1. Action of $\check{\mathfrak{g}}\otimes \mathrm{R}\Gamma(X)$ on Eisenstein series, elementary definition

1.0.1. Let k be an algebraically closed field, G reductive connected over k, $[G,G]$ simply-connected, $B\subset G$ a Borel, $B^-\subset G$ opposite Borel, $T=B\cap B^-$ a maximal torus, Λ coweights of T, $\check{\Lambda}$ weights. Let Λ^{pos} be the $\mathbb{Z}_+$-span of positive coroots, $\check{\Lambda}^+$ be the dominant weights.

For $\mu\in\Lambda^{pos}$ we have the locally closed embedding $j_{=\mu}:\mathrm{Bun}_B\times X^\mu\to\overline{\mathrm{Bun}}_B$, it is affine. Set $\overline{\mathrm{Bun}}_{B,=\mu}=\mathrm{Bun}_B\times X^\mu$. Together they form a stratification of $\overline{\mathrm{Bun}}_B$. Here X^μ is the moduli scheme of Λ^{pos}-valued divisors on X of degree μ. Denote by $\bar{\mathfrak{p}}:\overline{\mathrm{Bun}}_B\to\mathrm{Bun}_G$ the projection extending $\mathfrak{p}:\mathrm{Bun}_B\to\mathrm{Bun}_G$.

Let $j_{\ge\mu}:\overline{\mathrm{Bun}}_B\times X^\mu\to\overline{\mathrm{Bun}}_B$ be the map sending $(\mathcal{F}_T,\mathcal{F},\kappa,D\in X^\mu)$ to $(\mathcal{F}_T(-D),\mathcal{F},\kappa)$. This is a finite map extending $j_{=\mu}$.

Let $\overline{\mathrm{Bun}}_{B,\le\mu}\subset\overline{\mathrm{Bun}}_B$ be the open substack classifying $(\mathcal{F}_T,\mathcal{F},\kappa)$ such that for any $\check{\lambda}\in\check{\Lambda}^+$ the map $\kappa^{\check{\lambda}}:\mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T}\hookrightarrow\mathcal{V}^{\check{\lambda}}_{\mathcal{F}}$ has zeros of total degree $\le\langle\mu,\check{\lambda}\rangle$. So,

$$\overline{\mathrm{Bun}}_{B,\le\mu}=\cup_{\mu'\le\mu}\overline{\mathrm{Bun}}_{B,=\mu'}\tag{1}$$

Work in the usual setting of $\bar{\mathbb{Q}}_\ell$-sheaves, ℓ is distinct from the characteristic of k. We will repeat the construction of the action of $\check{\mathfrak{g}}\otimes\mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)$ on $\mathcal{K}=\bar{\mathfrak{p}}_!\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ from [3] in more elementary terms.

1.0.2. Let α be a simple coroot. By ([2], 1.12), one has

$$j^!_{=\alpha}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}\,\widetilde{\to}\,\mathrm{IC}[-1]\tag{2}$$

Note that $X^\alpha\,\widetilde{\to}\,X$. We need the following.

Proposition 1.0.3. *Let α be a simple coroot. There is a unique morphism $c:\mathrm{IC}[-1]\to j^!_{\ge\alpha}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ on $\overline{\mathrm{Bun}}_B\times X$ whose restriction to $\mathrm{Bun}_B\times X$ is the isomorphism (2).*

Proof. It is immediately reduced to Lemma 1.0.4 below. □

Lemma 1.0.4. *Let $\mu\in\Lambda^{pos}$. There is a unique morphism $c_\mu:\mathrm{IC}[-1]\to j^!_{\ge\alpha}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ on $\overline{\mathrm{Bun}}_{B,\le\mu}\times X$ whose restriction to $\mathrm{Bun}_B\times X$ is the isomorphism (2).*

Set

$$\mathcal{K}=\mathbb{D}\,\mathrm{RHom}_{\overline{\mathrm{Bun}}_{B,\le\mu}\times X}(\mathrm{IC}[-1],j^!_{\ge\alpha}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B})\,\widetilde{\to}\,\mathrm{R}\Gamma_c(\overline{\mathrm{Bun}}_{B,\le\mu}\times X,\mathrm{IC}\otimes j^*_{\ge\alpha}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B})[-1]$$

Set also

$$\overset{\circ}{\mathcal{K}}=\mathrm{R}\Gamma_c(\mathrm{Bun}_B\times X,\mathrm{IC}\otimes j^*_{\ge\alpha}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B})[-1]$$

Lemma 1.0.4 is an immediate consequence of the following.

