Assumptions on the sheaf theory for the 2nd joint paper with Dennis

0.0.1. k is algebraically closed of any characteristic, e is an algebraically closed field of characteristic zero. The notation DGCat stands for the category denoted DGCat_{cont} in [6].

We are given a right-lax symmetric monoidal functor

$$(\mathrm{Sch}^{aff}_{ft})^{op} \to \mathrm{DGCat}, S \mapsto Shv(S), \text{ and } (S_1 \xrightarrow{f} S_2) \text{ goes to } f^! : Shv(S_2) \to Shv(S_1)$$

Its right Kan extension under $(\mathrm{Sch}^{aff}_{ft})^{op} \subset (\mathrm{PreStk}_{lft})^{op}$ defines a functor

$$Shv : (\mathrm{PreStk}_{lft})^{op} \to \mathrm{DGCat}$$

It is assumed that the latter functor satisfies the etale descent for etale covers in PreStk_{lft}.

0.0.2. Probably the functor Shv should be defined in a larger category than PreStk_{lft}? For example, there should be $Shv(\mathrm{Hecke}^{loc}_{G,\mathrm{Ran}})$, though the latter is not locally of finite type. Indeed, for a closed $\mathfrak{L}^+(G)_{\mathrm{Ran}}$-equivariant subscheme $Y \subset \mathrm{Gr}_{G,\mathrm{Ran}}$ we may define $Shv((\mathfrak{L}^+(G)_{\mathrm{Ran}})\backslash Y)$ and pass to the colimit (or limit). Similarly, do we need $Shv(\mathrm{Hecke}^{loc}_{G,x})$?

Also, we need to make sense of invariants under $(\mathfrak{L}(N)_x, \chi_N)$, and $\mathfrak{L}(N)$ is not locally of finite type. At least, give a reference to Appendix C of [9].

0.0.3. For a map $f : Y_1 \to Y_2$ in PreStk_{lft} the left adjoint $f_!$ to $f^!$ is only partially defined in general (everywhere defined in the constructible context). If f is schematic open embedding, $f_* : Shv(Y_1) \to Shv(Y_2)$ is defined as the right adjoint to $f^! = f^*$. Moreover f_* satisfies the base change with respect to $g^!$ for $g : Y_2' \to Y_2$.

When we say f is ind-schematic, this means that f is ind-schematic of ind-finite type, as Shv was only defined for PreStk_{lft}. For f ind-schematic we have the functor $f_* : Shv(Y_1) \to Shv(Y_2)$. What is its definition? It has a partially defined left adjoint f^*. Is f^* always defined in the constructible context? For this f has to be of finite type, I think. For example, for $p : Y \to k$, where Y is an ind-scheme of ind-finite type the functor $p_* : Shv(Y) \to \mathrm{Vect}$ does not admit a left adjoint unless Y is a scheme of finite type (see [24], 1.2.7).

For f ind-schematic, f_* satisfies the base change formula with respect to $g^!$, where $g : Y_2' \to Y_2$. If f is ind-proper then $f_* = f_!$. My understanding is that this holds more generally for f pseudo-proper.

If f is etale then $f^! = f^*$ is the left adjoint of f_*.

The functor f_* should be defined more generally under the assumption that after a base change $S \to Y_2$ with $S \in \mathrm{Sch}^{aff}_{ft}$, $S \times_{Y_2} Y_1$ is an ind-algebraic stack. In this case f_* should also satisfy the base change formula with respect to $g^!$.

For example, the following is crucial: the category $Shv(B_{et}(e^{*,tors}))$ is monoidal for the convolution monoidal structure. For Y a prestack this is used to define a twist of $Shv(Y)$ by a $e^{*,tors}$-gerbe over Y.

0.0.4. For Y an ind-scheme we have an equivalence $\mathbb{D} : (Shv(Y)^c)^{op} \widetilde{\to} Shv(Y)^c$. Its definition is given in ([7], 7.1.3) for any $Y \in \mathrm{PreStk}_{lft}$ such that the diagonal $Y \to Y \times Y$ is pseudo-proper.

For an Artin stack Y locally of finite type with an affine diagonal we should define $Shv(Y)^{constr} \subset Shv(Y)$ as the full subcategory of objects that !-pull back to an object of $Shv(S)^c$ for any $S \to Y$, where $S \in \mathrm{Sch}^{aff}_{ft}$. Then by ([1], Appendix C),

$$\mathbb{D} : (Shv(Y)^{constr})^{op} \widetilde{\to} Shv(Y)^{constr} \tag{1}$$

is an equivalence. Indeed, we have $Shv(Y) \widetilde{\to} \lim_{S \to Y} Shv(S)$ taken over the category opposite to the one classifying smooth maps $S \to Y$ with $S \in \mathrm{Sch}^{aff}$. For $a : S \to Y$ smooth, we may use $a^!$ or a^* to test compactness, they differ by a shift. Then $Shv(Y)^{constr} \widetilde{\to} \lim_{S \to Y} Shv(S)^c$ in $\mathrm{DGCat}^{non-cocompl}$. Recall that $\mathrm{DGCat}^{non-cocompl}$ admits limits. This gives $(Shv(Y)^{constr})^{op} \widetilde{\to} \lim_{S \to Y} (Shv(S)^c)^{op}$ in $\mathrm{DGCat}^{non-cocompl}$. So, the Verdier duality for schemes of finity type gives the equivalence (1).

For $F_i \in Shv(Y)$ write $\mathcal{H}om_{Shv}(F_1, F_2) \in \mathrm{Vect}$ for the relative inner hom for the Vect-action on $Shv(Y)$. For ind-schemes or Artin stacks $\mathbb{D}$ satisfies the formula

$$\mathcal{H}om_{Shv}(\mathbb{D}(F_1), F_2) \widetilde{\to} \mathrm{R}\Gamma(Y, F_1 \otimes^! F_2)$$

for $F_1 \in Shv(Y)^{constr}$. This property charaterizes $\mathbb{D}(F_1)$ uniquely. For example see ([1], F.2.5, F.1.3, F.4).

0.0.5. For $Y \in \mathrm{PreStk}_{lft}$ and $F_i \in Shv(Y)$ write $\mathcal{H}om^!(F_1, F_2)$ for the relative inner hom in $Shv(Y)$ for the !-pointwise monoidal structure. For Y smooth of dimension n we get $\mathrm{R}\Gamma\, \mathcal{H}om^!(F_1, F_2)[-2n] \widetilde{\to} \mathcal{H}om_{Shv}(F_1, F_2)$.

In which generality the category $Shv(Y)$ admits a symmetric monoidal structure given by $(F_1, F_2) \mapsto F_1 \otimes F_2 = d^*(F_1 \boxtimes F_2)$ for the diagonal $d : Y \to Y \times Y$? This should be always the case in the constructible context, and we reserve the notation $\otimes$ for this tensor product structure on $Shv(Y)$.

If the monoidal structure on $Shv(Y)$ given by $\otimes$ exists, we reserve the notation $\mathcal{H}om(F_1, F_2)$ for the inner hom for $Shv(Y)$ for this monoidal structure.

Lemma 0.0.6. *In the constructible context the Verdier duality for a scheme Y of finite type (or an Artin stack locally of finite type with an affine diagonal) satisfies a stronger property: for $F_1 \in Shv(Y)^{constr}, F_2 \in Shv(Y)$,*

$$\mathcal{H}om(\mathbb{D}(F_1), F_2) \widetilde{\to} F_1 \otimes^! F_2$$

in $Shv(Y)$.

Proof. For a map $f : S \to Y$ with $S \in \mathrm{Sch}^{aff}$ let us construct an isomorphism $f^! \mathcal{H}om(\mathbb{D}(F_1), F_2) \widetilde{\to} f^!(F_1 \otimes^! F_2)$ in a way compatible with compositions $S' \to S$ for $S' \in Sch^{aff}$. We have

$$f^! \mathcal{H}om(\mathbb{D}(F_1), F_2) \widetilde{\to} \mathcal{H}om(\mathbb{D}(f^! F_1), f^! F_2) \widetilde{\to} (f^! F) \otimes^! (f^! F_2) \widetilde{\to} f^!(F_1 \otimes^! F_2)$$

as desired. □

For $F\in Shv(Y)^c$ the functor $Shv(Y)\to Shv(Y), G\mapsto \mathcal{H}om(F,G)$ preserves filtered colimits.

If we assume that there is an adjoint pair $p^*:\mathrm{Vect}\leftrightarrows Shv(Y):p_*$ for $p:Y\to\operatorname{Spec}k$ then given $F_i\in Shv(Y)$ we get

$$\mathcal{H}om_{Shv}(F_1,F_2)\widetilde{\to}\mathrm{R}\Gamma\,\mathcal{H}om(F_1,F_2)$$

0.0.7. For which maps $f:Y\to\operatorname{Spec}k$ the functor f^* is defined on e? If defined, it gives the constant sheaf on Y. This happens at least for algebraic stack locally of finite type (with an affine diagonal).

Let now Y be a scheme of finite type or an algebraic stack locally of finite type (with an affine diagobal). Assume we are in the constructible context and $F\in Shv(Y)^{constr}$. Then the functor $Shv(Y)\to Shv(Y), K\mapsto K\otimes^! F$ admits a continuous left adjoint given by $K\mapsto K\otimes(\mathbb{D}F)$. Indeed, for $L\in Shv(Y)$ we get

$$\mathcal{H}om(L,K\otimes^! F)\widetilde{\to}\mathcal{H}om(L,\mathcal{H}om(\mathbb{D}F,K))\widetilde{\to}\mathcal{H}om(L\otimes(\mathbb{D}F),K)$$

Recall that here $\mathcal{H}om$ denotes the inner hom in $(Shv(Y),\otimes)$.

Claim Let $X,Y\in Sch_{ft}$. Note that the exteriour product $h:Shv(X)\otimes Shv(Y)\to Shv(X\times Y)$ is a map of $Shv(X)\otimes Shv(Y)$-modules, where the action of $L\in Shv(X)$ (resp., of $L'\in Shv(Y)$) on $Shv(X\times Y)$ sends K to $(L\boxtimes\omega)\otimes^! K$ (resp., to $(\omega\boxtimes L')\otimes^! K$). So, its right adjoint h^R is a right-lax morphism of $Shv(X)\otimes Shv(Y)$-modules. In fact, this right-lax structure is strict.

Proof. Let $K\in Shv(X\times Y)$ and $F\in Shv(X)$. We must show that the natural map $(F\boxtimes\omega)\otimes h^R(K)\to h^R((F\boxtimes\omega)\otimes^! K)$ is an isomorphism in $Shv(X)\otimes Shv(Y)$. We may and do assume $F\in Shv(X)^c$. It is understood that $Shv(Y), Shv(X)$ is equipped with the $\otimes^!$-symmetric monoidal structures, so $Shv(X)\otimes Shv(Y)$ is also symmetric monoidal. By the above, the functor $Shv(Y)\otimes Shv(Y)\to Shv(Y)\otimes Shv(Y)$, $S\mapsto (F\boxtimes\omega)\otimes S$ admits a continuous left adjoint sending $K_1\boxtimes K_2$ to $(K_1\otimes\mathbb{D}(F))\boxtimes K_2$ for $K_i\in Shv(Y)$.

Now for $K_1\in Shv(X), K_2\in Shv(Y)$ we get

$$\mathrm{Map}_{Shv(X)\otimes Shv(Y)}(K_1\boxtimes K_2, h^R((F\boxtimes\omega)\otimes^! K)\widetilde{\to}\mathrm{Map}_{Shv(X\times Y)}(K_1\boxtimes K_2,(F\boxtimes\omega)\otimes^! K)$$
$$\widetilde{\to}\mathrm{Map}_{Shv(X\times Y)}((K_1\otimes(\mathbb{D}F))\boxtimes K_2, K)\widetilde{\to}\mathrm{Map}_{Shv(X)\otimes Shv(Y)}((K_1\otimes(\mathbb{D}F))\boxtimes K_2, h^R(K))$$
$$\widetilde{\to}\mathrm{Map}_{Shv(X)\otimes Shv(Y)}(K_1\boxtimes K_2,(F\boxtimes\omega)\otimes h^R(K))$$

Let us underline that in the above formulas $(F\boxtimes\omega)\otimes h^R(K)$ denotes the tensor product in the symmetric monoidal category $Shv(X)\otimes Shv(Y)$. □

Recall also that h^R coincides with $h^\vee$ with respect to the Verdier self-dualities, see ([21], Sect. 1.0.1).

0.0.8. If $i:Z'\to Z$ is a closed immersion and $F\in Shv(Z)$ satisfies $i^!F=0$ then F is in the essential image of $j_*:Shv(Z-Z')\to Shv(Z)$. Here $j:Z-Z'\to Z$. For $F\in Shv(Z)$ one as a fibre sequence

$$i_!i^!F\to F\to j_*j^!F$$

in $Shv(Z)$. In particular, if $M \in Shv(Z)$ satisfies $j^*M = 0$ then M is in the essential image of $i_!$.

