

WEAK ACTIONS OF LOOP GROUPS

SAM RASKIN

1. PROFINITE GROUPS

1.1. **Weak actions of $G(O)$.** Let G be an affine algebraic group.

This section revolves around the following key definition, whose meaning will be given in what follows.

Definition 1.1.1. The category of (DG) categories with a weak action of $G(O)$ is:

$$G(O)\text{-mod}_{weak} := \operatorname{colim}_n G_n\text{-mod}_{weak}.$$

Explanations are in order. First, $G(O) = \lim_n G_n$, where $G_n = \{\operatorname{Spec}(k[[t]]/t^{n+1}) \rightarrow G\}$.

Note that G_n is a finite type group over k , so we know what DG categories with weak G_n action are. I.e., $G_n\text{-mod}_{weak} = \operatorname{QCoh}(G_n)\text{-mod}$, where $\operatorname{QCoh}(G_n)$ is considered as an algebra in $\operatorname{DGCat}_{cont}$ under convolution.

The colimit in the above definition is taken in¹ $\operatorname{DGCat}_{cont}\text{-mod}$, where in this notation we are considering $\operatorname{DGCat}_{cont}$ as a (commutative) algebra in the symmetric monoidal category of cocomplete categories and functors commuting with arbitrary colimits. (Note that $\operatorname{DGCat}_{cont}\text{-mod}$ may be regarded as a higher categorical analogue of $\operatorname{DGCat}_{cont}$ itself.) The functors in this colimit are restriction maps.

Remark 1.1.2. The above definition is taken in analogy with the notion of *smooth representation* of a profinite group H . Indeed, for $H = \lim_i H_i$ with H_i finite, the abelian category of smooth representations of H is $\operatorname{colim} H_i\text{-mod}^\heartsuit$, where the colimit is taken in the category of cocomplete abelian categories and functors commuting with all colimits.

1.2. There is a tautological forgetful functor:

$$\operatorname{Oblv} : G(O)\text{-mod}_{weak} \rightarrow \operatorname{DGCat}_{cont}.$$

This functor upgrades to a functor:

$$G(O)\text{-mod}_{weak} \rightarrow \operatorname{QCoh}(G(O))\text{-mod}. \quad (1.2.1)$$

Warning 1.2.1. The main thing to be careful about in dealing with weak $G(O)$ -actions is that this functor is *not* conservative. (Indeed, this follows from [Gai] Theorem 2.2.3 and our analysis below.)

We typically omit the notation Oblv , and when we write $\mathcal{C} \in G(O)\text{-mod}_{weak}$ we also use $\mathcal{C}$ to denote what should be called $\operatorname{Oblv}(\mathcal{C})$. We say $G(O)$ acts weakly on $\mathcal{C}$, and so on.

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¹We are neglecting set-theoretic considerations here.

1.3. **Representations of $G(O)$.** Define $\text{Rep}(G(O))$ similarly:

$$\text{Rep}(G(O)) := \text{colim}_n \text{Rep}(G_n) \in \text{DGCat}_{\text{cont}} \quad (1.3.1)$$

under forgetful functors.

Each of the structural functors in the colimit (1.3.1) are symmetric monoidal, t -exact, and preserve compact objects. Therefore, we obtain:

Proposition 1.3.1. *$\text{Rep}(G(O))$ has a canonical symmetric monoidal structure $\otimes$ and t -structure such that the forgetful functor $\text{Oblv} : \text{Rep}(G(O)) \rightarrow \text{Vect}$ is symmetric monoidal and t -exact. Moreover, $\text{Rep}(G(O))$ is compactly generated with compact objects the bounded complexes of finite-dimensional representations. (In particular, $\text{Rep}(G(O))$ is rigid monoidal, i.e., its tensor unit is compact and compact objects are closed under the monoidal operation.)*

Warning 1.3.2. The natural functor $\text{Rep}(G(O)) \rightarrow \text{QCoh}(\mathbb{B}G(O))$ is an equivalence on bounded below derived categories but is *not* an equivalence.² Note that the right hand side is canonically equivalent to $\text{Fun}(G(O))\text{-mod}$, so $\text{Rep}(G(O))$ is something slightly more subtle.

