

T-3: TUTORIAL ON THE FUNCTOR OF WEAK INVARIANTS

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1. INTRODUCTION

Let G be a finite group, and let k be a field of characteristic zero. The group algebra $k[G]$ admits two mutually dual Hopf algebra structures. In the first case, thinking of $k[G]$ as functions on G , one defines the multiplication

$$\mathrm{Funct}(G, k) \otimes \mathrm{Funct}(G, k) \simeq \mathrm{Funct}(G \times G, k) \rightarrow \mathrm{Funct}(G, k)$$

to be $(f_1 \cdot f_2)(g) = f_1(g) f_2(g)$, which can be thought of as pointwise multiplication or pullback along $\Delta : G \rightarrow G \times G$. One defines the comultiplication

$$\mathrm{Funct}(G, k) \rightarrow \mathrm{Funct}(G \times G, k)$$

to be $(\Delta f)(g_1, g_2) = f(g_1 g_2)$, which is pullback along the multiplication map $m : G \times G \rightarrow G$. The antipode is given by $s(f)(g) = f(g^{-1})$. In this way, the coalgebra structure captures the group multiplication. Given a k -vector space V , making V a G -representation is equivalent to making V a $k[G]$ -comodule, for the above Hopf algebra structure on $k[G]$.

In the second case, thinking of $k[G]$ as distributions on G , one defines the multiplication

$$\mathrm{Dist}(G \times G, k) \rightarrow \mathrm{Dist}(G, k)$$

to be pushforward along the multiplication map $m : G \times G \rightarrow G$. One defines the comultiplication

$$\mathrm{Dist}(G, k) \rightarrow \mathrm{Dist}(G \times G, k)$$

to be pushforward along the diagonal map $\Delta : G \rightarrow G \times G$. The antipode is given by pushforward along the inversion map $s : G \rightarrow G$. Here, it is the algebra structure that captures the group multiplication. Given a k -vector space V , making V a G -representation is also equivalent to making V a $k[G]$ -module, for this second Hopf algebra structure on $k[G]$.

Furthermore, the two Hopf algebra structures are exchanged by the self-duality $k[G]^\vee \simeq k[G]$ induced by the canonical basis $\{e_g\}_{g \in G}$ of $k[G]$. Let $k[G]^{\mathrm{funct}}$ and $k[G]^{\mathrm{dist}}$ denote the first and second Hopf algebra structures, respectively. To go from a $k[G]^{\mathrm{funct}}$ -comodule structure on V to a $k[G]^{\mathrm{dist}}$ -module structure, one applies duality:

$$\mathrm{act} : k[G]^{\mathrm{dist}} \otimes V \rightarrow V \quad \Leftrightarrow \quad \mathrm{coact} : V \rightarrow k[G]^{\mathrm{funct}} \otimes V.$$

Now we shall think of G -representations as modules over $k[G]^{\mathrm{dist}}$. The trivial representation, denoted k , is one example. For any G -representation V , we have $V^G = \mathrm{Hom}_G(k, V)$ by definition. We can also define coinvariants $V_G = k \otimes_{k[G]} V$. When G is finite, Maschke's theorem implies that invariants and coinvariants coincide. When G is compact and V is a finite-dimensional real or complex vector space, Weyl's unitary trick also proves semisimplicity, from which it follows that invariants and coinvariants coincide.

In these notes, we pursue categorical analogues of these constructions. In order to replace vector spaces by categories, we also need to replace functions and distributions on G by categories of sheaves on G . (The totality of all vector spaces is a particular category, just as the totality of all functions – or all distributions – is a particular sheaf.) Our main goal will be statements to the effect that ‘invariants equal coinvariants,’ but we shall also do some explicit computations of invariants and coinvariants for categorical representations of geometric origin, i.e. those that arise from a geometric action of a group G on a scheme X .

The following analogies will help guide our expectations:

algebraic geometry	set theory
finite type schemes	finite sets
schemes	profinite sets
ind-schemes of ind-finite type	discrete sets
ind-schemes	locally profinite sets

We remark that locally profinite sets are also called ℓ -spaces and td -spaces. The precise interpretation is that, if X is a finite type scheme defined over $\text{Spec } \mathbb{F}_q$, then $X(\mathbb{F}_q)$ is a finite set, and similarly for the other rows. The particular case of schemes and profinite sets follows from the fact that any k -algebra is the colimit of its finitely-generated k -algebras, for any field k .

In particular, let G be an affine algebraic group of finite type. Then $G[[t]]$ is a scheme of infinite type and $G((t))$ is an ind-scheme not of ind-finite type. In accordance with the analogy, if G is defined over $\text{Spec } \mathbb{Z}$, then $G(\mathbb{F}_p)$ is a finite group, $G(\mathbb{Z}_p)$ is a profinite group, and $G(\mathbb{Q}_p)$ is a locally profinite group.

We will be able to prove that invariants and coinvariants coincide for finite type group schemes (analogous to finite groups) and even for a fairly general class of group schemes not of finite type (analogous to profinite groups, which are compact). We will also succeed for $G((t))$ where G is reductive, but the method doesn't generalize to other ind-pro groups. Notably, it fails for $N((t))$.

The organization and content of these notes closely follows the first half of [2].

2. ACTIONS BY FINITE TYPE GROUPS

2.1. Definitions. Let G be an affine group scheme of finite type. Since we are in characteristic zero, G is smooth. Let $\Delta : G \rightarrow G \times G$ be the diagonal map, let $m : G \times G \rightarrow G$ be the multiplication map, let $\iota : G \rightarrow G$ be the inversion map, and let $p : G \rightarrow \text{pt}$ be the projection.

2.1.1. Hopf algebras. The usual monoidal structure on $\text{QCoh}(G)$ is defined by Δ^* . This upgrades to a Hopf algebra structure, with comultiplication given by m^* and antipode given by ι^* . The existence of the antipode is necessary for the proof of Proposition 2.2.1. The validity of this Hopf algebra structure follows from the group axioms for G .

Applying naive duality for $\text{QCoh}(G)$, we obtain a different Hopf algebra structure: the multiplication is given by m_* and is called the convolution monoidal structure, the comultiplication is given by Δ_* , and antipode is still given by ι^* .

Since this is a functorial assignment of a Hopf algebra object in DGCat for each group scheme, and since p is a map of group schemes, we obtain that $p_* : \text{QCoh}(G) \rightarrow \text{Vect}$ respects the Hopf algebra structures. In particular, Vect is a module for $\text{QCoh}(G)$ with the convolution monoidal structure. The result of acting on $V \in \text{Vect}$ by $\mathcal{F} \in \text{QCoh}(G)$ is $\Gamma(G, \mathcal{F}) \otimes V$.

Analogous constructions are valid in the setting of $\mathcal{D}$ -modules. The category $\mathcal{D}(G)$ admits two mutually dual Hopf algebra structures, one given by $(\mathcal{D}(G), \Delta^!, m^!, \iota)$ and another given by $(\mathcal{D}(G), m_*, \Delta_*, \iota)$. As before, $p_* : \mathcal{D}(G) \rightarrow \text{Vect}$ is a map of Hopf algebras, and Vect is a module for $\mathcal{D}(G)$ with the convolution monoidal structure, and the action of $\mathcal{M} \in \mathcal{D}(G)$ is tensor by $\Gamma_{\text{dR}}(G, \mathcal{M})$.

2.1.2. Weak and strong actions. Let $\mathcal{C} \in \text{DGCat}$. By definition, a weak G -action on $\mathcal{C}$ is an action by $\text{QCoh}(G)$ with the convolution monoidal structure. Therefore, the category of weak G -modules is $\text{QCoh}(G)\text{-mod}$. Similarly, the category of strong G -modules is $\mathcal{D}(G)\text{-mod}$. These are often denoted $G\text{-mod}^w$ and $G\text{-mod}$, respectively.

Upon carrying out the same definitions for more general group prestacks $\mathcal{G}$, one may say that a strong action by $\mathcal{G}$ is the same as a weak action by $\mathcal{G}_{\text{dR}}$. Therefore, results about weak actions which are proven in sufficient generality also apply to strong actions.

2.1.3. *Invariants and coinvariants.* Let $\mathcal{C} \in \mathbf{QCoh}(G)\text{-}\mathbf{mod}$. We define the categories of invariants and coinvariants as follows:

$$\begin{aligned}\mathcal{C}^{G,w} &= \mathrm{Hom}_{\mathbf{QCoh}(G)}(\mathrm{Vect}, \mathcal{C}) \\ \mathcal{C}_{G,w} &= \mathrm{Vect} \underset{\mathbf{QCoh}(G)}{\otimes} \mathcal{C}.\end{aligned}$$

Similarly, if $\mathcal{C} \in \mathcal{D}(G)\text{-}\mathbf{mod}$, then define

$$\begin{aligned}\mathcal{C}^G &= \mathrm{Hom}_{\mathcal{D}(G)}(\mathrm{Vect}, \mathcal{C}) \\ \mathcal{C}_G &= \mathrm{Vect} \underset{\mathcal{D}(G)}{\otimes} \mathcal{C}.\end{aligned}$$

These assignments are functorial in $\mathcal{C}$, so we get functors

$$\begin{aligned}\mathrm{inv}^{G,w}, \mathrm{coinv}^{G,w} : \mathbf{QCoh}(G)\text{-}\mathbf{mod} &\rightrightarrows \mathrm{DGCat} \\ \mathrm{inv}^G, \mathrm{coinv}^G : \mathcal{D}(G)\text{-}\mathbf{mod} &\rightrightarrows \mathrm{DGCat}\end{aligned}$$

To carry out computations, it is convenient to use the fact that relative Homs are defined as totalizations and relative tensors as geometric realizations. Here is one example:

Lemma 2.1.4. *Let $\mathcal{X}$ be a prestack with G -action. Then $\mathbf{QCoh}(\mathcal{X})$ is a $\mathbf{QCoh}(G)$ -module, and $\mathbf{QCoh}(\mathcal{X})^{G,w} \simeq \mathbf{QCoh}(\mathcal{X}/G)$. The assignment*

$$\mathcal{X} \rightsquigarrow \mathbf{QCoh}(\mathcal{X}) \in \mathbf{QCoh}(G)\text{-}\mathbf{mod}$$

is functorial with respect to $*$ -pushforward and pullback along G -equivariant maps. Analogous statements hold for $\mathcal{D}(\mathcal{X})$ and $\mathcal{D}(G)$.

Proof. The quotient prestack $\mathcal{X}/G$ is the geometric realization in the category of prestacks of the simplicial system

$$\cdots G \times G \times \mathcal{X} \rightrightarrows G \times \mathcal{X} \rightrightarrows \mathcal{X}.$$

Applying $\mathbf{QCoh}$ with $*$ -pullbacks, we obtain a cosimplicial system whose terms are

$$\mathbf{QCoh}(G^{\times n} \times \mathcal{X}) \simeq \mathbf{QCoh}(G)^{\otimes n} \otimes \mathbf{QCoh}(\mathcal{X}),$$

where the equivalence follows because G is a scheme of finite type. Examining the transition maps shows that this is exactly a $(\mathbf{QCoh}(G), m^*)$ -comodule structure on $\mathbf{QCoh}(\mathcal{X})$. Applying duality, we get a $(\mathbf{QCoh}(G), m_*)$ -module structure on $\mathbf{QCoh}(\mathcal{X})$.

