TOPOLOGICAL CHIRAL CATEGORIES

Contents

1. Factorization algebras in topology

1.1. **Setup.**

1.1.1. Let M be a smooth manifold of dimension n. In this case, we can define a topological version of the notion of factorization algebra on M. The idea is that, as in algebraic geometry, we will consider sheaves on powers of M together with certain compatibility data. What makes the story topological is that we will restrict ourselves to sheaves which are locally constant along the stratification on powers of M given by diagonals. In particular, the sheaf on M will be locally constant.

1.1.2. Given a stratified topological space X, let $\mathrm{Shv}^!(X)$ denote the category of co-sheaves on X (valued in Vect), which are locally constant on each stratum. Note that in this case, Verdier duality gives an equivalence between sheaves and cosheaves [1]; thus we can think of objects of $\mathrm{Shv}^!(X)$ as sheaves on X but where the natural pullback functor is upper-!.

1.2. **Topological factorization algebras.**

1.2.1. Recall the notion of a (topological) factorization algebra, which we will informally define as follows:

Definition 1.2.2. *Let M be a (smooth) topological manifold. A factorization algebra on M is the data of:*

(1) *An object $\mathcal{A}^I \in \mathrm{Shv}^!(M^I)$ for each nonempty finite set I (where M^I is stratified by diagonals).*
(2) *(Ran's condition) A compatible family of isomorphisms*

$$\Delta^!_f(\mathcal{A}^J) \simeq \mathcal{A}^I$$

for every surjective map $f : J \twoheadrightarrow I$, where $\Delta_f : M^I \hookrightarrow M^J$ is the corresponding diagonal embedding.

Date: January 30, 2018.

[1] We will need to consider co-sheaves valued in more general categories which are not stable and thus Verdier duality does not hold

(3) *(Factorization condition) A compatible family of isomorphisms*

$$j^!(\mathcal{A}^I) \simeq j^!(\mathcal{A}^{I_0} \boxtimes \mathcal{A}^{I_1})$$

for every disjoint decomposition $I = I_1 \sqcup I_2$, where j is the inclusion of the open subset of X^I consisting of $(x_i)_{i \in I_1}$, $(y_j)_{j \in I_2}$ such that the sets $\{x_i\}$ and $\{y_j\}$ are disjoint.

1.2.3. For a manifold M, let $\mathrm{FactAlg}(M)$ denote the category of factorization algebras on M.

1.2.4. Consider the case that $M = \mathbb{R}$. In this case, a factorization algebra is an associative algebra. Namely, we have $\mathcal{A} \in \mathrm{Shv}^!(\mathbb{R}) \simeq \mathrm{Vect}$ is the underlying vector space of the algebra. On $\mathbb{R}^2$, we have a sheaf given by $\mathcal{A} \otimes \mathcal{A}$ away from the diagonal and $\mathcal{A}$ on the diagonal. Note that each stratum is contractible. Thus the data of such a sheaf on $\mathbb{R}^2$ is the data of specialization maps from the two dimensional strata to the diagonal:

$$m_1, m_2 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

Ran's condition, in particular, makes the sheaf on $\mathbb{R}^2$ equivariant with respect to swapping the two coordinates. It follows that $m_2 \simeq m_1 \circ \sigma$, where $\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the map swapping the two factors. Thus the data of the sheaf on $\mathbb{R}^2$ gives a binary operation on $\mathcal{A}$. The sheaves on higher powers of $\mathbb{R}$ give the data of (higher) associativity for this binary operation.

In the case of $\mathbb{R}^n$, topological factorization algebras are equivalent to E_n-algebras, i.e. algebras over the little disks operad. For our purposes, we can take this to be the definition:

Definition 1.2.5. *For $n \geq 0$, the category of E_n-algebras is given by*

$$E_n\text{-alg} := \mathrm{FactAlg}(\mathbb{R}^n).$$

1.2.6. Taking the limit over n, we can define E_∞-algebras as factorization algebras on $\mathbb{R}^\infty = \mathrm{colim}\, \mathbb{R}^n$. Note all the strata of $\mathrm{Ran}(\mathbb{R}^\infty)$, i.e. configurations of points in $\mathbb{R}^\infty$ are contractible. Combinatorially, we can assign to each stratum $\mathrm{Conf}_k(\mathbb{R}^\infty)$ a pointed finite set consisting of a basepoint and k additional elements. One can then see that the category $\mathrm{Shv}^!(\mathrm{Ran}(\mathbb{R}^\infty))$ is equivalent to the category of functors

$$\mathrm{Fin}_* \to \mathrm{Vect},$$

where Fin_* is the category of finite pointed sets. Moreover, it follows that E_∞-algebras as the same as commutative algebras (i.e. homotopy coherent algebras over the commutative operad).

A similar combinatorial construction shows that for any manifold M, we have a canonical functor

$$E_\infty\text{-alg} \to \mathrm{FactAlg}(M).$$

Geometrically, this construction is the pullback of the factorization algebra on $\mathbb{R}^\infty$ to a factorization algebra on M along an embedding $M \hookrightarrow \mathbb{R}^\infty$.

1.3. $O(n)$**-action.** We shall see that we can, in a certain sense, understand factorization algebras on arbitrary manifolds in terms of E_n-algebras, and in fact in terms of E_1 (=associative) algebras.

Note that there is an action of the orthogonal group $O(n)$ on the category of E_n-algebras (by acting on $\mathbb{R}^n$). Let $E_n\text{-alg}_{O(n)}$ denote the quotient category; it comes with a natural functor $E_n\text{-alg}_{O(n)} \to BO(n)$.

A key observation is that the data of a factorization algebra on M is local on M; this implies:

Theorem 1.3.1. *Let M be a smooth n-dimensional manifold. A factorization algebra on M is a locally constant family of factorization algebras $\mathcal{A}_x$ on the tangent spaces T_xM for all $x \in M$. More precisely, the category of factorization algebras on M is the category of lifts*

$$\begin{array}{ccc} & & E_n\text{-alg}_{O(n)} \\ & \nearrow & \downarrow \\ M & \longrightarrow & BO(n) \end{array}$$

In other words (up to the $O(n)$-action), we have that a factorization algebra on an n-dimensional manifold M is a family of E_n-algebras parametrized by M.

We will be particularly interested in factorization algebras on oriented manifolds (namely, Riemann surfaces). We have:

Corollary 1.3.2. *Let M be an oriented n-dimensional manifold. There is a natural functor*

$$E_n\text{-alg}_{SO(n)} \to \mathrm{FactAlg}(M).$$

1.3.3. Note that in the infinite case, the strata of $\mathrm{Ran}(\mathbb{R}^\infty)$ are contractible. Therefore, the action of the infinite orthogonal group O on the category of E_∞-algebras is trivial. In particular, this gives another description of the functor

$$E_\infty\text{-alg} \to \mathrm{FactAlg}(M),$$

for any manifold M.

1.4. Coefficients and additivity.

1.4.1. One of the advantages of working with topological spaces rather than algebraic varieties is that we can consider (co)-sheaves valued in any category whatsoever. For a stratified space X and a category $\mathcal{C}$, let $\mathrm{Shv}^!(X;\mathcal{C})$ denote the category of co-sheaves on X with values in $\mathcal{C}$. Moreover, if $\mathcal{C}$ is a symmetric monoidal category and M is a manifold, let $\mathrm{FactAlg}(M;\mathcal{C})$ denote the category of factorization algebras on M valued in $\mathcal{C}$ (the symmetric monoidal structure on $\mathcal{C}$ is needed to make sense of the factorization isomorphisms).

For a manifold M, the category $\mathrm{FactAlg}(M)$ carries a natural symmetric monoidal structure given by the !-tensor product on each M^I. In particular, by the above, we can consider factorization algebras with values in factorization algebras. A key result is the following:

Theorem 1.4.2. *Let M, N be two manifolds. We have a natural equivalence*

$$\mathrm{FactAlg}(M;\mathrm{FactAlg}(N)) \simeq \mathrm{FactAlg}(M\times N).$$

1.4.3. In particular, we have that E_2 algebras are equivalent to E_1 algebras in E_1 algebras. In the classical (i.e. not homotopical) setting, this implies that an E_2-algebra is just a commutative algebra:

Exercise 1.4.4. *Show that the category of monoids in the (ordinary) category of monoids is equivalent to the category of commutative monoids.*

2. Topological factorization categories

Given a manifold M, let $\mathrm{FactCat}(M) := \mathrm{FactAlg}(M;\mathrm{DGCat}_{\mathrm{cont}})$ denote the category of factorization categories on M. By the above discussion, we have that factorization categories on $\mathbb{R}$ are equivalent to monoidal DG categories.

