if $i \to j$ in I then $f_{ij} : Y_j \to Y_i$ is smooth, affine and surjective, and Y_i is a scheme of finite type. In this case $Shv(Y)$ is defined in [10] as $\lim_{i\in I^{op}} Shv(Y_i)$ via the maps $(f_{ij})_*$.

In the paper there are situations, where we have morphisms $h : Y \to S$, where S is an ind-scheme, and we want functors between $Shv(Y)$ and $Shv(S)$ attached to h. So, the above definition of $Shv(Y)$ for placid schemes should be "unified" with the definition of $Shv(Z)$ for prestacks Z locally of finite type. Namely, do we have certain full subcategory of PreStk, on which Shv is defined as a functor, and which contains both PreStk_{lft}, placid schemes, and is closed under colimits? Compare with [29].

0.0.25. Let now Z, Z' be placid schemes and $i : Z' \to Z$ a placid closed immersion. What is the dual of the adjoint pair $i_* : Shv(Z') \to Shv(Z) : i^!$?

We explain the dual of i_*. If $Z = \lim_{i\in I^{op}} Z_i$ and, assume for simplicity I has an initial object i_0 such that $Z' = Z'_{i_0} \times_{Z_{i_0}} Z$. So, $Z' = \lim_{i\in I^{op}} Z'_i$ with $Z'_i = Z_i \times_{Z_{i_0}} Z'_{i_0}$. For $i \to j$ in I let $f_{ij} : Z_j \to Z_i$ be the corresponding transition map. For the closed embeddings $i_i : Z'_i \to Z_i$ writing $Shv(Z) = \lim_{i\in I^{op}} Shv(Z_i)$ for $(f_{ij})_* : Shv(Z_j) \to Shv(Z_i)$ and similarly for $Shv(Z')$, the dual functor is given by the collection of functors $i_i^![2d_i - 2d'_i] : Shv(Z_i) \to Shv(Z'_i)$, here $d_i = \dim Z_i, d'_i = \dim Z'_i$ as locally constant functions, they form a morphism of the corresponding inverse systems. The number $d_i - d'_i$ does depend on i, and can be denoted $\mathrm{codim}_Z(Z') = d_i - d'_i$. So, the dual of $i_* : Shv(Z') \to Shv(Z)$ is $i^![2\,\mathrm{codim}_Z(Z')]$.

0.0.26. Let Z be a placid ind-scheme. Is $Shv(Z)$ canonically self-dual? Here is some answer.

Write $Z = \mathrm{colim}_{i\in I} Z_i$ with Z_i a placid scheme, I small filtered, and for $i \to j$ the map $f_{ij} : Z_i \to Z_j$ is a placid closed immersion. We have $Shv(Z) = \mathrm{colim}_{i\in I} Shv(Z_i)$ with respect to the transition functors $(f_{ij})_*$.

Consider the functor $I \to \mathrm{DGCat}_{cont}$, $i \mapsto Shv(Z_i)$, $(i \to j) \mapsto (f_{ij})_*$. By ([7], ch. I.1, 6.3.4), the colimit of this functor $\mathrm{colim}_{i\in I} Shv(Z_i) = Shv(Z)$ is dualizable, and $Shv(Z)^\vee \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)^\vee$, the limit of the dual functor.

Recall for each i the canonical self-duality on $Shv(Z_i)$ introduced in Sect. 0.0.22 of this file. It allows to rewrite $Shv(Z)^\vee \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)$, where the transition functors for $i\to j$ in I is $(f_{ij})^![2\operatorname{codim}_{Z_j}(Z_i)]$ in the notations of Section 0.0.25.

Pick an element $i_0\in I$. Consider for $i\to j$ in I a commutative diagram

$$\begin{array}{ccc} Shv(Z_j) & \stackrel{\otimes e[-2\operatorname{codim}_{Z_j}(Z_{i_0})]}{\to} & Shv(Z_j) \\ \downarrow f_{ij}^! & & \downarrow f_{ij}^![2\operatorname{codim}_{Z_j}(Z_i)] \\ Shv(Z_i) & \stackrel{\otimes e[-2\operatorname{codim}_{Z_i}(Z_{i_0})]}{\to} & Shv(Z_i) \end{array}$$

Indeed, we have $\operatorname{codim}_{Z_j}(Z_i)+\operatorname{codim}_{Z_i}(Z_{i_0})=\operatorname{codim}_{Z_j}(Z_{i_0})$. Passing to the limit over I^{op}, this provides an equivalence $Shv(Z)^\vee\widetilde{\to} Shv(Z)$.

This duality maybe depend on a choice of an element $i_0\in I$.

0.0.27. In Section 7.3.5 the perverse t-structure on $Shv_{\mathfrak{G}^G}((\overline{\operatorname{Bun}}_N^{\omega^\rho})_{\infty x})$ is mentioned without any definition. In the convention section a definition of the perverse t-structure for an ind-algebraic stack should be given. My understanding is as follows: if $Y=\operatorname{colim}_{i\in I} Y_i$ with Y_i an algebraic stack locally of finite type, I filtered then $Shv(Y)^{\le 0}$ should be the smallest full subcategory of $Shv(Y)$ containing $Shv(Y_i)^{\le 0}$ for any i, closed under extensions and small colimits. Then by (HA, 1.4.4.11), $Shv(Y)^{\le 0}$ is then presentable and defines an accessible t-structure on $Shv(Y)$. For $K\in Shv(Y)$ we have $K\in Shv(Y)^{\ge 0}$ iff for any i, the !-restriction of K to Y_i lies in $Shv(Y_i)^{\ge 0}$. As in the case of ind-schemes of ind-finite type, this t-structure is compatible with filtered colimits.

0.0.28. For a scheme of finite type S, the perverse t-structure on $Shv(S)$ is left complete (by [1], 1.1.4). This implies that for an Artin stack locally of finite type S the t-structure on $Shv(S)$ is left complete as in ([7], ch. I.3, 1.5.7), because for a smooth atlas $f: S'\to S$ with S' a scheme locally of finite type, $f^*[\dim f]$ is t-exact.

It should be clarified for which topologies on Sch_{ft} the functor $Shv: (\mathrm{Sch}_{ft})^{op}\to \mathrm{DGCat}_{cont}$ satisfies the descent, and a precise reference should be given. In particular, in ([12], proof of 4.2.7) you claim it satisfies the descent for the topology of finite surjective maps on Sch_{ft}. Give also a reference for the fact that it satisfies the étale descent. (For the proper descent this is Section 0.0.33 of this file). Sam claim we get this way h-descent, give accurate references. Add also it satisfies the smooth descent: if Y is a quasi-compact algebraic stack with a smooth cover $S\to Y$, where $S\in Sch_{ft}^{aff}$, if $S^\bullet$ is the Cech nerve of this map then $Shv(Y)\to Tot(Shv(S^\bullet))$ is an equivalence. Does etale descent automatically implies the smooth descent here?

Add also the following. For a map $f: Y\to Z$ in PreStk_{lft}, which is surjective on geometric points, $f^!$ is conservative.

Cite the following. If $Y\in\mathrm{PreStk}_{lft}$ is an algebraic stack then $Shv(Y)=\lim_{S\to Y} Shv(S)$, where the limit is taken over the opposite to the category of affine schemes smooth over Y, and morphisms are smooth maps between those ([1], C.1.1).

0.0.29. Say that for any $Y\in\mathrm{PreStk}_{lft}$, $Shv(Y)$ is compactly generated in the constructible context by ([1], C.1.1). What happens for $\mathcal{D}$-modules?

In Section 4.3.3 you claimed the existence of the equivalence $\mathbb{D} : (Shv(Y)^c)^{op} \tilde{\to} Shv(Y)^c$ for an algebraic stack of finite type. Explain that this is known only under the assumption that Y is locally a quotient of a scheme S of finite type by an affine algebraic group, give a reference!

0.0.30. For example, it should be said somewhere that if $Z = \lim_{i\in I^{op}} Z_i$ is a placid scheme, where I is filtered, Z_i is a scheme of finite type, with the transition maps affine smooth and surjective, then for $i \in I$ and the projection $ev_i : Z \to Z_i$ the functor $ev_i^* : Shv(Z_i) \to Shv(Z)$ is defined, this is the natural functor $ins_i : Shv(Z_i) \to \operatorname{colim}_{j\in I} Shv(Z_j)$. For the moment this is hidden in ([10], C.2.9).

0.0.31. *On exteriour product.* If $S_i \in \mathrm{Sch}_{ft}$, $F_i \in Shv(S_i)^{constr}$ then $F_1 \boxtimes F_2 \in Shv(S_1 \times S_2)^{constr}$ by definition, as *constr* means being bounded with ocnstructible cohomology.

Let $S \in \mathrm{Sch}_{ft}$, $Y \in \mathrm{PreStk}_{lft}$. The functor $Shv(S) \otimes Shv(Y) \to Shv(S \times Y)$, $(F, K) \mapsto F \boxtimes K$ is fully faithful and preserves compactness. We have to verify this in the constructible context, as for $\mathcal{D}$-modules this is an equivalence. Fully-faithfulness property is preserved by passing to the limit, and tensoring by $Shv(S)$ is the functor $\mathrm{DGCat}_{cont} \to \mathrm{DGCat}_{cont}$ preserving limits, as $Shv(S)$ is dualizable. This is why our functor is fully faithful.

To see that it admits a continuous right adjoint we use ([7], ch. I.1, 2.6.4). Write $Shv(Y) \tilde{\to} \lim_{T\to Y} Shv(T)$ over $(\mathrm{Sch}^{aff}_{ft})^{op}_{/Y}$. For each T the inclusion $i_T : Shv(S) \otimes Shv(T) \to Shv(S \times T)$ admits a continuous right adjoint i_T^R. Let $\alpha : T \to T'$ be a map in $(Sch^{aff}_{ft})_{/Y}$. In the constructible context, $\alpha^! : Shv(T') \to Shv(T)$ admits a left adjoint $\alpha_!$, and we have $i_{T'}(\mathrm{id} \otimes \alpha_!) \tilde{\to} (\mathrm{id} \times \alpha_!) i_T$. This gives an isomorphism $\mathrm{id} \otimes \alpha^! i^R_{T'} \tilde{\to} i^R_T (\mathrm{id} \times \alpha^!)$. By ([7], ch. I.1, 2.6.4), $i : Shv(S) \otimes Shv(Y) \to Shv(S \times Y)$ has a right adjoint i^R, and for any $(T \xrightarrow{b} Y) \in \mathrm{Sch}^{aff}_{ft})_{/Y}$ we have $(\mathrm{id} \otimes b^!) i^R \tilde{\to} i^R_T (\mathrm{id} \times b)^!$.

We check that i^R is continuous. Let $K \tilde{\to} \operatorname{colim}_{j\in J} K_j$ in $Shv(S \times Y)$. By ([21], 2.2.68), it suffices to show that for any $(T \xrightarrow{b} Y) \in \mathrm{Sch}^{aff}_{ft})_{/Y}$, $\mathrm{id} \otimes b^!$ sends our diagram to a colimit diagram. This is true, because i^R_T and $(\mathrm{id} \times b)^!$ are continuous.

Claim Let $X, Y, Z \in \mathrm{Sch}_{ft}$ with X proper. In the constructible context, the diagram commutes

$$\begin{array}{ccc}
Shv(X \times Y \times Z) & \stackrel{\boxtimes}{\leftarrow} & Shv(X) \otimes Shv(Y \times Z) \\
\downarrow \boxtimes^R & & \downarrow \boxtimes^R \\
Shv(X \times Y) \otimes Shv(Z) & \stackrel{\boxtimes}{\leftarrow} & Shv(X) \otimes Shv(Y) \otimes Shv(Z)
\end{array}$$

Proof. The left vertical arrow is $Shv(X \times Y) \otimes Shv(Z)$-linear by Section 0.0.7. Therefore, it suffices to calculate for $F \in Shv(Y \times Z)$ and the projection $q : X \times Y \times Z \to Y \times Z$ the object $\boxtimes^R(q^! F) \in Shv(X \times Y) \otimes Shv(Z)$. The functior $\boxtimes^R \circ q^!$ is right adjoint to the functor $Shv(X \times Y) \otimes Shv(Z) \xrightarrow{\boxtimes} Shv(X \times Y \times Z) \xrightarrow{q_!} Shv(Y \times Z)$. The latter functor identifies with the composition $Shv(X \times Y) \otimes Shv(Z) \xrightarrow{\bar{q}_! \otimes \mathrm{id}} Shv(Y) \otimes Shv(Z) \xrightarrow{\boxtimes} Shv(Y \times Z)$, because X is proper. Here $\bar{q} : X \times Y \to Y$ is the projection. So, $\boxtimes^R \circ q^!$

identifies with the functor $Shv(Y\times Z)\stackrel{\boxtimes^R}{\to} Shv(Y)\times Shv(Z)\stackrel{\bar{q}^!\otimes \mathrm{id}}{\to} Shv(X\times Y)\otimes Shv(Z)$. Our clailm follows. □

0.0.32. *Question.* Let $f: Y\to \operatorname{Spec} k$ be a scheme of finite type. In the constructible context does the functor $p_!: Shv(Y)\to \mathrm{Vect}$ preserve limits? Consider the dual functor $(p_!)^\vee: \mathrm{Vect}\to Shv(Y)$. Is the object $(p_!)^\vee(e)$ compact? If it was compact, the functor $p_!$ would preserve limits.

0.0.33. Let $Y, Z\in \mathrm{PreStk}_{lft}$ and $\pi: Y\to Z$ be proper, in particular, of finite type. Consider the Cech nerve $[\dots Y^3_Z \substack{\to\\ \to\\ \to} Y^2_Z \rightrightarrows Y]$ of π. Applying Shv, we get a cosimplicial category $\boldsymbol{\Delta}^{op}\to \mathrm{DGCat}_{cont}$, $[n]\mapsto Shv(Y^{n+1}_Z)$, here $Y^n_Z=Y\times_Z Y\times_Z\dots\times_Z Y$, the product of n copies. For $i\ge 0$ let $\partial_i: [i]\to [i+1]$ be the last face map, it avoids $i+1$. The corresponding map $p^{\partial_i}: Y^{i+1}_Z\to Y^i_Z$ is the projection, so $(p^{\partial_i})^!$ has a left adjoint $(p^{\partial_i})_! = (p^{\partial_i})_*$. By base change, this cosimplicial category satisfies the monadic Beck-Chevalley conditions, so

$$\underset{[n]\in\boldsymbol{\Delta}}{Tot}\, Shv(Y^{n+1}_Z)\,\widetilde{\to}\, \mathcal{A}-mod(Shv(Y)),$$

where $\mathcal{A}=(p_2)_*p_1^!$ for the projections $p_1, p_2: Y^2_Z\to Y$.

Now $\pi^!: Shv(Z)\to Shv(Y)$ has a left adjoint $\pi_!$, and the monad $\pi^!\pi_!$ acting on $Shv(Y)$ identifies with $\mathcal{A}$. We always have a natural functor $Shv(Z)\to \mathcal{A}-mod(Shv(Y))$. Assume in addition that $\pi: Y\to Z$ is surjective on k-points (and more generally, on field-valued points, we have to take into account generic points in particular). Then $\pi^!$ is conservative, so that $\pi^!$ satisfies the Beck-Chevalley theorem ([7], ch. I.1, 3.7.7), and the induced functor $Shv(Z)\to \mathcal{A}-mod(Shv(Y))$ is an equivalence. Thus, Shv satisfies the proper descent.

0.0.34. It seems the following is also needed. Consider the cartesian square (3), where all the maps are schematic quasi-compact say. Let $F\in Shv(X')$ such that $g'_!: Shv(X')\to Shv(Y')$ is defined on F and $f^*_Y g'_! F$ is defined. Then $f^*_X F$ and $g_! f^*_X F$ are both defined and we have a natural isomorphism $g_! f^*_X F\,\widetilde{\to}\, f^*_Y g'_! F$. Is this true?

I think this was used in ([11], proof of Prop. 2.8.2).

0.0.35. *Question.* Let Y be an ind-scheme of ind-finite type (or a classical algebraic stack locally of finite type). Let $U_i\subset Y$ be an open immersion for $i\in\mathbb{N}$ such that for $i<j$ we have $U_i\subset U_j$ and $\cup_i U_i=Y$. Is it true that $\operatorname{colim}_{i\in\mathbb{N}} U_i$ in PreStk identifies with Y?

Example: we may form a sequence of opens $U_i\subset \mathrm{Gr}_G$, where each U_i is of the form $\mathrm{Gr}_G-\cup_{i=1}^n \bar{S}^{\lambda_i}$ and

$$U_i\subset U_{i+1}\subset\dots$$

with $\cup_i U_i=\mathrm{Gr}_G$. We have $Shv(\mathrm{Gr}_G)\,\widetilde{\to}\,\lim_i Shv(U_i)$ anyway, as for any closed subscheme of finite type $S\subset \mathrm{Gr}_G$, $S\subset U_i$ for some i.

0.0.36. *Torsors under placid group-schemes.* Let $\mathcal{Y}_\alpha$ be an ind-scheme of ind-finite type functorial in $\alpha \in A^{op}$, where A is filtered, $\alpha_0 \in A$ is initial in A. Let $G = \lim_{\alpha\in A^{op}} G_\alpha$ be a placid group scheme, where G_α is a smooth group scheme of finite type, and for $\alpha \to \beta$ in A, $G_\beta \to G_\alpha$ is smooth, affine and surjective homomoprhism of group schemes. Assume $\mathcal{Y}_\alpha \to \mathcal{Y}_{\alpha_0}$ is a G_α-torsor. For $\alpha \to \beta$ in A, $\mathcal{Y}_\beta \to \mathcal{Y}_\alpha$ is G_β-equivariant.

Then we are in the setting of ([10], C.1.6), so we get a placid ind-scheme $\mathcal{Y}$ as follows. Write $\mathcal{Y}_{\alpha_0} = \operatorname{colim}_{i\in I} Y_i$, where Y_i is a scheme of finite type, I is filtered, and for $i \to j$, $Y_i \to Y_j$ is a closed immersion. Let $\mathcal{Z}_i = \lim_{\alpha\in A^{op}} Y_i \times_{\mathcal{Y}_{\alpha_0}} \mathcal{Y}_\alpha$, so $\mathcal{Z}_i$ is a placid scheme, and $\mathcal{Z}_i \to \mathcal{Z}_j$ is a placid closed immersion. So, $\mathcal{Y} := \operatorname{colim} \mathcal{Z}_i$ is a placid ind-scheme, and $Shv(\mathcal{Y}) \widetilde{\to} \lim_{\alpha\in A^{op}} Shv(\mathcal{Y}_\alpha)$ with respect to the functors $(f_{\alpha,\beta})_* : Shv(\mathcal{Y}_\beta) \to Shv(\mathcal{Y}_\alpha)$ for $\alpha \to \beta$ in A and $f_{\alpha,\beta} : \mathcal{Y}_\beta \to \mathcal{Y}_\alpha$. The group G acts on $\mathcal{Y}_\alpha$ for each α via the quotient $G \to G_\alpha$, this gives an action of $Shv(G)$ on $Shv(\mathcal{Y}_\alpha)$. The functors $(f_{\alpha,\beta})_*$ are morphisms of $Shv(G)$-modules, so $Shv(\mathcal{Y})$ can be seen as $\lim_{\alpha\in A^{op}} Shv(\mathcal{Y}_\alpha)$ taken in $Shv(G) - mod$.

