DAY III, TALK 3: SPECTRAL SIDE IN THE CLASSICAL CASE (GL-6)

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Contents

Summary

In the case classical limit, the 'naive' guess about the statement of the geometric Langlands conjecture is false. In this talk, we discuss how to correct this matter. The correction is required in both settings: global and local. The global case is easier: all objects are rigorously defined. In the local case, there is a framework, but it must be applied in a situation outside of its 'comfort zone', and it is not clear how to do this.

Since the talk is concerned with the classical limit, it is essentially independent from the rest of the conference.

1. Global correspondence

1.1. **Formulation.** Recall that the global quantum Langlands correspondence is expected to be an equivalence

$$\text{D-mod}(\text{Bun}_G)_\kappa \simeq \text{D-mod}(\text{Bun}_{\check{G}})_{-\check{\kappa}}.$$

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As $\kappa \to \infty$, the correspondence degenerates into the global classical Langlands correspondence

$$\mathrm{QCoh}(\mathrm{LocSys}_G) \simeq \text{D-mod}(\mathrm{Bun}_{\check{G}}).$$

(The critical twist on the right-hand side can safely be ignored.) However, the conjecture cannot possibly be correct as stated: the left-hand side is too small. To match the right-hand side, it needs to be enlarged as follows:

Conjecture 1. *There is an equivalence*

$$\mathrm{IndCoh}_{\mathrm{Nilp}^{glob}}(\mathrm{LocSys}_G) \simeq \text{D-mod}(\mathrm{Bun}_{\check{G}}).$$

(Of course, there is more to be said: the properties of this equivalence, why it is better than the original form, etc.) This was actually covered in more detail at the 2014 school in Jerusalem, the notes are available online (Or see [1] for all the technicalities). However, let us quickly review the general structure of this enlargement.

1.2. **Ind-coherent sheaves.** Let Z be a reasonable scheme. The compact objects in $\mathrm{QCoh}(Z)$ are perfect sheaves:

$$\mathrm{QCoh}(Z)^c = \mathrm{Perf}(Z).$$

The category is compactly generated, so

$$\mathrm{QCoh}(Z) = \mathrm{Ind}(\mathrm{Perf}(Z)).$$

Being a perfect sheaf is a kind of 'smallness' condition on a quasicoherent sheaf. However, there is another, weaker, smallness condition, which is sometimes more reasonable: the condition of being a *coherent* sheaf. (In the more 'classical' language, this means being bounded with coherent cohomology.) The two categories are different if Z is singular. By definition, the category of ind-coherent sheaves is the ind-completion of the category of coherent sheaves:

$$\mathrm{IndCoh}(Z) = \mathrm{Ind}(\mathrm{Coh}(Z)).$$

It is important to point out that although $\mathrm{Coh}(Z)$ starts as a subcategory of $\mathrm{QCoh}(Z)$, the category $\mathrm{IndCoh}(Z)$ is larger. The relation between the categories is summarized by two adjoint functors:

$$\Xi : \mathrm{QCoh}(Z) \rightleftarrows \mathrm{IndCoh}(Z) : \Psi.$$

Ξ is fully faithful; what is slightly misleading is that $\Psi|_{\mathrm{Coh}(Z)}$ is fully faithful as well; this is why we normally view $\mathrm{Coh}(Z)$ as a subcategory in $\mathrm{QCoh}(Z)$. However, $\Xi \circ \Psi|_{\mathrm{Coh}(Z)}$ is not the identity.

1.3. **Singular support.** Suppose now that Z is a local complete intersection (perhaps in the dg sense: *quasi-smooth*). In this case, there is a precise way to describe categories that are intermediate between $\mathrm{Perf}(Z)$ and $\mathrm{Coh}(Z)$.

