

Assumptions on the sheaf theory for the 2nd joint paper with Dennis¹

0.0.1. k is algebraically closed of any characteristic, e is an algebraically closed field of characteristic zero. The notation DGCat stands for the category denoted DGCat_{cont} in [7].

We are given a right-lax symmetric monoidal functor

$$(\mathrm{Sch}_{ft}^{aff})^{op} \rightarrow \mathrm{DGCat}, S \mapsto Shv(S), \text{ and } (S_1 \xrightarrow{f} S_2) \text{ goes to } f^! : Shv(S_2) \rightarrow Shv(S_1)$$

Its right Kan extension under $(\mathrm{Sch}_{ft}^{aff})^{op} \subset (\mathrm{PreStk}_{lft})^{op}$ defines a functor

$$Shv : (\mathrm{PreStk}_{lft})^{op} \rightarrow \mathrm{DGCat}$$

It is assumed that the latter functor satisfies the etale descent for etale covers in PreStk_{lft} .

0.0.2. Probably the functor Shv should be defined in a larger category than PreStk_{lft} ? For example, there should be $Shv(\mathrm{Hecke}_{G,\mathrm{Ran}}^{loc})$, though the latter is not locally of finite type. Indeed, for a closed $\mathfrak{L}^+(G)_{\mathrm{Ran}}$ -equivariant subscheme $Y \subset \mathrm{Gr}_{G,\mathrm{Ran}}$ we may define $Shv((\mathfrak{L}^+(G)_{\mathrm{Ran}}) \setminus Y)$ and pass to the colimit (or limit). Similarly, do we need $Shv(\mathrm{Hecke}_{G,x}^{loc})$?

Also, we need to make sense of invariants under $(\mathfrak{L}(N)_x, \chi_N)$, and $\mathfrak{L}(N)$ is not locally of finite type. At least, give a reference to Appendix C of [10].

0.0.3. For a map $f : Y_1 \rightarrow Y_2$ in PreStk_{lft} the left adjoint $f_!$ to $f^!$ is only partially defined in general (everywhere defined in the constructible context). If f is schematic open embedding, $f_* : Shv(Y_1) \rightarrow Shv(Y_2)$ is defined as the right adjoint to $f^! = f^*$. Moreover f_* satisfies the base change with respect to $g^!$ for $g : Y'_2 \rightarrow Y_2$.

When we say f is ind-schematic, this means that f is ind-schematic of ind-finite type, as Shv was only defined for PreStk_{lft} . For f ind-schematic we have the functor $f_* : Shv(Y_1) \rightarrow Shv(Y_2)$. What is its definition? It has a partially defined left adjoint f^* . Is f^* always defined in the constructible context? For this f has to be of finite type, I think. For example, for $p : Y \rightarrow k$, where Y is an ind-scheme of ind-finite type the functor $p_* : Shv(Y) \rightarrow \mathrm{Vect}$ does not admit a left adjoint unless Y is a scheme of finite type (see [25], 1.2.7).

For f ind-schematic, f_* satisfies the base change formula with respect to $g^!$, where $g : Y'_2 \rightarrow Y_2$. If f is ind-proper then $f_* = f_!$. My understanding is that this holds more generally for f pseudo-proper.

If f is etale then $f^! = f^*$ is the left adjoint of f_* .

The functor f_* should be defined more generally under the assumption that after a base change $S \rightarrow Y_2$ with $S \in \mathrm{Sch}_{ft}^{aff}$, $S \times_{Y_2} Y_1$ is an ind-algebraic stack. In this case f_* should also satisfy the base change formula with respect to $g^!$.

For example, the following is crucial: the category $Shv(B_{et}(e^{*,tors}))$ is monoidal for the convolution monoidal structure. For Y a prestack this is used to define a twist of $Shv(Y)$ by a $e^{*,tors}$ -gerbe over Y .

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0.0.4. For a scheme of finite type S we have $Shv(S)^{constr} \subset Shv(S)$ is the full category of bounded complexes with constructible cohomology sheaves, then $Shv(S)^{constr} = Shv(S)^c$, and it coincides with $D^b(\text{Perv}(S))$ in the constructible context. The subcategory $Shv(S)^{constr} \subset Shv(S)$ is closed under $\otimes, \otimes^!$. For Y an ind-scheme we have an equivalence $\mathbb{D} : (Shv(Y)^c)^{op} \xrightarrow{\sim} Shv(Y)^c$. Its definition is given in ([8], 7.1.3) for any $Y \in \text{PreStk}_{lft}$ such that the diagonal $Y \rightarrow Y \times Y$ is pseudo-proper.

For an Artin stack Y locally of finite type with an affine diagonal we should define $Shv(Y)^{constr} \subset Shv(Y)$ as the full subcategory of objects that $!$ -pull back to an object of $Shv(S)^c$ for any $S \rightarrow Y$, where $S \in \text{Sch}_{ft}^{aff}$. Then by ([1], Appendix C),

$$(1) \quad \mathbb{D} : (Shv(Y)^{constr})^{op} \xrightarrow{\sim} Shv(Y)^{constr}$$

is an equivalence. Indeed, we have $Shv(Y) \xrightarrow{\sim} \lim_{S \rightarrow Y} Shv(S)$ taken over the category opposite to the one classifying smooth maps $S \rightarrow Y$ with $S \in \text{Sch}^{aff}$. For $a : S \rightarrow Y$ smooth, we may use $a^!$ or a^* to test compactness, they differ by a shift. Then $Shv(Y)^{constr} \xrightarrow{\sim} \lim_{S \rightarrow Y} Shv(S)^c$ in $\text{DGCat}^{non-cocompl}$. Recall that $\text{DGCat}^{non-cocompl}$ admits limits. This gives $(Shv(Y)^{constr})^{op} \xrightarrow{\sim} \lim_{S \rightarrow Y} (Shv(S)^c)^{op}$ in $\text{DGCat}^{non-cocompl}$. So, the Verdier duality for schemes of finity type gives the equivalence (1).

For $F_i \in Shv(Y)$ write $\mathcal{H}om_{Shv}(F_1, F_2) \in \text{Vect}$ for the relative inner hom for the Vect-action on $Shv(Y)$. For ind-schemes or Artin stacks \mathbb{D} satisfies the formula

$$\mathcal{H}om_{Shv}(\mathbb{D}(F_1), F_2) \xrightarrow{\sim} \text{R}\Gamma(Y, F_1 \otimes^! F_2)$$

for $F_1 \in Shv(Y)^{constr}$. This property charaterizes $\mathbb{D}(F_1)$ uniquely. For example see ([1], F.2.5, F.1.3, F.4).

0.0.5. For $Y \in \text{PreStk}_{lft}$ and $F_i \in Shv(Y)$ write $\mathcal{H}om^!(F_1, F_2)$ for the relative inner hom in $Shv(Y)$ for the $!$ -pointwise monoidal structure. For Y smooth of dimension n we get $\text{R}\Gamma \mathcal{H}om^!(F_1, F_2)[-2n] \xrightarrow{\sim} \mathcal{H}om_{Shv}(F_1, F_2)$.

In which generality the category $Shv(Y)$ admits a symmetric monoidal structure given by $(F_1, F_2) \mapsto F_1 \otimes F_2 = d^*(F_1 \boxtimes F_2)$ for the diagonal $d : Y \rightarrow Y \times Y$? This should be always the case in the constructible context, and we reserve the notation \otimes for this tensor product structure on $Shv(Y)$.

If the monoidal structure on $Shv(Y)$ given by \otimes exists, we reserve the notation $\mathcal{H}om(F_1, F_2)$ for the inner hom for $Shv(Y)$ for this monoidal structure.

Lemma 0.0.6. *In the constructible context the Verdier duality for a scheme Y of finite type (or an Artin stack locally of finite type with an affine diagonal) satisfies a stronger property: for $F_1 \in Shv(Y)^{constr}, F_2 \in Shv(Y)$,*

$$\mathcal{H}om(\mathbb{D}(F_1), F_2) \xrightarrow{\sim} F_1 \otimes^! F_2$$

in $Shv(Y)$.

Proof. For a map $f : S \rightarrow Y$ with $S \in \text{Sch}^{aff}$ let us construct an isomorphism $f^! \mathcal{H}om(\mathbb{D}(F_1), F_2) \xrightarrow{\sim} f^!(F_1 \otimes^! F_2)$ in a way compatible with compositions $S' \rightarrow S$ for $S' \in \text{Sch}^{aff}$. We have

$$f^! \mathcal{H}om(\mathbb{D}(F_1), F_2) \xrightarrow{\sim} \mathcal{H}om(\mathbb{D}(f^! F_1), f^! F_2) \xrightarrow{\sim} (f^! F_1) \otimes^! (f^! F_2) \xrightarrow{\sim} f^!(F_1 \otimes^! F_2)$$

as desired. \square

For $F \in Shv(Y)^c$ the functor $Shv(Y) \rightarrow Shv(Y), G \mapsto \mathcal{H}om(F, G)$ preserves filtered colimits.

If we assume that there is an adjoint pair $p^* : Vect \rightleftarrows Shv(Y) : p_*$ for $p : Y \rightarrow Spec k$ then given $F_i \in Shv(Y)$ we get

$$\mathcal{H}om_{Shv}(F_1, F_2) \xrightarrow{\sim} R\Gamma \mathcal{H}om(F_1, F_2)$$

0.0.7. For which maps $f : Y \rightarrow Spec k$ the functor f^* is defined on e? If defined, it gives the constant sheaf on Y . This happens at least for algebraic stack locally of finite type (with an affine diagonal).

Let now Y be a scheme of finite type or an algebraic stack locally of finite type (with an affine diagonal). Assume we are in the constructible context and $F \in Shv(Y)^{const}$. Then the functor $Shv(Y) \rightarrow Shv(Y), K \mapsto K \otimes^! F$ admits a continuous left adjoint given by $K \mapsto K \otimes (\mathbb{D}F)$. Indeed, for $L \in Shv(Y)$ we get

$$\mathcal{H}om(L, K \otimes^! F) \xrightarrow{\sim} \mathcal{H}om(L, \mathcal{H}om(\mathbb{D}F, K)) \xrightarrow{\sim} \mathcal{H}om(L \otimes (\mathbb{D}F), K)$$

Recall that here $\mathcal{H}om$ denotes the inner hom in $(Shv(Y), \otimes)$.

Claim Let $X, Y \in Sch_{ft}$. Note that the exterior product $h : Shv(X) \otimes Shv(Y) \rightarrow Shv(X \times Y)$ is a map of $Shv(X) \otimes Shv(Y)$ -modules, where the action of $L \in Shv(X)$ (resp., of $L' \in Shv(Y)$) on $Shv(X \times Y)$ sends K to $(L \boxtimes \omega) \otimes^! K$ (resp., to $(\omega \boxtimes L') \otimes^! K$). So, its right adjoint h^R is a right-lax morphism of $Shv(X) \otimes Shv(Y)$ -modules. In fact, this right-lax structure is strict.

Proof. Let $K \in Shv(X \times Y)$ and $F \in Shv(X)$. We must show that the natural map $(F \boxtimes \omega) \otimes h^R(K) \rightarrow h^R((F \boxtimes \omega) \otimes^! K)$ is an isomorphism in $Shv(X) \otimes Shv(Y)$. We may and do assume $F \in Shv(X)^c$. It is understood that $Shv(Y), Shv(X)$ is equipped with the $\otimes^!$ -symmetric monoidal structures, so $Shv(X) \otimes Shv(Y)$ is also symmetric monoidal. By the above, the functor $Shv(Y) \otimes Shv(Y) \rightarrow Shv(Y) \otimes Shv(Y), S \mapsto (F \boxtimes \omega) \otimes S$ admits a continuous left adjoint sending $K_1 \boxtimes K_2$ to $(K_1 \otimes \mathbb{D}(F)) \boxtimes K_2$ for $K_i \in Shv(Y)$.

Now for $K_1 \in Shv(X), K_2 \in Shv(Y)$ we get

$$\begin{aligned} \text{Map}_{Shv(X) \otimes Shv(Y)}(K_1 \boxtimes K_2, h^R((F \boxtimes \omega) \otimes^! K)) &\xrightarrow{\sim} \text{Map}_{Shv(X \times Y)}(K_1 \boxtimes K_2, (F \boxtimes \omega) \otimes^! K) \\ &\xrightarrow{\sim} \text{Map}_{Shv(X \times Y)}((K_1 \otimes (\mathbb{D}F)) \boxtimes K_2, K) \xrightarrow{\sim} \text{Map}_{Shv(X) \otimes Shv(Y)}((K_1 \otimes (\mathbb{D}F)) \boxtimes K_2, h^R(K)) \\ &\xrightarrow{\sim} \text{Map}_{Shv(X) \otimes Shv(Y)}(K_1 \boxtimes K_2, (F \boxtimes \omega) \otimes h^R(K)) \end{aligned}$$

Let us underline that in the above formulas $(F \boxtimes \omega) \otimes h^R(K)$ denotes the tensor product in the symmetric monoidal category $Shv(X) \otimes Shv(Y)$. \square

Recall also that h^R coincides with h^\vee with respect to the Verdier self-dualities, see ([22], Sect. 1.0.1).

0.0.8. If $i : Z' \rightarrow Z$ is a closed immersion and $F \in Shv(Z)$ satisfies $i^! F = 0$ then F is in the essential image of $j_* : Shv(Z - Z') \rightarrow Shv(Z)$. Here $j : Z - Z' \rightarrow Z$. For $F \in Shv(Z)$ one as a fibre sequence

$$i_! i^! F \rightarrow F \rightarrow j_* j^! F$$

in $Shv(Z)$. In particular, if $M \in Shv(Z)$ satisfies $j^*M = 0$ then M is in the essential image of $i_!$.

0.0.9. Let S be an ind-scheme of ind-finite type. In the constructible context, the tensor product $\otimes^! : Shv(S) \otimes Shv(S) \rightarrow Shv(S)$ admits a continuous right adjoint. Indeed, for $F_i \in Shv(S)^c$ it suffices to show that $F_1 \otimes^! F_2$ is compact. For this, it suffices to show that $\mathbb{D}(\Delta^*(\mathbb{D}F_1 \boxtimes \mathbb{D}F_2))$ is compact, and in turn that $\Delta^*(\mathbb{D}F_1 \boxtimes \mathbb{D}F_2)$ is compact. This is true, because for $\Delta : S \rightarrow S \times S$, Δ^* has a continuous right adjoint Δ_* .

This is not the case for \mathcal{D} -modules, as far as I understand, because Δ^* is not always defined.

0.0.10. What are the t-structures on $Shv(Y)$ and under which assumptions and how they are defined? Perverse one, usual one?

For $Y \in Sch_{ft}$ there is a t-structure on $Shv(Y)$ that we think of as being perverse. It is important that this t-structure is accessible. It is also compatible with filtered colimits (this reduces to the fact that the t-structure on $Vect$ is compatible with filtered colimits).

The t-structure on $Shv(Y)$ for Y an ind-scheme is defined as follows. If $Y = \text{colim}_{i \in I} Y_i$ with I filtered and $Y_i \in Sch_{ft}$ then $Shv(Y)^{\leq 0} \subset Shv(Y)$ should be the smallest full subcategory containing $Shv(Y_i)^{\leq 0}$ for any i , closed under extensions and closed under small colimits. By (HA, 1.4.4.11), $Shv(Y)^{\leq 0}$ is then presentable and defines an accessible t-structure on Y . We use here the fact that $Shv(Y_i)$ is generated by a small set of objects.

Note that for an ind-scheme Y of ind-finite type $F \in Shv(Y)$ lies in $Shv(Y)^{\geq 0}$ iff for any closed subscheme $i : Y' \subset Y$ one has $i^!F \in Shv(Y')^{\geq 0}$. This implies that the t-structure on $Shv(Y)$ is compatible with filtered colimits.

You should also explain what is assumed about right or left completeness of the t-structure on $Shv(S)$ for $S \in Sch_{ft}$. Apparently, you assume it is right complete, as you want to use maps like $D^+(Shv(Y)^{\heartsuit}) \rightarrow Shv(Y)$?

For an algebraic stack with an affine diagonal Y we define the perverse t-structure on $Shv(Y)$ by

$$Shv(Y)^{\leq 0} \simeq \lim_{S \xrightarrow{\alpha} Y} Shv(S)^{\leq -\dim.\text{rel}(\alpha)},$$

where the limit is over the category whose objects are smooth maps $\alpha : S \rightarrow Y$ with $S \in Sch_{ft}$, and morphisms from (S, α) to (S', α') is a smooth map $S \rightarrow S'$ compatible with α, α' . The transition functors here are the $!$ -pullbacks. This defines an accessible t-structure by ([16], 1.4.4.11) or better by ([7], ch. I.3, Lemma 1.5.8). We have $Shv(Y)^{>0} \simeq \lim_{S \xrightarrow{\alpha} Y} Shv(S)^{> -\dim.\text{rel}(\alpha)}$ taken over the same category with the transition

functors being $!$ -pullbacks. This t-structure is compatible with filtered colimits and both left and right complete by *loc.cit.*

Claim. If Y is an algebraic stack with an affine diagonal then in the constructible context $Shv(Y)$ is right complete.

Proof. The t-structure on $Shv(Y)$ is accessible, so by ([21], 4.0.10) it suffices to show that for $L \in Shv(Y)$ the natural map $\text{colim}_n \tau^{\leq n} L \rightarrow L$ is an isomorphism. This

property is local in Zariski topology, so it suffices to show this is an isomorphism over any open quasi-compact substack $U \subset Y$.

For each U the category $Shv(U)$ is right complete. Indeed, we have an adjoint pair $ren_U : Shv(U) \rightleftarrows Shv(U)^{ren} : un - ren_U$ in $DGCat_{cont}$ as in ([1], F.5.3) with ren_U fully faithful. The t-structure on $Shv(U)^{ren}$ is right complete by ([21], 9.3.18). The t-structure on $Shv(U)$ is accessible, so by ([21], 4.0.10) it suffices to show that for $K \in Shv(U)$ the natural map $\text{colim}_n \tau^{\leq n} K \rightarrow K$ is an isomorphism in $Shv(U)$. To see this, let $K' = ren_U(K)$. Then the natural map $\text{colim}_n \tau^{\leq n} K' \rightarrow K'$ is an isomorphism in $Shv(U)^{ren}$. Since $un - ren_U$ is t-exact, $K \xrightarrow{\sim} un - ren_U(K')$ identifies with

$$\text{colim}_n un - ren_U(\tau^{\leq n} K') \xrightarrow{\sim} \text{colim}_n \tau^{\leq n}(un - ren_U(K')) \xrightarrow{\sim} \text{colim}_n \tau^{\leq n}(K)$$

We are done. \square

0.0.11. For $S \in \text{Sch}_{ft}$ mention that $Shv(S)$ is assumed compactly generated. So, for an ind-scheme of ind-finite type Y , $Shv(Y)$ is also compactly generated. Moreover, the Verdier duality provides an equivalence $Shv(Y)^\vee \xrightarrow{\sim} Shv(Y)$, which is an isomorphism of $Shv(Y)$ -modules. The corresponding map $Shv(Y) \otimes Shv(Y) \rightarrow \text{Vect}$ sends (F_1, F_2) to $\text{R}\Gamma(Y, F_1 \otimes^! F_2)$.

If now $f : Y_1 \rightarrow Y_2$ is a morphism of ind-schemes of ind-finite type then the dual to $f^! : Shv(Y_2) \rightarrow Shv(Y_1)$ identifies with $f_* : Shv(Y_1) \rightarrow Shv(Y_2)$.

If moreover, we are in the **constructible context**, since $(f_!, f^!)$ is an adjoint pair, its dual $((f^!)^\vee, (f_!)^\vee)$ is also an adjoint pair. So, the dual to $f_! : Shv(Y_1) \rightarrow Shv(Y_2)$ is the right adjoint to $f_* : Shv(Y_1) \rightarrow Shv(Y_2)$.

Assume $f : Y_1 \rightarrow Y_2$ schematic of finite type. In the constructible context, f_* has a left adjoint f^* , hence $(f^!, (f^*)^\vee)$ is an adjoint pair, so $f^!$ has a continuous right adjoint.

Example: let T be a split torus. Then e on $B(T)$ is not compact in the constructible context, that is $\text{R}\Gamma : Shv(B(T)) \rightarrow \text{Vect}$ is not continuous. So, this functor can not be the dual of $f^!$ for $f : B(T) \rightarrow \text{Spec } k$.

There is a projection formula for maps $f : Y \rightarrow Y'$, where Y is a quasi-compact classical algebraic stack with affine diagonal and Verdier compatible, it is formulated in ([2], B). This f_* satisfies the projection formula (even if not continuous).

0.0.12. Consider the 1-full subcategory $\text{PreStk}_{ind-sch} \subset \text{PreStk}_{lft}$, where we restrict 1-morphisms to be ind-schematic. Then we have a well-defined functor

$$Shv_{\text{PreStk}_{ind-sch}} : \text{PreStk}_{ind-sch} \rightarrow DGCat_{cont}$$

sending Y to $Shv(Y)$ and a morphism $f : Y \rightarrow Y'$ to $f_* : Shv(Y) \rightarrow Shv(Y')$. Moreover, this functor is right-lax symmetric monoidal, so sends algebras to algebras. So, if G is an algebra in $\text{PreStk}_{ind-sch}$, $(Shv(G), \star)$ will become a monoidal DG -category with the monoidal convolution structure.

So, we may talk about strong actions of $Shv(G)$ on some $C \in DGCat$, this is an object of $(Shv(G), \star) - \text{mod}(DGCat)$.

0.0.13. If G is an ind-scheme of ind-finite type, assume $m : G \times G \rightarrow G$ ind-proper. Then $(Shv(G), \star)$ is rigid for any sheaf theory. My understanding is that there is no hope for it to be rigid without the ind-properness assumption.

0.0.14. If G is a group ind-scheme of ind-finite type then $(Shv(G), m_*)$ is monoidal (convolution monoidal structure).

The functor $Shv(G) \otimes Shv(G) \rightarrow Shv(G \times G)$ sends a compact object $F_1 \otimes F_2$ to a compact object $F_1 \boxtimes F_2$.² So, this functor admits a continuous right adjoint. In the constructible context the functor $m_* : Shv(G \times G) \rightarrow Shv(G)$ admits a continuous right adjoint. Besides, the dual to m_* is the functor $m^!$. Thus, passing to the dual in $(Shv(G), m_*)$, in the constructible context we get a coalgebra $(Shv(G), m^!)$ in $DGCat_{cont}$. Recall that $(Shv(G), m_*) - mod \xrightarrow{\sim} (Shv(G), m^!) - comod$ (cf. [21]).

