NOTES ON QUANTUM PARAMETERS (GL-2)

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Contents

The space of quantum parameters

0.1. Defining Par_G°.

0.1.1. Throughout, we let k denote our ground field: an algebraically closed field of characteristic zero. Let $Q(\Lambda_T, k)^W$ be the vector space of W-invariant quadratic forms on Λ_T. Its elements identify with W-invariant, symmetric bilinear form κ on Λ_T via the formula:

$$\kappa(\lambda, \mu) := q(\lambda + \mu) - q(\lambda) - q(\mu).$$

Such forms in turn identify with G-invariant, symmetric bilinear forms κ on $\mathfrak{g}$. Suppose $\mathfrak{g}$ has simple factors $\mathfrak{g}_1, \cdots, \mathfrak{g}_r$ and center $\mathfrak{z}$. Then such forms are parametrized (non-canonically) by a product $(\mathbb{A}^1)^{\times r} \times \mathrm{Sym}^2(\mathfrak{z}^*)$.

0.1.2. From now on, we fix a smooth curve X (not necessarily projective). The space Par_G° is defined to be a product of:

– $Q(\Lambda_T, k)^W$ (or equivalently G-invariant, symmetric bilinear forms on $\mathfrak{g}$); and
– the space of extensions of $\mathcal{O}_X$-modules:

$$0 \to \omega_X \to E \to \mathfrak{z} \otimes \mathcal{O}_X \to 0. \tag{0.1}$$

It is clear that Par_G° is a smooth algebraic stack. We will denote its k-points by pairs (κ, E), where κ is a G-invariant symmetric bilinear form on $\mathfrak{g}$, and E is an extension as in (0.1).

Example 0.1. – The Killing form defines a quantum parameter $(\mathrm{Kil}_G, 0) \in \mathrm{Par}_G^\circ$, where 0 is understood as the trivial extension; the *critical level* is defined as

$$(\mathrm{crit}_G, 0) := (-\frac{1}{2}\mathrm{Kil}_G, 0) \in \mathrm{Par}_G^\circ.$$

Date: Winter, 2017.

– There are also some distinguished additional parameters E. Let $\check{Z}^\circ_G$ denote the torus dual to the connected component of the center Z°_G. Then for each $\check{Z}^\circ_G$-bundle $\mathcal{P}$, we may consider its Atiyah bundle:
$$0 \to \mathrm{Lie}(\check{Z}^\circ_G) \otimes \mathcal{O}_X \to \mathrm{At}(\mathcal{P}) \to \mathcal{T}_X \to 0.$$
Via the isomorphism $\mathrm{Lie}(\check{Z}^\circ_G) \xrightarrow{\sim} \mathfrak{z}^*$, we see that the monoidal dual $\mathrm{At}(\mathcal{P})^*$ defines an extension (0.1). The additional parameters arising this way are called *integral*.

Remark 0.2. The additional parameters necessarily arise on Levi subgroups of G. Namely, for each Levi subgroup M, a quantum parameter (κ, E) for G corresponds to a unique quantum parameter for M, such that the appropriately twisted $\mathcal{D}$-modules on Bun_M and Bun_G talk to each other (in a way that we will make precise in [Ja-4]). However, the passage of quantum parameters from G to B always introduces a nontrivial E-term.[1]

0.1.3. We now describe how to associate a Langlands dual parameter $(\check{\kappa}, \check{E})$ to a given one (κ, E) which is *not* critical[2]. Indeed, $\check{\kappa}$ is defined so that
$$\kappa - \mathrm{crit}_G \text{ and } \check{\kappa} - \mathrm{crit} \tag{0.2}$$
define mutually inverse maps between $\mathfrak{t}$ and $\check{\mathfrak{t}} \xrightarrow{\sim} \mathfrak{t}^*$. In order to define $\check{E}$, we note that under the isomorphism $\mathfrak{t} \xrightarrow{\sim} \check{\mathfrak{t}}$ defined by (0.2), the subspace $\mathfrak{z} \subset \mathfrak{t}$ passes to $\check{\mathfrak{z}} \subset \check{\mathfrak{t}}$. Hence the extension E induces an extension $\check{E}$.

0.2. What's in these notes?

0.2.1. The main goal of these notes is to make two constructions:

– We construct the functor:
$$\mathrm{Par}^\circ_G \to \mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_G), \quad (\kappa, E) \rightsquigarrow \mathcal{T}^{(\kappa,E)}_{\mathrm{Gr}_G}$$
where $\mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_G)$ is the category of factorization twistings on the affine Grassmannian Gr_G. This is achieved in §1.

– In fact, in the course of the construction we will also obtain factorization *multiplicative* twistings on the loop group $\mathcal{L}G$;

– We also construct the functor:
$$\mathrm{Par}^\circ_G \to \mathbf{Tw}(\mathrm{Bun}_G), \quad (\kappa, E) \rightsquigarrow \mathcal{T}^{(\kappa,E)}_{\mathrm{Bun}_G}$$
where $\mathbf{Tw}(\mathrm{Bun}_G)$ is the category of twistings on Bun_G. This is achieved in §2.

For the purpose of the workshop, only these are the necessary parts of the notes. For an audience uninterested in *global* geometric Langlands theory, even the materials in §2 can be ignored.

0.2.2. In §3, we explain some progress towards answering the question:

– *What is a natural class of geometric objects classified by* Par°_G*?*

The naïve guess would be either factorization twistings on Gr_G or twistings on Bun_G, but both of the functors above fail to be equivalences. Our hope is that Par°_G classifies factorization twistings on Gr_G that are *regular* in a certain sense. This would give an intrinsic meaning to Par°_G.

[1] A manifestation: in order to obtain a critically twisted $\mathcal{D}$-module on Bun_G by induction, one needs to start with a $\mathcal{D}$-module on Bun_T twisted by the "Tate line bundle," which corresponds to the parameter $(-\mathrm{crit}_G, \mathrm{At}(\omega_X^{\check{\rho}})^*) \in \mathrm{Par}^\circ_T$.

[2] i.e., the restriction of κ to any simple factor $\mathfrak{g}_i$ is *not* critical.

0.2.3. The space Par°_G has a natural compactification, denoted by Par_G. It includes points where the bilinear form κ "equals ∞." We will define Par_G and explain various constructions associated to it in §4.

The limiting behavior of categories appearing in geometric Langlands has long been noted by experts. Examples include:

- the Kazhdan-Lusztig category at level ∞ is stipulated to be Rep_G;
- the Whittaker category at level ∞ is stipulated to be $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{unr}}_G(\overset{\circ}{D}))$, where $\mathrm{Op}^{\mathrm{unr}}_G(\overset{\circ}{D})$ is the ind-scheme of unramified opers on the punctured formal disc.

The constructions in §4 turn these "stipulations" into precise statements regarding categories over Par_G, whose fibers at $(\mathfrak{g}^\infty, 0)$ identify with the expected ones.

1. Factorization twistings on Gr_G

The construction of factorization twistings on Gr_G follows the chart:

$$\mathrm{Par}^\circ_G \to \left\{\begin{matrix}\text{Lie-}*\text{ central}\\ \text{extension of }\mathfrak{g}_{\mathcal{D}}\end{matrix}\right\} \to \left\{\begin{matrix}\text{factorization central}\\ \text{extension of }\mathcal{L}\mathfrak{g}\\ \text{with splitting over }\mathcal{L}^+\mathfrak{g}\end{matrix}\right\}$$
$$\to \left\{\begin{matrix}\text{factorization multiplicative}\\ \text{twisting over }\mathcal{L}G\\ \text{with trivialization over }\mathcal{L}^+G\end{matrix}\right\} \to \left\{\begin{matrix}\text{factorization}\\ \text{twisting over }\mathrm{Gr}_G\end{matrix}\right\}$$

1.1. Lie-$*$ extensions of $\mathfrak{g}_{\mathcal{D}}$.

1.1.1. Let X be a smooth curve (but not necessarily proper). A *Lie-$*$ algebra* over X is a (right) $\mathcal{D}_X$-module $\mathcal{L}$ together with a morphism:

$$[-,-] : \mathcal{L} \boxtimes \mathcal{L} \to \Delta_{*,\mathrm{dR}}(\mathcal{L})$$

satisfying anti-symmetry and Jacobi identity.

