Second adjointness for loop groups

This talk is about the following conjecture:

Conj: For $G(K) \curvearrowright \mathcal{C}$ (strongly, possibly with level κ), the composition:

$$\mathcal{C}^{N(K)} \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}_{N^-(K)}$$

is an equivalence.

I personally think of this as the major missing ingredient in our understanding of loop group actions, and that a proof would be quite revealing.

In this talk, we will discuss evidence for this conjecture and try to give it more concrete meaning. We will take $\kappa=0$ throughout. Also, we expect this conjecture to hold factorizably, but we will (mostly) ignore this aspect as well.

Reformulation by inverting operators:

The following reformulation is convenient.

Suppose V is a vector space and $T \in End(V)$.
We usually invert T via:

$$V[T^{-1}] := \operatorname{colim}\left(V \xrightarrow{T} V \xrightarrow{T} \cdots\right)$$

but we also have:

$$V\langle T^{-1}\rangle := \lim\left(\cdots \xrightarrow{T} V \xrightarrow{T} V\right).$$

(The former is a left adjoint and the latter a right adjoint.) We have $V\langle T^{-1}\rangle \to V \to V[T^{-1}]$.

Moreover, this construction makes sense for any object in any category equipped with an endomorphism.

Now suppose $K \subseteq G(O)$ is a compact open subgroup admitting a <u>triangular decomposition</u>, i.e., for:

$$K_N := K \cap N(K), \quad K_{N^-} := K \cap N^-(K)$$

$$K_T := K \cap T(K)$$

we have:

$$K = K_N \cdot K_T \cdot K_{N^-}.$$

(This is true for cofinal $K \to id$, so is a harmless assumption.)

Define $\Theta : \mathcal{C}^K \to \mathcal{C}^K$ (for $G(K) \curvearrowright \mathcal{C}$)

as convolution with $j_*(\omega_{K \cdot -2\check{\rho}(t) \cdot K})$, i.e., apply the action of $-2\check{\rho}(t) \in G(K)$ and then $*$-average back to $\mathcal{C}^K$.

Lem: 1) $(\mathcal{C}_{N^-(K)})^{K_T} = \mathcal{C}^K[\Theta^{-1}]$.

2) $\mathcal{C}^{N(K)K_T} = \mathcal{C}^K\langle\Theta^{-1}\rangle$.

Proof: We have:

$$\mathcal{C}^K[\Theta^{-1}] :=$$

$$\begin{tikzcd}
\mathrm{colim}\big(\mathcal{C}^K \arrow[r, "\Theta"] \arrow[d, equal] & \mathcal{C}^K \arrow[r, "\Theta"] \arrow[d, "\wr\, a\check{p}(t)\cdot -"] & \cdots\big) \\
\mathcal{C}^K \arrow[r, "Av_*"] & \mathcal{C}^{Ad_{a\check{p}(t)}K} \arrow[r, "Av_*"] & \cdots
\end{tikzcd}$$

The bottom is:

$$\mathrm{colim}_{n, Av_*} \mathcal{C}^{Ad_{n\check{p}(t)}K} =$$

$$\mathrm{colim}_{n, Av_*} \mathcal{C}^{(Ad_{n\check{p}(t)}K_N)\cdot K_T \cdot Ad_{n\check{p}(t)}K_{N^-}} =$$

$$\mathrm{colim}_{n,m} \mathcal{C}^{(Ad_{m\check{p}(t)}K_N)\cdot K_T \cdot Ad_{n\check{p}(t)}K_{N^-}}$$

It is straightforward to see

$$\operatorname*{colim}_{m,\,Obl_v} \mathcal{C}^{\mathrm{Ad}_{m\check{\rho}(t)} K_N} \xrightarrow{\simeq} \mathcal{C}$$

from the fact that these subgroups converge to the identity. So the above is equivalent to:

$$\operatorname*{colim}_{n,\,Av_*} \mathcal{C}^{K_T \mathrm{Ad}_{n\check{\rho}(t)} K_{N^-}} =: (\mathcal{C}_{N^-(K)})^{K_T}.$$

(Note that $\mathrm{Ad}_{n\check{\rho}(t)} K_{N^-} \xrightarrow{n} N^-(K)$.)