For any ind-scheme of ind-finite type Y , Y is a cocommutative coalgebra in $PreStk_{lft}$ via the maps $Y \rightarrow Y \times Y$ and $Y \rightarrow \text{Spec } k$, hence a commutative algebra in $(PreStk_{lft})^{op}$. Applying the right-lax monoidal functor Shv , we get on $Shv(Y)$ a commutative algebra structure in $CAlg(DGCat_{cont})$. The product is $Shv(Y) \otimes Shv(Y) \rightarrow Shv(Y \times Y) \xrightarrow{\Delta^!} Shv(Y)$. We denote this algebra $(Shv(Y), \Delta^!)$. It makes sense for any sheaf theory. Applying the duality, we get a coalgebra structure on $Shv(Y)$, which we denote $(Shv(Y), \Delta_*)$ following [3]. Recall that this duality exchanges the functors Δ_* and $\Delta^!$.

Sam says $(Shv(G), \Delta^!, m^!)$ is probably not a Hopf algebra in the constructible context (only for \mathcal{D} -modules). Similarly for $(Shv(G), m_*, \Delta_*)$. For \mathcal{D} -modules this was explained in [3]. Though $(Shv(G), m_*) - mod$ is a symmetric monoidal category for \mathcal{D} -modules, this does not seem to be the case in the constructible context.

Sam's idea: if this was the case, consider the diagonal action of $(Shv(G), m_*)$ on $Shv(G) \otimes Shv(G)$. It is given by a map of algebras $h^R \circ \Delta_* : Shv(G) \rightarrow Shv(G) \otimes Shv(G)$, which is the coproduct. Here $h : Shv(G) \otimes Shv(G) \hookrightarrow Shv(G \times G)$ is the exterior product, and h^R is its right adjoint. Besides, $\Delta_* : Shv(G) \rightarrow Shv(G \times G)$ is a morphism in $Alg(DGCat_{cont})$. Is it true that h^R or h then becomes a morphism in $Shv(G) - mod$? Then we could consider the map between the invariants, hopefully to get a contradiction. We have in mind that $\Delta_* \omega_G$ is invariant under the diagonal action, but does not lie in the essential image of h , here $\Delta : G \rightarrow G \times G$ is the diagonal. Not clear.

0.0.15. If $Y \in PreStk_{lft}$ is equipped with a G -action then the action map $a : G \times Y \rightarrow Y$ is ind-schematic (isomorphic to the projection $Y \times G \rightarrow Y$). So, $(Shv(G), \star)$ acts on $Shv(Y)$ on the left via $F \in Shv(G), K \in Shv(Y) \mapsto a_*(F \boxtimes K)$. If $f : Y_1 \rightarrow Y_2$ is an ind-schematic morphism in $PreStk_{lft}$ commuting with G -actions then $f_* : Shv(Y_1) \rightarrow Shv(Y_2)$ is a map of $(Shv(G), \star)$ -modules. Besides, $f^!$ is a map of $(Shv(G), \star)$ -modules. Consider the prestack quotient $Y/G \in PreStk_{lft}$. The map $f : Y \rightarrow Y/G$ commutes with G -actions, where G acts trivially on Y/G . So, $f^! : Shv(Y/G) \rightarrow Shv(Y)$ is a map of $(Shv(G), \star)$ -modules. Thus, by ([25], 1.10.10) it induces a functor

$$(2) \quad Shv(Y/G) \rightarrow \text{Fun}_{(Shv(G), \star)}(\text{Vect}, Shv(Y))$$

Is it an equivalence?

²Is it true for any sheaf theory? In ([10], 1.2.5(b)) you mentioned this only for two sheaf theories, but not for constructible sheaves in the classical topology. I imagine this is a misprint there! You actually claim this for any placid ind-schemes Y_1, Y_2 in ([10], C.2.8), so I assume this is true for any sheaf theory.

0.0.16. In general the answer is not clear. Assume G smooth of finite type. Then this is an equivalence, as Lin Chen shows (there is a different proof in ([10], 1.4.5)). Here is his argument.

One shows that $Shv(Y/G) \xrightarrow{\sim} e\text{-comod}(Shv(Y))$ by verifying the comonadic Beck-Chevalley conditions. Here e is the constant sheaf on G , it is a coalgebra in $(Shv(G), \star)$, and we consider the corresponding category of comodules with the convolution action of $Shv(G)$ on $Shv(Y)$. The forgetful functor $e\text{-comod}(Shv(Y)) \rightarrow Shv(Y)$ is $f^!$ for $f : Y \rightarrow Y/G$. The self-functor underlying the comonad is $p_*a^* : Shv(Y) \rightarrow Shv(Y)$. It also identifies with a_*p^* , here $a : G \times Y \rightarrow Y$ is the action map, $p : G \times Y \rightarrow Y$ is the projection.

Since $Shv(G)$ is self-dual, $Shv(Y)^G$ identifies with the limit of

$$Shv(Y) \rightrightarrows Shv(G) \otimes Shv(Y) \rightrightarrows Shv(G)^{\otimes 2} \otimes Shv(Y) \dots$$

(For \mathcal{D} -modules, since $Shv(G)^{\otimes n} \otimes Shv(Y) \xrightarrow{\sim} Shv(G^n \times Y)$, this finishes the proof). Assume now we are in the constructible context.

The above cosimplicial diagram is also

$$Shv(Y) \rightrightarrows \text{Fun}(Shv(G), Shv(Y)) \rightrightarrows \text{Fun}(Shv(G)^{\otimes 2}, Shv(Y)) \dots$$

The functors $Shv(Y) \rightrightarrows \text{Fun}(Shv(G), Shv(Y))$ are: F goes to $(K \mapsto K * F)$, and F goes to $(K \mapsto R\Gamma(G, K) \otimes F)$. The second functor identifies via the Verdier duality with $Shv(Y) \rightarrow Shv(G) \otimes Shv(Y)$, $F \mapsto \omega_G \otimes F$. Its right adjoint is $p_*[-2n] \otimes \text{id} : Shv(G) \otimes Shv(Y) \rightarrow Shv(Y)$ for $p : G \rightarrow \text{Spec } k$, where $n = \dim G$.

The comonadic Beck-Chevalley condition for the above cosimplicial diagram holds, it is mentioned in [10], 1.4.6 without a proof. We also check this in bigger generality in Section 0.0.23 of this file.

The corresponding comonad on $Shv(Y)$ is $Shv(Y) \rightarrow \text{Fun}(Shv(G), Shv(Y)) \xrightarrow{T^0} Shv(Y)$, where the first functor sends F to $(K \mapsto K * F)$. Thus, this comonad sends F to $e * F$. We see that both comonads are the same.

0.0.17. Let G be a smooth group scheme of finite type, $Y \in \text{PreStk}_{\text{ft}}$. The equivalence $Shv(B(G)) \xrightarrow{\sim} \text{Fun}_{(Shv(G), \star)}(\text{Vect}, \text{Vect})$ given by (2) transforms the symmetric monoidal structure on $Shv(B(G))$ given by $\otimes^!$ to the composition monoidal structure on $\text{Fun}_{(Shv(G), \star)}(\text{Vect}, \text{Vect})$.

The projection $q : Y/G \rightarrow B(G)$ yields an action of $(Shv(B(G)), \otimes^!)$ on $Shv(Y/G)$. Namely, $K \in Shv(B(G))$ acts on $M \in Shv(Y/G)$ as $(q^!K) \otimes^! M$. Similarly, the monoidal category $\text{Fun}_{(Shv(G), \star)}(\text{Vect}, \text{Vect})$ acts on $\text{Fun}_{(Shv(G), \star)}(\text{Vect}, Shv(Y))$ by composition on the left. The equivalence (2) is compatible with these actions via the above monoidal equivalence

$$Shv(B(G)) \xrightarrow{\sim} \text{Fun}_{(Shv(G), \star)}(\text{Vect}, \text{Vect})$$

0.0.18. We need the following claim: for $Y \in \text{Sch}_{\text{ft}}$, its cohomology $C(Y)$ is bounded, and the dimension of each H^i is finite. It was used in ([10], B.3.1) to show that for a smooth group scheme of finite type H and $C \in Shv(H)\text{-mod}$, $C_H \rightarrow C^H$ is an

equivalence. In the constructible context this is automatic, because $p_* : Shv(Y) \rightarrow Vect$ for $p : Y \rightarrow Spec k$ admits a continuous right adjoint, and the constant sheaf e_Y is compact, so $p_*(e_Y)$ is also compact.

So, a suitable finiteness assumption on the functor p_* should be formulated which holds for any sheaf theory. How it is formulated?

0.0.19. Consider a cartesian square

$$(3) \quad \begin{array}{ccc} X & \xrightarrow{f_X} & X' \\ \downarrow g & & \downarrow g' \\ Y & \xrightarrow{f_Y} & Y', \end{array}$$

in $PreStk$, where all objects are placid ind-schemes. For which morphisms g' we have the functors $(g')^!, (g')^*$? When do we have the base change with respect to $(f_Y)_*$?

Lemma 0.0.20. *let $Y' \in Sch_{ft}$ and Y, X' be placid schemes over Y' , recall then X is also a placid scheme. Assume $Y \xrightarrow{\sim} \lim_{i \in I^{op}} Y_i$, where I is filtered, $f_{Y,i} : Y_i \rightarrow Y'$ is smooth, $Y_i \in Sch_{ft}$, and for $i \rightarrow j$ in I , $Y_j \rightarrow Y_i$ is smooth affine surjective morphism in Sch_{ft} . Then one has $f_Y^* g'_* \xrightarrow{\sim} g_* f_X^*$.*

Proof. 1) Assume first $g' : X' \rightarrow Y'$ a morphism in Sch_{ft} . Set $X_i = Y_i \times_{Y'} X'$ for $i \in I$, so $X \xrightarrow{\sim} \lim_{i \in I^{op}} X_i$. For each i we get a cartesian square

$$\begin{array}{ccc} X_i & \xrightarrow{f_{X,i}} & X' \\ \downarrow g_i & & \downarrow g' \\ Y_i & \xrightarrow{f_{Y,i}} & Y', \end{array}$$

So, $f_{Y,i}^* g'_* \xrightarrow{\sim} (g_i)_* f_{X,i}^*$ naturally. So, $(g_i)_*$ form a morphism of the corresponding colimit systems giving $g_* : Shv(X) \xrightarrow{\sim} \text{colim}_{i \in I} Shv(X_i) \rightarrow \text{colim}_{i \in I} Shv(Y_i) \xrightarrow{\sim} Shv(Y)$. The claim follows.

2) Let now $g' : X' \rightarrow Y'$ be any placid scheme over Y' . Write $X' \xrightarrow{\sim} \lim_{j \in J} X'_j$ with $X'_j \in Sch_{ft}$, J filtered, and for $j \rightarrow j'$ in J the map $X'_{j'} \rightarrow X'_j$ is smooth affine and surjective. Set $X_j = Y \times_{Y'} X'_j$ for $j \in J$. Then X_j is a placid scheme, and we get the diagram

$$\begin{array}{ccc} X_j & \xrightarrow{f_{X,j}} & X'_j \\ \downarrow g_j & & \downarrow g'_j \\ Y & \xrightarrow{f_Y} & Y', \end{array}$$

for $j \in J$. Note that $Shv(X) \xrightarrow{\sim} \lim_{j \in J^{op}} Shv(X_j)$ with respect to the $*$ -direct image transition functors. By 1), for each $j \in J$,

$$(4) \quad f_Y^* (g'_j)_* \xrightarrow{\sim} (g_j)_* f_{X,j}^*$$

naturally. The functors $f_{X,j}^*$ are compatible with the corresponding inverse systems and give in the limit over J^{op} the functor f_X^* . Pick any $j \in J$. Then g' is the composition $X' \xrightarrow{ev'_j} X'_j \xrightarrow{g'_j} Y'$. Since $(f_{X,j}^*)(ev'_j)_* \xrightarrow{\sim} (ev_j)_* f_X^*$ our claim follows from (4). \square

If $S \in \text{Sch}_{ft}$ and $Z \rightarrow S$ is a placid S -scheme, let $i : S' \rightarrow S$ be a map in Sch_{ft} and $h : Z' \rightarrow Z$ be obtained by base change. Then $h^! : \text{Shv}(Z) \rightarrow \text{Shv}(Z')$ is defined: write $Z \xrightarrow{\sim} \lim_{i \in I^{op}} Z_i$, where I is small filtered, $Z_i \in \text{Sch}_{ft}/S$, and for $i \rightarrow j$ in I the map $Z_j \rightarrow Z_i$ in Sch_{ft}/S is smooth affine surjective. Then let $Z'_i = Z_i \times_S S'$, let $h_i : Z'_i \rightarrow Z_i$ be the corresponding map. The functors $h_i^!$ are compatible with $*$ -pushforwards in the diagrams $\text{Shv}(Z) \xrightarrow{\sim} \lim_{i \in I^{op}} \text{Shv}(Z_i)$, $\text{Shv}(Z') \xrightarrow{\sim} \lim_{i \in I^{op}} \text{Shv}(Z'_i)$. In the limit they yield the functor $h^!$. For the projections $p_i : Z \rightarrow Z_i$, $p'_i : Z' \rightarrow Z'_i$ we get $h_i^!(p_i)_* \xrightarrow{\sim} (p'_i)_* h^!$ and $h^! p_i^* \xrightarrow{\sim} (p'_i)^* h_i^!$ canonically.

Lemma 0.0.21. *Let $S \in \text{Sch}_{ft}$, assume given a cartesian square in PreStk_S*

$$\begin{array}{ccc} Y & \xleftarrow{g} & Y' \\ \downarrow f & & \downarrow f' \\ Z & \xleftarrow{h} & Z' \end{array}$$

Assume I is a filtered category, and we are given a morphism $f_i : Y_i \rightarrow Z_i$ in $(\text{Sch}_{ft})/S$ functorial in $i \in I^{op}$, where f_i is smooth. We assume for $i \rightarrow j$ in I the transition maps $Y_j \rightarrow Y_i$, $Z_j \rightarrow Z_i$ are smooth affine surjective. We assume that $f : Y \rightarrow Z$ is obtained from f_i by passing to the limit over I^{op} . We assume $i : S' \rightarrow S$ is a map in Sch_{ft} , and $f' : Y' \rightarrow Z'$ is obtained from f by the base change $i : S' \rightarrow S$. Then $g^! f^ \xrightarrow{\sim} (f')^* h^!$ naturally. We do not assume here that the squares*

$$\begin{array}{ccc} Y_j & \rightarrow & Y_i \\ \downarrow f_j & & \downarrow f_i \\ Z_j & \rightarrow & Z_i \end{array}$$

are cartesian.

Proof. By definition, $f^* : \text{Shv}(Z) \rightarrow \text{Shv}(Y)$ is obtained by passing to the colimit over I in $f_i^* : \text{Shv}(Z_i) \rightarrow \text{Shv}(Y_i)$. Note that Y, Y', Z, Z' are placid S -schemes. Note that $h^!$ is obtained by passing to the colimit over I in $h_i^! : \text{Shv}(Z_i) \rightarrow \text{Shv}(Z'_i)$, and similarly for $g^!$. Recall that $\text{Shv}(Z) \xrightarrow{\sim} \text{colim}_{i \in I} \text{Shv}(Z_i)$.

For $i \in I$ and $K \in \text{Shv}(Z_i)$ we have $g_i^! f_i^* K \xrightarrow{\sim} (f'_i)^* h_i^! K$ canonically. Passing to the colimit over I , one gets the desired claim. \square

0.0.22. Let Z be a placid scheme written as $Z = \lim_{i \in I^{op}} Z_i$. For $i \rightarrow j$ in I let $f_{ij} : Z_j \rightarrow Z_i$ be the corresponding morphism, it is smooth of relative dimension d_{ij} , affine, surjective. Since $\text{Shv}(Z) \xrightarrow{\sim} \text{colim}_i \text{Shv}(Z_i)$ via the maps f_{ij}^* , $\text{Shv}(Z)$ is compactly generated, hence dualizable. By ([7], ch. I.1, 6.3.4), by applying the dualization functor to the functor

$$I \rightarrow \text{DGCat}_{cont}, i \mapsto \text{Shv}(Z_i), (i \rightarrow j) \mapsto f_{ij}^*,$$

we get a functor $I^{op} \rightarrow \text{DGCat}_{cont}, i \mapsto \text{Shv}(Z_i), (i \rightarrow j) \mapsto (f_{ij})_*[-2d_{ij}]$. Moreover,

$$\text{Shv}(Z)^\vee \xrightarrow{\sim} \lim_{i \in I^{op}} \text{Shv}(Z_i)$$

with respect to the transition maps $(f_{ij})_*[-2d_{ij}]$. Consider for $i \in I$ the isomorphism $Shv(Z_i) \xrightarrow{\otimes e[2d_i]} Shv(Z_i)$ with $d_i = \dim Z_i$. So, $d_{ij} = d_j - d_i$. The diagram commutes

$$\begin{array}{ccc} Shv(Z_j) & \xrightarrow{\otimes e[2d_j]} & Shv(Z_j) \\ \downarrow (f_{ij})_* & & \downarrow (f_{ij})_*[-2d_{ij}] \\ Shv(Z_i) & \xrightarrow{\otimes e[2d_i]} & Shv(Z_i) \end{array}$$

Passing to the limit over I^{op} , we obtain an equivalence $Shv(Z) \xrightarrow{\sim} Shv(Z)^\vee$. So, a possibility is to mention that for each placid scheme Z , $Shv(Z)$ is canonically self-dual. However, this self-duality is not compatible with the one for finite type schemes, so maybe it is not needed.

Example: assume $0 \in I$ is an initial object, let $K_0 \in Shv(Z_0)$. For the projection $f_0 : Z \rightarrow Z_0$ the image of $f_0^* K_0$ in $Shv(Z)^\vee$ under this duality is the composition $Shv(Z) \xrightarrow{(f_0)_*} Shv(Z_0) \rightarrow \text{Vect}$, where the second functor is $M \mapsto R\Gamma(Z_0, K_0 \otimes^! M)[2d_0]$.

0.0.23. Let G be a group scheme, which is a placid scheme, $C \in G - \text{mod}$. Consider the cosimplicial category defining C^G :

$$\text{Fun}(\text{Vect}, C) \rightrightarrows \text{Fun}(Shv(G), C) \rightrightarrows \text{Fun}(Shv(G)^{\otimes 2}, C) \dots$$

Let us show that it satisfies the comonadic Beck-Chevalley conditions.

The functor corresponding to the last face map $\partial_n : [n] \rightarrow [n+1]$ (its image avoids $n+1$) is the following functor F_n . We consider $Shv(G)^{\otimes n+1} \rightarrow Shv(G)^{\otimes n}$, $\text{id} \otimes R\Gamma$, and compose it with $\text{Fun}(\cdot, C)$. For $p : G \rightarrow \text{Spec } k$ the functor p_* has a left adjoint p^* . Let T_n be the functor obtained from $\text{id} \otimes p^* : Shv(G)^{\otimes n} \rightarrow Shv(G)^{\otimes n+1}$ by composing with $\text{Fun}(\cdot, C)$. Then T_n is the right adjoint to F_n . Let now $\alpha : [m] \rightarrow [n]$ be a map in Δ . Consider the corresponding diagram

$$\begin{array}{ccc} \text{Fun}(Shv(G)^{\otimes n}, C) & \xleftarrow{T_n} & \text{Fun}(Shv(G)^{\otimes n+1}, C) \\ \uparrow F_\alpha & & \uparrow F_{\alpha+1} \\ \text{Fun}(Shv(G)^{\otimes m}, C) & \xleftarrow{T_m} & \text{Fun}(Shv(G)^{\otimes m+1}, C) \end{array}$$

We show that it commutes. It suffices to prove this for α injective, because of the following. Let $\Delta_s \subset \Delta$ be the full subcategory with the same class of object, where we keep only injective maps. Then $\Delta_s^{op} \rightarrow \Delta^{op}$ is cofinal by ([15], 6.5.3.7). If $\alpha : [m] \rightarrow [n]$ is injective, and $0, n$ are in the image then the desired commutativity follows from the commutativity of

$$\begin{array}{ccc} Shv(G)^{\otimes n} & \xrightarrow{\text{id} \otimes p^*} & Shv(G)^{\otimes n+1} \\ \downarrow (m_\alpha)_* & & \downarrow (m_{\alpha+1})_* \\ Shv(G)^{\otimes m} & \xrightarrow{\text{id} \otimes p^*} & Shv(G)^{\otimes m+1}, \end{array}$$

where $(m_\alpha)_*$ is the product along α in the monoidal category $Shv(G)$.

If $\alpha : [n-1] \rightarrow [n]$ is the last face map then $\alpha+1 : [n] \rightarrow [n+1]$ avoids n . The functor $F_{\alpha+1}$ is the composition with $Shv(G)^{\otimes n+1} \rightarrow Shv(G)^{\otimes n}$, $K_1 \otimes \dots \otimes K_{n+1} \mapsto K_1 \otimes \dots \otimes K_{n-1} \otimes K_n * K_{n+1}$. In this case the desired commutativity follows from $K * e_G \xrightarrow{\sim} R\Gamma(G, K) \otimes e_G$.

If $\alpha : [n-1] \rightarrow [n]$ is injective and avoids 0 then F_α sends f to the functor

$$K_1 \otimes \dots \otimes K_n \mapsto K_1 * f(K_2 \otimes \dots \otimes K_n)$$

and the commutativity is tautological. So, it always hold.

By ([9], Lemma C.1.9), the functor $\text{oblv}_G : C^G \rightarrow C$ is comonadic, and the corresponding comonad on C is $C \rightarrow C, c \mapsto e_G * c$.

0.0.24. Consider a placid scheme $Y = \lim_{i \in I^{\text{op}}} Y_i$, where I is filtered, if $i \rightarrow j$ in I then $f_{ij} : Y_j \rightarrow Y_i$ is smooth, affine and surjective, and Y_i is a scheme of finite type. In this case $\text{Shv}(Y)$ is defined in [10] as $\lim_{i \in I^{\text{op}}} \text{Shv}(Y_i)$ via the maps $(f_{ij})_*$.