Let $\mathcal{L}$ be a Lie-$*$ algebra. Then an *$\mathcal{L}$-module* is a (right) $\mathcal{D}_X$-module $\mathcal{M}$ together with a morphism $\mathcal{L} \boxtimes \mathcal{M} \to \Delta_{*,\mathrm{dR}}(\mathcal{M})$ satisfying the cocycle condition.

Example 1.1. The $\mathcal{D}$-module $\mathfrak{g}_{\mathcal{D}} := \mathfrak{g} \underset{k}{\otimes} \mathcal{D}_X$ is a Lie-$*$ algebra with bracket induced from that of $\mathfrak{g}$. More precisely, $[\xi \otimes \mathbf{1}, \xi' \otimes \mathbf{1}] := [\xi, \xi']_{\mathfrak{g}} \otimes \mathbf{1}_{\mathcal{D}}$ where $\mathbf{1}_{\mathcal{D}}$ denotes the canonical symmetric section of $\Delta_{*,\mathrm{dR}}(\mathcal{D}_X)$.

1.1.2. Let $\mathcal{G}$ denote the group jet scheme of G_X. More precisely, we regard $\mathcal{O}_{G_X}$ as a Hopf algebra object in $\mathrm{QCoh}(X)$. The functor $\mathrm{Jet} : \mathrm{QCoh}(X) \to \mathcal{D}_X\text{-}\mathbf{Mod}^l$ has a symmetric monoidal structure. Hence $\mathcal{O}_{\mathcal{G}} := \mathrm{Jet}(\mathcal{O}_{G_X})$ is a Hopf algebra object in $\mathcal{D}_X\text{-}\mathbf{Mod}^l$.

The notion of $\mathcal{G}$-action on $\mathcal{M} \in \mathcal{D}_X\text{-}\mathbf{Mod}^r$ can be described by a morphism $\mathcal{M} \to (\mathcal{O}_{\mathcal{G}})^r \overset{!}{\otimes} \mathcal{M}$ satisfying the cocycle condition. Alternatively, it may be described as a functorial assignment to a test object $\mathcal{A} \in \mathcal{D}_X\text{-}\mathbf{Alg}^l$ with $g \in \mathrm{Maps}_{\mathcal{D}_X\text{-}\mathbf{Alg}^l}(\mathcal{O}_{\mathcal{G}}, \mathcal{A})$ of an endormophism of $\mathcal{M} \underset{\mathcal{D}_X}{\otimes} \mathcal{A}$. Note that the tautological isomorphism (the definition of Jet as a left adjoint):

$$\mathrm{Maps}_{\mathcal{D}_X\text{-}\mathbf{Alg}^l}(\mathcal{O}_{\mathcal{G}}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{O}_X\text{-}\mathbf{Alg}}(\mathcal{O}_{G_X}, \mathcal{A})$$

makes this description particularly simple.

Example 1.2. To describe the adjoint action of $\mathcal{G}$ on $\mathfrak{g}_{\mathcal{D}}$, we take a test obejct $\mathcal{A} \in \mathcal{D}_X\text{-}\mathbf{Alg}^l$ and an $\mathcal{A}$-section g of $\mathcal{G}$. Then the usual adjoint action gives rise to the endomorphism on $\mathfrak{g} \underset{k}{\otimes} \mathcal{A}$.

1.1.3. We define the category $\mathbf{CExt}(\mathfrak{g}_{\mathcal{D}})$ as classifying the following data:

– a central extension of Lie-$*$ algebras:

$$0 \to \omega_X \to \widehat{\mathfrak{g}}_{\mathcal{D}} \to \mathfrak{g}_{\mathcal{D}} \to 0. \tag{1.1}$$

– an extension of the $\mathcal{G}$-action on $\mathfrak{g}_{\mathcal{D}}$ to a $\mathcal{G}$-action on $\widehat{\mathfrak{g}}_{\mathcal{D}}$.

Remark 1.3. The $\mathcal{G}$-action on $\widehat{\mathfrak{g}}_{\mathcal{D}}$ is included in order to later pass from central extension of the Lie algebra $\mathcal{L}\mathfrak{g}$ to multiplicative twisting on $\mathcal{L}G$. Ignoring it will not cause any conceptual damage.

1.1.4. Given a quantum parameter $(\kappa, E) \in \mathrm{Par}^{\circ}_G$, we define an extension (1.1) as follows: as $\mathcal{D}$-modules it is the pushout along the action map $\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to \omega_X$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X & \longrightarrow & (\mathfrak{g}_s \underset{k}{\otimes} \mathcal{D}_X) \oplus (E \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X) & \longrightarrow & \mathfrak{g}_{\mathcal{D}} & \longrightarrow & 0 \\ & & \big\downarrow {\scriptstyle \mathrm{act}} & & \big\downarrow & & & & \\ & & \omega_X & \longrightarrow & \widehat{\mathfrak{g}}_{\mathcal{D}} & & & & \end{array}$$

where $\mathfrak{g}_s$ is the semisimple part of $\mathfrak{g}$. In other words, we have a direct sum decomposition:

$$\widehat{\mathfrak{g}}_{\mathcal{D}} \xrightarrow{\sim} (E_{\mathcal{D}} \sqcup_{(\omega_X)_{\mathcal{D}}} \omega_X) \oplus (\mathfrak{g}_s)_{\mathcal{D}} \tag{1.2}$$

where $(-)_{\mathcal{D}}$ on the right-hand-side means induced $\mathcal{D}_X$-modules.

The Lie-$*$ bracket on $\widehat{\mathfrak{g}}_{\mathcal{D}}$ is defined by the cocycle:

$$\mathfrak{g}_{\mathcal{D}} \boxtimes \mathfrak{g}_{\mathcal{D}} \to \Delta_{*,\mathrm{dR}}(\omega_X), \quad (\xi \otimes \mathbf{1}) \boxtimes (\xi' \otimes \mathbf{1}) \rightsquigarrow \kappa(\xi, \xi')\mathbf{1}'_{\omega},$$

where $\mathbf{1}'_{\omega}$ is the canonical *anti*-symmetric section of $\Delta_{*,\mathrm{dR}}(\omega_X)$.[3]

Remark 1.4. Here we are using the fact that $\widehat{\mathfrak{g}}_{\mathcal{D}}$ splits over $(\mathfrak{g}_s)_{\mathcal{D}}$, and the Lie-$*$ bracket $\mathfrak{g}_{\mathcal{D}} \boxtimes \mathfrak{g}_{\mathcal{D}} \to \Delta_{*,\mathrm{dR}}(\mathfrak{g}_{\mathcal{D}})$ lands in $\Delta_{*,\mathrm{dR}}(\mathfrak{g}_s)_{\mathcal{D}}$.

1.1.5. In order to construct the $\mathcal{G}$-action on $\widehat{\mathfrak{g}}_{\mathcal{D}}$, we take a test object $\mathcal{A} \in \mathcal{D}_X\text{-}\mathbf{Alg}^l$ and an $\mathcal{A}$-section g of $\mathcal{G}$. We ought to construct an endomorphism of $\widehat{\mathfrak{g}}_{\mathcal{D}} \underset{\mathcal{D}_X}{\otimes} \mathcal{A}$. According to the decomposition (1.2), the required endomorphism centralizes the $E_{\mathcal{D}} \sqcup_{(\omega_X)_{\mathcal{D}}} \omega_X$-summand, acts by adjoint on $(\mathfrak{g}_s)_{\mathcal{D}}$-summand, and introduces the image of the given section along:

$$\widehat{\mathfrak{g}}_{\mathcal{D}} \underset{\mathcal{D}_X}{\otimes} \mathcal{A} \to \mathfrak{g} \underset{k}{\otimes} \mathcal{A} \xrightarrow{\kappa(g^{-1}dg, -)} \omega_X \underset{\mathcal{D}_X}{\otimes} \mathcal{A}.$$

In other words, we have constructed a functor:

$$\mathrm{Par}^{\circ}_G \to \mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}), \quad (\kappa, E) \rightsquigarrow \mathfrak{g}_{\mathcal{D}}^{(\kappa,E)}. \tag{1.3}$$

We call $\mathfrak{g}_{\mathcal{D}}^{(\kappa,E)}$ the *Kac-Moody* Lie-$*$ algebra corresponding to the quantum parameter (κ, E).