The second point is similar. //

The projection:

$$\mathcal{C}^{N(K) K_T} \longrightarrow (\mathcal{C}_{N^-(K)})^{K_T}$$

is compatible with the above lemma.

<u>Example</u>: Suppose $K = I$ is the Iwahori subgroup $ev^{-1}(B)$ for $B \leq G$ our Borel and $ev : G(\mathcal{O}) \longrightarrow G$ the evaluation map.

Then it is well-known that Θ is an equivalence in this case. Therefore, we obtain:

$$\mathcal{C}^{N(K)T(\mathcal{O})} \xrightarrow{\simeq} \mathcal{C}^{I} \xrightarrow{\simeq} (\mathcal{C}_{N^-(K)})^{T(\mathcal{O})}$$
$$\cong$$
$$\mathcal{C}_{N^-(K)T(\mathcal{O})}.$$

<u>Rem</u>: For $K = \overset{\circ}{I}$, I have no idea what to do. (The middle map as above is not an equivalence here.)

<u>Rem</u>: It makes sense to ask if $\mathcal{C}^{N(K)T(\mathcal{O})} \xrightarrow{\simeq} \mathcal{C}_{N^-(K)T(\mathcal{O})}$ factorizably.

We expect this to be true, but the above argument is insufficient.

Relation to Bernstein's "second adjointness"

Suppose now that K is a locally compact non-Arch. field (e.g., $\mathbb{Q}_p$), and V is a smooth representation of $\mathbb{Q}_p$.

Let $\hat{V} := \lim\limits_{\substack{K \subseteq G(O) \\ \text{cpt open}}} V^K$ with averaging (i.e., integration/summing) maps as the structure maps. We write $v \in \hat{V}$ as $\{v_K\}$.

We write $\hat{V}^{N(K), \text{naive}}$ to mean invariant vectors, and $\hat{V}^{N(K)}$ to mean T-smooth vectors in $\hat{V}^{N(K), \text{naive}}$, i.e., $\operatorname{colim}\limits_{K_T \subseteq T(O)} \hat{V}^{N(K) K_T}$.

There is a projection map:

$$\hat{V}^{N(K)} \longrightarrow V_{N^-(K)}.$$

Indeed, for $K' \leq K \leq G(\mathcal{O})$ with triangular decompositions, note that $Av_{K'}^{K}(v_{K'}) = v_K$ (for $v \in \hat{V}$), and if $v \in \hat{V}^{N(K)K_T}$, then

$Av_{K'}^{K} v = Av_{K'_{N^-}}^{K_{N^-}} v$. This makes the claim evident.

<u>Thm</u> (Bernstein): This projection is an isomorphism.

For what it's worth, the outline of the proof is the following:

First, as in the categorical setting we have:

$$(V_{N^-(K)})^{K_T} = V^K[\Theta^{-1}]$$

$$\hat{V}^{N(K)K_T} = V^K\langle\Theta^{-1}\rangle.$$

Then the serious point is to show that $\mathrm{Ker}\,\Theta^n$ and $\mathrm{Coker}\,\Theta^n$ stabilize, i.e., $\mathrm{Ker}\,\Theta^{n} \xrightarrow{\simeq} \mathrm{Ker}\,\Theta^{n+1}$ and $\mathrm{Coker}\,\Theta^n \xrightarrow{\simeq} \mathrm{Coker}\,\Theta^{n+1}$ for $n \gg 0$.

In this case, it is easy to see both sides identify with $\mathrm{Image}(\Theta^n)$ for $n \gg 0$.

Bernstein shows this stabilization by a tricky argument.