Let us show that $\mathcal{Y} \widetilde{\to} \lim_{\alpha\in A^{op}} \mathcal{Y}_\alpha$ as prestacks. We have a natural map $\mathcal{Y} \to \lim_\alpha \mathcal{Y}_\alpha$. Let $S \in \mathrm{Sch}^{aff}$. Recall that for any n, $\tau_{\le n} \mathrm{Spc} \subset \mathrm{Spc}$ is stable under filtered colimits, so $\mathcal{Y}(S) \in Sets$ and an element of $\mathcal{Y}(S)$ comes from an element of $\mathcal{Z}_i(S)$ for some i (by [21], Cor. 13.1.14). So, an element of $\mathcal{Z}_i(S)$ is the same as an element of $\mathcal{Y}(S)$ whose image in $\mathcal{Y}_{\alpha_0}(S)$ lies in the subset $Y_i(S)$. The makes the claim manifest (and it holds more generally in the situation of ([10], C.1.6)).

Since $\mathrm{Ker}(G \to G_\alpha)$ is prounipotent for $\alpha \neq \alpha_0$, we get $Shv(\mathcal{Y}_\alpha)^G \widetilde{\to} Shv(\mathcal{Y}_\alpha)^{G_\alpha}$ for $\alpha \neq \alpha_0$ by ([25], 1.3.21). Now by Section 0.0.16 of this file, $Shv(\mathcal{Y}_\alpha)^{G_\alpha} \widetilde{\to} Shv(\mathcal{Y}_{\alpha_0})$ via the functor $f^*_{\alpha_0,\alpha} : Shv(\mathcal{Y}_{\alpha_0}) \to Shv(\mathcal{Y}_\alpha)$. So,

$$Shv(\mathcal{Y})^G \widetilde{\to} \lim_{\alpha\in A^{op}} Shv(\mathcal{Y}_\alpha)^G \widetilde{\to} Shv(\mathcal{Y}_{\alpha_0})$$

We could take the functors $f^!_{\alpha_0,\alpha}$ instead, but the two limits would be isomorphic.

We may strengthen the above as follows. Assume H is a placid group scheme, $G \subset H$ is a placid closed immersion, and a normal group subscheme with the cokernel K, here K is a smooth affine group scheme of finite type. Assume the G-action on $\mathcal{Y}$ is extended to a H-action. Then as above we get $Shv(\mathcal{Y})^H \widetilde{\to} Shv(\mathcal{Y}_{\alpha_0}/K)$.

0.0.37. Let $H \in Grp(\mathrm{PreStk})$ be a placid ind-scheme written as $H \widetilde{\to} \operatorname{colim}_{j\in J} H_j$, where H_j is a placid group scheme, and for $j \to j'$ in J the map $H_j \to H_{j'}$ is a placid closed immersion and a homomorphism of group schemes. Assume $j = 0$ is initial in J and let $G = H_0$. Then for any j, H_j/G is a scheme of finite type, so $H/G \widetilde{\to} \operatorname{colim}_{j\in J} H_j/G$, because colimits commute with colimits, so H/G is an ind-scheme of ind-finite type. Assume G prosmooth.

Write as in the previous section $G \widetilde{\to} \lim_{\alpha\in A^{op}} G_\alpha$, where G_α is a smooth group scheme of finite type, and for $\alpha \to \beta$ in A, $G_\beta \to G_\alpha$ is smooth, affine and surjective. Set $K_\alpha = Ker(G \to G_\alpha)$. For $\alpha \to \beta$ in A let $1 \to K_{\alpha,\beta} \to G_\beta \to G_\alpha \to 1$ be an exact sequence. Assume $K_{\alpha,\beta}$ is a unipotent group scheme. Then $K_\alpha \widetilde{\to} \lim_\beta K_{\alpha,\beta}$ is prounipotent.

Set $\mathcal{Y}_\alpha = H/K_\alpha$, we usually mean by this the etale sheafification of the prestack quotient. This is an ind-scheme of ind-finite type by the above, and for $\alpha \to \beta$ in A the map $\mathcal{Y}_\beta \to \mathcal{Y}_\alpha$ is a K_α/K_β-torsor. So, we are in the situation of the previous section,

α_0 is initial in A. We write $H/K_{\alpha_0} = \widetilde{\to} \operatorname{colim}_j H_j/K_{\alpha_0}$. So, $\mathcal{Y} \widetilde{\to} \lim_\alpha H/K_\alpha$. Note that $\lim_\beta (K_\alpha/K_\beta) \widetilde{\to} K_\alpha$. We get $\mathcal{Y} \widetilde{\to} \operatorname{colim}_j H_j \widetilde{\to} H$, because $\lim_\alpha H_j/K_\alpha \widetilde{\to} H_j$ for any j.

From $H \widetilde{\to} \lim_{\alpha \in A^{op}} H/K_\alpha$ we get $Shv(H) \widetilde{\to} \lim_\alpha Shv(H/K_\alpha)$. From the previous section we now get an equivalence $Shv(H/K_{\alpha_0}) \widetilde{\to} Shv(H)^{K_{\alpha_0}}$. Similarly, we may get $Shv(H/K_\alpha) \widetilde{\to} Shv(H)^{K_\alpha}$ for any α.

We have an action of G_α by right translations on H/K_α, and $(H/K_\alpha)/G_\alpha \widetilde{\to} H/G$. Now Section 0.0.16 gives $Shv(H/K_\alpha)^{G_\alpha} \widetilde{\to} Shv(H/G)$.

As in the previous subsection, we get $Shv(H/G) \widetilde{\to} Shv(H/K_\alpha)^{G_\alpha} \widetilde{\to} Shv(H)^G$ for any of the 4 sheaf theories (for $\mathcal{D}$-modules this is ([4], Lemma B.5.1).

Corollary 0.0.38. *Let $H \in Grp(\mathrm{PreStk})$ be a placid ind-scheme, $G \subset H$ be a closed placid prosmooth group subscheme. For any of the 4 sheaf theories $Shv(H/G) \widetilde{\to} Shv(H)^G$, where G acts on H by right translations.*

0.0.39. Let $G \in Grp(\mathrm{PreStk})$ be a placid ind-scheme, Y be a placid ind-scheme with a G-action. Then $Shv(Y)$ is equipped with a $Shv(G)$-action. Namely, for $K \in Shv(G), F \in Shv(Y)$ one has $K * F \widetilde{\to} a_*(K \boxtimes F)$ for the action map $a : G \times Y \to Y$.

0.0.40. Let $Y \to S$ be a map in Sch_{ft}, G be a placid group scheme over S acting on Y over S though its quotient $G \to G_0$ group scheme (smooth and of finite type over S) with a prounipotent kernel. We have canonically $Shv(Y)^G \widetilde{\to} Shv(Y)^{G_0}$ by ([25], 1.3.21). Consider the stack quotient Y/G (by which we mean etale sheafification of the prestack quotient). We define $Shv(Y/G)$ as $Shv(Y/G_0)$ in such a way that for $q : Y \to Y/G$ the functor $q^* : Shv(Y/G) \to Shv(Y)$ is defined as $q_0^* : Shv(Y/G_0) \to Shv(Y)$ for $q_0 : Y \to Y/G_0$. So, if $G \to G_1 \to G_0$ are given, where G_1 is another finite-dimensional quotient group scheme over S with $\mathrm{Ker}(G \to G_1)$ prounipotent then we identify $Shv(Y/G_1) \widetilde{\to} Shv(Y/G_0)$ via a^* for the natural map $a : Y/G_1 \to Y/G_0$. No shifts appear. Note that $Shv(Y/G_0)$ is compactly generated both for $\mathcal{D}$-modules and in the constructible context (for $\mathcal{D}$-modules this is true, as Y/G_0 is perfect [5]).

If $f : Y \to Y'$ is a G-equivariant map in $(Sch_{ft})_{/S}$ (we assume the G-action on both schemes factor through a finite dimensional quotient group scheme) then we have $f^! : Shv(Y'/G) \to Shv(Y/G)$.

We extend this definition to the case of an ind-scheme of ind-finite type Y over S equipped with a G-action over S as follows. Assume Y admits a presentation $Y \widetilde{\to} \operatorname{colim}_{i \in I} Y_i$, where Y_i is a G-invariant closed subscheme of finite type, I is filtered, and for $i \to j$ in I the map $Y_i \to Y_j$ is a closed immersion. Assume the G-action on Y_i factors through a quotient group scheme $G \to G_i$, where $G_i \to S$ is smooth, of finite type with $\mathrm{Ker}(G \to G_i)$ prounipotent. Then we have $Shv(Y_i/G)$ defined as above and set $Shv(Y/G) \widetilde{\to} \lim_{i \in I^{op}} Shv(Y_i/G)$ with respect to the !-restrictions. With this definition for $q : Y \to Y/G$ we get the functor $q^* : Shv(Y/G) \to Shv(Y)$, which is the limit over $i \in I^{op}$ of the functors $q_i^* : Shv(Y_i/G) \to Shv(Y_i)$ for $q_i : Y_i \to Y_i/G$. It also identifies with $\mathrm{oblv} : Shv(Y)^G \to Shv(Y)$. Note that for $i \to j$ in I the functor of !-restriction $Shv(Y_j/G) \to Shv(Y_i/G)$ admits a fully faithful left adjoint. So, $Shv(Y)^G \widetilde{\to} \operatorname{colim}_{i \in I} Shv(Y_i/G)$ with respect to the !-direct images. We see that for $\mathcal{D}$-modules or in the constructible context $Shv(Y)^G$ is compactly generated.

Let now H be a placid group ind-scheme over S, $G \subset H$ a closed placid group subscheme over S, so H/G is an ind-scheme of ind-finite type over S. Then the above assumption is satisfied for the G-action on H/G over S. So, $Shv(H/G)^G$ identifies with $Shv(G\backslash H/G)$. For $q : H/G \to G\backslash H/G$ the functor $q^* : Shv(G\backslash H/G) \to Shv(H/G)$ identifies with $\mathrm{oblv} : Shv(H/G)^G \to Shv(H/G)$.

Let again $Y \to S$ be a map in Sch_{ft} and G a placid group scheme over S. Assume that the action of G on Y factors though the finite-dimensional group scheme $G_0 \to S$ smooth over S, and let $G \to G_1 \to G_0$ be as above. Another way to realize $Shv(Y/G)$ is as the category $Shv(Y/G_0)$ via the identifications $a^*[\dim.\mathrm{rel}(a)] : Shv(Y/G_0) \widetilde{\to} Shv(Y/G_1)$ for every G_1 as above. Indeed, the equivalence

$$Shv(Y/G_0) \widetilde{\to} Shv(Y/G_0),\ K \mapsto K[-\dim(G_0/S)]$$

from the first model to the second one allows to identify them. The advantage of the second model is that the transition functors are t-exact for the perverse t-structure, so allow to equip $Shv(Y/G)$ with the perverse t-structure: this is the perverse t-structure on $Shv(Y/G_0)$.

For the second model for $q_0 : Y \to Y/G_0$ consider the functor $Shv(Y/G) = Shv(Y/G_0) \to Shv(Y)$ given by $q_0^*[\dim(G_0/S)]$, it is t-exact and compatible with the transition functors for the second model, so defines a functor $Shv(Y/G) \to Shv(Y)$ that we denote by $q^*[\dim.\mathrm{rel}(q)]$, this is just one symbol.

Let now Y be an ind-scheme of ind-finite type over S with a G-action and a presentation $Y \widetilde{\to} \mathrm{colim}_{i\in I} Y_i$, where I is small filtered, $Y_i \to S$ is a G-invariant closed subscheme of finite type in Y, and for $i \to j$ the map $h_{ij} : Y_i \to Y_j$ is a closed immersion. Assume G-action on Y_i factors through a quotient group scheme $G_i \to S$ smooth and of finite type over S, where $\mathrm{Ker}(G \to G_i)$ is a prounipotent group scheme over S. Equip each $Shv(Y_i/G)$ with the perverse t-structure. Then the !-pullbacks under $Y_i/G_0 \to Y_j/G_0$ are compatible with the transition functors for the second model, so define a functor $h_{ij}^! : Shv(Y_j/G) \to Shv(Y_i/G)$, which is left t-exact. It also commutes with the functors $q_i^*[\dim.\mathrm{rel}(q_i)]$ for $q_i : Y_i \to Y_i/G$. Recall that $Shv(Y/G) \widetilde{\to} \lim_{i\in I^{op}} Shv(Y_i/G)$ with respect to the functors $h_{ij}^!$. The limit over $i \in I^{op}$ of the functors $q_i^*[\dim.\mathrm{rel}(q_i)] : Shv(Y_i/G) \to Shv(Y_i)$ in the second model is denoted $\mathrm{oblv}[\dim.\mathrm{rel}] : Shv(Y)^G \to Shv(Y)$.

Each $h_{ij}^! : Shv(Y_j/G) \to Shv(Y_i/G)$ admits a left adjoint $(h_{ij})_! : Shv(Y_i/G) \to Shv(Y_j/G)$, and we may also write $Shv(Y/G) \widetilde{\to} \mathrm{colim}_{i\in I} Shv(Y_i/G)$ with the transition functors $(h_{ij})_!$. Now we may define the perverse t-structure on $Shv(Y/G)$ as in the case of an ind-scheme of ind-finite type. Namely, $K \in Shv(Y/G)$ lies in $Shv(Y/G)^{\geq 0}$ iff for any i, its !-restriction to Y_i/G lies in $Shv(Y_i/G)^{\geq 0}$. So, $Shv(Y/G)^{\geq 0} \widetilde{\to} \lim_{i\in I^{op}} Shv(Y_i/G)^{\geq 0}$, which shows that $Shv(Y/G)^{\geq 0}$ is presentable, so the t-structure is accessible. This t-structure is also compatible with filtered colimits.

In fact, if we identify the first and the second model of $Shv(Y)^G$ as above then the functors oblv for the first model becomes the functor $\mathrm{oblv}[\dim.\mathrm{rel}] : Shv(Y)^G \to Shv(Y)$ for the second one. So, this is just a matter of notations.

For the natural map $q : Y \to Y/G$ the functor $q^! : Shv(Y/G) \to Shv(Y)$ is also defined similarly, though q is not locally of finite type.

Namely, if $Y \in \mathrm{Sch}_{ft}$ we first consider a third model for $Shv(Y/G)$: for a quotient $G \to G_0$ as above such that G-action on Y factors through G_0, we identify the 3rd model with the second via the equivalences: $Shv(Y/G_0 \widetilde{\to} Shv(Y/G_0), K \mapsto K[2\dim G_0]$. For $G \to G_1 \to G_0$ let $a : Y/G_1 \to Y/G_0$ be the natural map. Under such equivalences the transition functor a^* for the first model becomes the transition functor $a^!$ for the 3rd model. Now for the third model we define $q^! : Shv(Y/G) \to Shv(Y)$ as $q_0^!$ for $q_0 : Y \to Y/G_0$.

This definition is similarly extended to ind-schemes of ind-finite type.

0.0.41. Let $Y \widetilde{\to} \mathrm{colim}_{i\in I} Y_i$ in PreStk, where I is filtered, Y_i is a scheme of finite type, and for $i \to j$ in I, $Y_i \to Y_j$ is a closed immersion, so Y is an ind-scheme of ind-finite type. Let $H \to G$ be a homomorphism of placid group schemes over $\mathrm{Spec}\, k$. Assume G acts on Y and the assumption of the previous subsection holds, that is, each Y_i is G-invariant, and on Y_i the group scheme G acts via a finite-dimensional quotient group scheme $G \to G_i$ with $\mathrm{Ker}(G \to G_i)$ prounipotent. We have a natural map of stack quotients $h : Y/H \to Y/G$. We have defined the categories $Shv(Y/G), Shv(Y/H)$ in the previous subsection. Then the functor $h^* : Shv(Y/G) \to Shv(Y/H)$ is defined, namely this is $\mathrm{oblv} : Shv(Y)^G \to Shv(Y)^H$.

0.0.42. If Y is a stack locally of finite type, a placid group scheme over Y should be defined as a group object $(G \to Y) \in Grp(\mathrm{PreStk}_{/Y})$ such that for any $S \to Y$ with $S \in \mathrm{Sch}^{aff}_{ft}$, $S \times_Y G$ is a placid group scheme over S.

Let $Z \to Y$ be a map in Stk_{lft} and G be a placid group scheme over Y acting on Z over Y. Write Z/G for the stack quotient of Z by G (etale sheafification of the prestack quotient), so $Z/G \to Y$. How do we define $Shv(Z)^G$?

First, for any $S \to Y$ with $S \in \mathrm{Sch}^{aff}_{ft}$ we have a monoidal category $Shv(S \times_Y G)$ defined in ([25], 1.3.7), it is an object of $Alg(Shv(Y) - mod)$. For a map $S' \stackrel{\alpha}{\to} S \to Y$ in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$ let $\beta : S' \times_Y G \to S \times_Y G$ be obtained by base change. As in ([25], 1.3.12), $\beta^! : Shv(S \times_Y G) \to Shv(S' \times_Y G)$ is monoidal, it is actually a morphism in $Alg(Shv(Y) - mod)$. To see this we used Lemma 0.0.43 below. So, we may understand

$$Shv(G) \widetilde{\to} \lim_{(S\to Y)\in((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op}} Shv(S \times_Y G)$$

as limit taken in $Alg(Shv(Y) - mod)$. We get a monoidal structure on $Shv(G)$ via the latter limit.

This is one more extension of our sheaf theory needed. In general, we can not write G as $\lim_{i\in I^{op}} G_i$, where $G_i \to Y$ is an affine group scheme "of finite type" over Y, I is filtered, and for $i \to j$ in I the map $G_j \to G_i$ is affine smooth surjective. This does not hold already for $\mathcal{L}^+(G) \to \mathrm{Ran}$, I think, where G is a reductive group.

We will see that the monoidal category $Shv(G)$ acts on $Shv(Z)$.

For each $S \to Y$ in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$, $S\times_Y G$ acts on $S\times_Y Z$ over S, so $Shv(S\times_Y G)$ acts on $Shv(S\times_Y Z)$ naturally. For a map $S' \stackrel{\alpha}{\to} S \to Y$ in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$ let $\bar{\alpha} : S'\times_Y Z \to S\times_Y Z$ be obtained from α by base change.

Let $Shv(S \times_Y G)$ act on $Shv(S' \times_Y Z)$ via the map $Shv(S \times_Y G) \to Shv(S' \times_Y G)$. Then $\bar{\alpha}^!$ commutes with $Shv(S \times_Y G)$-actions.

Recall that the sheafification is a left exact functor, so for the stack quotients we get $((S \times_Y Z)/(S \times_Y G)) \times_S S' \widetilde{\to} (S' \times_Y Z)/(S' \times_Y G)$ canonically.