1.2. Central extensions of $\mathcal{L}\mathfrak{g}$.

1.2.1. Let $\mathrm{QCoh}^{\mathrm{Tate}}(\mathrm{Ran})$ denote the category of Tate modules over Ran. In other words, each $\mathcal{M} \in \mathrm{QCoh}^{\mathrm{Tate}}(\mathrm{Ran})$ is an association:

$$S \in \mathbf{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}} \rightsquigarrow \text{a Tate } \mathcal{O}_S\text{-module } \mathcal{M}\big|_S$$

together with isomorphisms $\mathcal{M}\big|_T \xrightarrow{\sim} \mathcal{M}\big|_S \underset{\mathcal{O}_S}{\widehat{\otimes}} \mathcal{O}_T$ for any map $T \to S$ in $\mathbf{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$.

[3]Using the Cousin sequence:

$$0 \to \omega_{X^2} \to \omega_{X^2}(\infty\Delta) \to \Delta_{*,\mathrm{dR}}(\omega_X) \to 0,$$

the section $\mathbf{1}'_{\omega} \in \Delta_{*,\mathrm{dR}}(\omega_X)$ is expressed as the image of $dx \wedge dy/(x-y)^2$.

1.2.2. For any $S \in \mathbf{Sch}^{\mathrm{aff}}$ equipped with a map to Ran, i.e., an I-family of S-points x^I of X, we set D_{x^I} as the formal completion of $S \times X$ along $\Gamma := \cup_{i \in I} \Gamma_{x^i}$ as an affine scheme.[4] Let $\overset{\circ}{D}_{x^I}$ denote its localization away from Γ.

We define $\mathcal{L}\mathfrak{g}$ as a Lie algebra in $\mathrm{QCoh}^{\mathrm{Tate}}(\mathrm{Ran})$, whose value at $x^I : S \to \mathrm{Ran}$ is the Tate $\mathcal{O}_S$-module $\mathfrak{g}(\mathcal{K}_{x^I}) := \mathfrak{g} \otimes \Gamma(\overset{\circ}{D}_{x^I}, \mathcal{O})$. The Lie algebra $\mathcal{L}^+\mathfrak{g}$ is defined similarly, where we replace $\mathfrak{g}(\mathcal{K}_{x^I})$ by its lattice subalgebra $\mathfrak{g}(\mathcal{O}_{x^I}) := \mathfrak{g} \otimes \Gamma(D_{x^I}, \mathcal{O})$.

1.2.3. We define the category $\mathbf{CExt}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g})$ as classifying the following data:

- a central extension of Lie algebras in $\mathrm{QCoh}^{\mathrm{Tate}}(\mathrm{Ran})$:

$$0 \to \mathcal{O}_{\mathrm{Ran}} \to \widehat{\mathfrak{g}} \to \mathcal{L}\mathfrak{g} \to 0 \tag{1.4}$$

- an extension of the $\mathcal{L}G$-action on $\mathcal{L}\mathfrak{g}$ to $\widehat{\mathfrak{g}}$;
- a trivialization of the above data over the Lie subalgebra $\mathcal{L}^+\mathfrak{g} \hookrightarrow \mathcal{L}\mathfrak{g}$.

Let $\mathbf{CExt}^{\mathrm{fact}}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g})$ denote the categories of "linearly factorization" objects in $\mathbf{CExt}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g})$. In other words, an object of $\mathbf{CExt}^{\mathrm{fact}}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g})$ is an object $\widehat{\mathfrak{g}}$ of $\mathbf{CExt}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g})$ equipped with the following additional datum:

- there is an isomorphism of $\widehat{\mathfrak{g}}\big|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}}$ with the pushout:

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Ran}} \boxplus \mathcal{O}_{\mathrm{Ran}} & \longrightarrow & (\widehat{\mathfrak{g}} \boxplus \widehat{\mathfrak{g}}) \\ \big\downarrow{\scriptstyle \mathrm{add}} & & \\ \mathcal{O}_{\mathrm{Ran}} & & \end{array}$$

as central extensions of $\mathcal{L}\mathfrak{g}\big|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}} \xrightarrow{\sim} \mathcal{L}\mathfrak{g} \boxplus \mathcal{L}\mathfrak{g}$.

1.2.4. Fix a k-point $x \in X$, and let D_x and $\overset{\circ}{D}_x$ denote the formal, respectively punctured, disc around x. Recall the functor of de Rham cohomology of the parametrized formal (punctured) disc (see [BD04]):

$$\mathrm{H}^0_{\mathrm{dR}}(D_x, -),\ \mathrm{H}^0_{\mathrm{dR}}(\overset{\circ}{D}_x, -) : \mathcal{D}_X\text{-}\mathbf{Mod}^r_{\mathrm{coh}} \to \mathbf{Vect}^{\mathrm{Tate}}.$$

Furthermore, these functors carry Lie-$*$ algebras to Lie algebras in $\mathbf{Vect}^{\mathrm{Tate}}$.

Applying $\mathrm{H}^0_{\mathrm{dR}}(\overset{\circ}{D}_x, -)$ to the exact sequence (1.1), we obtain:

$$0 \to k \to \mathrm{H}^0_{\mathrm{dR}}(\overset{\circ}{D}_x, \widehat{\mathfrak{g}}_{\mathcal{D}}) \to \mathfrak{g}(\mathcal{K}_x) \to 0 \tag{1.5}$$

Lemma 1.5. *The sequence* (1.5) *remains exact.*

Proof. We need the vanishing statements $\mathrm{H}^{-1}_{\mathrm{dR}}(\overset{\circ}{D}_x, \mathfrak{g}_{\mathcal{D}})$ and $\mathrm{H}^1(\overset{\circ}{D}_x, \omega_X)$. The first follows from the freeness of $\mathfrak{g}_{\mathcal{D}}$ as a $\mathcal{D}_X$-module. The second follows from the affineness of $\overset{\circ}{D}_x$. □

The sequence (1.5) acquires the following additional structures:

- a canonical splitting over $\mathfrak{g}(\mathcal{O}_x) \hookrightarrow \mathfrak{g}(\mathcal{K}_x)$; indeed, this follows from applying $\mathrm{H}^0_{\mathrm{dR}}(D_x, -)$ to the exact sequence (1.1) and noting $\mathrm{H}^0_{\mathrm{dR}}(D_x, \omega_X) = 0$;
- an action of $\mathcal{L}_x G$ on the middle piece $\mathrm{H}^0_{\mathrm{dR}}(\overset{\circ}{D}_x, \widehat{\mathfrak{g}}_{\mathcal{D}})$ that extends its action on $\mathfrak{g}(\mathcal{K}_x)$.

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[4] i.e., $\operatorname*{colim}_{i \in I} \Gamma^{(i)}$ in the category of *affine* schemes.

1.2.5. We may repeat the above construction in family. This procedure defines a functor of k-linear groupoids:

$$\mathrm{H}^0_{\mathrm{dR}}(\overset{\circ}{D}, -) : \mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}) \to \mathbf{CExt}^{\mathrm{fact}}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g}). \tag{1.6}$$

Remark 1.6. The Kac-Moody object $\mathfrak{g}^{(\kappa,E)}_{\mathcal{D}}$ passes to a factorization extension of $\mathcal{L}\mathfrak{g}$ that we denote by $\widehat{\mathfrak{g}}^{(\kappa,E)}$. Note that its fiber at $x \in X$ is the familiar Kac-Moody extension of the loop algebra $\mathfrak{g}(\mathcal{K}_x)$.

