DUALITY FOR HEISENBERG ALGEBRAS AND GROUPS

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1. Outline

Let T/k be a torus with weight lattice Λ and coweight lattice $\check{\Lambda}$. Let $\check{T}$ be the dual torus.

Let κ be a level for T and let $\check{\kappa}$ be the dual level for $\check{T}$. (We will recall the meaning of such terms in what follows.)

Let e.g. $\widehat{\mathfrak{t}}_\kappa$ denote the corresponding Heisenberg algebra and let $\widehat{\mathfrak{t}}_\kappa\text{–}\mathsf{mod}$ denote the appropriate DG category of modules for it (always with the requirement that the central element acts by the unit).

We will show how Contou-Carrère's duality between $T(K)$ and $\check{T}(K)$ implies the following.

Theorem 1.0.1. *(1) There is a canonical equivalence of DG categories:*

$$\widehat{\mathfrak{t}}_\kappa\text{–}\mathsf{mod} \simeq \widehat{\check{\mathfrak{t}}}_{\check{\kappa}}\text{–}\mathsf{mod}.$$

(2) In (1), the natural level κ $T(K)$-action on the left hand side and the level $\check{\kappa}$ $\check{T}(K)$ action on the right hand side canonically commute.

(3) With respect to the above bimodule structure, $\widehat{\mathfrak{t}}_\kappa\text{–}\mathsf{mod}$ induces an equivalence of categories:

$$T(K)\text{–}\mathsf{mod}_\kappa \simeq \check{T}(K)\text{–}\mathsf{mod}_{-\check{\kappa}}$$

such that the diagram:

$$\begin{array}{ccc} T(K)\text{–}\mathsf{mod}_\kappa & \xrightarrow{\quad\simeq\quad} & \check{T}(K)\text{–}\mathsf{mod}_{-\check{\kappa}} \\ & {\scriptstyle \mathrm{Oblv}}\searrow \quad \swarrow{\scriptstyle (-)^{\check{T}(K),w}} & \\ & \mathsf{DGCat}_{cont} & \end{array}$$

commutes.

Comparing endomorphisms of these forgetful functors, we obtain:

Corollary 1.0.2. *There is a canonical equivalence of monoidal DG categories:*

$$D_\kappa(T(K)) \simeq \mathrm{HC}^{\mathrm{aff}}_{\check{T},\check{\kappa}}$$

where the right hand side denotes Harish-Chandra bimodules for the affine algebra $\widehat{\check{\mathfrak{t}}}_{\check{\kappa}}$.

For $\kappa = 0$, we obtain:

Corollary 1.0.3. *There is a canonical equivalence of monoidal DG categories:*

$$D(T(K)) \simeq \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}(\overset{\circ}{\mathcal{D}})).$$

(In fact, this equivalence is symmetric monoidal.)

Remark 1.0.4. Throughout these notes, we suppress renormalization issues.

Date: January 18, 2018.

2. Levels

2.1. We quickly review the formalism of levels in the abelian case (c.f. [Zha]).

A *naive level* for T is just a symmetric bilinear form κ on $\mathfrak{t}$. Recall that these are the traditional parameters for Heisenberg algebras. The problem for this definition is that it is not clear what the dual level $\check{\kappa}$ should be, or what is meant by an infinite level, etc.

Clearly we need to compactify the space of naive levels. We do this in the standard way: realize naive levels as certain maps $\kappa : \mathfrak{t} \to \mathfrak{t}^\vee = \check{\mathfrak{t}}$, replace this map by its graph, and then take a Hilbert scheme-style compactification.