Lemma 1.0.5. *Both $\mathcal{K},\overset{\circ}{\mathcal{K}}$ are placed in degrees ≤ 0. The natural map $\overset{\circ}{\mathcal{K}}\to\mathcal{K}$ induces an isomorphism $\mathrm{H}^0(\overset{\circ}{\mathcal{K}})\to\mathrm{H}^0(\mathcal{K})$.*

Proof. Calculate $\mathcal{K}$ via the stratification (1) of $\overline{\mathrm{Bun}}_{B,\le\mu}$. Denote by $\mathfrak{B}(\mu)$ an element of the free abelian semigroup generated by all positive coroots. For such $\mathfrak{B}(\mu)$ we have the finite map $i_{\mathfrak{B}(\mu)}:X^{\mathfrak{B}(\mu)}\to X^\mu$ as in [2]. Set

$$\mathcal{M}_\mu=\underset{\mathfrak{B}(\mu)}{\oplus}\,i_{\mathfrak{B}(\mu)*}\bar{\mathbb{Q}}_\ell$$

This is a $\bar{\mathbb{Q}}_\ell$-sheaf (placed in usual degree zero) on X^μ. By ([2], Cor. 4.7),

$$j^*_{=\mu'} \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \widetilde{\to} \mathrm{IC} \boxtimes \mathbb{D}\mathcal{M}_{\mu'}$$

So, the $*$-restriction of $\mathrm{IC} \otimes j^*_{\ge\alpha} \mathrm{IC}_{\overline{\mathrm{Bun}}_B}[-1]$ under $\mathrm{Bun}_B \times X^{\mu'} \times X^\alpha \hookrightarrow \overline{\mathrm{Bun}}_{B,\le\mu} \times X^\alpha$ is

$$\bar{\mathbb{Q}}_\ell[2\dim \mathrm{Bun}_B] \boxtimes (s^*\mathbb{D}\mathcal{M}_{\mu'+\alpha} \otimes (\mathbb{D}\mathcal{M}_{\mu'} \boxtimes \bar{\mathbb{Q}}_\ell))$$

Here $s : X^{\mu'} \times X^\alpha \to X^{\mu'+\alpha}$ is the sum map, and $\dim \mathrm{Bun}_B$ is a function of a connected component of Bun_B. The contribution of the above stratum is the sum over pairs $\mathfrak{B}(\mu'), \mathfrak{B}(\mu'+\alpha)$ of

$$\mathrm{R}\Gamma_c(X^{\mathfrak{B}(\mu'+\alpha)} \times_{X^{\mu'+\alpha}} (X^{\mathfrak{B}(\mu')} \times X^\alpha), \bar{\mathbb{Q}}_\ell)[2 \mid \mathfrak{B}(\mu') \mid +2 \mid \mathfrak{B}(\mu'+\alpha) \mid]$$

tensored by $\mathrm{R}\Gamma_c(\mathrm{Bun}_B, \bar{\mathbb{Q}}_\ell)[2\dim \mathrm{Bun}_B]$. Since $\dim X^{\mathfrak{B}(\mu'+\alpha)} \times_{X^{\mu'+\alpha}} (X^{\mathfrak{B}(\mu')} \times X^\alpha) \le \mid \mathfrak{B}(\mu'+\alpha) \mid$, the latter complex is placed in degrees ≤ 0, and the inequality is strict unless $\mu' = 0$. We are done. □

1.0.6. By adjointness, the map c of Proposition 1.0.3 yields a map also denoted $c : (j_{\ge\alpha})_! \mathrm{IC}[-1] \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$. Pushing forward via $\bar{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$, this yields a map

$$\mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell) \otimes \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \to \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \tag{3}$$

This is the action of $\check{\mathfrak{n}}_\alpha \otimes \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell) \subset \check{\mathfrak{g}} \otimes \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell)$ on the Eisenstein series $\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$. Here $\check{\mathfrak{n}}$ is the Lie algebra of the unipotent radical of $\check{B}$, $\check{\mathfrak{n}}_\alpha$ is the root subspace corresponding to α. According to [3] and [4], these maps generate an action of $\check{\mathfrak{n}} \otimes \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell)$.

The Verdier dual of (3) gives a map $\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \to \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \otimes \mathbb{D}\,\mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell)$. By adjointness, this map yields a morphism

$$\mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell) \otimes \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \to \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \tag{4}$$

This is the action of $\check{\mathfrak{n}}^-_{-\alpha} \otimes \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell)$ on $\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$.

The action

$$\check{\mathfrak{h}} \otimes \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell) \to \mathrm{RHom}(\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}, \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}) \tag{5}$$

is defined as follows. The algebra $\mathrm{R}\Gamma(\mathrm{Bun}_T, \bar{\mathbb{Q}}_\ell)$ acts on $\bar{\mathbb{Q}}_\ell$ on Bun_T, hence by functoriality a map

$$\mathrm{R}\Gamma(\mathrm{Bun}_T, \bar{\mathbb{Q}}_\ell) \to \mathrm{RHom}(\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}, \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B})$$

Pullback under the evaluation map $X \times \mathrm{Bun}_T \to B(T)$ gives a morphism of algebras $\mathrm{Sym}(\check{\mathfrak{h}}[-2]) \widetilde{\to} \mathrm{R}\Gamma(B(T), \bar{\mathbb{Q}}_\ell) \to \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell) \otimes \mathrm{R}\Gamma(X, \mathrm{Bun}_T)$. Here $\check{\mathfrak{h}}$ is Lie algebra of $\check{T}$ as a vector space over $\bar{\mathbb{Q}}_\ell$. The latter by adjointness gives a morphism

$$\check{\mathfrak{h}}[-2] \otimes \mathbb{D}\,\mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell) \to \mathrm{R}\Gamma(X, \mathrm{Bun}_T)$$

Since $(\mathbb{D}\,\mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell))[-2] \widetilde{\to} \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell)$, we get a morphism $\check{\mathfrak{h}} \otimes \mathrm{R}\Gamma(X, \bar{\mathbb{Q}}_\ell) \to \mathrm{R}\Gamma(X, \mathrm{Bun}_T)$, hence also (5) as a composition.