0.0.9. Let S be an ind-scheme of ind-finite type. In the constructible context, the tensor product $\otimes^! : Shv(S) \otimes Shv(S) \to Shv(S)$ admits a continuous right adjoint. Indeed, for $F_i \in Shv(S)^c$ it suffices to show that $F_1 \otimes^! F_2$ is compact. For this, it suffices to show that $\mathbb{D}(\vartriangle^* (\mathbb{D}F_1 \boxtimes \mathbb{D}F_2))$ is compact, and in turn that $\vartriangle^* (\mathbb{D}F_1 \boxtimes \mathbb{D}F_2))$ is compact. This is true, because for $\vartriangle : S \to S \times S$, $\vartriangle^*$ has a continuous right adjoint $\vartriangle_*$.

This is not the case for $\mathcal{D}$-modules, as far as I understand, because $\vartriangle^*$ is not always defined.

0.0.10. What are the t-structures on $Shv(Y)$ and under which assumptions and how they are defined? Perverse one, usual one?

For $Y \in \mathrm{Sch}_{ft}$ there is a t-structure on $Shv(Y)$ that we think of as being perverse. It is important that this t-structure is accessible. It is also compatible with filtered colimits (this reduces to the fact that the t-structure on Vect is compatible with filtered colimits).

The t-structure on $Shv(Y)$ for Y an ind-scheme is defined as follows. If $Y = \operatorname{colim}_{i \in I} Y_i$ with I filtered and $Y_i \in \mathrm{Sch}_{ft}$ then $Shv(Y)^{\le 0} \subset Shv(Y)$ should be the smallest full subcategory containing $Shv(Y_i)^{\le 0}$ for any i, closed under extensions and closed under small colimits. By (HA, 1.4.4.11), $Shv(Y)^{\le 0}$ is then presentable and defines an accessible t-structure on Y. We use here the fact that $Shv(Y_i)$ is generated by a small set of objects.

Note that for an ind-scheme Y of ind-finite type $F \in Shv(Y)$ lies in $Shv(Y)^{\ge 0}$ iff for any closed subscheme $i : Y' \subset Y$ one has $i^! F \in Shv(Y')^{\ge 0}$. This implies that the t-structure on $Shv(Y)$ is compatible with filtered colimits.

You should also explain what is assumed about right or left completeness of the t-structure on $Shv(S)$ for $S \in \mathrm{Sch}_{ft}$. Apparently, you assume it is right complete, as you want to use maps like $\mathrm{D}^+(Shv(Y)^\heartsuit) \to Shv(Y)$?

For an algebraic stack with an affine diagonal Y we define the perverse t-structure on $Shv(Y)$ by

$$Shv(Y)^{\le 0} \widetilde{\to} \lim_{S \overset{\alpha}{\to} Y} Shv(S)^{\le -\operatorname{dim.rel}(\alpha)},$$

where the limit is over the category whose objects are smooth maps $\alpha : S \to Y$ with $S \in \mathrm{Sch}_{ft}$, and morphisms from (S, α) to (S', α') is a smooth map $S \to S'$ compatible with α, α'. The transition functors here are the !-pullbacks. This defines an accessible t-structure by ([15], 1.4.4.11) or better by ([6], ch. I.3, Lemma 1.5.8). We have $Shv(Y)^{>0} \widetilde{\to} \lim_{S \overset{\alpha}{\to} Y} Shv(S)^{>-\operatorname{dim.rel}(\alpha)}$ taken over the same category with the transition functors being !-pullbacks. This t-structure is compatible with filtered colimits and both left and right complete by *loc.cit.*

Claim. If Y is an algebraic stack with an affine diagonal then in the constructible context $Shv(Y)$ is right complete.

Proof. The t-structure on $Shv(Y)$ is accessible, so by ([20], 4.0.10) it suffices to show that for $L \in Shv(Y)$ the natural map $\operatorname{colim}_n \tau^{\le n} L \to L$ is an isomorphism. This property is local in Zariski topology, so it suffices to show this is an isomorphism over any open quasi-compact substack $U \subset Y$.

For each U the category $Shv(U)$ is right complete. Indeed, we have an adjoint pair $ren_U : Shv(U) \leftrightarrows Shv(U)^{ren} : un - ren_U$ in DGCat_{cont} as in ([1], F.5.3) with ren_U fully faithful. The t-structure on $Shv(U)^{ren}$ is right complete by ([20], 9.3.18). The t-structure on $Shv(U)$ is accessible, so by ([20], 4.0.10) it suffices to show that for $K \in Shv(U)$ the natural map $\mathrm{colim}_n \tau^{\le n} K \to K$ is an isomorphism in $Shv(U)$. To see this, let $K' = ren_U(K)$. Then the natural map $\mathrm{colim}_n \tau^{\le n} K' \to K'$ is an isomorphism in $Shv(U)^{ren}$. Since $un - ren_U$ is t-exact, $K \widetilde{\to} un - ren_U(K')$ identifies with

$$\mathrm{colim}_n\, un - ren_U(\tau^{\le n} K') \widetilde{\to} \mathrm{colim}_n\, \tau^{\le n}(un - ren_U(K')) \widetilde{\to} \mathrm{colim}_n\, \tau^{\le n}(K)$$

We are done. □

0.0.11. For $S \in \mathrm{Sch}_{ft}$ mention that $Shv(S)$ is assumed compactly generated. So, for an ind-scheme of ind-finite type Y, $Shv(Y)$ is also compactly generated. Moreover, the Verdier duality provides an equivalence $Shv(Y)^\vee \widetilde{\to} Shv(Y)$. The corresponding map $Shv(Y) \otimes Shv(Y) \to \mathrm{Vect}$ sends (F_1, F_2) to $\mathrm{R}\Gamma(Y, F_1 \otimes^! F_2)$.

If now $f : Y_1 \to Y_2$ is a morphism of ind-schemes of ind-finite type then the dual to $f^! : Shv(Y_2) \to Shv(Y_1)$ identifies with $f_* : Shv(Y_1) \to Shv(Y_2)$.

If moreover, we are in the **constructible context**, since $(f_!, f^!)$ is an adjoint pair, its dual $((f^!)^\vee, (f_!)^\vee)$ is also an adjoint pair. So, the dual to $f_! : Shv(Y_1) \to Shv(Y_2)$ is the right adjoint to $f_* : Shv(Y_1) \to Shv(Y_2)$.

Assume $f : Y_1 \to Y_2$ schematic of finite type. In the conctructible context, f_* has a left adjoint f^*, hence $(f^!, (f^*)^\vee)$ is an adjoint pair, so $f^!$ has a continuous right adjoint.

Example: let T be a split torus. Then e on $B(T)$ is not compact in the constructible context, that is $\mathrm{R}\Gamma : Shv(B(T)) \to \mathrm{Vect}$ is not continuous. So, this functor can not be the dual of $f^!$ for $f : B(T) \to \mathrm{Spec}\, k$.

There is a projection formula for maps $f : Y \to Y'$, where Y is a quasi-compact classical algebraic stack with affine diagonal and Verdier compatible, it is formulated in ([2], B). This f_* satisfies the projection formula (even if not continuous).

0.0.12. Consider the 1-full subcategory $\mathrm{PreStk}_{ind-sch} \subset \mathrm{PreStk}_{lft}$, where we restrict 1-morphisms to be ind-schematic. Then we have a well-defined functor

$$Shv_{\mathrm{PreStk}_{ind-sch}} : \mathrm{PreStk}_{ind-sch} \to \mathrm{DGCat}_{cont}$$

sending Y to $Shv(Y)$ and a morphism $f : Y \to Y'$ to $f_* : Shv(Y) \to Shv(Y')$. Moreover, this functor is right-lax symmetric monoidal, so sends algebras to algebras. So, if G is an algebra in $\mathrm{PreStk}_{ind-sch}$, $(Shv(G), \star)$ will become a monoidal DG-category with the monoidal convolution structure.

So, we may talk about strong actions of $Shv(G)$ on some $C \in \mathrm{DGCat}$, this is an object of $(Shv(G), \star) - mod(\mathrm{DGCat})$.

0.0.13. If G is an ind-scheme of ind-finite type, assume $m : G \times G \to G$ ind-proper. Then $(Shv(G), \star)$ is rigid for any sheaf theory. My understanding is that there is no hope for it to be rigid without the ind-properness assumption.

0.0.14. If G is a group ind-scheme of ind-finite type then $(Shv(G), m_*)$ is monoidal (convolution monoidal structure).

The functor $Shv(G) \otimes Shv(G) \to Shv(G \times G)$ sends a compact object $F_1 \otimes F_2$ to a compact object $F_1 \boxtimes F_2$. [1] So, this functor admits a continuous right adjoint. In the contstructible context the functor $m_* : Shv(G \times G) \to Shv(G)$ admits a continuous right adjoint. Besides, the dual to m_* is the functor $m^!$. Thus, passing to the dual in $(Shv(G), m_*)$, in the constructible context we get a coalgebra $(Shv(G), m^!)$ in DGCat_{cont}. Recall that $(Shv(G), m_*) - mod \widetilde{\to} (Shv(G), m^!) - comod$ (cf. [20]).

For any ind-scheme of ind-finite type Y, Y is a cocommutative coalgebra in PreStk_{lft} via the maps $Y \to Y \times Y$ and $Y \to \operatorname{Spec} k$, hence a commutative algebra in $(\mathrm{PreStk}_{lft})^{op}$. Applying the right-lax monoidal functor Shv, we get on $Shv(Y)$ a commutative algebra structure in $CAlg(\mathrm{DGCat}_{cont})$. The product is $Shv(Y) \otimes Shv(Y) \to Shv(Y \times Y) \stackrel{\vartriangle^!}{\to} Shv(Y)$. We denote this algebra $(Shv(Y), \vartriangle^!)$. It makes sense for any sheaf theory. Applying the duality, we get a coalgebra structure on $Shv(Y)$, which we denote $(Shv(Y), \vartriangle_*)$ following [3]. Recall that this duality exchanges the functors $\vartriangle_*$ and $\vartriangle^!$.

Sam says $(Shv(G), \vartriangle^!, m^!)$ is probably not a Hopf algebra in the constructible context (only for $\mathcal{D}$-modules). Similarly for $(Shv(G), m_*, \vartriangle_*)$. For $\mathcal{D}$-modules this was explained in [3]. Though $(Shv(G), m_*) - mod$ is a symmetric monoidal category for $\mathcal{D}$-modules, this does not seem to be the case in the constructible context.

Sam's idea: if this was the case, consider the diagonal action of $(Shv(G), m_*)$ on $Shv(G) \otimes Shv(G)$. It is given by a map of algebras $h^R \circ \Delta_* : Shv(G) \to Shv(G) \otimes Shv(G)$, which is the coproduct. Here $h : Shv(G) \otimes Shv(G) \hookrightarrow Shv(G \times G)$ is the exteriour product, and h^R is its right adjoint. Besides, $\Delta_* : Shv(G) \to Shv(G \times G)$ is a morphism in $Alg(\mathrm{DGCat}_{cont})$. Is it true that h^R or h then becomes a morphism in $Shv(G) - mod$? Then we could consider the map between the invariants, hopefully to get a contradiction. We have in mind that $\Delta_* \omega_G$ is invariant under the diagonal action, but does not lie in the essential image of h, here $\Delta : G \to G \times G$ is the diagonal. Not clear.

0.0.15. If $Y \in \mathrm{PreStk}_{lft}$ is equipped with a G-action then the action map $a : G \times Y \to Y$ is ind-schematic (isomorphic to the projection $Y \times G \to Y$). So, $(Shv(G), \star)$ acts on $Shv(Y)$ on the left via $F \in Shv(G), K \in Shv(Y) \mapsto a_*(F \boxtimes K)$. If $f : Y_1 \to Y_2$ is an ind-schematic morphism in PreStk_{lft} commuting with G-actions then $f_* : Shv(Y_1) \to Shv(Y_2)$ is a map of $(Shv(G), \star)$-modules. Besides, $f^!$ is a map of $(Shv(G), \star)$-modules. Consider the prestack quotient $Y/G \in \mathrm{PreStk}_{lft}$. The map $f : Y \to Y/G$ commutes with G-actions, where G acts trivially on Y/G. So, $f^! : Shv(Y/G) \to Shv(Y)$ is a map of $(Shv(G), \star)$-modules. Thus, it induces a functor

$$Shv(Y/G) \to \mathrm{Fun}_{(Shv(G), \star)}(\mathrm{Vect}, Shv(Y)) \tag{2}$$

Is it an equivalence?

[1] Is it true for any sheaf theory? In ([9], 1.2.5(b)) you mentioned this only for two sheaf theories, but not for constructible sheaves in the classical topology. I imagine this is a misprint there! You actually claim this for any placid ind-schemes Y_1, Y_2 in ([9], C.2.8), so I assume this is true for any sheaf theory.

0.0.16. In general the answer is not clear. Assume G smooth of finite type. Then this is an equivalence, as Lin Chen shows (there is a different proof in ([9], 1.4.5)). Here is his argument.

One shows that $Shv(Y/G)\widetilde{\to} e-comod(Shv(Y))$ by verifying the comonadic Beck-Chevalley conditions. Here e is the constant sheaf on G, it is a coalgebra in $(Shv(G),\star)$, and we consider the corresponding category of comodules with the convolution action of $Shv(G)$ on $Shv(Y)$. The forgetful functor $e-comod(Shv(Y))\to Shv(Y)$ is $f^!$ for $f: Y\to Y/G$. The self-functor underlying the comonad is $p_*a^*: Shv(Y)\to Shv(Y)$. It also identifies with a_*p^*, here $a: G\times Y\to Y$ is the action map, $p: G\times Y\to Y$ is the projection.