In the paper there are situations, where we have morphisms $h : Y \rightarrow S$, where S is an ind-scheme, and we want functors between $\text{Shv}(Y)$ and $\text{Shv}(S)$ attached to h . So, the above definition of $\text{Shv}(Y)$ for placid schemes should be "unified" with the definition of $\text{Shv}(Z)$ for prestacks Z locally of finite type. Namely, do we have certain full subcategory of $\text{PreStk}_{\text{lft}}$, on which Shv is defined as a functor, and which contains both $\text{PreStk}_{\text{lft}}$, placid schemes, and is closed under colimits? Compare with [28].

0.0.25. Let now Z, Z' be placid schemes and $i : Z' \rightarrow Z$ a placid closed immersion. What is the dual of the adjoint pair $i_* : \text{Shv}(Z') \rightarrow \text{Shv}(Z) : i^!$?

We explain the dual of i_* . If $Z = \lim_{i \in I^{\text{op}}} Z_i$ and, assume for simplicity I has an initial object i_0 such that $Z' = Z'_{i_0} \times_{Z_{i_0}} Z$. So, $Z' = \lim_{i \in I^{\text{op}}} Z'_i$ with $Z'_i = Z_i \times_{Z_{i_0}} Z'_{i_0}$. For $i \rightarrow j$ in I let $f_{ij} : Z_j \rightarrow Z_i$ be the corresponding transition map. For the closed embeddings $i_i : Z'_i \rightarrow Z_i$ writing $\text{Shv}(Z) = \lim_{i \in I^{\text{op}}} \text{Shv}(Z_i)$ for $(f_{ij})_* : \text{Shv}(Z_j) \rightarrow \text{Shv}(Z_i)$ and similarly for $\text{Shv}(Z')$, the dual functor is given by the collection of functors $i_i^![2d_i - 2d'_i] : \text{Shv}(Z_i) \rightarrow \text{Shv}(Z'_i)$, here $d_i = \dim Z_i, d'_i = \dim Z'_i$ as locally constant functions, they form a morphism of the corresponding inverse systems. The number $d_i - d'_i$ does depend on i , and can be denoted $\text{codim}_Z(Z') = d_i - d'_i$. So, the dual of $i_* : \text{Shv}(Z') \rightarrow \text{Shv}(Z)$ is $i^![2 \text{codim}_Z(Z')]$.

0.0.26. Let Z be a placid ind-scheme. Is $\text{Shv}(Z)$ canonically self-dual? Here is some answer.

Write $Z = \text{colim}_{i \in I} Z_i$ with Z_i a placid scheme, I small filtered, and for $i \rightarrow j$ the map $f_{ij} : Z_i \rightarrow Z_j$ is a placid closed immersion. We have $\text{Shv}(Z) = \text{colim}_{i \in I} \text{Shv}(Z_i)$ with respect to the transition functors $(f_{ij})_*$.

Consider the functor $I \rightarrow \text{DGCat}_{\text{cont}}, i \mapsto \text{Shv}(Z_i), (i \rightarrow j) \mapsto (f_{ij})_*$. By ([7], ch. I.1, 6.3.4), the colimit of this functor $\text{colim}_{i \in I} \text{Shv}(Z_i) = \text{Shv}(Z)$ is dualizable, and $\text{Shv}(Z)^\vee \xrightarrow{\sim} \lim_{i \in I^{\text{op}}} \text{Shv}(Z_i)^\vee$, the limit of the dual functor.

Recall for each i the canonical self-duality on $\text{Shv}(Z_i)$ introduced in Sect. 0.0.22 of this file. It allows to rewrite $\text{Shv}(Z)^\vee \xrightarrow{\sim} \lim_{i \in I^{\text{op}}} \text{Shv}(Z_i)$, where the transition functors for $i \rightarrow j$ in I is $(f_{ij})^![2 \text{codim}_{Z_j}(Z_i)]$ in the notations of Section 0.0.25.

Pick an element $i_0 \in I$. Consider for $i \rightarrow j$ in I a commutative diagram

$$\begin{array}{ccc} \text{Shv}(Z_j) & \xrightarrow{\otimes e[-2 \text{codim}_{Z_j}(Z_{i_0})]} & \text{Shv}(Z_j) \\ \downarrow f_{ij}^! & & \downarrow f_{ij}^![2 \text{codim}_{Z_j}(Z_i)] \\ \text{Shv}(Z_i) & \xrightarrow{\otimes e[-2 \text{codim}_{Z_i}(Z_{i_0})]} & \text{Shv}(Z_i) \end{array}$$

Indeed, we have $\mathrm{codim}_{Z_j}(Z_i) + \mathrm{codim}_{Z_i}(Z_{i_0}) = \mathrm{codim}_{Z_j}(Z_{i_0})$. Passing to the limit over I^{op} , this provides an equivalence $Shv(Z)^\vee \xrightarrow{\sim} Shv(Z)$.

This duality maybe depend on a choice of an element $i_0 \in I$.

0.0.27. In Section 7.3.5 the perverse t-structure on $Shv_{\mathcal{G}\mathcal{G}}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty x})$ is mentioned without any definition. In the convention section a definition of the perverse t-structure for an ind-algebraic stack should be given. My understanding is as follows: if $Y = \mathrm{colim}_{i \in I} Y_i$ with Y_i an algebraic stack locally of finite type, I filtered then $Shv(Y)^{\leq 0}$ should be the smallest full subcategory of $Shv(Y)$ containing $Shv(Y_i)^{\leq 0}$ for any i , closed under extensions and small colimits. Then by (HA, 1.4.4.11), $Shv(Y)^{\leq 0}$ is then presentable and defines an accessible t-structure on $Shv(Y)$. For $K \in Shv(Y)$ we have $K \in Shv(Y)^{\geq 0}$ iff for any i , the !-restriction of K to Y_i lies in $Shv(Y_i)^{\geq 0}$. As in the case of ind-schemes of ind-finite type, this t-structure is compatible with filtered colimits.

0.0.28. For a scheme of finite type S , the perverse t-structure on $Shv(S)$ is left complete (by [1], 1.1.4). This implies that for an Artin stack locally of finite type S the t-structure on $Shv(S)$ is left complete as in ([7], ch. I.3, 1.5.7), because for a smooth atlas $f : S' \rightarrow S$ with S a scheme locally of finite type, $f^*[\mathrm{dim} f]$ is t-exact.

It should be clarified for which topologies on Sch_{ft} the functor $Shv : (\mathrm{Sch}_{ft})^{op} \rightarrow \mathrm{DGCat}_{cont}$ satisfies the descent, and a precise reference should be given. In particular, in ([12], proof of 4.2.7) you claim it satisfies the descent for the topology of finite surjective maps on Sch_{ft} . Give also a reference for the fact that it satisfies the étale descent. (For the proper descent this is Section 0.0.33 of this file). Sam claim we get this way h-descent, give accurate references. Add also it satisfies the smooth descent: if Y is a quasi-compact algebraic stack with a smooth cover $S \rightarrow Y$, where $S \in \mathrm{Sch}_{ft}^{aff}$, if S^\bullet is the Čech nerve of this map then $Shv(Y) \rightarrow \mathrm{Tot}(Shv(S^\bullet))$ is an equivalence. Does étale descent automatically implies the smooth descent here?

Add also the following. For a map $f : Y \rightarrow Z$ in PreStk_{lft} , which is surjective on geometric points, $f^!$ is conservative.

Cite the following. If $Y \in \mathrm{PreStk}_{lft}$ is an algebraic stack then $Shv(Y) = \lim_{S \rightarrow Y} Shv(S)$, where the limit is taken over the opposite to the category of affine schemes smooth over Y , and morphisms are smooth maps between those ([1], C.1.1).

0.0.29. Say that for any $Y \in \mathrm{PreStk}_{lft}$, $Shv(Y)$ is compactly generated in the constructible context by ([1], C.1.1). What happens for \mathcal{D} -modules?

In Section 4.3.3 you claimed the existence of the equivalence $\mathbb{D} : (Shv(Y)^c)^{op} \xrightarrow{\sim} Shv(Y)^c$ for an algebraic stack of finite type. Explain that this is known only under the assumption that Y is locally a quotient of a scheme S of finite type by an affine algebraic group, give a reference!

0.0.30. For example, it should be said somewhere that if $Z = \lim_{i \in I^{op}} Z_i$ is a placid scheme, where I is filtered, Z_i is a scheme of finite type, with the transition maps affine smooth and surjective, then for $i \in I$ and the projection $ev_i : Z \rightarrow Z_i$ the functor $ev_i^* : Shv(Z_i) \rightarrow Shv(Z)$ is defined, this is the natural functor $ins_i : Shv(Z_i) \rightarrow \mathrm{colim}_{j \in I} Shv(Z_j)$. For the moment this is hidden in ([10], C.2.9).

0.0.31. *On exterior product.* If $S_i \in \text{Sch}_{ft}$, $F_i \in \text{Shv}(S_i)^{\text{constr}}$ then $F_1 \boxtimes F_2 \in \text{Shv}(S_1 \times S_2)^{\text{constr}}$ by definition, as *constr* means being bounded with constructible cohomology.

Let $S \in \text{Sch}_{ft}$, $Y \in \text{PreStk}_{lft}$. The functor $\text{Shv}(S) \otimes \text{Shv}(Y) \rightarrow \text{Shv}(S \times Y)$, $(F, K) \mapsto F \boxtimes K$ is fully faithful and preserves compactness. We have to verify this in the constructible context, as for \mathcal{D} -modules this is an equivalence. Fully-faithfulness property is preserved by passing to the limit, and tensoring by $\text{Shv}(S)$ is the functor $\text{DGCat}_{\text{cont}} \rightarrow \text{DGCat}_{\text{cont}}$ preserving limits, as $\text{Shv}(S)$ is dualizable. This is why our functor is fully faithful.

To see that it admits a continuous right adjoint we use ([7], ch. I.1, 2.6.4). Write $\text{Shv}(Y) \xrightarrow{\sim} \lim_{T \rightarrow Y} \text{Shv}(T)$ over $(\text{Sch}_{ft}^{aff})_Y^{op}$. For each T the inclusion $i_T : \text{Shv}(S) \otimes \text{Shv}(T) \rightarrow \text{Shv}(S \times T)$ admits a continuous right adjoint i_T^R . Let $\alpha : T \rightarrow T'$ be a map in $(\text{Sch}_{ft}^{aff})_Y$. In the constructible context, $\alpha^! : \text{Shv}(T') \rightarrow \text{Shv}(T)$ admits a left adjoint $\alpha_!$, and we have $i_{T'}(\text{id} \otimes \alpha_!) \xrightarrow{\sim} (\text{id} \times \alpha_!)i_T$. This gives an isomorphism $\text{id} \otimes \alpha^! i_T^R \xrightarrow{\sim} i_{T'}^R(\text{id} \times \alpha^!)$. By ([7], ch. I.1, 2.6.4), $i : \text{Shv}(S) \otimes \text{Shv}(Y) \rightarrow \text{Shv}(S \times Y)$ has a right adjoint i^R , and for any $(T \xrightarrow{b} Y) \in (\text{Sch}_{ft}^{aff})_Y$ we have $(\text{id} \otimes b^!)i^R \xrightarrow{\sim} i_T^R(\text{id} \times b)^!$.

We check that i^R is continuous. Let $K \xrightarrow{\sim} \text{colim}_{j \in J} K_j$ in $\text{Shv}(S \times Y)$. By ([21], 2.2.68), it suffices to show that for any $(T \xrightarrow{b} Y) \in (\text{Sch}_{ft}^{aff})_Y$, $\text{id} \otimes b^!$ sends our diagram to a colimit diagram. This is true, because i_T^R and $(\text{id} \times b)^!$ are continuous.

Claim Let $X, Y, Z \in \text{Sch}_{ft}$ with X proper. In the constructible context, the diagram commutes

$$\begin{array}{ccc} \text{Shv}(X \times Y \times Z) & \xleftarrow{\boxtimes} & \text{Shv}(X) \otimes \text{Shv}(Y \times Z) \\ \downarrow \boxtimes^R & & \downarrow \boxtimes^R \\ \text{Shv}(X \times Y) \otimes \text{Shv}(Z) & \xleftarrow{\boxtimes} & \text{Shv}(X) \otimes \text{Shv}(Y) \otimes \text{Shv}(Z) \end{array}$$

Proof. The left vertical arrow is $\text{Shv}(X \times Y) \otimes \text{Shv}(Z)$ -linear by Section 0.0.7. Therefore, it suffices to calculate for $F \in \text{Shv}(Y \times Z)$ and the projection $q : X \times Y \times Z \rightarrow Y \times Z$ the object $\boxtimes^R(q^! F) \in \text{Shv}(X \times Y) \otimes \text{Shv}(Z)$. The functor $\boxtimes^R \circ q^!$ is right adjoint to the functor $\text{Shv}(X \times Y) \otimes \text{Shv}(Z) \xrightarrow{\boxtimes} \text{Shv}(X \times Y \times Z) \xrightarrow{q_!} \text{Shv}(Y \times Z)$. The latter functor identifies with the composition $\text{Shv}(X \times Y) \otimes \text{Shv}(Z) \xrightarrow{\bar{q} \otimes \text{id}} \text{Shv}(Y) \otimes \text{Shv}(Z) \xrightarrow{\boxtimes} \text{Shv}(Y \times Z)$, because X is proper. Here $\bar{q} : X \times Y \rightarrow Y$ is the projection. So, $\boxtimes^R \circ q^!$ identifies with the functor $\text{Shv}(Y \times Z) \xrightarrow{\boxtimes^R} \text{Shv}(Y) \otimes \text{Shv}(Z) \xrightarrow{\bar{q}^! \otimes \text{id}} \text{Shv}(X \times Y) \otimes \text{Shv}(Z)$. Our claim follows. \square

0.0.32. *Question.* Let $f : Y \rightarrow \text{Spec } k$ be a scheme of finite type. In the constructible context does the functor $p_! : \text{Shv}(Y) \rightarrow \text{Vect}$ preserve limits? Consider the dual functor $(p_!)^\vee : \text{Vect} \rightarrow \text{Shv}(Y)$. Is the object $(p_!)^\vee(e)$ compact? If it was compact, the functor $p_!$ would preserve limits.

0.0.33. Let $Y, Z \in \text{PreStk}_{lft}$ and $\pi : Y \rightarrow Z$ be proper, in particular, of finite type. Consider the Čech nerve $[\dots Y_Z^3 \rightrightarrows Y_Z^2 \rightrightarrows Y]$ of π . Applying Shv , we get a cosimplicial category $\Delta^{op} \rightarrow \text{DGCat}_{\text{cont}}$, $[n] \mapsto \text{Shv}(Y_Z^{n+1})$, here $Y_Z^n = Y \times_Z Y \times_Z \dots \times_Z Y$, the

product of n copies. For $i \geq 0$ let $\partial_i : [i] \rightarrow [i+1]$ be the last face map, it avoids $i+1$. The corresponding map $p^{\partial_i} : Y_Z^{i+1} \rightarrow Y_Z^i$ is the projection, so $(p^{\partial_i})^!$ has a left adjoint $(p^{\partial_i})_! = (p^{\partial_i})_*$. By base change, this cosimplicial category satisfies the monadic Beck-Chevalley conditions, so

$$\text{Tot}_{[n] \in \Delta} Shv(Y_Z^{n+1}) \xrightarrow{\sim} \mathcal{A} - \text{mod}(Shv(Y)),$$

where $\mathcal{A} = (p_2)_* p_1^!$ for the projections $p_1, p_2 : Y_Z^2 \rightarrow Y$.

Now $\pi^! : Shv(Z) \rightarrow Shv(Y)$ has a left adjoint $\pi_!$, and the monad $\pi^! \pi_!$ acting on $Shv(Y)$ identifies with \mathcal{A} . We always have a natural functor $Shv(Z) \rightarrow \mathcal{A} - \text{mod}(Shv(Y))$. Assume in addition that $\pi : Y \rightarrow Z$ is surjective on k -points (and more generally, on field-valued points, we have to take into account generic points in particular). Then $\pi^!$ is conservative, so that $\pi^!$ satisfies the Beck-Chevalley theorem ([7], ch. I.1, 3.7.7), and the induced functor $Shv(Z) \rightarrow \mathcal{A} - \text{mod}(Shv(Y))$ is an equivalence. Thus, Shv satisfies the proper descent.

0.0.34. It seems the following is also needed. Consider the cartesian square (3), where all the maps are schematic quasi-compact say. Let $F \in Shv(X')$ such that $g'_! : Shv(X') \rightarrow Shv(Y')$ is defined on F and $f_Y^* g'_! F$ is defined. Then $f_X^* F$ and $g_! f_X^* F$ are both defined and we have a natural isomorphism $g_! f_X^* F \xrightarrow{\sim} f_Y^* g'_! F$. Is this true?

I think this was used in ([11], proof of Prop. 2.8.2).

0.0.35. *Question.* Let Y be an ind-scheme of ind-finite type (or a classical algebraic stack locally of finite type). Let $U_i \subset Y$ be an open immersion for $i \in \mathbb{N}$ such that for $i < j$ we have $U_i \subset U_j$ and $\cup_i U_i = Y$. Is it true that $\text{colim}_{i \in \mathbb{N}} U_i$ in PreStk identifies with Y ?

Example: we may form a sequence of opens $U_i \subset \text{Gr}_G$, where each U_i is of the form $\text{Gr}_G - \cup_{i=1}^n \bar{S}^{\lambda_i}$ and

$$U_i \subset U_{i+1} \subset \dots$$

with $\cup_i U_i = \text{Gr}_G$. We have $Shv(\text{Gr}_G) \xrightarrow{\sim} \lim_i Shv(U_i)$ anyway, as for any closed subscheme of finite type $S \subset \text{Gr}_G$, $S \subset U_i$ for some i .

0.0.36. *Torsors under placid group-schemes.* Let \mathcal{Y}_α be an ind-scheme of ind-finite type functorial in $\alpha \in A^{op}$, where A is filtered, $\alpha_0 \in A$ is initial in A . Let $G = \lim_{\alpha \in A^{op}} G_\alpha$ be a placid group scheme, where G_α is a smooth group scheme of finite type, and for $\alpha \rightarrow \beta$ in A , $G_\beta \rightarrow G_\alpha$ is smooth, affine and surjective homomorphism of group schemes. Assume $\mathcal{Y}_\alpha \rightarrow \mathcal{Y}_{\alpha_0}$ is a G_α -torsor. For $\alpha \rightarrow \beta$ in A , $\mathcal{Y}_\beta \rightarrow \mathcal{Y}_\alpha$ is G_β -equivariant.

Then we are in the setting of ([10], C.1.6), so we get a placid ind-scheme \mathcal{Y} as follows. Write $\mathcal{Y}_{\alpha_0} = \text{colim}_{i \in I} Y_i$, where Y_i is a scheme of finite type, I is filtered, and for $i \rightarrow j$, $Y_i \rightarrow Y_j$ is a closed immersion. Let $\mathcal{Z}_i = \lim_{\alpha \in A^{op}} Y_i \times_{\mathcal{Y}_{\alpha_0}} \mathcal{Y}_\alpha$, so \mathcal{Z}_i is a placid scheme, and $\mathcal{Z}_i \rightarrow \mathcal{Z}_j$ is a placid closed immersion. So, $\mathcal{Y} := \text{colim} \mathcal{Z}_i$ is a placid ind-scheme, and $Shv(\mathcal{Y}) \xrightarrow{\sim} \lim_{\alpha \in A^{op}} Shv(\mathcal{Y}_\alpha)$ with respect to the functors $(f_{\alpha, \beta})_* : Shv(\mathcal{Y}_\beta) \rightarrow Shv(\mathcal{Y}_\alpha)$ for $\alpha \rightarrow \beta$ in A and $f_{\alpha, \beta} : \mathcal{Y}_\beta \rightarrow \mathcal{Y}_\alpha$. The group G acts on \mathcal{Y}_α for each α via the quotient $G \rightarrow G_\alpha$, this gives an action of $Shv(G)$ on $Shv(\mathcal{Y}_\alpha)$. The functors $(f_{\alpha, \beta})_*$ are morphisms of $Shv(G)$ -modules, so $Shv(\mathcal{Y})$ can be seen as $\lim_{\alpha \in A^{op}} Shv(\mathcal{Y}_\alpha)$ taken in $Shv(G) - \text{mod}$.

Let us show that $\mathcal{Y} \rightrightarrows \lim_{\alpha \in A^{op}} \mathcal{Y}_\alpha$ as prestacks. We have a natural map $\mathcal{Y} \rightarrow \lim_{\alpha} \mathcal{Y}_\alpha$. Let $S \in \text{Sch}^{aff}$. Recall that for any n , $\tau_{\leq n} \text{Spc} \subset \text{Spc}$ is stable under filtered colimits, so $\mathcal{Y}(S) \in \text{Sets}$ and an element of $\mathcal{Y}(S)$ comes from an element of $\mathcal{Z}_i(S)$ for some i (by [21], Cor. 13.1.14). So, an element of $\mathcal{Z}_i(S)$ is the same as an element of $\mathcal{Y}(S)$ whose image in $\mathcal{Y}_{\alpha_0}(S)$ lies in the subset $Y_i(S)$. This makes the claim manifest (and it holds more generally in the situation of ([10], C.1.6)).

Since $\text{Ker}(G \rightarrow G_\alpha)$ is pronipotent for $\alpha \neq \alpha_0$, we get $Shv(\mathcal{Y}_\alpha)^G \rightrightarrows Shv(\mathcal{Y}_\alpha)^{G_\alpha}$ for $\alpha \neq \alpha_0$ by ([25], 1.3.21). Now by Section 0.0.16 of this file, $Shv(\mathcal{Y}_\alpha)^{G_\alpha} \rightrightarrows Shv(\mathcal{Y}_{\alpha_0})$ via the functor $f_{\alpha_0, \alpha}^* : Shv(\mathcal{Y}_{\alpha_0}) \rightarrow Shv(\mathcal{Y}_\alpha)$. So,

$$Shv(\mathcal{Y})^G \rightrightarrows \lim_{\alpha \in A^{op}} Shv(\mathcal{Y}_\alpha)^G \rightrightarrows Shv(\mathcal{Y}_{\alpha_0})$$

We could take the functors $f_{\alpha_0, \alpha}^!$ instead, but the two limits would be isomorphic.