Consider the ∞-category $AssAlg + Mod(\mathrm{DGCat}_{cont})$ defined in ([7], ch. I.1, 3.5.4). By ([16], 3.2.2.5), it admits limits and the projection $AssAlg + Mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves limits. We obtained a functor

$$((\mathrm{Sch}_{ft}^{aff})_{/Y})^{op} \to AssAlg + Mod(\mathrm{DGCat}_{cont})$$

sending $S \to Y$ to the pair $Shv(S \times_Y G), Shv(S \times_Y Z)$. So, the limit of the latter functor is an object of $AssAlg + Mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$. In other words, $Shv(G)$ act on $Shv(Z)$ naturally, and we may consider the invariants

$$Shv(Z)^{Shv(G)} = \mathrm{Fun}_{Shv(G)}(Shv(Y), Shv(Z)) \in Shv(Y) - mod$$

Question. Can we rewrite the above as limit of $Shv(S \times_Y Z)^{Shv(S \times_Y G)}$ over S? More precisely, for $(S \to Y) \in (\mathrm{Sch}_{ft}^{aff})_{/Y}$, let $q_S : S \times_Y G \to S$ be the projection. By ([25], 1.3.16), we have canonically

$$Shv(S \times_Y Z)^{Shv(S \times_Y G)} \widetilde{\to} q_S^*\omega - comod(Shv(S \times_Y Z))$$

If $\alpha : S' \to S$ is a morphism in $(\mathrm{Sch}_{ft}^{aff})_{/Y}$ then $\beta^! q_S^*\omega \widetilde{\to} q_{S'}^*\omega$ as coalgebras in $Shv(S' \times_Y G)$. This is by definition of the functors $q_S^*, q_{S'}^*$. So, $\bar{\alpha}^! : Shv(S \times_Y Z) \to Shv(S' \times_Y Z)$ induces a functor between the comodule categories

$$q_S^*\omega - comod(Shv(S \times_Y Z)) \to q_S^*\omega - comod(Shv(S' \times_Y Z)) = q_{S'}^*\omega - comod(Shv(S' \times_Y Z))$$

Now we may consider

$$\lim_{S \to Y} q_S^*\omega - comod(Shv(S \times_Y Z))$$

taken in DGCat_{cont} over the category $((\mathrm{Sch}_{ft}^{aff})_{/Y})^{op}$. Is it equivalent to $Shv(Z)^{Shv(G)}$?

Lemma 0.0.43. *Let $S' \to S$ be a map in Sch_{ft}. Let $f : Y \to Z$ be a morphism of placid schemes over S, let $f' : Y' \to Z'$ be obtained from f by the base change $\alpha : S' \to S$. Write $\alpha_Y : Y' \to Y$ and $\alpha_Z : Z' \to Z$ for the obtained maps. Then for $K \in Shv(Y)$ one has canonically $\alpha_Z^! f_* K \widetilde{\to} f'_* \alpha_Y^! K$.*

Proof. Write $Y \widetilde{\to} \lim_{i \in I^{op}} Y_i$, where I is filtered, Y_i is a scheme of finite type over S, and for $i \to i'$ in I the map $Y_{i'} \to Y_i$ is affine smooth surjective (over S), and similarly for $Z \widetilde{\to} \lim_{j \in J^{op}} Z_j$. These are presentations from a definition of a placid scheme. Let Y'_i, Z'_j be obtained from Y_i, Z_j by base change $S' \to S$, so $Y' \widetilde{\to} \lim_{i \in I^{op}} Y'_i$ and $Z' \widetilde{\to} \lim_{j \in J^{op}} Z'_j$.

It suffices to establish the desired isomorphism after applying $(ev'_j)_* : Shv(Z') \to Shv(Z'_j)$ for each $j \in J$, here $ev'_j : Z' \to Z'_j$ is the projection. Pick $i \in I$ such that the composition $Y \to Z \to Z_j$ factors through $\bar{f} : Y_i \to Z_j$. By base change under $S' \to S$

we get a cartesian square

$$\begin{array}{ccc} Y_i & \stackrel{\bar f}{\to} & Z_j \\ \uparrow \alpha_{Y_i} & & \uparrow \alpha_{Z_j} \\ Y'_i & \stackrel{\bar f'}{\to} & Z'_j \end{array}$$

Let $(ev_i)_*: Shv(Y)\to Shv(Y_i)$ be the direct image under $ev_i: Y\to Y_i$. The key point is the base change isomorphism $\alpha^!_{Z_j}\bar f_*\widetilde{\to}\bar f'_*\alpha^!_{Y_i}$. We get

$$(ev'_j)_*\alpha^!_Z f_*K\widetilde{\to}\alpha^!_{Z_j}(ev_j)_*f_*K\widetilde{\to}\alpha^!_{Z_j}\bar f_*(ev_i)_*K\widetilde{\to}\bar f'_*\alpha^!_{Y_i}(ev_i)_*K$$
$$\widetilde{\to}\bar f'_*(ev_i)_*\alpha^!_Y K\widetilde{\to}(ev'_j)_*f'_*\alpha^!_Y K$$

We are done. □

Corollary: let $\alpha: S'\to S$ be a map in Sch_{ft}, $Y\to S$ be a placid ind-scheme over S and $Y'\to S'$ be obtained by base change. Let $\bar\alpha: Y'\to Y$ be the natural map. Then $\bar\alpha^!: Shv(Y)\to Shv(Y')$ is well-defined.

Proof. Write $Y\widetilde{\to}\underset{i\in I}{\mathrm{colim}} Y_i$, where I is small filtered category, Y_i is a placid S-scheme, and for $i\to j$ in I the map $Y_i\to Y_j$ is a placid closed immersion over S. Write $Y'_i = Y_i\times_S S'$. For each $i\in I$ let $\bar\alpha_i: Y'_i\to Y_i$ be the correspnding map. The functors $\bar\alpha^!_i$ are compatible with $*$-pushforwards in the corresponding inductive systems, passing to the colimit in $\bar\alpha^!_i: Shv(Y_i)\to Shv(Y'_i)$, we get the desired $\bar\alpha^!$. These functors are also compatible with the !-pullbacks, so we can also pass to the limit. □

0.0.44. Let us be in the situation of Section 0.0.42. Probably the only case we need satisfies the following additional assumption that we make. For any $S\to Y$ in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$, $S\times_Y Z$ may be written as $S\times_Y Z\widetilde{\to}\mathrm{colim}_{i\in I} Z_{S,i}$, where I is filtered, and $Z_{S,i}\to S$ is a scheme of finite type such that for any $i\to j$ in I, the map $Z_{S,i}\to Z_{S,j}$ is a closed immersion. Moreover, for each $i\in J$, $Z_{S,i}$ is stable under the action of $S\times_Y G$, and the latter acts through a finite-dimensional quotient scheme $G_{S,i}$ over S. In particular, $S\times_Y Z$ is a ind-scheme of ind-finite type over S.

In this setting one may define the category $Shv(Z/G)$ in one more way. Namely, for $S\to Y$ as above, in Section 0.0.40 we have defined the category $Shv((S\times_Y Z)/(S\times_Y G))$ together with the functor $q_S^*: Shv((S\times_Y Z)/(S\times_Y G))\to Shv(S\times_Y Z)$ for the projection $q_S: S\times_Y Z\to(S\times_Y Z)/(S\times_Y G)$. Recall that q_S^* identifies with oblv : $Shv(S\times_Y Z)^{S\times_Y G}\to Shv(S\times_Y Z)$.

Let now $\alpha: S'\to S$ be a map in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$. Let $\bar\alpha: S'\times_Y Z\to S\times_Y Z$ and $\beta: S'\times_Y G\to S\times_Y G$ be obtained by base change from α. Pick a finite-dimensional quotient group scheme $S\times_Y G\to G_S$ such that $S\times_Y G$-action on $S\times_Y Z$ factors through G_S. Let $G_{S'} = G_S\times_S S'$. We have the cartesian square

$$\begin{array}{ccc} S'\times_Y Z & \stackrel{\bar\alpha}{\to} & S\times_Y Z \\ \downarrow h' & & \downarrow h \\ (S'\times_Y Z)/G_{S'} & \stackrel{\tilde\alpha}{\to} & (S\times_Y Z)/G_S \end{array}$$

The functors $\tilde{\alpha}^! : Shv((S\times_Y Z)/G_S)\to Shv((S'\times_Y Z)/G_{S'})$ can be seen by definition as the functors that fit into a commuttaive diagram

$$\begin{array}{ccc} Shv(S\times_Y Z) & \stackrel{\bar{\alpha}^!}{\to} & Shv(S'\times_Y Z)\\ \uparrow \text{oblv} & & \uparrow \text{oblv}\\ Shv(S\times_Y Z)^{S\times_Y G} & \stackrel{\tilde{\alpha}^!}{\to} & Shv(S'\times_Y Z)^{S'\times_Y G}\end{array}$$

This way we get a functor $((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op}\to Shv(Y)-mod$, $S\mapsto Shv(S\times_Y Z)^{S\times_Y G}$. Finally, we may consider

$$\lim_{S\to Y} Shv(S\times_Y Z)^{S\times_Y G}$$

in $Shv(Y)-mod$ taken over $((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op}$. This should be our definition of $Shv(Z/G)$ I think.

0.0.45. For S a scheme of finite type consider the perverse t-structure on S. The functor $\mathrm{H}^i : Shv(S)\to Shv(S)^{\heartsuit}$ preserves products, is this correct? This was used to conclude that your $QLisse(S)$ for S smooth is left complete.

For a scheme of finite type S any object of $Shv(S)^{constr}$ is bounded.

0.0.46. Let Y be a classical algebraic stack locally of finite type with an affine diagonal. Then the truncation functors for the perverse t-structure $\tau^{\le n},\tau^{\ge n}$ preserve the subcategory $Shv(Y)^{constr}\subset Shv(Y)$, so we get a t-structure on $Shv(Y)^{constr}$.

0.0.47. Let Y be a classical quasi-compact algebraic stack with an affine diagonal. Let $F\in Shv(Y)^{constr}$ then F is bounded. Indeed, pick a smooth covering $f : S\to Y$, where $S\in \mathrm{Sch}^{aff}_{ft}$. Since $f^*[\dim.\mathrm{rel}(f)]$ is t-exact and conservative, it suffices to show that f^*F is bounded. However, any compact object in $Shv(S)$ is bounded.

0.0.48. Let $S\in \mathrm{Sch}_{ft}$, $K_i\in Shv(S)$ and $E\in Lisse(E)$, that is, E is dualizable with respect to the $\otimes$-monoidal structure on $Shv(S)$. Recall that $\mathcal{H}om(K_1,K_2)\in Shv(S)$ denotes the inner hom with respect to the $\otimes$-monoidal structure on $Shv(S)$. Then by ([16], 4.6.2.1) we get $\mathcal{H}om(K_1\otimes E, K_2)\widetilde{\to}\mathcal{H}om(K_1, E^\vee\otimes K_2)$ with $E^\vee=\mathcal{H}om(E,e)$.

0.0.49. Recall the following from ([13], A.1.7). Let $Corr(\mathrm{PreStk}_{lft})_{ind-sch,all}$ be the category of correspondences, whose objects are prestacks locally of finite type $\mathcal{Y}$, and a morphism from $\mathcal{Y}_1$ to $\mathcal{Y}_2$ is a diagram $\mathcal{Y}_1\stackrel{g}{\leftarrow}\mathcal{Y}_{12}\stackrel{f}{\to}\mathcal{Y}_2$ with g any and f ind-schematic of ind-finite type. Then in the constructible context we get a functor $Shv_{Corr} : Corr(\mathrm{PreStk}_{lft})_{ind-sch,all}\to \mathrm{DGCat}_{cont}$ sending $\mathcal{Y}$ to $Shv(\mathcal{Y})$, and sending the above morphism to the functor $f_*g^! : Shv(\mathcal{Y}_1)\to Shv(\mathcal{Y}_2)$. Then the functor Shv_{Corr} possesses a natural right-lax symmetric monoidal structure, see ([7], Vol. 2, Chapter 3, Sect. 6.1), where $Corr(\mathrm{PreStk}_{lft})_{ind-sch,all}$ is a symmetric monoidal category with respect to the level-wise product.

In particular, this means that given $f_i : Y_i\to Z_i$ ind-schematic of ind-finite type in PreStk_{lft} and $K_i\in Shv(Y_i)$, we have

$$(f_1\times f_2)_*(K_1\boxtimes K_2)\widetilde{\to}((f_1)_*K_1)\boxtimes((f_2)_*K_2)$$

Let $\mathcal{H}$ be a groupoid acting on $\mathcal{Y}$ in PreStk_{lft} given by a functor $\boldsymbol{\Delta}^{op} \to \mathrm{PreStk}_{lft}$ such that the action map $m : \mathcal{H} \times_{t,\mathcal{Y},s} \mathcal{H} \to \mathcal{H}$ is ind-schematic of ind-finite type, here $s, t : H \to \mathcal{Y}$ are source and targets maps. We get a monoidal structure on $Shv(\mathcal{H})$ with the product given by $(K_1, K_2) \mapsto m_* q^!(K_1 \boxtimes K_2)$ for $q : \mathcal{H} \times_{\mathcal{Y}} \mathcal{H} \to \mathcal{H} \times \mathcal{H}$. Let $\alpha : \mathcal{Y} \to \mathcal{H}$ be the map corresponding to $[1] \to [0]$ in $\boldsymbol{\Delta}$. Then $\alpha_* \omega_{\mathcal{Y}}$ is the unit of $Shv(\mathcal{H})$. Moreover, the functor $\alpha_* : (Shv(\mathcal{Y}), \otimes^!) \to Shv(\mathcal{H})$ is monoidal. Indeed, $\mathcal{H} \in Alg(Corr(\mathrm{PreStk}_{lft})_{ind-sch,all})$, so we just apply a right-lax monoidal functior Shv_{Corr}. Moreover,

$$(\mathcal{H}, \mathcal{Y}) \in Alg + module(Corr(\mathrm{PreStk}_{lft})_{ind-sch,all})$$

Namely, write $\mathrm{pr}, \mathrm{act} : \mathcal{H} \to \mathcal{Y}$ for the two maps from $\mathcal{H}$ to $\mathcal{Y}$ given by $0, 1 : [0] \to [1]$. Then the action map from $\mathcal{H} \times \mathcal{Y}$ to $\mathcal{Y}$ is given by the correspondence $\mathcal{H} \times \mathcal{Y} \stackrel{\mathrm{id,pr}}{\leftarrow} \mathcal{H} \stackrel{\mathrm{act}}{\to} \mathcal{Y}$. Applying Shv_{Corr}, we see that $Shv(\mathcal{Y}) \in Shv(\mathcal{H}) - mod(\mathrm{DGCat}_{cont})$.

The whole Section A.1 of [13] can be advised as a reference on the generalities about the sheaf theories.

More generally, we may define the category of relative groupoids Grpd / PreStk /Sch and the corresponding functor

$$\mathrm{Grpd} / \mathrm{PreStk} / \mathrm{Sch} \to Alg(\mathrm{Corr}(\mathrm{PreStk}_{lft}))$$

as in ([26], 1.4.48).

0.0.50. Let $H = \lim_{i \in I^{op}} H_i$ be a placid group scheme, here $I \in 1 - \mathrm{Cat}$ is filtered, if $i \in I$ then H_i is a group scheme of finite type, and for $i \to j$ in I, $H_j \to H_i$ is a smooth affine surjective morphism, a homomorphism of group schemes. Let $i \in I$ and $K_i \hookrightarrow H_i$ be a closed group subscheme, set $K = H \times_{H_i} K_i$. So, $K \hookrightarrow H$ is a placid closed immersion. Let $L = \mathrm{Ker}(H \to H_i)$. Then the natural map $H/K \to H_i/K_i$ is an isomorphism. Indeed, $L = \mathrm{Ker}(K \to K_i)$, and L acts trivially on H/K. So, the H-action on H/K by left translations factors through a transitive H_i-action, and the stabilizer of $K/K \in H/K$ is $K/L \widetilde{\to} K_i$.

0.0.51. *Application.* Let H be a smooth affine group scheme of finite type, $F = k((t)), O = k[[t]]$. Then $H(F)$ is a placid ind-scheme. It could be defined in two equivalent ways. Let for $n \geq 1$, $K_n = \mathrm{Ker}(H(\mathcal{O}) \to H(\mathcal{O}/t^n))$. Set $K_0 = H(\mathcal{O})$, so $\dots K_2 \subset K_1 \subset K_0$. Then $H(F)/K_n$ is an ind-scheme of ind-finite type. For $n < m$ we have the map $H(F)/K_m \to H(F)/K_n$, which is schematic, smooth affine and surjective. It is actually a torsor under K_n/K_m. So, we are in the situation of Section 0.0.36 for $A = \mathbb{Z}_{\geq 0}$. For $\alpha \in A$, $\mathcal{Y}_\alpha = H(F)/K_\alpha$, $G_\alpha = H(\mathcal{O}/t^\alpha)$. This gives $G = \lim_\alpha G_\alpha = H(\mathcal{O})$. Then we may define $H(F)$ as $\lim_{\alpha \in A^{op}} \mathcal{Y}_\alpha$, where the limit is taken in PreStk. By Section 0.0.36, for $n \leq m$ we have the projection $f_{n,m} : H(F)/K_m \to H(F)/K_n$ and the adjoint pair $f_{n,m}^* : Shv(H(F)/K_n) \leftrightarrows Shv(H(F)/K_m) : (f_{n,m})_*$. We may view $Shv(H(F)$ as $\lim_{n \in A^{op}} Shv(H(F)/K_n)$ in DGCat_{cont} with respect to $(f_{n,m})_*$. For $n > 0$ the group scheme K_n is prounipotent.

For the $H(\mathcal{O})$-action by right translations on $H(F)$ by Section 0.0.36 one gets

$$Shv(H(F))^{H(\mathcal{O})} \widetilde{\to} Shv(\mathrm{Gr}_H)$$

0.0.52. Let Y be placid scheme written as $Y\widetilde{\to}\lim_{i\in I^{op}} Y_i$, where I is small filtered category, for $i\to j$ in I the map $Y_j\to Y_i$ is smooth affine surjective morphism in Sch_{ft}. Let $S\in\mathrm{Sch}_{ft}$. For the projection $f_i: Y\to Y_i$ the diagram commutes

$$\begin{array}{ccc} Shv(Y)\otimes Shv(S) & \to & Shv(Y\times S)\\ \downarrow {\scriptstyle (f_i)_*\otimes \mathrm{id}} & & \downarrow {\scriptstyle (f_i\times\mathrm{id})_*}\\ Shv(Y_i)\otimes Shv(S) & \to & Shv(Y_i\times S), \end{array}$$

where the horizontal arrows are exteriour products. Indeed, $Shv(S)$ is dualizable, so $\lim_{i\in I^{op}} Shv(Y_i)\otimes Shv(S)\widetilde{\to} Shv(Y)\otimes Shv(S)$, where the limit is taken with respect to $(f_i)_*\otimes\mathrm{id}$.