1.0.7. Write Bun_T^λ for the component of Bun_T classifying $\mathcal{F}_T$ such that for any $\check{\lambda}$, $\langle\lambda,\check{\lambda}\rangle = \det \mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T}$. This gives $\overline{\mathrm{Bun}}_B^\lambda, \mathrm{Bun}_B^\lambda$ and so on. The map $j_{\ge\mu}$ induces a map $\overline{\mathrm{Bun}}_B^{\lambda+\mu}\times X^\mu \to \overline{\mathrm{Bun}}_B^\lambda$.

Write for brevity IC^λ for $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^\lambda}$. The object $\bar{\mathfrak{p}}_*\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ is Λ-graded, where the λ-component is $\bar{\mathfrak{p}}_*\mathrm{IC}^\lambda$. We see that (3) gives a map $\mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)\otimes\bar{\mathfrak{p}}_*\mathrm{IC}^{\lambda+\alpha}\to\bar{\mathfrak{p}}_*\mathrm{IC}^\lambda$. Similarly, (4) gives a map $\mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)\otimes\bar{\mathfrak{p}}_*\mathrm{IC}^\lambda\to\bar{\mathfrak{p}}_*\mathrm{IC}^{\lambda+\alpha}$.

1.1. Let us generalize Proposition 1.0.3 to the case of any $\alpha\in\Lambda^{pos}$. By ([2], 4.7), one has

$$j^!_{=\alpha}\mathrm{IC}_{\overline{\mathrm{Bun}}_B}\widetilde{\to}\mathrm{IC}\boxtimes\mathcal{M}_\alpha \tag{6}$$

Proposition 1.1.1. *Let $\alpha\in\Lambda^{pos}$. There is a unique morphism $c:\mathrm{IC}\boxtimes\mathcal{M}_\alpha\to j^!_{\ge\alpha}\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ over $\overline{\mathrm{Bun}}_B\times X^\alpha$ extending (6).*

Take $\mu\in\Lambda^{pos}$. As above we extend this c to $\overline{\mathrm{Bun}}_{B,\le\mu}\times X^\alpha$ first. Set

$$\mathcal{K}=\mathbb{D}\,\mathrm{RHom}_{\overline{\mathrm{Bun}}_{B,\le\mu}\times X^\alpha}(\mathrm{IC}\boxtimes\mathcal{M}_\alpha, j^!_{\ge\alpha}\mathrm{IC}_{\overline{\mathrm{Bun}}_B})\widetilde{\to}$$
$$\mathrm{R}\Gamma_c(\overline{\mathrm{Bun}}_{B,\le\mu}\times X^\alpha,(\mathrm{IC}\boxtimes\mathcal{M}_\alpha)\otimes j^*_{\ge\alpha}\mathrm{IC}_{\overline{\mathrm{Bun}}_B})$$

Let also

$$\mathring{\mathcal{K}}=\mathrm{R}\Gamma_c(\mathrm{Bun}_B\times X^\alpha,(\mathrm{IC}\boxtimes\mathcal{M}_\alpha)\otimes j^*_{\ge\alpha}\mathrm{IC}_{\overline{\mathrm{Bun}}_B})$$

Proof of Lemma 1.0.5 for any α. Calculate $\mathcal{K}$ via the same stratification of $\overline{\mathrm{Bun}}_{B,\le\mu}$. Let $\mu'\le\mu$, $\mu'\in\Lambda^{pos}$. The $*$-restriction of

$$(\mathrm{IC}\boxtimes\mathcal{M}_\alpha)\otimes j^*_{\ge\alpha}\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$$

to $\overline{\mathrm{Bun}}_{B,=\mu'}\times X^\alpha=\mathrm{Bun}_B\times X^{\mu'}\times X^\alpha$ becomes

$$\bar{\mathbb{Q}}_\ell[2\dim\mathrm{Bun}_B]\boxtimes((\mathbb{D}\mathcal{M}_{\mu'}\boxtimes\mathcal{M}_\alpha)\otimes s^*\mathbb{D}\mathcal{M}_{\mu'+\alpha})$$

Here $s:X^{\mu'}\times X^\alpha\to X^{\mu'+\alpha}$ is the sum map. The complex $\mathrm{R}\Gamma_c(\mathrm{Bun}_B,\bar{\mathbb{Q}}_\ell)[2\dim\mathrm{Bun}_B]$ is placed in degrees ≤ 0. Let us show that $\mathrm{R}\Gamma_c(X^{\mu'}\times X^\alpha,(\mathbb{D}\mathcal{M}_{\mu'}\boxtimes\mathcal{M}_\alpha)\otimes s^*\mathbb{D}\mathcal{M}_{\mu'+\alpha})$ is placed in degrees ≤ 0, and in fact in degrees ≤ -2 unless $\mu'=0$. The latter complex is a sum over triples $\mathfrak{B}(\mu'),\mathfrak{B}(\alpha),\mathfrak{B}(\mu'+\alpha)$ of

$$\mathrm{R}\Gamma((X^{\mathfrak{B}(\mu')}\times X^{\mathfrak{B}(\alpha)})\times_{X^{\mu'+\alpha}}X^{\mathfrak{B}(\mu'+\alpha)},\bar{\mathbb{Q}}_\ell)[2\mid\mathfrak{B}(\mu')\mid+2\mid\mathfrak{B}(\mu'+\alpha)\mid]$$