Since $Shv(G)$ is self-dual, $Shv(Y)^G$ identifies with the limit of

$$Shv(Y)\rightrightarrows Shv(G)\otimes Shv(Y) \overset{\to}{\underset{\to}{\to}} Shv(G)^{\otimes 2}\otimes Shv(Y)\dots$$

(For $\mathcal{D}$-modules, since $Shv(G)^{\otimes n}\otimes Shv(Y)\widetilde{\to} Shv(G^n\times Y)$, this finishes the proof). Assume now we are in the constructible context.

The above cosimplicial diagram is also

$$Shv(Y)\rightrightarrows \mathrm{Fun}(Shv(G), Shv(Y)) \overset{\to}{\underset{\to}{\to}} \mathrm{Fun}(Shv(G)^{\otimes 2}, Shv(Y))\dots$$

The functors $Shv(Y)\rightrightarrows \mathrm{Fun}(Shv(G), Shv(Y))$ are: F goes to $(K\mapsto K*F)$, and F goes to $(K\mapsto \mathrm{R}\Gamma(G,K)\otimes F)$. The second functor identifies via the Verdier duality with $Shv(Y)\to Shv(G)\otimes Shv(Y)$, $F\mapsto \omega_G\otimes F$. Its right adjoint is $p_*[-2n]\otimes\mathrm{id}: Shv(G)\otimes Shv(Y)\to Shv(Y)$ for $p: G\to\operatorname{Spec} k$, where $n=\dim G$.

The comonadic Beck-Chevalley condition for the above cosimplicial diagram holds, it is mentioned in [9], 1.4.6 without a proof. We also check this in bigger generality in Section 0.0.23 of this file.

The corresponding comonad on $Shv(Y)$ is $Shv(Y)\to \mathrm{Fun}(Shv(G), Shv(Y))\overset{T^0}{\to} Shv(Y)$, where the first functor sends F to $(K\mapsto K*F)$. Thus, this comonad sends F to $e*F$. We see that both comonads are the same.

0.0.17. Let G be a smooth group scheme of finite type, $Y\in \mathrm{PreStk}_{lft}$. The equivalence $Shv(B(G))\widetilde{\to}\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect},\mathrm{Vect})$ given by (2) transforms the symmetric monoidal structure on $Shv(B(G))$ given by $\otimes^!$ to the composition monoidal structure on $\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect},\mathrm{Vect})$.

The projection $q: Y/G\to B(G)$ yields an action of $(Shv(B(G)),\otimes^!)$ on $Shv(Y/G)$. Namely, $K\in Shv(B(G))$ acts on $M\in Shv(Y/G)$ as $(q^!K)\otimes^! M$. Similarly, the monoidal category $\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect},\mathrm{Vect})$ acts on $\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect}, Shv(Y))$ by composition on the left. The equivalence (2) is compatible with these actions via the above monoidal equivalence

$$Shv(B(G))\widetilde{\to}\mathrm{Fun}_{(Shv(G),\star)}(\mathrm{Vect},\mathrm{Vect})$$

0.0.18. We need the following claim: for $Y\in\mathrm{Sch}_{ft}$, its cohomology $C(Y)$ is bounded, and the dimension of each H^i is finite. It was used in ([9], B.3.1) to show that for a smooth group scheme of finite type H and $C\in Shv(H)-mod$, $C_H\to C^H$ is an equivalence. In the constructible context this is automatic, because $p_*: Shv(Y) \to \mathrm{Vect}$ for $p: Y \to \operatorname{Spec} k$ admits a continuous right adjoint, and the constant sheaf e_Y is compact, so $p_*(e_Y)$ is also compact.

So, a suitable finiteness assumption on the functor p_* should be formulated which holds for any sheaf theory. How it is formulated?

0.0.19. Consider a cartesian square

$$\begin{array}{ccc} X & \overset{f_X}{\to} & X' \\ \downarrow g & & \downarrow g' \\ Y & \overset{f_Y}{\to} & Y', \end{array} \tag{3}$$

in PreStk, where all objects are placid ind-schemes. For which morphisms g' we have the functors $(g')^!, (g')^*$? When do we have the base change with respect to $(f_Y)_*$?

Lemma 0.0.20. *let $Y' \in Sch_{ft}$ and Y, X' be placid schemes over Y', recall then X is also a placid scheme. Assume $Y \widetilde{\to} \lim_{i\in I^{op}} Y_i$, where I is filtered, $f_{Y,i}: Y_i \to Y'$ is smooth, $Y_i \in Sch_{ft}$, and for $i \to j$ in I, $Y_j \to Y_i$ is smooth affine surjective morphism in Sch_{ft}. Then one has $f_Y^* g'_* \widetilde{\to} g_* f_X^*$.*

Proof. 1) Assume first $g': X' \to Y'$ a morphism in Sch_{ft}. Set $X_i = Y_i \times_{Y'} X'$ for $i \in I$, so $X \widetilde{\to} \lim_{i\in I^{op}} X_i$. For each i we get a cartesian square

$$\begin{array}{ccc} X_i & \overset{f_{X,i}}{\to} & X' \\ \downarrow g_i & & \downarrow g' \\ Y_i & \overset{f_{Y,i}}{\to} & Y', \end{array}$$

So, $f_{Y,i}^* g'_* \widetilde{\to} (g_i)_* f_{X,i}^*$ naturally. So, $(g_i)_*$ form a morphism of the corresponding colmit systems giving $g_*: Shv(X) \widetilde{\to} \operatorname{colim}_{i\in I} Shv(X_i) \to \operatorname{colim}_{i\in I} Shv(Y_i) \widetilde{\to} Shv(Y)$. The claim follows.

2) Let now $g': X' \to Y'$ be any placid scheme over Y'. Write $X' \widetilde{\to} \lim_{j\in J} X'_j$ with $X'_j \in Sch_{ft}$, J filtered, and for $j \to j'$ in J the map $X'_{j'} \to X'_j$ is smooth affine and surjective. Set $X_j = Y \times_{Y'} X'_j$ for $j \in J$. Then X_j is a placid scheme, and we get the diagram

$$\begin{array}{ccc} X_j & \overset{f_{X,j}}{\to} & X'_j \\ \downarrow g_j & & \downarrow g'_j \\ Y & \overset{f_Y}{\to} & Y', \end{array}$$

for $j \in J$. Note that $Shv(X) \widetilde{\to} \lim_{j\in J^{op}} Shv(X_j)$ with respect to the $*$-direct image transition functors. By 1), for each $j \in J$,

$$f_Y^*(g'_j)_* \widetilde{\to} (g_j)_* f_{X,j}^* \tag{4}$$

naturally. The functors $f_{X,j}^*$ are compatible with the corresponding inverse systems and give in the limit over J^{op} the functor f_X^*. Pick any $j \in J$. Then g' is the composition $X' \overset{ev'_j}{\to} X'_j \overset{g'_j}{\to} Y'$. Since $(f_{X,j}^*)(ev'_j)_* \widetilde{\to} (ev_j)_* f_X^*$ our claim follows from (4). $\square$

Lemma 0.0.21. *Let $S \in \mathrm{Sch}_{ft}$, assume given a cartesian square in* $\mathrm{PreStk}_{/S}$

$$\begin{array}{ccc} Y & \overset{g}{\leftarrow} & Y' \\ \downarrow f & & \downarrow f' \\ Z & \overset{h}{\leftarrow} & Z' \end{array}$$

Assume I is a filtered category, and we are given a morphism $f_i : Y_i \to Z_i$ in $(\mathrm{Sch}_{ft})_{/S}$ functorial in $i \in I^{op}$, where f_i is smooth. We assume for $i \to j$ in I the transition maps $Y_j \to Y_i$, $Z_j \to Z_i$ are smooth affine surjective. We assume that $f : Y \to Z$ is obtained from f_i by passing to the limit over I^{op}. We assume $i : S' \hookrightarrow S$ is a closed immersion, and $f' : Y' \to Z'$ is obtained from f by the base change $S' \to S$. Then $g^! f^* \widetilde{\to} (f')^* h^!$ naturally. We do not assume here that the squares

$$\begin{array}{ccc} Y_j & \to & Y_i \\ \downarrow f_j & & \downarrow f_i \\ Z_j & \to & Z_i \end{array}$$

are cartesian.

Proof. By definition, $f^* : Shv(Z) \to Shv(Y)$ is obtained by passing to the limit over I^{op} in $f_i^* : Shv(Z_i) \to Shv(Y_i)$. Note that h, g are placid closed immersions, and Y, Y', Z, Z' are placid S-schemes. Note that $h^!$ is obtained by passing to the limit over I^{op} in $h_i^! : Shv(Z_i) \to Shv(Z_i')$, and similarly for $g^!$. Since $Shv(Z) \widetilde{\to} \mathrm{colim}_{i \in I} Shv(Z_i)$, it suffices to prove that for any $i \in I$ and $K \in Shv(Z_i)$ the desired isomorphism $g^! f^* K \widetilde{\to} (f')^* h^! K$ holds.

We have $g_i^! f_i^* K \widetilde{\to} (f_i')^* h_i^! K$ canonically. Applying ev_i^* to both sides, one gets the desired claim. □

0.0.22. Let Z be a placid scheme written as $Z = \lim_{i \in I^{op}} Z_i$. For $i \to j$ in I let $f_{ij} : Z_j \to Z_i$ be the corresponding morphism, it is smooth of relative dimension d_{ij}, affine, surjective. Since $Shv(Z) \widetilde{\to} \mathrm{colim}_i Shv(Z_i)$ via the maps f_{ij}^*, $Shv(Z)$ is compactly generated, hence dualizable. By ([6], ch. I.1, 6.3.4), by applying the dualization functor to the functor

$$I \to \mathrm{DGCat}_{cont}, i \mapsto Shv(Z_i),\ (i \to j) \mapsto f_{ij}^*,$$

we get a functor $I^{op} \to \mathrm{DGCat}_{cont}, i \mapsto Shv(Z_i)$, $(i \to j) \mapsto (f_{ij})_*[-2d_{ij}]$. Moreover,

$$Shv(Z)^\vee \widetilde{\to} \lim_{i \in I^{op}} Shv(Z_i)$$

with respect to the transition maps $(f_{ij})_*[-2d_{ij}]$. Consider for $i \in I$ the isomorphism $Shv(Z_i) \overset{\otimes e[2d_i]}{\to} Shv(Z_i)$ with $d_i = \dim Z_i$. So, $d_{ij} = d_j - d_i$. The diagram commutes

$$\begin{array}{ccc} Shv(Z_j) & \overset{\otimes e[2d_j]}{\to} & Shv(Z_j) \\ \downarrow (f_{ij})_* & & \downarrow (f_{ij})_*[-2d_{ij}] \\ Shv(Z_i) & \overset{\otimes e[2d_i]}{\to} & Shv(Z_i) \end{array}$$

Passing to the limit over I^{op}, we obtain an equivalence $Shv(Z) \widetilde{\to} Shv(Z)^\vee$. So, a possibility is to mention that for each placid scheme Z, $Shv(Z)$ is canonically self-dual.

However, this self-duality is not compatible with the one for finite type schemes, so maybe it is not needed.

Example: assume $0 \in I$ is an initial object, let $K_0 \in Shv(Z_0)$. For the projection $f_0 : Z \to Z_0$ the image of $f_0^* K_0$ in $Shv(Z)^\vee$ under this duality is the composition $Shv(Z) \stackrel{(f_0)_*}{\to} Shv(Z_0) \to \mathrm{Vect}$, where the second functor is $M \mapsto \mathrm{R}\Gamma(Z_0, K_0 \otimes^! M)[2d_0]$.

0.0.23. Let G be a group scheme, which is a placid scheme, $C \in G-mod$. Consider the cosimplicial category defining C^G:

$$\mathrm{Fun}(\mathrm{Vect}, C) \rightrightarrows \mathrm{Fun}(Shv(G), C) \substack{\to\\ \to\\ \to} \mathrm{Fun}(Shv(G)^{\otimes 2}, C) \dots$$

Let us show that it satisfies the comonadic Beck-Chevalley conditions.