We may strengthen the above as follows. Assume H is a placid group scheme, $G \subset H$ is a placid closed immersion, and a normal group subscheme with the cokernel K , here K is a smooth affine group scheme of finite type. Assume the G -action on \mathcal{Y} is extended to a H -action. Then as above we get $Shv(\mathcal{Y})^H \rightrightarrows Shv(\mathcal{Y}_{\alpha_0}/K)$.

0.0.37. Let $H \in \text{Grp}(\text{PreStk})$ be a placid ind-scheme written as $H \rightrightarrows \text{colim}_{j \in J} H_j$, where H_j is a placid group scheme, and for $j \rightarrow j'$ in J the map $H_j \rightarrow H_{j'}$ is a placid closed immersion and a homomorphism of group schemes. Assume $j = 0$ is initial in J and let $G = H_0$. Then for any j , H_j/G is a scheme of finite type, so $H/G \rightrightarrows \text{colim}_{j \in J} H_j/G$, because colimits commute with colimits, so H/G is an ind-scheme of ind-finite type. Assume G prosmooth.

Write as in the previous section $G \rightrightarrows \lim_{\alpha \in A^{op}} G_\alpha$, where G_α is a smooth group scheme of finite type, and for $\alpha \rightarrow \beta$ in A , $G_\beta \rightarrow G_\alpha$ is smooth, affine and surjective. Set $K_\alpha = \text{Ker}(G \rightarrow G_\alpha)$. For $\alpha \rightarrow \beta$ in A let $1 \rightarrow K_{\alpha, \beta} \rightarrow G_\beta \rightarrow G_\alpha \rightarrow 1$ be an exact sequence. Assume $K_{\alpha, \beta}$ is a unipotent group scheme. Then $K_\alpha \rightrightarrows \lim_{\beta} K_{\alpha, \beta}$ is pronipotent.

Set $\mathcal{Y}_\alpha = H/K_\alpha$, we usually mean by this the etale sheafification of the prestack quotient. This is an ind-scheme of ind-finite type by the above, and for $\alpha \rightarrow \beta$ in A the map $\mathcal{Y}_\beta \rightarrow \mathcal{Y}_\alpha$ is a K_α/K_β -torsor. So, we are in the situation of the previous section, α_0 is initial in A . We write $H/K_{\alpha_0} = \rightrightarrows \text{colim}_j H_j/K_{\alpha_0}$. So, $\mathcal{Y} \rightrightarrows \lim_{\alpha} H/K_\alpha$. Note that $\lim_{\beta} (K_\alpha/K_\beta) \rightrightarrows K_\alpha$. We get $\mathcal{Y} \rightrightarrows \text{colim}_j H_j \rightrightarrows H$, because $\lim_{\alpha} H_j/K_\alpha \rightrightarrows H_j$ for any j .

From $H \rightrightarrows \lim_{\alpha \in A^{op}} H/K_\alpha$ we get $Shv(H) \rightrightarrows \lim_{\alpha} Shv(H/K_\alpha)$. From the previous section we now get an equivalence $Shv(H/K_{\alpha_0}) \rightrightarrows Shv(H)^{K_{\alpha_0}}$. Similarly, we may get $Shv(H/K_\alpha) \rightrightarrows Shv(H)^{K_\alpha}$ for any α .

We have an action of G_α by right translations on H/K_α , and $(H/K_\alpha)/G_\alpha \rightrightarrows H/G$. Now Section 0.0.16 gives $Shv(H/K_\alpha)^{G_\alpha} \rightrightarrows Shv(H/G)$.

As in the previous subsection, we get $Shv(H/G) \rightrightarrows Shv(H/K_\alpha)^{G_\alpha} \rightrightarrows Shv(H)^G$ for any of the 4 sheaf theories (for \mathcal{D} -modules this is ([4], Lemma B.5.1)).

Corollary 0.0.38. *Let $H \in \text{Grp}(\text{PreStk})$ be a placid ind-scheme, $G \subset H$ be a closed placid prosmooth group subscheme. For any of the 4 sheaf theories $Shv(H/G) \rightrightarrows Shv(H)^G$, where G acts on H by right translations.*

0.0.39. Let $G \in \text{Grp}(\text{PreStk})$ be a placid ind-scheme, Y be a placid ind-scheme with a G -action. Then $\text{Shv}(Y)$ is equipped with a $\text{Shv}(G)$ -action. Namely, for $K \in \text{Shv}(G)$, $F \in \text{Shv}(Y)$ one has $K * F \xrightarrow{\sim} a_*(K \boxtimes F)$ for the action map $a : G \times Y \rightarrow Y$.

0.0.40. Let $Y \rightarrow S$ be a map in Sch_{ft} , G be a placid group scheme over S acting on Y over S though its quotient $G \rightarrow G_0$ group scheme (smooth and of finite type over S) with a prounipotent kernel. We have canonically $\text{Shv}(Y)^G \xrightarrow{\sim} \text{Shv}(Y)^{G_0}$ by ([25], 1.3.21). Consider the stack quotient Y/G (by which we mean etale sheafification of the prestack quotient). We define $\text{Shv}(Y/G)$ as $\text{Shv}(Y/G_0)$ in such a way that for $q : Y \rightarrow Y/G$ the functor $q^* : \text{Shv}(Y/G) \rightarrow \text{Shv}(Y)$ is defined as $q_0^* : \text{Shv}(Y/G_0) \rightarrow \text{Shv}(Y)$ for $q_0 : Y \rightarrow Y/G_0$. So, if $G \rightarrow G_1 \rightarrow G_0$ are given, where G_1 is another finite-dimensional quotient group scheme over S with $\text{Ker}(G \rightarrow G_1)$ prounipotent then we identify $\text{Shv}(Y/G_1) \xrightarrow{\sim} \text{Shv}(Y/G_0)$ via a^* for the natural map $a : Y/G_1 \rightarrow Y/G_0$. No shifts appear. Note that $\text{Shv}(Y/G_0)$ is compactly generated both for \mathcal{D} -modules and in the constructible context (for \mathcal{D} -modules this is true, as Y/G_0 is perfect [5]).

If $f : Y \rightarrow Y'$ is a G -equivariant map in $(\text{Sch}_{ft})/S$ (we assume the G -action on both schemes factor through a finite dimensional quotient group scheme) then we have $f^! : \text{Shv}(Y'/G) \rightarrow \text{Shv}(Y/G)$.

We extend this definition to the case of an ind-scheme of ind-finite type Y over S equipped with a G -action over S as follows. Assume Y admits a presentation $Y \xrightarrow{\sim} \text{colim}_{i \in I} Y_i$, where Y_i is a G -invariant closed subscheme of finite type, I is filtered, and for $i \rightarrow j$ in I the map $Y_i \rightarrow Y_j$ is a closed immersion. Assume the G -action on Y_i factors through a quotient group scheme $G \rightarrow G_i$, where $G_i \rightarrow S$ is smooth, of finite type with $\text{Ker}(G \rightarrow G_i)$ prounipotent. Then we have $\text{Shv}(Y_i/G)$ defined as above and set $\text{Shv}(Y/G) \xrightarrow{\sim} \lim_{i \in I^{op}} \text{Shv}(Y_i/G)$ with respect to the $!$ -restrictions. With this definition for $q : Y \rightarrow Y/G$ we get the functor $q^* : \text{Shv}(Y/G) \rightarrow \text{Shv}(Y)$, which is the limit over $i \in I^{op}$ of the functors $q_i^* : \text{Shv}(Y_i/G) \rightarrow \text{Shv}(Y_i)$ for $q_i : Y_i \rightarrow Y_i/G$. It also identifies with $\text{oblv} : \text{Shv}(Y)^G \rightarrow \text{Shv}(Y)$. Note that for $i \rightarrow j$ in I the functor of $!$ -restriction $\text{Shv}(Y_j/G) \rightarrow \text{Shv}(Y_i/G)$ admits a fully faithful left adjoint. So, $\text{Shv}(Y)^G \xrightarrow{\sim} \text{colim}_{i \in I} \text{Shv}(Y_i/G)$ with respect to the $!$ -direct images. We see that for \mathcal{D} -modules or in the constructible context $\text{Shv}(Y)^G$ is compactly generated.

Let now H be a placid group ind-scheme over S , $G \subset H$ a closed placid group subscheme over S , so H/G is an ind-scheme of ind-finite type over S . Then the above assumption is satisfied for the G -action on H/G over S . So, $\text{Shv}(H/G)^G$ identifies with $\text{Shv}(G \backslash H/G)$. For $q : H/G \rightarrow G \backslash H/G$ the functor $q^* : \text{Shv}(G \backslash H/G) \rightarrow \text{Shv}(H/G)$ identifies with $\text{oblv} : \text{Shv}(H/G)^G \rightarrow \text{Shv}(H/G)$.

Let again $Y \rightarrow S$ be a map in Sch_{ft} and G a placid group scheme over S . Assume that the action of G on Y factors through the finite-dimensional group scheme $G_0 \rightarrow S$ smooth over S , and let $G \rightarrow G_1 \rightarrow G_0$ be as above. Another way to realize $\text{Shv}(Y/G)$ is as the category $\text{Shv}(Y/G_0)$ via the identifications $a^*[\dim.\text{rel}(a)] : \text{Shv}(Y/G_0) \xrightarrow{\sim} \text{Shv}(Y/G_1)$ for every G_1 as above. Indeed, the equivalence

$$\text{Shv}(Y/G_0) \xrightarrow{\sim} \text{Shv}(Y/G_0), K \mapsto K[-\dim(G_0/S)]$$

from the first model to the second one allows to identify them. The advantage of the second model is that the transition functors are t-exact for the perverse t-structure, so

allow to equip $Shv(Y/G)$ with the perverse t-structure: this is the perverse t-structure on $Shv(Y/G_0)$.

For the second model for $q_0 : Y \rightarrow Y/G_0$ consider the functor $Shv(Y/G) = Shv(Y/G_0) \rightarrow Shv(Y)$ given by $q_0^*[\dim(G_0/S)]$, it is t-exact and compatible with the transition functors for the second model, so defines a functor $Shv(Y/G) \rightarrow Shv(Y)$ that we denote by $q^*[\dim.\text{rel}(q)]$, this is just one symbol.

Let now Y be an ind-scheme of ind-finite type over S with a G -action and a presentation $Y \rightrightarrows \text{colim}_{i \in I} Y_i$, where I is small filtered, $Y_i \rightarrow S$ is a G -invariant closed subscheme of finite type in Y , and for $i \rightarrow j$ the map $h_{ij} : Y_i \rightarrow Y_j$ is a closed immersion. Assume G -action on Y_i factors through a quotient group scheme $G_i \rightarrow S$ smooth and of finite type over S , where $\text{Ker}(G \rightarrow G_i)$ is a prounipotent group scheme over S . Equip each $Shv(Y_i/G)$ with the perverse t-structure. Then the $!$ -pullbacks under $Y_i/G_0 \rightarrow Y_j/G_0$ are compatible with the transition functors for the second model, so define a functor $h_{ij}^! : Shv(Y_j/G) \rightarrow Shv(Y_i/G)$, which is left t-exact. It also commutes with the functors $q_i^*[\dim.\text{rel}(q_i)]$ for $q_i : Y_i \rightarrow Y_i/G$. Recall that $Shv(Y/G) \rightrightarrows \lim_{i \in I^{op}} Shv(Y_i/G)$ with respect to the functors $h_{ij}^!$. The limit over $i \in I^{op}$ of the functors $q_i^*[\dim.\text{rel}(q_i)] : Shv(Y_i/G) \rightarrow Shv(Y_i)$ in the second model is denoted $\text{oblv}[\dim.\text{rel}] : Shv(Y)^G \rightarrow Shv(Y)$.

Each $h_{ij}^! : Shv(Y_j/G) \rightarrow Shv(Y_i/G)$ admits a left adjoint $(h_{ij})_! : Shv(Y_i/G) \rightarrow Shv(Y_j/G)$, and we may also write $Shv(Y/G) \rightrightarrows \text{colim}_{i \in I} Shv(Y_i/G)$ with the transition functors $(h_{ij})_!$. Now we may define the perverse t-structure on $Shv(Y/G)$ as in the case of an ind-scheme of ind-finite type. Namely, $K \in Shv(Y/G)$ lies in $Shv(Y/G)^{\geq 0}$ iff for any i , its $!$ -restriction to Y_i/G lies in $Shv(Y_i/G)^{\geq 0}$. So, $Shv(Y/G)^{\geq 0} \rightrightarrows \lim_{i \in I^{op}} Shv(Y_i/G)^{\geq 0}$, which shows that $Shv(Y/G)^{\geq 0}$ is presentable, so the t-structure is accessible. This t-structure is also compatible with filtered colimits.

In fact, if we identify the first and the second model of $Shv(Y)^G$ as above then the functors oblv for the first model becomes the functor $\text{oblv}[\dim.\text{rel}] : Shv(Y)^G \rightarrow Shv(Y)$ for the second one. So, this is just a matter of notations.

For the natural map $q : Y \rightarrow Y/G$ the functor $q^! : Shv(Y/G) \rightarrow Shv(Y)$ is also defined similarly, though q is not locally of finite type.

Namely, if $Y \in \text{Sch}_{ft}$ we first consider a third model for $Shv(Y/G)$: for a quotient $G \rightarrow G_0$ as above such that G -action on Y factors through G_0 , we identify the 3rd model with the second via the equivalences: $Shv(Y/G_0) \xrightarrow{\sim} Shv(Y/G_0)$, $K \mapsto K[2 \dim G_0]$. For $G \rightarrow G_1 \rightarrow G_0$ let $a : Y/G_1 \rightarrow Y/G_0$ be the natural map. Under such equivalences the transition functor a^* for the first model becomes the transition functor $a^!$ for the 3rd model. Now for the third model we define $q^! : Shv(Y/G) \rightarrow Shv(Y)$ as $q_0^!$ for $q_0 : Y \rightarrow Y/G_0$.

This definition is similarly extended to ind-schemes of ind-finite type.

0.0.41. Let $Y \rightrightarrows \text{colim}_{i \in I} Y_i$ in PreStk , where I is filtered, Y_i is a scheme of finite type, and for $i \rightarrow j$ in I , $Y_i \rightarrow Y_j$ is a closed immersion, so Y is an ind-scheme of ind-finite type. Let $H \rightarrow G$ be a homomorphism of placid group schemes over $\text{Spec } k$. Assume G acts on Y and the assumption of the previous subsection holds, that is, each Y_i is G -invariant, and on Y_i the group scheme G acts via a finite-dimensional quotient group scheme $G \rightarrow G_i$ with $\text{Ker}(G \rightarrow G_i)$ prounipotent. We have a natural map of stack

quotients $h : Y/H \rightarrow Y/G$. We have defined the categories $Shv(Y/G), Shv(Y/H)$ in the previous subsection. Then the functor $h^* : Shv(Y/G) \rightarrow Shv(Y/H)$ is defined, namely this is $oblv : Shv(Y)^G \rightarrow Shv(Y)^H$.

0.0.42. If Y is a stack locally of finite type, a placid group scheme over Y should be defined as a group object $(G \rightarrow Y) \in Grp(\text{PreStk}/Y)$ such that for any $S \rightarrow Y$ with $S \in \text{Sch}_{ft}^{aff}$, $S \times_Y G$ is a placid group scheme over S .

Let $Z \rightarrow Y$ be a map in Stk_{lft} and G be a placid group scheme over Y acting on Z over Y . Write Z/G for the stack quotient of Z by G (etale sheafification of the prestack quotient), so $Z/G \rightarrow Y$. How do we define $Shv(Z)^G$?

First, for any $S \rightarrow Y$ with $S \in \text{Sch}_{ft}^{aff}$ we have a monoidal category $Shv(S \times_Y G)$ defined in ([25], 1.3.7), it is an object of $\text{Alg}(Shv(Y) - \text{mod})$. For a map $S' \xrightarrow{\alpha} S \rightarrow Y$ in $(\text{Sch}_{ft}^{aff})/Y$ let $\beta : S' \times_Y G \rightarrow S \times_Y G$ be obtained by base change. As in ([25], 1.3.12), $\beta^! : Shv(S \times_Y G) \rightarrow Shv(S' \times_Y G)$ is monoidal, it is actually a morphism in $\text{Alg}(Shv(Y) - \text{mod})$. To see this we used Lemma 0.0.43 below. So, we may understand

$$Shv(G) \xrightarrow{\sim} \lim_{(S \rightarrow Y) \in ((\text{Sch}_{ft}^{aff})/Y)^{op}} Shv(S \times_Y G)$$

as limit taken in $\text{Alg}(Shv(Y) - \text{mod})$. We get a monoidal structure on $Shv(G)$ via the latter limit.

This is one more extension of our sheaf theory needed. In general, we can not write G as $\lim_{i \in I^{op}} G_i$, where $G_i \rightarrow Y$ is an affine group scheme "of finite type" over Y , I is filtered, and for $i \rightarrow j$ in I the map $G_j \rightarrow G_i$ is affine smooth surjective. This does not hold already for $\mathcal{L}^+(G) \rightarrow \text{Ran}$, I think, where G is a reductive group.

We will see that the monoidal category $Shv(G)$ acts on $Shv(Z)$.

For each $S \rightarrow Y$ in $(\text{Sch}_{ft}^{aff})/Y$, $S \times_Y G$ acts on $S \times_Y Z$ over S , so $Shv(S \times_Y G)$ acts on $Shv(S \times_Y Z)$ naturally. For a map $S' \xrightarrow{\alpha} S \rightarrow Y$ in $(\text{Sch}_{ft}^{aff})/Y$ let $\bar{\alpha} : S' \times_Y Z \rightarrow S \times_Y Z$ be obtained from α by base change.

Let $Shv(S \times_Y G)$ act on $Shv(S' \times_Y Z)$ via the map $Shv(S \times_Y G) \rightarrow Shv(S' \times_Y G)$. Then $\bar{\alpha}^!$ commutes with $Shv(S \times_Y G)$ -actions.

Recall that the sheafification is a left exact functor, so for the stack quotients we get $((S \times_Y Z)/(S \times_Y G)) \times_S S' \xrightarrow{\sim} (S' \times_Y Z)/(S' \times_Y G)$ canonically.

Consider the ∞ -category $\text{AssAlg} + \text{Mod}(\text{DGCat}_{cont})$ defined in ([7], ch. I.1, 3.5.4). By ([16], 3.2.2.5), it admits limits and the projection $\text{AssAlg} + \text{Mod}(\text{DGCat}_{cont}) \rightarrow \text{DGCat}_{cont}$ preserves limits. We obtained a functor

$$((\text{Sch}_{ft}^{aff})/Y)^{op} \rightarrow \text{AssAlg} + \text{Mod}(\text{DGCat}_{cont})$$

sending $S \rightarrow Y$ to the pair $Shv(S \times_Y G), Shv(S \times_Y Z)$. So, the limit of the latter functor is an object of $\text{AssAlg} + \text{Mod}(\text{DGCat}_{cont}) \rightarrow \text{DGCat}_{cont}$. In other words, $Shv(G)$ act on $Shv(Z)$ naturally, and we may consider the invariants

$$Shv(Z)^{Shv(G)} = \text{Fun}_{Shv(G)}(Shv(Y), Shv(Z)) \in Shv(Y) - \text{mod}$$

Question. Can we rewrite the above as limit of $Shv(S \times_Y Z)^{Shv(S \times_Y G)}$ over S ? More precisely, for $(S \rightarrow Y) \in (\text{Sch}_{ft}^{aff})/Y$, let $q_S : S \times_Y G \rightarrow S$ be the projection. By ([25],

1.3.16), we have canonically

$$Shv(S \times_Y Z)^{Shv(S \times_Y G)} \xrightarrow{\sim} q_S^* \omega - comod(Shv(S \times_Y Z))$$

If $\alpha : S' \rightarrow S$ is a morphism in $(Sch_{ft}^{aff})/Y$ then $\beta^! q_S^* \omega \xrightarrow{\sim} q_{S'}^* \omega$ as coalgebras in $Shv(S' \times_Y G)$. This is by definition of the functors $q_S^*, q_{S'}^*$. So, $\bar{\alpha}^! : Shv(S \times_Y Z) \rightarrow Shv(S' \times_Y Z)$ induces a functor between the comodule categories

$$q_S^* \omega - comod(Shv(S \times_Y Z)) \rightarrow q_{S'}^* \omega - comod(Shv(S' \times_Y Z)) = q_{S'}^* \omega - comod(Shv(S' \times_Y Z))$$

Now we may consider

$$\lim_{S \rightarrow Y} q_S^* \omega - comod(Shv(S \times_Y Z))$$

taken in $DGCat_{cont}$ over the category $((Sch_{ft}^{aff})/Y)^{op}$. Is it equivalent to $Shv(Z)^{Shv(G)}$?

Lemma 0.0.43. *Let $S' \rightarrow S$ be a map in Sch_{ft} . Let $f : Y \rightarrow Z$ be a morphism of placid schemes over S , let $f' : Y' \rightarrow Z'$ be obtained from f by the base change $\alpha : S' \rightarrow S$. Write $\alpha_Y : Y' \rightarrow Y$ and $\alpha_Z : Z' \rightarrow Z$ for the obtained maps. Then for $K \in Shv(Y)$ one has canonically $\alpha_Z^! f_* K \xrightarrow{\sim} f'_* \alpha_Y^! K$.*

Proof. Write $Y \xrightarrow{\sim} \lim_{i \in I^{op}} Y_i$, where I is filtered, Y_i is a scheme of finite type over S , and for $i \rightarrow i'$ in I the map $Y_{i'} \rightarrow Y_i$ is affine smooth surjective (over S), and similarly for $Z \xrightarrow{\sim} \lim_{j \in J^{op}} Z_j$. These are presentations from a definition of a placid scheme. Let Y'_i, Z'_j be obtained from Y_i, Z_j by base change $S' \rightarrow S$, so $Y' \xrightarrow{\sim} \lim_{i \in I^{op}} Y'_i$ and $Z' \xrightarrow{\sim} \lim_{j \in J^{op}} Z'_j$.