Since $\dim(X^{\mathfrak{B}(\mu')}\times X^{\mathfrak{B}(\alpha)})\times_{X^{\mu'+\alpha}}X^{\mathfrak{B}(\mu'+\alpha)}\le\mathfrak{B}(\mu'+\alpha)$, the latter complex is placed in degrees ≤ 2, and actually in degrees ≤ -2 unless $\mu'=0$. We are done. $\square$

Proposition 1.1.1 is proved.

1.1.2. By adjointness, the map c of Proposition 1.1.1 yields a morphism also denoted $c:(j_{\ge\alpha})_!(\mathrm{IC}\boxtimes\mathcal{M}_\alpha)\to\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$. Pushing forward via $\bar{\mathfrak{p}}:\overline{\mathrm{Bun}}_B\to\mathrm{Bun}_G$, this yields a map

$$\mathrm{R}\Gamma(X^\alpha,\mathcal{M}_\alpha)\otimes\bar{\mathfrak{p}}_!\mathrm{IC}_{\overline{\mathrm{Bun}}_B}\to\bar{\mathfrak{p}}_!\mathrm{IC}_{\overline{\mathrm{Bun}}_B} \tag{7}$$

1.1.3. If α is any, not necessarily simple, coroot of G, then there is a distinguished element $\mathfrak{B}(\alpha)=\alpha$. For the corresponding map $i_{\mathfrak{B}(\alpha)}: X^{\mathfrak{B}(\alpha)}\to X^\alpha$ we get the direct summand $\mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)\subset \mathrm{R}\Gamma(X^\alpha,\mathcal{M}_\alpha)$. Restricting (7) to this direct summand, we get the action

$$\mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)\otimes \bar{\mathfrak{p}}_!\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}\to \bar{\mathfrak{p}}_!\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B} \tag{8}$$

of $\check{\mathfrak{n}}_\alpha\otimes \mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)$ on $\bar{\mathfrak{p}}_!\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$.

The Verdier dual of (8) gives a map $\bar{\mathfrak{p}}_*\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}\to \bar{\mathfrak{p}}_*\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}\otimes \mathbb{D}\,\mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)$. By adjointness, this map yields a morphism

$$\mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)\otimes \bar{\mathfrak{p}}_*\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}\to \bar{\mathfrak{p}}_*\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$$

This is the action of $\check{\mathfrak{n}}^-_{-\alpha}\otimes \mathrm{R}\Gamma(X,\bar{\mathbb{Q}}_\ell)$ on $\bar{\mathfrak{p}}_*\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$.

1.1.4. Question. Clarify the structure of $\underset{\alpha\in\Lambda^{pos}}{\oplus}\mathrm{R}\Gamma(X^\alpha,\mathcal{M}_\alpha)$, see [1].

2. An analog for a parabolic P

2.1. Let $M\subset G$ be a standard Levi corresponding to $\mathcal{I}_M\subset\mathcal{I}$, here $\mathcal{I}$ is the set of vertices of the Dynkin diagram. We take P and P^- the corresponding standard parabolics with Levi M, so that $M=P\cap P^-$.

Write $\Lambda_{G,P}$ for the quotient of Λ by the $\mathbb{Z}$-span of $\alpha_i, i\in\mathcal{I}_M$. Let $\Lambda^{pos}_{G,P}\subset\Lambda_{G,P}$ be the $\mathbb{Z}_+$-span of $\alpha_i, i\in\mathcal{I}-\mathcal{I}_M$.

For $\theta\in\Lambda^{pos}_{G,P}$ let X^θ be the scheme of $\Lambda^{pos}_{G,P}$-valued divisors of degree θ on X. For $\theta\in\Lambda^{pos}_{G,P}$ we have the locally closed immersion $j_{P,=\theta}: X^\theta\times\mathrm{Bun}_P\hookrightarrow\overline{\mathrm{Bun}}_P$ sending

$$(\mathcal{F}_{M/[M,M]},\mathcal{F},\kappa,D\in X^\theta)$$

to $(\mathcal{F}_{M/[M,M](-D)},\mathcal{F},\kappa)$. Set $\overline{\mathrm{Bun}}_{P,=\theta}=X^\theta\times\mathrm{Bun}_P$. They form a stratification of $\overline{\mathrm{Bun}}_P$ indexed by $\Lambda^{pos}_{G,P}$. For $\theta\in\Lambda^{pos}_{G,P}$ let

$$j_{P,\ge\theta}: X^\theta\times\overline{\mathrm{Bun}}_P\hookrightarrow\overline{\mathrm{Bun}}_P$$

be the map sending $(\mathcal{F}_{M/[M,M]},\mathcal{F},\kappa,D\in X^\theta)$ to $(\mathcal{F}_{M/[M,M](-D)},\mathcal{F},\kappa)$. This is a finite map extending $j_{P,=\theta}$.