Let now $f: Y\to Z$ be a morphism of placid schemes. The above shows that the diagram commutes

$$\begin{array}{ccc} Shv(Y)\otimes Shv(S) & \to & Shv(Y\times S)\\ \downarrow {\scriptstyle f_*\otimes \mathrm{id}} & & \downarrow {\scriptstyle (f\times\mathrm{id})_*}\\ Shv(Z)\otimes Shv(S) & \to & Shv(Z\times S), \end{array}$$

In turn, this show that the above diagram still commutes if we only assume that S is a placid scheme also. Finally, if $f: Y\to Z, f': Y'\to Z'$ are morphisms of placid schemes, the diagram commutes

$$\begin{array}{ccc} Shv(Y)\otimes Shv(Y') & \to & Shv(Y\times Y')\\ \downarrow {\scriptstyle f_*\otimes (f')_*} & & \downarrow {\scriptstyle (f\times f')_*}\\ Shv(Z)\otimes Shv(Z') & \to & Shv(Z\times Z'), \end{array}$$

0.0.53. Let $S\in\mathrm{Sch}_{ft}$, let Y be a placid S-scheme. We claim that in the constructible context, the symmetric monoidal structure $\otimes: Shv(Y)\otimes Shv(Y)\to Shv(Y)$ is well defined.

Indeed, write $Y\widetilde{\to}\lim_{i\in I^{op}} Y_i$, where I is small filtered, Y_i is a S-scheme of finite type, for $i\to j$ in I the map $Y_j\to Y_i$ is smooth affine surjective.

Let $p_{ij}: Y_j\to Y_i$ be the transition map for $\alpha: i\to j$ in I. Then $p_{ij}^*: Shv(Y_i)\to Shv(Y_j)$ is a map in $CAlg(\mathrm{DGCat}_{cont})$, and $CAlg(\mathrm{DGCat}_{cont})\to\mathrm{DGCat}_{cont}$ preserves filtered colimits, so $Shv(Y)\widetilde{\to}\mathrm{colim}_{i\in I} Shv(Y_i)$ could be veiwed as colimit in $CAlg(\mathrm{DGCat}_{cont})$. Note that e_Y is the unit.

0.0.54. Let $i: Z\to Y$ be a placid closed immersion of placid S-schemes, where $S\in\mathrm{Sch}_{ft}$. Then $i^*: Shv(Z)\to Shv(Y)$ is defined naturally in the constructible context. Namely, write $Y\widetilde{\to}\lim_{i\in I^{op}} Y_i$, where I is small filtered, Y_i is a S-scheme of finite type, for $i\to j$ in I the map $Y_j\to Y_i$ is smooth affine surjective. We may assume that $i_0\in I$ is initial, $i_0: Z_0\subset Y_0$ is a closed subscheme, and $Z=Y\times_{Y_0} Z_0$. Then for any i we have a closed immersion $i_i: Z_i\hookrightarrow Y_i$ obtained from i_0 by base change. Then the functors $i_i^*: Shv(Y_i)\to Shv(Z_i)$ are compatible with the $*$-pullbacks in the transition systems, so in the colimit yield $i^*: Shv(Y)\to Shv(Z)$.

Lemma 0.0.55. *In the coinstructible context for $K\in Shv(Z), L\in Shv(Y)$ one has the projection formula $(i_!K)\otimes L\widetilde{\to} i_!(K\otimes i^*L)$ in $Shv(Y)$ canonically.*

Proof. This is a particular case of base change established in Lemma 0.0.58. □

0.0.56. Let now $\mathcal{Y}$ be a placid ind-scheme over S written as $\mathcal{Y}\widetilde{\to}\operatorname{colim}_{i\in I}\mathcal{Y}_i$, where I is small filtered, $\mathcal{Y}_i$ is a placid S-scheme, and for $i\to j$ in I, $\mathcal{Y}_i\to\mathcal{Y}_j$ is a placid closed immersion. Then we get the category $\lim_{i\in I^{op}} Shv(\mathcal{Y}_i)$ with respect to the $*$-pullbacks.

0.0.57. Let I be a small filtered category. Assume given a functor $I^{op}\times[1]\to \mathrm{Sch}_{ft}$, $i\mapsto (Z_i\stackrel{f_i}{\to} Y_i)$. Assume that for $i\to j$ in I the transition maps $Y_j\to Y_i$ and $Z_j\to Z_i$ are smooth affine surjective. Set $Z=\lim_{i\in I^{op}} Z_i, Y=\lim_{i\in I^{op}} Y_i$. Let $f:Z\to Y$ be obtained from f_i by passing to the limit over I^{op}. Then the functor $f^*: Shv(Y)\to Shv(Z)$ is well-defined in the constructible context. In the case of $\mathcal{D}$-modules, we assume in addition that each f_i is smooth. Then f^* is defined.

Indeed, for each i we have $f_i^*: Shv(Y_i)\to Shv(Z_i)$ compatible with the transition $*$-pullbacks, and f^* is obtained by passing to the colimit.

For example, if Y is a placid scheme then for the diagonal $f:Y\to Y\times Y$ the functor $f^*: Shv(Y\times Y)\to Shv(Y)$ is defined in the constructible context.

Lemma 0.0.58. *Assume given the cartesian square of placid schemes*

$$\begin{array}{ccc} Z' & \stackrel{f'}{\to} & Y' \\ \downarrow g_Z & & \downarrow g_Y \\ Z & \stackrel{f}{\to} & Y, \end{array}$$

where the vertical arrows are placid closed immersions, and f is obtained as in Section 0.0.57. Then the same holds for f', and one has canonically $f^*(g_Y)_!\widetilde{\to}(g_Z)_!(f')^*$ as functors $Shv(Y')\to Shv(Z)$.

Proof. Pick a functor $I^{op}\times[1]\to\mathrm{Sch}_{ft}$, $i\mapsto(Z_i\stackrel{f_i}{\to}Y_i)$ as in Section 0.0.57, so f is obtained by passing to the limit from $Z_i\stackrel{f_i}{\to}Y_i$ over I^{op}. We may assume that $0\in I$ is initial, and we have a closed immersion $Y'_0\hookrightarrow Y_0$ such that $Y'=Y\times_{Y_0}Y'_0$. By Lemma 0.0.43, $g_Y^! f_*\widetilde{\to} f'_* g_Z^!$. Our claim is obtained by passing to left adjoints. □

0.1. **Verdier compatible algebraic stacks.**

0.1.1. For ([2], A.2.2). Let Y be quasi-compact classical algebraic stack with an affine diagonal, which is Verdier compatible. They claim there that for $f:S\to Y$ a scheme of finite type over Y, the objects f_*K with $K\in Shv(S)^c$ are compact and generate $Shv(Y)$. Indeed, $Shv(Y)^c$ is the Karoubi closure (that is, idempotent closure) of the smallest stable subcategory generated by objects of the form $f_!(K)$ with $K\in Shv(S)^c$. This implies the claim, see ([21], 9.2.27).

0.1.2. For ([2], A.2.3). Let Y, Y' be a quasi-compact classical algebraic stacks with affine diagonals, which are Verdier compatible. Let $f:Y\to Y'$ be a morphism. Recall that $f_\blacktriangle: Shv(Y)\to Shv(Y')$ is defined as the continuous extension of the functor $f_*: Shv(Y)^c\to Shv(Y')^c\subset Shv(Y')$. The fact that f_* preserves compact objects is explained in ([2], A.2.2).

Let Z be another algebraic stack locally of finite type with an affine diagonal, which is Verdier compatible. Then we have the following.

Lemma 0.1.3. *For $K \in Shv(Y), F \in Shv(Z)$ we have canonically*

$$(f_{\blacktriangle}K) \boxtimes F \widetilde{\to} (f \times \mathrm{id})_{\blacktriangle}(K \boxtimes F)$$

Besides, for $L \in Shv(Y')$ we get $(f_{\blacktriangle}K) \otimes^! L \widetilde{\to} f_{\blacktriangle}(K \otimes^! f^! L)$.

Proof. 1) Both sides for any F fixed preserve colimits as a functor of K. Therefore, it suffices to prove this for K of the form $K = g_*K'$, where $g : S \to Y$ is a morphism, $S \in \mathrm{Sch}_{ft}$ and $K' \in Shv(S)^c$, as such objects generate $Shv(Y)$. Moreover, we may assume $F \in Shv(Z)^c$. Then $f_{\blacktriangle}K \widetilde{\to} f_*K \widetilde{\to} (fg)_*K'$, and $(f \times \mathrm{id})_{\blacktriangle}(K \boxtimes F) \widetilde{\to} (f \times \mathrm{id})_*(K \boxtimes F)$, and $K \boxtimes F \widetilde{\to} (g \times \mathrm{id})_*(K' \boxtimes F)$, because $g \times \mathrm{id}$ is schematic. Now $(fg \times \mathrm{id})_*(K' \boxtimes F) \widetilde{\to} ((fg)_*K') \boxtimes F$, because fg is schematic. The first claim follows.

2) For the second, note that both sides preserve colimits separately in each variable, so we may assume K of the form $K = g_*K'$, where $g : S \to Y$ is a morphism, $S \in \mathrm{Sch}_{ft}$ and $K' \in Shv(S)^c$. Then $f_{\blacktriangle}K \widetilde{\to} f_*K \widetilde{\to} (fg)_*K'$. We may also assume $L \in Shv(Y')^{constr}$. We have the cartesian squares

$$\begin{array}{ccc}
S & \to & S \times Y' \\
\downarrow g & & \downarrow g\times\mathrm{id} \\
Y & \stackrel{\Gamma_f}{\to} & Y \times Y' \\
\downarrow & & \downarrow f\times\mathrm{id} \\
Y' & \stackrel{\triangle}{\to} & Y' \times Y'
\end{array}$$

Now f_* satisfies the base change againts !-pullbacks, so

$$(f_{\blacktriangle}K) \otimes^! L \widetilde{\to} \triangle^! (fg \times \mathrm{id})_*(K' \boxtimes L) \widetilde{\to} (fg)_*(K' \otimes^! (fg)^! L) \widetilde{\to} f_{\blacktriangle}(K \otimes^! f^! L),$$

because $f_{\blacktriangle}g_* \widetilde{\to} (fg)_*$. Indeed, g and fg are schematic. □

Recall the self-duality

$$(5) \qquad Shv(Y) \otimes Shv(Y) \to \mathrm{Vect}, \quad (K_1, K_2) \mapsto C^{\cdot}_{\blacktriangle}(Y, K_1 \otimes^! K_2)$$

from ([2], A.4.1). Under this self-duality, for $f : Y \to Y'$ as above the dual of the functor $f^! : Shv(Y') \to Shv(Y)$ is the functor $f_{\blacktriangle} : Shv(Y) \to Shv(Y')$, this follows from the above projection formula.

For $K \in Shv(Y)^c, K' \in Shv(Y')^c$ we have $\mathbb{D}(K \boxtimes K') \widetilde{\to} (\mathbb{D}K) \boxtimes (\mathbb{D}K')$ naturally. Now as in ([22], Sect. 1.0.1) one shows that the dual $h^{\vee}$ of the exteriour product functor $h : Shv(Y) \otimes Shv(Y') \to Shv(Y \times Y')$ with respect to the above dualities identifies with the right adjoint $h^R : Shv(Y \times Y') \to Shv(Y) \otimes Shv(Y')$.

So, the unit of the self-duality (5) is the object $h^R(\triangle_{\blacktriangle}\, \omega_Y)$, where $\triangle : Y \to Y \times Y$ is the diagonal.

0.1.4. For algebraic stacks locally of finite type (with affine diagonal) we always have a $(!, *)$-base change in the constructible context, this is mentioned in ([2], A.1.8) in particular.

0.1.5. Let $f : Y \to Y'$ be a morphism of algebraic stacks as in Section 0.1.2. For $F \in Shv(Y), K \in Shv(Y')$ we have a natural transformation functorial in K, F

$$(f_\blacktriangle F) \otimes K \to f_\blacktriangle (F \otimes f^* K)$$

This comes from ([2], Section A.3.3-A.3.4).

The following is also useful. For $K_1, K_2 \in Shv(Y')$ there is a natural transformation

$$f^*(K_1 \otimes^! K_2) \to (f^! K_1) \otimes^! (f^* K_2)$$

Indeed, it comes from the natural map $K_1 \otimes^! K_2 \to K_1 \otimes^! (f_* f^* K_2)$ and the projection formula for f_*.

Similarly, we have a natural map $f^* K_1 \otimes f^! K_2 \to f^!(K_1 \otimes K_2)$.

0.1.6. Let Z, Y, Y' be algebraic stacks as in Section 0.1.2, and $f : Y \to Y'$ be a morphism. For $K \in Shv(Z), F \in Shv(Y)$ we have canonically

$$(\mathrm{id} \times f)_*(K \boxtimes F) \widetilde{\to} K \boxtimes f_* F$$

Indeed, this is a particular case of the projection formula for $f \times \mathrm{id} : Z \times Y \to Z \times Y'$, as $K \boxtimes F \widetilde{\to} ((\mathrm{id} \times f)^! (K \boxtimes \omega_{Y'})) \otimes^! p_2^! F$, so $(\mathrm{id} \times f)_*(K \boxtimes F) \widetilde{\to} (K \boxtimes \omega_{Y'}) \otimes^! (\mathrm{id} \times f)_* p_2^! F$.

0.1.7. Let $f : Y \to Y'$ be a morphism of algebraic stacks as in Section 0.1.2. For $F \in Shv(Y), K \in Shv(Y')$ we have a natural transformation functorial in K, F

$$(f_* F) \otimes K \to f_*(F \otimes f^* K)$$

Indeed, it comes from $f^*((f_* F) \otimes K) \to F \otimes f^* K$.

There is a Verdier dual version of this map. Namely, a natural transformation

$$f_!(F \otimes^! f^! K) \to (f_! F) \otimes^! K$$

It comes from the evident map $F \otimes^! f^! K \to f^!((f_! F) \otimes^! K)$.

0.1.8. For a cartesian square of any algebraic stacks locally of finite type

$$\begin{array}{ccc} Y_1' & \xrightarrow{f'} & Y_2' \\ \downarrow g_1 & & \downarrow g_2 \\ Y_1 & \xrightarrow{f} & Y_2 \end{array}$$

we have the natural transformation

$$g_2^* \circ f_* \to f'_* \circ g_1^*$$

arising by adjointness from $f_*(g_1)_* \widetilde{\to} (g_2)_* f'_*$. Besides, the base change isomorphism $g_2^* \circ f_! \widetilde{\to} f'_! \circ g_1^*$ gives by adjointness a natural transformation

$$g_1^* \circ f^! \to (f')^! \circ g_2^*$$

Similarly, we have a natural transformation $f'_! g_1^! \to g_2^! f_!$.

0.1.9. Let $f : Y \to Y'$ be a morphism of algebraic stacks as in Section 0.1.2. Let us construct a natural morphism functiorial in $L \in Shv(Y), K, M \in Shv(Y')$

$$((f^! K) \otimes^! L) \otimes f^* M \to f^!(K \otimes M) \otimes^! L$$

We have a natural map $f^! K \otimes f^* M \to f^!(K \otimes M)$ by Section 0.1.5. So, it suffices to construct a natural map $(f^! K \otimes^! L) \otimes f^* M \to (f^! K \otimes f^* M) \otimes^! L$. It comles from the next observation.

Lemma 0.1.10. *Let Y be an algebraic stack as in Section 0.1.2. For $K_1, K_2, L \in Shv(Y)$ there is a natural map $(K_1 \otimes^! K_2) \otimes L \to (K_1 \otimes L) \otimes^! K_2$.*

Proof. 1) First, assume $K_2 \in Shv(Y)^{constr}$. Then

$$(K_1 \otimes^! K_2) \otimes L \widetilde{\to} \mathcal{H}om(\mathbb{D}K_2, K_1) \otimes L$$

and $(K_1 \otimes L) \otimes^! K_2 \widetilde{\to} \mathcal{H}om(\mathbb{D}K_2, K_1 \otimes L)$, here $\mathcal{H}om$ is the inner hom in $(Shv(Y), \otimes)$. The desired morphism comes from the natural map $(\mathbb{D}K_2) \otimes \mathcal{H}om(\mathbb{D}K_2, K_1) \otimes L \to K_1 \otimes L$. The so obtained morphisms are functorial in K_2. Now if $K_2 \in Shv(Y)$ is written as $K_2 \widetilde{\to} \operatorname{colim}_{i \in I} K_2^i$ with I small filtered and $K_2^i \in Shv(Y)^{constr}$ then the desired morphism of obtained by passing to the colimit over $i \in I$ in the diagram $(K_1 \otimes^! K_2^i) \otimes L \to (K_1 \otimes L) \otimes^! K_2^i$.
2) Simplier argument. Consider the cartesian square

$$\begin{array}{ccc} Y & \stackrel{\vartriangle}{\to} & Y \times Y \\ \downarrow \vartriangle & & \downarrow \vartriangle \times \mathrm{id} \\ Y \times Y & \stackrel{\mathrm{id} \times \vartriangle}{\to} & Y \times Y \times Y \end{array}$$

and apply the natural transformation $\vartriangle^* (\mathrm{id} \times \vartriangle)^! \to \vartriangle^! (\vartriangle \times \mathrm{id})^*$. ☐

The map in the above lemma is not an isomorphism in general. For example, let $i : Y' \hookrightarrow Y$ be a closed immersion. Taking $K_2 = i_* \omega_{Y'}$, the above map reduces to a morphism $i^! K_1 \otimes i^* L \to i^!(K_1 \otimes L)$ on Y', which is usually not an isomorphism. For example, if $i : \operatorname{Spec} k \to Y$ is a closed point on a smooth curve Y and $K_1 = e_X$ this is a map $i^* L[-2] \to i^! L$.

0.2. Let $f_1 : Y_1 \to Z_1, f_2 : Y_2 \to Z_2$ be morphisms of algebraic stacks locally of finite type (with an affine diagonal). Assume we are in the constructible context. Then for any $F_i \in Shv(Y_i)$ we have

$$(f_1 \times f_2)_!(F_1 \boxtimes F_2) \widetilde{\to} ((f_1)_! F_1) \boxtimes ((f_2)_! F_2)$$

in $Shv(Z_1 \times Z_2)$.

Proof. 1) For morphisms between schemes of finite type this is just the $(_!, ^*)$-projection formula, because $(f_1 \times f_2)_! = (f_1 \times \mathrm{id})_! (\mathrm{id} \times f_2)_!$.

2) Now we prove this under the additional assumption that $Y_i \in \mathrm{Sch}_{ft}$. It suffices to establish this after any base change by $h_1 \times h_2 : S_1 \times S_2 \to Z_1 \times Z_2$, where $S_i \in \mathrm{Sch}_{ft}^{aff}$ and h_i are smooth. This follows from the $(!, *)$-base change.