2.1. E_2-categories and braided monoidal categories.

2.1.1. Consider a factorization category on $\mathbb{R}^2$. By Sect. 1.4.3 and the above discussion, this is equivalent to a category $\mathcal{C}$ together with two monoidal structures

$$\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}, \quad \text{and}$$
$$\odot : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$$

such that $\odot$ is a monoidal functor with respect to $\otimes$, i.e. there are natural isomorphisms

$$(X_1\otimes Y_1)\odot(X_2\otimes Y_2)\simeq(X_1\odot X_2)\otimes(Y_1\odot Y_2).$$

We have,

$$\mathbb{1}_\otimes \simeq \mathbb{1}_\otimes\otimes\mathbb{1}_\otimes \simeq (\mathbb{1}_\otimes\odot\mathbb{1}_\odot)\otimes(\mathbb{1}_\odot\odot\mathbb{1}_\otimes)\simeq(\mathbb{1}_\otimes\otimes\mathbb{1}_\odot)\odot(\mathbb{1}_\odot\otimes\mathbb{1}_\otimes)\simeq\mathbb{1}_\odot\odot\mathbb{1}_\odot\simeq\mathbb{1}_\odot.$$

Thus, we have an isomorphism between the units of the two monoidal structures. From now on, we will denote the object $\mathbb{1}$ as the unit of both monoidal structures.

Now, for $X, Y \in \mathcal{C}$, consider

$$X \otimes Y \simeq (X \odot \mathbb{1}) \otimes (\mathbb{1} \odot Y) \simeq (X \otimes \mathbb{1}) \odot (Y \otimes \mathbb{1}) \simeq X \odot Y.$$

Thus we obtain an isomorphism between the two tensor structures.

Moreover, we have

$$X \otimes Y \simeq (\mathbb{1} \odot X) \otimes (Y \odot \mathbb{1}) \simeq Y \otimes X.$$

Thus, we obtain a braiding morphism

$$b_{X,Y} : X \otimes Y \to Y \to X.$$

One can check that if $\mathcal{C}$ is an ordinary category then the above structures give an equivalence between E_2-categories and braided monoidal categories.

2.2. **Ribbon twist.** By the above discussion, given a braided monoidal category $\mathcal{C}$, we can consider the corresponding factorization category on $\mathbb{R}^2$. We will be interested in similarly constructing factorization categories on other Riemann surfaces. To do that, we need to understand the abstract $SO(2)$-action on the category of braided monoidal categories in more concrete terms.

We can describe the action of $SO(2)$ in the following terms. Let us adopt the following convention. We will describe an E_2-category as a triple $(\mathcal{C}, \otimes_1, \otimes_2)$ where $\mathcal{C}$ is the underlying category, $\otimes^1$ is the monoidal structure obtained by restricting to the positively oriented x-axis and $\otimes^2$ is the monoidal structure obtained by restricting to the positively oriented y-axis.

Now, given an E_2-category $(\mathcal{C}, \otimes, \dot{\ })$ for every element of $SO(2)$, we have another E_2-category. The underlying category will be the same but the monoidal structures will be different. In the case of rotation by 90, 180 and 270 degrees, the corresponding E_2-categories are given, respectively, by $(\mathcal{C}, \odot, \otimes^{op})$, $(\mathcal{C}, \otimes^{op}, \odot^{op})$, and $(\mathcal{C}, \odot^{op}, \otimes)$. Moreover, given a path between two elements of $SO(2)$, we obtain an equivalence between the corresponding E_2-categories. In particular, the path around the circle clockwise gives a functor of braided monoidal categories

$$T : (\mathcal{C}, \otimes) \to (\mathcal{C}, \otimes)$$

whose underlying functor is the identity functor and the monoidal structure on T is given by the square of the braiding

$$b^2 = b_{Y,X} b_{X,Y} : X \otimes Y \to X \otimes Y.$$

It follows that an $SO(2)$-equivariant structure on a braided monoidal category is a natural equivalence of braided monoidal functors between T and the identity.