Let $H^{-1}T^*Z$ be the shifted cotangent bundle to Z. It is a 'vector bundle' over Z (but not really, its fiber is a vector space of variable dimension) that is non-zero over the singular locus of Z. To any coherent sheaf $F \in \mathrm{Coh}(Z)$ we can assign its *singular support*, which is a closed conical subset $\mathrm{SingSupp}(F) \subset H^{-1}T^*Z$. Conversely, given any closed conical subset $C \subset H^{-1}T^*Z$, we obtain a full subcategories

$$\mathrm{Coh}_C(Z) := \{F \in \mathrm{Coh}(Z) : \mathrm{SingSupp}(F) \subset C\}$$

and

$$\mathrm{IndCoh}_C(Z) := \mathrm{Ind}(\mathrm{Coh}_C(Z)).$$

All of the above constructions extend to stacks, and we apply them to $\mathrm{LocSys}_G(X)$, which is quasi-smooth. It turns out that

$$H^{-1}T^* \,\mathrm{LocSys}_G(X) = \{(L, A) : L \in \mathrm{LocSys}_G(X), A \in \Gamma(X_{dR}, \mathfrak{g}_L)\};$$

that is, the 'shifted covector' A is a (horizontal) infinitesimal symmetry of the local system L. We now put

$$\mathrm{Nilp}^{glob} := \{(L, A) : A \text{ is nilpotent}\} \subset H^{-1}T^* \,\mathrm{LocSys}_G(X).$$

This is a closed conical subset, so the category

$$\mathrm{IndCoh}_{\mathrm{Nilp}^{glob}}(\mathrm{LocSys}_G(X))$$

on the left-hand side of Conjecture 1 makes sense.

2. Local correspondence: the stack of local systems

2.1. **Formulation.** Now let us look at the local version. Recall that the quantum local correspondence is expected to be an equivalence of two 2-categories

$$\mathfrak{L}(G) - \mathrm{mod}_\kappa \simeq \mathfrak{L}(\check{G}) - \mathrm{mod}_{-\check{\kappa}}\,.$$

As $\kappa \to \infty$, we may 'naively' expect the following.

Conjecture 2 (Wrong)**.** *There is an equivalence of 2-categories*

$$\mathrm{ShvCat}(\mathrm{LocSys}_G(\mathring{\mathrm{D}})) \simeq \mathfrak{L}(\check{G}) - \mathrm{mod}\,.$$

Here $\mathring{\mathrm{D}}$ is the punctured formal disk, $\mathrm{LocSys}_G(\mathring{\mathrm{D}})$ is the stack of local systems on $\mathring{\mathrm{D}}$, and $\mathrm{ShvCat}(\mathrm{LocSys}_G(\mathring{\mathrm{D}}))$ is the 2-category of sheaves of categories over it.

2.2. **Spherical example.** Before trying to make sense of all of this, here is an example. Let the marked point $* \in \mathrm{LocSys}_G(\mathring{\mathrm{D}})$ correspond to the trivial local system. As a stack, $* = BG$ (the trivial local system has automorphisms). For this reason, the full subcategory

$$\mathcal{C} := \mathrm{ShvCat}(\mathrm{LocSys}_G(\mathring{\mathrm{D}}))_* \subset \mathrm{LocSys}_G(\mathring{\mathrm{D}}) - \mathrm{ShvCat}$$

consisting of categories supported over $*$ is equivalent to $\mathrm{ShvCat}(BG)$. (However, see Remark 3.)

On the other side of the correspondence, $\mathcal{C}$ corresponds to the full subcategory

$$\check{\mathcal{C}} := \mathfrak{L}\check{G} - \mathrm{mod}^{sph}$$

of *spherical* representations $\mathfrak{L}\check{G}$-modules; that is, representations that are generated by $\mathfrak{L}^+\check{G}$-invariants. As in the classical theory, such representations are actually modules over the corresponding Hecke algebra (well, monoidal category), which in this case is the spherical Hecke category

$$\mathcal{H} := \text{D-mod}(\mathrm{Gr}_{\check{G}})^{\mathfrak{L}^+(\check{G})}$$

of $\mathfrak{L}^+(\check{G})$ bi-invariant $\mathcal{D}$-modules on $\mathfrak{L}(\check{G})$.

Thanks to the Satake equivalence, we have

$$\mathcal{H} \simeq \mathrm{Rep}(G).$$

Note that $\mathrm{Rep}(G) = \mathrm{QCoh}(BG)$. Now, the stack BG is 1-affine (this means there is an equivalence $\mathrm{ShvCat}(BG) \simeq \mathrm{QCoh}(BG) - \mathrm{mod}$). This completes the proof of the 'spherical' part of the local classical geometric Langlands correspondence: it is given by the chain of equivalences

$$\mathcal{C} \simeq \mathrm{ShvCat}(BG) \simeq \mathrm{QCoh}(BG) - \mathrm{mod} = \mathrm{Rep}(G) - \mathrm{mod} \simeq \mathcal{H} - \mathrm{mod} \simeq \check{\mathcal{C}}.$$

As we see, modulo some general claims, it reduces to the Satake equivalence.