What properties does the graph Γ_κ of $\kappa : \mathfrak{t} \to \mathfrak{t}$ have? Clearly it is a subspace of $\mathfrak{t} \times \check{\mathfrak{t}}$. The fact that κ is symmetric is equivalent to its graph being Lagrangian[1] for the symplectic form on $\mathfrak{t} \times \check{\mathfrak{t}}$:

$$((\xi, \lambda), (\psi, \mu)) = \mu(\xi) - \lambda(\psi).$$

Therefore, we define a *level* κ for T to be a Lagrangian subspace $\Gamma_\kappa \subseteq \mathfrak{t} \times \mathfrak{t}^\vee$. Now this notion is self-dual: the level $\check{\kappa}$ for $\check{T}$ has $\sigma(\Gamma_\kappa) = \Gamma_{\check{\kappa}}$ for σ the natural (up to sign) symplectic isomorphism:

$$\sigma : \mathfrak{t} \times \check{\mathfrak{t}} \simeq \check{\mathfrak{t}} \times \mathfrak{t}.$$

Remark 2.1.1. Geometrically, the space of levels for T is a partial flag variety for the symplectic group of $\mathfrak{t} \times \check{\mathfrak{t}}$ and naive levels for T constitute its open Bruhat cell.

2.2. Note that naive levels (for T) form a commutative group (scheme) under addition. Moreover, they act on $\mathfrak{t} \times \mathfrak{t}^\vee$ by symplectic automorphisms. Indeed, bilinear forms on $\mathfrak{t}$ are maps $\kappa : \mathfrak{t} \to \mathfrak{t}^\vee$, and these define automorphisms of $\mathfrak{t} \times \mathfrak{t}^\vee$ via $\begin{pmatrix} \mathrm{id} & 0 \\ \kappa & \mathrm{id} \end{pmatrix}$; it is immediate to see κ is symmetric if and only if this automorphism is symplectic.

In particular, for κ a naive level and κ' an arbitrary level, we may write $\kappa + \kappa'$ to be the image of κ' under the above automorphism; if κ' is a naive level, this coincides with usual addition of symmetric bilinear forms.

2.3. We now define the Heisenberg algebra $\widehat{\mathfrak{t}}_\kappa$ for a level κ.

Equip $\mathfrak{t} \times \check{\mathfrak{t}}$ with the symmetric form:

$$\langle (\xi, \lambda), (\psi, \mu) \rangle = \frac{1}{2}\big(\mu(\xi) + \lambda(\psi)\big).$$

Now Γ_κ inherits a symmetric form[2] and therefore $\Gamma_\kappa((t))$ inherits an alternating form:

$$\mathrm{Res}\langle -, d(-) \rangle$$

in the usual way. This defines a Heisenberg central extension:

$$0 \to k \to \widehat{\mathfrak{t}}_\kappa \to \Gamma_\kappa((t)) \to 0.$$

By definition, this exact sequence of vector space is equipped with a splitting, and then the above alternating form is taken as a Lie algebraic 2-cocycle.

[1] This follows from the general fact: if $S : V \to W$ is a morphism of finite-dimensional vector spaces and $S^\vee : W^\vee \to V^\vee$ the dual map, then $\Gamma_{S^\vee} \subseteq W^\vee \times V^\vee$ is obtained from $\Gamma_S \subseteq V \times W$ by taking $\Gamma_S^\perp$ and pulling it back by the inversion map along either factor $W^\vee$ or $V^\vee$.

[2] We remark that $\langle -, - \rangle$ restricted to Γ_κ can be calculated as:

$$\langle (\xi, \lambda), (\psi, \mu) \rangle = \mu(\xi) = \lambda(\psi).$$

Example 2.3.1. If κ is a naive level, then:

$$\Gamma_\kappa \hookrightarrow \mathfrak{t} \times \check{\mathfrak{t}} \xrightarrow{p_1} \mathfrak{t}$$

is an isomorphism, so $\widehat{\mathfrak{t}}_\kappa$ is a central extension of $\mathfrak{t}((t))$. The corresponding alternating form in this case may be written as $\mathrm{Res}(\kappa(-, d(-)))$.