Definition 2.2.1. *Let $\mathcal{C}$ be a braided monoidal category. A ribbon twist on $\mathcal{C}$ is a natural transformation of braided monoidal functors between T and the identity functor. Explicitly, it consists of an isomorphism*

$$\theta_X : X \to X$$

for each $X \in \mathcal{C}$ such that for every $X, Y \in \mathcal{C}$ the diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{b^2} & X \otimes Y \\ {\scriptstyle \theta_{X\otimes Y}}\downarrow & & \downarrow{\scriptstyle \theta_X \otimes \theta_Y} \\ X \otimes Y & \xrightarrow{\mathrm{id}} & X \otimes Y \end{array}$$

commutes.

Note that any symmetric monoidal category has a canonical ribbon twist given by the identity natural transformation.

2.2.2. Thus we have that given a braided monoidal category with a ribbon twist (alias: balanced monoidal category), we can construct a factorization algebra on any Riemann surface.

2.3. $Req_q(T)$ **via twisted sheaves.**

2.3.1. *Twisted sheaves.* Given a topological space X, we can consider the (DG-)category of locally constant sheaves on X.

In what follows, we will consider a twisted version of this category. Let k be the ring of coefficients and let $\mathcal{G}erb_k = B^2\mathbb{G}_m(k)$ denote the (2-)Picard groupoid of $\mathbb{G}_m$-gerbes over Spec(k). A $\mathbb{G}_m$-gerbe on Spec(k) gives a twisted version of the category of k-modules. These categories assemble into a sheaf of categories over $\mathcal{G}erb_k$.

Recall that if X is a topological space then a k-gerbe on X, i.e. a map

$$\eta : X \to \mathcal{G}erb_k,$$

allows to consider the category $\mathrm{Shv}_\eta(X)$ of twisted locally constant sheaves on X (with coefficients in k). Formally, this category is defined as the global sections of the sheaf of categories pulled back from $\mathcal{G}erb_k$.

Informally, an object $\mathcal{F}$ of $\mathrm{Shv}_\eta(X)$ is given by the following data. For each $x \in X$, an object $\mathcal{F}_x \in k\text{-mod}_{\eta(x)}$ in the category of $\eta(x)$-twisted k-modules and for each path γ from x to y, an isomorphism between $\gamma_*(\mathcal{F}_x)$ and $\mathcal{F}_y$ together with higher coherence data for homotopies of paths (and homotopies between homotopies, etc).

2.3.2. *From spaces to categories.* Now, suppose that M is a manifold and X is an M-factorization space, i.e. a factorization algebra valued in topological spaces. In this case, we can build a factorization algebra on M valued in categories by passing to (locally constant) sheaves on M.

We can also consider a twisted version of this construction. Slightly abusing notation, we will denote by $\mathcal{G}erb_k$ the factorization space on M with fibers $\mathcal{G}erb_k$ and factorization structure induced from the symmetric monoidal (=E_∞) structure on $\mathcal{G}erb_k$.

Definition 2.3.3. *Let X be a factorization space on a manifold M. A factorization gerbe on X is a map of factorization spaces*

$$\eta : X \to \mathcal{G}erb_k$$

Now, given a factorization gerbe η on a factorization space X, the category $\mathrm{Shv}_\eta(X)$ of η-twisted sheaves on X has a canonical structure of a factorization category on M.

2.3.4. $Rep_q(T)$. We will apply the above construction in the following case. Let Λ be a lattice (which we will regard as corresponding to an algebraic torus T). Regarding Λ as an abelian group, we can form the corresponding factorization space on $\mathbb{R}^2$. Note that this factorization space is given by the $\mathbb{C}$-points of the algebraic factorization space $\mathrm{Gr}_{\check{T}}$ over the affine line $\mathbb{A}^1$.