The functor corresponding to the last face map $\partial_n : [n] \to [n+1]$ (its image avoids $n+1$) is the following functor F_n. We consider $Shv(G)^{\otimes n+1} \to Shv(G)^{\otimes n}$, $\mathrm{id} \otimes \mathrm{R}\Gamma$, and compose it with $\mathrm{Fun}(\cdot, C)$. For $p : G \to \mathrm{Spec}\, k$ the functor p_* has a left adjoint p^*. Let T_n be the functor obtained from $\mathrm{id} \otimes p^* : Shv(G)^{\otimes n} \to Shv(G)^{\otimes n+1}$ by composing with $\mathrm{Fun}(\cdot, C)$. Then T_n is the right adjoint to F_n. Let now $\alpha : [m] \to [n]$ be a map in $\boldsymbol{\Delta}$. Consider the corresponding diagram

$$\begin{array}{ccc} \mathrm{Fun}(Shv(G)^{\otimes n}, C) & \stackrel{T_n}{\leftarrow} & \mathrm{Fun}(Shv(G)^{\otimes n+1}, C) \\ \uparrow F_\alpha & & \uparrow F_{\alpha+1} \\ \mathrm{Fun}(Shv(G)^{\otimes m}, C) & \stackrel{T_m}{\leftarrow} & \mathrm{Fun}(Shv(G)^{\otimes m+1}, C) \end{array}$$

We show that it commutes. It suffices to prove this for α injective, becase of the following. Let $\boldsymbol{\Delta}_s \subset \boldsymbol{\Delta}$ be the full subcategory with the same class of object, where we keep only injective maps. Then $\boldsymbol{\Delta}_s^{op} \to \boldsymbol{\Delta}^{op}$ is cofinal by ([14], 6.5.3.7). If $\alpha : [m] \to [n]$ is injective, and $0, n$ are in the image then the desired commutativity follows from the commutativity of

$$\begin{array}{ccc} Shv(G)^{\otimes n} & \stackrel{\mathrm{id} \otimes p^*}{\to} & Shv(G)^{\otimes n+1} \\ \downarrow (m_\alpha)_* & & \downarrow (m_{\alpha+1})_* \\ Shv(G)^{\otimes m} & \stackrel{\mathrm{id} \otimes p^*}{\to} & Shv(G)^{\otimes m+1}, \end{array}$$

where $(m_\alpha)_*$ is the product along α in the monoidal category $Shv(G)$.

If $\alpha : [n-1] \to [n]$ is the last face map then $\alpha + 1 : [n] \to [n+1]$ avoids n. The functor $F_{\alpha+1}$ is the composition with $Shv(G)^{\otimes n+1} \to Shv(G)^{\otimes n}$, $K_1 \otimes \ldots \otimes K_{n+1} \mapsto K_1 \otimes \ldots \otimes K_{n-1} \otimes K_n * K_{n+1}$. In this case the desired commutativity follows from $K * e_G \widetilde{\to} \mathrm{R}\Gamma(G, K) \otimes e_G$.

If $\alpha : [n-1] \to [n]$ is injective and avoids 0 then F_α sends f to the functor

$$K_1 \otimes \ldots \otimes K_n \mapsto K_1 * f(K_2 \otimes \ldots \otimes K_n)$$

and the commutativity is tautological. So, it always hold.

By ([8], Lemma C.1.9), the functor $\mathrm{oblv}_G : C^G \to C$ is comonadic, and the corresponding comonad on C is $C \to C, c \mapsto e_G * c$.

0.0.24. Consider a placid scheme $Y = \lim_{i\in I^{op}} Y_i$, where I is filtered, if $i \to j$ in I then $f_{ij} : Y_j \to Y_i$ is smooth, affine and surjective, and Y_i is a scheme of finite type. In this case $Shv(Y)$ is defined in [9] as $\lim_{i\in I^{op}} Shv(Y_i)$ via the maps $(f_{ij})_*$.

In the paper there are situations, where we have morphisms $h : Y \to S$, where S is an ind-scheme, and we want functors between $Shv(Y)$ and $Shv(S)$ attached to h. So, the above definition of $Shv(Y)$ for placid schemes should be "unified" with the definition of $Shv(Z)$ for prestacks Z locally of finite type. Namely, do we have certain full subcategory of PreStk, on which Shv is defined as a functor, and which contains both PreStk_{lft}, placid schemes, and is closed under colimits? Compare with [27].

0.0.25. Let now Z, Z' be placid schemes and $i : Z' \to Z$ a placid closed immersion. What is the dual of the adjoint pair $i_* : Shv(Z') \to Shv(Z) : i^!$?

We explain the dual of i_*. If $Z = \lim_{i\in I^{op}} Z_i$ and, assume for simplicity I has an initial object i_0 such that $Z' = Z'_{i_0} \times_{Z_{i_0}} Z$. So, $Z' = \lim_{i\in I^{op}} Z'_i$ with $Z'_i = Z_i \times_{Z_{i_0}} Z'_{i_0}$. For $i \to j$ in I let $f_{ij} : Z_j \to Z_i$ be the corresponding transition map. For the closed embeddings $i_i : Z'_i \to Z_i$ writing $Shv(Z) = \lim_{i\in I^{op}} Shv(Z_i)$ for $(f_{ij})_* : Shv(Z_j) \to Shv(Z_i)$ and similarly for $Shv(Z')$, the dual functor is given by the collection of functors $i_i^![2d_i - 2d'_i] : Shv(Z_i) \to Shv(Z'_i)$, here $d_i = \dim Z_i, d'_i = \dim Z'_i$ as locally constant functions, they form a morphism of the corresponding inverse systems. The number $d_i - d'_i$ does depend on i, and can be denoted $\mathrm{codim}_Z(Z') = d_i - d'_i$. So, the dual of $i_* : Shv(Z') \to Shv(Z)$ is $i^![2\,\mathrm{codim}_Z(Z')]$.

0.0.26. Let Z be a placid ind-scheme. Is $Shv(Z)$ canonically self-dual? Here is some answer.

Write $Z = \mathrm{colim}_{i\in I} Z_i$ with Z_i a placid scheme, I small filtered, and for $i \to j$ the map $f_{ij} : Z_i \to Z_j$ is a placid closed immersion. We have $Shv(Z) = \mathrm{colim}_{i\in I} Shv(Z_i)$ with respect to the transition functors $(f_{ij})_*$.

Consider the functor $I \to \mathrm{DGCat}_{cont}$, $i \mapsto Shv(Z_i)$, $(i \to j) \mapsto (f_{ij})_*$. By ([6], ch. I.1, 6.3.4), the colimit of this functor $\mathrm{colim}_{i\in I} Shv(Z_i) = Shv(Z)$ is dualizable, and $Shv(Z)^\vee \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)^\vee$, the limit of the dual functor.

Recall for each i the canonical self-duality on $Shv(Z_i)$ introduced in Sect. 0.0.22 of this file. It allows to rewrite $Shv(Z)^\vee \widetilde{\to} \lim_{i\in I^{op}} Shv(Z_i)$, where the transition functors for $i \to j$ in I is $(f_{ij})^![2\,\mathrm{codim}_{Z_j}(Z_i)]$ in the notations of Section 0.0.25.

Pick an element $i_0 \in I$. Consider for $i \to j$ in I a commutative diagram

$$\begin{array}{ccc} Shv(Z_j) & \stackrel{\otimes e[-2\,\mathrm{codim}_{Z_j}(Z_{i_0})]}{\to} & Shv(Z_j) \\ \downarrow f_{ij}^! & & \downarrow f_{ij}^![2\,\mathrm{codim}_{Z_j}(Z_i)] \\ Shv(Z_i) & \stackrel{\otimes e[-2\,\mathrm{codim}_{Z_i}(Z_{i_0})]}{\to} & Shv(Z_i) \end{array}$$

Indeed, we have $\mathrm{codim}_{Z_j}(Z_i) + \mathrm{codim}_{Z_i}(Z_{i_0}) = \mathrm{codim}_{Z_j}(Z_{i_0})$. Passing to the limit over I^{op}, this provides an equivalence $Shv(Z)^\vee \widetilde{\to} Shv(Z)$.

This duality maybe depend on a choice of an element $i_0 \in I$.

0.0.27. In Section 7.3.5 the perverse t-structure on $Shv_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ is mentioned without any definition. In the convention section a definition of the perverse t-structure for an ind-algebraic stack should be given. My understanding is as follows: if $Y =$

$\mathrm{colim}_{i\in I} Y_i$ with Y_i an algebraic stack locally of finite type, I filtered then $Shv(Y)^{\le 0}$ should be the smallest full subcategory of $Shv(Y)$ containing $Shv(Y_i)^{\le 0}$ for any i, closed under extensions and small colimits. Then by (HA, 1.4.4.11), $Shv(Y)^{\le 0}$ is then presentable and defines an accessible t-structure on $Shv(Y)$. For $K \in Shv(Y)$ we have $K \in Shv(Y)^{\ge 0}$ iff for any i, the !-restriction of K to Y_i lies in $Shv(Y_i)^{\ge 0}$. As in the case of ind-schemes of ind-finite type, this t-structure is compatible with filtered colimits.

0.0.28. For a scheme of finite type S, the perverse t-structure on $Shv(S)$ is left complete (by [1], 1.1.4). This implies that for an Artin stack locally of finite type S the t-structure on $Shv(S)$ is left complete as in ([6], ch. I.3, 1.5.7), because for a smooth atlas $f : S' \to S$ with S a scheme locally of finite type, $f^*[\dim f]$ is t-exact.

It should be clarified for which topologies on Sch_{ft} the functor $Shv : (\mathrm{Sch}_{ft})^{op} \to \mathrm{DGCat}_{cont}$ satisfies the descent, and a precise reference should be given. In particular, in ([11], proof of 4.2.7) you claim it satisfies the descent for the topology of finite surjective maps on Sch_{ft}. Give also a reference for the fact that it satisfies the étale descent. (For the proper descent this is Section 0.0.33 of this file). Sam claim we get this way h-descent, give accurate references. Add also it satisfies the smooth descent: if Y is a quasi-compact algebraic stack with a smooth cover $S \to Y$, where $S \in Sch^{aff}_{ft}$, if $S^\bullet$ is the Cech nerve of this map then $Shv(Y) \to Tot(Shv(S^\bullet))$ is an equivalence. Does etale descent automatically implies the smooth descent here?

Add also the following. For a map $f : Y \to Z$ in PreStk_{lft}, which is surjective on geometric points, $f^!$ is conservative.

Cite the following. If $Y \in \mathrm{PreStk}_{lft}$ is an algebraic stack then $Shv(Y) = \lim_{S\to Y} Shv(S)$, where the limit is taken over the opposite to the category of affine schemes smooth over Y, and morphisms are smooth maps between those ([1], C.1.1).

0.0.29. Say that for any $Y \in \mathrm{PreStk}_{lft}$, $Shv(Y)$ is compactly generated in the constructible context by ([1], C.1.1). What happens for $\mathcal{D}$-modules?

In Section 4.3.3 you claimed the existence of the equivalence $\mathbb{D} : (Shv(Y)^c)^{op} \widetilde{\to} Shv(Y)^c$ for an algebraic stack of finite type. Explain that this is known only under the assumption that Y is locally a quotient of a scheme S of finite type by an affine algebraic group, give a reference!

0.0.30. For example, it should be said somewhere that if $Z = \lim_{i\in I^{op}} Z_i$ is a placid scheme, where I is filtered, Z_i is a scheme of finite type, with the transition maps affine smooth and surjective, then for $i \in I$ and the projection $ev_i : Z \to Z_i$ the functor $ev_i^* : Shv(Z_i) \to Shv(Z)$ is defined, this is the natural functor $ins_i : Shv(Z_i) \to \mathrm{colim}_{j\in I} Shv(Z_j)$. For the moment this is hidden in ([9], C.2.9).

0.0.31. *On exteriour product.* Let $S \in \mathrm{Sch}_{ft}$, $Y \in \mathrm{PreStk}_{lft}$. The functor $Shv(S) \otimes Shv(Y) \to Shv(S\times Y)$, $(F, K) \mapsto F\boxtimes K$ is fully faithful and preserves compactness. We have to verify this in the constructible context, as for $\mathcal{D}$-modules this is an equivalence. Fully-faithfulness property is preserved by passing to the limit, and tensoring by $Shv(S)$ is the functor $\mathrm{DGCat}_{cont} \to \mathrm{DGCat}_{cont}$ preserving limits, as $Shv(S)$ is dualizable. This is why our functor is fully faithful.

To see that it admits a continuous right adjoint we use ([6], ch. I.1, 2.6.4). Write $Shv(Y)\widetilde{\to}\lim_{T\to Y} Shv(T)$ over $(\mathrm{Sch}^{aff}_{ft})^{op}_{/Y}$. For each T the inclusion $i_T : Shv(S)\otimes Shv(T)\to Shv(S\times T)$ admits a continuous right adjoint i^R_T. Let $\alpha : T\to T'$ be a map in $(Sch^{aff}_{ft})_{/Y}$. In the constructible context, $\alpha^! : Shv(T')\to Shv(T)$ admits a left adjoint $\alpha_!$, and we have $i_{T'}(\mathrm{id}\otimes\alpha_!)\widetilde{\to}(\mathrm{id}\times\alpha_!)i_T$. This gives an isomorphism $\mathrm{id}\otimes\alpha^! i^R_{T'}\widetilde{\to} i^R_T(\mathrm{id}\times\alpha^!)$. By ([6], ch. I.1, 2.6.4), $i : Shv(S)\otimes Shv(Y)\to Shv(S\times Y)$ has a right adjoint i^R, and for any $(T\stackrel{b}{\to} Y)\in \mathrm{Sch}^{aff}_{ft})_{/Y}$ we have $(\mathrm{id}\otimes b^!)i^R\widetilde{\to} i^R_T(\mathrm{id}\times b)^!$.

We check that i^R is continuous. Let $K\widetilde{\to}\mathrm{colim}_{j\in J} K_j$ in $Shv(S\otimes Y)$. By ([20], 2.2.68), it suffices to show that for any $(T\stackrel{b}{\to} Y)\in \mathrm{Sch}^{aff}_{ft})_{/Y}$, $\mathrm{id}\otimes b^!$ sends our diagram to a colimit diagram. This is true, because i^R_T and $(\mathrm{id}\times b)^!$ are continuous.