2.4. **Duality for Heisenberg algebras.** The above constructions were completely symmetric in $\mathfrak{t}$ and $\check{\mathfrak{t}}$. Therefore, $\widehat{\mathfrak{t}}_\kappa \simeq \widehat{\check{\mathfrak{t}}}_{\check{\kappa}}$ as central extensions of $\Gamma_\kappa((t)) \simeq \Gamma_{\check{\kappa}}((t))$.

This proves Theorem 1.0.1 (1).

3. Group actions

3.1. In this section, we (formulate and) show that $T(K)$ acts with level κ on $\widehat{\mathfrak{t}}_\kappa$–$\mathsf{mod}$.

3.2. **Digression: twisted G-actions.** Fix G an algebraic group. We wish to give a general format for discussing various notions of G-action on DG categories.

3.3.

Definition 3.3.1. A *twist* for G is the datum of:

- A group inf-scheme $\widetilde{G}$
- A homomorphism $G \to \widetilde{G}$ which is a nil-isomorphism (i.e., $G \xrightarrow{\simeq} \widetilde{G}$ is an isomorphism when evaluated on reduced schemes).
- A central extension:

$$1 \to \mathbb{B}\mathbb{G}_m \to \widetilde{G}' \to \widetilde{G} \to 1$$

  equipped with a splitting over G.

A *split twisting* is a twisting, plus an extension of the splitting $G \to \widetilde{G}'$ to $\widetilde{G}$. I.e., it is just the datum $G \to \widetilde{G}$.

Remark 3.3.2. Above, we allow $\widetilde{G}$ to be in the world of derived algebraic geometry for aesthetic reasons. This is not needed for our examples.

Example 3.3.3. (1) $\widetilde{G} = G$ defines a split twisting.
(2) $\widetilde{G} = G_{dR}$ defines another split twisting.
(3) If $\mathfrak{h} \subseteq \mathfrak{g}$ is a normal subalgebra, $\widetilde{G} = G/\exp(\mathfrak{h})$ generalizes both of the above examples. (Here $\exp(\mathfrak{h})$ is the formal group associated with $\mathfrak{h}$.)

3.4. Suppose we are given a twist as above; we denote this datum by τ.

We then define G–mod_τ, the category of cocomplete DG categories with *τ-twisted G-action*, as follows.

First, the twisting is split. Then we take G–$\mathsf{mod}_\tau = \mathsf{IndCoh}(\widetilde{G})$–$\mathsf{mod}$, where the right hand side is equipped with the convolution monoidal structure.

In general, note that a $\mathbb{B}\mathbb{G}_m$-action on $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is equivalent to giving a decomposition $\mathcal{C} = \prod_{n\in\mathbb{Z}} \mathcal{C}_n$. (The sheaves $\mathcal{O}(n)$ on $\mathbb{B}\mathbb{G}_m$ define mutually commuting idempotents.) Then we take G–mod_τ as the full subcategory of $\mathsf{IndCoh}(\widetilde{G}')$–$\mathsf{mod}$ where $\mathcal{C} = \mathcal{C}_1$ in the above notation.

Example 3.4.1. Our examples are for split twists. If $\widetilde{G} = G$, then G–$\mathsf{mod}_\tau = G$–mod_{weak}. If $\widetilde{G} = G_{dR}$, then G–$\mathsf{mod}_\tau = G$–$\mathsf{mod}(:= G$–$\mathsf{mod}_{strong})$.

Remark 3.4.2. Roughly, $G\text{–}\mathsf{mod}_\tau$ is $\mathsf{ShvCat}_{/\mathbb{B}\widetilde{G}}$ in the split case. For any prestack $\mathcal{Y}$, a $\mathbb{G}_m$ 3-gerbe[3] defines a "twisted" version of $\mathsf{ShvCat}_{/\mathcal{Y}}$; and our central extension is equivalent to specifying such a 3-gerbeon $\mathbb{B}\widetilde{G}$ (split over $\mathbb{B}G$).