3) For any prestack Y_1 we get

$$F_i \widetilde{\to} \operatorname{colim}_{S_i \xrightarrow{g_i} Y_i} (g_i)_!(g_i^!)F_1,$$

where the colimit is over $\mathrm{Sch}_{ft}^{aff}/Y$. So,

$$F_1 \boxtimes F_2 \widetilde{\to} \operatorname{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} ((g_1)_!(g_1^!)F_1) \boxtimes ((g_2)_!(g_2^!)F_2) \widetilde{\to} \operatorname{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} (g_1 \times g_2)_!(g_1 \times g_2)^!(F_1 \boxtimes F_2),$$

where the second isomorphism uses 2). So,

$$(f_1 \times f_2)_!(F_1 \boxtimes F_2) \widetilde{\to} \operatorname{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} (f_1 g_1 \times f_2 g_2)_!(g_1^! F_1 \boxtimes g_2^! F_2) \widetilde{\to}$$

$$\operatorname{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} ((f_1 g_1)_! g_1^! F_1) \boxtimes ((f_2 g_2)_! g_2^! F_2),$$

where the last isomorphism used 2). The latter expression identifies with

$$\operatorname{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} f_{1!}(g_{1!} g_1^! F_1) \boxtimes f_{2!}(g_{2!} g_2^! F_2) \widetilde{\to} \operatorname{colim}_{S_1 \xrightarrow{g_1} Y_1} f_{1!}(g_{1!} g_1^! F_1) \boxtimes f_{2!}(\operatorname{colim}_{S_2 \xrightarrow{g_2} Y_2} g_{2!} g_2^! F_2)$$

The latter identifies with $((f_1)_! F_1) \boxtimes ((f_2)_! F_2)$. □

0.2.1. As a corollary, let Y be an algebraic stack locally of finite type (with an affine diagonal). Then $\mathrm{R}\Gamma_c : (Shv(Y), \otimes^!) \to \mathrm{Vect}$ is left-lax symmetric monoidal, so sends cocommutative coalgebras to cocommutative coalgebras. So, $\mathrm{R}\Gamma_c(Y, \omega)$ is a cocommutative coalgebra in Vect. Moreover, ω becomes an object of $\mathrm{R}\Gamma_c(Y, \omega) - comod(Shv(Y), \otimes^!)$ via the natural adjunction map act : $\omega_Y \to \mathrm{R}\Gamma_c(Y, \omega) \otimes \omega_Y$. So, for any $F \in Shv(Y)$, F gets a coaction of $\mathrm{R}\Gamma_c(Y, \omega)$ just by applying $\bullet \otimes^! F$ to the previous action map. The functor $\mathrm{R}\Gamma_c$ extends to a functor $Shv(Y) \to \mathrm{R}\Gamma_c(Y, \omega) - comod(\mathrm{Vect})$ naturally, so the composition with oblv : $\mathrm{R}\Gamma_c(Y, \omega) - comod(\mathrm{Vect}) \to \mathrm{Vect}$ is $\mathrm{R}\Gamma_c(Y, \bullet)$.

0.2.2. Assume we are in the constructible context. Let Z, Y be algebraic stacks locally of finite type (with affine diagonals), and $p : Z \to Y$ any morphism, maybe not representable. Then for $K \in Shv(Y), L \in Shv(Z)$ one has canonically

$$p_* \mathcal{H}om(p^* K, L) \widetilde{\to} \mathcal{H}om(K, p_* L)$$

in $Shv(Y)$. We underline that here p_* is maybe discontinuous if p is not representable.

Proof. Let us first assume $K \in Shv(Y)^{constr}$. Then

$$p_* \mathcal{H}om(p^* K, L) \widetilde{\to} p_*(p^!(\mathbb{D}K) \otimes^! L) \widetilde{\to} \mathbb{D}(K) \otimes^! p_* L \widetilde{\to} \mathcal{H}om(K, p_* L)$$

by the projection formula.

Let now K be any, pick a presentation $K \widetilde{\to} \operatorname{colim}_{i \in I} K_i$ with $K_i \in Shv(Y)^{constr}$. Then p_* preserves limits, so we get

$$p_* \mathcal{H}om(p^* K, L) \widetilde{\to} \lim_{i \in I^{op}} p_* \mathcal{H}om(p^* K_i, L) \widetilde{\to} \lim_{i \in I^{op}} \mathcal{H}om(K_i, p_* L)$$

$$\widetilde{\to} \mathcal{H}om(\operatorname{colim}_{i \in I} K_i, p_* L)$$

□

I wonder if one may replace here p_* by $p_\blacktriangle$.

0.2.3. Let I be small filtered, $I \to \mathrm{Stk}$, $i \mapsto Y_i$ be the functor such that Y_i is an algebraic stack locally of finite type, for $i \to j$ in I, $Y_i \to Y_j$ is a closed immersion, $Y = \mathrm{colim}_i Y_i$ in Stk. Then the functor $\mathrm{R}\Gamma : Shv(Y) \to \mathrm{Vect}$ is defined by passing to the colimit in the functors $\mathrm{R}\Gamma : Shv(Y_i) \to \mathrm{Vect}$ with respect to the maps $(f_{ij})_* : Shv(Y_i) \to Shv(Y_j)$. Usually, $\mathrm{R}\Gamma : Shv(Y) \to \mathrm{Vect}$ does not have a left adjoint.

A corollary of this: let $K \in Shv(Y)$, write $K \widetilde{\to} \mathrm{colim}_{i\in I}(i_i)_* i_i^! K$, where $i_i : Y_i \to Y$ is the natural map. Assume that for each i the functor $Shv(Y) \to \mathrm{Vect}$, $F \mapsto \mathrm{R}\Gamma(Y, ((i_i)_* i_i^! K) \otimes^! F)$ is continuous. Then $Shv(Y) \to \mathrm{Vect}, F \mapsto \mathrm{R}\Gamma(Y, F \otimes^! K)$ is also continuous.

0.2.4. Let $S \in \mathrm{Sch}_{ft}$, let $f : Y \to S$ be a placid scheme over S. Then we have an action of $(Shv(S), \otimes^!)$ on $Shv(Y)$ such that $K \in Shv(S)$ sends $F \in Shv(Y)$ to $b^!(F \boxtimes K)$ for $b : Y \to Y \times S$.

The same structure is obtained as follows. Write $Y \widetilde{\to} \lim_{i\in I^{op}} Y_i$ with Y_i a scheme of finite type, for $i \to j$ in I the map $f_{ij} : Y_j \to Y_i$ is smooth affine surjective. Then for $i \to j$ in I, $(f_{ij})_* : Shv(Y_j) \to Shv(Y_i)$ is a map in $Shv(S) - mod$, so $Shv(Y) \widetilde{\to} \lim_{i\in I^{op}} Shv(Y_i)$ may be understood in $Shv(S) - mod$.

If $h : Y \to Z$ is any morphism of placid schemes over S then $h_* : Shv(Y) \to Shv(Z)$ is a map in $Shv(S) - mod$. Indeed, write $Z = \lim\limits_{j\in J^{op}} Z_j$, where J is small filtered, $Z_j \in (\mathrm{Sch}_{ft})_{/S}$, and for any $j \to j'$ in J the map $Z_{j'} \to Z_j$ in $(\mathrm{Sch}_{ft})_{/S}$ is smooth affine surjective. It suffices to show that for $g_j : Z \to Z_j$ the functor $(g_j h)_* : Shv(Y) \to Shv(Z_j)$ is $Shv(S)$-linear. However, there is $i \in I$ such that h factors through $Y \to Y_i \stackrel{\bar{h}}{\to} Z_j$. Our claim follows from the fact that $\bar{h}_* : Shv(Y_i) \to Shv(Z_j)$ is $Shv(S)$-linear.

Remark: let $S \in \mathrm{Sch}_{ft}$, I a small filtered category, $I^{op} \to (\mathrm{PreStk}_{lft})_{/S}$, $i \mapsto Y_i$ a functor such that for $i \to j$ in I the map $f_{ij} : Y_j \to Y_i$ is smooth of some relative dimension d, affine, surjective. Let $Y = \lim\limits_{i\in I^{op}} Y_i$ in PreStk. Define $Shv(Y) = \mathrm{colim}_{i\in I} Shv(Y_i)$ in $Shv(S) - mod$ with respect to the functors $f_{ij}^* : Shv(Y_i) \to Shv(Y_j)$. Let $0 \in I$ be initial. Then we get the structure functor $f_0^* : Shv(Y_0) \to Shv(Y)$ for $f_0 : Y \to Y_0$. It is clear that f_0^* is $Shv(S)$-linear.

0.2.5. A property of the constructible context. Let S, X be schemes of finite type. Let $K_1, K_2 \in Shv(S)$. Then for the projection $q : S \times X \to S$ we have

$$q_* \mathcal{H}om(q^* K_1, q^* K_2) \widetilde{\to} \mathcal{H}om(K_1, K_2) \otimes \mathrm{R}\Gamma(X, e)$$

in $Shv(S)$, where $\mathcal{H}om$ denotes the local $\mathcal{H}om$ over the corresponding scheme. This isomorphism is compatible with compositions: given $K_i \in Shv(S)$ for $i = 1, 2, 3$ the composition

$$\mathcal{H}om(K_1, K_2) \otimes \mathcal{H}om(K_2, K_3) \to \mathcal{H}om(K_1, K_3)$$

via the above isomorphism corresponds to the composition

$$\mathcal{H}om(q^* K_1, q^* K_2) \otimes \mathcal{H}om(q^* K_2, q^* K_3) \to \mathcal{H}om(q^* K_1, q^* K_3)$$

In particular, we get an isomorphism of algebras in $Shv(S)$

$$q_* \mathcal{H}om(q^* K, q^* K) \widetilde{\to} \mathcal{H}om(K, K) \otimes \mathrm{R}\Gamma(X, e)$$

0.2.6. For $S \in \mathrm{Sch}_{ft}$ for our sheaf theories, $Shv(S)$ is never rigid. For example, in the constructible context if S is smooth then for a k-point $s \in S$, $\delta_s \in Shv(S)$ is not dualizable, though compact.

In the constructible context the following is **not known**: given a map of schemes of finite type $f : S \to T$, is it true that $Shv(S)$ is. dualizable as a $Shv(T)$-module? Here $Shv(T)$ acts via the monoidal functor $f^!$.

The functor $Shv : (\mathrm{Sch}_{ft}^{aff})^{op} \to \mathrm{DGCat}_{cont}$ satisfies both Zarizki descent and proper descent, hence h-descent.

A useful thing: if $f : Y_1 \to Y_2$ in PreStk_{lft} is an isomorphism in the h-topology then $f^! : Shv(Y_2) \to Shv(Y_1)$ is an isomorphism.

0.2.7. Let Y be a placid scheme written as $Y \widetilde{\to} \lim_{i \in I^{op}} Y_i$, where I is small filtered, $Y_i \in \mathrm{Sch}_{ft}$, for $i \to j$ in I, $f_{ij} : Y_j \to Y_i$ is smooth affine surjective. Assume $0 \in I$ is initial, and for any $i \to j$ in I, $f_{ij} : Y_j \to Y_i$ is a generalized affine fibration of rank $\dim Y_j - \dim Y_i$ (locally constant function on Y_i). Let $p : Y \to Y_0$ be the natural map. Assume we are in the constructible context. Then $p^* : Shv(Y_0) \to Shv(Y)$ admits a left adjoint $(p^*)^L$. The natural map $(p^*)^L p^* \to \mathrm{id}$ is an isomorphism. The dual $((p^*)^L)^\vee$ identifies with the right adjoint to p_* via the self-dualities of $Shv(Y), Shv(Y_0)$ appearing in ([22], 1.2.11).

Proof. For $i \in I$ let $f_i : Y_i \to Y_0$ be the map f_{0i}. For $i \in I$ with $i \neq 0$ the functors $((f_i)_![2\dim Y_i - 2\dim Y_0], f_i^*)$ form an adjojnt pair. The system of functors $(f_i)_![2\dim Y_i - 2\dim Y_0] : Shv(Y_i) \to Shv(Y_0)$ is compatible with the transition functors in $Shv(Y) \widetilde{\to} \mathrm{colim}_{j \in I} Shv(Y_j)$ with $*$-pullbacks, so in the colimit over $j \in I$ we get a functor $\mathrm{colim}_{j \in I} Shv(Y_j) \to Shv(Y_0)$. By ([21], 9.2.6), this is the left adjoint to p^*. We used that for $i \in I$, $(f_{ij})_* f_j^* \widetilde{\to} f_i^*$ naturally.

By ([22], 1.2.11), the dual of $p^* : Shv(Y_0) \to Shv(Y)$ identifies naturally with $p_* : Shv(Y) \to Shv(Y_0)$. So, the dual of the adjoint pair $((p^*)^L, p^*)$ is $(p_*, ((p^*)^L)^\vee)$. □

This situation happens offen. For example, if G is an affine smooth algebraic group of finite type and $\mathcal{O} = k[[t]]$ then $G(\mathcal{O})$ is a placid scheme satisfying the above. So, for $p : G(\mathcal{O}) \to G$ in the constructible context we have an adjoint pair $(p^*)^L : Shv(G(\mathcal{O})) \leftrightarrows Shv(G) : p^*$ with p^* fully faithful. In particular, for $q : G(\mathcal{O}) \to \mathrm{Spec}\, k$ the functor q^* has a left adjoint $(q^*)^L$.

0.2.8. Assume for this subsection we are in the constructible context. Let G be a group scheme of finite type. Then $Shv(G)$ is equipped with the monoidal structure given by $K_1 *^! K_2 = m_!(K_1 \boxtimes K_2)$, where $m : G \times G \to G$ is the product map, $K_i \in Shv(G)$. Now for $Y \in \mathrm{PreStk}_{lft}$ with a G-action, $(Shv(G), *^!)$ acts on $Shv(Y)$ so that $K \in Shv(G)$ acts on $F \in Shv(Y)$ as $a_!(K \boxtimes F)$, where $a : G \times Y \to Y$ is the action map.

Consider the projections

$$\mathrm{pr}_G : G \times Y \to G, \mathrm{pr}_Y : G \times Y \to Y.$$

Assume G smooth. Let L be a character local system on G in the usual sense, that is, for $m : G \times G \to G$ we are given $m^* L \widetilde{\to} L \boxtimes L$, and a map $L \to i_* e$ for the unit $i : \mathrm{Spec}\, k \to G$ with the usual properties. Let $\mathfrak{a}^R : Shv(Y) \to Shv(Y)$ be the comonad given by $K \mapsto L * K$, where we use the usual action (not the !-one). Then the functor

$\mathfrak{a}^R$ admits a left adjoint $\mathfrak{a} : Shv(Y) \to Shv(Y)$, which is automatically a monad in $\mathrm{Fun}_{e,cont}(Shv(Y), Shv(Y))$. One has an equivalence

$$\mathfrak{a} - mod(Shv(Y)) \widetilde{\to} \mathfrak{a}^R - comod(Shv(Y)) \tag{6}$$

commuting with the oblivion functors to $Shv(Y)$. In particular, $\mathrm{oblv} : Shv(Y)^{G,L} \to Shv(Y)$ admits a left adjoint $\mathrm{ind} : Shv(Y) \to \mathfrak{a} - mod(Shv(Y))$.

Proof. For $K, M \in Shv(Y)$ one has

$$\mathcal{H}om((\mathrm{pr}_Y)_!(\mathrm{pr}_G^* L^{-1} \otimes a^*K)[2\dim G], M) \widetilde{\to} \mathcal{H}om(\mathrm{pr}_G^* L^{-1} \otimes a^*K, e \boxtimes M) \widetilde{\to}$$
$$\mathcal{H}om(a^*K, L \boxtimes M) \widetilde{\to} \mathcal{H}om(K, a_*(L \boxtimes M)) \widetilde{\to} \mathcal{H}om(K, L * M)$$

So, the functor $\mathfrak{a} : Shv(Y) \to Shv(Y)$ given by $\mathfrak{a}(K) = (\mathrm{pr}_Y)_!(\mathrm{pr}_G^* L^{-1} \otimes a^*K)[2\dim G]$ is left adjoint to $\mathfrak{a}^R$. The fact that $\mathfrak{a}$ is monad and the equivalence (6) follow from ([21], 9.2.62). □

0.2.9. Let $S \in \mathrm{Sch}_{ft}$, $f : Y \to S$ be an ind-scheme of ind-finite type over S. Assume $Y \widetilde{\to} \mathrm{colim}_{i\in I} Y_i$, where I is small filtered, $Y_i \subset Y$ is a closed subscheme, $Y_i \in \mathrm{Sch}_{ft}$, for $i \to j$ in I, $Y_i \hookrightarrow Y_j$. Assume also each map $f_i : Y_i \to S$ is smooth.

Let $F \in Shv(S)^c, K \in Shv(Y)$. Then one has canonically

$$\mathcal{H}om(f^!F, K) \widetilde{\to} \mathcal{H}om(f^!(e_S), f^!(\mathbb{D}F) \otimes^! K)$$

Here $\mathcal{H}om \in \mathrm{Vect}$ is the relative inner hom for the Vect-action.

Proof. **Step 1**. First, assume $f : Y \to S$ is a map in Sch_{ft} with f smooth of relative dimension d. Then $f^! = f^*[2d]$, so the LHS is

$$\mathcal{H}om(f^*F, K[-2d]) \widetilde{\to} \mathcal{H}om(F, f_*K[-2d]) \widetilde{\to} \mathrm{R}\Gamma(S, (\mathbb{D}F) \otimes^! f_*K[-2d])$$
$$\widetilde{\to} \mathrm{R}\Gamma(Y, K[-2d] \otimes^! f^!(\mathbb{D}F))$$

The RHS identifies with

$$\mathcal{H}om(f^*(e_S), f^!(\mathbb{D}F) \otimes^! K[-2d]) \widetilde{\to} \mathrm{R}\Gamma(Y, f^!(\mathbb{D}F) \otimes^! K[-2d])$$

We are done.

Step 2. Let $i_i : Y_i \to Y$ be the inclusion. Write $f^!F \widetilde{\to} \mathrm{colim}_{i\in I}(i_i)_* f_i^! F$, so the LHS becomes

$$\lim_{i\in I^{op}} \mathcal{H}om(f_i^!F, (i_i)^!K) \widetilde{\to} \lim_{i\in I^{op}} \mathcal{H}om(f_i^!(e_S), f_i^!(\mathbb{D}F) \otimes^! (i_i)^!K)$$

by Step 1. Write $f^!(e_S) \widetilde{\to} \mathrm{colim}_{i\in I}(i_i)_! f_i^!(e_S)$ then the RHS of the latter expression becomes

$$\lim_{i\in I^{op}} \mathcal{H}om(f_i^!(e_S), (i_i)^!(f^!(\mathbb{D}F) \otimes^! K)) \widetilde{\to} \lim_{i\in I^{op}} \mathcal{H}om((i_i)_! f_i^!(e_S), f^!(\mathbb{D}F) \otimes^! K)$$
$$\widetilde{\to} \mathcal{H}om(f^!(e_S), f^!(\mathbb{D}F) \otimes^! K)$$

as desired. □

In the sense of ULA property in its form given by Dennis in ([12], 1.6.3) this says that ω_Y is ULA with respect to the $Shv(S)$-action on $Shv(Y)$.

0.2.10. A generality: let $S \in \mathrm{Sch}_{ft}$, $U \to S$ be a smooth unipotent group scheme over S. Then for $f : B(U) \to S$ the functor $f^* : Shv(S) \widetilde{\to} Shv(B(U))$ is an equivalence.

Let $U_1 \to U_2$ be a homomorphism of smooth unipotent group schemes over S. Take $Y = U_2/U_1$, the stack quotient over S. Let $a : Y \to S$ be the natural map. Then $a^* a_*$ is left t-exact.

0.3. **Addition for any sheaf theory.**

Work in any of our 4 sheaf theory for this subsection.