1.3. A quick tour of twistings.

1.3.1. Suppose $\mathcal{A}$ is a commutative group prestack. Write $\mathrm{B}^2\mathcal{A}$ for the twofold classifying prestack of $\mathcal{A}$ (without sheafification). The groupoid $\mathrm{Maps}(\mathcal{Y}, \mathrm{B}^2\mathcal{A})$ classifies $\mathcal{A}$-gerbes on $\mathcal{Y}$, which is neutral on every $S \in \mathbf{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$. We define the Picard category[5] of $\mathcal{A}$-*twistings* on $\mathcal{Y}$ as:

$$\mathbf{Tw}^{\mathcal{A}}(\mathcal{Y}) := \mathrm{Fib}(\mathrm{Maps}(\mathcal{Y}_{\mathrm{dR}}, \mathrm{B}^2\mathcal{A}) \to \mathrm{Maps}(\mathcal{Y}, \mathrm{B}^2\mathcal{A})).$$

In other words, a twisting on $\mathcal{Y}$ is a $\mathcal{A}$-gerbe on $\mathcal{Y}_{\mathrm{dR}}$ together with a trivialization of its pullback to $\mathcal{Y}$.

Lemma 1.7. *The morphism $\mathcal{A}_{\widehat{\{1\}}} \to \mathcal{A}$ induces an equivalence $\mathbf{Tw}^{\mathcal{A}_{\widehat{\{1\}}}}(\mathcal{Y}) \xrightarrow{\sim} \mathbf{Tw}^{\mathcal{A}}(\mathcal{Y})$.*

Applying the Lemma to $\mathcal{A} = \mathbb{G}_m$ and $\mathbb{G}_a$, and using the identification $\widehat{\mathbb{G}}_m \xrightarrow{\sim} \widehat{\mathbb{G}}_a$, we obtain:

$$\mathbf{Tw}^{\mathbb{G}_m}(\mathcal{Y}) \xleftarrow{\sim} \mathbf{Tw}^{\widehat{\mathbb{G}}_m}(\mathcal{Y}) \xrightarrow{\sim} \mathbf{Tw}^{\widehat{\mathbb{G}}_a}(\mathcal{Y}) \xrightarrow{\sim} \mathbf{Tw}^{\mathbb{G}_m}(\mathcal{Y}). \tag{1.7}$$

We let $\mathbf{Tw}(\mathcal{Y})$ be one of the categories in (1.7); we call its objects simply as *twistings*.

Remark 1.8. One can deduce from the equivalences in (1.7) another form of flexibility in the definition of a twisting. Namely, instead of $\mathrm{B}^2\mathbb{G}_m$ we may use its sheafified versions $\mathrm{B}^2_{\mathrm{Zar}}\mathbb{G}_m$ or $\mathrm{B}^2_{\text{ét}}\mathbb{G}_m$.

Remark 1.9. For $\mathcal{Y} = Y$ a classical scheme of finite type, twistings have been studied under the names *twisted differential operators* (TDOs), or *Picard algebroids*. We refer the reader to [BB93] for their definitions.

1.3.2. Aside from twistings, we may also consider *gerbes*; by this term we do not mean $\mathcal{A}$-gerbes as mentioned before, but something more "topological", akin to the gerbes in analytic topology or $\mathbb{Z}/l\mathbb{Z}$-gerbes in characteristic p. In our setting, we write:

$$\mathbf{Ge}(\mathcal{Y}) := \mathrm{Maps}(\mathcal{Y}_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\mathbb{G}_m).$$

In particular, there is a forgetful functor $\mathbf{Tw}(\mathcal{Y}) \to \mathbf{Ge}(\mathcal{Y})$.

Given $\mathcal{G} \in \mathrm{Maps}(\mathcal{Y}_{\mathrm{dR}}, \mathrm{B}^2\mathbb{G}_m)$, we may form the twisted category $\mathcal{D}\text{-}\mathbf{Mod}^{\mathcal{G}}(\mathcal{Y})$. If $\mathcal{G}$ arises from a twisting $\mathcal{T}$, then we have a forgetful functor:

$$\mathrm{oblv} : \mathcal{D}\text{-}\mathbf{Mod}^{\mathcal{T}}(\mathcal{Y}) := \mathcal{D}\text{-}\mathbf{Mod}^{\mathcal{G}}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}).$$

Remark 1.10. There is a sequence of maps, where the stack $\mathbf{Pic}(\mathcal{Y})$ identifies with the fiber of the second map:

$$\mathbf{Pic}(\mathcal{Y}) \to \mathbf{Tw}(\mathcal{Y}) \to \mathbf{Ge}(\mathcal{Y}).$$

A notable feature of this sequence is that it relates data of three different kinds: algebro-geometric, differential-geometric, and topological.

Remark 1.11. Of course, $\mathbf{Ge}(\mathcal{Y})$ as defined above is not purely topological. For example, the trivial object in $\mathbf{Ge}(\mathbb{A}^1)$ has the exponential local system as a nontrivial automorphism. We will return to this question in §3.

[5] i.e., one can form product of twistings.

1.4. Multiplicative twistings on $\mathcal{L}G$.

1.4.1. Let H be a group prestack locally of finite type. We use $\mathfrak{h}$ to denote its Lie algebra, and we have an equivalence $\exp(\mathfrak{h}) \xrightarrow{\sim} H_{\widehat{\{1\}}}$. It follows that we have an exact sequence of group prestacks:

$$1 \to \exp(\mathfrak{h}) \to H \to H_{\mathrm{dR}} \to 1.$$

In other words, H_{dR} is the quotient of the simplicial system $\cdots \rightrightarrows H \times \exp(\mathfrak{h}) \rightrightarrows H$. The H-action on $\exp(\mathfrak{h})$ upgrades this simplicial system into one in $\mathrm{Grp}(\mathbf{PStk})$. Hence its quotient inherits a group structure, identified with the one on H_{dR}.

1.4.2. Let $\mathbf{CExt}(\mathfrak{h})$ denote the category of central extensions:

$$0 \to k \to \widehat{\mathfrak{h}} \to \mathfrak{h} \to 0$$

together with an H-action on $\widehat{\mathfrak{h}}$ that extends the adjoint action on $\mathfrak{h}$.

Lemma 1.12. *There is an equivalence of categories:*

$$\mathbf{CExt}(\mathfrak{h}) \xrightarrow{\sim} \mathbf{Tw}^{\mathrm{mult}}(H). \tag{1.8}$$

We build the functor (1.8) as follows. We interpret an object of $\mathbf{CExt}(\mathfrak{h})$ as an H-equivariant map $\exp(\mathfrak{h}) \to \mathrm{B}\,\widehat{\mathbb{G}}_m$ of group prestacks, which gives rise to a map of simplicial systems in $\mathrm{Grp}(\mathbf{PStk})$:

$$\begin{array}{ccccc} \cdots & \rightrightarrows & H \times \exp(\mathfrak{h}) & \rightrightarrows & H \\ & & \downarrow & & \downarrow \\ \cdots & \rightrightarrows & \mathrm{B}\,\widehat{\mathbb{G}}_m & \rightrightarrows & \mathrm{pt} \end{array}$$

Taking quotient, we obtain a morphism $H_{\mathrm{dR}} \to \mathrm{B}^2\,\widehat{\mathbb{G}}_m$ of *group* prestacks together with a trivialization over H.

Remark 1.13. If we disregard the H-action on $\widehat{\mathfrak{h}}$ in defining central extensions of $\mathfrak{h}$, we would still obtain a twisting on H, but it will not come equipped with a multiplicative structure.