Remark 3. I cheated several times in the above argument. First of all, $*$ is not an isolated point, so the category $\mathcal{C}$ is not just $\mathrm{ShvCat}(BG)$ (equivalently, the group of automorphisms of the trivial local system has some derived directions). Secondly, the Hecke category is more complicated than $\mathrm{Rep}(G)$; the complexity is captured by the derived Satake equivalence. Finally, the statement of Conjecture 2 needs a 'singular support correction' (Conjecture 11).

It is a nice exercise to verify that these cheats actually cancel out.

2.3. **Stack of local systems on the punctured disk.** Let us now look at the stack $\mathrm{LocSys}(\overset{\circ}{\mathrm{D}})$. (The group G is going to be fixed, so we omit it.) Technically, it can be defined as the quotient of connection matrices by gauge equivalence:

$$\mathrm{LocSys}(\overset{\circ}{\mathrm{D}}) = \{d + \mathfrak{g}((t))dt\}/\mathfrak{L}G.$$

The point is, this quotient is scary. Well, perhaps not scarier than most things at this conference, but we are trying to treat it as an algebraic stack. For instance, the problem of classifying connections (the most familiar case is $G = \mathrm{GL}(n)$, when it is known as the Turrittin-Levelt classification) describes $\mathbb{k}$-points $\mathrm{LocSys}(\overset{\circ}{\mathrm{D}})(\mathbb{k})$. However, the description is clearly 'stratum-by-stratum', and it seems completely impossible to understand the geometry of how the strata attach to each other. For instance, I have only a very vague idea of what $\mathrm{LocSys}(\overset{\circ}{\mathrm{D}})(\mathbb{A}^1)$ looks like.

Despite this, it turns out that *the category of quasicoherent sheaves* on $\mathrm{LocSys}(\overset{\circ}{\mathrm{D}})$ is very well behaved.

Theorem 4 (Sam Raskin, [3])**.** *For any reductive group G,*

- *The category* $\mathrm{QCoh}(\mathrm{LocSys}(\overset{\circ}{\mathrm{D}}))$ *is compactly generated;*
- *The stack* $\mathrm{LocSys}(\overset{\circ}{\mathrm{D}})$ *is 'weakly 1-affine' in the sense that the localization functor*

$$\mathrm{QCoh}(\mathrm{LocSys}(\overset{\circ}{\mathrm{D}})) - \mathrm{mod} \to \mathrm{ShvCat}(\mathrm{LocSys}(\overset{\circ}{\mathrm{D}}))$$

is fully faithful.

Because of Theorem 4, we should not abandon all hope concerning Conjecture 2.

3. Local correspondence: singular support

Unfortunately, Conjecture 2 cannot hold as stated. It needs a 'singular support' correction, similar to the correction included in Conjecture 1.

3.1. **Singular support of categories over a smooth space.** Inspired by Theorem 4, we will first pretend that $\mathrm{LocSys}(\overset{\circ}{\mathrm{D}})$ is a smooth space. (The stack $\mathrm{LocSys}(\overset{\circ}{\mathrm{D}})$ is indeed smooth, the pretense is ignoring its non-algebraic nature.) In this case, there is a theory of singular support for categories (but there is no reference on this so far, although certain points are similar to the work of Kapustin-Rozansky-Saulina).

Here is a summary. Let Z be a smooth variety. We have the 2-category $\mathrm{ShvCat}(Z)$ of categories over Z. It turns out that Z is 1-affine, so $\mathrm{ShvCat}(Z) \simeq \mathrm{QCoh}(Z) - \mathrm{mod}$. For any isotropic conical closed subset

$$C \subset T^*Z$$

(no shift, the usual cotangent bundle!) we will define the 2-category $\mathrm{ShvCat}(Z)_C$ such that $\mathrm{ShvCat}(Z)_0 = \mathrm{ShvCat}(Z)$ and $\mathrm{ShvCat}(Z)_{C_1} \subset \mathrm{ShvCat}(Z)_{C_2}$ (full embedding) for $C_1 \subset C_2$.