Let $\overline{\mathrm{Bun}}_{P,\le\theta}\subset\overline{\mathrm{Bun}}_P$ be the open substack classifying $(\mathcal{F}_{M/[M,M]},\mathcal{F},\kappa)$ such that for any $\check{\lambda}\in\check{\Lambda}_{G,P}\cap\check{\Lambda}^+$ the map $\kappa^{\check{\lambda}}:\mathcal{L}^{\check{\lambda}}_{\mathcal{F}_{M/[M,M]}}\to\mathcal{V}^{\check{\lambda}}_{\mathcal{F}}$ has zeros of total degree $\le\langle\theta,\check{\lambda}\rangle$. So, $\overline{\mathrm{Bun}}_{P,\le\theta}=\underset{\theta'\le\theta}{\cup}\mathrm{Bun}_{P,=\theta'}$.

2.2. Let $\mathfrak{u}(P)$ denote the Lie algebra of the unipotent radical of $\check{P}$. For $V\in\mathrm{Rep}(\check{M})$ write V_θ for the direct summand of V on which $Z(\check{M})$ acts by θ. Here $Z(\check{M})$ is the center of $\check{M}$, it is connected because $[M,M]$ is simply-connected.

Let $\Theta=\{\theta\in\Lambda_{G,P}\mid\mathfrak{u}(P)_\theta\ne 0\}$. Then $\Theta\subset\Lambda^{pos}_{G,P}$. Let $\mathbb{Z}_+\Theta$ denote the free abelian semigroup with base Θ. The inclusion $\Theta\subset\Lambda^{pos}_{G,P}$ extends to a homomorphism of semigroups $\mathbb{Z}_+\Theta\to\Lambda^{pos}_{G,P}$. An element of $\mathbb{Z}_+\Theta$ over $\theta\in\Lambda^{pos}_{G,P}$ is denoted by $\mathfrak{B}(\theta)$ as in [2]. We denote then by $X^{\mathfrak{B}(\theta)}$ the scheme of $\mathbb{Z}_+\Theta$-valued divisors on X of degree $\mathfrak{B}(\theta)$.

The above homomorphism of semigroups yields a finite map $i_{P,\mathfrak{B}(\theta)} : X^{\mathfrak{B}(\theta)} \to X^{\theta}$. Recall that for $\theta \in \Theta$, $\mathfrak{u}(P)_\theta$ is an irreducible $\check{M}$-module.

As in [2], we denote by $\overline{\mathrm{Loc}}$ the functor from $\mathbb{Z}$-graded vector spaces to sheaves on $\operatorname{Spec} k$ given by $V \mapsto \oplus_n V_n[-n]$ (we ignore Tate twists). For a $\mathbb{Z}$-graded vector space V, viewing $\overline{\mathrm{Loc}}(V)$ as a constant complex on X, one gets the pure complex $\overline{\mathrm{Loc}}(V)^{(n)}$ on $X^{(n)}$, these are S_n-invariants in $sum_! : \overline{\mathrm{Loc}}(V)^{\boxtimes n}$ for $sum : X^n \to X^{(n)}$.

Let $f \in \operatorname{Lie} M$ be the principal nilpotent defined in ([2], Section 7.1), for $V \in \operatorname{Rep}(\check{M})$ the notation $V^f = \{v \in V \mid fv = 0\}$ is from *loc.cit.* The dual of V^f identifies with $(V^*)^e = \{v \in V \mid ev = 0\}$.

We identify the dual of $\check{\mathfrak{u}}(P)$ with $\check{\mathfrak{u}}(P^-)$ as $\check{M}$-modules, the latter is the Lie algebra of the unipotent radical of $\check{P}^-$. Under this isomorphism, the dual of the direct summand $\check{\mathfrak{u}}(P)_\theta$ (resp., $\check{\mathfrak{u}}(P)^f_\theta$) identifies with $\check{\mathfrak{u}}(P^-)_{-\theta}$ (resp., $\check{\mathfrak{u}}(P^-)^e_{-\theta}$).

2.2.1. For $\theta \in \Lambda^{pos}_{G,P}$ denote by $\mathbb{D}(\mathcal{M}_{P,\theta})$ the direct sum over $\mathfrak{B}(\theta) = \sum\limits_{\theta_m \in \Theta} n_m\theta_m$ of the direct images under $i_{P,\mathfrak{B}(\theta)} : X^{\mathfrak{B}(\theta)} \to X^\theta$ of $S^{\mathfrak{B}(\theta)}[2 \mid \mathfrak{B}(\theta) \mid]$, where

$$S^{\mathfrak{B}(\theta)} = (\underset{\theta_m \in \Theta}{\boxtimes} (\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P)^f_{\theta_m})^{(n_m)})$$

Let $\mathcal{M}_{P,\theta}$ be the Verdier dual of $\mathbb{D}(\mathcal{M}_{P,\theta})$. Given $\mathfrak{B}(\theta)$ for $\theta \in \Lambda^{pos}_{G,P}$, set

$$S^{\mathfrak{B}(\theta)}_- = (\underset{\theta_m \in \Theta}{\boxtimes} (\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P^-)^e_{-\theta_m})^{(n_m)})$$