1.4.3. We now turn to the case of the loop group. Let $\mathbf{Tw}^{\mathrm{mult}}_{/\mathcal{L}^+G}(\mathcal{L}G)$ denote the fiber of $\mathbf{Tw}^{\mathrm{mult}}(\mathcal{L}G) \to \mathbf{Tw}^{\mathrm{mult}}(\mathcal{L}^+G)$. The analogue of Lemma 1.12 provides an equivalence of categories:

$$\mathbf{CExt}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g}) \xrightarrow{\sim} \mathbf{Tw}^{\mathrm{mult}}_{/\mathcal{L}^+G}(\mathcal{L}G).$$

Let $\mathbf{Tw}^{\mathrm{mult,fact}}_{/\mathcal{L}^+G}(\mathcal{L}G)$ denote the category of factorization objects in $\mathbf{Tw}^{\mathrm{mult}}_{/\mathcal{L}^+G}(\mathcal{L}G)$. In other words, an object of $\mathbf{Tw}^{\mathrm{mult,fact}}_{/\mathcal{L}^+G}(\mathcal{L}G)$ is a multiplicative twisting $\mathcal{T}$ on $\mathcal{L}G$ together with isomorphisms:

$$\mathcal{T}\big|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}} \xrightarrow{\sim} \mathcal{T} \boxtimes \mathcal{T}$$

Then we have an equivalence of categories:

$$\mathbf{CExt}^{\mathrm{fact}}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g}) \xrightarrow{\sim} \mathbf{Tw}^{\mathrm{mult,fact}}_{/\mathcal{L}^+G}(\mathcal{L}G). \tag{1.9}$$

1.5. Twistings on Gr_G.

1.5.1. Suppose $H \to G$ is a morphism of group prestacks. Given a multiplicative twisting $\mathcal{T}$ on G equipped with a trivialization on H, we obtain an H-equivariant twisting on G, i.e., a twisting on G/H. Indeed, the H-equivariance data of $\mathcal{T}$ comes from restricting the multiplicative data of $\mathcal{T}$ to the simplicial system $G \times H^{n-1} \hookrightarrow G^n$.

1.5.2. The above procedure defines a functor:

$$\mathbf{Tw}^{\text{mult,fact}}_{/\mathcal{L}^+G}(\mathcal{L}G) \to \mathbf{Tw}^{\text{fact}}(\text{Gr}_G). \tag{1.10}$$

Summarizing, we have a chain of functors:

$$\mathcal{T}^{(-,-)}_{\text{Gr}_G} : \text{Par}^{\circ}_G \xrightarrow{(1.3)} \mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}) \xrightarrow{(1.6)} \mathbf{CExt}^{\text{fact}}_{/\mathcal{L}^+\mathfrak{g}}(\mathcal{L}\mathfrak{g}) \xrightarrow{(1.9)} \mathbf{Tw}^{\text{mult,fact}}_{/\mathcal{L}^+G}(\mathcal{L}G) \xrightarrow{(1.10)} \mathbf{Tw}^{\text{fact}}(\text{Gr}_G).$$

This composition gives rise to the factorization twisting $\mathcal{T}^{(\kappa,E)}_{\text{Gr}_G}$ corresponding to the quantum parameter (κ, E).

2. Twistings on Bun_G

The construction of twistings on Bun_G follows the chart:

$$\text{Par}^{\circ}_G \to \left\{ \begin{matrix} \text{Lie-}* \text{ central} \\ \text{extension of } \mathfrak{g}_{\mathcal{D}} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{twisting on } \text{Bun}_{G,\infty x} \\ \text{acted on by } \mathcal{L}^+_x G \end{matrix} \right\} \to \left\{ \begin{matrix} \text{twisting} \\ \text{on } \text{Bun}_G \end{matrix} \right\}.$$

2.1. Twistings on $\text{Bun}_{G,\infty x}$.

2.1.1. Fix $x \in X$. Let $\text{Bun}_{G,\infty x}$ denote the prestack classifying a G-bundle $\mathcal{P}_G$ together with a trivialization $\alpha : \mathcal{P}_G\big|_{D_x} \xrightarrow{\sim} \mathcal{P}^0_G$. It is represented by a scheme (albeit of ∞-type.) The canonical projection $\text{Bun}_{G,\infty x} \to \text{Bun}_G$ realizes $\text{Bun}_{G,\infty x}$ as an $\mathcal{L}^+_x G$-torsor over Bun_G. Furthermore, the $\mathcal{L}^+_x G$-action on $\text{Bun}_{G,\infty x}$ extends to a full $\mathcal{L}_x G$-action.

Given any point $(\mathcal{P}_G, \alpha)$ of $\text{Bun}_{G,\infty x}$, we have an exact sequence:

$$0 \to \Gamma(X - x, \mathfrak{g}_{\mathcal{P}_G}) \xrightarrow{\text{Res}} \mathfrak{g}(\mathcal{K}_x) \to \mathcal{T}_{\text{Bun}_{G,\infty x}}\big|_{(\mathcal{P}_G,\alpha)} \to 0$$

where the restriction map is defined using α. The second map encodes the infinitesimal action of $\mathcal{L}_x G$ on $\text{Bun}_{G,\infty x}$.

2.1.2. Recall the category $\mathbf{CExt}(\mathfrak{g}_{\mathcal{D}})$ of §1.1.3. We now describe a functor:

$$\mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}) \to \mathbf{Tw}(\text{Bun}_{G,\infty x}). \tag{2.1}$$

Indeed, given an object $\widehat{\mathfrak{g}}_{\mathcal{D}}$ of $\mathbf{CExt}(\mathfrak{g}_{\mathcal{D}})$, we first consider its twist by the universal G-bundle over $\text{Bun}_{G,\infty x} \times X$. This procedure defines a central extension of Lie-$*$ algebras:

$$0 \to \mathcal{O}_{\text{Bun}_{G,\infty x}} \boxtimes \omega_X \to (\widehat{\mathfrak{g}}_{\mathcal{D}})_{\mathcal{P}_G} \to (\mathfrak{g}_{\mathcal{P}_G})_{\mathcal{D}} \to 0 \tag{2.2}$$

over $\text{Bun}_{G,\infty x} \times X$ (relative to $\text{Bun}_{G,\infty x}$). Now, applying the functors $\text{H}^0_{\text{dR}}(\overset{\circ}{D}_x, -)$ and $\text{H}^0_{\text{dR}}(X - x, -)$ on (2.2) and using $\text{H}^0_{\text{dR}}(X - x, \omega_X) = 0$, we obtain an exact sequence of Lie algebroids on $\text{Bun}_{G,\infty x}$, together with a splitting:

$$\begin{array}{ccccccccc} & & & & \Gamma(X - x, \mathfrak{g}_{\mathcal{P}_G}) & & & & \\ & & & \swarrow^{\widehat{\gamma}} & \downarrow^{\gamma} & & & & \\ 0 \longrightarrow & \mathcal{O}_{\text{Bun}_{G,\infty x}} & \longrightarrow & \text{H}^0_{\text{dR}}(\overset{\circ}{D}_x, \widehat{\mathfrak{g}}_{\mathcal{D}}) & \longrightarrow & \mathfrak{g}(\mathcal{K}_x) \otimes \mathcal{O}_{\text{Bun}_{G,\infty x}} & \longrightarrow 0 \end{array}$$

Lemma 2.1. *The image of $\widehat{\gamma}$ is a Lie algebroid ideal.*

Proof. This follows from the fact that $\widehat{\gamma}$ is a morphism of $\mathcal{L}_x G$-equivariant $\mathcal{O}_{\text{Bun}_{G,\infty x}}$-modules. □

Taking the cokernels of $\widehat{\gamma}$ and γ, we obtain a central extension of Lie algebroids:

$$0 \to \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \mathrm{Coker}(\widehat{\gamma}) \to \mathrm{Coker}(\gamma) \to 0 \tag{2.3}$$

Using the identification $\mathrm{Coker}(\gamma) \xrightarrow{\sim} \mathcal{T}_{\mathrm{Bun}_{G,\infty x}}$, we see that (2.3) defines a Picard algebroid, hence a twisting on $\mathrm{Bun}_{G,\infty x}$.