It suffices to establish the desired isomorphism after applying $(ev'_j)_* : Shv(Z') \rightarrow Shv(Z'_j)$ for each $j \in J$, here $ev'_j : Z' \rightarrow Z'_j$ is the projection. Pick $i \in I$ such that the composition $Y \rightarrow Z \rightarrow Z_j$ factors through $\bar{f} : Y_i \rightarrow Z_j$. By base change under $S' \rightarrow S$ we get a cartesian square

$$\begin{array}{ccc} Y_i & \xrightarrow{\bar{f}} & Z_j \\ \uparrow \alpha_{Y_i} & & \uparrow \alpha_{Z_j} \\ Y'_i & \xrightarrow{\bar{f}'} & Z'_j \end{array}$$

Let $(ev_i)_* : Shv(Y) \rightarrow Shv(Y_i)$ be the direct image under $ev_i : Y \rightarrow Y_i$. The key point is the base change isomorphism $\alpha_{Z_j}^! \bar{f}_* \xrightarrow{\sim} \bar{f}'_* \alpha_{Y_i}^!$. We get

$$\begin{aligned} (ev'_j)_* \alpha_Z^! f_* K &\xrightarrow{\sim} \alpha_{Z_j}^! (ev_j)_* f_* K \xrightarrow{\sim} \alpha_{Z_j}^! \bar{f}_* (ev_i)_* K \xrightarrow{\sim} \bar{f}'_* \alpha_{Y_i}^! (ev_i)_* K \\ &\xrightarrow{\sim} \bar{f}'_* (ev_i)_* \alpha_Y^! K \xrightarrow{\sim} (ev'_j)_* f'_* \alpha_Y^! K \end{aligned}$$

We are done. \square

Corollary: let $\alpha : S' \rightarrow S$ be a map in Sch_{ft} , $Y \rightarrow S$ be a placid ind-scheme over S and $Y' \rightarrow S'$ be obtained by base change. Let $\bar{\alpha} : Y' \rightarrow Y$ be the natural map. Then $\bar{\alpha}^! : Shv(Y) \rightarrow Shv(Y')$ is well-defined.

Proof. Write $Y \xrightarrow{\sim} \text{colim}_{i \in I} Y_i$, where I is small filtered category, Y_i is a placid S -scheme, and for $i \rightarrow j$ in I the map $Y_i \rightarrow Y_j$ is a placid closed immersion over S . Write $Y'_i = Y_i \times_S S'$. For each $i \in I$ let $\bar{\alpha}_i : Y'_i \rightarrow Y_i$ be the corresponding map. The functors

$\bar{\alpha}_i^!$ are compatible with $*$ -pushforwards in the corresponding inductive systems, passing to the colimit in $\bar{\alpha}_i^! : Shv(Y_i) \rightarrow Shv(Y_i')$, we get the desired $\bar{\alpha}^!$. These functors are also compatible with the $!$ -pullbacks, so we can also pass to the limit. \square

0.0.44. Let us be in the situation of Section 0.0.42. Probably the only case we need satisfies the following additional assumption that we make. For any $S \rightarrow Y$ in $(Sch_{ft}^{aff})/Y$, $S \times_Y Z$ may be written as $S \times_Y Z \xrightarrow{\sim} \text{colim}_{i \in I} Z_{S,i}$, where I is filtered, and $Z_{S,i} \rightarrow S$ is a scheme of finite type such that for any $i \rightarrow j$ in I , the map $Z_{S,i} \rightarrow Z_{S,j}$ is a closed immersion. Moreover, for each $i \in I$, $Z_{S,i}$ is stable under the action of $S \times_Y G$, and the latter acts through a finite-dimensional quotient scheme $G_{S,i}$ over S . In particular, $S \times_Y Z$ is a ind-scheme of ind-finite type over S .

In this setting one may define the category $Shv(Z/G)$ in one more way. Namely, for $S \rightarrow Y$ as above, in Section 0.0.40 we have defined the category $Shv((S \times_Y Z)/(S \times_Y G))$ together with the functor $q_S^* : Shv((S \times_Y Z)/(S \times_Y G)) \rightarrow Shv(S \times_Y Z)$ for the projection $q_S : S \times_Y Z \rightarrow (S \times_Y Z)/(S \times_Y G)$. Recall that q_S^* identifies with $\text{oblv} : Shv(S \times_Y Z)^{S \times_Y G} \rightarrow Shv(S \times_Y Z)$.

Let now $\alpha : S' \rightarrow S$ be a map in $(Sch_{ft}^{aff})/Y$. Let $\bar{\alpha} : S' \times_Y Z \rightarrow S \times_Y Z$ and $\beta : S' \times_Y G \rightarrow S \times_Y G$ be obtained by base change from α . Pick a finite-dimensional quotient group scheme $S \times_Y G \rightarrow G_S$ such that $S \times_Y G$ -action on $S \times_Y Z$ factors through G_S . Let $G_{S'} = G_S \times_S S'$. We have the cartesian square

$$\begin{array}{ccc} S' \times_Y Z & \xrightarrow{\bar{\alpha}} & S \times_Y Z \\ \downarrow h' & & \downarrow h \\ (S' \times_Y Z)/G_{S'} & \xrightarrow{\tilde{\alpha}} & (S \times_Y Z)/G_S \end{array}$$

The functors $\tilde{\alpha}^! : Shv((S \times_Y Z)/G_S) \rightarrow Shv((S' \times_Y Z)/G_{S'})$ can be seen by definition as the functors that fit into a commutative diagram

$$\begin{array}{ccc} Shv(S \times_Y Z) & \xrightarrow{\tilde{\alpha}^!} & Shv(S' \times_Y Z) \\ \uparrow \text{oblv} & & \uparrow \text{oblv} \\ Shv(S \times_Y Z)^{S \times_Y G} & \xrightarrow{\tilde{\alpha}^!} & Shv(S' \times_Y Z)^{S' \times_Y G} \end{array}$$

This way we get a functor $((Sch_{ft}^{aff})/Y)^{op} \rightarrow Shv(Y) - mod$, $S \mapsto Shv(S \times_Y Z)^{S \times_Y G}$. Finally, we may consider

$$\lim_{S \rightarrow Y} Shv(S \times_Y Z)^{S \times_Y G}$$

in $Shv(Y) - mod$ taken over $((Sch_{ft}^{aff})/Y)^{op}$. This should be our definition of $Shv(Z/G)$ I think.

0.0.45. For S a scheme of finite type consider the perverse t-structure on S . The functor $H^i : Shv(S) \rightarrow Shv(S)^\heartsuit$ preserves products, is this correct? This was used to conclude that your $QLisse(S)$ for S smooth is left complete.

For a scheme of finite type S any object of $Shv(S)^{constr}$ is bounded.

0.0.46. Let Y be a classical algebraic stack locally of finite type with an affine diagonal. Then the truncation functors for the perverse t-structure $\tau^{\leq n}, \tau^{\geq n}$ preserve the subcategory $Shv(Y)^{constr} \subset Shv(Y)$, so we get a t-structure on $Shv(Y)^{constr}$.

0.0.47. Let Y be a classical quasi-compact algebraic stack with an affine diagonal. Let $F \in Shv(Y)^{constr}$ then F is bounded. Indeed, pick a smooth covering $f : S \rightarrow Y$, where $S \in Sch_{ft}^{aff}$. Since $f^*[\dim.\text{rel}(f)]$ is t-exact and conservative, it suffices to show that f^*F is bounded. However, any compact object in $Shv(S)$ is bounded.

0.0.48. Let $S \in Sch_{ft}$, $K_i \in Shv(S)$ and $E \in Lisse(E)$, that is, E is dualizable with respect to the \otimes -monoidal structure on $Shv(S)$. Recall that $\mathcal{H}om(K_1, K_2) \in Shv(S)$ denotes the inner hom with respect to the \otimes -monoidal structure on $Shv(S)$. Then by ([16], 4.6.2.1) we get $\mathcal{H}om(K_1 \otimes E, K_2) \xrightarrow{\sim} \mathcal{H}om(K_1, E^\vee \otimes K_2)$ with $E^\vee = \mathcal{H}om(E, e)$.

0.0.49. Recall the following from ([13], A.1.7). Let $Corr(\text{PreStk}_{lft})_{ind-sch,all}$ be the category of correspondences, whose objects are prestacks locally of finite type \mathcal{Y} , and a morphism from \mathcal{Y}_1 to \mathcal{Y}_2 is a diagram $\mathcal{Y}_1 \xleftarrow{g} \mathcal{Y}_{12} \xrightarrow{f} \mathcal{Y}_2$ with g any and f ind-schematic of ind-finite type. Then in the constructible context we get a functor $Shv_{Corr} : Corr(\text{PreStk}_{lft})_{ind-sch,all} \rightarrow \text{DGCat}_{cont}$ sending \mathcal{Y} to $Shv(\mathcal{Y})$, and sending the above morphism to the functor $f_*g^! : Shv(\mathcal{Y}_1) \rightarrow Shv(\mathcal{Y}_2)$. Then the functor Shv_{Corr} possesses a natural right-lax symmetric monoidal structure, see ([7], Vol. 2, Chapter 3, Sect. 6.1), where $Corr(\text{PreStk}_{lft})_{ind-sch,all}$ is a symmetric monoidal category with respect to the level-wise product.

In particular, this means that given $f_i : Y_i \rightarrow Z_i$ ind-schematic of ind-finite type in PreStk_{lft} and $K_i \in Shv(Y_i)$, we have

$$(f_1 \times f_2)_*(K_1 \boxtimes K_2) \xrightarrow{\sim} ((f_1)_*K_1) \boxtimes ((f_2)_*K_2)$$

Let \mathcal{H} be a groupoid acting on \mathcal{Y} in PreStk_{lft} given by a functor $\Delta^{op} \rightarrow \text{PreStk}_{lft}$ such that the action map $m : \mathcal{H} \times_{t, \mathcal{Y}, s} \mathcal{H} \rightarrow \mathcal{H}$ is ind-schematic of ind-finite type, here $s, t : H \rightarrow \mathcal{Y}$ are source and targets maps. We get a monoidal structure on $Shv(\mathcal{H})$ with the product given by $(K_1, K_2) \mapsto m_*q^!(K_1 \boxtimes K_2)$ for $q : \mathcal{H} \times_{\mathcal{Y}} \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}$. Let $\alpha : \mathcal{Y} \rightarrow \mathcal{H}$ be the map corresponding to $[1] \rightarrow [0]$ in Δ . Then $\alpha_*\omega_{\mathcal{Y}}$ is the unit of $Shv(\mathcal{H})$. Moreover, the functor $\alpha_* : (Shv(\mathcal{Y}), \otimes^!) \rightarrow Shv(\mathcal{H})$ is monoidal. Indeed, $\mathcal{H} \in Alg(Corr(\text{PreStk}_{lft})_{ind-sch,all})$, so we just apply a right-lax monoidal functor Shv_{Corr} . Moreover,

$$(\mathcal{H}, \mathcal{Y}) \in Alg + module(Corr(\text{PreStk}_{lft})_{ind-sch,all})$$

Namely, write $\text{pr}, \text{act} : \mathcal{H} \rightarrow \mathcal{Y}$ for the two maps from \mathcal{H} to \mathcal{Y} given by $0, 1 : [0] \rightarrow [1]$. Then the action map from $\mathcal{H} \times \mathcal{Y}$ to \mathcal{Y} is given by the correspondence $\mathcal{H} \times \mathcal{Y} \xleftarrow{\text{id}, \text{pr}} \mathcal{H} \xrightarrow{\text{act}} \mathcal{Y}$. Applying Shv_{Corr} , we see that $Shv(\mathcal{Y}) \in Shv(\mathcal{H}) - mod(\text{DGCat}_{cont})$.

The whole Section A.1 of [13] can be advised as a reference on the generalities about the sheaf theories.

More generally, we may define the category of relative groupoids $\text{Grpd} / \text{PreStk} / \text{Sch}$ and the corresponding functor

$$\text{Grpd} / \text{PreStk} / \text{Sch} \rightarrow Alg(Corr(\text{PreStk}_{lft}))$$

as in ([26], 1.4.48).

0.0.50. Let $H = \lim_{i \in I^{op}} H_i$ be a placid group scheme, here $I \in 1 - \text{Cat}$ is filtered, if $i \in I$ then H_i is a group scheme of finite type, and for $i \rightarrow j$ in I , $H_j \rightarrow H_i$ is a smooth affine surjective morphism, a homomorphism of group schemes. Let $i \in I$ and $K_i \hookrightarrow H_i$ be a closed group subscheme, set $K = H \times_{H_i} K_i$. So, $K \hookrightarrow H$ is a placid closed immersion. Let $L = \text{Ker}(H \rightarrow H_i)$. Then the natural map $H/K \rightarrow H_i/K_i$ is an isomorphism. Indeed, $L = \text{Ker}(K \rightarrow K_i)$, and L acts trivially on H/K . So, the H -action on H/K by left translations factors through a transitive H_i -action, and the stabilizer of $K/K \in H/K$ is $K/L \xrightarrow{\sim} K_i$.

0.0.51. *Application.* Let H be a smooth affine group scheme of finite type, $F = k((t))$, $O = k[[t]]$. Then $H(F)$ is a placid ind-scheme. It could be defined in two equivalent ways. Let for $n \geq 1$, $K_n = \text{Ker}(H(O) \rightarrow H(O/t^n))$. Set $K_0 = H(O)$, so $\dots K_2 \subset K_1 \subset K_0$. Then $H(F)/K_n$ is an ind-scheme of ind-finite type. For $n < m$ we have the map $H(F)/K_m \rightarrow H(F)/K_n$, which is schematic, smooth affine and surjective. It is actually a torsor under K_n/K_m . So, we are in the situation of Section 0.0.36 for $A = \mathbb{Z}_{\geq 0}$. For $\alpha \in A$, $\mathcal{Y}_\alpha = H(F)/K_\alpha$, $G_\alpha = H(O/t^\alpha)$. This gives $G = \lim_\alpha G_\alpha = H(O)$. Then we may define $H(F)$ as $\lim_{\alpha \in A^{op}} \mathcal{Y}_\alpha$, where the limit is taken in PreStk . By Section 0.0.36, for $n \leq m$ we have the projection $f_{n,m} : H(F)/K_m \rightarrow H(F)/K_n$ and the adjoint pair $f_{n,m}^* : \text{Shv}(H(F)/K_n) \rightleftarrows \text{Shv}(H(F)/K_m) : (f_{n,m})_*$. We may view $\text{Shv}(H(F))$ as $\lim_{n \in A^{op}} \text{Shv}(H(F)/K_n)$ in DGCat_{cont} with respect to $(f_{n,m})_*$. For $n > 0$ the group scheme K_n is prounipotent.

For the $H(O)$ -action by right translations on $H(F)$ by Section 0.0.36 one gets

$$\text{Shv}(H(F))^{H(O)} \xrightarrow{\sim} \text{Shv}(\text{Gr}_H)$$

0.0.52. Let Y be placid scheme written as $Y \xrightarrow{\sim} \lim_{i \in I^{op}} Y_i$, where I is small filtered category, for $i \rightarrow j$ in I the map $Y_j \rightarrow Y_i$ is smooth affine surjective morphism in Sch_{ft} . Let $S \in \text{Sch}_{ft}$. For the projection $f_i : Y \rightarrow Y_i$ the diagram commutes

$$\begin{array}{ccc} \text{Shv}(Y) \otimes \text{Shv}(S) & \rightarrow & \text{Shv}(Y \times S) \\ \downarrow (f_i)_* \otimes \text{id} & & \downarrow (f_i \times \text{id})_* \\ \text{Shv}(Y_i) \otimes \text{Shv}(S) & \rightarrow & \text{Shv}(Y_i \times S), \end{array}$$

where the horizontal arrows are exterior products. Indeed, $\text{Shv}(S)$ is dualizable, so $\lim_{i \in I^{op}} \text{Shv}(Y_i) \otimes \text{Shv}(S) \xrightarrow{\sim} \text{Shv}(Y) \otimes \text{Shv}(S)$, where the limit is taken with respect to $(f_i)_* \otimes \text{id}$.

Let now $f : Y \rightarrow Z$ be a morphism of placid schemes. The above shows that the diagram commutes

$$\begin{array}{ccc} \text{Shv}(Y) \otimes \text{Shv}(S) & \rightarrow & \text{Shv}(Y \times S) \\ \downarrow f_* \otimes \text{id} & & \downarrow (f \times \text{id})_* \\ \text{Shv}(Z) \otimes \text{Shv}(S) & \rightarrow & \text{Shv}(Z \times S), \end{array}$$

In turn, this show that the above diagram still commutes if we only assume that S is a placid scheme also. Finally, if $f : Y \rightarrow Z, f' : Y' \rightarrow Z'$ are morphisms of placid schemes, the diagram commutes

$$\begin{array}{ccc} \text{Shv}(Y) \otimes \text{Shv}(Y') & \rightarrow & \text{Shv}(Y \times Y') \\ \downarrow f_* \otimes (f')_* & & \downarrow (f \times f')_* \\ \text{Shv}(Z) \otimes \text{Shv}(Z') & \rightarrow & \text{Shv}(Z \times Z'), \end{array}$$

0.0.53. Let $S \in \text{Sch}_{ft}$, let Y be a placid S -scheme. We claim that in the constructible context, the symmetric monoidal structure $\otimes : \text{Shv}(Y) \otimes \text{Shv}(Y) \rightarrow \text{Shv}(Y)$ is well defined.

Indeed, write $Y \xrightarrow{\sim} \lim_{i \in I^{op}} Y_i$, where I is small filtered, Y_i is a S -scheme of finite type, for $i \rightarrow j$ in I the map $Y_j \rightarrow Y_i$ is smooth affine surjective.

Let $p_{ij} : Y_j \rightarrow Y_i$ be the transition map for $\alpha : i \rightarrow j$ in I . Then $p_{ij}^* : \text{Shv}(Y_i) \rightarrow \text{Shv}(Y_j)$ is a map in $\text{CAlg}(\text{DGCat}_{cont})$, and $\text{CAlg}(\text{DGCat}_{cont}) \rightarrow \text{DGCat}_{cont}$ preserves filtered colimits, so $\text{Shv}(Y) \xrightarrow{\sim} \text{colim}_{i \in I} \text{Shv}(Y_i)$ could be viewed as colimit in $\text{CAlg}(\text{DGCat}_{cont})$. Note that e_Y is the unit.

0.0.54. Let $i : Z \rightarrow Y$ be a placid closed immersion of placid S -schemes, where $S \in \text{Sch}_{ft}$. Then $i^* : \text{Shv}(Z) \rightarrow \text{Shv}(Y)$ is defined naturally in the constructible context. Namely, write $Y \xrightarrow{\sim} \lim_{i \in I^{op}} Y_i$, where I is small filtered, Y_i is a S -scheme of finite type, for $i \rightarrow j$ in I the map $Y_j \rightarrow Y_i$ is smooth affine surjective. We may assume that $i_0 \in I$ is initial, $i_0 : Z_0 \subset Y_0$ is a closed subscheme, and $Z = Y \times_{Y_0} Z_0$. Then for any i we have a closed immersion $i_i : Z_i \hookrightarrow Y_i$ obtained from i_0 by base change. Then the functors $i_i^* : \text{Shv}(Y_i) \rightarrow \text{Shv}(Z_i)$ are compatible with the $*$ -pullbacks in the transition systems, so in the colimit yield $i^* : \text{Shv}(Y) \rightarrow \text{Shv}(Z)$.

Lemma 0.0.55. *In the coinstructible context for $K \in \text{Shv}(Z), L \in \text{Shv}(Y)$ one has the projection formula $(i_! K) \otimes L \xrightarrow{\sim} i_!(K \otimes i^* L)$ in $\text{Shv}(Y)$ canonically.*

Proof. This is a particular case of base change established in Lemma 0.0.58. \square

0.0.56. Let now \mathcal{Y} be a placid ind-scheme over S written as $\mathcal{Y} \xrightarrow{\sim} \text{colim}_{i \in I} \mathcal{Y}_i$, where I is small filtered, \mathcal{Y}_i is a placid S -scheme, and for $i \rightarrow j$ in I , $\mathcal{Y}_i \rightarrow \mathcal{Y}_j$ is a placid closed immersion. Then we get the category $\lim_{i \in I^{op}} \text{Shv}(\mathcal{Y}_i)$ with respect to the $*$ -pullbacks.

0.0.57. Let I be a small filtered category. Assume given a functor $I^{op} \times [1] \rightarrow \text{Sch}_{ft}$, $i \mapsto (Z_i \xrightarrow{f_i} Y_i)$. Assume that for $i \rightarrow j$ in I the transition maps $Y_j \rightarrow Y_i$ and $Z_j \rightarrow Z_i$ are smooth affine surjective. Set $Z = \lim_{i \in I^{op}} Z_i, Y = \lim_{i \in I^{op}} Y_i$. Let $f : Z \rightarrow Y$ be obtained from f_i by passing to the limit over I^{op} . Then the functor $f^* : \text{Shv}(Y) \rightarrow \text{Shv}(Z)$ is well-defined in the constructible context. In the case of \mathcal{D} -modules, we assume in addition that each f_i is smooth. Then f^* is defined.

Indeed, for each i we have $f_i^* : \text{Shv}(Y_i) \rightarrow \text{Shv}(Z_i)$ compatible with the transition $*$ -pullbacks, and f^* is obtained by passing to the colimit.

For example, if Y is a placid scheme then for the diagonal $f : Y \rightarrow Y \times Y$ the functor $f^* : \text{Shv}(Y \times Y) \rightarrow \text{Shv}(Y)$ is defined in the constructible context.

Lemma 0.0.58. *Assume given the cartesian square of placid schemes*

$$\begin{array}{ccc} Z' & \xrightarrow{f'} & Y' \\ \downarrow g_Z & & \downarrow g_Y \\ Z & \xrightarrow{f} & Y, \end{array}$$

where the vertical arrows are placid closed immersions, and f is obtained as in Section 0.0.57. Then the same holds for f' , and one has canonically $f^*(g_Y)_! \xrightarrow{\sim} (g_Z)_!(f')^*$ as functors $\text{Shv}(Y') \rightarrow \text{Shv}(Z)$.