By flat descent, $\mathbf{QCoh}(\mathcal{X}/G)$ is the totalization of the above cosimplicial system. On the other hand, $\mathrm{Hom}_{\mathbf{QCoh}(G)}(\mathrm{Vect}, \mathbf{QCoh}(\mathcal{X}))$ is the totalization of the cosimplicial system

$$\mathrm{Hom}(\mathrm{Vect}, \mathbf{QCoh}(\mathcal{X})) \rightrightarrows \mathrm{Hom}(\mathbf{QCoh}(G), \mathbf{QCoh}(\mathcal{X})) \rightrightarrows \mathrm{Hom}(\mathbf{QCoh}(G) \otimes \mathbf{QCoh}(G), \mathbf{QCoh}(\mathcal{X})) \cdots$$

obtained from $(\mathbf{QCoh}(G), m_*)$. Applying duality, we have

$$\mathrm{Hom}(\mathbf{QCoh}(G)^{\otimes n}, \mathbf{QCoh}(\mathcal{X})) \simeq \mathbf{QCoh}(G)^{\otimes n} \otimes \mathbf{QCoh}(\mathcal{X}),$$

and this cosimplicial system coincides with the previous one. □

Corollary 2.1.5. (i) *We have*

$$\begin{aligned}\mathrm{End}_{\mathbf{QCoh}(G)}(\mathrm{Vect}) &\simeq \mathrm{Rep}(G) \\ \mathrm{End}_{\mathrm{Rep}(G)}(\mathrm{Vect}) &\simeq \mathbf{QCoh}(G),\end{aligned}$$

so that Vect is a bimodule for $\mathbf{QCoh}(G)$ and $\mathrm{Rep}(G)$.

(ii) *The functor $\mathrm{inv}^{G,w}$ factors as follows:*

$$\begin{array}{ccc} \mathbf{QCoh}(G)\text{-}\mathbf{mod} & \xrightarrow{\mathrm{inv}^{G,w,\mathrm{enh}}} & \mathrm{Rep}(G)\text{-}\mathbf{mod} \\ & \searrow \mathrm{inv}^{G,w} & \downarrow \mathrm{oblv} \\ & & \mathrm{DGCat} \end{array}$$

An analogous statement holds for $\mathrm{coinv}^{G,w}$.

(iii) We have an adjoint pair

$$\mathrm{rec} : \mathrm{Rep}(G)\text{-}\mathbf{mod} \rightleftarrows \mathrm{QCoh}(G)\text{-}\mathbf{mod} : \mathrm{inv}^{G,w,\mathrm{enh}}$$

$$\text{where } \mathrm{rec} = \mathrm{Vect} \underset{\mathrm{Rep}(G)}{\otimes} -.$$

Proof. The first equation follows from Lemma 2.1.4 upon taking $\mathcal{X} = \mathrm{pt}$. The second equation follows from the self-duality of $\mathrm{Vect} \in \mathrm{Rep}(G)\text{-}\mathbf{mod}$ and the fibered diagram

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{pt}/G \end{array}$$

Point (ii) follows immediately from (i). Point (iii) is tensor-Hom adjunction for the bimodule Vect . $\square$

2.1.6. Averaging and projection. Let $\mathcal{C} \in \mathrm{QCoh}(G)\text{-}\mathbf{mod}$. Applying the functors $\mathrm{Hom}_{\mathrm{QCoh}(G)}(-, \mathcal{C})$ and $- \underset{\mathrm{QCoh}(G)}{\otimes} \mathcal{C}$ to the adjunction

$$p^* : \mathrm{Vect} \rightleftarrows \mathrm{QCoh}(G) : p_*$$

gives the adjunctions

$$\begin{aligned} \mathrm{oblv}^{G,w} : \mathcal{C}^{G,w} &\rightleftarrows \mathcal{C} : \mathrm{Av}_*^{G,w} \\ (\mathrm{pr}_G)^L : \mathcal{C}_{G,w} &\rightleftarrows \mathcal{C} : \mathrm{pr}_G \end{aligned}$$

Applying Lemma 2.1.4 to the map $p : G \rightarrow \mathrm{pt}$ which is equivariant under the action of $G \times G$, we obtain that p^* and p_* are both maps of $\mathrm{QCoh}(G)$ -bimodules. Therefore, all four functors defined above are maps of $\mathrm{QCoh}(G)$ -modules, where $\mathrm{QCoh}(G)$ acts trivially on $\mathcal{C}^{G,w}$ and $\mathcal{C}_{G,w}$.

Let $\mathcal{E} \in \mathrm{DGCat}$. By tensor-Hom adjunction for Vect as a $(\mathrm{Vect}, \mathrm{QCoh}(G))$ -bimodule, we have

$$(1) \quad \mathrm{Hom}(\mathcal{C}_{G,w}, \mathcal{E}) \simeq \mathrm{Hom}_{\mathrm{QCoh}(G)}(\mathcal{C}, \mathrm{triv}(\mathcal{E})),$$

where $\mathrm{triv}(\mathcal{E})$ is $\mathcal{E}$ equipped with the trivial $\mathrm{QCoh}(G)$ -action. Setting $\mathcal{E} = \mathcal{C}^{G,w}$, we obtain from the functor $\mathrm{Av}_*^{G,w}$ a functor

$$\theta_{G,w}(\mathcal{C}) : \mathcal{C}_{G,w} \rightarrow \mathcal{C}^{G,w}.$$

This is functorial in $\mathcal{C}$, so we get a natural transformation

$$\theta_{G,w} : \mathrm{coinv}^{G,w} \rightarrow \mathrm{inv}^{G,w}.$$

We have entirely similar constructions for $\mathcal{D}(G)$ in place of $\mathrm{QCoh}(G)$. In making the definitions, we continue to use $p^* : \mathrm{Vect} \rightarrow \mathcal{D}(G)$ rather than $p^! : \mathrm{Vect} \rightarrow \mathcal{D}(G)$. The former is defined since G is smooth. In particular, we get a natural transformation θ_G .

2.2. Sheaves of categories on classifying prestacks. The following proposition says that $\mathcal{C} \in \mathrm{QCoh}(G)\text{-}\mathbf{mod}$ can be recovered from $\mathcal{C}^{G,w}$, which is surprising because the analogous statement for ordinary representations of G is false. The discrepancy arises because BG is 1-affine but not 0-affine.

The proposition also says that the datum of $\mathcal{C}$ is equivalent to the datum of a sheaf of categories on BG , but this is not surprising: one categorical level down, it corresponds to the fact that a G -representation on a vector space is equivalent to a quasicoherent sheaf on BG .

Proposition 2.2.1. *We have a commuting diagram of equivalences:*

$$\begin{array}{ccc} & \mathrm{ShvCat}(BG) & \\ \Gamma(\mathrm{pt}, -) \swarrow & & \searrow \Gamma(BG, -) \\ \mathrm{QCoh}(G)\text{-}\mathbf{mod} & \begin{array}{c} \xleftarrow{\mathrm{rec}} \\ \xrightarrow{\mathrm{inv}^{G,w,\mathrm{enh}}} \end{array} & \mathrm{Rep}(G)\text{-}\mathbf{mod} \end{array}$$

Proof. This follows from 10.2 of [3] and the fact that BG is 1-affine, cf. Theorem 2.3.7 in [2]. We shall indicate the main ideas.

Let $B^\bullet G$ denote the simplicial system

$$\cdots G \times G \rightrightarrows G \rightrightarrows \text{pt},$$

so that BG is its geometric realization in the category of prestacks. By descent, we have

$$\text{ShvCat}(BG) \simeq \text{Tot}(\text{ShvCat}(B^\bullet G)).$$

Since each $G^{\times n}$ is 1-affine, this is also $\text{Tot}(\text{QCoh}(B^\bullet G)\text{-}\mathbf{mod})$.

Appendix E of [3] proves a general result about Hopf algebra objects which implies the existence of an equivalence making the following diagram commute:

$$\begin{array}{ccc} \text{QCoh}(G)\text{-}\mathbf{comod} & \xrightarrow{\simeq} & \text{Tot}(\text{co-Bar}^\bullet(\text{QCoh}(G))\text{-}\mathbf{mod}) \\ & \searrow \text{oblv} & \swarrow \text{ev}_0 \\ & \text{DGCat} & \end{array}$$

By definition, the functor ev_0 extracts the underlying $\text{co-Bar}^0(\text{QCoh}(G))\text{-}\mathbf{mod}$ term of the compatible system. Since $\text{QCoh}(G)^{\otimes n} \simeq \text{QCoh}(G^{\times n})$, we have

$$\text{co-Bar}^\bullet(\text{QCoh}(G)) \simeq \text{QCoh}(B^\bullet G).$$

Since the self-duality of $\text{QCoh}(G)$ exchanges the standard and convolution monoidal structures, we have

$$(\text{QCoh}(G), m^*)\text{-}\mathbf{comod} \simeq (\text{QCoh}(G), m_*)\text{-}\mathbf{mod}.$$

Putting these facts together, we conclude that

$$\text{ShvCat}(BG) \simeq \text{QCoh}(G)\text{-}\mathbf{mod},$$

with the rightward functor being given by $\Gamma(\text{pt}, -)$ at the level of the underlying DG category. It is straightforward to check that $\Gamma(\text{pt}, -)$ identifies the symmetric monoidal structures of its source and target.

On the other hand, the functor $\Gamma(BG, -)$ is an equivalence because BG is 1-affine. Moreover, this functor is representable:

$$\Gamma(BG, -) \simeq \text{Hom}_{\text{ShvCat}(BG)}(\text{QCoh}_{/BG}, -).$$

Under $\Gamma(\text{pt}, -)$, the sheaf $\text{QCoh}_{/BG}$ goes to $\text{Vect} \in \text{QCoh}(G)\text{-}\mathbf{mod}$. This shows that the triangle with $\text{inv}^{G,w,\text{enh}}$ commutes, so $\text{inv}^{G,w,\text{enh}}$ is an equivalence. Then rec , being the left adjoint, is also an equivalence, and the triangle with rec commutes. $\square$

Corollary 2.2.2. *For $p : BG \rightarrow \text{pt}$, we have a commuting diagram of adjoint pairs*

$$\begin{array}{ccc} \text{ShvCat}(BG) & \xrightarrow{\Gamma(\text{pt}, -)} & \text{QCoh}(G)\text{-}\mathbf{mod} \\ \text{cores}_p \uparrow \downarrow \text{coind}_p & & \text{triv} \uparrow \downarrow \text{inv}^{G,w} \\ \text{DGCat} & \xrightarrow{\quad} & \text{DGCat} \end{array}$$

Furthermore, the functor $\text{coinv}^{G,w}$ is left adjoint to triv .