2.2. **Group action on twistings.**

2.2.1. In the local case, we obtained $\mathcal{T}^{(\kappa,E)}_{\mathrm{Gr}_G}$ from a twisting on the loop group using its multiplicative structure. The global analogue of the loop group is $\mathrm{Bun}_{G,\infty x}$, which has no multiplicative structure. Thus, in order to "descend" the twisting (2.3) to Bun_G, we need to make sense of a $\mathcal{L}^+_x\mathcal{O}$-action on (2.3).

2.2.2. Let $Y \in \mathbf{Sch}$ be acted on by some group scheme H. We first describe what it means for a Lie algebroid $\mathcal{L}$ on Y to be *acted on* by H. The required data are as follows:

– an H-equivariance structure on the underlying $\mathcal{O}_Y$-module of $\mathcal{L}$;
– a morphism $\eta : \mathfrak{h} \otimes \mathcal{O}_Y \to \mathcal{L}$ of H-equivariant $\mathcal{O}_Y$-modules.

They are supposed to satisfy a (rather long) list of conditions:

– the H-equivariance structure on (the underling $\mathcal{O}_Y$-module of) $\mathcal{L}$ is compatible with its Lie bracket;
– the anchor map σ of $\mathcal{L}$ intertwines the H-equivariance on $\mathcal{L}$ and $\mathcal{T}_Y$;
– the composition:
$$\mathfrak{h} \otimes \mathcal{O}_Y \xrightarrow{\eta} \mathcal{L} \xrightarrow{\sigma} \mathcal{T}_Y$$
identifies with the infinitesimal action of H on Y;
– η is compatible with the Lie bracket on $\mathcal{L}$ in the following sense: given $\xi \in \mathfrak{h} \otimes \mathcal{O}_Y$ and $l \in \mathcal{L}$, there holds:
$$[\eta(\xi), l] = \xi \cdot l \in \mathcal{L} \tag{2.4}$$
where $\xi \cdot l$ denotes the infinitesimal action coming from the H-equivariance structure.

Let $\mathbf{LieAlgd}^H(Y)$ the category of Lie algebroids on Y acted on by H. The notion of Picard algebroids acted on by H is completely analogous.

2.2.3. We will now build a functor

$$\mathbf{Q}^H : \mathbf{LieAlgd}^H(Y) \to \mathbf{LieAlgd}(Y/H).$$

We install the assumption that H acts freely on Y, and the general case will follow from smooth descent of Lie algebroids.

Remark 2.2. Under this assumption, η is necessarily injective.

Given a Lie algebroid $\mathcal{L}$ acted on by H, we consider the $\mathcal{O}_Y$-module $\mathrm{Coker}(\eta)$. It inherits an H-equivariant structure, and thus descends to an $\mathcal{O}_{Y/H}$-module $\mathcal{L}_0$. We set:

$$\mathbf{Q}^H(\mathcal{L}) := \mathcal{L}_0, \quad [l_0, \tilde{l}_0] = [\pi^{-1}l_0, \pi^{-1}\tilde{l}_0].$$

where $\pi^{-1}l_0 \in \mathrm{Coker}(\eta)$. In order to show that the Lie bracket is well-defined, we need the vanishing of $[\eta(\xi), l]$ for all H-invariant sections $l \in L$. However, this readily follows from the identity (2.4). The analogous construction for Picard algebroids defines a functor:

$$\mathbf{Q}^H : \mathbf{PicAlgd}^H(Y) \to \mathbf{PicAlgd}(Y/H).$$

2.2.4. The above constructions has a conceptual interpretation in terms of twistings.[6] Let us represent a twisting $\mathcal{T} \in \mathbf{Tw}(Y)$ by its total space, regarded as a $\mathrm{B}\,\widehat{\mathbb{G}}_m$-torsor $\widehat{Y}$ over Y_{dR}, trivialized over Y:

$$\begin{array}{ccc} & & \widehat{Y} \\ & \nearrow & \downarrow \mathrm{B}\,\widehat{\mathbb{G}}_m \\ Y & \longrightarrow & Y_{\mathrm{dR}} \end{array}$$

Then a *strong H-action* on $\mathcal{T}$ is an extension of the H-action on Y to a H_{dR}-action on $\widehat{Y}$, such that the projection $\widehat{Y} \to Y_{\mathrm{dR}}$ is H_{dR}-equivariant. Write $\mathbf{Tw}^H(Y)$ for the category of twistings equipped with a strong H-action.

We now define a functor:

$$\mathbf{Q}^H_{\mathrm{geom}} : \mathbf{Tw}^H(Y) \to \mathbf{Tw}(Y/H) \tag{2.5}$$

which sends an object $\mathcal{T} \in \mathbf{Tw}^H(Y)$ to the twisting represented by the diagram:

$$\begin{array}{ccc} & & \widehat{Y}/H_{\mathrm{dR}} \\ & \nearrow & \downarrow \mathrm{B}\,\widehat{\mathbb{G}}_m \\ Y/H & \longrightarrow & (Y/H)_{\mathrm{dR}} \end{array}$$

2.2.5. Recall the equivalence of categories:

$$\mathbf{Tw}(Y) \xrightarrow{\sim} \mathbf{PicAlgd}(Y), \quad \mathcal{T} \rightsquigarrow \mathcal{T}_{Y/\widehat{Y}}.$$

where $\mathcal{T}_{Y/\widehat{Y}}$ denotes the relative tangent complex.

Lemma 2.3. *The above equivalence upgrades to an equivalence $\mathbf{Tw}^H(Y) \to \mathbf{PicAlgd}^H(Y)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{Tw}^H(Y) & \xrightarrow{\sim} & \mathbf{PicAlgd}^H(Y) \\ \downarrow \mathbf{Q}^H & & \downarrow \mathbf{Q}^H_{\mathrm{geom}} \\ \mathbf{Tw}(Y/H) & \xrightarrow{\sim} & \mathbf{PicAlgd}(Y/H) \end{array}$$

2.3. Twistings on Bun_G.

2.3.1. We now upgrade the functor (2.1) to the $\mathcal{L}^+_x G$-equivariant category:

$$\mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}) \to \mathbf{PicAlgd}^{\mathcal{L}^+_x G}(\mathrm{Bun}_{G,\infty x}) \xrightarrow{\sim} \mathbf{Tw}^{\mathcal{L}^+_x G}(\mathrm{Bun}_{G,\infty x}). \tag{2.6}$$

Indeed, the $\mathcal{L}^+_x G$-equivariance structure on (2.3) is clear. The morphism

$$\eta : \mathfrak{g}(\mathcal{O}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \mathrm{H}^0_{\mathrm{dR}}(\mathring{D}_x, \widehat{\mathfrak{g}}_{\mathcal{D}})$$

arises from applying $\mathrm{H}^0_{\mathrm{dR}}(D_x, -)$ to the exact sequence (2.2).

2.3.2. The construction of twistings on Bun_G is the following composition:

$$\mathcal{T}^{(-,-)}_{\mathrm{Bun}_G} : \mathrm{Par}^\circ_G \xrightarrow{(1.3)} \mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}) \xrightarrow{(2.6)} \mathbf{Tw}^{\mathcal{L}^+_x G}(\mathrm{Bun}_{G,\infty x}) \xrightarrow{\mathbf{Q}^{\mathcal{L}^+_x G}} \mathbf{Tw}(\mathrm{Bun}_G).$$

sending (κ, E) to the twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{Bun}_G}$. Instead of choosing $x \in X$, we could have chosen arbitrarily many points $x^I \subset X$ and repeated the above construction. One can show that an inclusion of subsets $x^I \subset x^J$ produces isomorphic twistings. In particular, this argument shows:

Lemma 2.4. *The twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{Bun}_G}$ is independent of the choice of $x \in X$.*

[6] Strictly speaking, we won't need this point of view for our applications.

2.3.3. We now compare this functor to the one from §1. Let $p : \mathrm{Gr}_G \to \mathrm{Bun}_G$ denote the projection map.