0.3.1. Let

$$\begin{array}{ccc} Y' & \overset{i'}{\hookrightarrow} & Z' \\ \downarrow & & \downarrow \alpha_Z \\ Y & \overset{i}{\hookrightarrow} & Z \end{array}$$

be a cartesian square in PreStk_{lft} such that i is a closed immersion in Sch_{ft}. Then the natural functor $Shv(Z') \otimes_{Shv(Z)} Shv(Y) \to Shv(Y')$ is an equivalence. Here we view $Shv(Z)$ with the $\otimes^!$-monoidal structure.

Proof. First, for $\mathcal{D}$-modules this is true without the assumption that i is a closed immersion by ([14], Section 1.6.4). Assume now we are in the constructible context.

Consider the comonad $\mathcal{A} := i'_*(i')^!$ on $Shv(Z')$, it is $Shv(Z')$-linear, hence given by the coalgebra $i'_*\omega$. The functor $i'_* : Shv(Y') \to Shv(Z')$ is comonadic. Indeed, i'_* has a left adjoint, hence preserves limits, and i'_* is fully faithful.

In fact, since i'_* is fully faithful, we obtain that $Shv(Y') \subset Shv(Z')$ is the full subcategory of those $K \in Shv(Z')$ for which the counit map $\mathcal{A}(K) \to K$ is an isomorphism.

We claim that the composition $Shv(Z') \otimes_{Shv(Z)} Shv(Y) \to Shv(Y') \to Shv(Z')$ is described similarly with the same comonad. Indeed, consider the adjoint pair $i_! : Shv(Y) \leftrightarrows Shv(Z) : i^!$ in $Shv(Z) - mod$. We have the comonad $i_!\omega$ on $Shv(Z)$ with $Shv(Y) \widetilde{\to} (i_!\omega) - comod(Shv(Z))$. After the base change $\cdot \otimes_{Shv(Z)} Shv(Z')$, our adjoint pair becomes

$$L : Shv(Y) \otimes_{Shv(Z)} Shv(Z') \leftrightarrows Shv(Z') : R$$

in $Shv(Z') - mod$ with $LR \widetilde{\to} \alpha_Z^! i_* \omega$. As a coalgebra in $Shv(Z')$ it coincides with $i'_*\omega$. Note that L is fully faithful, because $\mathrm{id} \to RL$ is an isomorphism, so L is conservative. Finally, $Shv(Y) \otimes_{Shv(Z)} Shv(Z') \subset Shv(Z')$ is the full subcategory of $K \in Shv(Z')$ for which the natural map $LR(K) \to K$ is an isomorphism. We are done.

Note also that $Shv(Y)$ is self-dual in $Shv(Z) - mod$, so by ([21], 9.2.57), the map L rewrites as

$$\mathrm{Fun}_{Shv(Z)}(Shv(Y), Shv(Z')) \to \mathrm{Fun}_{Shv(Z)}(Shv(Z), Shv(Z')) \tag{7}$$

given by the composition with $i^! : Shv(Z) \to Shv(Y)$. Indeed, the dual of i_* is $i^!$ for the standard self-dualities. It is not clear here if (7) preserves limits, as the limits are not "computed pointwise"! $\square$

0.3.2. For $X \in \mathrm{Sch}_{ft}$ let $j : U \hookrightarrow X$ an open subscheme. Equip $Shv(X)$ with the $\otimes^!$-symmetric monoidal structure. The adjoint pair $j^* : Shv(X) \leftrightarrows Shv(U) : j_*$ is in $Shv(X) - mod$, and j_* is right-lax nonunital symmetric monoidal. So $j_*\omega \in CAlg(Shv(X))$. The functor $j_* : Shv(U) \to Shv(X)$ factors naturally through $Shv(U) \to (j_*\omega) - mod(Shv(X))$. Now $Shv(U) \widetilde{\to} (j_*\omega) - mod(Shv(X))$, this is the image of the action of the idempotent $(j_*\omega)$ on $Shv(X)$, cf. ([21], 9.2.74). Here $j_*\omega$ is an idempotent commutative algebra in $Shv(X)$ in the sense of ([16], 4.8.2.8).

If $M \in Shv(X) - mod$ then we get an adjoint pair $j^* : M \leftrightarrows M \otimes_{Shv(X)} Shv(U) : j_*$ in DGCat_{cont}, and the right adjoint is monadic. So,

$$M \otimes_{Shv(X)} Shv(U) \widetilde{\to} (j_*\omega) - mod(M)$$

Recall that $\mathrm{oblv} : (j_*\omega) - mod(M) \to M$ is fully faithful, and its image is the image of the action of $j_*\omega$ on M.

This gives that for any map $X' \to X$ in PreStk_{lft} for $U' = U \times_X X'$ we get

$$Shv(X') \otimes_{Shv(X)} Shv(U) \widetilde{\to} Shv(U').$$

Let now $A \in coAlg(Shv(X))$. By ([21], 9.2.60), let $M = A - comod(Shv(X)) \in Shv(X) - mod$, and we have an adjoint pair in $Shv(X) - mod$

$$\mathrm{oblv} : A - comod(Shv(X)) \leftrightarrows Shv(X) : coind. \tag{8}$$

Applying $\otimes_{Shv(X)} Shv(U)$, one gets the adjoint pair $l : M \otimes_{Shv(X)} Shv(U) \leftrightarrows Shv(U) : r$ in $Shv(U) - mod$. The comonad $lr : Shv(U) \to Shv(U)$ is given by $A_U \in coAlg(Shv(U))$, the restriction of A to U.

Lemma 0.3.3. *l is comonadic, so*

$$(A - comod(Shv(X))) \otimes_{Shv(X)} Shv(U) \widetilde{\to} A_U - comod(Shv(U))$$

Proof. Consider the diagram, where the horizontal functors are fully faithful

$$\begin{array}{ccc} M \otimes_{Shv(X)} Shv(U) & \overset{j_*}{\hookrightarrow} & M \\ \downarrow l & & \downarrow \mathrm{oblv} \\ Shv(U) & \overset{j_*}{\hookrightarrow} & Shv(X) \end{array}$$

It shows that l is conservative. Let now V be a simplicial object of $(M \otimes_{Shv(X)} Shv(U))^{op}$ such that $l(V)$ is split in $Shv(U)^{op}$. Then $j_* l(V) \widetilde{\to} \mathrm{oblv}(j_*V)$ is split in $Shv(X)^{op}$. By ([16], 4.7.3.5), j_*V admits a colimit in M^{op}, and $\mathrm{oblv} : M^{op} \to Shv(X)^{op}$ preserves this colimit. Since $M \otimes_{Shv(X)} Shv(U)$ has all limits and colimits, V admits a colimit in $(M \otimes_{Shv(X)} Shv(U))^{op}$, and j_* preserves this colimit in M^{op}. Now $l(V) \to l(\mathrm{colim}\, V)$ is a diagram in $Shv(U)^{op}$, which becomes a colimit diagram in $Shv(X)^{op}$ after applying j_*. Hence, it is also a colimit diagram in $Shv(U)^{op}$. That is, l preserves the colimit of V. By ([16], 4.7.3.5), l is comonadic. □

0.3.4. For $X \in Sch_{ft}$ let $i : Z \to X$ be a closed subscheme. The dual pair $i_! : Shv(Z) \leftrightarrows Shv(X) : i^!$ takes place in $Shv(X) - mod$, the coresponding comonad is $i_*\omega \in coAlg(Shv(X))$. The functor $i_! : Shv(Z) \to Shv(X)$ is comonadic, this is easy using the full ([16], 4.7.3.5). So, $Shv(Z) \widetilde{\to} (i_*\omega) - comod(Shv(X))$.

In turn, $(i_*\omega)-mod(Shv(X)^{op})$ is the image of the localization functor $i_*\omega\otimes^!\cdot : Shv(X)^{op}\to Shv(X)^{op}$ by ([21], 9.2.74). So, $Shv(Z)$ is the full subcategory of those $K\in Shv(X)$, for which the map $i_*\omega\to\omega$ tensored by K becomes an isomorphism.

Let $M\in Shv(X)-mod$. We get an adjoint pair $i_! : M\otimes_{Shv(X)} Shv(Z)\leftrightarrows M : i^!$ in DGCat_{cont}, and $i_!$ is fully faithful. For the same reasons, $i_!$ is comonadic, so $M\otimes_{Shv(X)} Shv(Z)\,\widetilde{\to}\,(i_*\omega)-comod(M)$. The image of $i_!$ is the full subcategory of those $K\in M$ for which the map $i_*\omega\to\omega$ tensored by K becomes an an isomorphism (where now $\otimes$ stand for the $Shv(X)$-action on M). Again, $i_*\omega\in ComCoAlg(Shv(X))$ is an idempotent coalgebra ([16], 4.2.4.10).

Let now $A\in coAlg(Shv(X),\otimes^!)$. Consider the adjoint pair (8) in $Shv(X)-mod$. Let $M=A-comod(Shv(X))$. Applying $\otimes_{Shv(X)}Shv(Z)$, one gets the adjoint pair

$$l : M\otimes_{Shv(X)} Shv(Z)\leftrightarrows Shv(Z) : r$$

in $Shv(Z)-mod$. The comonad $lr : Shv(Z)\to Shv(Z)$ is given by tensoring with $A_Z := i^!A\in coAlg(Shv(Z))$.

Lemma 0.3.5. *l is comonadic, so*

$$(A-comod(Shv(X))\otimes_{Shv(X)} Shv(Z)\,\widetilde{\to}\,A_Z-comod(Shv(Z))$$

Proof. Consider the diagram, where the horizontal functors are fully faithful

$$\begin{array}{ccc} M\otimes_{Shv(X)} Shv(Z) & \overset{i_*}{\hookrightarrow} & M \\ \downarrow l & & \downarrow \mathrm{oblv} \\ Shv(Z) & \overset{i_*}{\hookrightarrow} & Shv(X) \end{array}$$

It shows that l is conservative.

Let now V be a simplicial object of $(M\otimes_{Shv(X)} Shv(Z))^{op}$ such that $l(V)$ is split in $Shv(Z)^{op}$. Then $i_*l(V)\,\widetilde{\to}\,\mathrm{oblv}\, i_*(V)$ is split in $Shv(X)^{op}$. So, i_*V admits a colimit in M^{op}, and $\mathrm{oblv} : M^{op}\to Shv(X)^{op}$ preserves this colimit. Write W for the colimit of i_*V in M^{op}, so $i_*V\to W$ is a colimits diagram in M^{op}. Since $i^! : M^{op}\to (M\otimes_{Shv(X)} Shv(Z))^{op}$ preserves colimits, $i^!i_*V\to i^!W$ is a colimit diagram in $(M\otimes_{Shv(X)} Shv(Z))^{op}$. Note that $li^!\,\widetilde{\to}\, i^!\,\mathrm{oblv}$. Further, $\mathrm{oblv}(i_*V)\to\mathrm{oblv}(W)$ is a colimit diagram in $Shv(X)^{op}$, hence $i^!\,\mathrm{oblv}(i_*V)\to i^!\,\mathrm{oblv}(W)$ is a colimit diagram in $Shv(Z)^{op}$. The latter diagram is nothing but the desired diagram $l(i^!i_*V)\to l(i^!W)$. Thus, the colimit of V in $(M\otimes_{Shv(X)} Shv(Z))^{op}$ is preserved by l. By ([16], 4.7.3.5), l is comonadic. □

We propose the following generalization.

Lemma 0.3.6. *Let $i : Z\hookrightarrow X$ be a closed immersion in Sch_{ft}, let $Y\to X$ be a map in PreStk_{lft}, set $Y'=Y\times_X Z$. Let*

$$A\in \mathrm{Fun}_{Shv(X)}(Shv(Y), Shv(Y))$$

be a $Shv(X)$-linear continuous comonad, set

$$A_Z = A\otimes\mathrm{id} : Shv(Y)\otimes_{Shv(X)} Shv(Z)\to Shv(Y)\otimes_{Shv(X)} Shv(Z).$$

Then $A_Z \in \mathrm{Fun}_{Shv(Z)}(Shv(Y'), Shv(Y'))$ *is a* $Shv(Z)$*-linear continuous comonad. Moreover, one has canonically*

$$(A - comod(Shv(Y))) \otimes_{Shv(X)} Shv(Z) \widetilde{\to} A_Z - comod(Shv(Y')). \tag{9}$$

Proof. The natural functor (9) is constructed in ([21], 9.2.75) in bigger generality. Set $M = A - comod(Shv(Y))$. Consider the commutative diagram, where the horizontal functors are fully faithful

$$\begin{array}{ccc} M \otimes_{Shv(X)} Shv(Z) & \overset{i_*}{\hookrightarrow} & M \\ \downarrow l & & \downarrow \mathrm{oblv} \\ Shv(Y) \otimes_{Shv(X)} Shv(Z) & \overset{i_*}{\to} & Shv(Y) \end{array}$$

Here l is obtained by base change from oblv. This diagram shows that l s conservatve. Recall the low arrow identifies with $i_* : Shv(Y') \to Shv(Y)$ by Section 0.3.1.

Let us verify that l is comonadic by ([16], 4.7.3.5). Exactly the same argument as in the previous lemma applies. □

0.3.7. Let $Z \hookrightarrow X \overset{j}{\hookleftarrow} U$ be a diagram in Sch_{ft}, where i is a closed immersion and j is the complement open. Let $C \in Shv(X) - mod$. Then for any $c \in C$ we get a fibre sequence $i_! i^! c \to c \to j_* j^* c$ in C, where the functors $i_!, i^!, j_*, j^*$ are are in the previous subsections. It is obtained by tensoring the fibre sequence $i_*\omega_Z \to \omega_X \to j_*\omega_U$ by c.

Applying $\mathrm{Fun}_{Shv(X)}(C, \cdot)$ to the adjoint pair $j^* : Shv(X) \leftrightarrows Shv(U) : j_*$ in $Shv(X)-mod$, one gets an adjoint pair

$$\mathrm{Fun}_{Shv(X)}(C, Shv(X)) \leftrightarrows \mathrm{Fun}_{Shv(X)}(C, Shv(U))$$

in $Shv(X) - mod$, where the right adjoint is fully faithful (hence monadic). The corresponding monad is given by the action of the algebra $j_*\omega$. So,

$$\mathrm{Fun}_{Shv(X)}(C, Shv(U)) \widetilde{\to} j_*\omega - mod(\mathrm{Fun}_{Shv(X)}(C, Shv(X)))$$

By ([21], 9.2.74), $\mathrm{Fun}_{Shv(X)}(C, Shv(U))$ is just the image of the action of the idempotent $j_*\omega$ on $\mathrm{Fun}_{Shv(X)}(C, Shv(X))$. From Section 0.3.2 we conclude that

$$\mathrm{Fun}_{Shv(U)}(C \otimes_{Shv(X)} Shv(U), Shv(U)) \widetilde{\to} \mathrm{Fun}_{Shv(X)}(C, Shv(U)) \widetilde{\to} \\ \mathrm{Fun}_{Shv(X)}(C, Shv(X)) \otimes_{Shv(X)} Shv(U)$$

Applying $\mathrm{Fun}_{Shv(X)}(C, \cdot)$ to the adjoint pair $i_! : Shv(Z) \leftrightarrows Shv(X) : i^!$ in $Shv(X) - mod$, we get an adjoint pair in $Shv(X) - mod$

$$\mathrm{Fun}_{Shv(X)}(C, Shv(Z)) \leftrightarrows \mathrm{Fun}_{Shv(X)}(C, Shv(X)),$$

where the left adjoint is fully faithful. Using this fully faithfulness, we show as above that the left adjoint is comonadic, so

$$\mathrm{Fun}_{Shv(Z)}(C \otimes_{Shv(X)} Shv(Z), Shv(Z)) \widetilde{\to} \mathrm{Fun}_{Shv(X)}(C, Shv(Z)) \widetilde{\to} \\ (i_!\omega) - comod(\mathrm{Fun}_{Shv(X)}(C, Shv(X)))$$

As in Section 0.3.4,

$$(i_!\omega) - comod(\mathrm{Fun}_{Shv(X)}(C, Shv(X))) \widetilde{\to} \mathrm{Fun}_{Shv(X)}(C, Shv(X)) \otimes_{Shv(X)} Shv(Z)$$

We conclude that

$$\mathrm{Fun}_{Shv(Z)}(C\otimes_{Shv(X)} Shv(Z), Shv(Z))\widetilde{\to}\mathrm{Fun}_{Shv(X)}(C, Shv(X))\otimes_{Shv(X)} Shv(Z)$$

0.3.8. Let $j_i : U_i \hookrightarrow S$ are open subsets in $S \in \mathrm{Sch}_{ft}$ for $i = 1, 2$ with $U = U_1 \cap U_2$ and $U_1 \cup U_2 = S$. Then the square is cocartesian in $Shv(S) - mod$

$$\begin{array}{ccc} Shv(U_1) & \stackrel{(j_1)_*}{\to} & Shv(S) \\ \uparrow & & \uparrow (j_2)_* \\ Shv(U) & \to & Shv(U_2), \end{array}$$

where all the functors are given by $*$-direct images. Indeed, this follows from Zariski descent for sheaves of categories on S. Namely, this diagram after restriction to each U_i becomes cocartesian.

0.4. **More about the constructible context.** For this subsection we work in the constructible context.

0.4.1. Let $S \in \mathrm{Sch}_{ft}$. The functor $\mathbb{D} : Shv(S)^c\widetilde{\to} Shv(S)^{c,op}$ is an equivalence of symmetric monoidal categories, where the LHS is equipped with the $\otimes^!$-monoidal structure, and the RHS is eqiupped with the $\otimes$-monoidal structure.

Proof: we have an isomorphism $K_1 \otimes^! K'\widetilde{\to}\mathbb{D}(\mathbb{D}(K)\otimes\mathbb{D}(K'))$ in $Shv(S)^c$ functorial in $K, K' \in Shv(S)$. $\square$

0.4.2. If $Y \in \mathrm{Sch}_{ft}$ then the right adjoint m^R to the product $m : Shv(Y)\otimes Shv(Y) \to Shv(Y)$ is not a morphism of $Shv(Y)$-bimodules, that is, the right-lax structure is not strict in general. Indeed, for $K \in Shv(Y), L \in Shv(Y)^c$ the canonical map $(L \boxtimes \omega) \otimes m^R(K) \to m^R(K \otimes^! L)$ is not an isomorphism in general. Indeed, given $M_i \in Shv(Y)$ the map

$$\mathcal{H}om(M_1 \boxtimes M_2, (L \boxtimes \omega) \otimes m^R(K)) \to \mathcal{H}om(M_1 \boxtimes M_2, m^R(K \otimes^! L))$$

is not an isomorphism in general. This reduces to the fact the map $(M_1\otimes^! M_2)\otimes\mathbb{D}(L) \to (M_1 \otimes \mathbb{D}(L)) \otimes^! M_2$ is not an equivalence in general, cf. Section 0.1.10.

0.4.3. Let $i : Z \to Y$ be a closed immersion in Sch_{ft}. Recall that $i^! : Shv(Y) \to Shv(Z)$ has a continuous right adjoint $(i^!)^R$. Then $(i^!)^R$ is fully faithful! Indeed, pass to the right adjoints in $i^! i_! \widetilde{\to} \mathrm{id}$.