Remark 3.4.3. The splitting of $\widetilde{G}'$ over G implies that the functor:

$$G\text{–}\mathsf{mod}_\tau \xrightarrow{\mathcal{C}\mapsto\mathcal{C}^{G,w}} \mathsf{Vect}$$

is well-defined and conservative.

3.5. We have the following characterization of twists.

Proposition 3.5.1. *The following categories are canonically equivalent.*

(1) Split twists for G.
(2) Lie algebroids on $\mathbb{B}G$.
(3) A DG Lie algebra $\mathfrak{h}$ equipped with an action of G and a G-equivariant homomorphism $\iota : \mathfrak{h} \to \mathfrak{g}$ such that the action of $\mathfrak{h}$ on its via ι and the infinitesimal $\mathfrak{g}$ action coincides with the adjoint action, plus higher homotopical data if $\mathfrak{h}$ is not classical.

Proof. The equivalence of the first two points follows from [GR], while the third is essentially [BB] §1.8.4.

The constructions go as follows. For a twisting $G \to \widetilde{G}$, $\mathbb{B}G \times_{\mathbb{B}\widetilde{G}} \mathbb{B}G$ is a formal groupoid on $\mathbb{B}G$, and the corresponding Lie algebroid is the desired one. The corresponding Lie algebra $\mathfrak{h}$ is the Lie algebra of the formal group $\mathrm{Ker}(G \to \widetilde{G})$, (and the morphism ι is the tautological map). □

We have the following analogue in general.

Proposition 3.5.2. *The following categories are canonically equivalent.*

(1) Twists for G.
(2) Lie algebroids L on $\mathbb{B}G$ equipped with a central extension:[4]

$$0 \to \mathcal{O}_{\mathbb{B}G} \to L' \to L \to 0.$$

(3) A central extension:

$$0 \to k \to \mathfrak{h}' \xrightarrow{\pi} \mathfrak{h} \to 0$$

of DG Lie algebras acted on by G (with G acting trivially on k), and a G-equivariant morphism $\iota : \mathfrak{h} \to \mathfrak{g}$, such that ι and $\pi \circ \iota$ (compatibly) satisfy the hypotheses of Proposition 3.5.1 (3)*.*

Remark 3.5.3. In the third perspective above, ignoring higher homotopical issues, a τ-twisted G-action on $\mathcal{C}$ is an action of G and a G-equivariant trivialization of the induced action of[5] $\mathfrak{h}'$, and such that the two trivializations of the action of the Lie algebra $k = \mathrm{Lie}(\mathbb{G}_m)$ corresponds to the canonical action of $\mathbb{B}\widehat{\mathbb{G}_m}$ on $\mathcal{C}$ (through $\mathbb{B}\mathbb{G}_m$, with $\mathcal{C} = \mathcal{C}_1$ as above).

[3]Analogous to how a 2-gerbe defines a twisted version of QCoh and a 1-gerbe defines a twisted version of functions.

[4]Since our examples are classical, we are lazy in writing triangles in the derived category as short exact sequences.

[5]Here by an action of a Lie algebra $\mathfrak{h}$ on $\mathcal{C}$, we mean an action of the monoidal category $\mathsf{IndCoh}(\exp(\mathfrak{h}))$. Equivalently, this is a morphism of Lie algebras $\mathfrak{h} \to HH^\bullet(\mathcal{C})[1]$.

3.6. **Twisted Harish-Chandra data.** The main example of the above structure for our purposes is the following.

Suppose $A \in \mathsf{Vect}$ is an algebra equipped with an action of G. Suppose moreover that we are given a G-equivariant map of (DG) Lie algebras $\mathfrak{h}' \to A$ satisfying the usual (twisted, and appropriately homotopical) Harish-Chandra conditions. Then $A\text{–}\mathsf{mod}$ has a natural τ-twisted G-action.