So, $\mathcal{M}_{P,\theta} \widetilde{\to} \underset{\mathfrak{B}(\theta)}{\oplus} (i_{P,\mathfrak{B}(\theta)})_! S^{\mathfrak{B}(\theta)}_-$. We used that $\mathbb{D}(S^{\mathfrak{B}(\theta)}_-) \widetilde{\to} S^{\mathfrak{B}(\theta)}[2 \mid \mathfrak{B}(\theta) \mid]$. The complex $S^{\mathfrak{B}(\theta)}$ is placed in usual degrees ≤ 0.

By ([2], 7.2), for $\theta \in \Lambda^{pos}_{G,P}$ one has

$$(j_{P,=\theta})^* \, \mathrm{IC}_{\overline{\mathrm{Bun}}_P} \widetilde{\to} \mathbb{D}(\mathcal{M}_{P,\theta}) \boxtimes \mathrm{IC}_{\mathrm{Bun}_P}$$

and

$$(j_{P,=\theta})^! \, \mathrm{IC}_{\overline{\mathrm{Bun}}_P} \widetilde{\to} (\mathcal{M}_{P,\theta}) \boxtimes \mathrm{IC}_{\mathrm{Bun}_P} \tag{9}$$

Question 1. Let $\theta \in \Lambda^{pos}_{G,P}$. Is it true that there is a natural morphism $c : (\mathcal{M}_{P,\theta}) \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P} \to (j_{P,\ge\theta})^! \, \mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ over $X^\theta \times \overline{\mathrm{Bun}}_P$ extending (9) over $X^\theta \times \mathrm{Bun}_P$?

Pick $\mu \in \Lambda^{pos}_{G,P}$. As above, it suffices for this μ to show that (9) extends naturally to a morphism over $X^\theta \times \overline{\mathrm{Bun}}_{P,\le\mu}$. Set

$$\mathcal{K} = \mathbb{D}\, \mathrm{RHom}_{X^\theta \times \overline{\mathrm{Bun}}_{P,\le\mu}}(\mathcal{M}_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P}, (j_{P,\ge\theta})^! \, \mathrm{IC}_{\overline{\mathrm{Bun}}_P}) \widetilde{\to} \mathrm{R\Gamma}_c(X^\theta \times \overline{\mathrm{Bun}}_{P,\le\mu}, (\mathcal{M}_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P}) \otimes (j_{P,\ge\theta})^* \, \mathrm{IC}_{\overline{\mathrm{Bun}}_P})$$

and

$$\overset{\circ}{\mathcal{K}} = \mathrm{R\Gamma}_c(X^\theta \times \mathrm{Bun}_P, (\mathcal{M}_{P,\theta} \boxtimes \mathrm{IC}) \otimes (j_{P,\ge\theta})^* \, \mathrm{IC}_{\overline{\mathrm{Bun}}_P})$$

It is no longer true that $\overset{\circ}{\mathcal{K}}$ is placed in degrees ≤ 0. The reason is that $\mathrm{RHom}(\mathcal{M}_{P,\theta}, \mathcal{M}_{P,\theta})$ is not placed in degrees ≥ 0. This already happens in the example of Remark 2.2.2. The same proof of of Lemma 1.0.5 does not work in this case.

False proof of Lemma 1.0.5 in this case. Calculate $\mathcal{K}$ via the stratification of $\overline{\mathrm{Bun}}_{P,\le\mu}$ by $\overline{\mathrm{Bun}}_{P,=\mu'}$, $\mu' \le \mu$. Let $\mu' \in \Lambda^{pos}_{G,P}$ with $\mu' \le \mu$. The $*$-restriction of $(\mathcal{M}_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P}) \otimes (j_{P,\ge\theta})^* \mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ to $X^\theta \times \overline{\mathrm{Bun}}_{P,=\mu'} = X^\theta \times X^{\mu'} \times \mathrm{Bun}_P$ is

$$((\mathcal{M}_{P,\theta} \boxtimes \mathbb{D}\mathcal{M}_{P,\mu'}) \otimes s^*\mathbb{D}\mathcal{M}_{P,\theta+\mu'}) \boxtimes \bar{\mathbb{Q}}_\ell[2\dim \mathrm{Bun}_P]$$

Here $s : X^\theta \times X^\mu \to X^{\theta+\mu'}$ is the sum map. The complex $\mathrm{R}\Gamma_c(\mathrm{Bun}_P, \bar{\mathbb{Q}}_\ell)[2\dim \mathrm{Bun}_P]$ is placed in degrees ≤ 0. Let us show that

$$\mathrm{R}\Gamma(X^\theta \times X^{\mu'}, ((\mathcal{M}_{P,\theta} \boxtimes \mathbb{D}\mathcal{M}_{P,\mu'}) \otimes s^*\mathbb{D}\mathcal{M}_{P,\theta+\mu'})$$

is placed in degrees ≤ 0. The latter complex is a sum over $\mathfrak{B}(\theta), \mathfrak{B}(\mu'), \mathfrak{B}(\theta+\mu')$ of

$$\mathrm{R}\Gamma((X^{\mathfrak{B}(\theta)} \times X^{\mathfrak{B}(\mu')}) \times_{X^{\theta+\mu'}} X^{\mathfrak{B}(\theta+\mu')}, (S_-^{\mathfrak{B}(\theta)} \boxtimes S^{\mathfrak{B}(\mu')}) \boxtimes S^{\mathfrak{B}(\theta+\mu')})[2\mid\mathfrak{B}(\mu')\mid + 2\mid\mathfrak{B}(\theta+\mu')\mid]$$