Proof. Pick a functor $I^{op} \times [1] \rightarrow \text{Sch}_{ft}$, $i \mapsto (Z_i \xrightarrow{f_i} Y_i)$ as in Section 0.0.57, so f is obtained by passing to the limit from $Z_i \xrightarrow{f_i} Y_i$ over I^{op} . We may assume that $0 \in I$ is initial, and we have a closed immersion $Y'_0 \hookrightarrow Y_0$ such that $Y' = Y \times_{Y_0} Y'_0$. By Lemma 0.0.43, $g_Y^! f_* \xrightarrow{\sim} f'_* g_Z^!$. Our claim is obtained by passing to left adjoints. \square

0.1. Verdier compatible algebraic stacks.

0.1.1. For ([2], A.2.2). Let Y be quasi-compact classical algebraic stack with an affine diagonal, which is Verdier compatible. They claim there that for $f : S \rightarrow Y$ a scheme of finite type over Y , the objects $f_* K$ with $K \in \text{Shv}(S)^c$ are compact and generate $\text{Shv}(Y)$. Indeed, $\text{Shv}(Y)^c$ is the Karoubi closure (that is, idempotent closure) of the smallest stable subcategory generated by objects of the form $f_!(K)$ with $K \in \text{Shv}(S)^c$. This implies the claim, see ([21], 9.2.27).

0.1.2. For ([2], A.2.3). Let Y, Y' be a quasi-compact classical algebraic stacks with affine diagonals, which are Verdier compatible. Let $f : Y \rightarrow Y'$ be a morphism. Recall that $f_\bullet : \text{Shv}(Y) \rightarrow \text{Shv}(Y')$ is defined as the continuous extension of the functor $f_* : \text{Shv}(Y)^c \rightarrow \text{Shv}(Y')^c \subset \text{Shv}(Y')$.

Let Z be another algebraic stack locally of finite type with an affine diagonal, which is Verdier compatible. Then we have the following.

Lemma 0.1.3. *For $K \in \text{Shv}(Y), F \in \text{Shv}(Z)$ we have canonically*

$$(f_\bullet K) \boxtimes F \xrightarrow{\sim} (f \times \text{id})_\bullet (K \boxtimes F)$$

Besides, for $L \in \text{Shv}(Y')$ we get $(f_\bullet K) \otimes^! L \xrightarrow{\sim} f_\bullet (K \otimes^! f^! L)$.

Proof. 1) Both sides for any F fixed preserve colimits as a functor of K . Therefore, it suffices to prove this for K of the form $K = g_* K'$, where $g : S \rightarrow Y$ is a morphism, $S \in \text{Sch}_{ft}$ and $K' \in \text{Shv}(S)^c$, as such objects generate $\text{Shv}(Y)$. Moreover, we may assume $F \in \text{Shv}(Z)^c$. Then $f_\bullet K \xrightarrow{\sim} f_* K \xrightarrow{\sim} (fg)_* K'$, and $(f \times \text{id})_\bullet (K \boxtimes F) \xrightarrow{\sim} (f \times \text{id})_* (K \boxtimes F)$, and $K \boxtimes F \xrightarrow{\sim} (g \times \text{id})_* (K' \boxtimes F)$, because $g \times \text{id}$ is schematic. Now $(fg \times \text{id})_* (K' \boxtimes F) \xrightarrow{\sim} ((fg)_* K') \boxtimes F$, because fg is schematic. The first claim follows.

2) For the second, note that both sides preserve colimits separately in each variable, so we may assume K of the form $K = g_* K'$, where $g : S \rightarrow Y$ is a morphism, $S \in \text{Sch}_{ft}$ and $K' \in \text{Shv}(S)^c$. Then $f_\bullet K \xrightarrow{\sim} f_* K \xrightarrow{\sim} (fg)_* K'$. We may also assume $L \in \text{Shv}(Y')^{\text{constr}}$. We have the cartesian squares

$$\begin{array}{ccc} S & \rightarrow & S \times Y' \\ \downarrow g & & \downarrow g \times \text{id} \\ Y & \xrightarrow{\Gamma_f} & Y \times Y' \\ \downarrow & & \downarrow f \times \text{id} \\ Y' & \xrightarrow{\Delta} & Y' \times Y' \end{array}$$

Now f_* satisfies the base change againts $!$ -pullbacks, so

$$(f_\bullet K) \otimes^! L \xrightarrow{\sim} \Delta^! (fg \times \text{id})_* (K' \boxtimes L) \xrightarrow{\sim} (fg)_* (K' \otimes^! (fg)^! L) \xrightarrow{\sim} f_\bullet (K \otimes^! f^! L),$$

because $f_\bullet g_* \xrightarrow{\sim} (fg)_*$. Indeed, g and fg are schematic. \square

Recall the self-duality

$$(5) \quad Shv(Y) \otimes Shv(Y) \rightarrow \text{Vect}, \quad (K_1, K_2) \mapsto C_{\blacktriangle}(Y, K_1 \otimes^! K_2)$$

from ([2], A.4.1). Under this self-duality, for $f : Y \rightarrow Y'$ as above the dual of the functor $f^! : Shv(Y') \rightarrow Shv(Y)$ is the functor $f_{\blacktriangle} : Shv(Y) \rightarrow Shv(Y')$, this follows from the above projection formula.

For $K \in Shv(Y)^c, K' \in Shv(Y')^c$ we have $\mathbb{D}(K \boxtimes K') \xrightarrow{\sim} (\mathbb{D}K) \boxtimes (\mathbb{D}K')$ naturally. Now as in ([22], Sect. 1.0.1) one shows that the dual h^\vee of the exterior product functor $h : Shv(Y) \otimes Shv(Y') \rightarrow Shv(Y \times Y')$ with respect to the above dualities identifies with the right adjoint $h^R : Shv(Y \times Y') \rightarrow Shv(Y) \otimes Shv(Y')$.

So, the unit of the self-duality (5) is the object $h^R(\Delta_{\blacktriangle} \omega_Y)$, where $\Delta : Y \rightarrow Y \times Y$ is the diagonal.

0.1.4. For algebraic stacks locally of finite type (with affine diagonal) we always have a $(!, *)$ -base change in the constructible context, this is mentioned in ([2], A.1.8) in particular.

0.1.5. Let $f : Y \rightarrow Y'$ be a morphism of algebraic stacks as in Section 0.1.2. For $F \in Shv(Y), K \in Shv(Y')$ we have a natural transformation functorial in K, F

$$(f_{\blacktriangle} F) \otimes K \rightarrow f_{\blacktriangle}(F \otimes f^* K)$$

This comes from ([2], Section A.3.3-A.3.4).

The following is also useful. For $K_1, K_2 \in Shv(Y')$ there is a natural transformation

$$f^*(K_1 \otimes^! K_2) \rightarrow (f^! K_1) \otimes^! (f^* K_2)$$

Indeed, it comes from the natural map $K_1 \otimes^! K_2 \rightarrow K_1 \otimes^! (f_* f^* K_2)$ and the projection formula for f_* .

Similarly, we have a natural map $f^* K_1 \otimes f^! K_2 \rightarrow f^!(K_1 \otimes K_2)$.

0.1.6. Let Z, Y, Y' be algebraic stacks as in Section 0.1.2, and $f : Y \rightarrow Y'$ be a morphism. For $K \in Shv(Z), F \in Shv(Y)$ we have canonically

$$(\text{id} \times f)_*(K \boxtimes F) \xrightarrow{\sim} K \boxtimes f_* F$$

Indeed, this is a particular case of the projection formula for $f \times \text{id} : Z \times Y \rightarrow Z \times Y'$, as $K \boxtimes F \xrightarrow{\sim} ((\text{id} \times f)^!(K \boxtimes \omega_{Y'})) \otimes^! p_2^! F$, so $(\text{id} \times f)_*(K \boxtimes F) \xrightarrow{\sim} (K \boxtimes \omega_{Y'}) \otimes^! (\text{id} \times f)_* p_2^! F$.

0.1.7. Let $f : Y \rightarrow Y'$ be a morphism of algebraic stacks as in Section 0.1.2. For $F \in Shv(Y), K \in Shv(Y')$ we have a natural transformation functorial in K, F

$$(f_* F) \otimes K \rightarrow f_*(F \otimes f^* K)$$

Indeed, it comes from $f^*((f_* F) \otimes K) \rightarrow F \otimes f^* K$.

There is a Verdier dual version of this map. Namely, a natural transformation

$$f_!(F \otimes^! f^! K) \rightarrow (f_! F) \otimes^! K$$

It comes from the evident map $F \otimes^! f^! K \rightarrow f^!((f_! F) \otimes^! K)$.

0.1.8. For a cartesian square of any algebraic stacks locally of finite type

$$\begin{array}{ccc} Y'_1 & \xrightarrow{f'} & Y'_2 \\ \downarrow g_1 & & \downarrow g_2 \\ Y_1 & \xrightarrow{f} & Y_2 \end{array}$$

we have the natural transformation

$$g_2^* \circ f_* \rightarrow f'_* \circ g_1^*$$

arising by adjointness from $f_*(g_1)_* \xrightarrow{\sim} (g_2)_* f'_*$. Besides, the base change isomorphism $g_2^* \circ f_! \xrightarrow{\sim} f'_! \circ g_1^*$ gives by adjointness a natural transformation

$$g_1^* \circ f^! \rightarrow (f')^! \circ g_2^*$$

Similarly, we have a natural transformation $f'_! g_1^! \rightarrow g_2^! f_!$.

0.1.9. Let $f : Y \rightarrow Y'$ be a morphism of algebraic stacks as in Section 0.1.2. Let us construct a natural morphism functorial in $L \in Shv(Y)$, $K, M \in Shv(Y')$

$$((f^! K) \otimes^! L) \otimes f^* M \rightarrow f^! (K \otimes M) \otimes^! L$$

We have a natural map $f^! K \otimes f^* M \rightarrow f^! (K \otimes M)$ by Section 0.1.5. So, it suffices to construct a natural map $(f^! K \otimes^! L) \otimes f^* M \rightarrow (f^! K \otimes f^* M) \otimes^! L$. It comes from the next observation.

Lemma 0.1.10. *Let Y be an algebraic stack as in Section 0.1.2. For $K_1, K_2, L \in Shv(Y)$ there is a natural map $(K_1 \otimes^! K_2) \otimes L \rightarrow (K_1 \otimes L) \otimes^! K_2$.*

Proof. 1) First, assume $K_2 \in Shv(Y)^{constr}$. Then

$$(K_1 \otimes^! K_2) \otimes L \xrightarrow{\sim} \mathcal{H}om(\mathbb{D}K_2, K_1) \otimes L$$

and $(K_1 \otimes L) \otimes^! K_2 \xrightarrow{\sim} \mathcal{H}om(\mathbb{D}K_2, K_1 \otimes L)$, here $\mathcal{H}om$ is the inner hom in $(Shv(Y), \otimes)$. The desired morphism comes from the natural map $(\mathbb{D}K_2) \otimes \mathcal{H}om(\mathbb{D}K_2, K_1) \otimes L \rightarrow K_1 \otimes L$. The so obtained morphisms are functorial in K_2 . Now if $K_2 \in Shv(Y)$ is written as $K_2 \xrightarrow{\sim} \text{colim}_{i \in I} K_2^i$ with I small filtered and $K_2^i \in Shv(Y)^{constr}$ then the desired morphism is obtained by passing to the colimit over $i \in I$ in the diagram $(K_1 \otimes^! K_2^i) \otimes L \rightarrow (K_1 \otimes L) \otimes^! K_2^i$.

2) Simplier argument. Consider the cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{\Delta} & Y \times Y \\ \downarrow \Delta & & \downarrow \Delta \times \text{id} \\ Y \times Y & \xrightarrow{\text{id} \times \Delta} & Y \times Y \times Y \end{array}$$

and apply the natural transformation $\Delta^* (\text{id} \times \Delta)^! \rightarrow \Delta^! (\Delta \times \text{id})^*$. \square

The map in the above lemma is not an isomorphism in general. For example, let $i : Y' \hookrightarrow Y$ be a closed immersion. Taking $K_2 = i_* \omega_{Y'}$, the above map reduces to a morphism $i^! K_1 \otimes i^* L \rightarrow i^! (K_1 \otimes L)$ on Y' , which is usually not an isomorphism. For example, if $i : \text{Spec } k \rightarrow Y$ is a closed point on a smooth curve Y and $K_1 = e_X$ this is a map $i^* L[-2] \rightarrow i^! L$.

0.2. Let $f_1 : Y_1 \rightarrow Z_1, f_2 : Y_2 \rightarrow Z_2$ be morphisms of algebraic stacks locally of finite type (with an affine diagonal). Assume we are in the constructible context. Then for any $F_i \in Shv(Y_i)$ we have

$$(f_1 \times f_2)_!(F_1 \boxtimes F_2) \xrightarrow{\sim} ((f_1)_! F_1) \boxtimes ((f_2)_! F_2)$$

in $Shv(Z_1 \times Z_2)$.

Proof. 1) For morphisms between schemes of finite type this is just the $(!, *)$ -projection formula, because $(f_1 \times f_2)_! = (f_1 \times \text{id})_!(\text{id} \times f_2)_!$.

2) Now we prove this under the additional assumption that $Y_i \in \text{Sch}_{ft}$. It suffices to establish this after any base change by $h_1 \times h_2 : S_1 \times S_2 \rightarrow Z_1 \times Z_2$, where $S_i \in \text{Sch}_{ft}^{aff}$ and h_i are smooth. This follows from the $(!, *)$ -base change.

3) For any prestack Y_1 we get

$$F_i \xrightarrow{\sim} \text{colim}_{S_i \xrightarrow{g_i} Y_i} (g_i)_!(g_i^! F_i),$$

where the colimit is over Sch_{ft}^{aff}/Y . So,

$$F_1 \boxtimes F_2 \xrightarrow{\sim} \text{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} ((g_1)_!(g_1^! F_1) \boxtimes ((g_2)_!(g_2^! F_2)) \xrightarrow{\sim} \text{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} (g_1 \times g_2)_!(g_1 \times g_2)^!(F_1 \boxtimes F_2),$$

where the second isomorphism uses 2). So,

$$\begin{aligned} (f_1 \times f_2)_!(F_1 \boxtimes F_2) &\xrightarrow{\sim} \text{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} (f_1 g_1 \times f_2 g_2)_!(g_1^! F_1 \boxtimes g_2^! F_2) \xrightarrow{\sim} \\ &\quad \text{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} ((f_1 g_1)_! g_1^! F_1) \boxtimes ((f_2 g_2)_! g_2^! F_2), \end{aligned}$$

where the last isomorphism used 2). The latter expression identifies with

$$\text{colim}_{S_1 \xrightarrow{g_1} Y_1, S_2 \xrightarrow{g_2} Y_2} f_{1!}(g_1^! F_1) \boxtimes f_{2!}(g_2^! F_2) \xrightarrow{\sim} \text{colim}_{S_1 \xrightarrow{g_1} Y_1} f_{1!}(g_1^! F_1) \boxtimes f_{2!}(\text{colim}_{S_2 \xrightarrow{g_2} Y_2} g_2^! F_2)$$

The latter identifies with $((f_1)_! F_1) \boxtimes ((f_2)_! F_2)$. \square

0.2.1. As a corollary, let Y be an algebraic stack locally of finite type (with an affine diagonal). Then $\text{R}\Gamma_c : (Shv(Y), \otimes^!) \rightarrow \text{Vect}$ is left-lax symmetric monoidal, so sends cocommutative coalgebras to cocommutative coalgebras. So, $\text{R}\Gamma_c(Y, \omega)$ is a cocommutative coalgebra in Vect . Moreover, ω becomes an object of $\text{R}\Gamma_c(Y, \omega) - \text{comod}(Shv(Y), \otimes^!)$ via the natural adjunction map $\text{act} : \omega_Y \rightarrow \text{R}\Gamma_c(Y, \omega) \otimes \omega_Y$. So, for any $F \in Shv(Y)$, F gets a coaction of $\text{R}\Gamma_c(Y, \omega)$ just by applying $\bullet \otimes^! F$ to the previous action map. The functor $\text{R}\Gamma_c$ extends to a functor $Shv(Y) \rightarrow \text{R}\Gamma_c(Y, \omega) - \text{comod}(\text{Vect})$ naturally, so the composition with $\text{oblv} : \text{R}\Gamma_c(Y, \omega) - \text{comod}(\text{Vect}) \rightarrow \text{Vect}$ is $\text{R}\Gamma_c(Y, \bullet)$.

0.2.2. Assume we are in the constructible context. Let Z, Y be algebraic stacks locally of finite type (with affine diagonals), and $p : Z \rightarrow Y$ any morphism, maybe not representable. Then for $K \in Shv(Y), L \in Shv(Z)$ one has canonically

$$p_* \mathcal{H}om(p^* K, L) \xrightarrow{\sim} \mathcal{H}om(K, p_* L)$$

in $Shv(Y)$. We underline that here p_* is maybe discontinuous if p is not representable.

Proof. Let us first assume $K \in Shv(Y)^{constr}$. Then

$$p_* \mathcal{H}om(p^* K, L) \xrightarrow{\sim} p_*(p^! (\mathbb{D} K) \otimes^! L) \xrightarrow{\sim} \mathbb{D}(K) \otimes^! p_* L \xrightarrow{\sim} \mathcal{H}om(K, p_* L)$$

by the projection formula.

Let now K be any, pick a presentation $K \xrightarrow{\sim} \text{colim}_{i \in I} K_i$ with $K_i \in Shv(Y)^{constr}$. Then p_* preserves limits, so we get

$$\begin{aligned} p_* \mathcal{H}om(p^* K, L) &\xrightarrow{\sim} \lim_{i \in I^{op}} p_* \mathcal{H}om(p^* K_i, L) \xrightarrow{\sim} \lim_{i \in I^{op}} \mathcal{H}om(K_i, p_* L) \\ &\xrightarrow{\sim} \mathcal{H}om(\text{colim}_{i \in I} K_i, p_* L) \end{aligned}$$

□

I wonder if one may replace here p_* by p_Δ .

0.2.3. Let I be small filtered, $I \rightarrow \text{Stk}$, $i \mapsto Y_i$ be the functor such that Y_i is an algebraic stack locally of finite type, for $i \rightarrow j$ in I , $Y_i \rightarrow Y_j$ is a closed immersion, $Y = \text{colim}_i Y_i$ in Stk . Then the functor $R\Gamma : Shv(Y) \rightarrow \text{Vect}$ is defined by passing to the colimit in the functors $R\Gamma : Shv(Y_i) \rightarrow \text{Vect}$ with respect to the maps $(f_{ij})_* : Shv(Y_i) \rightarrow Shv(Y_j)$. Usually, $R\Gamma : Shv(Y) \rightarrow \text{Vect}$ does not have a left adjoint.

A corollary of this: let $K \in Shv(Y)$, write $K \xrightarrow{\sim} \text{colim}_{i \in I} (i_i)_* i_i^! K$, where $i_i : Y_i \rightarrow Y$ is the natural map. Assume that for each i the functor $Shv(Y) \rightarrow \text{Vect}$, $F \mapsto R\Gamma(Y, ((i_i)_* i_i^! K) \otimes^! F)$ is continuous. Then $Shv(Y) \rightarrow \text{Vect}$, $F \mapsto R\Gamma(Y, F \otimes^! K)$ is also continuous.

0.2.4. Let $S \in \text{Sch}_{ft}$, let $f : Y \rightarrow S$ be a placid scheme over S . Then we have an action of $(Shv(S), \otimes^!)$ on $Shv(Y)$ such that $K \in Shv(S)$ sends $F \in Shv(Y)$ to $b^!(F \boxtimes K)$ for $b : Y \rightarrow Y \times S$.

The same structure is obtained as follows. Write $Y \xrightarrow{\sim} \lim_{i \in I^{op}} Y_i$ with Y_i a scheme of finite type, for $i \rightarrow j$ in I the map $f_{ij} : Y_j \rightarrow Y_i$ is smooth affine surjective. Then for $i \rightarrow j$ in I , $(f_{ij})_* : Shv(Y_j) \rightarrow Shv(Y_i)$ is a map in $Shv(S) - \text{mod}$, so $Shv(Y) \xrightarrow{\sim} \lim_{i \in I^{op}} Shv(Y_i)$ may be understood in $Shv(S) - \text{mod}$.

If $h : Y \rightarrow Z$ is any morphism of placid schemes over S then $h_* : Shv(Y) \rightarrow Shv(Z)$ is a map in $Shv(S) - \text{mod}$. Indeed, write $Z = \lim_{j \in J^{op}} Z_j$, where J is small filtered, $Z_j \in (\text{Sch}_{ft})/S$, and for any $j \rightarrow j'$ in J the map $Z_{j'} \rightarrow Z_j$ in $(\text{Sch}_{ft})/S$ is smooth affine surjective. It suffices to show that for $g_j : Z \rightarrow Z_j$ the functor $(g_j h)_* : Shv(Y) \rightarrow Shv(Z_j)$ is $Shv(S)$ -linear. However, there is $i \in I$ such that h factors through $Y \rightarrow Y_i \xrightarrow{\bar{h}} Z_j$. Our claim follows from the fact that $\bar{h}_* : Shv(Y_i) \rightarrow Shv(Z_j)$ is $Shv(S)$ -linear.

Remark: let $S \in \text{Sch}_{ft}$, I a small filtered category, $I^{op} \rightarrow (\text{PreStk}_{lft})/S$, $i \mapsto Y_i$ a functor such that for $i \rightarrow j$ in I the map $f_{ij} : Y_j \rightarrow Y_i$ is smooth of some relative dimension d , affine, surjective. Let $Y = \lim_{i \in I^{op}} Y_i$ in PreStk . Define $\text{Shv}(Y) = \text{colim}_{i \in I} \text{Shv}(Y_i)$ in $\text{Shv}(S) - \text{mod}$ with respect to the functors $f_{ij}^* : \text{Shv}(Y_i) \rightarrow \text{Shv}(Y_j)$. Let $0 \in I$ be initial. Then we get the structure functor $f_0^* : \text{Shv}(Y_0) \rightarrow \text{Shv}(Y)$ for $f_0 : Y \rightarrow Y_0$. It is clear that f_0^* is $\text{Shv}(S)$ -linear.