1.4. We now have the following convenient description of $G(O)\text{-mod}_{\text{weak}}$.

Proposition 1.4.1. *There is an equivalence:*

$$G(O)\text{-mod}_{\text{weak}} \xrightarrow{\mathcal{C} \mapsto \mathcal{C}^{G(O),w}} \text{Rep}(G(O))\text{-mod}.$$

Proof. For each n , we have the equivalence:

$$G_n\text{-mod}_{\text{weak}} \xrightarrow{\mathcal{C} \mapsto \mathcal{C}^{G_n,w}} \text{Rep}(G_n)\text{-mod}.$$

Then because $\text{Rep}(G(O)) = \text{colim}_n \text{Rep}(G_n)$ is a colimit under monoidal functors, $\text{Rep}(G(O))\text{-mod} = \lim_n \text{Rep}(G_n)\text{-mod}$ on general grounds. The standard formalism “colimit under left adjoints equals a limit under right adjoints” shows $G(O)\text{-mod}_{\text{weak}} = \lim_n G_n\text{-mod}_{\text{weak}}$ under weak invariants functors, giving the claim. $\square$

Remark 1.4.2. The main formal difference between $\text{QCoh}(G(O))$ and $\text{Rep}(G(O))$ above is that while both are colimits of their G_n counterparts, $\text{Rep}(G(O))$ is a colimit under monoidal functors while $\text{QCoh}(G(O))$ is not (rather, it is a limit under monoidal functors).

1.5. **Some formal remarks.** We see that the functor $G(O)\text{-mod}_{\text{weak}} \xrightarrow{(-)^{G(O),w}} \text{DGCat}_{\text{cont}}$ is conservative, unlike the naive forgetful functor. Moreover, it commutes with colimits and is a $\text{DGCat}_{\text{cont}}$ -linear functor, which may be taken as a “invariants = coinvariants” statement in this context.

By contrast, one can show that the natural functor $G(O)\text{-mod}_{\text{weak}} \rightarrow \text{QCoh}(G(O))\text{-mod}$ admits fully-faithful left and right adjoints, and that the two induced functors:

$$\text{QCoh}(G(O))\text{-mod} \hookrightarrow G(O)\text{-mod}_{\text{weak}} \xrightarrow{(-)^{G(O),w}} \text{DGCat}_{\text{cont}}$$

correspond to the natural definitions of weak coinvariants and invariants on the left hand side. Therefore, the failure of (1.2.1) to be an equivalence can be understood as a failure of invariants and coinvariants to coincide for $\text{QCoh}(G(O))\text{-mod}$.

²To see this: show that the right hand side has is left complete for its t -structure, and show that the left hand side is not by showing that the map $\oplus_n k[n] \rightarrow \prod_n k[n]$ is not an equivalence.

1.6. Examples. Recall that the category $G(O)\text{-mod}$ of categories with a strong $G(O)$ -action was previously defined as $D(G(O))\text{-mod}$.

Exercise 1.6.1. Show that in contrast to the weak setting, the functor $\text{colim } G_n\text{-mod} \xrightarrow{\cong} D(G(O))\text{-mod}$.

From the exercise, we obtain a natural forgetful functor:

$$G(O)\text{-mod} \rightarrow G(O)\text{-mod}_{\text{weak}}.$$

Explicitly, this functor can be understood as follows.

By Proposition 1.4.1, for $\mathcal{C} \in D(G(O))\text{-mod}$, it suffices to define $\mathcal{C}^{G(O),w} \in \text{Rep}(G(O))$. Let $K_n := \text{Ker}(G(O) \rightarrow G_n)$. Then the above construction gives:

$$\mathcal{C}^{G(O),w} := \text{colim}(\mathcal{C}^{K_n})^{G_n,w} \in \text{DGCat}_{\text{cont}}$$

with its natural $\text{Rep}(G(O))$ -module structure. (This may be also be taken as a definition of the forgetful functor, if one likes.)