Remark 5. So if you want, put $\mathcal{A} := \varinjlim \mathrm{ShvCat}(Z)_C$, and treat $\mathrm{ShvCat}(Z)_C \subset \mathcal{A}$ as the full 2-subcategory of 'categories over Z with singular support in C'.

Here is the definition of $\mathrm{ShvCat}(Z)_C$ for special class of subsets C.

Definition 6. Let Y be another smooth variety, and let $p : Y \to Z$ be a proper map. Consider the fiber product

$$Y \times_Z Y.$$

It has a natural structure of a groupoid over Y; it yields a convolution product on the category $\mathrm{IndCoh}(Y \times_Z Y)$ of ind-coherent sheaves. Put

$$\mathrm{ShvCat}(Z)_C := \mathrm{IndCoh}(Y \times_Z Y) - \mathrm{mod},$$

where $C = N_Y^\vee \subset T^*Z$ is the 'conormal bundle' to Y defined as follows:

$$N^\vee Y := \{(z \in Z, \xi \in T^*Z) : \xi \perp dp(T_yY) \text{ for some } y \in p^{-1}(z)\}.$$

The distinction between indcoherent and quasicoherent sheaves is crucial here: if we consider $\mathrm{QCoh}(Y \times_Z Y)$-modules instead (again, using the convolution structure), a version of proper descent will apply: the 2-category $\mathrm{QCoh}(Y \times_Z Y) - \mathrm{mod}$ is equivalent to the full 2-subcategory

$$\mathrm{ShvCat}(Z)_{p(Y)} \subset \mathrm{ShvCat}(Z)$$

of sheaves of categories supported over $p(Y) \subset Z$, so that we don't get anything new.

Remark 7. This can also be viewed as a kind of Morita equivalence: the full 2-subcategory $\mathrm{ShvCat}(Z)_{p(Y)}$ is generated by

$$\mathrm{QCoh}(Y) \in \mathrm{QCoh}(Z) - \quad \mathrm{mod} \ = \mathrm{ShvCat}(Z),$$

where $\mathrm{QCoh}(Y)$ is considered as a $\mathrm{QCoh}(Z)$-module under p^*, and

$$\mathrm{QCoh}(Y \times_Z Y) = \mathrm{Hom}_{\mathrm{QCoh}(Z)}(\mathrm{QCoh}(Y), \mathrm{QCoh}(Y)).$$

Note also that while Y and Z are smooth, the fiber product $Y \times_Z Y$ is only quasi-smooth: this is why the category $\mathrm{IndCoh}(Y \times_Z Y)$ is larger than $\mathrm{QCoh}(Y \times_Z Y)$. Using the notion of singular support, we can measure exactly the difference between the two categories. It is easy to see that $H^{-1}T^*(Y \times_Z Y)$ naturally embeds into $T^*Z \times_Z Y \times_Z Y$; the embedding identifies it with the set of collections (z, ξ, y_1, y_2) such that $y_1, y_2 \in Y$, $p(y_1) = p(y_2) = z \in Z$, $\xi \in T_z^*Z$, and $\xi \perp dpT_{y_i}Y$ for $i = 1, 2$.

In particular, we see that the conormal bundle $N^\vee Z$ is equal to the projection of $H^{-1}T^*(Y \times_Z Y)$. We can use this observation to define $\mathrm{ShvCat}(Z)_C$ for arbitrary isotropic C by decreasing the monoidal category:

Definition 8. Let $C \subset T^*Z$ be arbitrary conical isotropic closed subset. Choose smooth Y equipped with a projection $p : Y \to Z$ such that $C \subset N^\vee Y$. Put

$C' := \operatorname{Sing} H^{-1}T*(Y\times_Z Y)\cap C\times_Z Y\times_Z Y \subset \operatorname{Sing} H^{-1}T*(Y\times_Z Y) \subset T^*Z\times_Z Y\times_Z Y.$