0.2.5. A property of the constructible context. Let S, X be schemes of finite type. Let $K_1, K_2 \in \text{Shv}(S)$. Then for the projection $q : S \times X \rightarrow S$ we have

$$q_* \mathcal{H}om(q^* K_1, q^* K_2) \xrightarrow{\sim} \mathcal{H}om(K_1, K_2) \otimes \text{R}\Gamma(X, e)$$

in $\text{Shv}(S)$, where $\mathcal{H}om$ denotes the local $\mathcal{H}om$ over the corresponding scheme. This isomorphism is compatible with compositions: given $K_i \in \text{Shv}(S)$ for $i = 1, 2, 3$ the composition

$$\mathcal{H}om(K_1, K_2) \otimes \mathcal{H}om(K_2, K_3) \rightarrow \mathcal{H}om(K_1, K_3)$$

via the above isomorphism corresponds to the composition

$$\mathcal{H}om(q^* K_1, q^* K_2) \otimes \mathcal{H}om(q^* K_2, q^* K_3) \rightarrow \mathcal{H}om(q^* K_1, q^* K_3)$$

In particular, we get an isomorphism of algebras in $\text{Shv}(S)$

$$q_* \mathcal{H}om(q^* K, q^* K) \xrightarrow{\sim} \mathcal{H}om(K, K) \otimes \text{R}\Gamma(X, e)$$

0.2.6. For $S \in \text{Sch}_{ft}$ for our sheaf theories, $\text{Shv}(S)$ is never rigid. For example, in the constructible context if S is smooth then for a k -point $s \in S$, $\delta_s \in \text{Shv}(S)$ is not dualizable, though compact.

In the constructible context the following is **not known**: given a map of schemes of finite type $f : S \rightarrow T$, is it true that $\text{Shv}(S)$ is dualizable as a $\text{Shv}(T)$ -module? Here $\text{Shv}(T)$ acts via the monoidal functor $f^!$.

The functor $\text{Shv} : (\text{Sch}_{ft}^{aff})^{op} \rightarrow \text{DGCat}_{cont}$ satisfies both Zariski descent and proper descent, hence h-descent.

A useful thing: if $f : Y_1 \rightarrow Y_2$ in PreStk_{lft} is an isomorphism in the h-topology then $f^! : \text{Shv}(Y_2) \rightarrow \text{Shv}(Y_1)$ is an isomorphism.

0.2.7. Let Y be a placid scheme written as $Y \xrightarrow{\sim} \lim_{i \in I^{op}} Y_i$, where I is small filtered, $Y_i \in \text{Sch}_{ft}$, for $i \rightarrow j$ in I , $f_{ij} : Y_j \rightarrow Y_i$ is smooth affine surjective. Assume $0 \in I$ is initial, and for any $i \rightarrow j$ in I , $f_{ij} : Y_j \rightarrow Y_i$ is a generalized affine fibration of rank $\dim Y_j - \dim Y_i$ (locally constant function on Y_i). Let $p : Y \rightarrow Y_0$ be the natural map. Assume we are in the constructible context. Then $p^* : \text{Shv}(Y_0) \rightarrow \text{Shv}(Y)$ admits a left adjoint $(p^*)^L$. The natural map $(p^*)^L p^* \rightarrow \text{id}$ is an isomorphism. The dual $((p^*)^L)^\vee$ identifies with the right adjoint to p_* via the self-dualities of $\text{Shv}(Y), \text{Shv}(Y_0)$ appearing in ([22], 1.2.11).

Proof. For $i \in I$ let $f_i : Y_i \rightarrow Y_0$ be the map f_{0i} . For $i \in I$ with $i \neq 0$ the functors $((f_i)_! [2 \dim Y_i - 2 \dim Y_0], f_i^*)$ form an adjoint pair. The system of functors $(f_i)_! [2 \dim Y_i - 2 \dim Y_0] : \text{Shv}(Y_i) \rightarrow \text{Shv}(Y_0)$ is compatible with the transition functors in $\text{Shv}(Y) \xrightarrow{\sim} \text{colim}_{j \in I} \text{Shv}(Y_j)$ with $*$ -pullbacks, so in the colimit over $j \in I$ we get a

functor $\operatorname{colim}_{j \in I} Shv(Y_j) \rightarrow Shv(Y_0)$. By ([21], 9.2.6), this is the left adjoint to p^* . We used that for $i \in I$, $(f_{ij})_* f_j^* \xrightarrow{\sim} f_i^*$ naturally.

By ([22], 1.2.11), the dual of $p^* : Shv(Y_0) \rightarrow Shv(Y)$ identifies naturally with $p_* : Shv(Y) \rightarrow Shv(Y_0)$. So, the dual of the adjoint pair $((p^*)^L, p^*)$ is $(p_*, ((p^*)^L)^\vee)$. \square

This situation happens often. For example, if G is an affine smooth algebraic group of finite type and $\mathcal{O} = k[[t]]$ then $G(\mathcal{O})$ is a placid scheme satisfying the above. So, for $p : G(\mathcal{O}) \rightarrow G$ in the constructible context we have an adjoint pair $(p^*)^L : Shv(G(\mathcal{O})) \rightleftarrows Shv(G) : p^*$ with p^* fully faithful. In particular, for $q : G(\mathcal{O}) \rightarrow \operatorname{Spec} k$ the functor q^* has a left adjoint $(q^*)^L$.

0.2.8. Assume for this subsection we are in the constructible context. Let G be a group scheme of finite type. Then $Shv(G)$ is equipped with the monoidal structure given by $K_1 *^! K_2 = m_!(K_1 \boxtimes K_2)$, where $m : G \times G \rightarrow G$ is the product map, $K_i \in Shv(G)$. Now for $Y \in \operatorname{PreStk}_{lft}$ with a G -action, $(Shv(G), *^!)$ acts on $Shv(Y)$ so that $K \in Shv(G)$ acts on $F \in Shv(Y)$ as $a_!(K \boxtimes F)$, where $a : G \times Y \rightarrow Y$ is the action map.

Consider the projections

$$\operatorname{pr}_G : G \times Y \rightarrow G, \operatorname{pr}_Y : G \times Y \rightarrow Y.$$

Assume G smooth. Let L be a character local system on G in the usual sense, that is, for $m : G \times G \rightarrow G$ we are given $m^* L \xrightarrow{\sim} L \boxtimes L$, and a map $L \rightarrow i_* e$ for the unit $i : \operatorname{Spec} k \rightarrow G$ with the usual properties. Let $\mathfrak{a}^R : Shv(Y) \rightarrow Shv(Y)$ be the comonad given by $K \mapsto L * K$, where we use the usual action (not the $!$ -one). Then the functor \mathfrak{a}^R admits a left adjoint $\mathfrak{a} : Shv(Y) \rightarrow Shv(Y)$, which is automatically a monad in $\operatorname{Fun}_{e, cont}(Shv(Y), Shv(Y))$. One has an equivalence

$$(6) \quad \mathfrak{a} - \operatorname{mod}(Shv(Y)) \xrightarrow{\sim} \mathfrak{a}^R - \operatorname{comod}(Shv(Y))$$

commuting with the oblivion functors to $Shv(Y)$. In particular, $\operatorname{oblv} : Shv(Y)^{G, L} \rightarrow Shv(Y)$ admits a left adjoint $\operatorname{ind} : Shv(Y) \rightarrow \mathfrak{a} - \operatorname{mod}(Shv(Y))$.

Proof. For $K, M \in Shv(Y)$ one has

$$\begin{aligned} \mathcal{H}om((\operatorname{pr}_Y)_!(\operatorname{pr}_G^* L^{-1} \otimes a^* K)[2 \dim G], M) &\xrightarrow{\sim} \mathcal{H}om(\operatorname{pr}_G^* L^{-1} \otimes a^* K, e \boxtimes M) \xrightarrow{\sim} \\ &\mathcal{H}om(a^* K, L \boxtimes M) \xrightarrow{\sim} \mathcal{H}om(K, a_*(L \boxtimes M)) \xrightarrow{\sim} \mathcal{H}om(K, L * M) \end{aligned}$$

So, the functor $\mathfrak{a} : Shv(Y) \rightarrow Shv(Y)$ given by $\mathfrak{a}(K) = (\operatorname{pr}_Y)_!(\operatorname{pr}_G^* L^{-1} \otimes a^* K)[2 \dim G]$ is left adjoint to \mathfrak{a}^R . The fact that \mathfrak{a} is monad and the equivalence (6) follow from ([21], 9.2.62). \square

0.2.9. Let $S \in \operatorname{Sch}_{ft}$, $f : Y \rightarrow S$ be an ind-scheme of ind-finite type over S . Assume $Y \xrightarrow{\sim} \operatorname{colim}_{i \in I} Y_i$, where I is small filtered, $Y_i \subset Y$ is a closed subscheme, $Y_i \in \operatorname{Sch}_{ft}$, for $i \rightarrow j$ in I , $Y_i \hookrightarrow Y_j$. Assume also each map $f_i : Y_i \rightarrow S$ is smooth.

Let $F \in Shv(S)^c$, $K \in Shv(Y)$. Then one has canonically

$$\mathcal{H}om(f^! F, K) \xrightarrow{\sim} \mathcal{H}om(f^!(e_S), f^!(\mathbb{D}F) \otimes^! K)$$

Here $\mathcal{H}om \in \operatorname{Vect}$ is the relative inner hom for the Vect -action.

Proof. Step 1. First, assume $f : Y \rightarrow S$ is a map in Sch_{ft} with f smooth of relative dimension d . Then $f^! = f^*[2d]$, so the LHS is

$$\begin{aligned} \mathcal{H}om(f^*F, K[-2d]) &\xrightarrow{\sim} \mathcal{H}om(F, f_*K[-2d]) \xrightarrow{\sim} \text{R}\Gamma(S, (\mathbb{D}F) \otimes^! f_*K[-2d]) \\ &\xrightarrow{\sim} \text{R}\Gamma(Y, K[-2d] \otimes^! f^!(\mathbb{D}F)) \end{aligned}$$

The RHS identifies with

$$\mathcal{H}om(f^*(e_S), f^!(\mathbb{D}F) \otimes^! K[-2d]) \xrightarrow{\sim} \text{R}\Gamma(Y, f^!(\mathbb{D}F) \otimes^! K[-2d])$$

We are done.

Step 2. Let $i_i : Y_i \rightarrow Y$ be the inclusion. Write $f^!F \xrightarrow{\sim} \text{colim}_{i \in I} (i_i)_* f_i^!F$, so the LHS becomes

$$\lim_{i \in I^{op}} \mathcal{H}om(f_i^!F, (i_i)^!K) \xrightarrow{\sim} \lim_{i \in I^{op}} \mathcal{H}om(f_i^!(e_S), f_i^!(\mathbb{D}F) \otimes^! (i_i)^!K)$$

by Step 1. Write $f^!(e_S) \xrightarrow{\sim} \text{colim}_{i \in I} (i_i)_! f_i^!(e_S)$ then the RHS of the latter expression becomes

$$\begin{aligned} \lim_{i \in I^{op}} \mathcal{H}om(f_i^!(e_S), (i_i)^!(f^!(\mathbb{D}F) \otimes^! K)) &\xrightarrow{\sim} \lim_{i \in I^{op}} \mathcal{H}om((i_i)_! f_i^!(e_S), f^!(\mathbb{D}F) \otimes^! K) \\ &\xrightarrow{\sim} \mathcal{H}om(f^!(e_S), f^!(\mathbb{D}F) \otimes^! K) \end{aligned}$$

as desired. \square

In the sense of ULA property in its form given by Dennis in ([12], 1.6.3) this says that ω_Y is ULA with respect to the $\text{Shv}(S)$ -action on $\text{Shv}(Y)$.

0.2.10. A generality: let $S \in \text{Sch}_{ft}$, $U \rightarrow S$ be a smooth unipotent group scheme over S . Then for $f : B(S) \rightarrow S$ the functor $f^* : \text{Shv}(S) \xrightarrow{\sim} \text{Shv}(B(S))$ is an equivalence.

Let $U_1 \rightarrow U_2$ be a homomorphism of smooth unipotent group schemes over S . Take $Y = U_2/U_1$, the stack quotient over S . Let $a : Y \rightarrow S$ be the natural map. Then a^*a_* is left t-exact.

0.3. Addition for any sheaf theory. Work in any of our 4 sheaf theory for this subsection.

0.3.1. Let

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & Z' \\ \downarrow & & \downarrow \alpha_Z \\ Y & \xrightarrow{i} & Z \end{array}$$

be a cartesian square in PreStk_{lft} such that i is a closed immersion in Sch_{ft} . Then the natural functor $\text{Shv}(Z') \otimes_{\text{Shv}(Z)} \text{Shv}(Y) \rightarrow \text{Shv}(Y')$ is an equivalence. Here we view $\text{Shv}(Z)$ with the $\otimes^!$ -monoidal structure.

Proof. First, for \mathcal{D} -modules this is true without the assumption that i is a closed immersion by ([14], Section 1.6.4). Assume now we are in the constructible context.

Consider the comonad $\mathcal{A} := i'_*(i')^!$ on $\text{Shv}(Z')$, it is $\text{Shv}(Z')$ -linear, hence given by the coalgebra $i'_*\omega$. The functor $i'_* : \text{Shv}(Y) \rightarrow \text{Shv}(Z)$ is comonadic. Indeed, i'_* has a left adjoint, hence preserves limits, and i'_* is fully faithful.

In fact, since i'_* is fully faithful, we obtain that $Shv(Y') \subset Shv(Z')$ is the full subcategory of those $K \in Shv(Z')$ for which the counit map $\mathcal{A}(K) \rightarrow K$ is an isomorphism.

We claim that the composition $Shv(Z') \otimes_{Shv(Z)} Shv(Y) \rightarrow Shv(Y') \rightarrow Shv(Z')$ is described similarly with the same comonad. Indeed, consider the adjoint pair $i_! : Shv(Y) \rightleftarrows Shv(Z) : i^!$ in $Shv(Z) - mod$. We have the comonad $i_! \omega$ on $Shv(Z)$ with $Shv(Y) \xrightarrow{\sim} (i_! \omega) - comod(Shv(Z))$. After the base change $\cdot \otimes_{Shv(Z)} Shv(Z')$, our adjoint pair becomes

$$L : Shv(Y) \otimes_{Shv(Z)} Shv(Z') \rightleftarrows Shv(Z') : R$$

in $Shv(Z') - mod$ with $LR \xrightarrow{\sim} \alpha_Z^! i_* \omega$. As a coalgebra in $Shv(Z')$ it coincides with $i'_* \omega$. Note that L is fully faithful, because $\text{id} \rightarrow RL$ is an isomorphism, so L is conservative. Finally, $Shv(Y) \otimes_{Shv(Z)} Shv(Z') \subset Shv(Z')$ is the full subcategory of $K \in Shv(Z')$ for which the natural map $LR(K) \rightarrow K$ is an isomorphism. We are done.

Note also that $Shv(Y)$ is self-dual in $Shv(Z) - mod$, so by ([21], 9.2.57), the map L rewrites as

$$(7) \quad \text{Fun}_{Shv(Z)}(Shv(Y), Shv(Z')) \rightarrow \text{Fun}_{Shv(Z)}(Shv(Z), Shv(Z'))$$

given by the composition with $i^! : Shv(Z) \rightarrow Shv(Y)$. Indeed, the dual of i_* is $i^!$ for the standard self-dualities. It is not clear here if (7) preserves limits, as the limits are not "computed pointwise"! \square

0.3.2. For $X \in \text{Sch}_{ft}$ let $j : U \hookrightarrow X$ an open subscheme. Equip $Shv(X)$ with the $\otimes^!$ -symmetric monoidal structure. The adjoint pair $j^* : Shv(X) \rightleftarrows Shv(U) : j_*$ is in $Shv(X) - mod$, and j_* is right-lax nonunital symmetric monoidal. So $j_* \omega \in \text{CAlg}(Shv(X))$. The functor $j_* : Shv(U) \rightarrow Shv(X)$ factors naturally through $Shv(U) \rightarrow (j_* \omega) - mod(Shv(X))$. Now $Shv(U) \xrightarrow{\sim} (j_* \omega) - mod(Shv(X))$, this is the image of the action of the idempotent $(j_* \omega)$ on $Shv(X)$, cf. ([21], 9.2.74). Here $j_* \omega$ is an idempotent commutative algebra in $Shv(X)$ in the sense of ([16], 4.8.2.8).

If $M \in Shv(X) - mod$ then we get an adjoint pair $j^* : M \rightleftarrows M \otimes_{Shv(X)} Shv(U) : j_*$ in DGCat_{cont} , and the right adjoint is monadic. So,

$$M \otimes_{Shv(X)} Shv(U) \xrightarrow{\sim} (j_* \omega) - mod(M)$$

Recall that $\text{oblv} : (j_* \omega) - mod(M) \rightarrow M$ is fully faithful, and its image is the image of the action of $j_* \omega$ on M .

Let now $A \in \text{coAlg}(Shv(X))$. By ([21], 9.2.60), let $M = A - comod(Shv(X)) \in Shv(X) - mod$, and we have an adjoint pair in $Shv(X) - mod$

$$(8) \quad \text{oblv} : A - comod(Shv(X)) \rightleftarrows Shv(X) : \text{coind}.$$

Applying $\otimes_{Shv(X)} Shv(U)$, one gets the adjoint pair $l : M \otimes_{Shv(X)} Shv(U) \rightleftarrows Shv(U) : r$ in $Shv(U) - mod$. The comonad $lr : Shv(U) \rightarrow Shv(U)$ is given by $A_U \in \text{coAlg}(Shv(U))$, the restriction of A to U .

Lemma 0.3.3. *l is comonadic, so*

$$(A - comod(Shv(X))) \otimes_{Shv(X)} Shv(U) \xrightarrow{\sim} A_U - comod(Shv(U))$$

Proof. Consider the diagram, where the horizontal functors are fully faithful

$$\begin{array}{ccc} M \otimes_{Shv(X)} Shv(U) & \xrightarrow{j_*} & M \\ \downarrow l & & \downarrow \text{oblv} \\ Shv(U) & \xrightarrow{j_*} & Shv(X) \end{array}$$

It shows that l is conservative. Let now V be a simplicial object of $(M \otimes_{Shv(X)} Shv(U))^{\text{op}}$ such that $l(V)$ is split in $Shv(U)^{\text{op}}$. Then $j_* l(V) \xrightarrow{\sim} \text{oblv}(j_* V)$ is split in $Shv(X)^{\text{op}}$. By ([16], 4.7.3.5), $j_* V$ admits a colimit in M^{op} , and $\text{oblv} : M^{\text{op}} \rightarrow Shv(X)^{\text{op}}$ preserves this colimit. Since $M \otimes_{Shv(X)} Shv(U)$ has all limits and colimits, V admits a colimit in $(M \otimes_{Shv(X)} Shv(U))^{\text{op}}$, and j_* preserves this colimit in M^{op} . Now $l(V) \rightarrow l(\text{colim } V)$ is a diagram in $Shv(U)^{\text{op}}$, which becomes a colimit diagram in $Shv(X)^{\text{op}}$ after applying j_* . Hence, it is also a colimit diagram in $Shv(U)^{\text{op}}$. That is, l preserves the colimit of V . By ([16], 4.7.3.5), l is comonadic. \square

0.3.4. For $X \in Sch_{ft}$ let $i : Z \rightarrow X$ be a closed subscheme. The dual pair $i_! : Shv(Z) \rightleftharpoons Shv(X) : i^!$ takes place in $Shv(X) - \text{mod}$, the corresponding comonad is $i_* \omega \in \text{coAlg}(Shv(X))$. The functor $i_! : Shv(Z) \rightarrow Shv(X)$ is comonadic, this is easy using the full ([16], 4.7.3.5). So, $Shv(Z) \xrightarrow{\sim} (i_* \omega) - \text{comod}(Shv(X))$.

In turn, $(i_* \omega) - \text{mod}(Shv(X)^{\text{op}})$ is the image of the localization functor $i_* \omega \otimes^! \cdot : Shv(X)^{\text{op}} \rightarrow Shv(X)^{\text{op}}$ by ([21], 9.2.74). So, $Shv(Z)$ is the full subcategory of those $K \in Shv(X)$, for which the map $i_* \omega \rightarrow \omega$ tensored by K becomes an isomorphism.

Let $M \in Shv(X) - \text{mod}$. We get an adjoint pair $i_! : M \otimes_{Shv(X)} Shv(Z) \rightleftharpoons M : i^!$ in $\text{DGCat}_{\text{cont}}$, and $i_!$ is fully faithful. For the same reasons, $i_!$ is comonadic, so $M \otimes_{Shv(X)} Shv(Z) \xrightarrow{\sim} (i_* \omega) - \text{comod}(M)$. The image of $i_!$ is the full subcategory of those $K \in M$ for which the map $i_* \omega \rightarrow \omega$ tensored by K becomes an isomorphism (where now \otimes stand for the $Shv(X)$ -action on M). Again, $i_* \omega \in \text{ComCoAlg}(Shv(X))$ is an idempotent coalgebra ([16], 4.2.4.10).

Let now $A \in \text{coAlg}(Shv(X), \otimes^!)$. Consider the adjoint pair (8) in $Shv(X) - \text{mod}$. Let $M = A - \text{comod}(Shv(X))$. Applying $\otimes_{Shv(X)} Shv(Z)$, one gets the adjoint pair

$$l : M \otimes_{Shv(X)} Shv(Z) \rightleftharpoons Shv(Z) : r$$

in $Shv(Z) - \text{mod}$. The comonad $lr : Shv(Z) \rightarrow Shv(Z)$ is given by tensoring with $A_Z := i^! A \in \text{coAlg}(Shv(Z))$.

Lemma 0.3.5. l is comonadic, so

$$(A - \text{comod}(Shv(X))) \otimes_{Shv(X)} Shv(Z) \xrightarrow{\sim} A_Z - \text{comod}(Shv(Z))$$

Proof. Consider the diagram, where the horizontal functors are fully faithful

$$\begin{array}{ccc} M \otimes_{Shv(X)} Shv(Z) & \xrightarrow{i_*} & M \\ \downarrow l & & \downarrow \text{oblv} \\ Shv(Z) & \xrightarrow{i_*} & Shv(X) \end{array}$$

It shows that l is conservative.