1.7. As a particular case, define $\mathfrak{g}[[t]]\text{-mod}$ as $D(G(O))^{G(O),w}$ using the above notation.

The following exercise, which we leave to the reader, gives the basic properties of this construction.

- Exercise 1.7.1.* (1) Show that $\mathfrak{g}[[t]]\text{-mod} = \text{colim } \mathfrak{g}[[t]]/t^{n+1}\text{-mod}$ under the forgetful functors.
 (2) Show that $\mathfrak{g}[[t]]\text{-mod}$ has a natural t -structure and that the forgetful functor $\mathfrak{g}[[t]]\text{-mod} \rightarrow \text{Vect}$ is conservative on $\mathfrak{g}[[t]]\text{-mod}^+$ but not in general.³
 (3) Show that $\mathfrak{g}[[t]]\text{-mod}^+ = D^+(\mathfrak{g}[[t]]\text{-mod}^\heartsuit)$.
 (4) Show that $\mathfrak{g}[[t]]\text{-mod}$ is compactly generated with compact generators the enveloping algebras $U(\mathfrak{g}[[t]]/t^{n+1}) \in \mathfrak{g}[[t]]\text{-mod}^\heartsuit$.

Note that $G(O)$ obviously acts strongly on $\mathfrak{g}[[t]]\text{-mod}$.

Exercise 1.7.2. Show that $\mathfrak{g}[[t]]\text{-mod}$ is self-dual as a $G(O)$ -module, and that the functors:

$$\begin{aligned} G(O)\text{-mod} &\rightarrow \text{DGCat}_{\text{cont}} \\ \mathcal{C} &\mapsto \mathcal{C} \otimes_{D(G(O))} \mathfrak{g}[[t]]\text{-mod} \\ \mathcal{C} &\mapsto \text{Hom}_{G(O)\text{-mod}}(\mathfrak{g}[[t]]\text{-mod}, \mathcal{C}) \\ \mathcal{C} &\mapsto \mathcal{C}^{G(O),w} \end{aligned}$$

all coincide.

1.8. Here is a different sort of example.

Suppose $G(O)$ acts on an indscheme X and that $X = \text{colim } X_i$ such that for every i , $G(O)$ acts on X_i through G_n for $n \gg 0$.

Then clearly $\text{QCoh}(X_i) \in G_n\text{-mod}_{\text{weak}}$ for $n \gg 0$, making $\text{QCoh}(X_i)$ an object of $G(O)\text{-mod}_{\text{weak}}$ in a natural way. The structural functors in the limit $\text{QCoh}(X) = \lim_i \text{QCoh}(X_i)$ are then $G(O)$ -equivariant, so $\text{QCoh}(X)$ is also naturally an object of $G(O)\text{-mod}_{\text{weak}}$.

³Hint: show that $\mathfrak{g}[[t]]\text{-mod}$ is naturally self-dual as a DG category and that the object in $\mathfrak{g}[[t]]\text{-mod} = \mathfrak{g}[[t]]\text{-mod}^\vee$ corresponding to the forgetful functor “lies in cohomological degree $-\infty$,” i.e., is in $\mathfrak{g}[[t]]\text{-mod}^{\leq -n}$ for all n .

1.9. Finally, one last example of more specific interest.

Following [Ras], say a $G(O)$ -action on an indscheme X is *tame* if the natural $\mathrm{QCoh}(G(O))$ -action on $\mathrm{QCoh}(X)$ satisfies “invariants = coinvariants” in the natural sense. Then as in §1.5, there is a natural construction of an object $\mathrm{QCoh}(X) \in G(O)\text{-mod}_{\mathrm{weak}}$ (take either the left or the right adjoint). This happens e.g. if $K_n \subseteq G(O)$ acts freely on X .