<u>Rem</u>: The above analysis can be modified to work for complexes of vector spaces. Maybe it gives a hint for the categorical setting.

The case of SL_2:

In this case, we get a "Radon transform".
More generally, suppose V/k is a finite dimensional vector space (dim $V = 2$ with a symplectic form $V \simeq V^\vee$ will correspond to SL_2).

For any prestack $\mathcal{Y}$, there are two D-module categories: $D^!(\mathcal{Y})$ and $D^*(\mathcal{Y})$. The former has upper-! functors and the latter has lower-$*$ functors; for "nice" spaces, the two are identified.

Now in the above setting, let
$$R = \{ v \in V, \lambda \in V^\vee \mid \lambda(v) = 1 \}.$$
We have the correspondence:

$$\begin{tikzcd} & \mathcal{R} \arrow[dl] \arrow[dr] & \\ V\backslash 0 & & V^\vee\backslash 0 \end{tikzcd}$$

Note that $\mathcal{R}$ is affine, while the other two terms are not (for $\dim V > 1$).
Taking loop spaces, we obtain a functor:

$$D^!((V\backslash 0)(K)) \longrightarrow D^!(\mathcal{R}(K)) \simeq$$
$$D^*(\mathcal{R}(K)) \longrightarrow D^*((V^\vee\backslash 0)(K)).$$

The isomorphism is because $\mathcal{R}(K)$ is "nice", while $(V\backslash 0)(K)$ is completely horrible (it's not an indscheme, etc.).

Conj: The above functor is an equivalence.

Exercise: Show that the Sl_2 case reduces to this claim (for $\dim V = 2$).

Rem: In the function-theoretic setting, there is a nice paper[1] of Jonathan Wang with a similar perspective.
For him, $D^!((V\setminus 0)(K))$ is replaced by functions on $V(K)$ bounded away from ∞, while $D^*((V^\vee\setminus 0)(K))$ is replaced by functions bounded away from zero.

Finite-dimensional setting

Here we have:

Exercise: Show that $D(V\setminus 0) \xrightarrow{\text{pull}} D(\mathcal{R}) \xrightarrow{\text{push}} D(V^*\setminus 0)$ is not an equivalence if $\dim V > 1$.
(E.g., show using functoriality in V that this functor does not preserve compact objects.)

1. J. Wang, "Radon inversion formulas over local fields."

This is one of the major difficulties: I don't know what finitedimensional statement I should prove.

Finally, a question:
What is the analogue for the classical limit, with LocSys?
The above does not translate directly to a reasonable conjecture.
But we should have some of equivalence:

$$QCoh\left(\left\{\begin{matrix}\text{Gauge forms}\\ \text{for } B \text{ on}\\ \mathring{D}\end{matrix}\right\}\right)_{B(K),w} \simeq QCoh\left(\left\{\begin{matrix}\text{Gauge forms}\\ \text{for } B^- \text{ on}\\ \mathring{D}\end{matrix}\right\}\right)^{B(K)/m}.$$

I think moreover that for Sl_2, it should involve the correspondence:

$$\mathcal{R}^{cl} = \left\{ \lambda \in \mathbb{A}^1 + \begin{tikzcd} & & 0 \arrow[d] & & \\ & & \sigma^\vee \arrow[d] & & \\ 0 \arrow[r] & \sigma \arrow[r] \arrow[dr, "\lambda"'] & \mathcal{E} \arrow[r] \arrow[d] & \sigma^\vee \arrow[r] & 0 \\ & & \sigma \arrow[d] & & \\ & & 0 & & \end{tikzcd} \right\} / \mathbb{G}_m$$

for $\mathbb{G}_m$ acting on the factor λ, and $\mathcal{E}$ a rank 2 local system on $\mathring{D}$ w/ trivialized ~~determinant~~ determinant, σ rank 1, and these sequences compatible with the trivialization of $\det \mathcal{E}$.

This space is not as nice as $\mathcal{R}$, but at least it evokes Simon Riche's thesis.