Let now V be a simplicial object of $(M \otimes_{Shv(X)} Shv(Z))^{op}$ such that $l(V)$ is split in $Shv(Z)^{op}$. Then $i_*l(V) \xrightarrow{\sim} \text{oblv } i_*(V)$ is split in $Shv(X)^{op}$. So, i_*V admits a colimit in M^{op} , and $\text{oblv} : M^{op} \rightarrow Shv(X)^{op}$ preserves this colimit. Write W for the colimit of i_*V in M^{op} , so $i_*V \rightarrow W$ is a colimits diagram in M^{op} . Since $i^! : M^{op} \rightarrow (M \otimes_{Shv(X)} Shv(Z))^{op}$ preserves colimits, $i^!i_*V \rightarrow i^!W$ is a colimit diagram in $(M \otimes_{Shv(X)} Shv(Z))^{op}$. Note that $li^! \xrightarrow{\sim} i^! \text{oblv}$. Further, $\text{oblv}(i_*V) \rightarrow \text{oblv}(W)$ is a colimit diagram in $Shv(X)^{op}$, hence $i^! \text{oblv}(i_*V) \rightarrow i^! \text{oblv}(W)$ is a colimit diagram in $Shv(Z)^{op}$. The latter diagram is nothing but the desired diagram $l(i^!i_*V) \rightarrow l(i^!W)$. Thus, the colimit of V in $(M \otimes_{Shv(X)} Shv(Z))^{op}$ is preserved by l . By ([16], 4.7.3.5), l is comonadic. \square

We propose the following generalization.

Lemma 0.3.6. *Let $i : Z \hookrightarrow X$ be a closed immersion in Sch_{ft} , let $Y \rightarrow X$ be a map in PreStk_{lft} , set $Y' = Y \times_X Z$. Let*

$$A \in \text{Fun}_{Shv(X)}(Shv(Y), Shv(Y))$$

be a $Shv(X)$ -linear continuous comonad, set

$$A_Z = A \otimes \text{id} : Shv(Y) \otimes_{Shv(X)} Shv(Z) \rightarrow Shv(Y) \otimes_{Shv(X)} Shv(Z).$$

Then $A_Z \in \text{Fun}_{Shv(Z)}(Shv(Y'), Shv(Y'))$ is a $Shv(Z)$ -linear continuous comonad. Moreover, one has canonically

$$(9) \quad (A - \text{comod}(Shv(Y))) \otimes_{Shv(X)} Shv(Z) \xrightarrow{\sim} A_Z - \text{comod}(Shv(Y')).$$

Proof. The natural functor (9) is constructed in ([21], 9.2.75) in bigger generality. Set $M = A - \text{comod}(Shv(Y))$. Consider the commutative diagram, where the horizontal functors are fully faithful

$$\begin{array}{ccc} M \otimes_{Shv(X)} Shv(Z) & \xrightarrow{i_*} & M \\ \downarrow l & & \downarrow \text{oblv} \\ Shv(Y) \otimes_{Shv(X)} Shv(Z) & \xrightarrow{i_*} & Shv(Y) \end{array}$$

Here l is obtained by base change from oblv . This diagram shows that l is conservative. Recall the low arrow identifies with $i_* : Shv(Y') \rightarrow Shv(Y)$ by Section 0.3.1.

Let us verify that l is comonadic by ([16], 4.7.3.5). Exactly the same argument as in the previous lemma applies. \square

0.3.7. Let $Z \hookrightarrow X \xleftarrow{j} U$ be a diagram in Sch_{ft} , where i is a closed immersion and j is the complement open. Let $C \in Shv(X) - \text{mod}$. Then for any $c \in C$ we get a fibre sequence $i^!i^!c \rightarrow c \rightarrow j_*j^*c$ in C , where the functors $i^!, i^!, j_*, j^*$ are in the previous subsections. It is obtained by tensoring the fibre sequence $i_*\omega_Z \rightarrow \omega_X \rightarrow j_*\omega_U$ by c .

Applying $\text{Fun}_{Shv(X)}(C, \cdot)$ to the adjoint pair $j^* : Shv(X) \rightleftarrows Shv(U) : j_*$ in $Shv(X) - \text{mod}$, one gets an adjoint pair

$$\text{Fun}_{Shv(X)}(C, Shv(X)) \rightleftarrows \text{Fun}_{Shv(X)}(C, Shv(U))$$

in $Shv(X) - mod$, where the right adjoint is fully faithful (hence monadic). The corresponding monad is given by the action of the algebra $j_*\omega$. So,

$$\mathrm{Fun}_{Shv(X)}(C, Shv(U)) \xrightarrow{\sim} j_*\omega - mod(\mathrm{Fun}_{Shv(X)}(C, Shv(X)))$$

By ([21], 9.2.74), $\mathrm{Fun}_{Shv(X)}(C, Shv(U))$ is just the image of the action of the idempotent $j_*\omega$ on $\mathrm{Fun}_{Shv(X)}(C, Shv(X))$. From Section 0.3.2 we conclude that

$$\begin{aligned} \mathrm{Fun}_{Shv(U)}(C \otimes_{Shv(X)} Shv(U), Shv(U)) &\xrightarrow{\sim} \mathrm{Fun}_{Shv(X)}(C, Shv(U)) \xrightarrow{\sim} \\ &\mathrm{Fun}_{Shv(X)}(C, Shv(X)) \otimes_{Shv(X)} Shv(U) \end{aligned}$$

Applying $\mathrm{Fun}_{Shv(X)}(C, \cdot)$ to the adjoint pair $i_! : Shv(Z) \rightleftarrows Shv(X) : i^!$ in $Shv(X) - mod$, we get an adjoint pair in $Shv(X) - mod$

$$\mathrm{Fun}_{Shv(X)}(C, Shv(Z)) \rightleftarrows \mathrm{Fun}_{Shv(X)}(C, Shv(X)),$$

where the left adjoint is fully faithful. Using this fully faithfulness, we show as above that the left adjoint is comonadic, so

$$\begin{aligned} \mathrm{Fun}_{Shv(Z)}(C \otimes_{Shv(X)} Shv(Z), Shv(Z)) &\xrightarrow{\sim} \mathrm{Fun}_{Shv(X)}(C, Shv(Z)) \xrightarrow{\sim} \\ &(i_!\omega) - comod(\mathrm{Fun}_{Shv(X)}(C, Shv(X))) \end{aligned}$$

As in Section 0.3.4,

$$(i_!\omega) - comod(\mathrm{Fun}_{Shv(X)}(C, Shv(X))) \xrightarrow{\sim} \mathrm{Fun}_{Shv(X)}(C, Shv(X)) \otimes_{Shv(X)} Shv(Z)$$

We conclude that

$$\mathrm{Fun}_{Shv(Z)}(C \otimes_{Shv(X)} Shv(Z), Shv(Z)) \xrightarrow{\sim} \mathrm{Fun}_{Shv(X)}(C, Shv(X)) \otimes_{Shv(X)} Shv(Z)$$

0.3.8. Let $j_i : U_i \hookrightarrow S$ are open subsets in $S \in \mathrm{Sch}_{ft}$ for $i = 1, 2$ with $U = U_1 \cap U_2$ and $U_1 \cup U_2 = S$. Then the square is cocartesian in $Shv(S) - mod$

$$\begin{array}{ccc} Shv(U_1) & \xrightarrow{(j_1)_*} & Shv(S) \\ \uparrow & & \uparrow (j_2)_* \\ Shv(U) & \rightarrow & Shv(U_2), \end{array}$$

where all the functors are given by $*$ -direct images. Indeed, this follows from Zariski descent for sheaves of categories on S . Namely, this diagram after restriction to each U_i becomes cocartesian.

0.4. More about the constructible context. For this subsection we work in the constructible context.

0.4.1. Let $S \in \mathrm{Sch}_{ft}$. The functor $\mathbb{D} : Shv(S)^c \xrightarrow{\sim} Shv(S)^{c, op}$ is an equivalence of symmetric monoidal categories, where the LHS is equipped with the $\otimes^!$ -monoidal structure, and the RHS is equipped with the \otimes -monoidal structure.

Proof: we have an isomorphism $K_1 \otimes^! K' \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(K) \otimes \mathbb{D}(K'))$ in $Shv(S)^c$ functorial in $K, K' \in Shv(S)$. \square

0.4.2. Consider a diagram $S_1 \rightarrow Y \leftarrow S_2$ in Sch_{ft} . Equip $\text{Shv}(Y)$ with the $\otimes^!$ -symmetric monoidal structure. Then $\text{Shv}(S_1) \otimes_{\text{Shv}(Y)} \text{Shv}(S_2)$ is dualizable in DGCat_{cont} by ([7], I.1, 6.3.4). Indeed, the product functor $\text{Shv}(Y) \otimes \text{Shv}(Y) \xrightarrow{m} \text{Shv}(Y)$ and action maps $\text{Shv}(Y) \otimes \text{Shv}(S_i) \rightarrow \text{Shv}(S_i)$ admits continuous right adjoints, and for any $n \geq 0$, $\text{Shv}(S_1) \otimes \text{Shv}(Y)^{\otimes n} \otimes \text{Shv}(S_2)$ is compactly generated.

It is easy to see that $\text{Shv}(S_1) \otimes_{\text{Shv}(Y)} \text{Shv}(S_2)$ is compactly generated by objects of the form $K_1 \boxtimes K_2$ with $K_i \in \text{Shv}(S_i)^c$.

Let $C_{\Delta^{op}} : \Delta^{op} \rightarrow \text{DGCat}_{cont}$ be the diagram

$$[n] \mapsto \text{Shv}(S_1) \otimes \text{Shv}(Y)^{\otimes n} \otimes \text{Shv}(S_2)$$

such that $\text{colim } C_{\Delta^{op}} \xrightarrow{\sim} \text{Shv}(S_1) \otimes_{\text{Shv}(Y)} \text{Shv}(S_2)$ in DGCat_{cont} by definition. Then we may pass to continuous right adjoints in $C_{\Delta^{op}}$ and get the functor $C_{\Delta}^R : \Delta \rightarrow \text{DGCat}_{cont}$, so that $\lim C_{\Delta}^R \xrightarrow{\sim} \text{Shv}(S_1) \otimes_{\text{Shv}(Y)} \text{Shv}(S_2)$ in DGCat_{cont} . The projection

$$\delta^R := \text{ev}_0 : \lim C_{\Delta}^R \rightarrow \text{Shv}(S_1) \otimes \text{Shv}(S_2)$$

has a left adjoint $\delta := \text{ins}_0 : \text{Shv}(S_1) \otimes \text{Shv}(S_2) \rightarrow \text{Shv}(S_1) \otimes_{\text{Shv}(Y)} \text{Shv}(S_2)$. So, by ([16], 4.7.5.1), the functor δ^R is monadic and

$$\text{Shv}(S_1) \otimes_{\text{Shv}(Y)} \text{Shv}(S_2) \xrightarrow{\sim} (\delta^R \delta) - \text{mod}(\text{Shv}(S_1) \otimes \text{Shv}(S_2)).$$

Lemma 0.4.3. *The dual $(\text{Shv}(S_1) \otimes_{\text{Shv}(Y)} \text{Shv}(S_2))^{\vee}$ identifies with*

$$\text{Shv}(S_1) \otimes_{(\text{Shv}(Y), \otimes)} \text{Shv}(S_2),$$

where now $\text{Shv}(Y)$ is equipped with the \otimes -symmetric monoidal structure, and the action maps are given as compositions

$$\text{Shv}(Y) \otimes \text{Shv}(S_i) \rightarrow \text{Shv}(Y \times S_i) \xrightarrow{\Gamma_i^*} \text{Shv}(S_i).$$

Here $\Gamma_i : S_i \rightarrow S_i \times Y$ is the graph of the map $S_i \rightarrow Y$. We used the canonical self-dualities on $\text{Shv}(S_i), \text{Shv}(Y)$.

Proof. The right adjoint to the composition

$$\text{Shv}(S) \otimes \text{Shv}(Y) \xrightarrow{h} \text{Shv}(S \times Y) \xrightarrow{\Gamma_i^!} \text{Shv}(S)$$

is the composition $h^R \circ (\Gamma_i^!)^R$, and $h^R \xrightarrow{\sim} h^{\vee}$, $(\Gamma_i^!)^R \xrightarrow{\sim} (\Gamma_i^*)^{\vee}$ canonically. Similarly for the product map

$$\text{Shv}(Y) \otimes \text{Shv}(Y) \xrightarrow{h} \text{Shv}(Y \times Y) \xrightarrow{\Delta^!} \text{Shv}(Y)$$

its right adjoint is $h^R \circ (\Delta^!)^R$, and $h^R \xrightarrow{\sim} h^{\vee}$, $(\Delta^!)^R \xrightarrow{\sim} (\Delta^*)^{\vee}$. The claim follows as in ([7], I.1, 6.3.4). \square

0.4.4. It is easy to see that $\text{Shv}(S_1) \otimes_{(\text{Shv}(Y), \otimes)} \text{Shv}(S_2)$ is compactly generated by objects of the form $K_1 \boxtimes K_2$ with $K_i \in \text{Shv}(S_i)^c$.

0.4.5. Let

$$\mathcal{F} : Shv(S_1) \otimes_{Shv(Y)} Shv(S_2) \rightarrow Shv(S_1 \times_Y S_2)$$

be the natural map coming from $K_1 \boxtimes K_2 \mapsto q^!(K_1 \boxtimes K_2)$ for $q : S_1 \times_Y S_2 \rightarrow Y_1 \times Y_2$. For $K_i \in Shv(S_i)^c$ the object $\mathcal{F}(K_1 \boxtimes K_2) \in Shv(S_1 \times_Y S_2)^c$, so \mathcal{F} has a continuous right adjoint.

We also have a natural functor

$$\mathcal{F}' : Shv(S_1) \otimes_{(Shv(Y), \otimes)} Shv(S_2) \rightarrow Shv(S_1 \times_Y S_2)$$

coming from $K_1 \boxtimes K_2 \mapsto q^*(K_1 \boxtimes K_2)$. For $K_i \in Shv(S_i)^c$ the object $\mathcal{F}'(K_1 \boxtimes K_2) \in Shv(S_1 \times_Y S_2)^c$, so \mathcal{F}' has a continuous right adjoint.

The dual of \mathcal{F} is the functor

$$\mathcal{F}^\vee : Shv(S_1 \times_Y S_2) \rightarrow Shv(S_1) \otimes_{(Shv(Y), \otimes)} Shv(S_2)$$

The dual of \mathcal{F}' is the functor

$$\mathcal{F}'^\vee : Shv(S_1 \times_Y S_2) \rightarrow Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$$

0.4.6. Write $\delta : Shv(S_1) \otimes Shv(S_2) \rightarrow Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$ and

$$\delta_\otimes : Shv(S_1) \otimes Shv(S_2) \rightarrow Shv(S_1) \otimes_{(Shv(Y), \otimes)} Shv(S_2)$$

for the natural functors. Let δ^R be the right adjoint to δ . By construction, we get $(\delta^R)^\vee \xrightarrow{\sim} \delta_\otimes$.

Lemma 0.4.7. *In the situation of Section 0.4.2 one has canonically $\mathcal{F}^\vee \xrightarrow{\sim} (\mathcal{F}')^R$ and $\mathcal{F}'^\vee \xrightarrow{\sim} \mathcal{F}^R$, where R stands for the right adjoint.*

Proof. Let $C_{\Delta^{op}} : \Delta^{op} \rightarrow \mathbf{DGCat}_{cont}$ be the functor giving rise to

$$Shv(S_1) \otimes_{Shv(Y)} Shv(S_2)$$

in its colimit by definition. It sends $[n]$ to $Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2)$. For a map $\alpha : [i] \rightarrow [j]$ in Δ we have the transition functor $\alpha_{ij} : C_{\Delta^{op}}(j) \rightarrow C_{\Delta^{op}}(i)$ in this diagram.

Write

$$f_n : Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2) \rightarrow Shv(S_1 \times_Y S_2)$$

for the composition of ins_n with \mathcal{F} . We have the adjoint pairs (f_n, f_n^R) and $(\mathcal{F}, \mathcal{F}^R)$ in \mathbf{DGCat}_{cont} . We get the adjoint pairs $((f_n^R)^\vee, (f_n)^\vee)$ and $((\mathcal{F}^R)^\vee, \mathcal{F}^\vee)$ in \mathbf{DGCat}_{cont} .

Denote by

$$C_{\Delta}^R : \Delta \rightarrow \mathbf{DGCat}_{cont}$$

the functor obtained from $C_{\Delta^{op}}$ by passing to the right adjoints. Denote by $C_{\Delta}^\vee : \Delta \rightarrow \mathbf{DGCat}_{cont}$ the functor obtained from $C_{\Delta^{op}}$ by passing to the duals. Denote by

$$(C_{\Delta}^R)^\vee : \Delta^{op} \rightarrow \mathbf{DGCat}_{cont}$$

the functor obtained from C_{Δ}^R by passing to the duals. Recall that

$$\mathrm{colim}(C_{\Delta}^\vee)^L \xrightarrow{\sim} \lim C_{\Delta}^\vee \xrightarrow{\sim} Shv(S_1) \otimes_{(Shv(Y), \otimes)} Shv(S_2)$$

canonically, where $(C_\Delta^\vee)^L : \Delta^{op} \rightarrow \text{DGCat}_{cont}$ is obtained from C_Δ^\vee by passing to the left adjoints. Recall that

$$(C_\Delta^\vee)^L \rightrightarrows (C_\Delta^R)^\vee.$$

The functor $\mathcal{F}^R : Shv(S_1 \times_Y S_2) \rightarrow \lim C_\Delta^R$ is obtained from the compatible system of functors f_n^R for $[n] \in \Delta$.

The functor

$$\mathcal{F}^\vee : Shv(S_1 \times_Y S_2) \rightarrow Shv(S_1) \otimes_{(Shv(Y), \otimes)} Shv(S_2) \rightrightarrows \lim C_\Delta^\vee$$

is obtained from the compatible system of functors f_n^\vee , $[n] \in \Delta$. So, the functor

$$(\mathcal{F}^\vee)^L : \text{colim}(C_\Delta^\vee)^L \rightarrow Shv(S_1 \times_Y S_2)$$

is obtained from the compatible system of functors $(f_n^\vee)^L$.

Consider the functor $D_{\Delta^{op}} : \Delta^{op} \rightarrow \text{DGCat}_{cont}$ such that

$$\text{colim } D_{\Delta^{op}} \rightrightarrows Shv(S_1) \otimes_{(Shv(Y), \otimes)} Shv(S_2)$$

by definition. It sends $[n]$ to $Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2)$. It suffices to show that the composition

$$Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2) \rightarrow Shv(S_1) \otimes_{(Shv(Y), \otimes)} Shv(S_2) \xrightarrow{\mathcal{F}'} Shv(S_1 \times_Y S_2)$$

for any n identifies with the functor $(f_n^\vee)^L \rightrightarrows (f_n^R)^\vee$. This is easy as in Lemma 0.4.3.

Namely, we have a natural map $\tau_n : S_1 \times_Y S_2 \rightarrow S_1 \times Y^n \times S_2$ coming from $(S_1 \times_Y S_2) \rightrightarrows (S_1 \times Y^n \times S_2) \times_{Y^{n+2}} Y \rightrightarrows S_1 \times_Y S_2$. Then f_n is the composition

$$Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2) \rightarrow Shv(S_1 \times Y^n \times S_2) \xrightarrow{\tau_n^!} Shv(S_1 \times_Y S_2)$$

So, $f_n^R = (\Delta^\vee) \circ (\tau_n^*)^{vee}$ as desired. \square

0.4.8. In the situation of Section 0.4.2 note that Y is naturally a cocommutative coalgebra in Sch_{ft} , the coproduct being the diagonal map $Y \rightarrow Y \times Y$. Besides, S_1 (resp., S_2) is a Y -comodule, the coaction map is $\Gamma_i : S_i \rightarrow S_i \times Y$, the graph of the map $S_i \rightarrow Y$. We get the morphism $S_1 \times_Y S_2 \rightarrow \lim_{[n] \in \Delta} S_1 \times Y^n \times S_2$, the corresponding version of the bar complex. It yields after applying Shv the morphisms

$$Shv(S_1) \otimes_{Shv(Y)} Shv(S_2) \xrightarrow{\mathcal{F}_1} C := \text{colim}_{[n] \in \Delta^{op}} Shv(S_1 \times Y^n \times S_2) \xrightarrow{\mathcal{F}_2} Shv(S_1 \times_Y S_2),$$

so $\mathcal{F} \rightrightarrows \mathcal{F}_2 \circ \mathcal{F}_1$. Let $\delta_1 : Shv(S_1 \times S_2) \rightarrow C$ be the natural map. Note that C is compactly generated by the images of $K \in Shv(S_1 \times S_2)^c$ under δ_1 .

Let $\tilde{C}_{\Delta^{op}} : \Delta^{op} \rightarrow \text{DGCat}_{cont}$ be the functor sending $[n]$ to $Shv(S_1 \times Y^n \times S_2)$ so that $C = \text{colim } \tilde{C}_{\Delta^{op}}$. We may pass to continuous right adjoints in $\tilde{C}_{\Delta^{op}}$ and get the functor $\tilde{C}_\Delta^R : \Delta \rightarrow \text{DGCat}_{cont}$, so $\lim \tilde{C}_\Delta^R \rightrightarrows C$. In particular, the right adjoint δ_1^R of δ_1 is continuous.

For each $[n] \in \Delta$ the exterior product $g_n : Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2) \rightarrow Shv(S_1 \times Y^n \times S_2)$ is fully faithful and has a continuous right adjoint g_n^R . So, the right adjoint \mathcal{F}_1^R of \mathcal{F}_1 is obtained by passing to the limit over $[n] \in \Delta$ in the functors $g_n^R : Shv(S_1 \times Y^n \times S_2) \rightarrow Shv(S_1) \otimes Shv(Y)^{\otimes n} \otimes Shv(S_2)$ in DGCat_{cont} . So, \mathcal{F}_1^R is continuous (cf. [21], 9.2.39).

Clearly, if $K \in Shv(S_1 \times S_2)^c$ then $\mathcal{F}_2(\delta_1(K)) \in Shv(S_1 \times_Y S_2)^c$, so \mathcal{F}_2 has a continuous right adjoint \mathcal{F}_2^R . Besides \mathcal{F}_2^R is conservative.

Since Δ^{op} is sifted, from ([16], 3.2.3.1) we see that $C \in Calg(DGCat_{cont})$, and δ_1 is a map in $Calg(DGCat_{cont})$ naturally. Besides, $\mathcal{F}_1, \mathcal{F}_2$ are naturally morphisms in $Calg(DGCat_{cont})$.

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