The main technical results of [Ras] say this also happens for the gauge action of $G(O)$ on gauge forms $\mathfrak{g}((t))dt$ even though there is no obvious geometric explanation (e.g., K_n does not act freely on $t^{-1}\mathfrak{g}[[t]]dt$ for $G = GL_2$, and if we use the adjoint action instead of the gauge action the claim is just false).

2. GROUP INDSCHEMES

2.1. We now suppose G is reductive and consider weak $G(K)$ -actions.

Remark 2.1.1. The reductivity hypothesis is to ensure $G(K)/G(O)$ is ind-proper. More generally, I am comfortable with weak actions of e.g. the formal completion of $\mathbb{G}_a(O)$ in $\mathbb{G}_a(K)$, which may be treated in exactly the same method as we use in this section. But I prefer to avoid weak $N(K)$ -actions, which continue to make me uncomfortable.

2.2. The downsides of the “ind-direction” are different from the “pro-direction.” On the one hand, working with modules over QCoh (or IndCoh) gives the correct answer (up to correcting for the pro-direction). However, weak invariants are no longer conservative.

Example 2.2.1. As a toy model, let V be a vector space thought of as an additive group indscheme. Then $\mathrm{IndCoh}(V_0^\wedge) \simeq \mathrm{Rep}(V^\vee)$ as monoidal categories, where the LHS is given the convolution monoidal structure and the right hand side is the usual tensor product of representations. This gives an equivalence:

$$\mathrm{IndCoh}(V_0^\wedge)\text{-mod} \simeq V^\vee\text{-mod}_{\mathrm{weak}}$$

and implies that for V infinite-dimensional, the weak invariants functor is not conservative.

2.3. Following the above principle, we will define $G(K)\text{-mod}_{\mathrm{weak}}$ such that the forgetful functor to $G(O)\text{-mod}_{\mathrm{weak}}$ is monadic.

To do this properly, we observe that $\mathrm{End}_{\mathrm{DGCat}_{\mathrm{cont}}\text{-mod}}(G(O)\text{-mod}_{\mathrm{weak}}) = G(O) \times G(O)\text{-mod}_{\mathrm{weak}}$, and as in §1.8, $\mathrm{QCoh}(G(K))$ naturally has weak $G(O)$ -bimodule structure (in the sense of the “correct” definition from §1). Moreover, the group structure on $G(K)$ and the IndCoh (or left adjoint) pushforward endows the corresponding endofunctor on $G(O)\text{-mod}_{\mathrm{weak}}$ with a monad structure.

Remark 2.3.1. There are forgetful functors:

$$G(K)\text{-mod}_{\mathrm{weak}} \rightarrow G(O)\text{-mod}_{\mathrm{weak}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$$

with the first map conservative and the second map not. For $\mathcal{C} \in G(K)\text{-mod}_{\mathrm{weak}}$, we continue to lazily write $\mathcal{C}$ for its image in $\mathrm{DGCat}_{\mathrm{cont}}$, and sometimes say “ $G(K)$ acts weakly on $\mathcal{C}$ ” in this circumstance. (E.g., we say $G(K)$ acts weakly on Vect , but we remember that $\mathrm{Vect}^{G(O),w} = \mathrm{Rep}(G(O))$ as previously defined.)

2.4. Somewhat more concretely, define $\mathrm{QCoh}(\mathrm{Gr}_G)^{G(O),w}$ analogously to $\mathrm{Rep}(G(O))$. I.e., this category is compactly generated by complexes such that the underlying quasicohherent sheaf on Gr_G is compact (i.e., it is obtained by pushforward from a bounded complex of coherent sheaves on a closed subscheme).

Then $\mathrm{QCoh}(\mathrm{Gr}_G)^{G(O),w}$ is a monoidal category under convolution, and the functor:

$$G(K)\text{-mod}_{weak} \xrightarrow{(-)^{G(O),w}} \text{QCoh}(\text{Gr}_G)^{G(O),w}\text{-mod}$$

is an equivalence.