Proof. Since coind_p is just $\Gamma(BG, -)$, the square involving the right adjoints commutes by Proposition 2.2.1. Since the horizontal maps are equivalences, commutativity is preserved when passing to left adjoints. The last statement is equation (1). $\square$

2.3. Invariants equal coinvariants – weak actions.

Proposition 2.3.1. *The functor $\theta_{G,w}$ is a natural equivalence.*

Proof. The functor $\text{coinv}^{G,w} = \text{Vect} \otimes_{\text{QCoh}(G)} -$ is colimit-preserving. The functor $\text{inv}^{G,w} = \text{Hom}_{\text{QCoh}(G)}(\text{Vect}, -)$, while not obviously colimit-preserving, can be seen to be so by using Proposition 2.2.1 and the fact that the forgetful functor $\text{Rep}(G)\text{-mod} \rightarrow \text{DGCat}$ preserves limits and colimits.

Therefore, in order to show that $\theta_{G,w}(\mathcal{C}) : \mathcal{C}_{G,w} \rightarrow \mathcal{C}^{G,w}$ is an equivalence, we may realize $\mathcal{C}$ as the colimit of a diagram whose terms are all $\text{QCoh}(G)$, and thereby reduce to the case $\mathcal{C} = \text{QCoh}(G)$. But

$$\text{QCoh}(G)_{G,w} = \text{Vect} \otimes_{\text{QCoh}(G)} \text{QCoh}(G) \simeq \text{Vect}$$

and $\text{QCoh}(G)^{G,w} \simeq \text{QCoh}(\text{pt})$ by Lemma 2.1.4. □

Corollary 2.3.2. (i) *We have*

$$\begin{aligned} \text{Vect} \otimes_{\text{QCoh}(G)} \text{Vect} &\simeq \text{Rep}(G) \\ \text{Vect} \otimes_{\text{Rep}(G)} \text{Vect} &\simeq \text{QCoh}(G). \end{aligned}$$

(ii) *Vect is self-dual in $\text{QCoh}(G)\text{-mod}$ and $\text{Rep}(G)\text{-mod}$.*

Proof. The first equation follows from Corollary 2.1.5(i) and Proposition 2.3.1:

$$\text{Vect} \otimes_{\text{QCoh}(G)} \text{Vect} \simeq \text{Hom}_{\text{QCoh}(G)}(\text{Vect}, \text{Vect}) \simeq \text{Rep}(G).$$

The second equation was proven in the course of proving Corollary 2.1.5(i).

To show (ii), we use the results of 2.4. We make substitutions as follows:

$$\begin{array}{ccc} \begin{array}{c} T \\ \downarrow q \\ X \xrightarrow{p} Y \end{array} & \text{equals} & \begin{array}{c} \text{pt}/G \\ \downarrow q \\ \text{pt} \xrightarrow{p} \text{pt}/G \end{array} \end{array}$$

Point (i) states exactly that the conclusion of Proposition 2.4.4 holds. Since q is the identity map, it satisfies base change against any map, so Proposition 2.4.5 applies, and (ii) follows. □

2.4. Convolution monoidal structures. In doing computations and proofs with specific modules, there are essentially only two tools available to us: results stating that invariants equal coinvariants, and diagram chasing. In this section, we carry out the diagram chasing arguments in a very general setting. What is important is not the results themselves but rather their proofs, which are very formal and rely only on base change and adjunction properties. We shall apply the results in whatever settings allow their proofs to go through. For example, ‘proper’ is needed only to ensure that $*$ -pushforward is left adjoint to $!$ -pullback, so weaker hypotheses such as ‘ind-inf-proper’ would also suffice.

In this section, $p : X \rightarrow Y$ is a map of prestacks, and $\Delta : X \rightarrow X \times X$ is the diagonal map.

A helpful example to keep in mind is when $Y = \text{pt}$ and X is a disjoint union of finitely many points. Then, if X consists of n points, a sheaf on X is just an n -tuple of vector spaces. Upon decategorifying by replacing each vector space by a real number (corresponding to its ‘dimension’), the monoidal category $\text{IndCoh}(X \times_Y X)$ is just the algebra of $n \times n$ matrices, $\text{IndCoh}(X)$ is just the space of length- n column vectors, and so on. All of the statements in this section have analogues for ordinary matrices.

2.4.1. The Čech nerve construction gives a groupoid $X^\bullet/Y$ which looks like

$$\cdots \quad X \times_Y X \times_Y X \rightrightarrows X \times_Y X \rightrightarrows X,$$

where the maps are all given by appropriate projections. For any $T \rightarrow Y$, the relative space $X \times_Y T \rightarrow X$ acquires an action by this groupoid, in the sense that we have a map of simplicial objects where all the squares are fibered:

$$\begin{array}{ccc} \cdots & X \times_Y X \times_Y T & \rightrightarrows X \times_Y T \\ & \downarrow & \downarrow \\ \cdots & X \times_Y X & \rightrightarrows X \end{array}$$

2.4.2. Applying the functor of $\mathrm{IndCoh}(-)$ with $*$ -pushforward covariance to the diagram $X^\bullet/Y$ endows

$$\mathrm{IndCoh}(X \times_Y X) \in \mathrm{IndCoh}(X)\text{-}\mathbf{bimod}$$

with the structure of algebra object with respect to the usual monoidal structure on $\mathrm{IndCoh}(X)\text{-}\mathbf{bimod}$. Since the forgetful functor

$$\mathrm{oblv} : \mathrm{IndCoh}(X)\text{-}\mathbf{bimod} \rightarrow \mathrm{DGCat}$$

is right lax monoidal, we obtain a monoidal structure on $\mathrm{IndCoh}(X \times_Y X)$, which is the convolution monoidal structure. Concretely, the multiplication map is given by pull-push along this correspondence:

$$\begin{array}{ccc} (X \times_Y X) \times (X \times_Y X) & & \\ \swarrow \scriptstyle \mathrm{id} \times \Delta \times \mathrm{id} & & \\ & X \times_Y X \times_Y X & \\ \searrow \scriptstyle \mathrm{id} \times p \times \mathrm{id} & & \\ & X \times_Y X & \end{array}$$

Given a map $T \rightarrow Y$, the action of $X^\bullet/Y$ on $X \times_Y T \rightarrow X$ endows

$$\mathrm{IndCoh}(X \times_Y T) \in \mathrm{IndCoh}(X)\text{-}\mathbf{mod}$$

with the structure of module over $\mathrm{IndCoh}(X \times_Y X)$. By similar reasoning as above, we may forget the $\mathrm{IndCoh}(X)$ -module structures and obtain the analogous statement for the underlying DG categories.

The left and right $\mathrm{IndCoh}(X \times_Y X)$ -actions on $\mathrm{IndCoh}(X)$ are intertwined by the monoidal equivalence

$$\mathrm{IndCoh}(X \times_Y X) \simeq \mathrm{IndCoh}(X \times_Y X)^{\mathrm{rev-mult}}$$

given by the switching map $\sigma : X \times_Y X \rightarrow X \times_Y X$.

Pushforward along the diagonal map $\Delta_{X/Y} : X \rightarrow X \times_Y X$ induces a monoidal functor $\Delta_{X/Y,*} : \mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X \times_Y X)$. The left and right $\mathrm{IndCoh}(X)$ -actions on $\mathrm{IndCoh}(X \times_Y X)$ are simply given by restricting the left and right multiplication actions on $\mathrm{IndCoh}(X \times_Y X)$ along $\Delta_{X/Y,*}$.

Lemma 2.4.3. *Let $\mathcal{O}$ be a monoidal category, and let $f : \mathcal{B} \rightarrow \mathcal{A}$ be a map of algebra objects therein. Let $\mathcal{M}$ and $\mathcal{N}$ be right and left modules for $\mathcal{A}$. Restricting along f , they are also right and left modules for $\mathcal{B}$. The two simplicial systems*

$$\cdots \quad \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{M} \rightrightarrows \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{N} \rightrightarrows \mathcal{M} \otimes \mathcal{N}$$

and

$$\cdots \quad \mathcal{M} \otimes_{\mathcal{B}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{M} \rightrightarrows \mathcal{M} \otimes_{\mathcal{B}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{N} \rightrightarrows \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}$$

have equivalent geometric realizations.

Proof. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ denote the two geometric realizations. There is a map $\mathcal{R}_1 \rightarrow \mathcal{R}_2$ induced from the termwise maps

$$\mathcal{M} \otimes \mathcal{A} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes_{\mathcal{B}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{N}.$$

There is also a map $\mathcal{R}_2 \rightarrow \mathcal{R}_1$ defined as follows. First, expand the second simplicial system into a bigger diagram by expressing each relative tensor over $\mathcal{B}$ as a colimit. Each term in the bigger diagram looks like

$$\mathcal{M} \otimes (?) \otimes \cdots \otimes (?) \otimes \mathcal{N}$$

where each $(?)$ is either $\mathcal{B}$ or $\mathcal{A}$. Via the maps f and id_A , each such term maps to a corresponding term

$$\mathcal{M} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A} \otimes \mathcal{N}$$

in the first simplicial system. This gives a natural transformation from the bigger diagram to the first simplicial system, and taking colimits gives the desired map $\mathcal{R}_2 \rightarrow \mathcal{R}_1$.

These two maps are mutually inverse, and the result follows. $\square$

Proposition 2.4.4. *Assume that p is proper. For any two maps $T_1 \rightarrow Y$ and $T_2 \rightarrow Y$, we have*

$$\text{IndCoh}(T_1 \times_Y X) \underset{\text{IndCoh}(X \times_Y X)}{\otimes} \text{IndCoh}(X \times_Y T_2) \simeq \text{IndCoh}(T_1 \times_Y T_2).$$

Proof. Pulling back along $X^\bullet/Y \rightarrow Y$, the map $T_1 \times_Y T_2 \rightarrow Y$ gives rise to the following simplicial system, denoted $T_1 \times_Y X^\bullet \times_Y T_2$, where all the squares are fibered:

$$\begin{array}{ccc} \cdots & T_1 \times_Y X \times_Y X \times_Y T_2 & \rightrightarrows T_1 \times_Y X \times_Y T_2 \\ & \downarrow & \downarrow \\ \cdots & X \times_Y X & \rightrightarrows X \end{array}$$

Each horizontal map is base-changed from $p : X \rightarrow Y$ and is proper by assumption. Therefore, for each horizontal map, $*$ -pushforward is left adjoint to $!$ -pullback.

By descent, we have

$$\text{IndCoh}(T_1 \times_Y T_2) \simeq \text{Tot}(\text{IndCoh}(T_1 \times_Y X^\bullet \times_Y T_2))$$

where the transition maps for the cosimplicial system are given by $!$ -pullback.