0.4.4. Consider a diagram $S_1 \stackrel{\beta_1}{\to} Y \stackrel{\beta_2}{\leftarrow} S_2$ in Sch_{ft}. Equip $Shv(Y)$ with the $\otimes^!$-symmetric monoidal structure. Then $Shv(S_1)\otimes_{Shv(Y)}Shv(S_2)$ is dualizable in DGCat_{cont} by ([7], I.1, 6.3.4). Indeed, the product functor $Shv(Y) \otimes Shv(Y) \stackrel{m}{\to} Shv(Y)$ and action maps $Shv(Y) \otimes Shv(S_i) \to Shv(S_i)$ admits continuous right adjoints, and for any $n \geq 0$, $Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2)$ is compactly generated.

It is easy to see that $Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$ is compactly generated by objects of the form $K_1 \boxtimes K_2$ with $K_i \in Shv(S_i)^c$.

Let $C_{\triangle^{op}} : \boldsymbol{\Delta}^{op} \to \mathrm{DGCat}_{cont}$ be the diagram

$$[n] \mapsto Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2)$$

such that $\operatorname{colim} C_{\triangle^{op}} \widetilde{\to} Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$ in DGCat_{cont} by definition. Then we may pass to continuous right adjoints in $C_{\triangle^{op}}$ and get the functor $C^R_{\triangle} : \boldsymbol{\Delta} \to \mathrm{DGCat}_{cont}$, so that $\lim C^R_{\triangle} \widetilde{\to} Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$ in DGCat_{cont}. The projection

$$\delta^R := ev_0 : \lim C^R_{\triangle} \to Shv(S_1) \otimes Shv(S_2)$$

has a left adjoint $\delta := ins_0 : Shv(S_1) \otimes Shv(S_2) \to Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$. So, by ([16], 4.7.5.1), the functor δ^R is monadic and

$$Shv(S_1) \otimes_{Shv(Y)} Shv(S_2) \widetilde{\to} (\delta^R \delta) - mod(Shv(S_1) \otimes Shv(S_2)).$$

Lemma 0.4.5. *The dual* $(Shv(S_1) \otimes_{Shv(Y)} Shv(S_2))^{\vee}$ *identifies with*

$$Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2),$$

where now $Shv(Y)$ *is equipped with the* $\otimes$*-symmetric monoidal structure, and the action maps are given as compositions*

$$Shv(Y) \otimes Shv(S_i) \to Shv(Y \times S_i) \stackrel{\Gamma_i^*}{\to} Shv(S_i).$$

Here $\Gamma_i : S_i \to S_i \times Y$ *is the graph of the map* $S_i \to Y$. *We used the canonical self-dualities on* $Shv(S_i), Shv(Y)$.

Proof. The right adjoint to the composition

$$Shv(S) \otimes Shv(Y) \stackrel{h}{\to} Shv(S \times Y) \stackrel{\Gamma_i^!}{\to} Shv(S)$$

is the composition $h^R \circ (\Gamma_i^!)^R$, and $h^R \widetilde{\to} h^{\vee}$, $(\Gamma_i^!)^R \widetilde{\to} (\Gamma_i^*)^{\vee}$ canonically. Similarly for the product map

$$Shv(Y) \otimes Shv(Y) \stackrel{h}{\to} Shv(Y \times Y) \stackrel{\triangle^!}{\to} Shv(Y)$$

its right adjoint is $h^R \circ (\triangle^!)^R$, and $h^R \widetilde{\to} h^{\vee}$, $(\triangle^!)^R \widetilde{\to} (\triangle^*)^{\vee}$. The claim follows as in ([7], I.1, 6.3.4). ☐

0.4.6. It is easy to see that $Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2)$ is compactly generated by objects of the form $K_1 \boxtimes K_2$ with $K_i \in Shv(S_i)^c$.

0.4.7. Let

$$\mathcal{F} : Shv(S_1) \otimes_{Shv(Y)} Shv(S_2) \to Shv(S_1 \times_Y S_2)$$

be the natural map coming from $K_1 \boxtimes K_2 \mapsto q^!(K_1 \boxtimes K_2)$ for $q : S_1 \times_Y S_2 \to Y_1 \times Y_2$. For $K_i \in Shv(S_i)^c$ the object $\mathcal{F}(K_1 \boxtimes K_2) \in Shv(S_1 \times_Y S_2)^c$, so $\mathcal{F}$ has a continuous right adjoint.

We also have a natural functor

$$\mathcal{F}' : Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2) \to Shv(S_1 \times_Y S_2)$$

coming from $K_1 \boxtimes K_2 \mapsto q^*(K_1 \boxtimes K_2)$. For $K_i \in Shv(S_i)^c$ the object $\mathcal{F}'(K_1 \boxtimes K_2) \in Shv(S_1 \times_Y S_2)^c$, so $\mathcal{F}'$ has a continuous right adjoint.

The dual of $\mathcal{F}$ is the functor

$$\mathcal{F}^{\vee} : Shv(S_1 \times_Y S_2) \to Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2)$$

The dual of $\mathcal{F}'$ is the functor

$$\mathcal{F}'^{\vee} : Shv(S_1 \times_Y S_2) \to Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$$

0.4.8. Write $\delta : Shv(S_1) \otimes Shv(S_2) \to Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$ and

$$\delta_{\otimes} : Shv(S_1) \otimes Shv(S_2) \to Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2)$$

for the natural functors. Let δ^R be the right adjoint to δ. By construction, we get $(\delta^R)^{\vee} \widetilde{\to} \delta_{\otimes}$.

Lemma 0.4.9. *In the situation of Section 0.4.4 one has canonically* $\mathcal{F}^{\vee} \widetilde{\to} (\mathcal{F}')^R$ *and* $\mathcal{F}'^{\vee} \widetilde{\to} \mathcal{F}^R$*, where* R *stands for the right adjoint.*

Proof. Let $C_{\boldsymbol{\Delta}^{op}} : \boldsymbol{\Delta}^{op} \to \mathrm{DGCat}_{cont}$ be the functor giving rise to

$$Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$$

in its colimit by definition. It sends $[n]$ to $Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2)$. For a map $\alpha : [i] \to [j]$ in $\boldsymbol{\Delta}$ we have the transition functor $\alpha_{ij} : C_{\boldsymbol{\Delta}^{op}}(j) \to C_{\boldsymbol{\Delta}^{op}}(i)$ in this diagram.

Write

$$f_n : Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2) \to Shv(S_1 \times_Y S_2)$$

for the composition of ins_n with $\mathcal{F}$. We have the adjoint pairs (f_n, f_n^R) and $(\mathcal{F}, \mathcal{F}^R)$ in DGCat_{cont}. We get the adjoint pairs $((f_n^R)^{\vee}, (f_n)^{\vee})$ and $((\mathcal{F}^R)^{\vee}, \mathcal{F}^{\vee})$ in DGCat_{cont}.

Denote by

$$C^R_{\boldsymbol{\Delta}} : \boldsymbol{\Delta} \to \mathrm{DGCat}_{cont}$$

the functor obtained from $C_{\boldsymbol{\Delta}^{op}}$ by passing to the right adjoints. Denote by $C^{\vee}_{\boldsymbol{\Delta}} : \boldsymbol{\Delta} \to \mathrm{DGCat}_{cont}$ the functor obtained from $C_{\boldsymbol{\Delta}^{op}}$ by passing to the duals. Denote by

$$(C^R_{\boldsymbol{\Delta}})^{\vee} : \boldsymbol{\Delta}^{op} \to \mathrm{DGCat}_{cont}$$

the functor obtained from $C^R_{\boldsymbol{\Delta}}$ by passing to the duals. Recall that

$$\mathrm{colim}(C^{\vee}_{\boldsymbol{\Delta}})^L \widetilde{\to} \lim C^{\vee}_{\boldsymbol{\Delta}} \widetilde{\to} Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2)$$

canonically, where $(C^{\vee}_{\boldsymbol{\Delta}})^L : \boldsymbol{\Delta}^{op} \to \mathrm{DGCat}_{cont}$ is obtained from $C^{\vee}_{\boldsymbol{\Delta}}$ by passing to the left adjoints. Recall that

$$(C^{\vee}_{\boldsymbol{\Delta}})^L \widetilde{\to} (C^R_{\boldsymbol{\Delta}})^{\vee}.$$

The functor $\mathcal{F}^R : Shv(S_1 \times_Y S_2) \to \lim C^R_{\boldsymbol{\Delta}}$ is obtained from the compatible system of functors f_n^R for $[n] \in \boldsymbol{\Delta}$.

The functor

$$\mathcal{F}^{\vee} : Shv(S_1 \times_Y S_2) \to Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2) \widetilde{\to} \lim C^{\vee}_{\boldsymbol{\Delta}}$$

is obtained from the compatible system of functors $f_n^{\vee}$, $[n] \in \boldsymbol{\Delta}$. So, the functor

$$(\mathcal{F}^{\vee})^L : \mathrm{colim}(C^{\vee}_{\boldsymbol{\Delta}})^L \to Shv(S_1 \times_Y S_2)$$

is obtained from the compatible system of functors $(f_n^{\vee})^L$.

Consider the functor $D_{\boldsymbol{\Delta}^{op}} : \boldsymbol{\Delta}^{op} \to \mathrm{DGCat}_{cont}$ such that

$$\mathrm{colim}\, D_{\boldsymbol{\Delta}^{op}} \widetilde{\to} Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2)$$

by definition. It sends $[n]$ to $Shv(S_1)\otimes Shv(Y)^{\otimes n}\otimes Shv(S_2)$. It suffices to show that the composition

$$Shv(S_1)\otimes Shv(Y)^{\otimes n}\otimes Shv(S_2)\to Shv(S_1)\otimes_{(Shv(Y),\otimes)} Shv(S_2)\stackrel{\mathcal{F}'}{\to} Shv(S_1\times_Y S_2)$$

for any n identifies with the functor $(f_n^{\vee})^L\widetilde{\to}(f_n^R)^{\vee}$. This is easy as in Lemma 0.4.5.

Namely, we have a natural map $\tau_n : S_1\times_Y S_2\to S_1\times Y^n\times S_2$ coming from

$$(S_1\times_Y S_2)\widetilde{\to}(S_1\times Y^n\times S_2)\times_{Y^{n+2}} Y\widetilde{\to} S_1\times_Y S_2.$$

Then f_n is the composition

$$Shv(S_1)\otimes Shv(Y)^{\otimes n}\otimes Shv(S_2)\to Shv(S_1\times Y^n\times S_2)\stackrel{\tau_n^!}{\to} Shv(S_1\times_Y S_2)$$

So, $f_n^R=(\vartriangle^{\vee})\circ(\tau_n^*)^{\vee}$ as desired. □

0.4.10. In the situation of Section 0.4.4 note that Y is naturally a cocommutative coalgebra in Sch_{ft}, the coproduct being the diagonal map $Y\to Y\times Y$. Besides, S_1 (resp., S_2) is a Y-comodule, the coaction map is $\Gamma_i : S_i\to S_i\times Y$, the graph of the map $S_i\to Y$. We get the morphism $S_1\times_Y S_2\to \lim\limits_{[n]\in\boldsymbol{\Delta}} S_1\times Y^n\times S_2$, the corresponding version of the bar complex. It yields after applying Shv and !-pullbacks the morphisms

$$Shv(S_1)\otimes_{Shv(Y)} Shv(S_2)\stackrel{\mathcal{F}_1}{\to} C:=\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{op}} Shv(S_1\times Y^n\times S_2)\stackrel{\mathcal{F}_2}{\to} Shv(S_1\times_Y S_2),$$

so $\mathcal{F}\widetilde{\to}\mathcal{F}_2\circ\mathcal{F}_1$. Let $\delta_1 : Shv(S_1\times S_2)\to C$ be the natural map. Note that C is compactly generated by the images of $K\in Shv(S_1\times S_2)^c$ under δ_1.

Let $\tilde{C}_{\vartriangle^{op}} : \boldsymbol{\Delta}^{op}\to \mathrm{DGCat}_{cont}$ be the functor sending $[n]$ to $Shv(S_1\times Y^n\times S_2)$ with the transition being the !-pullbacks, so $C=\operatorname{colim}\tilde{C}_{\vartriangle^{op}}$. We may pass to continuous right adjoints in $\tilde{C}_{\vartriangle^{op}}$ and get the functor $\tilde{C}^R_{\vartriangle} : \boldsymbol{\Delta}\to\mathrm{DGCat}_{cont}$, so $\lim\tilde{C}^R_{\vartriangle}\widetilde{\to}C$. In particular, the right adjoint δ_1^R of δ_1 is continuous.

For each $[n]\in\boldsymbol{\Delta}$ the exterior product $g_n : Shv(S_1)\otimes Shv(Y)^{\otimes n}\otimes Shv(S_2)\to Shv(S_1\times Y^n\times S_2)$ is fully faithful and has a continuous right adjoint g_n^R. So, the right adjoint $\mathcal{F}_1^R$ of $\mathcal{F}_1$ is obtained by passing to the limit over $[n]\in\boldsymbol{\Delta}$ in the functors $g_n^R : Shv(S_1\times Y^n\times S_2)\to Shv(S_1)\otimes Shv(Y)^{\otimes n}\otimes Shv(S_2)$ in DGCat_{cont}. So, $\mathcal{F}_1^R$ is continuous (cf. [21], 9.2.39).

Clearly, if $K\in Shv(S_1\times S_2)^c$ then $\mathcal{F}_2(\delta_1(K))\in Shv(S_1\times_Y S_2)^c$, so $\mathcal{F}_2$ has a continuous right adjoint $\mathcal{F}_2^R$. Besides $\mathcal{F}_2^R$ is conservative.

Since $\boldsymbol{\Delta}^{op}$ is sifted, from ([16], 3.2.3.1) we see that $C\in CAlg(\mathrm{DGCat}_{cont})$, and δ_1 is a map in $CAlg(\mathrm{DGCat}_{cont})$ naturally. Besides, $\mathcal{F}_1,\mathcal{F}_2$ are naturally morphisms in $CAlg(\mathrm{DGCat}_{cont})$.

0.4.11. Assume for what follows that Y is separated. Does the simplicial category $\tilde{C}_{\vartriangle^{op}}$ satisfy the monadic Beck-Chevalley conditions?

For $\alpha : [i]\to[j]$ in $\boldsymbol{\Delta}$ write

$$T^{\alpha} : S_1\times Y^i\times S_2\to S_1\times Y^j\times S_2$$

for the corresponding morphism, so the transition functor in $\tilde{C}_{\vartriangle^{op}}$ is $(T^{\alpha})^!$. For $i\geq 0$ denote by $\partial_i : [i]\to[i+1]$ the last face map sending k to k for $0\leq k\leq i$. Then $(T^{\partial_i})^!$

has the left adjoint $(T^{\partial_i})_!$. Recall that $\alpha+1:[i+1]\to[j+1]$ is the map given by $(\alpha+1)(k)=\alpha(k)$ for $0\le k\le i$ and $(\alpha+1)(i+1)=j+1$.

Proposition 0.4.12. *The simplicial category $\tilde{C}_{\triangle^{op}}$ satisfies the monadic Beck-Chevalley conditions.*

Proof. We have to check that for given map α as above the diagram commutes

$$\begin{array}{ccc} Shv(S_1\times Y^j\times S_2) & \stackrel{(T^\alpha)^!}{\to} & Shv(S_1\times Y^i\times S_2)\\ \downarrow (T^{\partial_j})_! & & \downarrow (T^{\partial_i})_!\\ Shv(S_1\times Y^{j+1}\times S_2) & \stackrel{(T^{\alpha+1})^!}{\to} & Shv(S_1\times Y^{i+1}\times S_2). \end{array}$$

Then T^{∂_i} is a closed immersion. The square

$$\begin{array}{ccc} S_1\times Y^j\times S_2 & \stackrel{T^\alpha}{\leftarrow} & S_1\times Y^i\times S_2\\ \downarrow T^{\partial_j} & & \downarrow T^{\partial_i}\\ S_1\times Y^{j+1}\times S_2 & \stackrel{T^{\alpha+1}}{\leftarrow} & S_1\times Y^{i+1}\times S_2 \end{array}$$

is always cartesian. So, we get $(T^{\partial_j})_!(T^\alpha)^!\widetilde{\to}(T^{\alpha+1})^!(T^{\partial_i})_!$ canonically. □

Write T^s, T^t for the maps $S_1\times S_2\to S_1\times Y\times S_2$ attached to maps $[0]\to[1]$ with the images $0,1$ respectively. Here 's' and 't' stand for source and target. These maps are given for $s_i\in S_i$ by

$$T^s(s_1,s_2)=(s_1,\beta_2(s_2),s_2),\quad T^t(s_1,s_2)=(s_1,\beta_1(s_1),s_2).$$

By ([9], C.1.6), the cosimplicial category $\tilde{C}^R_\triangle$ also satisfies the monadic Beck-Chevalley conditions. Now by ([9], C.1.8) or ([16], 4.7.5.2) we conclude that $\delta_1^R: C\to Shv(S_1\times S_2)$ is monadic, and the monad $\mathcal{A}=\delta_1^R\delta_1$ identifies with the composition

$$Shv(S_1\times S_2)\stackrel{((T^t)^!)^R}{\to} Shv(S_1\times Y\times S_2)\stackrel{(T^s)^!}{\to} Shv(S_1\times S_2)$$

The left adjoint to $\mathcal{A}$ is the endofunctor $\mathcal{A}^L=(T^t)^!(T^s)_!$ on $Shv(S_1\times S_2)$. The square is cartesian

$$\begin{array}{ccc} S_1\times S_2 & \stackrel{T^s}{\to} & S_1\times Y\times S_2\\ \uparrow q & & \uparrow T^t\\ S_1\times_Y S_2 & \stackrel{q}{\to} & S_1\times S_2. \end{array}$$

So, $(T^t)^!(T^s)_!\widetilde{\to} q_!q^!$. Here $q_!q^!L\widetilde{\to} q_*\omega\otimes^! L$ for $L\in Shv(S_1\times S_2)$. So, the functor $\mathcal{A}$ identifies with the inner hom functor $\underline{\mathcal{H}om}^!(q_*\omega, \_): Shv(S_1\times S_2)\to Shv(S_1\times S_2)$ with respect to the $\otimes^!$-symmetric monidal structure on $Shv(S_1\times S_2)$.

Recall that

$$Shv(S_1\times_Y S_2)\widetilde{\to}\mathcal{A}^L-comod(Shv(S_1\times S_2))$$

canonically as in Section 0.3.1.

The coproduct $\mathcal{A}^L\to\mathcal{A}^L\circ\mathcal{A}^L$ is an isomorphism, so passing to the right adjoints we see that the product $\mathcal{A}\circ\mathcal{A}\to\mathcal{A}$ is an equivalence.

0.4.13. For any injective map $\alpha : [i] \to [j]$ in $\boldsymbol{\Delta}$ the corresponding transition map

$$T^\alpha : S_1 \times Y^i \times S_2 \to S_1 \times Y^j \times S_2$$

in the above diagram $\boldsymbol{\Delta} \to \mathrm{Sch}_{ft}$ is a closed immersion, as Y is assumed separated. So,

$$((T^\alpha)^!)^R : Shv(S_1 \times Y^i \times S_2) \to Shv(S_1 \times Y^j \times S_2)$$

is fully faithful by Section 0.4.3.