2.5. Again, we have a conservative forgetful functor $G(K)\text{-mod} \rightarrow G(K)\text{-mod}_{weak}$.

3. INVARIANTS AND COINVARIANTS

3.1. In this section, we define invariants and coinvariants for weak $G(K)$ -modules. We will compare the two, and in the process see semi-infinite cohomology arising.

We retain the hypothesis that G is reductive.

3.2. It is immediate to see $G(K)$ acts “trivially” on Vect , and more generally, we have a functor $\text{triv} : \text{DGCat}_{cont} \rightarrow G(K)\text{-mod}_{weak}$.

The functor triv admits left and right adjoints $(-)_{G(K),w}$ and $(-)^{G(K),w}$. E.g., this invariants functor can be computed as Hom from $\text{Vect} \in G(K)\text{-mod}_{weak}$. The coinvariants functor is given by:

$$\mathcal{C}_{G(K),w} = \mathcal{C}^{G(O),w} \otimes_{\text{QCoh}(\text{Gr}_G)^{G(O),w}} \text{Rep}(G(O)).$$

Exercise 3.2.1. Use ind-properness of Gr_G to show that the functor $(-)^{G(K),w} : G(K)\text{-mod}_{weak} \rightarrow \text{DGCat}_{cont}$ is a morphism in $\text{DGCat}_{cont}\text{-mod}$ (i.e., this functor commutes with colimits and is a morphism of DGCat_{cont} -module categories).

3.3. We claim that $G(K)\text{-mod}_{weak}$ is dualizable as an object of $\text{DGCat}_{cont}\text{-mod}$. Indeed, this is true for $G_n\text{-mod}_{weak}$ (since it is modules over an algebra), for $G(O)\text{-mod}_{weak}$ (since it is a colimit under left adjoints), and then for $G(K)\text{-mod}_{weak}$ (since the forgetful functor to $G(O)\text{-mod}_{weak}$ is monadic).

Let $G(K)\text{-mod}_{weak}^\vee$ denote the dual DGCat_{cont} -module category. By the exercise, there are *two* natural functors:

$$G(K)\text{-mod}_{weak} \simeq G(K)\text{-mod}_{weak}^\vee = \text{Hom}_{\text{DGCat}_{cont}\text{-mod}}(G(K)\text{-mod}_{weak}, \text{DGCat}_{cont})$$

given by the formulae:

$$\begin{aligned} \mathcal{C} &\mapsto \left(\mathcal{D} \mapsto (\mathcal{C} \otimes \mathcal{D})_{G(K),w} \right) \\ \mathcal{C} &\mapsto \left(\mathcal{D} \mapsto (\mathcal{C} \otimes \mathcal{D})^{G(K),w} \right). \end{aligned}$$

Lemma 3.3.1. *Each of the above functors is an equivalence.*

Corollary 3.3.2. *There is an automorphism:*

$$G(K)\text{-mod}_{weak} \simeq G(K)\text{-mod}_{weak} \in \text{DGCat}_{cont}\text{-mod}$$

intertwining invariants and coinvariants.

Remark 3.3.3. We denote this equivalence by $\mathcal{C} \mapsto \mathcal{C} \otimes \chi$, i.e., we have the formula:

$$\mathcal{C}^{G(K),w} \simeq (\mathcal{C} \otimes \chi)_{G(K),w}.$$

Here we are thinking of χ as indicating a character of $G(K)$, i.e., a weak action on Vect . This is justified by the fact that the equivalence of Corollary 3.3.2 is manifestly an equivalence of

$G(K)\text{-mod}_{weak}$ -module categories (where $G(K)\text{-mod}_{weak}$ has the symmetric monoidal structure with e.g. the trivial action on $\mathbf{Vect}$ being the unit object).

Exercise 3.3.4. Show that for $\chi \in G(K)\text{-mod}_{weak}$ as above, the DG category underlying χ genuinely is isomorphic to $\mathbf{Vect}$. Explain how this isomorphism depends on the choice of parahoric subgroup $G(O) \subseteq G(K)$.