Claim Let $X, Y, Z\in \mathrm{Sch}_{ft}$ with X proper. In the constructible context, the diagram commutes

$$\begin{array}{ccc} Shv(X\times Y\times Z) & \stackrel{\boxtimes}{\leftarrow} & Shv(X)\otimes Shv(Y\times Z)\\ \downarrow\boxtimes^R & & \downarrow\boxtimes^R\\ Shv(X\times Y)\otimes Shv(Z) & \stackrel{\boxtimes}{\leftarrow} & Shv(X)\otimes Shv(Y)\otimes Shv(Z)\end{array}$$

Proof. The left vertical arrow is $Shv(X\times Y)\otimes Shv(Z)$-linear by Section 0.0.7. Therefore, it suffices to calculate for $F\in Shv(Y\times Z)$ and the projection $q : X\times Y\times Z\to Y\times Z$ the object $\boxtimes^R(q^! F)\in Shv(X\times Y)\otimes Shv(Z)$. The functior $\boxtimes^R\circ q^!$ is right adjoint to the functor $Shv(X\times Y)\otimes Shv(Z)\stackrel{\boxtimes}{\to} Shv(X\times Y\times Z)\stackrel{q_!}{\to} Shv(Y\times Z)$. The latter functor identifies with the composition $Shv(X\times Y)\otimes Shv(Z)\stackrel{\bar q_!\otimes\mathrm{id}}{\to} Shv(Y)\otimes Shv(Z)\stackrel{\boxtimes}{\to} Shv(Y\times Z)$, because X is proper. Here $\bar q : X\times Y\to Y$ is the projection. So, $\boxtimes^R\circ q^!$ identifies with the functor $Shv(Y\times Z)\stackrel{\boxtimes^R}{\to} Shv(Y)\times Shv(Z)\stackrel{\bar q^!\otimes\mathrm{id}}{\to} Shv(X\times Y)\otimes Shv(Z)$. Our clailm follows. □

0.0.32. *Question.* Let $f : Y\to \mathrm{Spec}\, k$ be a scheme of finite type. In the constructible context does the functor $p_! : Shv(Y)\to \mathrm{Vect}$ preserve limits? Consider the dual functor $(p_!)^\vee : \mathrm{Vect}\to Shv(Y)$. Is the object $(p_!)^\vee(e)$ compact? If it was compact, the functor $p_!$ would preserve limits.

0.0.33. Let $Y, Z\in \mathrm{PreStk}_{lft}$ and $\pi : Y\to Z$ be proper, in particular, of finite type. Consider the Cech nerve $[\dots Y^3_Z \substack{\to\\\to\\\to} Y^2_Z \rightrightarrows Y]$ of π. Applying Shv, we get a cosimplicial category $\boldsymbol{\Delta}^{op}\to \mathrm{DGCat}_{cont}$, $[n]\mapsto Shv(Y^{n+1}_Z)$, here $Y^n_Z = Y\times_Z Y\times_Z\dots\times_Z Y$, the product of n copies. For $i\ge 0$ let $\partial_i : [i]\to[i+1]$ be the last face map, it avoids $i+1$. The corresponding map $p^{\partial_i} : Y^{i+1}_Z\to Y^i_Z$ is the projection, so $(p^{\partial_i})^!$ has a left adjoint $(p^{\partial_i})_! = (p^{\partial_i})_*$. By base change, this cosimplicial category satisfies the monadic Beck-Chevalley conditions, so

$$\underset{[n]\in\boldsymbol{\Delta}}{Tot}\, Shv(Y^{n+1}_Z)\widetilde{\to}\mathcal{A}-mod(Shv(Y)),$$

where $\mathcal{A} = (p_2)_* p^!_1$ for the projections $p_1, p_2 : Y^2_Z\to Y$.

Now $\pi^! : Shv(Z) \to Shv(Y)$ has a left adjoint $\pi_!$, and the monad $\pi^!\pi_!$ acting on $Shv(Y)$ identifies with $\mathcal{A}$. We always have a natural functor $Shv(Z) \to \mathcal{A} - mod(Shv(Y))$. Assume in addition that $\pi : Y \to Z$ is surjective on k-points. Then, as far as I understand, $\pi^!$ is conservative, so that $\pi^!$ satisfies the Beck-Chevalley theorem ([6], ch. I.1, 3.7.7), and the induced functor $Shv(Z) \to \mathcal{A} - mod(Shv(Y))$ is an equivalence. Thus, Shv satisfies the proper descent.

0.0.34. It seems the following is also needed. Consider the cartesian square (3), where all the maps are schematic quasi-compact say. Let $F \in Shv(X')$ such that $g'_! : Shv(X') \to Shv(Y')$ is defined on F and $f_Y^* g'_! F$ is defined. Then $f_X^* F$ and $g_! f_X^* F$ are both defined and we have a natural isomorphism $g_! f_X^* F \widetilde{\to} f_Y^* g'_! F$. Is this true?

I think this was used in ([10], proof of Prop. 2.8.2).

0.0.35. *Question.* Let Y be an ind-scheme of ind-finite type (or a classical algebraic stack locally of finite type). Let $U_i \subset Y$ be an open immersion for $i \in \mathbb{N}$ such that for $i < j$ we have $U_i \subset U_j$ and $\cup_i U_i = Y$. Is it true that $\operatorname{colim}_{i \in \mathbb{N}} U_i$ in PreStk identifies with Y?

Example: we may form a sequence of opens $U_i \subset \mathrm{Gr}_G$, where each U_i is of the form $\mathrm{Gr}_G - \cup_{i=1}^n \bar{S}^{\lambda_i}$ and

$$U_i \subset U_{i+1} \subset \dots$$

with $\cup_i U_i = \mathrm{Gr}_G$. We have $Shv(\mathrm{Gr}_G) \widetilde{\to} \lim_i Shv(U_i)$ anyway, as for any closed subscheme of finite type $S \subset \mathrm{Gr}_G$, $S \subset U_i$ for some i.

0.0.36. *Torsors under placid group-schemes.* Let $\mathcal{Y}_\alpha$ be an ind-scheme of ind-finite type functorial in $\alpha \in A^{op}$, where A is filtered, $\alpha_0 \in A$ is initial in A. Let $G = \lim_{\alpha \in A^{op}} G_\alpha$ be a placid group scheme, where G_α is a smooth group scheme of finite type, and for $\alpha \to \beta$ in A, $G_\beta \to G_\alpha$ is smooth, affine and surjective. Assume $\mathcal{Y}_\alpha \to \mathcal{Y}_{\alpha_0}$ is a G_α-torsor. For $\alpha \to \beta$ in A, $\mathcal{Y}_\beta \to \mathcal{Y}_\alpha$ is G_β-equivariant.

Then we are in the setting of ([9], C.1.6), so we get a placid ind-scheme $\mathcal{Y}$ as follows. Write $\mathcal{Y}_{\alpha_0} = \operatorname{colim}_{i \in I} Y_i$, where Y_i is a scheme of finite type, I is filtered, and for $i \to j$, $Y_i \to Y_j$ is a closed immersion. Let $\mathcal{Z}_i = \lim_{\alpha \in A^{op}} Y_i \times_{\mathcal{Y}_{\alpha_0}} \mathcal{Y}_\alpha$, so $\mathcal{Z}_i$ is a placid scheme, and $\mathcal{Z}_i \to \mathcal{Z}_j$ is a placid closed immersion. So, $\mathcal{Y} := \operatorname{colim} \mathcal{Z}_i$ is a placid ind-scheme, and $Shv(\mathcal{Y}) \widetilde{\to} \lim_{\alpha \in A^{op}} Shv(\mathcal{Y}_\alpha)$ with respect to the functors $(f_{\alpha,\beta})_* : Shv(\mathcal{Y}_\beta) \to Shv(\mathcal{Y}_\alpha)$ for $\alpha \to \beta$ in A and $f_{\alpha,\beta} : \mathcal{Y}_\beta \to \mathcal{Y}_\alpha$. The group G acts on $\mathcal{Y}_\alpha$ for each α via the quotient $G \to G_\alpha$, this gives an action of $Shv(G)$ on $Shv(\mathcal{Y}_\alpha)$. The functors $(f_{\alpha,\beta})_*$ are morphisms of $Shv(G)$-modules, so $Shv(\mathcal{Y})$ can be seen as $\lim_{\alpha \in A^{op}} Shv(\mathcal{Y}_\alpha)$ taken in $Shv(G) - mod$.

Let us show that $\mathcal{Y} \widetilde{\to} \lim_{\alpha \in A^{op}} \mathcal{Y}_\alpha$ as prestacks. We have a natural map $\mathcal{Y} \to \lim_\alpha \mathcal{Y}_\alpha$. Let $S \in \mathrm{Sch}^{aff}$. Recall that for any n, $\tau_{\leq n} \mathrm{Spc} \subset \mathrm{Spc}$ is stable under filtered colimits, so $\mathcal{Y}(S) \in Sets$ and an element of $\mathcal{Y}(S)$ comes from an element of $\mathcal{Z}_i(S)$ for some i (by [20], Cor. 13.1.14). So, an element of $\mathcal{Z}_i(S)$ is the same as an element of $\mathcal{Y}(S)$ whose image in $\mathcal{Y}_{\alpha_0}(S)$ lies in the subset $Y_i(S)$. The makes the claim manifest (and it holds more generally in the situation of ([9], C.1.6)).

Since $\operatorname{Ker}(G \to G_\alpha)$ is prounipotent for $\alpha \neq \alpha_0$, we get $Shv(\mathcal{Y}_\alpha)^G \widetilde{\to} Shv(\mathcal{Y}_\alpha)^{G_\alpha}$ for $\alpha \neq \alpha_0$ by ([24], 1.3.21). Now by Section 0.0.16 of this file, $Shv(\mathcal{Y}_\alpha)^{G_\alpha} \widetilde{\to} Shv(\mathcal{Y}_{\alpha_0})$ via the functor $f^*_{\alpha_0,\alpha} : Shv(\mathcal{Y}_{\alpha_0}) \to Shv(\mathcal{Y}_\alpha)$. So,

$$Shv(\mathcal{Y})^G \tilde{\to} \lim_{\alpha\in A^{op}} Shv(\mathcal{Y}_\alpha)^G \tilde{\to} Shv(\mathcal{Y}_{\alpha_0})$$

We could take the functors $f^!_{\alpha_0,\alpha}$ instead, but the two limits would be isomorphic.

We may strengthen the above as follows. Assume H is a placid group scheme, $G \subset H$ is a placid closed immersion, and a normal group subscheme with the cokernel K, here K is a smooth affine group scheme of finite type. Assume the G-action on $\mathcal{Y}$ is extended to a H-action. Then as above we get $Shv(\mathcal{Y})^H \tilde{\to} Shv(\mathcal{Y}_{\alpha_0}/K)$.

0.0.37. Let $H \in Grp(\mathrm{PreStk})$ be a placid ind-scheme written as $H \tilde{\to} \mathrm{colim}_{j\in J} H_j$, where H_j is a placid group scheme, and for $j \to j'$ in J the map $H_j \to H_{j'}$ is a placid closed immersion and a homomorphism of group schemes. Assume $j = 0$ is initial in J and let $G = H_0$. Then for any j, H_j/G is a scheme of finite type, so $H/G \tilde{\to} \mathrm{colim}_{j\in J} H_j/G$, because colimits commute with colimits, so H/G is an ind-scheme of ind-finite type.

Write as in the previous section $G \tilde{\to} \lim_{\alpha\in A^{op}} G_\alpha$, where G_α is a smooth group scheme of finite type, and for $\alpha \to \beta$ in A, $G_\beta \to G_\alpha$ is smooth, affine and surjective. Set $K_\alpha = Ker(G \to G_\alpha)$. For $\alpha \to \beta$ in A let $1 \to K_{\alpha,\beta} \to G_\beta \to G_\alpha \to 1$ be an exact sequence. Assume $K_{\alpha,\beta}$ is a unipotent group scheme. Then $K_\alpha \tilde{\to} \lim_\beta K_{\alpha,\beta}$ is prounipotent.

Set $\mathcal{Y}_\alpha = H/K_\alpha$, we usually mean by this the etale sheafification of the prestack quotient. This is an ind-scheme of ind-finite type by the above, and for $\alpha \to \beta$ in A the map $\mathcal{Y}_\beta \to \mathcal{Y}_\alpha$ is a K_α/K_β-torsor. So, we are in the situation of the previous section, α_0 is initial in A. We write $H/K_{\alpha_0} = \tilde{\to} \mathrm{colim}_j H_j/K_{\alpha_0}$. So, $\mathcal{Y} \tilde{\to} \lim_\alpha H/K_\alpha$. Note that $\lim_\beta(K_\alpha/K_\beta) \tilde{\to} K_\alpha$. We get $\mathcal{Y} \tilde{\to} \mathrm{colim}_j H_j \tilde{\to} H$, because $\lim_\alpha H_j/K_\alpha \tilde{\to} H_j$ for any j.

From $H \tilde{\to} \lim_{\alpha\in A^{op}} H/K_\alpha$ we get $Shv(H) \tilde{\to} \lim_\alpha Shv(H/K_\alpha)$. From the previous section we now get an equivalence $Shv(H/K_{\alpha_0}) \tilde{\to} Shv(H)^{K_{\alpha_0}}$. Similarly, we may get $Shv(H/K_\alpha) \tilde{\to} Shv(H)^{K_\alpha}$ for any α.