In particular, this applies for $A = U'(\mathfrak{h})$, the twisted enveloping algebra of $\mathfrak{h}'$. We denote the category of modules in this case by $\mathfrak{h}\text{–}\mathsf{mod}'$. Then $\mathfrak{h}\text{–}\mathsf{mod}'$ is characterized by a universal property: the functor on $G\text{–}\mathsf{mod}_\tau$ it corepresents is $(-)^{G,w}$.

3.7. **Generalization to the infinite-dimensional setting.** Recall from [Ras] that there is a good notion of weak $T(K)$-action on $\mathcal{C}$. The same is true for Tate Lie algebras. We freely use the generalization of the above formalism for G replaced by a group like $T(K)$ and $\mathfrak{h}'$ (and $\mathfrak{h}$) being (classical) Tate Lie algebras.

3.8. **Back to Heisenberg algebras.** We now return to the setting of §2. Suppose κ is a level for T.

We obtain a datum as in Proposition 3.5.2 by the following construction.

Take $G = T(K)$, $\mathfrak{h} = \Gamma_\kappa((t))$, and $\mathfrak{h}' = \widehat{\mathfrak{t}}_\kappa$. The map $\iota : \mathfrak{h} \to \mathfrak{g}$ is the composition:

$$\Gamma_\kappa((t)) \hookrightarrow \mathfrak{t}((t)) \oplus \check{\mathfrak{t}}((t)) \xrightarrow{p_1} \mathfrak{t}((t)).$$

We let $T(K)$ act on $\Gamma_\kappa((t))$ trivially.

However, the action on $\widehat{\mathfrak{t}}_\kappa$ is[6] non-trivial. Specifying such an action on our central extension is equivalent to giving a homomorphism $T(K) \to \mathrm{Hom}(\Gamma_\kappa((t)), k)$. We have:

$$\mathrm{Hom}(\Gamma_\kappa((t)), k) = \Gamma_\kappa^\vee((t))dt = \big((\mathfrak{t} \times \check{\mathfrak{t}})/\Gamma_\kappa\big)((t))dt.$$

We then obtain the desired map from the homomorphism $d\log : T(K) \to \mathfrak{t}((t))dt$ and the composition:

$$\mathfrak{t} \hookrightarrow \mathfrak{t} \times \check{\mathfrak{t}} \to (\mathfrak{t} \times \check{\mathfrak{t}})/\Gamma_\kappa.$$

Example 3.8.1. If $\kappa = 0$ (in particular, κ is a naive level). Then the above action of $T(K)$ on $\widehat{\mathfrak{t}}_\kappa$ is non-trivial.

Example 3.8.2. If $\check{\kappa} = 0$, then the above action corresponds to the gauge action of $T(K)$ on $\mathfrak{t}$-valued 1-forms.

3.9. Above, we defined a twisting for $T(K)$ for any level κ. In what follows, we refer to τ-twisted $T(K)$ actions as (strong, if you like) actions of $T(K)$ with level κ.

For general reasons, $T(K)$ acts on $\widehat{\mathfrak{t}}_\kappa\text{–}\mathsf{mod}$ with level κ.

3.10. **Commutation.** We now prove Theorem 1.0.1 (2).

For tori T_1 and T_2, there is an obvious operation taking a level κ_i for T_i and producing a level $\kappa_1 \boxtimes \kappa_2$ for $T_1 \times T_2$. Note that commuting T_1 and T_2 actions with levels κ_i is equivalent to a $T_1 \times T_2$-action with level $\kappa_1 \boxtimes \kappa_2$.

Now we observe that the above construction of the level κ $T(K)$-action on $\widehat{\mathfrak{t}}_\kappa\text{–}\mathsf{mod}$ was symmetric in T and $\check{T}$.