Remark 3.3.5. It is straightforward to show that a weak action of $G(K)$ on $\mathbf{Vect}$ (or any $\mathbb{G}_m \times \mathbb{Z}$ -gerbe) is equivalent to a homomorphism $G(K) \rightarrow \mathbb{Z} \ltimes \mathbb{B}\mathbb{G}_m$, where the action of $1 \in \mathbb{Z}$ on $\mathbb{B}\mathbb{G}_m$ is the inversion automorphism.

In our example with χ , it is easy to see that the projection $G(K) \rightarrow \mathbb{Z}$ is trivial (e.g., by showing χ factors through $G_{ad}(K)$, so this datum is equivalent to a central extension:

$$1 \rightarrow (\mathbb{G}_m \times \mathbb{Z}) \rightarrow \widehat{G(K)} \xrightarrow{\pi} G(K) \rightarrow 1.$$

(Roughly, tensoring with the line corresponding to the $\mathbb{G}_m$ -torsor $\pi^{-1}(g)$ is the automorphism of $\mathbf{Vect}$ defined by g .)

Then one can show that this extension is the Tate canonical extension. In particular, at the level of Lie algebras, this is the Kac-Moody extension defined by the Killing form.

3.4. Suppose now that κ is a level. Unlike the rest of the conference, we do not incorporate a critical twist in discussing Kac-Moody algebras.⁴

Then as for $\kappa = 0$, we have a forgetful functor $G(K)\text{-mod}_{\kappa} \rightarrow G(K)\text{-mod}_{weak}$. In particular, we may consider $D_{\kappa}(G(K))$ as equipped with a commuting weak action of $G(K)$ (on the right) and a level κ strong action of $G(K)$ on the left.

Define $\widehat{\mathfrak{g}}_{\kappa}\text{-mod}$ as $D_{\kappa}(G(K))^{G(K),w}$. Note that $G(K)$ acts (strongly) on $\widehat{\mathfrak{g}}_{\kappa}\text{-mod}$ with level κ .

Exercise 3.4.1. Formulate and prove a version of Exercise 1.7.1.

3.5. We claim that $\widehat{\mathfrak{g}}_{\kappa}\text{-mod}$ is dualizable with dual $\widehat{\mathfrak{g}}_{-\kappa+2\text{-crit}}$. (In particular, when κ is critical, the category is self-dual.)

Indeed, suppose $\mathcal{C} \in G(K)\text{-mod}_{weak}$ is dualizable as an object in this symmetric monoidal category. Then we have:

$$(\mathcal{C}^{G(K),w})^{\vee} = (\mathcal{C}^{\vee})_{G(K),w} = (\mathcal{C}^{\vee} \otimes \chi^{-1})^{G(K),w}.$$

We apply this to $\mathcal{C} = D_{\kappa}(G(K))$, noting that $\mathcal{C}^{\vee} = D_{-\kappa}(G(K))$ and apply Remark 3.3.5.

Example 3.5.1. We have the canonical object $k \in \mathfrak{g}((t))\text{-mod} = \widehat{\mathfrak{g}}_{2\text{-crit}}\text{-mod}^{\vee}$, i.e., there is a canonical functor:

$$\widehat{\mathfrak{g}}_{2\text{-crit}}\text{-mod} \rightarrow \mathbf{Vect}.$$

This functor is the functor of *semi-infinite cohomology*. (The above may be taken as a convenient definition of this functor, or the reader may take on the somewhat cumbersome exercise of comparing the two.)

Remark 3.5.2. The above constructions with Kac-Moody algebras do not really require the existence of the group $G(K)$. Instead, one should run a version of §1 with pro-formal groups, and then adapt the material of the previous two sections accordingly.

⁴That critical twist is included elsewhere for good reasons, and we fully endorse that notation. But we are essentially witnessing its origins presently, so it is convenient to be more naive.

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