On the other hand, by Lemma 2.4.3, the LHS is the geometric realization of the simplicial system

$$\begin{aligned} \cdots \quad \text{IndCoh}(T_1 \times_Y X) \underset{\text{IndCoh}(X)}{\otimes} \text{IndCoh}(X \times_Y X) \underset{\text{IndCoh}(X)}{\otimes} \text{IndCoh}(X \times_Y T_2) \\ \Rightarrow \text{IndCoh}(T_1 \times_Y X) \underset{\text{IndCoh}(X)}{\otimes} \text{IndCoh}(X \times_Y T_2). \end{aligned}$$

Under the equivalences

$$\text{IndCoh}(T_1 \times_Y X) \underset{\text{IndCoh}(X)}{\otimes} \text{IndCoh}(X \times_Y T_2) \simeq \text{IndCoh}(T_1 \times_Y X \times_Y T_2)$$

$$\text{IndCoh}(T_1 \times_Y X) \underset{\text{IndCoh}(X)}{\otimes} \text{IndCoh}(X \times_Y X) \underset{\text{IndCoh}(X)}{\otimes} \text{IndCoh}(X \times_Y T_2) \simeq \text{IndCoh}(T_1 \times_Y X \times_Y X \times_Y T_2),$$

and so on, this simplicial system is exactly

$$\text{Real}(\text{IndCoh}(T_1 \times_Y X^\bullet \times_Y T_2))$$

where the transition maps are given by $*$ -pushforward (because the multiplication on $\text{IndCoh}(X \times_Y X)$ and action on $\text{IndCoh}(X \times_Y T)$ relative to $\text{IndCoh}(X)$ were both defined via p_*). Now the result follows from Proposition 2.5.7 in [5], which says that, under appropriate hypotheses, a limit of a system of categories is equivalent to a colimit of the system obtained by passing to the left adjoints. $\square$

Proposition 2.4.5. *Let $q : T \rightarrow Y$ be a map, and assume that*

- (i) *The conclusion of Proposition 2.4.4 holds for $T_1 = T_2 = T$.*
- (ii) *The diagonal map $\Delta_{T/Y} : T \rightarrow T \times_Y T$ belongs to a class of maps $\mathcal{P}_1$ which is closed under pullback.*
- (iii) *The diagonal map $\Delta_T : T \rightarrow T \times T$ belongs to a class of maps $\mathcal{P}_2$ which is closed under pullback.*
- (iv) *The base change property is satisfied for $*$ -pushforward along morphisms in $\mathcal{P}_1$ and $!$ -pullback along morphisms in $\mathcal{P}_2$.*

Then $\text{IndCoh}(T \times_Y X)$ and $\text{IndCoh}(X \times_Y T)$ are mutually dual as $\text{IndCoh}(X \times_Y X)$ -modules.

Proof. We will explicitly construct the duality datum

$$\begin{aligned} \text{Vect} &\rightarrow \text{IndCoh}(T \times_Y X) \underset{\text{IndCoh}(X \times_Y X)}{\otimes} \text{IndCoh}(X \times_Y T) \\ \text{IndCoh}(X \times_Y T) \otimes \text{IndCoh}(T \times_Y X) &\rightarrow \text{IndCoh}(X \times_Y X). \end{aligned}$$

By assumption (i), we have

$$\text{IndCoh}(T \times_Y X) \underset{\text{IndCoh}(X \times_Y X)}{\otimes} \text{IndCoh}(X \times_Y T) \simeq \text{IndCoh}(T \times_Y T)$$

To construct the unit map, let $\pi : T \rightarrow \text{pt}$ be the projection, and do pull-push along this correspondence:

$$\begin{array}{ccc} & \text{pt} & \\ & \swarrow \pi & \\ & T & \\ & \swarrow \Delta_{T/Y} & \\ T \times_Y T & & \end{array}$$

For the counit, do pull-push along this correspondence:

$$\begin{array}{ccc} (X \times_Y T) \times (T \times_Y X) & & \\ \swarrow \text{id} \times \Delta_T \times \text{id} & & \\ X \times_Y T \times_Y X & & \\ \swarrow \text{id} \times q \times \text{id} & & \\ X \times_Y X & & \end{array}$$

It is straightforward to check that this respects the left and right actions by $\text{IndCoh}(X \times_Y X)$.

To check that this defines a valid duality datum, consider the following iterated correspondence, where the square is fibered:

$$\begin{array}{ccccc} & & X \times_Y T & & \\ & & \swarrow \text{id} \times \pi & & \\ & & (X \times_Y T) \times T & & \\ & \swarrow \text{id} \times \Delta_{T/Y} & & \swarrow \text{id} \times \Delta_T & \\ (X \times_Y T) \times (T \times_Y T) & & & & X \times_Y T \\ & \swarrow \text{id} \times \Delta_T \times \text{id} & & \swarrow \text{id} \times \Delta_{T/Y} & \\ & & X \times_Y T \times_Y T & & \\ & \swarrow \text{id} \times q \times \text{id} & & & \\ & & X \times_Y T & & \end{array}$$

The outer correspondence is the identity, so the result follows from base change for the fibered square, which uses the assumptions (ii), (iii), and (iv). $\square$

Proposition 2.4.6. *Let $q : T \rightarrow Y$ be a map, and assume that*

- (i) *The conclusion of Proposition 2.4.4 holds for $T_1 = T_2 = T$.*
- (ii) *The conclusion of Proposition 2.4.5 holds.*

Then

$$\text{End}_{\text{IndCoh}(X \times_Y X)}(\text{IndCoh}(X \times_Y T)) \simeq \text{IndCoh}(T \times_Y T)$$

as monoidal categories.

Proof. By the assumptions (i) and (ii), we have

$$\begin{aligned} \mathrm{End}_{\mathrm{IndCoh}(X \times_Y X)}(\mathrm{IndCoh}(X \times_Y T)) &\simeq \mathrm{IndCoh}(T \times_Y X) \otimes_{\mathrm{IndCoh}(X \times_Y X)} \mathrm{IndCoh}(X \times_Y T) \\ &\simeq \mathrm{IndCoh}(T \times_Y X). \end{aligned}$$

It is straightforward to check the compatibility of the monoidal structures. $\square$

The following result deserves mention because it implies that $\mathrm{HCh}(G)$ is rigid and $\mathcal{D}(G[[t]] \backslash G((t))/G[[t]])$ is semi-rigid. But we will not need it for establishing that various module categories are self-dual since we can do that directly with Proposition 2.4.5.

Theorem 2.4.7. *Assume that p is proper and $\Delta : X \rightarrow X \times X$ is smooth. Then $\mathrm{IndCoh}(X \times_Y X)$ is semi-rigid. If moreover $*$ -pushforward along $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is quasi-proper, then $\mathrm{IndCoh}(X \times_Y X)$ is rigid.*

Proof. The first statement is Theorem 6.2 of [1], but we provide a different proof.

The multiplication map is push-pull along this correspondence

$$\begin{array}{ccc} X \times_Y X \times X \times_Y X & & \\ & \swarrow \Delta & \\ & X \times_Y X \times_Y X & \\ & \swarrow \mathrm{pr}_{13} & \\ X \times_Y X & & \end{array}$$

sending $\mathcal{M} \boxtimes \mathcal{N}$ to $\mathrm{pr}_{13,*} \Delta^!(\mathcal{M} \boxtimes \mathcal{N})$. Since pr_{13} is base-changed from p , it is proper, so its right adjoint is $\mathrm{pr}_{13}^!$. Since Δ is base-changed from the diagonal map of X , it is smooth, so its right adjoint is $\Delta_*[2 \dim X]$. Therefore, the right adjoint of the multiplication map is $\Delta_* \mathrm{pr}_{13}^![2 \dim X]$. This functor is continuous by definition.

To complete the proof, we need to show that this right adjoint mult^R is a map of $\mathcal{D}(X \times_Y X)$ -bimodule categories. In other words, we need to give canonical identifications

$$(2) \quad \mathrm{mult}(\mathcal{M} \boxtimes \mathrm{mult}^R(\mathcal{N})) \simeq \mathrm{mult}^R(\mathrm{mult}(\mathcal{M} \boxtimes \mathcal{N}))$$

and similarly for the right action. To this end, consider the iterated correspondence

$$\begin{array}{ccccc} (X \times_Y X) \times (X \times_Y X) & & & & \\ & \swarrow \Delta & & & \\ & X \times_Y X \times_Y X & & & \\ & \swarrow \mathrm{pr}_{13} & & \swarrow \mathrm{pr}_{124} & \\ X \times_Y X & & & X \times_Y X \times_Y X \times_Y X & \\ & \swarrow \mathrm{pr}_{13} & & \swarrow \mathrm{pr}_{134} & \\ & X \times_Y X \times_Y X & & & \\ & \swarrow \Delta & & & \\ (X \times_Y X) \times (X \times_Y X) & & & & \end{array}$$

where the square is fibered. Iterated push-pull along the two small correspondences therefore coincides with push-pull along the big (outer) correspondence. By definition, the iterated push-pull coincides with the RHS of (2) shifted by $2 \dim X$:

$$\mathrm{mult}^R(\mathrm{mult}(\mathcal{M} \boxtimes \mathcal{N})) \simeq \Delta_* \mathrm{pr}_{13}^! \mathrm{pr}_{13,*} \Delta^!(\mathcal{M} \boxtimes \mathcal{N})[2 \dim X].$$

Next, consider the iterated correspondence

$$\begin{array}{ccccc}
 & & (X \times_Y X) \times (X \times_Y X) & & \\
 & \swarrow \text{id} \times \text{pr}_{13} & & \nwarrow \Delta & \\
 & & (X \times_Y X) \times (X \times_Y X \times_Y X) & & \\
 & \swarrow \text{id} \times \Delta & & \nwarrow \Delta & \\
 (X \times_Y X) \times (X \times_Y X) \times (X \times_Y X) & & & & X \times_Y X \times_Y X \times_Y X \\
 & \swarrow \Delta \times \text{id} & & \nwarrow \Delta & \\
 & & (X \times_Y X \times_Y X) \times (X \times_Y X) & & \\
 & \swarrow \text{pr}_{13} \times \text{id} & & & \\
 & & (X \times_Y X) \times (X \times_Y X) & &
 \end{array}$$

where the square is fibered. Again, iterated push-pull coincides with push-pull along the outer correspondence. By definition, the iterated push-pull coincides with the LHS of (2) shifted by $2 \dim X$:

$$\text{mult}(\mathcal{M} \boxtimes \text{mult}^R(\mathcal{N})) \simeq (\text{pr}_{13} \times \text{id})_*(\Delta \times \text{id})^!(\text{id} \times \Delta)_*(\text{id} \times \text{pr}_{13})^!(\mathcal{M} \boxtimes \mathcal{N})[2 \dim X].$$

But the two outer correspondences are the same, and the result follows.

For the second statement, the unit is given by push-pull along

$$\begin{array}{ccc}
 \text{pt} & \xleftarrow{\pi} & X \\
 & \searrow \Delta_{X/Y} & \\
 X \times_Y X & &
 \end{array}$$

Since we are using IndCoh , $\mathcal{O}_X = \pi^* k$ is compact. Now if $\Delta_{X/Y,*}$ sends compacts to compacts, we obtain that the unit is compact, so the category is rigid. $\square$

2.5. Harish-Chandra bimodules. Having proven in Proposition 2.3.1 that weak G invariants and coinvariants coincide, we now pursue the analogous statement for strong G -actions. We shall bootstrap from the case of weak G -actions: given $\mathcal{C} \in \mathcal{D}(G)\text{-mod}$, we explain how to compute $\mathcal{C}^G$ given $\mathcal{C}^{G,w}$ as a module over $\text{HCh}(G)$, see Proposition 2.6.1. The reason why the strategy of bootstrapping from weak to strong actions works is that the ‘discrepancy’ is given by $\text{pt}/G \rightarrow \text{pt}/G_{\text{dR}}$, and this map is inf-proper. The same principle, applied to the ind-proper map $\text{pt}/G[[t]] \rightarrow \text{pt}/G((t))$, will allow us to bootstrap in an analogous way from strong $G[[t]]$ -actions to strong $G((t))$ actions, see Lemma 3.4.3.