More precisely, we can instead take $G = T(K) \times \check{T}(K)$, $\mathfrak{h}$ and $\mathfrak{h}'$ as before, ι as the map $\Gamma_\kappa((t)) \hookrightarrow \mathfrak{t}((t)) \times \check{\mathfrak{t}}((t))$, and the $T(K) \times \check{T}(K)$-action on $\widehat{\mathfrak{t}}_\kappa$ coming $d\log$ (along both factors now).

[6]Necessarily, if $\widehat{\mathfrak{t}}_\kappa$ is non-abelian.

4. Duality

4.1. We now treat the last point of Theorem 1.0.1.

4.2. **Contou-Carrère review.** Recall that there is a canonical bimultiplicative pairing:

$$T(K) \times \check{T}(K) \to \mathbb{G}_m.$$

We recall the construction in what follows.

4.3. Suppose V is a Tate vector space. Recall that there is a (trivial) $\mathbb{G}_m$ (2-)gerbe $\det(V)$. E.g., it can be defined as the groupoid of lattices $L \subseteq V$ with morphisms $L_1 \to L_2$ in this category given by points in the relative determinant line of L_1 and L_2.[7]

This construction behaves well in families. In particular, if a group indscheme G acts on V, then G acts on $\det(V)$. Therefore, we obtain a homomorphism $G \to \mathbb{B}\mathbb{G}_m = \mathrm{Aut}(\det(V))$.

Applying this for $G = \mathbb{G}_m(K)$ and its standard action (by multiplication) on $V = K$, we obtain $\mathbb{G}_m(K) \to \mathbb{B}\mathbb{G}_m$. This defines a central extension of $\mathbb{G}_m(K)$ by $\mathbb{G}_m$, and its commutator induces a bimultiplicative pairing:

$$(-,-) : \mathbb{G}_m(K) \times \mathbb{G}_m(K) \to \mathbb{G}_m.$$

This is the Contou-Carrère pairing (or "tame symbol"), and the desired pairing for $T = \mathbb{G}_m$; we refer to [BBE] for more details.

For a general torus, there is clearly a unique pairing characterized by:

$$(\check{\lambda}(f), \mu(g)) = (f,g)^{\check{\lambda}(\mu)}$$

for $f, g \in \mathbb{G}_m(K)$.

4.4. In particular, we obtain a canonical (invertible) function on $T(K) \times \check{T}(K)$. This induces a map (with continuous dual on the left hand side):

$$\mathrm{Fun}(T(K))^\vee \to \mathrm{Fun}(\check{T}(K)).$$

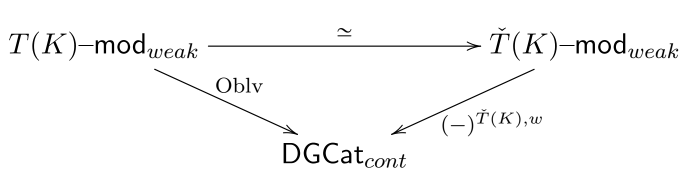
This map is well-known to be an isomorphism.

Exercise 4.4.1. Show this by filtering $T(K)$ as a group scheme.

Exercise 4.4.2. Show that the above induces an equivalence $\mathsf{QCoh}(T(K))^\heartsuit \simeq \mathsf{Rep}(\check{T}(K))^\heartsuit$, and that this equivalence extends to derived categories.

(Part of the exercise is defining the t-structure on QCoh to make this result true.)

Exercise 4.4.3. Show that the above equivalence extends to an equivalence $T(K)\text{–}\mathsf{mod}_{weak} \simeq \check{T}(K)\text{–}\mathsf{mod}_{weak}$ such that the diagram:



commutes. Show that *neither* of the vertical arrows is conservative.

[7]There is a nicer K-theoretic construction. Roughly, the DG category of Tate vector spaces is the pushout of Vect with Vect^{op} along $\mathsf{Vect}^c \simeq \mathsf{Vect}^{c,op}$ (for $\mathsf{Vect}^c \subseteq \mathsf{Vect}$ the subcategory of bounded complexes of finite-dimensional vector spaces). Then $K(\mathsf{Vect}) = K(\mathsf{Vect}^{op})$ are both trivial, and we are then using the determinant map $K(\mathsf{Vect}^c) \to \mathbb{B}k^\times$, and similarly in families.