In what follows, we will consider our coefficients to be $k = \mathbb{C}$, the complex numbers.

Suppose that we have a symmetric bilinear form

$$\kappa : \Lambda \times \Lambda \to \mathbb{C},$$

and let

$$q = \exp(\kappa) : \Lambda \times \Lambda \to \mathbb{C}^*.$$

In this case, we can define a factorization gerbe. By definition and the above considerations, a factorization gerbe is a braided monoidal functor

$$\eta_q : \Lambda \to \mathcal{G}erbe_k.$$

This functor is defined as follows. For each $x \in \Lambda$, $\eta_q(x)$ is the trivial gerbe. Moreover, for each $x, y \in \Lambda$, the monoidal isomorphism

$$\eta_q(x) \otimes \eta_q(y) \simeq \eta_q(x + y)$$

is the identity isomorphism of the trivial gerbe. It remains to specify the braiding compatibility, i.e. a natural transformation making the diagram

$$\begin{array}{ccc} \eta_q(x) \otimes \eta_q(y) & \longrightarrow & \eta_q(x+y) \\ \downarrow & & \downarrow \\ \eta_q(y) \otimes \eta_q(x) & \longrightarrow & \eta_q(y+x) \end{array}$$

commute, for every pair $x, y \in \Lambda$. In the diagram above every gerbe is trivial and every map is the identity isomorphism. In other words, we are required to give an isomorphism between the identity isomorphim of the trivial gerbe and itself for every pair of objects $x, y \in \Lambda$. The space of such isomorphisms is given by k^*, and we declare it to be $q(x, y)$.

Moreover, the gerbe η_q comes equipped with a natural $SO(2)$-equivariant structure. By above, such a structure is defined by an isomorphism from the trivial gerbe to itself for every element $x \in \Lambda$, satisfying a compatibility condition. We take this isomorphism to be $q(x, x) \in k^*$.

Passing to twisted sheaves, we obtain a braided monoidal category $\mathrm{Shv}_{\eta_q}(\Lambda)$. Since η_q assigns to every element $x \in \Lambda$ the trivial gerbe, and to every pair $x, y \in \Lambda$ the identity isomorphism, we have that the underlying monoidal category of $\mathrm{Shv}_{\eta_q}(\Lambda)$ is given by $Vect^\Lambda$ the monoidal category of Λ-graded vector spaces. In particular, $\mathrm{Shv}_{\eta_q}(\Lambda)$ is generated by objects k_x, $x \in \Lambda$ with

$$k_x \otimes k_y \simeq k_{x+y}$$

Unwinding the definitions, we have that the braiding

$$k_x \otimes k_y \to k_y \otimes k_x$$

in $\mathrm{Shv}_{\eta_q}(\Lambda)$ is given by multiplication by $q(x, y)$. In other words, $\mathrm{Shv}_{\eta_q}(\Lambda)$ is the braided monoidal category Vect_q^Λ.

Moreover, the $SO(2)$-equivariant structure on the factorization gerbe η_q gives a ribbon twist on the category Vect_q^Λ. Unwinding the definitions, this ribbon twist is defined by isomorphisms

$$k_x \simeq k_x$$

given by multiplication by $q(x, x)$.

3. Categorified Riemann-Hilbert correspondence

3.1. **Riemann-Hilbert for categories.**

3.1.1. Recall that the Riemann-Hilbert correspondence provides a correspondence between sheaves on an algebraic variety in the topological context (i.e. constructible sheaves) and D-modules.

Namely, given a variety X over $\mathbb{C}$, the Riemann-Hilbert correspondence is an equivalence of symmetric monoidal categories

$$\text{D-mod}_{\text{rh}}(X) \simeq \mathrm{Shv}^!_{\text{constr}}(X),$$

where $\text{D-mod}_{\text{rh}}(X)$ is the subcategory of D-modules on X consisting of regular holonomic D-modules and $\mathrm{Shv}^!_{\text{constr}}$ is the category of sheaves on $X(\mathbb{C})$ which are constructible with respect to some algebraic stratification.