Let $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ be the subcategory, where we keep all objects, but only injective maps. Recall that $\boldsymbol{\Delta}_s^{op} \to \boldsymbol{\Delta}^{op}$ is cofinal by ([15], 6.5.3.7). So, $C \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}_s} \tilde{C}^R_\triangle$ in DGCat_{cont}, all the transition functors in the latter diagram being fully faithful.

Write

$$\kappa_n : S_1 \times_Y S_2 \to S_1 \times Y^n \times S_2$$

for the map $(s_1, s_2) \mapsto (s_1, \beta_1(s_1), \ldots, \beta_1(s_1), s_2)$, where $\beta_i : S_i \to Y$ are the maps from Section 0.4.4. Note that

$$\kappa_\bullet : S_1 \times_Y S_2 \to S_1 \times Y^\bullet \times S_2$$

is a morphism of cosimplicial objects in Sch_{ft}.

The functor $\mathcal{F}_2$ can be seen as $\operatorname{colim}_{[n]\in\boldsymbol{\Delta}^{op}} \kappa_n^!$, where

$$\kappa_n^! : Shv(S_1 \times Y^n \times S_2) \to Shv(S_1 \times_Y S_2).$$

We claim that $\mathcal{F}_2^R : Shv(S_1 \times_Y S_2) \to C$ is fully faithful. Indeed, $\mathcal{F}_2^R$ is obtained by passing to the limit over $[n] \in \boldsymbol{\Delta}_s$ in the fully faithful functors

$$(\kappa_n^!)^R : Shv(S_1 \times_Y S_2) \to Shv(S_1 \times Y^n \times S_2).$$

0.4.14. Consider the evaluation map $\delta_1^R : C = \lim_{[n]\in\boldsymbol{\Delta}_s} \tilde{C}^R_\triangle \to Shv(S_1 \times S_2)$. We claim that δ_1^R is fully faithful. This is a consequence of the following.

Lemma 0.4.15. *Let $\boldsymbol{\Delta}_s \to \mathrm{DGCat}_{cont}$, $[n] \mapsto D_n$ be a diagram, set $D = \lim_{[n]\in\boldsymbol{\Delta}_s} D_n$ in DGCat_{cont}. Assume all the transition maps in this diagram are fully faithful. Let $ev_0 : D \to D_0$ be the evaluation for $[0] \in \boldsymbol{\Delta}$. Then ev_0 is fully faihtful.*

Proof. Take $a, b \in D$. For $[n] \in \boldsymbol{\Delta}$ let a_n, b_n be their images in D_n. We have $\mathrm{Map}_D(a, b) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}_s} \mathrm{Map}_{D_n}(a_n, b_n)$ in Spc. The diagram $\boldsymbol{\Delta}_s \to \mathrm{Spc}$, $[n] \mapsto \mathrm{Map}_{D_n}(a_n, b_n)$ is isomorphic to the constant diagram with value $\mathrm{Map}_{D_0}(a_0, b_0)$, because for any map $[n] \to [m]$ in $\boldsymbol{\Delta}$ the functor $D_n \to D_m$ is fully faithful. So,

$$\mathrm{Map}_D(a, b) \widetilde{\to} \lim_{[n]\in\boldsymbol{\Delta}_s} \mathrm{Map}_{D_0}(a_0, b_0) \widetilde{\to} \mathrm{Map}_{D_0}(a_0, b_0).$$

□

Lemma 0.4.16. *The essential image of δ_1^R is the full subcategory $q_! : Shv(S_1 \times_Y S_2) \hookrightarrow Shv(S_1 \times S_2)$. So, $\mathcal{F}_2$ is an equivalence.*

Proof. Since δ_1^R is fully faithful, we see that C is the full subcategory of those $K \in Shv(S_1 \times S_2)$ fpr which the natural map $K \to \delta_1^R \delta_1(K)$ is an isomorphism. In other words, the natural map $K \to \underline{\mathcal{H}om}^!(q_*\omega, K)$ in $Shv(S_1 \times S_2)$ is an isomorphism. The latter condition means that we have an isomorphism

$$\mathcal{H}om(L, K) \widetilde{\to} \mathcal{H}om(q_*\omega \otimes^! L, K)$$

in Vect functorial in $L \in Shv(S_1 \times S_2)$. Let $j : \mathcal{U} \hookrightarrow S_1 \times S_2$ be the open complement to $q : S_1 \times_Y S_2 \to S_1 \times S_2$. Our condition is equivalent to the property: if $L' \in Shv(\mathcal{U})$ then $\mathcal{H}om(j_* L', K) = 0$. Thus, the essential image of δ_1^R is the full subcategory $q_! : Shv(S_1 \times_Y S_2) \hookrightarrow Shv(S_1 \times S_2)$. □

0.4.17. In our situation for the graph $\Gamma_i : S_i \to S_i \times Y$ of β_i the functor $(\Gamma_i^!)^R : Shv(S_i) \to Shv(S_i \times Y)$ is not $Shv(Y)$-linear in general. This happens already when $S_2 = \operatorname{Spec} k$, so that $\Gamma_2 : \operatorname{Spec} k \to Y$ is a closed point, and $\Gamma_2^! M \otimes \Gamma_2^* N \to \Gamma_2^!(M \otimes N)$ is not an isomorphism in general for $M, N \in Shv(Y)$.

If $S_2 = \operatorname{Spec} k$ then the action map act : $Shv(Y) \otimes Shv(S_2) \to Shv(S_2)$ becomes $\Gamma_2^!$. So, act^R is not $Shv(Y)$-linear in general. So, we can not apply ([9], C.2.2(i)) to conclude that the diagram $C_\Delta^R : \boldsymbol{\Delta} \to \mathrm{DGCat}_{cont}$ satisfies the monadic Beck-Chevallet conditions. Here $\lim C_\Delta^R \widetilde{\to} Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$.

Theorem 0.4.18. *Consider the case when $\beta_1 : S_1 \to Y$ is finite. In this case the canonical map*

$$\mathcal{F} : Shv(S_1) \otimes_{Shv(Y)} Shv(S_2) \to Shv(S_1 \times_Y S_2)$$

is an equivalence.

Proof. In this case $(\beta_1)_*$ is conservative, so is monadic and $Shv(S_1) \widetilde{\to} (\beta_1)_* e{-}mod(Shv(Y))$. Here we viewed $Shv(Y)$ as equipped with the $\otimes$-symmetric monoidal structure, so that $K \mapsto (\beta_1)_* e \otimes K$ is the action of the commutative algebra $(\beta_1)_* e \in CAlg(Shv(Y), \otimes)$. This monad is $(Shv(Y), \otimes)$-linear, so in this case

$$\beta_1^* \otimes \mathrm{id} : Shv(S_2) :\leftrightarrows Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2) : (\beta_1)_* \otimes \mathrm{id}$$

is an adjoint pair in $(Shv(S_2), \otimes) - mod$. By ([7], ch. I.1, 8.5.7), $(\beta_1) \otimes \mathrm{id}$ is monadic and

$$Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2) \widetilde{\to} \beta_2^*(\beta_1)_! e - mod(Shv(S_2))$$

Now consider $\beta_1 \times \mathrm{id} : S_1 \times_Y S_2 \to S_2$ obtained by base change from β_1. This is a finite morphism, so as above we get $(\beta_1 \times \mathrm{id})_! e \widetilde{\to} \beta_2^*(\beta_1)_! e \in CAlg(Shv(S_2), \otimes)$, and $\beta_1 \times \mathrm{id} : Shv(S_1 \times_Y S_2) \to Shv(Y)$ is monadic with

$$Shv(S_1 \times_Y S_2) \widetilde{\to} \beta_2^*(\beta_1)_! e - mod(Shv(S_2)$$

Thus, the canonical map

$$Shv(S_1) \otimes_{(Shv(Y),\otimes)} Shv(S_2) \to Shv(S_1 \times_Y S_2)$$

is an equivalence!

Now dualize it using Lemma 0.4.5, this gives the fact that $\mathcal{F}$ is an equivalence. □

As an application, we get the following.

Corollary 0.4.19. *Let $f : Y \to Z$ be a finite morphism in Sch_{ft}. Then in the constructible context $Shv(Y)$ is canonically self-dual in $Shv(Z)-mod$.*

The counit is the functor $c : Shv(Y) \otimes_{Shv(Z)} Shv(Y) \to Shv(Z)$ sending $K_1, K_2 \in Shv(Y)$ to $f_*(K_1 \otimes^! K_2)$. The unit is the functor

$$u : Shv(Z) \to Shv(Y \times_Z Y) \widetilde{\to} Shv(Y) \times_{Shv(Z)} Shv(Y)$$

sending K to $(q^!K) \otimes^! \vartriangle_* \omega$, where $q : Y \times_Z Y \to Z$ is the projection, and $\vartriangle : Y \to Y \times_Z Y$ is the diagonal.

Proof. It is immediate to check that the compositions

$$Shv(Y) \stackrel{u\otimes \mathrm{id}}{\to} Shv(Y) \otimes_{Shv(Z)} Shv(Y) \otimes_{Shv(Z)} Shv(Y) \stackrel{\mathrm{id}\otimes c}{\to} Shv(Y)$$

and

$$Shv(Y) \stackrel{\mathrm{id}\otimes u}{\to} Shv(Y) \otimes_{Shv(Z)} Shv(Y) \otimes_{Shv(Z)} Shv(Y) \stackrel{c\otimes \mathrm{id}}{\to} Shv(Y)$$

are equivalences. □

0.5. **Addition about algebraic stacks.**

0.5.1. Work in the constructible context[3]. Let $Y \to S$ be a map in Sch_{ft}, G be a smooth group scheme of finite type over S. Write Y/G for the stack quotient over S. Let $q : Y \to Y/G$ be the natural map

Recall that $Shv(Y/G) \widetilde{\to} \mathcal{A}-comod(Shv(Y))$, where $\mathcal{A} = q^*q_* : Shv(Y) \to Shv(Y)$. In the constructible context by ([2], A.2.7), each cohomology group of $\mathrm{R}\Gamma(Y/G, e)$ is finite-dimensional. I imagine the same holds for $\mathcal{D}$-modules, maybe see [5]. We want to check that the degree wise dual $\mathbb{D}\,\mathrm{R}\Gamma(Y/G, e)$ identifies canonically with $\mathrm{R}\Gamma_c(Y/G, \omega)$. Let us identify $\mathrm{oblv} : \mathcal{A}-comod(Shv(Y)) \to Shv(Y)$ with q^*.

By ([21], 9.2.59), we have $e \widetilde{\to} \lim_{[n]\in \boldsymbol{\Delta}} \mathcal{A}^{n+1}(q^*e)$ taken in $Shv(Y/G)$. Here for any $[n] \in \boldsymbol{\Delta}$, $\mathcal{A}^{n+1}(q^*e) \in Shv(Y/G)^c$, and $e \in Shv(Y/G)^{constr}$. So, the limit

$$e \widetilde{\to} \lim_{[n]\in \boldsymbol{\Delta}} \mathcal{A}^{n+1}(q^*e)$$

is also a limit in $Shv(Y/G)^{constr}$. Applying $\mathbb{D} : (Shv(Y/G)^{constr})^{op} \widetilde{\to} Shv(Y/G)^{constr}$, we get

$$\omega \widetilde{\to} \mathbb{D}(e) \widetilde{\to} \operatorname*{colim}_{[n]\in \boldsymbol{\Delta}^{op}} \mathbb{D}(\mathcal{A}^{n+1}(q^*e)) \tag{10}$$

in $Shv(Y/G)^{constr}$. Let $\mathcal{B} = q^!q_!$, here we used already the constructible context to assure that $q_!$ exists. Then

$$\mathbb{D}(\mathcal{A}^{n+1}(q^*e)) \widetilde{\to} \mathcal{B}^{n+1}\omega_Y$$

in $Shv(Y/G)^{constr}$.

Let $m \geq 0$. The functor $\tau^{\leq m} : \mathrm{Vect} \to \mathrm{Vect}^{\leq m}$ preserves limits, it is a right adjoint. Besides, $\mathrm{R}\Gamma : Shv(Y/G) \to \mathrm{Vect}$ preserves limits. So,

$$\mathrm{R}\Gamma(Y/G, e) \widetilde{\to} \lim_{[n]\in \boldsymbol{\Delta}} \mathrm{R}\Gamma(Y/G, \mathcal{A}^{n+1}(q^*e))$$

[3] How to extend the result below to the case of $\mathcal{D}$-modules?

in Vect and

$$\tau^{\leq m}\,\mathrm{R}\Gamma(Y/G,e)\,\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\tau^{\leq m}\,\mathrm{R}\Gamma(Y/G,\mathcal{A}^{n+1}(q^*e))$$

taken in $\mathrm{Vect}^{\leq m}$. It actually takes place in $(\mathrm{Vect}^c)\cap(\mathrm{Vect}^{\leq m})$, because each cohomology group of both $\mathrm{R}\Gamma(Y/G,e)$ and of $\mathrm{R}\Gamma(Y/G,\mathcal{A}^{n+1}(q^*e))$ is finite-dimensional.

To continue, we have to know that the colimit (10) can also be undserstood in $Shv(Y/G)$. The functor $q^! : Shv(Y/G)\to Shv(Y)$ has a left adjoint $q_!$. Since $q^!$ is conservative and preserves geometric realizations, $q^!$ is monadic and

$$Shv(Y/G)\,\widetilde{\to}\,\mathcal{B}-mod(Shv(Y)).$$

The functor $\mathrm{oblv}:\mathcal{B}-mod(Shv(Y))\to Shv(Y)$ is identified with $q^!$. Now by ([16], 4.7.2.7),

$$\omega\,\widetilde{\to}\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{op}}\mathcal{B}^{n+1}(\omega_Y)$$

in $Shv(Y/G)$ as desired.

Finally, we get

$$\mathrm{R}\Gamma_c(Y/G,\omega)\,\widetilde{\to}\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{op}}\mathrm{R}\Gamma_c(Y/G,\mathcal{B}^{n+1}(\omega_Y))$$

in Vect. Since $\tau^{\geq -m}:\mathrm{Vect}\to\mathrm{Vect}^{\geq -m}$ preserves colimits, we get

$$\tau^{\geq -m}\,\mathrm{R}\Gamma_c(Y/G,\omega)\,\widetilde{\to}\operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{op}}\tau^{\geq -m}\,\mathrm{R}\Gamma_c(Y/G,\mathcal{B}^{n+1}(\omega_Y))$$

in $\mathrm{Vect}^{\geq -m}$, and in fact the latter diagram in entire contained in $\mathrm{Vect}^{\geq -m}\cap\mathrm{Vect}^c$.

Dualizing, we get

$$\mathbb{D}(\tau^{\geq -m}\,\mathrm{R}\Gamma_c(Y/G,\omega))\,\widetilde{\to}\lim_{[n]\in\boldsymbol{\Delta}}\tau^{\leq m}\mathbb{D}\,\mathrm{R}\Gamma_c(Y/G,\mathcal{B}^{n+1}(\omega_Y))$$

in $\mathrm{Vect}^{\leq m}\cap\mathrm{Vect}^c$. Now

$$\mathbb{D}\,\mathrm{R}\Gamma_c(Y/G,\mathcal{B}^{n+1}(\omega_Y))\,\widetilde{\to}\,\mathrm{R}\Gamma(Y/G,\mathcal{A}^{n+1}(e_Y))$$

We are done. How to extend this to the case of $\mathcal{D}$-modules?

0.6. **More about pseudo-schemes.**

0.6.1. Work either with $\mathcal{D}$-modules or in the constructible context. Let Y be a pseudo-scheme in the sense of ([27], Section F), that is, $Y\,\widetilde{\to}\,\mathrm{colim}_{i\in I}Y_i$, where $Y_i\in\mathrm{Sch}_{ft}$, and for $i\to j$ in I the map $Y_i\to Y_j$ is proper. Recall that $Shv(Y)$ is canonically self-dual with the counit $Shv(Y)\otimes Shv(Y)\to\mathrm{Vect}$, $K_1\otimes K_2\mapsto\mathrm{R}\Gamma(Y,K_1\otimes^! K_2)$.

Let $h: Shv(Y)\otimes Shv(Y)\to Shv(Y\times Y)$ be the exteriour product. Then h is fully faithful, and its right adjoint h^R is continuous. Let us show that h^R identifies with $h^\vee$ via the above self-duality. Recall that $Shv(Y)$ is compactly generated by objects of the form $(f_i)_!K$ for $i\in I, K\in Shv(Y_i)^c$. Here $f_i: Y_i\to Y$ is the natural map, it is pseudo-proper by ([8], 7.4.2).

Let $i,j\in I$. We claim that the image of $L\in Shv(Y)\otimes Shv(Y)$ under

$$(11)\quad Shv(Y)\otimes Shv(Y)\,\widetilde{\to}\,Shv(Y)^\vee\otimes Shv(Y)^\vee\stackrel{(f_{i!})^\vee\otimes(f_{j!})^\vee}{\to}Shv(Y_i)^\vee\otimes Shv(Y_j)^\vee$$
$$\widetilde{\to}\,\mathrm{Fun}_{k,bi-ex}(Shv(Y_i)^c\times Shv(Y_j)^c,\mathrm{Vect})$$

is the functor

$$(K_i, K_j) \mapsto \mathcal{H}om_{Shv(Y)\otimes Shv(Y)}(((f_i)_!\mathbb{D}K_i) \otimes ((f_j)_!\mathbb{D}K_j), L).$$

Indeed, t suffices to show this for L of the form $L_1 \boxtimes L_2$ with $L_1, L_2 \in Shv(Y)$, in this case this follows from ([7], I.1, 10.5.8). Composing h^R with (11) we get the functor sending $E \in Shv(Y^2)$ to the functor

$$(K_i, K_j) \mapsto \mathcal{H}om_{Shv(Y^2)}((f_i \times f_j)_!((\mathbb{D}K_i) \boxtimes (\mathbb{D}K_j)), E)$$

We used here the pseudo-properness of f_i, f_j giving by base change

$$(f_i \times f_j)_!((\mathbb{D}K_i) \boxtimes (\mathbb{D}K_j)) \widetilde{\to} ((f_i)_!\mathbb{D}K_i) \boxtimes ((f_j)_!\mathbb{D}K_j).$$

The composition

$$Shv(Y^2) \widetilde{\to} Shv(Y^2)^\vee \stackrel{h^\vee}{\to} Shv(Y)^\vee \otimes Shv(Y)^\vee \stackrel{(f_{i!})^\vee \otimes (f_{j!})^\vee}{\to} Shv(Y_i)^\vee \otimes Shv(Y_j)^\vee$$
$$\widetilde{\to} \operatorname{Fun}_{k,bi-ex}(Shv(Y_i)^c \times Shv(Y_j)^c, \operatorname{Vect})$$

sends E to the functor

$$(K_i, K_j) \mapsto \mathcal{H}om_{Shv(Y_i \times Y_j)}(\mathbb{D}(K_i \boxtimes K_j), (f_i \times f_j)^! E).$$

We are done, because $Shv(Y)^\vee \otimes Shv(Y)^\vee \widetilde{\to} \lim_{i,j} Shv(Y_i)^\vee \otimes Shv(Y_j)^\vee$.

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