2.5.1. In the setting of Section 2.4, we make substitutions as follows:

$$\begin{array}{ccc}
 \text{pt} & & T \\
 \downarrow q & \text{equals} & \downarrow q \\
 \text{pt}/G \xrightarrow{p} \text{pt}/G_{\text{dR}} & & X \xrightarrow{p} Y
 \end{array}$$

We define

$$\text{HCh}(G) := \text{IndCoh}(X \times_Y X) \simeq \text{IndCoh}(G \backslash G_{\text{dR}}/G),$$

with the convolution monoidal structure. According to 2.4.2,

$$\text{Rep}(G) \simeq \text{IndCoh}(X)$$

is a module for $\text{HCh}(G)$, and we have a monoidal functor $\text{Rep}(G) \rightarrow \text{HCh}(G)$ given by $*$ -pushforward along the map $\text{pt}/G \simeq G \backslash G/G \rightarrow G \backslash G_{\text{dR}}/G$. It should be mentioned at this point that the convolution monoidal structure on $\text{Rep}(G)$ defined via its realization as $\text{QCoh}(G \backslash G/G)$ coincides with the standard one.

On the other side, we have

$$\mathcal{D}(G) \simeq \text{IndCoh}(T \times_Y T)$$

where the convolution monoidal structures on both sides are identified. This time,

$$\text{Vect} \simeq \text{IndCoh}(T)$$

is a module for $\mathcal{D}(G)$, and we have a monoidal functor $\text{Vect} \rightarrow \mathcal{D}(G)$ given by $*$ -pushforward along $\text{pt} \rightarrow G_{\text{dR}}$.

Furthermore, we define

$$\mathfrak{g}\text{-mod} := \text{IndCoh}(X \times_Y T) \simeq \text{IndCoh}(G \backslash G_{\text{dR}}) \simeq \text{IndCoh}(\text{pt}/G_{\hat{1}}),$$

where the last equivalence follows from the exact sequence $G_{\hat{1}} \rightarrow G \rightarrow G_{\text{dR}}$. From 2.4.2, we see that $\mathfrak{g}\text{-mod}$ is a bialgebra for $\text{HCh}(G)$ and $\mathcal{D}(G)$.

The $\mathcal{D}(G)$ -action on $\mathfrak{g}\text{-mod}$ restricts to an action by $\text{QCoh}(G)$. By Lemma 2.1.4 applied to IndCoh in place of QCoh , we have

$$\begin{aligned} \mathfrak{g}\text{-mod}^{G,w} &\simeq \text{IndCoh}(G \backslash G_{\text{dR}}/G) \\ &\simeq \text{HCh}(G). \end{aligned}$$

Deequivariantization, i.e. the statement that rec is inverse to $\text{inv}^{G,w,\text{enh}}$, gives

$$(3) \quad \text{Vect} \underset{\text{Rep}(G)}{\otimes} \text{HCh}(G) \simeq \mathfrak{g}\text{-mod}.$$

2.5.2. We seek to render the results of 2.4 to the present setting.

Proposition 2.5.3. *(i) We have*

$$\begin{aligned} \mathfrak{g}\text{-mod} \underset{\text{HCh}(G)}{\otimes} \mathfrak{g}\text{-mod} &\simeq \mathcal{D}(G) \\ \text{Rep}(G) \underset{\text{HCh}(G)}{\otimes} \mathfrak{g}\text{-mod} &\simeq \text{Vect} \\ \text{Rep}(G) \underset{\text{HCh}(G)}{\otimes} \text{Rep}(G) &\simeq \mathcal{D}(\text{pt}/G) \end{aligned}$$

(ii) $\mathfrak{g}\text{-mod}$ and $\text{Rep}(G)$ are self-dual in $\text{HCh}(G)\text{-mod}$.

(iii) We have

$$\begin{aligned} \text{End}_{\text{HCh}(G)}(\mathfrak{g}\text{-mod}) &\simeq \mathcal{D}(G) \\ \text{End}_{\text{HCh}(G)}(\text{Rep}(G)) &\simeq \mathcal{D}(\text{pt}/G). \end{aligned}$$

(iv) $\text{HCh}(G)$ is rigid.

Proof. (i) follows from Proposition 2.4.4, which applies because $p : \text{pt}/G \rightarrow \text{pt}/G_{\text{dR}}$ is inf-proper.

For (ii), we check the hypotheses of Proposition 2.4.5. The case of $\mathfrak{g}\text{-mod}$ corresponds to $T = \text{pt}$, in which case Δ_T is the identity, so the desired base change property holds. The case of $\text{Rep}(G)$ corresponds to $T = \text{pt}/G_{\text{dR}}$, in which case $\Delta_{T/Y}$ is the identity, so the desired base change property still holds.

In view of Proposition 2.4.6, (iii) follows from (i) and (ii).

(iv) follows from Proposition 2.4.7 since p is inf-proper and pt/G has smooth diagonal. $\square$

Next, we focus on $\mathcal{D}(G)$ -actions, so we reverse the roles of X and T .

Proposition 2.5.4. *Let $\mathcal{C} \in \mathcal{D}(G)\text{-mod}$. We have*

$$\text{Hom}_{\mathcal{D}(G)}(\mathfrak{g}\text{-mod}, \mathcal{C}) \simeq \mathcal{C}^{G,w} \simeq \mathcal{C}_{G,w} \simeq \mathfrak{g}\text{-mod} \underset{\mathcal{D}(G)}{\otimes} \mathcal{C}$$

Proof. Lemma 2.1.4 applied to $\mathcal{X} = G_{\text{dR}}$ tells us that

$$\mathfrak{g}\text{-mod} = \text{IndCoh}(G \backslash G_{\text{dR}}) \simeq \mathcal{D}(G)^{G,w}.$$

Since weak invariants and coinvariants coincide, this is also $\mathcal{D}(G)_{G,w} = \mathcal{D}(G) \underset{\text{QCoh}(G)}{\otimes} \text{Vect}$. Now

$$\begin{aligned} \mathcal{C}_{G,w} &= \text{Vect} \underset{\text{QCoh}(G)}{\otimes} \mathcal{C} \\ &\simeq \text{Vect} \underset{\text{QCoh}(G)}{\otimes} \mathcal{D}(G) \underset{\mathcal{D}(G)}{\otimes} \mathcal{C} \\ &\simeq \mathfrak{g}\text{-mod} \underset{\mathcal{D}(G)}{\otimes} \mathcal{C}. \end{aligned}$$

For the other equivalence, apply tensor-Hom adjunction to $\mathcal{D}(G)$ viewed as a bimodule for $\mathcal{D}(G)$ and $\text{QCoh}(G)$:

$$\begin{aligned} \mathcal{C}^{G,w} &\simeq \text{Hom}_{\text{QCoh}(G)}(\text{Vect}, \mathcal{C}) \\ &\simeq \text{Hom}_{\text{QCoh}(G)}(\text{Vect}, \text{Hom}_{\mathcal{D}(G)}(\mathcal{D}(G), \mathcal{C})) \\ &\simeq \text{Hom}_{\mathcal{D}(G)}(\text{Vect} \underset{\text{QCoh}(G)}{\otimes} \mathcal{D}(G), \mathcal{C}) \\ &\simeq \text{Hom}_{\mathcal{D}(G)}(\mathfrak{g}\text{-mod}, \mathcal{C}). \end{aligned} \quad \square$$

Proposition 2.5.5. (i) *We have*

$$\begin{aligned} \mathfrak{g}\text{-mod} \underset{\mathcal{D}(G)}{\otimes} \mathfrak{g}\text{-mod} &\simeq \text{HCh}(G) \\ \mathfrak{g}\text{-mod} \underset{\mathcal{D}(G)}{\otimes} \text{Vect} &\simeq \text{Rep}(G). \end{aligned}$$

(ii) *$\mathfrak{g}\text{-mod}$ is self-dual in $\mathcal{D}(G)\text{-mod}$.*

(iii) *We have $\text{End}_{\mathcal{D}(G)}(\mathfrak{g}\text{-mod}) \simeq \text{HCh}(G)$.*

Proof. (i) follows from Proposition 2.5.4.

For $T = \text{pt}/G$ and $X = \text{pt}$, (i) verifies the conclusion of Proposition 2.4.4. Therefore, in order to apply Proposition 2.4.5, we need only observe that $\Delta_T : \text{pt}/G \rightarrow \text{pt}/G \times \text{pt}/G$ is affine, so the base change property is satisfied. The conclusion is that $\mathfrak{g}\text{-mod}$ is self-dual.

Using Proposition 2.4.6, (iii) follows from (i) and (ii). $\square$

For sake of completeness, we mention at this point the following result, even though the proof relies on the fact that strong invariants and coinvariants coincide.

Corollary 2.5.6. *We have*

$$\text{Vect} \underset{\mathcal{D}(G)}{\otimes} \text{Vect} \simeq \text{End}_{\mathcal{D}(G)}(\text{Vect}) \simeq \mathcal{D}(\text{pt}/G),$$

and Vect is self-dual in $\mathcal{D}(G)\text{-mod}$.

Proof, assuming Proposition 2.6.2. The first equivalence follows from the fact that strong invariants and coinvariants coincide. The second equivalence follows from the $\mathcal{D}$ -module analogue of Lemma 2.1.4. The composition of the two equivalences verifies the conclusion of Proposition 2.4.4, in the $\mathcal{D}$ -module setting, for $X = \text{pt}$, $Y = \text{pt}/G$ and $T_1 = T_2 = Y$. Then Proposition 2.4.5 applies because $\Delta_{T/Y}$ is the identity map, so the requisite base change property holds. Therefore Vect is self-dual in $\mathcal{D}(G)\text{-mod}$. $\square$

Proposition 2.5.7. *In the following diagram of adjoint pairs, all four squares commute:*

$$\begin{array}{ccc}
\mathrm{HCh}(G)\text{-}\mathbf{mod} & \begin{array}{c} \xleftarrow{- \otimes_{\mathrm{HCh}(G)} \mathfrak{g}\text{-mod}} \\ \xrightarrow{\mathrm{Hom}_{\mathcal{D}(G)}(\mathfrak{g}\text{-mod}, -)} \end{array} & \mathcal{D}(G)\text{-}\mathbf{mod} \\
\begin{array}{c} \uparrow \text{ind} \\ \downarrow \text{res} \end{array} & & \begin{array}{c} \uparrow \text{ind} \\ \downarrow \text{res} \end{array} \\
\mathrm{Rep}(G)\text{-}\mathbf{mod} & \begin{array}{c} \xleftarrow{\mathrm{rec}} \\ \xrightarrow{\mathrm{inv}^{G,w,\mathrm{enh}}} \end{array} & \mathrm{QCoh}(G)\text{-}\mathbf{mod}
\end{array}$$

Proof. Proposition 2.5.4 implies that the square involving res and Hom commutes (up to a natural isomorphism). To check that the square involving res and $\otimes$ commutes (up to a natural isomorphism), let $\mathcal{N} \in \mathrm{HCh}(G)\text{-}\mathbf{mod}$. Using (3), we have

$$\begin{aligned}
\mathcal{N} \otimes_{\mathrm{HCh}(G)} \mathfrak{g}\text{-mod} &\simeq \mathcal{N} \otimes_{\mathrm{HCh}(G)} \mathrm{HCh}(G) \otimes_{\mathrm{Rep}(G)} \mathrm{Vect} \\
&\simeq \mathcal{N} \otimes_{\mathrm{Rep}(G)} \mathrm{Vect}.
\end{aligned}$$

That the upper horizontal maps are equivalences follows from Lemma 2.5.8, applied to $A = \mathcal{D}(G)$, $B = \mathrm{HCh}(G)$, and $N = M = \mathfrak{g}\text{-mod}$. The requisite facts about duality and endomorphisms follow from Propositions 2.5.3 and 2.5.5.