For this reason, we write the above equivalence as:

$$(-)^{T(K),w} : T(K)\text{–}\mathsf{mod}_{weak} \to \check{T}(K)\text{–}\mathsf{mod}_{weak}.$$

4.5. We now complete the proof of Theorem 1.0.1.

First, note that by T $\widehat{\mathfrak{t}}_\kappa$–$\mathsf{mod}$ is a bimodule for the left action of $T(K)$ with level κ and the right action of $\check{T}(K)$ with level $-\check{\kappa}$ (the sign is because we have exchanged left and right actions). Therefore, we obtain the functor:

$$\check{T}(K)\text{–}\mathsf{mod}_{-\check{\kappa}} \to T(K)\text{–}\mathsf{mod}_{\kappa}$$
$$\check{\mathcal{C}} \mapsto \widehat{\mathfrak{t}}_\kappa\text{–}\mathsf{mod} \underset{T(K),-\check{\kappa}}{\otimes} \check{\mathcal{C}} = \check{\mathcal{C}}^{\check{T}(K),w}.$$

(Here we are writing tensor product of left and right modules.)

This functor fits into the diagram:

$$\begin{array}{ccc} \check{T}(K)\text{–}\mathsf{mod}_{-\check{\kappa}} & \xrightarrow{(-)^{\check{T}(K),w}} & T(K)\text{–}\mathsf{mod}_{\kappa} \\ \downarrow{\scriptstyle \mathrm{Oblv}} & & \downarrow{\scriptstyle \mathrm{Oblv}} \\ \check{T}(K)\text{–}\mathsf{mod}_{weak} & \xrightarrow{(-)^{\check{T}(K),w}} & T(K)\text{–}\mathsf{mod}_{weak} \end{array} . \qquad (4.5.1)$$

Remark 4.5.1. Note that the bottom arrow of this diagram is an equivalence and that the vertical arrows are conservative.

4.6. Note that we have the standard duality $\widehat{\mathfrak{t}}_\kappa\text{–}\mathsf{mod}^\vee \simeq \widehat{\mathfrak{t}}_{-\kappa}\text{–}\mathsf{mod}$. Moreover, this upgrades to a duality of bimodules with appropriate levels.

Therefore, it follows that our functor $(-)^{T(\check{K}),w} : \check{T}(K)\text{–}\mathsf{mod}_{-\check{\kappa}} \to T(K)\text{–}\mathsf{mod}_{\kappa}$ has a right (and actually, simultaenously left) adjoint given by the formula $(-)^{T(K),w}$. Moreover, this functor also makes the diagram:

$$\begin{array}{ccc} \check{T}(K)\text{–}\mathsf{mod}_{-\check{\kappa}} & \xleftarrow{(-)^{T(K),w}} & T(K)\text{–}\mathsf{mod}_{\kappa} \\ \downarrow{\scriptstyle \mathrm{Oblv}} & & \downarrow{\scriptstyle \mathrm{Oblv}} \\ \check{T}(K)\text{–}\mathsf{mod}_{weak} & \xleftarrow{(-)^{T(K),w}} & T(K)\text{–}\mathsf{mod}_{weak} \end{array}$$

commute. Moreover, this diagram commutes compatibly with the adjunctions and the diagram (4.5.1). Then the formal observations of Remark 4.5.1 imply the claim.

Exercise 4.6.1. Show that, in contrast to Exercise 4.4.3, the functors Oblv and $(-)^{T(K),w}$ *are* conservative on $T(K)\text{–}\mathsf{mod}_\kappa$. (Disclaimer: I only really checked this for $T = \mathbb{G}_m$.)

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