Consider the full subcategory

$$\operatorname{IndCoh}_{C'}(Y \times_Z Y) \subset \operatorname{IndCoh}(Y \times_Z Y).$$

The convolution product turns it into a monoidal category, and we put

$$\operatorname{ShvCat}(Z)_C := \operatorname{IndCoh}_{C'}(Y \times_Z Y) - \mathrm{mod}\,.$$

Remark 9. The embedding $\operatorname{IndCoh}_{C'}(Y \times_Z Y) \subset \operatorname{IndCoh}(Y \times_Z Y)$ admits a right adjoint $\Psi_{C'}$. The unit object in the category $\operatorname{IndCoh}(Y \times_Z Y)$ is $\Delta_*\omega_Y$, while the unit object of $\operatorname{IndCoh}_{C'}(Y \times_Z Y)$ is $\Psi_{C'}(\Delta_*\omega_Y)$. In fact,

$$\operatorname{IndCoh}_{C'}(Y \times_Z Y) = \operatorname{IndCoh}_C(Y \times_Z Y) \star \Psi_{C'}(\Delta_*\omega_Y).$$

Theorem 10. *The* 2*-category* $\operatorname{ShvCat}(Z)_C$ *depends only on* C*, and not on the choice of* Y*. It also has the properties announced above: if* $C_1 \subset C_2$*, there is a natural full embedding* $\operatorname{ShvCat}(Z)_{C_1} \hookrightarrow \operatorname{ShvCat}(Z)_{C_2}$*, while for the zero section, we have* $\operatorname{ShvCat}(Z)_0 = \operatorname{ShvCat}(Z)$*.*

3.2. **Simple example.** Suppose $Z = \operatorname{Spec} \Bbbk[t]$, and $Y = \operatorname{Spec} \Bbbk$ embedded into Z as the point 0. In this case, $C = N^\vee Y = T_0^*Z$.

The fiber product $Y \times_Z Y$ is the dg-scheme $\operatorname{Spec} \Bbbk[\epsilon]$, $\deg(\epsilon) = -1$. The category of $\operatorname{IndCoh}(Y \times_Z Y)$ can be understood via the Koszul transform: Let $B = \operatorname{Hom}_{\Bbbk[\epsilon]}(\Bbbk, \Bbbk)$ be the Koszul dual ring; explicitly, $B = \Bbbk[\eta]$ for $\deg(\eta) = 2$. The category $\operatorname{IndCoh}(Y \times_Z Y)$ (resp. $\operatorname{QCoh}(Y \times_Z Y)$) is identified with $B - \mathrm{mod}$ (resp. with the full subcategory of $B - \mathrm{mod}$ consisting of modules supported at zero).

The identification intertwines the convolution product with the tensor product of B-modules. We have the following equivalences of categories:

- $\operatorname{ShvCat}(Z)$ is the 2-category of $\Bbbk[t]$-linear categories;
- $\operatorname{ShvCat}(Z)_0$ is the 2-category of $\Bbbk[t]$-linear categories supported at 0. By definition, the category $\mathcal{C}$ is supported at 0 if
$$(\Bbbk[t, t^{-1}] - \mathrm{mod}) \otimes_{\Bbbk[t]-\mathrm{mod}} \mathcal{C} = 0.$$
one can check that this is equivalent to the vanishing
$$\Bbbk[t, t^{-1}] \otimes c = \varinjlim(c \xrightarrow{t} c \xrightarrow{t} c \xrightarrow{t} \dots) = 0 \qquad \text{for all } c \in C.$$
- Using the Koszul transform, we can alternatively view $\operatorname{ShvCat}(Z)_0$ as the category of $\Bbbk[\eta]$-linear categories supported at 0. The correspondence sends the $\Bbbk[t]$-linear category $\mathcal{C}$ to
$$(\Bbbk - \mathrm{mod}) \otimes_{\Bbbk[t]-\mathrm{mod}} \mathcal{C}.$$
- $\operatorname{ShvCat}(Z)_C$ is the 2-category of all $\Bbbk[\eta]$-linear categories.

3.3. **Formulation of the local geometric Langlands correspondence.** We can now correct Conjecture 2; however, this requires applying the techniques sketched in this section to $\operatorname{LocSys}_G(\mathring{\mathrm{D}})$; at this point, it is not clear how to make this rigorous.