We have an action of G_α by right translations on H/K_α, and $(H/K_\alpha)/G_\alpha \tilde{\to} H/G$. Now Section 0.0.16 gives $Shv(H/K_\alpha)^{G_\alpha} \tilde{\to} Shv(H/G)$.

As in the previous subsection, we get $Shv(H/G) \tilde{\to} Shv(H/K_\alpha)^{G_\alpha} \tilde{\to} Shv(H)^G$ for any of the 4 sheaf theories (for $\mathcal{D}$-modules this is ([4], Lemma B.5.1).

Corollary 0.0.38. *Let $H \in Grp(\mathrm{PreStk})$ be a placid ind-scheme, $G \subset H$ be a closed placid group subscheme. For any of the 4 sheaf theories one has $Shv(H/G) \tilde{\to} Shv(H)^G$, where G acts on H by right translations.*

0.0.39. Let $G \in Grp(\mathrm{PreStk})$ be a placid ind-scheme, Y be a placid ind-scheme with a G-action. Then $Shv(Y)$ is equipped with a $Shv(G)$-action. Namely, for $K \in Shv(G), F \in Shv(Y)$ one has $K * F \tilde{\to} a_*(K \boxtimes F)$ for the action map $a : G \times Y \to Y$.

0.0.40. Let $Y \to S$ be a map in Sch_{ft}, G be a placid group scheme over S acting on Y over S though its finite-dimensional quotient $G \to G_0$ with a prounipotent kernel. We have canonically $Shv(Y)^G \tilde{\to} Shv(Y)^{G_0}$ by ([24], 1.3.21). Consider the stack quotient Y/G (by which we mean etale sheafification of the prestack quotient). We define $Shv(Y/G)$ as $Shv(Y/G_0)$ in such a way that for $q : Y \to Y/G$ the functor $q^* : Shv(Y/G) \to Shv(Y)$ is defined as $q_0^* : Shv(Y/G_0) \to Shv(Y)$ for $q_0 : Y \to Y/G_0$. So, if $G \to G_1 \to G_0$ are given, where G_1 is another finite-dimensional quotient group scheme over S with $\mathrm{Ker}(G \to G_1)$ prounipotent then we identify $Shv(Y/G_1) \widetilde{\to} Shv(Y/G_0)$ via a^* for the natural map $a : Y/G_1 \to Y/G_0$. No shifts appear. If $f : Y \to Y'$ is a G-equivariant map in $(Sch_{ft})_{/S}$ (we assume the G-action on both schemes factor through a finite dimensional quotient group scheme) then we have $f^! : Shv(Y'/G) \to Shv(Y/G)$.

We extend this definition to the case of an ind-scheme of ind-finite type Y over S equipped with a G-action over S as follows. Assume Y admits a presentation $Y \widetilde{\to} \mathrm{colim}_{i\in I} Y_i$, where Y_i is a G-invariant closed subscheme of finite type, I is filtered, and for $i \to j$ in I the map $Y_i \to Y_j$ is a closed immersion. Assume the G-action on Y_i factors through a quotient group scheme $G \to G_i$, where $G_i \to S$ is of finite type with $\mathrm{Ker}(G \to G_i)$ prounipotent. Then we have $Shv(Y_i/G)$ defined as above and set $Shv(Y/G) \widetilde{\to} \lim_{i\in I^{op}} Shv(Y_i/G)$. With this definition for $q : Y \to Y/G$ we get the functor $q^* : Shv(Y/G) \to Shv(Y)$, which is the limit over $i \in I^{op}$ of the functors $q_i^* : Shv(Y_i/G) \to Shv(Y_i)$ for $q_i : Y_i \to Y_i/G$. It also identifies with $\mathrm{oblv} : Shv(Y)^G \to Shv(Y)$.

Let now H be a placid group ind-scheme over S, $G \subset H$ a closed placid group subscheme over S, so H/G is an ind-scheme of ind-finite type over S. Then the above assumption is satisfied for the G-action on H/G over S. So, $Shv(H/G)^G$ identifies with $Shv(G\backslash H/G)$. For $q : H/G \to G\backslash H/G$ the functor $q^* : Shv(G\backslash H/G) \to Shv(H/G)$ identifies with $\mathrm{oblv} : Shv(H/G)^G \to Shv(H/G)$.

Let again $Y \to S$ be a map in Sch_{ft} and G a placid group scheme over S. Assume that the action of G on Y factors though the finite-dimensional group scheme $G_0 \to S$, and let $G \to G_1 \to G_0$ be as above. Another way to realize $Shv(Y/G)$ is as the category $Shv(Y/G_0)$ via the identifications $a^*[\dim.\mathrm{rel}(a)] : Shv(Y/G_1) \widetilde{\to} Shv(Y/G_0)$ for every G_1 as above. Indeed, the equivalence $Shv(Y/G_0) \widetilde{\to} Shv(Y/G_0)$, $K \mapsto K[-\dim(G_0/S)]$ from the first model to the second one allows to identify them. The advantage of the second model is that the transition functors are t-exact for the perverse t-structure, so allow to equip $Shv(Y/G)$ with the perverse t-structure: this is the perverse t-structure on $Shv(Y/G_0)$.

For the second model for $q_0 : Y \to Y/G_0$ consider the functor $Shv(Y/G) = Shv(Y/G_0) \to Shv(Y)$ given by $q_0^*[\dim(G_0/S)]$, it is t-exact and compatible with the transition functors for the second model, so defines a functor $Shv(Y/G) \to Shv(Y)$ that we denote by $q^*[\dim.\mathrm{rel}(q)]$, this is just one symbol.

Let now Y be an ind-scheme of ind-finite type over S with a G-action and a presentation $Y \widetilde{\to} \mathrm{colim}_{i\in I} Y_i$, where I is filtered, $Y_i \to S$ is a G-invariant closed subscheme of finite type in Y, and for $i \to j$ the map $h_{ij} : Y_i \to Y_j$ is a closed immersion. Assume G-action on Y_i factors through a quotient group scheme $G_i \to S$ of finite type over S, where $\mathrm{Ker}(G \to G_i)$ is a prounipotent group scheme over S. Equip each $Shv(Y_i/G)$ with the perverse t-structure. Then the !-pullbacks under $Y_i/G_0 \to Y_j/G_0$ are compatible with the transition functors for the second model, so define a functor $h_{ij}^! : Shv(Y_j/G) \to$

$Shv(Y_i/G)$, which is left t-exact. It also commutes with the funtors $q_i^*[\dim.\mathrm{rel}(q_i)]$ for $q_i : Y_i \to Y_i/G$. Recall that $Shv(Y/G)\widetilde{\to}\lim_{i\in I^{op}} Shv(Y_i/G)$ with respect to the functors $h_{ij}^!$. Each $h_{ij}^! : Shv(Y_j/G) \to Shv(Y_i/G)$ admits a left adjoint $(h_{ij})_! : Shv(Y_i/G) \to Shv(Y_j/G)$, and we may also write $Shv(Y/G)\widetilde{\to}\operatorname{colim}_{i\in I} Shv(Y_i/G)$ with the transition functors $(h_{ij})_!$. Now we may define the perverse t-structure on $Shv(Y/G)$ as in the case of an ind-scheme of ind-finite type. Namely, $K \in Shv(Y/G)$ lies in $Shv(Y/G)^{\geq 0}$ iff for any i, its !-restriction to Y_i/G lies in $Shv(Y_i/G)^{\geq 0}$. So, $Shv(Y/G)^{\geq 0}\widetilde{\to}\lim_{i\in I^{op}} Shv(Y_i/G)^{\geq 0}$, which shows that $Shv(Y/G)^{\geq 0}$ is presentable, so the t-structure is accessible. This t-structure is also compatible with filtered colimits.

0.0.41. Let $Y\widetilde{\to}\operatorname{colim}_{i\in I} Y_i$ in PreStk, where I is filtered, Y_i is a scheme of finite type, and for $i \to j$ in I, $Y_i \to Y_j$ is a closed immersion, so Y is an ind-scheme of ind-finite type. Let $H \to G$ be a homomorphism of placid group schemes over $\operatorname{Spec} k$. Assume G acts on Y and the assumption of the previous subsection holds, that is, each Y_i is G-invariant, and on Y_i the group scheme G acts via a finite-dimensional quotient group scheme $G \to G_i$ with $\operatorname{Ker}(G \to G_i)$ prounipotent. We have a natural map of stack quotients $h : Y/H \to Y/G$. We have defined the categories $Shv(Y/G), Shv(Y/H)$ in the previous subsection. Then the functor $h^* : Shv(Y/G) \to Shv(Y/H)$ is defined, namely this is $\mathrm{oblv} : Shv(Y)^G \to Shv(Y)^H$.

0.0.42. If Y is a stack locally of finite type, a placid group scheme over Y should be defined as a group object $(G \to Y) \in Grp(\mathrm{PreStk}_{/Y})$ such that for any $S \to Y$ with $S \in \mathrm{Sch}_{ft}^{aff}$, $S \times_Y G$ is a placid group scheme over S.

Let $Z \to Y$ be a map in Stk_{lft} and G be a placid group scheme over Y acting on Z over Y. Write Z/G for the stack quotient of Z by G (etale sheafification of the prestack quotient), so $Z/G \to Y$. How do we define $Shv(Z)^G$?

First, for any $S \to Y$ with $S \in \mathrm{Sch}_{ft}^{aff}$ we have a monoidal category $Shv(S \times_Y G)$ defined in ([24], 1.3.7), it is an object of $Alg(Shv(Y) - mod)$. For a map $S' \stackrel{\alpha}{\to} S \to Y$ in $(\mathrm{Sch}_{ft}^{aff})_{/Y}$ let $\beta : S' \times_Y G \to S \times_Y G$ be obtained by base change. As in ([24], 1.3.12), $\beta^! : Shv(S \times_Y G) \to Shv(S' \times_Y G)$ is monoidal, it is actually a morphism in $Alg(Shv(Y) - mod)$. To see this we used Lemma 0.0.43 below. So, we may understand

$$Shv(G)\widetilde{\to} \lim_{(S\to Y)\in((\mathrm{Sch}_{ft}^{aff})_{/Y})^{op}} Shv(S \times_Y G)$$

as limit taken in $Alg(Shv(Y) - mod)$. We get a monoidal structure on $Shv(G)$ via the latter limit.

This is one more extension of our sheaf theory needed. In general, we can not write G as $\lim_{i\in I^{op}} G_i$, where $G_i \to Y$ is an affine group scheme "of finite type" over Y, I is filtered, and for $i \to j$ in I the map $G_j \to G_i$ is affine smooth surjective. This does not hold already for $\mathcal{L}^+(G) \to \mathrm{Ran}$, I think, where G is a reductive group.

We will see that the monoidal category $Shv(G)$ acts on $Shv(Z)$.

For each $S \to Y$ in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$, $S \times_Y G$ acts on $S \times_Y Z$ over S, so $Shv(S \times_Y G)$ acts on $Shv(S \times_Y Z)$ naturally. For a map $S' \stackrel{\alpha}{\to} S \to Y$ in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$ let $\bar{\alpha} : S' \times_Y Z \to S \times_Y Z$ be obtained from α by base change.

Let $Shv(S \times_Y G)$ act on $Shv(S' \times_Y Z)$ via the map $Shv(S \times_Y G) \to Shv(S' \times_Y G)$. Then $\bar{\alpha}^!$ commutes with $Shv(S \times_Y G)$-actions.

Recall that the sheafification is a left exact functor, so for the stack quotients we get $((S \times_Y Z)/(S \times_Y G)) \times_S S' \widetilde{\to} (S' \times_Y Z)/(S' \times_Y G)$ canonically.

Consider the ∞-category $AssAlg + Mod(\mathrm{DGCat}_{cont})$ defined in ([6], ch. I.1, 3.5.4). By ([15], 3.2.2.5), it admits limits and the projection $AssAlg + Mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$ preserves limits. We obtained a functor

$$((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op} \to AssAlg + Mod(\mathrm{DGCat}_{cont})$$

sending $S \to Y$ to the pair $Shv(S \times_Y G), Shv(S \times_Y Z)$. So, the limit of the latter functor is an object of $AssAlg + Mod(\mathrm{DGCat}_{cont}) \to \mathrm{DGCat}_{cont}$. In other words, $Shv(G)$ act on $Shv(Z)$ naturally, and we may consider the invariants

$$Shv(Z)^{Shv(G)} = \mathrm{Fun}_{Shv(G)}(Shv(Y), Shv(Z)) \in Shv(Y) - mod$$

Question. Can we rewrite the above as limit of $Shv(S \times_Y Z)^{Shv(S \times_Y G)}$ over S? More precisely, for $(S \to Y) \in (\mathrm{Sch}^{aff}_{ft})_{/Y}$, let $q_S : S \times_Y G \to S$ be the projection. By ([24], 1.3.16), we have canonically

$$Shv(S \times_Y Z)^{Shv(S \times_Y G)} \widetilde{\to} q_S^* \omega - comod(Shv(S \times_Y Z))$$

If $\alpha : S' \to S$ is a morphism in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$ then $\beta^! q_S^* \omega \widetilde{\to} q_{S'}^* \omega$ as coalgebras in $Shv(S' \times_Y G)$. This is by definition of the functors $q_S^*, q_{S'}^*$. So, $\bar{\alpha}^! : Shv(S \times_Y Z) \to Shv(S' \times_Y Z)$ induces a functor between the comodule categories

$$q_S^* \omega - comod(Shv(S \times_Y Z)) \to q_S^* \omega - comod(Shv(S' \times_Y Z)) = q_{S'}^* \omega - comod(Shv(S' \times_Y Z))$$

Now we may consider

$$\lim_{S \to Y} q_S^* \omega - comod(Shv(S \times_Y Z))$$

taken in DGCat_{cont} over the category $((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op}$. Is it equivalent to $Shv(Z)^{Shv(G)}$?