3.1.2. We can upgrade the above symmetric monoidal equivalence to give a Riemann-Hilbert correspondence between constructible sheaves of categories on X and crystals of categories.

Recall that for a variety X, the prestack X_{dR} is 1-affine, i.e. we have that crystals of categories on X are modules over $\text{D-mod}(X)$.

In the case of constructible sheaves, we have the following analogous key statement:

Theorem 3.1.3. *Let X be a stratified topological space. There is a natural equivalence of categories*

$$\mathrm{Shv}^!(X; \mathrm{DGCat}_{\text{cont}}) \simeq \mathrm{Shv}^!(X)\text{-mod}.$$

In other words constructible (co-)sheaves of categories are given by modules over the category of constructible sheaves.

3.1.4. Suppose that X is an algebraic variety and we have a sheaf of categories on X locally constant along an algebraic stratification. By the theorem, this is equivalent to a category $\mathcal{C}$ together with an action of $\mathrm{Shv}^!(X)$. Suppose that $\mathcal{C}$ is induced from a category with an action of

$$\mathrm{Shv}^!_{\mathrm{constr}}(X) \subset \mathrm{Shv}^!(X),$$

i.e.

$$\mathcal{C} \simeq \mathcal{C}_{\mathrm{constr}} \otimes_{\mathrm{Shv}^!_{\mathrm{constr}}(X)} \mathrm{Shv}^!(X).$$

In this case, using the Riemann-Hilbert correspondence, we can define the corresponding crystal of categories as:

$$\mathcal{C}_{\mathrm{constr}} \otimes_{\mathrm{D\text{-}mod_{rh}}(X)} \mathrm{D\text{-}mod}(X).$$

3.2. **Factorization categories: topological vs algebraic.**

3.2.1. Let us now apply the above discussion to the case of factorization categories over algebraic curves. As an immediate consequence of Theorem 3.1.3, we have:

Corollary 3.2.2. *Let M be a manifold. There is an equivalence between the category of factorization categories on M and the category consisting of:*

- *For each $I \in \mathrm{fSet}$, a category $\mathcal{A}_I$ with an action of $\mathrm{Shv}^!(X^I)$.*
- *(Ran's condition) for every surjective map $f : I \to J$ an equivalence of $\mathrm{Shv}^!(X^J)$-module categories*
$$\Delta^!_f(\mathcal{A}_I) := \mathcal{A}_I \otimes_{\mathrm{Shv}^!(\mathrm{X^I})} \mathrm{Shv}^!(X^J) \simeq \mathrm{Shv}^!(X^I).$$
- *Factorization isomorphisms.*

3.2.3. Now, suppose that $\mathcal{C}$ is a ribbon braided monoidal category. Let X be an algebraic curve and let $\mathcal{C}_I$ for $I \in \mathrm{fSet}$ denote the corresponding factorization category on $X(\mathbb{C})$.

In the case that $\mathcal{C}$ is given by Ind of a braided monoidal category (i.e. it is generated by a braided monoidal category of compact objects), we have that

$$\mathcal{C}_I \simeq \mathcal{C}_{\mathrm{constr,I}} \otimes_{\mathrm{Shv}^!_{\mathrm{constr}}(X^I)} \mathrm{Shv}^!(X^I)$$

functorially in I and therefore we can define the algebraic factorization category

$$\mathcal{C}_{\mathrm{constr,I} \otimes_{\mathrm{D\text{-}mod_{rh}}}(X^I)} \mathrm{D\text{-}mod}(X^I)$$

corresponding to $\mathcal{C}$.

3.2.4. In particular, we can apply the above construction to $\mathrm{Vect}^{\Lambda}_q$. Observing that the factorization space corresponding to Λ is the complex points of $\mathrm{Gr}_{\check{T}}$ and that the factorizable gerbe η_q is given by the complex points of the factorization gerbe given by the symmetric bilinear form κ, we have that the factorization category

$$\mathrm{D\text{-}mod}_k(\mathrm{Gr}_{\check{T}})$$

corresponds to $\mathrm{Vect}^{\Lambda}_q$ under the above Riemann-Hilbert correspondence.