Lemma 2.5. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Par}^\circ_G & \xrightarrow{\mathcal{T}^{(-,-)}_{\mathrm{Gr}_G}} & \mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_G) \\ \downarrow{\scriptstyle \mathcal{T}^{(-,-)}_{\mathrm{Bun}_G}} & & \downarrow{\scriptstyle \mathrm{oblv}} \\ \mathbf{Tw}(\mathrm{Bun}_G) & \xrightarrow{p^*} & \mathbf{Tw}(\mathrm{Gr}_G). \end{array}$$

2.3.4. *Examples.* We mention three line bundles that are particularly important for us. The twistings associated to them correspond to specific choices of quantum parameters.

- the line bundle $\mathcal{L}_{\det}$ on Bun_G, whose fiber at $\mathcal{P}_G$ is $\det \mathrm{R}\,\Gamma(X, \mathfrak{g}_{\mathcal{P}_G}[1])$, corresponds to the parameter $(\mathrm{Kil}, 0) \in \mathrm{Par}^\circ_G$;
- the line bundle $\mathcal{L}_{T,\mathrm{Tate}(\mathfrak{n})}$ on Bun_T, whose fiber at $\mathcal{P}_T$ is $\det \mathrm{R}\,\Gamma(X, \mathfrak{n}_{\mathcal{P}_T}[1])$, corresponds to the parameter $(-\mathrm{crit}_G, \mathrm{At}(\omega_X^{\check{\rho}})^*) \in \mathrm{Par}^\circ_T$;
- the line bundle $\mathcal{L}_{\mathcal{P}_{\check{T}}}$ that is Fourier-Mukai dual to some $\mathcal{P}_{\check{T}} \in \mathrm{Bun}_{\check{T}}$, corresponds to the parameter $(0, \mathrm{At}(\mathcal{P}_{\check{T}})^*) \in \mathrm{Par}^\circ_T$.

Remark 2.6. One frequently normalizes the line bundles $\mathcal{L}_{\det}$ and $\mathcal{L}_{T,\mathrm{Tate}(\mathfrak{n})}$, which amounts to tensoring them by a specific line. Note, however, that twistings associated to $\mathcal{L}$ and $\mathcal{L} \underset{k}{\otimes} l$ are canonically isomorphic.

3. What do quantum parameters parametrize?

3.1. Regular gerbes/twistings.

3.1.1. The role of *gerbes* over X in the de Rham setting is played by $\mathbb{G}_m$-gerbes on X_{dR}, i.e., morphisms $X_{\mathrm{dR}} \to \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m$. However, this notion is slightly inadequate as an analogue of $\mathbb{Z}/l$-gerbes in characteristic p, or analytic gerbes over $\mathbb{C}$—the latter notions are purely "topological" but the former is not.

Example 3.1. When $X = \mathbb{A}^1$, the neutral $\mathbb{G}_m$-gerbe on X_{dR} has a nontrivial automorphism, given by the exponential local system.

We introduce the notion of *regularity* to cure this problem. It amounts to allowing only regular singular local systems as transition functions of the given gerbe.

3.1.2. Consider $\mathbf{Pic}_\nabla$ as a functor $(\mathbf{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathbf{Gpd}$, defined by:

$$\mathbf{Pic}_\nabla(S) := \mathrm{Maps}(S_{\mathrm{dR}}, \mathrm{B}_{\text{ét}}\,\mathbb{G}_m),$$

i.e., $\mathbf{Pic}_\nabla(S)$ is the groupoid of line bundles on S together with a flat connection. Let $\mathbf{Pic}^{\mathrm{reg}}_\nabla$ denote the subgroupoid of line bundles with flat connections which are *regular singular.* We set:

$$\mathbf{Ge}^{\mathrm{reg}}(S) := \mathrm{Maps}(S, \mathrm{B}_{\text{ét}}\,\mathbf{Pic}^{\mathrm{reg}}_\nabla).$$

For a general prestack $\mathcal{Y}$, we set $\mathbf{Ge}^{\mathrm{reg}}(\mathcal{Y}) := \lim\limits_{S \to \mathcal{Y}} \mathbf{Ge}^{\mathrm{reg}}(S)$.

3.1.3. Note that there is natural transformation:

$$\mathbf{Ge}^{\mathrm{reg}} \to \mathrm{Maps}((-)_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m) \tag{3.1}$$

induced from $\mathrm{B}\,\mathbf{Pic}^{\mathrm{reg}}_{\nabla} \to \mathrm{Maps}((-)_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m)$ upon sheafification, which in turn arises from:

$$\mathrm{pt}\,/\,\mathrm{Maps}(S, \mathbf{Pic}_{\nabla}) \xrightarrow{\sim} \mathrm{pt}\,/\,\mathrm{Maps}(S_{\mathrm{dR}}, \mathrm{B}_{\text{ét}}\,\mathbb{G}_m) \to \mathrm{Maps}(S_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m).$$

Remark 3.2. The functor $\mathbf{Ge}^{\mathrm{reg}}(S) \to \mathrm{Maps}(S_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m)$ is in general neither fully faithful, nor essentially surjective.

3.1.4. We define $\mathbf{Tw}^{\mathrm{reg}}$ as the fiber of the composition:

$$\mathbf{Ge}^{\mathrm{reg}} \xrightarrow{(3.1)} \mathrm{Maps}((-)_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m) \to \mathrm{Maps}(-, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m).$$

Thus we have a functor $\mathbf{Tw}^{\mathrm{reg}} \to \mathbf{Tw}$, which is also neither fully faithful nor essentially surjective.

3.1.5. One of the main consequences of the definitions is the following "purity" lemma:

Lemma 3.3. *Let $Z \hookrightarrow X$ be an embedding of smooth schemes such that $\mathrm{codim}_X(Z) = 1$. Then:*
- *the fiber of $\mathbf{Ge}^{\mathrm{reg}}(X) \to \mathbf{Ge}^{\mathrm{reg}}(X - Z)$ identifies with $k/\mathbb{Z}$;*[7]
- *the fiber of $\mathbf{Tw}^{\mathrm{reg}}(X) \to \mathbf{Tw}^{\mathrm{reg}}(X - Z)$ identifies with k.*

The tautological map $\mathbf{Tw}^{\mathrm{reg}} \to \mathbf{Ge}^{\mathrm{reg}}$ has fiber $\mathbf{Pic}$, the moduli stack of line bundles. We observe that the sequence:

$$\mathbf{Pic}(X) \to \mathbf{Tw}^{\mathrm{reg}}(X) \to \mathbf{Ge}^{\mathrm{reg}}(X) \tag{3.2}$$

is a fiber sequence of Picard stacks when X is a smooth *curve*. Indeed, we only need to show that $\mathbf{Tw}^{\mathrm{reg}}(X) \to \mathbf{Ge}^{\mathrm{reg}}(X)$ is surjective on π_0, which follows from $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) = 0$.

3.2. Parametrizations.

3.2.1. Let $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$ (respectively $\mathbf{Tw}^{\mathrm{reg,fact}}(\mathrm{Gr}_G)$, $\mathbf{Ge}^{\mathrm{reg,fact}}(\mathrm{Gr}_G)$) denote the Picard stack of *factorization* line bundles (respectively regular twistings, gerbes) on Gr_G. We will now describe these Picard stacks more explicitly.

In order to do so, we first explain a paradigm:

$$\left\{\begin{matrix}\text{factorization gadgets} \\ \text{over } \mathrm{Gr}_G\end{matrix}\right\} \to \left\{\begin{matrix}W\text{-invariant} \\ \text{quadratic forms on } \Lambda_T\end{matrix}\right\}.$$

3.2.2. Consider the "combinatorial" affine Grassmannian:

$$\mathrm{Gr}_{T,\mathrm{comb}} := \operatorname*{colim}_{(I,\lambda^I)} X^I$$

where the index is taken over $I \in \mathbf{fSet}$, $\lambda^I : I \to \Lambda_T$, and we have a morphism $(I, \lambda^I) \to (J, \lambda^J)$ whenever $I \twoheadrightarrow J$ and λ^J is obtained from λ^I by "summing up the preimage."