Lemma 0.0.43. *Let $S' \to S$ be a map in Sch_{ft}. Let $f : Y \to Z$ be a morphism of placid schemes over S, let $f' : Y' \to Z'$ be obtained from f by the base change $\alpha : S' \to S$. Write $\alpha_Y : Y' \to Y$ and $\alpha_Z : Z' \to Z$ for the obtained maps. Then for $K \in Shv(Y)$ one has canonically $\alpha_Z^! f_* K \widetilde{\to} f'_* \alpha_Y^! K$.*

Proof. Write $Y \widetilde{\to} \lim_{i \in I^{op}} Y_i$, where I is filtered, Y_i is a scheme of finite type over S, and for $i \to i'$ in I the map $Y_{i'} \to Y_i$ is affine smooth surjective (over S), and similarly for $Z \widetilde{\to} \lim_{j \in J^{op}} Z_j$. These are presentations from a definition of a placid scheme. Let Y'_i, Z'_j be obtained from Y_i, Z_j by base change $S' \to S$, so $Y' \widetilde{\to} \lim_{i \in I^{op}} Y'_i$ and $Z' \widetilde{\to} \lim_{j \in J^{op}} Z'_j$.

It suffices to establish the desired isomorphism after applying $(ev'_j)_* : Shv(Z') \to Shv(Z'_j)$ for each $j \in J$, here $ev'_j : Z' \to Z'_j$ is the projection. Pick $i \in I$ such that the composition $Y \to Z \to Z_j$ factors through $\bar{f}: Y_i \to Z_j$. By base change under $S' \to S$ we get a cartesian square

$$\begin{array}{ccc} Y_i & \stackrel{\bar{f}}{\to} & Z_j \\ \uparrow \alpha_{Y_i} & & \uparrow \alpha_{Z_j} \\ Y'_i & \stackrel{\bar{f}'}{\to} & Z'_j \end{array}$$

Let $(ev_i)_*: Shv(Y) \to Shv(Y_i)$ be the direct image under $ev_i: Y \to Y_i$. The key point is the base change isomorphism $\alpha^!_{Z_j}\bar{f}_* \widetilde{\to} \bar{f}'_*\alpha^!_{Y_i}$. We get

$$(ev'_j)_*\alpha^!_Z f_*K \widetilde{\to} \alpha^!_{Z_j}(ev_j)_*f_*K \widetilde{\to} \alpha^!_{Z_j}\bar{f}_*(ev_!)_*K \widetilde{\to} \bar{f}'_*\alpha^!_{Y_i}(ev_i)_*K$$
$$\widetilde{\to} \bar{f}'_*(ev_i)_*\alpha^!_Y K \widetilde{\to} (ev'_j)_*f'_*\alpha^!_Y K$$

We are done. □

0.0.44. Let us be in the situation of Section 0.0.42. Probably the only case we need satisfies the following additional assumption that we make. For any $S \to Y$ in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$, $S \times_Y Z$ may be written as $S \times_Y Z \widetilde{\to} \mathrm{colim}_{i\in I} Z_{S,i}$, where I is filtered, and $Z_{S,i} \to S$ is a scheme of finite type such that for any $i \to j$ in I, the map $Z_{S,i} \to Z_{S,j}$ is a closed immersion. Moreover, for each $i \in J$, $Z_{S,i}$ is stable under the action of $S \times_Y G$, and the latter acts through a finite-dimensional quotient scheme $G_{S,i}$ over S. In particular, $S \times_Y Z$ is a ind-scheme of ind-finite type over S.

In this setting one may define the category $Shv(Z/G)$ in one more way. Namely, for $S \to Y$ as above, in Section 0.0.40 we have defined the category $Shv((S\times_Y Z)/(S\times_Y G))$ together with the functor $q_S^*: Shv((S \times_Y Z)/(S \times_Y G)) \to Shv(S \times_Y Z)$ for the projection $q_S: S \times_Y Z \to (S \times_Y Z)/(S \times_Y G)$. Recall that q_S^* identifies with $\mathrm{oblv}: Shv(S \times_Y Z)^{S\times_Y G} \to Shv(S \times_Y Z)$.

Let now $\alpha: S' \to S$ be a map in $(\mathrm{Sch}^{aff}_{ft})_{/Y}$. Let $\bar{\alpha}: S' \times_Y Z \to S \times_Y Z$ and $\beta: S' \times_Y G \to S \times_Y G$ be obtained by base change from α. Pick a finite-dimensional quotient group scheme $S \times_Y G \to G_S$ such that $S \times_Y G$-action on $S \times_Y Z$ factors through G_S. Let $G_{S'} = G_S \times_S S'$. We have the cartesian square

$$\begin{array}{ccc} S' \times_Y Z & \stackrel{\bar{\alpha}}{\to} & S \times_Y Z \\ \downarrow h' & & \downarrow h \\ (S' \times_Y Z)/G_{S'} & \stackrel{\tilde{\alpha}}{\to} & (S \times_Y Z)/G_S \end{array}$$

The functors $\tilde{\alpha}^!: Shv((S \times_Y Z)/G_S) \to Shv((S' \times_Y Z)/G_{S'})$ can be seen by definition as the functors that fit into a commuttaive diagram

$$\begin{array}{ccc} Shv(S \times_Y Z) & \stackrel{\bar{\alpha}^!}{\to} & Shv(S' \times_Y Z) \\ \uparrow \mathrm{oblv} & & \uparrow \mathrm{oblv} \\ Shv(S \times_Y Z)^{S\times_Y G} & \stackrel{\tilde{\alpha}^!}{\to} & Shv(S' \times_Y Z)^{S'\times_Y G} \end{array}$$

This way we get a functor $((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op} \to Shv(Y) - mod$, $S \mapsto Shv(S \times_Y Z)^{S\times_Y G}$. Finally, we may consider

$$\lim_{S\to Y} Shv(S \times_Y Z)^{S\times_Y G}$$

in $Shv(Y)-mod$ taken over $((\mathrm{Sch}^{aff}_{ft})_{/Y})^{op}$. This should be our definition of $Shv(Z/G)$ I think.

0.0.45. For S a scheme of finite type consider the perverse t-structure on S. The functor $\mathrm{H}^i: Shv(S)\to Shv(S)^{\heartsuit}$ preserves products, is this correct? This was used to conclude that your $QLisse(S)$ for S smooth is left complete.

For a scheme of finite type S any object of $Shv(S)^{constr}$ is bounded.

0.0.46. Let Y be a classical algebraic stack locally of finite type with an affine diagonal. Then the truncation functors for the perverse t-structure $\tau^{\le n}, \tau^{\ge n}$ preserve the subcategory $Shv(Y)^{constr}\subset Shv(Y)$, so we get a t-structure on $Shv(Y)^{constr}$.

0.0.47. Let Y be a classical quasi-compact algebraic stack with an affine diagonal. Let $F\in Shv(Y)^{constr}$ then F is bounded. Indeed, pick a smooth covering $f: S\to Y$, where $S\in \mathrm{Sch}^{aff}_{ft}$. Since $f^*[\dim.\mathrm{rel}(f)]$ is t-exact and conservative, it suffices to show that f^*F is bounded. However, any compact object in $Shv(S)$ is bounded.

0.0.48. Let $S\in \mathrm{Sch}_{ft}$, $K_i\in Shv(S)$ and $E\in Lisse(E)$, that is, E is dualizable with respect to the $\otimes$-monoidal structure on $Shv(S)$. Recall that $\mathcal{H}om(K_1, K_2)\in Shv(S)$ denotes the inner hom with respect to the $\otimes$-monoidal structure on $Shv(S)$. Then by ([15], 4.6.2.1) we get $\mathcal{H}om(K_1\otimes E, K_2)\widetilde{\to}\mathcal{H}om(K_1, E^\vee\otimes K_2)$ with $E^\vee=\mathcal{H}om(E,e)$.

0.0.49. Recall the following from ([12], A.1.7). Let $Corr(\mathrm{PreStk}_{lft})_{ind-sch,all}$ be the category of correspondences, whose objects are prestacks locally of finite type $\mathcal{Y}$, and a morphism from $\mathcal{Y}_1$ to $\mathcal{Y}_2$ is a diagram $\mathcal{Y}_1 \stackrel{g}{\leftarrow} \mathcal{Y}_{12}\stackrel{f}{\to}\mathcal{Y}_2$ with g any and f ind-schematic of ind-finite type. Then in the constructible context we get a functor $Shv_{Corr}: Corr(\mathrm{PreStk}_{lft})_{ind-sch,all}\to \mathrm{DGCat}_{cont}$ sending $\mathcal{Y}$ to $Shv(\mathcal{Y})$, and sending the above morphism to the functor $f_*g^!: Shv(\mathcal{Y}_1)\to Shv(\mathcal{Y}_2)$. Then the functor Shv_{Corr} possesses a natural right-lax symmetric monoidal structure, see ([6], Chapter 3, Sect. 6.1), where $Corr(\mathrm{PreStk}_{lft})_{ind-sch,all}$ is a symmetric monoidal category with respect to the level-wise product.

In particular, this means that given $f_i: Y_i\to Z_i$ ind-schematic of ind-finite type in PreStk_{lft} and $K_i\in Shv(Y_i)$, we have

$$(f_1\times f_2)_*(K_1\boxtimes K_2)\widetilde{\to}((f_1)_*K_1)\boxtimes((f_2)_*K_2)$$

If $\mathcal{H}$ is a groupoid acting on $\mathcal{Y}$ in PreStk_{lft} given by a functor $\boldsymbol{\Delta}^{op}\to \mathrm{PreStk}_{lft}$ such that the action map $m:\mathcal{H}\times_{\mathcal{Y}}\mathcal{H}\to\mathcal{H}$ is ind-affine of ind-finite type, we get a monoidal structure on $Shv(\mathcal{H})$ with the product given by $(K_1, K_2)\mapsto m_*q^!(K_1\boxtimes K_2)$ for $q:\mathcal{H}\times_{\mathcal{Y}}\mathcal{H}\to\mathcal{H}\times\mathcal{H}$. Let $\alpha:\mathcal{Y}\to\mathcal{H}$ be the map corresponding to $[1]\to[0]$ in $\boldsymbol{\Delta}$. Then $\alpha_*\omega_{\mathcal{Y}}$ is the unit of $Shv(\mathcal{H})$. Moreover, the functor $\alpha_*: (Shv(\mathcal{Y}),\otimes^!)\to Shv(\mathcal{H})$ is monoidal. Indeed, $\mathcal{H}\in Alg(Corr(\mathrm{PreStk}_{lft})_{ind-sch,all})$, so we just apply a right-lax monoidal functior Shv_{Corr}. Moreover,

$$(\mathcal{H},\mathcal{Y})\in Alg+module(Corr(\mathrm{PreStk}_{lft})_{ind-sch,all})$$

Namely, write $\mathrm{pr},\mathrm{act}:\mathcal{H}\to\mathcal{Y}$ for the two maps from $\mathcal{H}$ to $\mathcal{Y}$ given by $0,1:[0]\to[1]$. Then the action map from $\mathcal{H}\times\mathcal{Y}$ to $\mathcal{Y}$ is given by the correspondence $\mathcal{H}\times\mathcal{Y}\stackrel{\mathrm{id},\mathrm{pr}}{\leftarrow}\mathcal{H}\stackrel{\mathrm{act}}{\to}\mathcal{Y}$. Applying Shv_{Corr}, we see that $Shv(\mathcal{Y})\in Shv(\mathcal{H})-mod(\mathrm{DGCat}_{cont})$.

The whole Section A.1 of [12] can be advised as a reference on the generalities about the sheaf theories.

More generally, we may define the category of relative groupoids Grpd / PreStk /Sch and the corresponding functor

$$\mathrm{Grpd} / \mathrm{PreStk} / \mathrm{Sch} \to Alg(\mathrm{Corr}(\mathrm{PreStk}_{lft}))$$

as in ([25], 1.4.48).

0.1. **Verdier compatible algebraic stacks.**

0.1.1. For ([2], A.2.2). Let Y be quasi-compact classical algebraic stack with an affine diagonal, which is Verdier compatible. They claim there that for $f : S \to Y$ a scheme of finite type over Y, the objects f_*K with $K \in Shv(S)^c$ are compact and generate $Shv(Y)$. Indeed, $Shv(Y)^c$ is the Karoubi closure (that is, idempotent closure) of the smallest stable subcategory generated by objects of the form $f_!(K)$ with $K \in Shv(S)^c$. This implies the claim, see ([20], 9.2.27).

0.1.2. For ([2], A.2.3). Let Y, Y' be a quasi-compact classical algebraic stacks with affine diagonals, which are Verdier compatible. Let $f : Y \to Y'$ be a morphism. Recall that $f_\blacktriangle : Shv(Y) \to Shv(Y')$ is defined as the continuous extension of the functor $f_* : Shv(Y)^c \to Shv(Y')^c \subset Shv(Y')$.