We deduce the commutativity of the remaining two squares (involving ind rather than res) by passing to left adjoints. $\square$

Lemma 2.5.8. *Let $\mathbf{O}$ be a monoidal category, and let A, B be algebra objects in $\mathbf{O}$. Let*

$$\begin{aligned}
M &\in (A, B)\text{-bimod} \\
N &\in (B, A)\text{-bimod}
\end{aligned}$$

such that M and N are mutually dual as A -modules and as B -modules. Furthermore, assume that

$$\begin{aligned}
\mathrm{End}_A(M) &\simeq B \\
\mathrm{End}_B(M) &\simeq A.
\end{aligned}$$

Then the adjoint functors

$$A\text{-mod} \begin{array}{c} \xleftarrow{M \otimes_B -} \\ \xrightarrow{\mathrm{Hom}_A(M, -)} \end{array} B\text{-mod}$$

are actually equivalences.

Proof. For $Q \in B\text{-mod}$, the unit map is

$$Q \rightarrow \mathrm{Hom}_A(M, M \otimes_B Q).$$

This is an equivalence because it coincides with the composition

$$\begin{aligned}
Q &\simeq B \otimes_B Q \\
&\simeq \mathrm{Hom}_A(M, M) \otimes_B Q \\
&\simeq \mathrm{Hom}_A(M, M \otimes_B Q)
\end{aligned}$$

where the last equivalence uses that M is dualizable in $A\text{-mod}$.

Similarly, for $P \in A\text{-mod}$, the counit map is

$$M \otimes_B \mathrm{Hom}_A(M, P) \rightarrow P.$$

This is an equivalence because it coincides with the composition

$$\begin{aligned} M \otimes_B \operatorname{Hom}_A(M, P) &\simeq M \otimes_B N \otimes_A P \\ &\simeq \operatorname{Hom}_B(M, M) \otimes_A P \\ &\simeq A \otimes_A P \\ &\simeq P, \end{aligned}$$

where we have used that N is dual to M both in $A\text{-mod}$ and in $B\text{-mod}$. $\square$

2.6. Invariants equal coinvariants – strong actions. The following lemma says that it is $\operatorname{Rep}(G)$, viewed as a bimodule for $\operatorname{HCh}(G)$ and $\operatorname{Rep}(G)$, that takes us from weak invariants to strong invariants. One could formulate an analogous result in the generality of Section 2.4, involving $\operatorname{IndCoh}(X)$ in place of $\operatorname{Rep}(G)$ and $\operatorname{IndCoh}(X \times_Y X)$ in place of $\operatorname{HCh}(G)$. Rather than pursue this generalization, we merely remark that Lemma 3.4.3 is another special case.

Lemma 2.6.1. *We have*

$$\begin{aligned} \mathcal{C}^G &\simeq \operatorname{Hom}_{\operatorname{HCh}(G)}(\operatorname{Rep}(G), \operatorname{Hom}_{\mathcal{D}(G)}(\mathfrak{g}\text{-mod}, \mathcal{C})) \\ \mathcal{C}_G &\simeq \operatorname{Rep}(G) \otimes_{\operatorname{HCh}(G)} \mathfrak{g}\text{-mod} \otimes_{\mathcal{D}(G)} \mathcal{C}. \end{aligned}$$

Proof. By tensor-Hom adjunction applied to $\mathfrak{g}\text{-mod}$ as a bimodule for $\mathcal{D}(G)$ and $\operatorname{HCh}(G)$, we have

$$\operatorname{Hom}_{\operatorname{HCh}(G)}(\operatorname{Rep}(G), \operatorname{Hom}_{\mathcal{D}(G)}(\mathfrak{g}\text{-mod}, \mathcal{C})) \simeq \operatorname{Hom}_{\mathcal{D}(G)}(\operatorname{Rep}(G) \otimes_{\operatorname{HCh}(G)} \mathfrak{g}\text{-mod}, \mathcal{C}).$$

Therefore, both equations follow from Proposition 2.5.3(i). $\square$

The following result is Theorem 2.3.12 of [2].

Proposition 2.6.2. *The functor θ_G is a natural equivalence.*

Proof. We have

$$\begin{aligned} \mathcal{C}_G &\simeq \operatorname{Rep}(G) \otimes_{\operatorname{HCh}(G)} \mathfrak{g}\text{-mod} \otimes_{\mathcal{D}(G)} \mathcal{C} \\ &\simeq \operatorname{Rep}(G) \otimes_{\operatorname{HCh}(G)} \mathcal{C}_{G,w} \\ &\simeq \operatorname{Hom}_{\operatorname{HCh}(G)}(\operatorname{Rep}(G), \mathcal{C}_{G,w}) \\ &\simeq \operatorname{Hom}_{\operatorname{HCh}(G)}(\operatorname{Rep}(G), \mathcal{C}^{G,w}) \\ &\simeq \mathcal{C}^G \end{aligned}$$

by Lemma 2.6.1 and the self-duality of $\operatorname{Rep}(G)$ established in Proposition 2.5.3(ii). This is functorial in $\mathcal{C}$ so it gives a natural equivalence $\operatorname{inv}^G \simeq \operatorname{coinv}^G$. In particular, the former commutes with colimits since the latter does.

Now one needs to check that this equivalence coincides with $\theta_G(\mathcal{C})$. But the fact that both sides commute with colimits allows us to reduce to the case $\mathcal{C} = \mathcal{D}(G)$, which is straightforward. $\square$

3. ACTIONS BY IND-PRO-GROUPS

In the remaining part of these notes, we indicate the necessary modifications needed to define the action of more general group prestacks on a category. We shall work exclusively in the setting of strong actions, i.e. with $\mathcal{D}$ -modules, and we will prove ‘invariants equal coinvariants’ statements for $G[[t]]$ and $G((t))$, see Corollary 3.2.2 and Proposition 3.4.5, respectively. Our method of proof for $G[[t]]$ generalizes immediately to pro-groups which are of the form $G \ltimes N$ where G is finite type and N is pro-unipotent. The argument in this generality is carried out in Section 4.2 of [2], and the key idea is that pro-unipotent groups are cohomologically contractible. The latter fact is why these proofs rely on using $\mathcal{D}$ -modules rather than, say, ind-coherent sheaves.

However, the method of proof for $G((t))$ does not readily generalize to other ind-pro-groups, because it specifically relies on the fact that $G((t))/G[[t]]$ is ind-proper. In particular, it is *not* true that ‘invariants equal coinvariants’ for $N((t))$ when N is unipotent. Therefore, it is all the more remarkable that Whittaker invariants and coinvariants do coincide.

3.1. Definitions.

3.1.1. *Hopf algebras.* There is a good theory of $\mathcal{D}$ -modules on infinite-dimensional schemes, developed in [6]. To any prestack $\mathcal{Y}$, one assigns two categories $\mathcal{D}^!(\mathcal{Y})$ and $\mathcal{D}^*(\mathcal{Y})$. The category $\mathcal{D}^!(\mathcal{Y})$ is contravariant in $\mathcal{Y}$ under $!$ -pullback, and $\mathcal{D}^*(\mathcal{Y})$ is covariant in $\mathcal{Y}$ under $*$ -pushforward. So we have functors

$$\begin{aligned}\mathcal{D}^! &: \text{PreStk}^{\text{op}} \rightarrow \text{DGCat} \\ \mathcal{D}^* &: \text{PreStk} \rightarrow \text{DGCat}.\end{aligned}$$

In the case that $\mathcal{Y} = \mathcal{G}$ is a group ind-pro-scheme, e.g. $G((t))$, with multiplication map $m : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and diagonal map Δ , we have that $\mathcal{D}^*(\mathcal{G})$ is a Hopf algebra with multiplication m_* and comultiplication Δ_* , and $\mathcal{D}^!(\mathcal{G})$ is a Hopf algebra with multiplication $\Delta^!$ and comultiplication $m^!$. One uses the fact that $\mathcal{G}$ is an ind-pro-scheme to show that $\mathcal{D}^*(\mathcal{G}^{\times n}) \simeq \mathcal{D}^*(\mathcal{G})^{\otimes n}$, see Lemma 3.4.5 in [2].

When Verdier duality applies, it says that $\mathcal{D}^*(\mathcal{Y})$ is dual to $\mathcal{D}^!(\mathcal{Y})$, so that $*$ -pushforward and $!$ -pullback correspond to one another. Here, this means that the two Hopf algebra structures are exchanged under duality.

3.1.2. *Strong actions.* For a category $\mathcal{C} \in \text{DGCat}$, the datum of a strong $\mathcal{G}$ -action on $\mathcal{C}$ is defined to be an action of the monoidal category $\mathcal{D}^*(\mathcal{G})$. The category of modules for $\mathcal{D}^*(\mathcal{G})$ is denoted $\mathcal{D}^*(\mathcal{G})\text{-mod}$, or just $\mathcal{G}\text{-mod}$.

3.1.3. *Invariants and coinvariants.* The map $p : \mathcal{G} \rightarrow \text{pt}$ is a map of group prestacks, so it induces a monoidal functor $p_* : \mathcal{D}^*(\mathcal{G}) \rightarrow \text{Vect}$. In particular, we have the trivial representation $\text{Vect} \in \mathcal{D}^*(\mathcal{G})\text{-mod}$, and we can define the functors of invariants and coinvariants in analogy with the finite type case:

$$\begin{aligned}\text{inv}^{\mathcal{G}} &= \text{Hom}_{\mathcal{D}^*(\mathcal{G})}(\text{Vect}, -) \\ \text{coinv}^{\mathcal{G}} &= \text{Vect} \otimes_{\mathcal{D}^*(\mathcal{G})} -.\end{aligned}$$

Since p_* is also a map of $\mathcal{D}^*(\mathcal{G})$ -modules, for $\mathcal{C} \in \mathcal{D}^*(\mathcal{G})\text{-mod}$, we obtain functors

$$\begin{aligned}\text{oblv}^{\mathcal{G}} : \mathcal{C}^{\mathcal{G}} &\rightarrow \mathcal{C} \\ \text{pr}_{\mathcal{G}} : \mathcal{C} &\rightarrow \mathcal{C}_{\mathcal{G}}.\end{aligned}$$

Unfortunately, the functors $\text{Av}_*^{\mathcal{G}}$ and $(\text{pr}_{\mathcal{G}})^L$ fail to generalize to the present setting, since p_* need not admit a left adjoint. Proposition 6.18.1 from [6] does not apply because $\mathcal{G} \rightarrow \text{pt}$ is not, in general, placid. In particular, $G((t)) \rightarrow \text{pt}$ is not placid.