This is not true! The proof does not work. □

Remark 2.2.2. *i) The sheaf* $\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P)^f_\theta)$ *on* $\mathrm{Spec}\, k$ *is placed in degrees* ≤ 0. *Indeed, for a* $\mathfrak{sl}_2$*-module* V, V^f *is the space of lowest weight vectors in* V.

ii) The sheaf $\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P)^f_\theta)$ *on* $\mathrm{Spec}\, k$ *is not always placed in one degree. For example, take* $G = \mathrm{GL}_4$, M *the standard parabolic corresponding to the simple root* $e_2 - e_3$, *so* $M \tilde{\to} \mathrm{GL}_2 \times \mathrm{GL}_2$. *Let* $E = E_1 \oplus E_2$ *be the decomposition of the standard representation* E *of* $\check{G}$ *preserved by* $\check{M}$. *Then* $\check{\mathfrak{u}}(P) \tilde{\to} E_2^* \otimes E_1$ *is an irreducible* $\check{M}$*-module. Let* $\alpha = e_2 - e_3$, *let* θ *be the image of* α *in* $\Lambda_{G,P}$. *Then* $\check{\mathfrak{u}}(P)_\theta \ne 0$, *the center* $Z(\check{M})$ *acts on* $\check{\mathfrak{u}}(P)$ *by* θ. *The subspace* $\check{\mathfrak{u}}(P)^f_\theta$ *is 2-dimensional. One gets* $\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P)^f_\theta) \tilde{\to} \bar{\mathbb{Q}}_\ell[2] \oplus \bar{\mathbb{Q}}_\ell$.

iii) The sheaf $\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P)^f_\theta)$ *on* $\mathrm{Spec}\, k$ *may be placed in degrees* < 0. *For example, take* $n \ge 2$, $G = \mathbb{S}\mathrm{pin}_{2n}$, *and* P *the preimage of the Siegel parabolic of* $\mathbb{S}\mathbb{O}_{2n}$. *The Lie algebra is* $\mathrm{Lie}\,\check{M} \tilde{\to} \mathfrak{gl}_n$. *Let* E *be the standard representation of* $\mathrm{Lie}\,\check{M}$. *Then* $\check{\mathfrak{u}}(P) \tilde{\to} \wedge^2 E$. *Denote by* V^m *the* $m+1$*-dimensional irreducible representation of* $\mathfrak{sl}_2$. *As a representation of principal* $\mathfrak{sl}_2$ *of* $\mathrm{Lie}\,\check{M}$ *becomes* $\check{\mathfrak{u}}(P) \tilde{\to} \wedge^2 V^{n-1} \tilde{\to} V^{2n-4} + V^{2n-8} + \dots$ *according to ([5], ex. 11.31). It contains* V^0 *iff* n *is even. So, for* n *odd,* $\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P)^f_\theta)$ *is placed in degrees* < 0.

2.2.3. One is tempted to do the following. Assume for simplicity P is a maximal parabolic, and α is the unique simple coroot which is not a coroot of M. Let θ be the image of α in $\Lambda^{pos}_{G,P}$. Assume also $\Theta = \{\theta\}$. For any $\mu \in \Lambda^{pos}_{G,P}$ now $\mathfrak{B}(\mu)$ denotes the unique partition of μ, and $i_{P,\mathfrak{B}(\mu)} : X^{\mathfrak{B}(\mu)} \to X^\mu$ is an isomorphism.

Let $m \ge 0$ be the lowest degree such that $\mathrm{H}^m(\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P^-)^e_{-\theta}) \ne 0$. Equivalently, $-m$ is the top cohomological degree of $\overline{\mathrm{Loc}}(\check{\mathfrak{u}}(P)^f_\theta)$. Then $X^\theta = X$, and $\mathcal{M}_{P,\theta}$ contains the direct summand

$$\mathcal{M}^0_{P,\theta} := \mathrm{H}^m(S_-^{\mathfrak{B}(\theta)})[-m]$$

This is a constant sheaf on X placed in the usual degree m.

Proposition 2.2.4. *There is a unique morphism* $c : \mathcal{M}^0_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P} \to (j_{P,\ge\theta})^! \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ *over* $X^\theta \times \overline{\mathrm{Bun}}_P$ *extending the composition*

$$\mathcal{M}^0_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P} \hookrightarrow (\mathcal{M}_{P,\theta}) \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P} \to (j_{P,\ge\theta})^! \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$$

over $X^\theta \times \mathrm{Bun}_P$.