Similar to the global case, it is easy to check that

$$T^* \operatorname{LocSys}_G(\mathring{\mathrm{D}}) = \{(L, A) : L \in \operatorname{LocSys}_G(\mathring{\mathrm{D}}), A \in \Gamma(X_{dR}, \mathfrak{g}_L)\};$$

that is, the covector A is a (horizontal) infinitesimal symmetry of the local system L. We now put

$$\mathrm{Nilp}^{loc} := \{(L, A) : A \text{ is nilpotent}\} \subset T^* \mathrm{LocSys}_G(\mathring{\mathrm{D}}).$$

This is a closed conical isotropic subset, and we can now state the following conjecture.

Conjecture 11. *There is an equivalence of 2-categories*

$$\mathrm{ShvCat}(\mathrm{LocSys}_G(\mathring{\mathrm{D}}))_{\mathrm{Nilp}^{loc}} \simeq \mathfrak{L}(\check{G}) - \mathrm{mod}\,.$$

4. Affine Hecke algebra

We can reinterpret the results of Bezrukavnikov [2] in this language. The main result of [2] can be stated as follows.

Let $I \subset \mathfrak{L}^+\check{G}$ be the Iwahori subgroup, so that it fits into the pullback square

$$\begin{tikzcd} I \arrow[r] \arrow[d] & \mathfrak{L}^+\check{G} \arrow[d] \\ \check{B} \arrow[r] & \check{G}. \end{tikzcd}$$

Consider the corresponding Hecke category

$$\mathfrak{H} := \text{D-mod}(I\backslash \mathfrak{L}\check{G}/I),$$

which may be viewed as the geometrization of the affine Hecke algebra. The monoidal structure on $\mathfrak{H}$ is given by the convolution of $\mathcal{D}$-modules.

Theorem 12 (Bezrukavnikov)**.** *There is a monoidal equivalence*

$$\mathfrak{H} \simeq \mathrm{IndCoh}((\mathrm{Spr} \times_{\mathfrak{g}} \mathrm{Spr})/G);$$

here $\mathrm{Spr} = T^*(G/B)$ *is the Springer variety.*

The right-hand side of Theorem 12 fits into the framework of singular support for categories. Indeed, put $Z := \mathfrak{g}/G$, and $Y = \mathrm{Spr}/G$. (Note that we are using the framework in the setting of smooth stacks rather than varieties.) The natural map $p : Y \to Z$ is proper. Let us identify $\mathfrak{g}$ with its dual; we then have

$$T * Z = \{(z, \xi) \in \mathfrak{g} \times \mathfrak{g} : [\xi, z] = 0\}/G$$

and

$$N^\vee Y = \{(z, \xi) : [\xi, z] = 0, \xi \text{ and } z \text{ are nilpotent.}\}$$

In this way, we get a corollary of Theorem 12:

Corollary 13. *There is an equivalence of* 2*-categories*

$$\mathfrak{H} - \mathrm{mod} \simeq \mathrm{ShvCat}(\mathfrak{g}/G)_{N^\vee Y}.$$

Similar to Section 2.2, $\mathfrak{H} - \mathrm{mod}$ identifies with the full subcategory of $\mathfrak{L}\check{G} - \mathrm{mod}$ consisting of categories that are generated by I-invariants. On the other side of the correspondence, there is a natural map

$$\mathfrak{g}/G \to \mathrm{LocSys}_G(\mathring{\mathrm{D}}) : x \mapsto d + x\frac{dt}{t}.$$

The map is an isomorphism on the formal neighborhood of the nilpotent cone of $\mathfrak{g}$. We can now see that Corollary 13 in fact verifies Conjecture 11 on the formal neighborhood of local systems with regular singularity and nilpotent residue.

References

[1] D. Arinkin and D. Gaitsgory, Singular support of coherent sheaves and the geometric Langlands conjecture, Selecta Mathematica 21, no. 1 (2015): 1–199.

[2] R. Bezrukavnikov, On two geometric realizations of an affine Hecke algebra. Publ. Math. Inst. Hautes Études Sci. 123 (2016): 1–67.

[3] S. Raskin, On the notion of spectral decomposition in local geometric Langlands. arXiv:1511.01378.