However, $G[[t]] \rightarrow \text{pt}$ is placid, and in this case p_* does admit a left adjoint p^* , so the functors $\text{Av}_*^{G[[t]]}$ and $(\text{pr}_{G[[t]]})^L$ do exist in this setting. The same is true whenever $\mathcal{G}$ is a pro-algebraic group.

For general $\mathcal{G}$, equation (1) continues to hold, so we obtain a natural transformation $\theta_{\mathcal{G}} : \text{coinv}^{\mathcal{G}} \rightarrow \text{inv}^{\mathcal{G}}$.

Remark 3.1.4. There is in general no analogue of Proposition 2.2.1 in the ind-infinite type setting. To identify $\text{IndCoh}(\mathcal{G})$ -modules with sheaves of categories over $B\mathcal{G}$, we need to know that $\text{Loc}_{\mathcal{G}}$ is fully faithful, see 10.1.1 of [3]. In addition, to identify sheaves of categories over $B\mathcal{G}$ with $\text{IndCoh}(\text{pt}/\mathcal{G})$, we would need to know that $\text{pt}/\mathcal{G}$ is 1-affine. This fails for $\mathcal{G} = G((t))$, in which case $\text{IndCoh}(\text{pt}/G((t))) \simeq \text{Vect}$.

3.2. Invariants equal coinvariants for $G[[t]]$.

Lemma 3.2.1. (i) Let N be a pro-unipotent group scheme, not necessarily of finite type. Then oblv^N and $(\text{pr}_N)^L$ are fully faithful. Therefore $\mathcal{C}^N$ is a colocalization of $\mathcal{C}$.
(ii) The functor θ_N is a natural equivalence.

Proof. When N is pro-unipotent, the counit map $p_*p^* \rightarrow \text{Id}_{\text{Vect}}$ is an equivalence. This is essentially because N is an iterated extension of copies of $\mathbb{A}^1$ and is therefore contractible. This step fails in the setting of weak actions, since it is true for de Rham pushforward of $\mathcal{D}$ -modules but not $*$ -pushforward of (quasi or ind) coherent sheaves. The unit of the adjunction $(\text{oblv}^N, \text{Av}_*^N)$ is obtained from the counit for (p_*p^*) , so it is an equivalence. Likewise for the counit of the adjunction $((\text{pr}_N)^L, \text{pr}_N)$. This proves (i).

For (ii), $\text{pr}_N \circ \text{oblv}^N$ provides an inverse, cf. Theorem 4.3.2 from [2]. $\square$

Corollary 3.2.2. The functor $\theta_{G[[t]]}$ is a natural equivalence.

Proof. Let $G_1 \subset G[[t]]$ be the first congruence subgroup. Since G_1 is pro-unipotent, for $\mathcal{C} \in \mathcal{D}^*(G[[t]])\text{-mod}$, Lemma 3.2.1 tells us that $\mathcal{C}^{G_1} \simeq \mathcal{C}_{G_1}$. According to Lemma 2.3.14 of [2], $\mathcal{C}^{G_1}$ has an action by $G = G[[t]]/G_1$ so that $\mathcal{C}^{G[[t]]} \simeq (\mathcal{C}^{G_1})^G$, and similarly for coinvariants. Therefore, the result follows from the fact that invariants and coinvariants for G coincide. $\square$

3.3. Realization via actions by finite type groups. The purpose of this section is to explain a few corollaries of Lemma 3.2.1, all to the effect that strong invariance with respect to a unipotent group is a property rather than extra data. The idea, as always, is to use the fact that unipotent groups are cohomologically contractible, i.e. the de Rham pushforward $\Gamma_{\text{dR}}(N, -)$ is a colocalization.

Let N be a pro-unipotent group scheme, not necessarily of finite type. This means that we can find a chain

$$\cdots \subset N_k \subset \cdots \subset N_1 \subset N = N_0$$

of normal subgroups of N such that each N/N_k is finite type and $N = \text{colim}_k N_k = \lim_k N/N_k$. Given $\mathcal{C} \in N\text{-mod}$, we have functors

$$\text{Av}_*^{N_i/N_j} : \mathcal{C}^{N_j} \hookrightarrow \mathcal{C}^{N_i} : \text{oblv}^{N_i/N_j}$$

for $i \leq j$. Since N_i/N_j is unipotent, oblv^{N_i/N_j} is fully faithful, so $\mathcal{C}^{N_i}$ is a colocalization of $\mathcal{C}^{N_j}$.

Proposition 3.3.1. We have

$$\mathcal{C} \simeq \text{colim}_{\text{oblv}^{N_i/N_j}} \mathcal{C}^{N_i} \simeq \lim_{\text{Av}_*^{N_i/N_j}} \mathcal{C}^{N_i}.$$

Proof. First, for $i \leq j$, let $p_{ij} : N/N_j \rightarrow N/N_i$ and $p_i : N \rightarrow N/N_i$ be the quotient maps, so that

$$\mathcal{D}^*(N) \simeq \lim_{p_{ij,*}} \mathcal{D}(N/N_i).$$

Let $\delta_1 \in \mathcal{D}^*(N)$ denote the skyscraper at the identity of N . We claim that $\delta_1 \simeq \text{colim}_i p_i^* p_{i,*} \delta_1$. Indeed, for another $\mathcal{M} \in \mathcal{D}^*(N)$, a compatible system of maps $p_i^* p_{i,*} \delta_1 \rightarrow \mathcal{M}$ is equivalent to a compatible system of maps $p_{i,*} \delta_1 \rightarrow p_{i,*} \mathcal{M}$, and the latter is exactly $\text{Hom}(\delta_1, \mathcal{M})$ by the construction of categorical limits.

Now, let $c \in \mathcal{C}$ be any object. Since the action functor

$$\star : \mathcal{D}(N) \otimes \mathcal{C} \rightarrow \mathcal{C}$$

is continuous, the functor $- \star c$ preserves colimits, so we have

$$\begin{aligned} c &\simeq \delta_1 \star c \\ &\simeq \text{colim}_i (p_i^* p_{i,*} \delta_1 \star c) \\ &\simeq \text{oblv}^{N_i}(\text{Av}^{N_i}(c)). \end{aligned}$$

In other words, each object c is the colimit in $\mathcal{C}$ of its projections to each full subcategory $\mathcal{C}^{N_i}$.

At this point, it follows formally that $\mathcal{C} \simeq \text{colim}_i \mathcal{C}^{N_i}$. Indeed, given a compatible system of functors $F_i : \mathcal{C}^{N_i} \rightarrow \mathcal{E}$ for some other $\mathcal{E} \in \text{DGCat}$, we get a functor

$$F_\infty : \mathcal{C}' \rightarrow \mathcal{E}$$

where $\mathcal{C}' = \bigcup_{i=1}^{\infty} \mathcal{C}^{N_i}$ is a full subcategory of $\mathcal{C}$ which is not closed under colimits. Then, we define $F : \mathcal{C} \rightarrow \mathcal{E}$ as $\text{LKE}_{\mathcal{C}' \hookrightarrow \mathcal{C}}$. This is forced by the requirements that F be continuous and that it agrees with F_i on each full subcategory $\mathcal{C}^{N_i}$.

For any $c \in \mathcal{C}$, we have a functor $\mathbb{N} \rightarrow \mathcal{C}$ which sends $i \mapsto \text{oblv}(\text{Av}^{N_i}(c))$. Since this functor is cofinal for $\mathcal{C}' \hookrightarrow \mathcal{C}$, we have $F(c) \simeq \text{colim}_i \text{oblv}(\text{Av}^{N_i}(c))$. In particular, the restriction of F to each full subcategory $\mathcal{C}^{N_i}$ does indeed agree with F_i . This completes the proof that $\mathcal{C} \simeq \text{colim}_i \mathcal{C}^{N_i}$.

Since the inclusions $\text{oblv}^{N_i/N_j} : \mathcal{C}^{N_i} \hookrightarrow \mathcal{C}^{N_j}$ are left adjoints, by Proposition 2.5.7 in [5], we may replace the colimit by a limit over the right adjoints $\text{Av}_*^{N_i/N_j}$, and the result follows. $\square$

Let G_i be the i -th congruence subgroup of $G[[t]]$, so that $G_0 = G[[t]]$. In what follows, restrict to $i, j \geq 1$ so that G_i is pro-unipotent. For $i \leq j$, we have functors

$$\text{inv}^{G_i/G_j} = \text{coinv}^{G_i/G_j} : \mathcal{D}(G[[t]]/G_j)\text{-}\mathbf{mod} \rightarrow \mathcal{D}(G[[t]]/G_i)\text{-}\mathbf{mod},$$

as well as

$$\text{inv}^{G_i} = \text{coinv}^{G_i} : \mathcal{D}(G[[t]])\text{-}\mathbf{mod} \rightarrow \mathcal{D}(G[[t]]/G_i)\text{-}\mathbf{mod}.$$

Corollary 3.3.2. *These functors yield an equivalence*

$$\mathcal{D}(G[[t]])\text{-}\mathbf{mod} \simeq \lim_i \mathcal{D}(G[[t]]/G_i)\text{-}\mathbf{mod}.$$

Proof. The functors inv^{G_i} yield a map

$$\mathcal{D}(G[[t]])\text{-}\mathbf{mod} \rightarrow \lim_i \mathcal{D}(G[[t]]/G_i)\text{-}\mathbf{mod}.$$

This assigns to each $\mathcal{C} \in \mathcal{D}(G[[t]])\text{-}\mathbf{mod}$ the system $\mathcal{C}^{G_i} \in \mathcal{D}(G[[t]]/G_i)\text{-}\mathbf{mod}$ compatible under the functors inv^{G_i/G_j} .

To define the inverse map, suppose we are given a compatible system $\mathcal{C}_i \in \mathcal{D}(G[[t]]/G_i)\text{-}\mathbf{mod}$. We obtain the $\mathcal{D}(G)$ -module

$$\mathcal{C} := \text{colim}_{\text{oblv}^{G_i/G_j}} \mathcal{C}_i \simeq \lim_{\text{Av}_*^{G_i/G_j}} \mathcal{C}_i,$$

where the limit and colimit are evaluated in DGCat .

Proposition 3.3.1 tells us that one composition is an equivalence. To check the other composition, let $\mathcal{C}$ be as defined in the previous paragraph, and observe that $\mathcal{C}^{G_i} \simeq \mathcal{C}_i$ as full subcategories of $\mathcal{C}$, since each oblv is fully faithful (each G_i is pro-unipotent). $\square$

Just for fun, here is a result along a similar theme:

Proposition 3.3.3. *Let G be an ind-pro-group and let $N \subset G$ be a normal pro-unipotent subgroup. We have an adjunction*

$$\text{inv}^N : \mathcal{D}(G)\text{-}\mathbf{mod} \rightleftarrows \mathcal{D}(G/N)\text{-}\mathbf{mod} : \text{res}.$$

Furthermore, res is fully faithful, so this is a colocalization.