Let Z be another algebraic stack locally of finite type with an affine diagonal, which is Verdier compatible. Then we have the following.

Lemma 0.1.3. *For $K \in Shv(Y), F \in Shv(Z)$ we have canonically*

$$(f_\blacktriangle K) \boxtimes F \widetilde{\to} (f \times \mathrm{id})_\blacktriangle (K \boxtimes F)$$

Besides, for $L \in Shv(Y')$ we get $(f_\blacktriangle K) \otimes^! L \widetilde{\to} f_\blacktriangle(K \otimes^! f^! L)$.

Proof. 1) Both sides for any F fixed preserve colimits as a functor of K. Therefore, it suffices to prove this for K of the form $K = g_*K'$, where $g : S \to Y$ is a morphism, $S \in \mathrm{Sch}_{ft}$ and $K' \in Shv(S)^c$, as such objects generate $Shv(Y)$. Moreover, we may assume $F \in Shv(Z)^c$. Then $f_\blacktriangle K \widetilde{\to} f_*K \widetilde{\to} (fg)_*K'$, and $(f \times \mathrm{id})_\blacktriangle (K \boxtimes F) \widetilde{\to} (f \times \mathrm{id})_*(K \boxtimes F)$, and $K \boxtimes F \widetilde{\to} (g \times \mathrm{id})_*(K' \boxtimes F)$, because $g \times \mathrm{id}$ is schematic. Now $(fg \times \mathrm{id})_*(K' \boxtimes F) \widetilde{\to} ((fg)_*K') \boxtimes F$, because fg is schematic. The first claim follows.

2) For the second, note that both sides preserve colimits separately in each variable, so we may assume K of the form $K = g_*K'$, where $g : S \to Y$ is a morphism, $S \in \mathrm{Sch}_{ft}$ and $K' \in Shv(S)^c$. Then $f_\blacktriangle K \widetilde{\to} f_*K \widetilde{\to} (fg)_*K'$. We may also assume $L \in Shv(Y')^{constr}$. We have the cartesian squares

$$\begin{array}{ccc} S & \to & S \times Y' \\ \downarrow g & & \downarrow g\times\mathrm{id} \\ Y & \stackrel{\Gamma_f}{\to} & Y \times Y' \\ \downarrow & & \downarrow f\times\mathrm{id} \\ Y' & \stackrel{\vartriangle}{\to} & Y' \times Y' \end{array}$$

Now f_* satisfies the base change againts !-pullbacks, so

$$(f_\blacktriangle K) \otimes^! L \widetilde{\to} \vartriangle^! (fg \times \mathrm{id})_*(K' \boxtimes L) \widetilde{\to} (fg)_*(K' \otimes^! (fg)^! L) \widetilde{\to} f_\blacktriangle(K \otimes^! f^! L),$$

because $f_\blacktriangle g_* \widetilde{\to} (fg)_*$. Indeed, g and fg are schematic. □

Recall the self-duality

$$Shv(Y) \otimes Shv(Y) \to \mathrm{Vect}, \quad (K_1, K_2) \mapsto C^\cdot_\blacktriangle(Y, K_1 \otimes^! K_2) \tag{5}$$

from ([2], A.4.1). Under this self-duality, for $f : Y \to Y'$ as above the dual of the functor $f^! : Shv(Y') \to Shv(Y)$ is the functor $f_\blacktriangle : Shv(Y) \to Shv(Y')$, this follows from the above projection formula.

For $K \in Shv(Y)^c, K' \in Shv(Y')^c$ we have $\mathbb{D}(K \boxtimes K') \widetilde{\to} (\mathbb{D}K) \boxtimes (\mathbb{D}K')$ naturally. Now as in ([21], Sect. 1.0.1) one shows that the dual $h^\vee$ of the exteriour product functor $h : Shv(Y) \otimes Shv(Y') \to Shv(Y \times Y')$ with respect to the above dualities identifies with the right adjoint $h^R : Shv(Y \times Y') \to Shv(Y) \otimes Shv(Y')$.

So, the unit of the self-duality (5) is the object $h^R(\triangle_\blacktriangle \omega_Y)$, where $\triangle \colon Y \to Y \times Y$ is the diagonal.

0.1.4. For algebraic stacks locally of finite type (with affine diagonal) we always have a $(_!, ^*)$-base change in the constructible context, this is mentioned in ([2], A.1.8) in particular.

0.1.5. Let $f : Y \to Y'$ be a morphism of algebraic stacks as in Section 0.1.2. For $F \in Shv(Y), K \in Shv(Y')$ we have a natural transformation functorial in K, F

$$(f_\blacktriangle F) \otimes K \to f_\blacktriangle(F \otimes f^* K)$$

This comes from ([2], Section A.3.3-A.3.4).

The following is also useful. For $K_1, K_2 \in Shv(Y')$ there is a natural transformation

$$f^*(K_1 \otimes^! K_2) \to (f^! K_1) \otimes^! (f^* K_2)$$

Indeed, it comes from the natural map $K_1 \otimes^! K_2 \to K_1 \otimes^! (f_* f^* K_2)$ and the projection formula for f_*.

Similarly, we have a natural map $f^* K_1 \otimes f^! K_2 \to f^!(K_1 \otimes K_2)$.

0.1.6. Let Z, Y, Y' be algebraic stacks as in Section 0.1.2, and $f : Y \to Y'$ be a morphism. For $K \in Shv(Z), F \in Shv(Y)$ we have canonically

$$(\mathrm{id} \times f)_*(K \boxtimes F) \widetilde{\to} K \boxtimes f_* F$$

Indeed, this is a particular case of the projection formula for $f \times \mathrm{id} : Z \times Y \to Z \times Y'$, as $K \boxtimes F \widetilde{\to} ((\mathrm{id} \times f)^! (K \boxtimes \omega_{Y'})) \otimes^! p_2^! F$, so $(\mathrm{id} \times f)_*(K \boxtimes F) \widetilde{\to} (K \boxtimes \omega_{Y'}) \otimes^! (\mathrm{id} \times f)_* p_2^! F$.

0.1.7. Let $f : Y \to Y'$ be a morphism of algebraic stacks as in Section 0.1.2. For $F \in Shv(Y), K \in Shv(Y')$ we have a natural transformation functorial in K, F

$$(f_* F) \otimes K \to f_*(F \otimes f^* K)$$

Indeed, it comes from $f^*((f_* F) \otimes K) \to F \otimes f^* K$.

There is a Verdier dual version of this map. Namely, a natural transformation

$$f_!(F \otimes^! f^! K) \to (f_! F) \otimes^! K$$

It comes from the evident map $F \otimes^! f^! K \to f^!((f_! F) \otimes^! K)$.

0.1.8. For a cartesian square of any algebraic stacks locally of finite type

$$\begin{array}{ccc} Y_1' & \stackrel{f'}{\to} & Y_2' \\ \downarrow g_1 & & \downarrow g_2 \\ Y_1 & \stackrel{f}{\to} & Y_2 \end{array}$$

we have the natural transformation

$$g_2^* \circ f_* \to f'_* \circ g_1^*$$

arising by adjointness from $f_*(g_1)_* \widetilde{\to} (g_2)_* f'_*$. Besides, the base change isomorphism $g_2^* \circ f_! \widetilde{\to} f'_! \circ g_1^*$ gives by adjointness a natural transformation

$$g_1^* \circ f^! \to (f')^! \circ g_2^*$$

Similarly, we have a natural transformation $f'_! g_1^! \to g_2^! f_!$.

0.1.9. Let $f : Y \to Y'$ be a morphism of algebraic stacks as in Section 0.1.2. Let us construct a natural morphism functiorial in $L \in Shv(Y), K, M \in Shv(Y')$

$$((f^! K) \otimes^! L) \otimes f^* M \to f^!(K \otimes M) \otimes^! L$$

We have a natural map $f^! K \otimes f^* M \to f^!(K \otimes M)$ by Section 0.1.5. So, it suffices to construct a natural map $(f^! K \otimes^! L) \otimes f^* M \to (f^! K \otimes f^* M) \otimes^! L$. It comles from the next observation.

Lemma 0.1.10. *Let Y be an algebraic stack as in Section 0.1.2. For $K_1, K_2, L \in Shv(Y)$ there is a natural map $(K_1 \otimes^! K_2) \otimes L \to (K_1 \otimes L) \otimes^! K_2$.*

Proof. 1) First, assume $K_2 \in Shv(Y)^{constr}$. Then

$$(K_1 \otimes^! K_2) \otimes L \widetilde{\to} \mathcal{H}om(\mathbb{D}K_2, K_1) \otimes L$$

and $(K_1 \otimes L) \otimes^! K_2 \widetilde{\to} \mathcal{H}om(\mathbb{D}K_2, K_1 \otimes L)$, here $\mathcal{H}om$ is the inner hom in $(Shv(Y), \otimes)$. The desired morphism comes from the natural map $(\mathbb{D}K_2) \otimes \mathcal{H}om(\mathbb{D}K_2, K_1) \otimes L \to K_1 \otimes L$. The so obtained morphisms are functorial in K_2. Now if $K_2 \in Shv(Y)$ is written as $K_2 \widetilde{\to} \operatorname{colim}_{i \in I} K_2^i$ with I small filtered and $K_2^i \in Shv(Y)^{constr}$ then the desired morphism of obtained by passing to the colimit over $i \in I$ in the diagram $(K_1 \otimes^! K_2^i) \otimes L \to (K_1 \otimes L) \otimes^! K_2^i$.

2) Simplier argument. Consider the cartesian square

$$\begin{array}{ccc} Y & \stackrel{\triangle}{\to} & Y \times Y \\ \downarrow \triangle & & \downarrow \triangle \times \mathrm{id} \\ Y \times Y & \stackrel{\mathrm{id} \times \triangle}{\to} & Y \times Y \times Y \end{array}$$

and apply the natural transformation $\triangle^* \, (\mathrm{id} \times \triangle)^! \to \triangle^! \, (\triangle \times \mathrm{id})^*$.

□

References

[1] D. Arinkin, D. Gaitsgory, D. Kazhdan, S. Raskin, N. Rozenblyum, Y. Varshavsky, The stack of local systems with restricted variation and geometric Langlands theory with nilpotent singular support, arXiv:2010.01906, arxiv version 2

[2] D. Arinkin, D. Gaitsgory, D. Kazhdan, S. Raskin, N. Rozenblyum, Y. Varshavsky, Duality for automorphic sheaves with nilpotent singular support, arxiv version 2

[3] D. Beraldo, Loop group actions on categories and Whittaker invariants, Adv. in Math. 322 (2017), 565 - 636

[4] L. Chen, Nearby cycles on Drinfeld-Gaitsgory-Vinberg interpolation grassmanian and long intertwining functor, arXiv:2008.09349

[5] D. Gaitsgory, Frenkel, D-modules on the affine Grassmannian and representations of affine Kac-Moody algebras, arxiv

[6] D. Gaitsgory, N. Rozenblyum, A study in derived algebraic geometry, book version on Dennis homepage

[7] D. Gaitsgory, The Atiyah-Bott formula for the cohomology of the moduli space of bundles on a curve, arxiv version 2 (24 June 2019)

[8] D. Gaitsgory, Sheaves of categories and the notion of 1-affineness, arxiv

[9] D. Gaitsgory, The local and global versions of the Whittaker category

[10] D. Gaitsgory, The semi-infinite intersection cohomoogy sheaf, arxiv version 5 (and Advances of Math.)

[11] D. Gaitsgory, The semi-infinite intersection cohomology sheaf-II: the Ran space version, arxiv

[12] D. Gaitsgory, D. Kazhdan, N. Rozenblyum, Y. Varshavsky, A toy model for the Drinfeld-Lafforgue shtuka construction, arXiv:1908.05420

[13] D. Gaitsgory, S. Lysenko, Parameters and duality for the metaplectic geometric Langlands theory, Selecta Math., (2018) Vol. 24, Issue 1, 227-301. Corrected version: of August 30, 2020 on my webpage

[14] J. Lurie, Higher topos theory

[15] J. Lurie, Higher algebra, September 18, 2017

[16] J. Lurie, Derived Algebraic Geometry I: Stable ∞-Categories

[17] J. Lurie, Derived Algebraic Geometry II: Noncommutative Algebra, arxiv

[18] J. Lurie, Derived Algebraic Geometry III: Commutative Algebra, arXiv:math/0703204

[19] J. Lurie, $(\infty, 2)$-Categories and the Goodwillie Calculus I, arXiv: 0905.0462

[20] S. Lysenko, Comments to Gaitsgory Lurie Tamagawa, my homepage

[21] S. Lysenko, Comments to: Gaitsgory, The local and global versions of the Whittaker category

[22] S. Lysenko, Twisted Whittaker models for metaplectic groups II, working file

[23] S. Lysenko, Comments to 1st joint paper with Dennis

[24] S. Lysenko, Comments to small FLE

[25] S. Lysenko, Comments to: The stack of local systems with restricted variation and geometric Langlands theory with nilpotent singular support

[26] S. Raskin, Chiral categories, his homepage, version 4 Sept. 2019

[27] S. Raskin, $\mathcal{D}$-modules on infinite-dimensional varieties