Pick any $\mu \in \Lambda^{pos}_{G,P}$. Set

$$\mathcal{K} = \mathbb{D}\,\mathrm{RHom}_{X^\theta \times \overline{\mathrm{Bun}}_{P,\le\mu}}(\mathcal{M}^0_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P}, (j_{P,\ge\theta})^! \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P}) \,\widetilde{\to}\, \mathrm{R}\Gamma_c(X^\theta \times \overline{\mathrm{Bun}}_{P,\le\mu}, (\mathcal{M}^0_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P}) \otimes (j_{P,\ge\theta})^* \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P})$$

and

$$\mathring{\mathcal{K}} = \mathrm{R}\Gamma_c(X^\theta \times \mathrm{Bun}_P, (\mathcal{M}^0_{P,\theta} \boxtimes \mathrm{IC}) \otimes (j_{P,\ge\theta})^* \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P}) \,\widetilde{\to}\, \mathbb{D}\,\mathrm{RHom}_{X^\theta \times \mathrm{Bun}_P}(\mathcal{M}^0_{P,\theta} \boxtimes \mathrm{IC}, \mathcal{M}_{P,\theta} \boxtimes \mathrm{IC})$$

Proof of Proposition 2.2.4. We check that the analog of Lemma 1.0.5 holds in this case. Calculate $\mathcal{K}$ via the stratification of $\overline{\mathrm{Bun}}_{P,\le\mu}$ by $\overline{\mathrm{Bun}}_{P,=\mu'}$, $\mu' \le \mu$. Let $\mu' \in \Lambda^{pos}_{G,P}$ with $\mu' \le \mu$. The $*$-restriction of

$$(\mathcal{M}^0_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P}) \otimes (j_{P,\ge\theta})^* \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$$

to $X^\theta \times \overline{\mathrm{Bun}}_{P,=\mu'} = X^\theta \times X^{\mu'} \times \mathrm{Bun}_P$ is

$$((\mathcal{M}^0_{P,\theta} \boxtimes \mathbb{D}\mathcal{M}_{P,\mu'}) \otimes s^*\mathbb{D}\mathcal{M}_{P,\theta+\mu'}) \boxtimes \bar{\mathbb{Q}}_\ell[2\dim \mathrm{Bun}_P]$$

Here $s : X^\theta \times X^\mu \to X^{\theta+\mu'}$ is the sum map. The complex $\mathrm{R}\Gamma_c(\mathrm{Bun}_P, \bar{\mathbb{Q}}_\ell)[2\dim \mathrm{Bun}_P]$ is placed in degrees ≤ 0. Let us show that

$$\mathrm{R}\Gamma(X^\theta \times X^{\mu'}, ((\mathcal{M}^0_{P,\theta} \boxtimes \mathbb{D}\mathcal{M}_{P,\mu'}) \otimes s^*\mathbb{D}\mathcal{M}_{P,\theta+\mu'})$$

is placed in degrees ≤ 0, and actually in degrees ≤ -2 unless $\mu' = 0$. The latter complex is

$$\mathrm{R}\Gamma(X^\theta \times X^{\mu'}, (\mathcal{M}^0_{P,\theta} \boxtimes S^{\mathfrak{B}(\mu')}) \otimes s^* S^{\mathfrak{B}(\theta+\mu')})[2 \mid \mathfrak{B}(\mu') \mid +2 \mid \mathfrak{B}(\theta+\mu') \mid]$$

If $\mu' = r\theta$ then $\mid \mathfrak{B}(\mu') \mid = r, \mid \mathfrak{B}(\theta+\mu') \mid = r+1$, and the latter complex is placed in usual degrees $\le -2rm - 2r$. We are done. □

2.2.5. By adjointness, the map c from Proposition 2.2.4 yields a morphism also denoted $c : (j_{P,\theta})_! \mathcal{M}^0_{P,\theta} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_P} \to \mathrm{IC}_{\overline{\mathrm{Bun}}_P}$. Pushing forward via $\bar{\mathfrak{p}} : \overline{\mathrm{Bun}}_P \to \mathrm{Bun}_G$, this yields a morphism

$$\mathrm{R}\Gamma(X, \mathcal{M}^0_{P,\theta}) \otimes \mathfrak{p}_* \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P} \to \mathfrak{p}_* \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P} \tag{10}$$

For $\mu \in \Lambda_{G,P}$ let Bun^μ_M denote the component cassifying $\mathcal{F}_M$ such that the induced $M/[M,M]$-torsor is of degree μ. Let $\mathrm{Bun}^\mu_P, \overline{\mathrm{Bun}}^\mu_P$ be the corresponding components. Write IC^μ for $\mathrm{IC}_{\overline{\mathrm{Bun}}^\mu_P}$. View $\mathfrak{p}_* \,\mathrm{IC}_{\overline{\mathrm{Bun}}_P}$ as $\Lambda_{G,P}$-graded, where the μ-component is $\bar{\mathfrak{p}}_! \,\mathrm{IC}^\mu$. Then (10) gives a map

$$\mathrm{R}\Gamma(X, \mathcal{M}^0_{P,\theta}) \otimes \mathfrak{p}_* \,\mathrm{IC}^{\mu+\theta} \to \mathfrak{p}_* \,\mathrm{IC}^\mu$$

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