Proof. The functor res is restriction along the colocalization $p_* : \mathcal{D}(G) \rightarrow \mathcal{D}(G/N)$, which is monoidal for the convolution monoidal structures. Let $\mathcal{C} \in \mathcal{D}(G)\text{-}\mathbf{mod}$ and $\mathcal{E} \in \mathcal{D}(G/N)\text{-}\mathbf{mod}$. First, since N is pro-nilpotent, invariants and coinvariants coincide, so

$$\begin{aligned} \text{Vect}_{\mathcal{D}(N)} \otimes \mathcal{D}(G) &\simeq \mathcal{D}(G)^N \\ &\simeq \mathcal{D}(G/N) \end{aligned}$$

by a variant of Lemma 2.1.4. Next, applying tensor-Hom adjunction for $\mathcal{D}(G)$ as a $(\mathcal{D}(G), \mathcal{D}(N))$ -bimodule, we obtain

$$\begin{aligned}\mathcal{C}^N &= \mathrm{Hom}_{\mathcal{D}(N)}(\mathrm{Vect}, \mathcal{C}) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(N)}(\mathrm{Vect}, \mathrm{Hom}_{\mathcal{D}(G)}(\mathcal{D}(G), \mathcal{C})) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(G)}(\mathrm{Vect}_{\mathcal{D}(N)} \otimes_{\mathcal{D}(N)} \mathcal{D}(G), \mathcal{C}) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(G)}(\mathcal{D}(G/N), \mathcal{C}).\end{aligned}$$

Now, applying tensor-Hom adjunction for $\mathcal{D}(G/N)$ as a bimodule for $\mathcal{D}(G)$ and $\mathcal{D}(G/N)$, we obtain

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(G/N)}(\mathcal{E}, \mathcal{C}^N) &\simeq \mathrm{Hom}_{\mathcal{D}(G)}(\mathcal{E} \otimes_{\mathcal{D}(G/N)} \mathcal{D}(G/N), \mathcal{C}) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(G)}(\mathcal{E}, \mathcal{C}).\end{aligned}$$

This proves the adjunction.

To finish the proof, we need to show that the counit map is an equivalence. This follows from a slightly more general statement: for $\mathcal{E} \in \mathrm{DGCat}$, equipped with the trivial N -action, the functor

$$\mathrm{oblv}^N : \mathcal{E}^N \hookrightarrow \mathcal{E}$$

is an equivalence. Indeed, since this functor is fully faithful, we need only show that it is essentially surjective. Now, for any object $e \in \mathcal{E}$, we have

$$\begin{aligned}\mathrm{Av}_*^N(e) &= p^* k \star e \\ &\simeq p_* p^* k \otimes e \\ &\simeq e\end{aligned}$$

where $p : N \rightarrow \mathrm{pt}$ denotes the projection, and $\star$ denotes the (trivial) action of $\mathcal{D}(N)$ on $\mathcal{E}$. □

3.4. Invariants equal coinvariants for $G((t))$.

3.4.1. We apply 2.4 with substitutions as follows:

$$\begin{array}{ccc} \begin{array}{c} T \\ \downarrow q \\ X \xrightarrow{p} Y \end{array} & \text{equals} & \begin{array}{c} \mathrm{pt} \\ \downarrow q \\ \mathrm{pt}/G[[t]] \xrightarrow{p} \mathrm{pt}/G((t)) \end{array} \end{array}$$

In addition, we work in the setting of $\mathcal{D}$ -modules rather than IndCoh .

We have

$$\begin{aligned}\mathcal{D}(X \times_Y X) &\simeq \mathcal{D}(G[[t]] \setminus G((t))/G[[t]]) \\ \mathcal{D}(T \times_Y T) &\simeq \mathcal{D}(G((t)))\end{aligned}$$

with the convolution monoidal structures on both sides. We also have the bimodule

$$\mathcal{D}(T \times_Y X) \simeq \mathcal{D}(G((t))/G[[t]]) \simeq \mathcal{D}(\mathrm{Gr}_G).$$

Lemma 3.4.2. *For $\mathcal{C} \in \mathcal{D}(G((t)))-\mathbf{mod}$, we have*

$$\begin{aligned}\mathcal{C}^{G[[t]]} &\simeq \mathrm{Hom}_{\mathcal{D}(G((t)))}(\mathcal{D}(G((t))/G[[t]]), \mathcal{C}) \\ \mathcal{C}_{G[[t]]} &\simeq \mathcal{D}(G((t))/G[[t]]) \otimes_{\mathcal{D}(G((t)))} \mathcal{C}.\end{aligned}$$

Proof. Since invariants and coinvariants for $G[[t]]$ coincide,

$$\begin{aligned}\mathrm{Vect} \otimes_{\mathcal{D}(G[[t]])} \mathcal{D}(G((t))) &\simeq \mathcal{D}(G((t)))^{G[[t]]} \\ &\simeq \mathcal{D}(G((t))/G[[t]]).\end{aligned}$$

Now tensor-Hom adjunction gives

$$\begin{aligned}
\mathcal{C}^{G[t]} &= \mathrm{Hom}_{\mathcal{D}(G[t])}(\mathrm{Vect}, \mathcal{C}) \\
&\simeq \mathrm{Hom}_{\mathcal{D}(G[t])}(\mathrm{Vect}, \mathrm{Hom}_{\mathcal{D}(G((t)))}(\mathcal{D}(G((t))), \mathcal{C})) \\
&\simeq \mathrm{Hom}_{\mathcal{D}(G((t)))}(\mathrm{Vect}_{\mathcal{D}(G[t])} \otimes_{\mathcal{D}(G[t])} G((t)), \mathcal{C}) \\
&\simeq \mathrm{Hom}_{\mathcal{D}(G((t)))}(\mathcal{D}(G((t))/G[t]), \mathcal{C}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{C}_{G[t]} &= \mathrm{Vect}_{\mathcal{D}(G[t])} \otimes_{\mathcal{D}(G[t])} \mathcal{C} \\
&\simeq \mathrm{Vect}_{\mathcal{D}(G[t])} \otimes_{\mathcal{D}(G[t])} \mathcal{D}(G((t))) \otimes_{\mathcal{D}(G((t)))} \mathcal{C} \\
&\simeq \mathcal{D}(G((t))/G[t]) \otimes_{\mathcal{D}(G((t)))} \mathcal{C}.
\end{aligned}$$

□

Lemma 3.4.3. *For $\mathcal{C} \in \mathcal{D}(G((t))\text{-mod}$, we have*

$$\begin{aligned}
\mathcal{C}^{\mathcal{D}(G((t)))} &\simeq \mathrm{Hom}_{\mathcal{D}(G[t] \setminus G((t))/G[t])}(\mathcal{D}(\mathrm{pt}/G[t]), \mathcal{C}^{G[t]}) \\
\mathcal{C}_{\mathcal{D}(G((t)))} &\simeq \mathcal{D}(\mathrm{pt}/G[t]) \otimes_{\mathcal{D}(G[t] \setminus G((t))/G[t])} \mathcal{C}_{G[t]}.
\end{aligned}$$

Proof. First, we claim that

$$\mathcal{D}(\mathrm{pt}/G[t]) \otimes_{\mathcal{D}(G[t] \setminus G((t))/G[t])} \mathcal{D}(G((t))/G[t]) \simeq \mathrm{Vect}.$$

This follows from the $\mathcal{D}$ -module version of Proposition 2.4.4 with the following substitutions: $X \rightarrow Y$ is $\mathrm{pt}/G[t] \rightarrow \mathrm{pt}/G((t))$, $T_1 = \mathrm{pt}/G((t))$, and $T_2 = \mathrm{pt}$. The proposition applies because $\mathrm{pt}/G[t] \rightarrow \mathrm{pt}/G((t))$ is ind-proper.

Next, we use Lemma 3.4.2 along with tensor-Hom adjunction for $\mathcal{D}(G((t))/G[t])$ as a bimodule for $\mathcal{D}(G((t)))$ and $\mathcal{D}(G[t] \setminus G((t))/G[t])$:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}(G[t] \setminus G((t))/G[t])}(\mathcal{D}(\mathrm{pt}/G[t]), \mathcal{C}^{G[t]}) &\simeq \mathrm{Hom}_{\mathcal{D}(G[t] \setminus G((t))/G[t])}(\mathcal{D}(\mathrm{pt}/G[t]), \mathrm{Hom}_{\mathcal{D}(G((t)))}(\mathcal{D}(G((t))/G[t]), \mathcal{C})) \\
&\simeq \mathrm{Hom}_{\mathcal{D}(G((t)))}(\mathcal{D}(\mathrm{pt}/G[t]) \otimes_{\mathcal{D}(G[t] \setminus G((t))/G[t])} \mathcal{D}(G((t))/G[t]), \mathcal{C}) \\
&\simeq \mathrm{Hom}_{\mathcal{D}(G((t)))}(\mathrm{Vect}, \mathcal{C}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{D}(\mathrm{pt}/G[t]) \otimes_{\mathcal{D}(G[t] \setminus G((t))/G[t])} \mathcal{C}^{G[t]} &\simeq \mathcal{D}(\mathrm{pt}/G[t]) \otimes_{\mathcal{D}(G[t] \setminus G((t))/G[t])} \mathcal{D}(G((t))/G[t]) \otimes_{\mathcal{D}(G((t)))} \mathcal{C} \\
&\simeq \mathrm{Vect}_{\mathcal{D}(G((t)))} \otimes_{\mathcal{D}(G((t)))} \mathcal{C}.
\end{aligned}$$

□

Lemma 3.4.4. (i) $\mathcal{D}(G((t))/G[t])$ is self-dual in $\mathcal{D}(G[t] \setminus G((t))/G[t])\text{-mod}$.

(ii) $\mathcal{D}(G[t] \setminus G((t))/G[t])$ is semi-rigid.

Proof. (i) follows from Proposition 2.4.5. Note that Proposition 2.4.4 applies since $\mathrm{pt}/G[t] \rightarrow \mathrm{pt}/G((t))$ is ind-proper. The base change property holds because $T = \mathrm{pt}$ so Δ_T is the identity.

(ii) follows from Theorem 2.4.7 since $\mathrm{pt}/G[t] \rightarrow \mathrm{pt}/G((t))$ is ind-proper and $\mathrm{pt}/G((t))$ has placid diagonal.

□

Proposition 3.4.5. *The functor $\theta_{G((t))}$ is a natural equivalence.*

Proof. By Lemma 3.4.4(i), Lemma 3.4.3, and the fact that invariants and coinvariants coincide for $G[t]$, we have

$$\begin{aligned}
\mathcal{C}^{G((t))} &\simeq \mathrm{Hom}_{\mathcal{D}(G[t] \setminus G((t))/G[t])}(\mathcal{D}(\mathrm{pt}/G[t]), \mathcal{C}^{G[t]}) \\
&\simeq \mathcal{D}(\mathrm{pt}/G[t]) \otimes_{\mathcal{D}(G[t] \setminus G((t))/G[t])} \mathcal{C}_{G[t]} \\
&\simeq \mathcal{C}_{G((t))}.
\end{aligned}$$

This is functorial in $\mathcal{C}$ so it gives a natural equivalence $\mathrm{inv}^{G((t))} \simeq \mathrm{coinv}^{G((t))}$. To check that this equivalence coincides with $\theta_{G((t))}$, one emulates the proof of Proposition 2.6.2. $\square$

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