Given each pair (I, λ^I), we have a morphism $X^I \to \mathrm{Gr}_{T,X^I}$ sending $(x_1, \cdots, x_{|I|})$ to the T-bundle $\mathcal{O}(\sum_i \lambda^{(i)} x_i)$ together with its tautological trivialization. Hence we have a morphisms:

$$\mathrm{Gr}_{T,\mathrm{comb}} \to \mathrm{Gr}_T \to \mathrm{Gr}_G \tag{3.3}$$

of prestacks over $\mathrm{Ran}(X)$. Now, given a factorization gadget on Gr_G, we obtain a factorization gadget on $\mathrm{Gr}_{T,\mathrm{comb}}$ via pulling back along (3.3).

[7] $k/\mathbb{Z}$ (and later k) is regarded as a discrete groupoid.

3.2.3. Suppose the said gadget is a line bundle; we denote its factor corresponding to $(I, \lambda^I) : X^I \hookrightarrow \mathrm{Gr}_{T,\mathrm{comb}}$ by $\mathcal{L}^{(\lambda^I)}$. The factorization data over $\mathrm{Gr}_{T,\mathrm{comb}}$ supplies us with an isomorphism:

$$\mathcal{L}^{(\lambda,\mu)}\big|_{X^2-\Delta} \xrightarrow{\sim} \mathcal{L}^{(\lambda)} \boxtimes \mathcal{L}^{(\mu)}.$$

Thus, $\mathcal{L}^{(\lambda,\mu)} \otimes (\mathcal{L}^{(\lambda)} \boxtimes \mathcal{L}^{(\mu)})^{-1}$ is a line bundle on X^2 trivialized away from Δ. This supplies us with an integer, denoted by $\kappa(\lambda, \mu)$. One then checks that κ defines a W-invariant quadratic form on Λ_T; this procedure defines a functor:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \to Q(\Lambda_T, \mathbb{Z})^W. \tag{3.4}$$

3.2.4. When the factorization gadget in question is a regular twisting, or a regular gerbe, we appeal to Lemma 3.3 to obtain functors organized in the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) & \longrightarrow & \mathbf{Tw}^{\mathrm{reg,fact}}(\mathrm{Gr}_G) & \longrightarrow & \mathbf{Ge}^{\mathrm{reg,fact}}(\mathrm{Gr}_G) \\ \downarrow & & \downarrow & & \downarrow \\ Q(\Lambda_T, \mathbb{Z})^W & \longrightarrow & Q(\Lambda_T, k)^W & \longrightarrow & Q(\Lambda_T, k/\mathbb{Z})^W \end{array}$$

3.2.5. Consider first a semisimple, simply connected group $\widetilde{G}$, with maximal torus $\widetilde{T}$.

Lemma 3.4. *The functor* (3.4) *is an isomorphism* $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\widetilde{G}}) \xrightarrow{\sim} Q(\Lambda_{\widetilde{T}}, \mathbb{Z})^W$.

Thus, given $q \in Q(\Lambda_{\widetilde{T}}, \mathbb{Z})^W$, we may call its preimage under (3.4) the factorization line bundle $\mathcal{L}^{(q)} \in \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\widetilde{G}})$ *associated* to q. Via pulling back along:

$$X^{(\widetilde{\lambda})} \hookrightarrow \mathrm{Gr}_{\widetilde{T},\mathrm{comb}} \to \mathrm{Gr}_{\widetilde{T}} \to \mathrm{Gr}_{\widetilde{G}},$$

we obtain a system of line bundles $\mathcal{L}^{(\widetilde{\lambda})}$ on X together with isomorphisms:

$$c_{\tilde{\lambda},\tilde{\mu}} : \mathcal{L}^{(\tilde{\lambda}+\tilde{\mu})} \xrightarrow{\sim} \mathcal{L}^{(\tilde{\lambda})} \otimes \mathcal{L}^{(\tilde{\mu})} \otimes \omega_X^{\kappa(\tilde{\lambda},\tilde{\mu})},$$

satisfying a κ-twisted commutativity condition: $\sigma_{12} \circ c_{\tilde{\lambda},\tilde{\mu}} = (-1)^{\kappa(\tilde{\lambda},\tilde{\mu})} \circ c_{\tilde{\mu},\tilde{\lambda}}$.

3.2.6. For a more general reductive group G, we denote by $\widetilde{G}_{\mathrm{der}}$ the universal cover of its derived subgroup G_{der}. Denote by $\widetilde{T}_{\mathrm{der}}$ the preimage of T in $\widetilde{G}_{\mathrm{der}}$. Consider the Picard stack $\mathrm{Par}_G^{\circ}(\mathbf{Pic})$ of data $(q, \mathcal{L}^{(\lambda)}, \varphi)$ where:

– $q \in Q(\Lambda_T, \mathbb{Z})^W$ (whose associated symmetric bilinear form is denoted by κ);
– $\mathcal{L}^{(\lambda)}$ is a system of line bundles on X indexed by $\lambda \in \Lambda_T$, together with isomorphisms:

$$\mathcal{L}^{(\lambda+\mu)} \xrightarrow{\sim} \mathcal{L}^{(\lambda)} \otimes \mathcal{L}^{(\mu)} \otimes \omega_X^{\kappa(\lambda,\mu)}$$

satisfying a κ-twisted commutativity condition;
– φ is an isomorphism of $\mathcal{L}^{(\lambda)}\big|_{\Lambda_{\widetilde{T}_{\mathrm{der}}}}$ with the system of line bundle $\mathcal{L}^{(\tilde{\lambda})}$ associated to $q\big|_{\Lambda_{\widetilde{T}_{\mathrm{der}}}}$ in the sense of §3.2.5 (applied to $\widetilde{G}_{\mathrm{der}}$).

Since $\pi_1(G) \xrightarrow{\sim} \Lambda_T/\Lambda_{\widetilde{T}_{\mathrm{der}}}$, there is a fiber sequence:

$$\mathrm{Hom}(\pi_1(G), \mathbf{Pic}(X)) \to \mathrm{Par}_G^{\circ}(\mathbf{Pic}) \to Q(\Lambda_T, \mathbb{Z})^W,$$

which does not split in general.

Remark 3.5. The notation $\mathrm{Par}_G^{\circ}(\mathbf{Pic})$ alludes to the fact that it is the parameter space of factorization line bundles on Gr_G. For $G = T$ a torus, it is known as θ-data (see [BD04]).

The procedure of pulling back to the combinatorial affine Grassmannians $\mathrm{Gr}_{T,\mathrm{comb}}$ and $\mathrm{Gr}_{\widetilde{T}_{\mathrm{der}},\mathrm{comb}}$ defines a functor $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \to \mathrm{Par}^\circ_G(\mathbf{Pic})$.

3.2.7. One may replicate the above definition for regular twistings and obtain a Picard stack $\mathrm{Par}^\circ_G(\mathbf{Tw}^{\mathrm{reg}})$ that fits into a fiber sequence:

$$\mathrm{Hom}(\pi_1(G), \mathbf{Tw}^{\mathrm{reg}}(X)) \to \mathrm{Par}^\circ_G(\mathbf{Tw}^{\mathrm{reg}}) \to Q(\Lambda_T, k)^W. \tag{3.5}$$

Unlike the previous situation, however, the construction of §1 provides a splitting of (3.5):[8]

$$Q(\Lambda_T, k)^W \to \mathbf{Tw}^{\mathrm{reg,fact}}(\mathrm{Gr}_G) \to \mathrm{Par}^\circ_G(\mathbf{Tw}^{\mathrm{reg}}).$$

On the other hand, $\mathbf{Tw}^{\mathrm{reg}}(X) \xrightarrow{\sim} \Gamma(X, \omega_X^{\mathrm{reg}}[1])$, where ω_X^{reg} is the subsheaf of ω_X, consisting of differential forms with poles of order ≤ 1 at $\overline{X} - X$ for any compactification $\overline{X}$ of X. Thus,

$$\begin{aligned}\mathrm{Hom}(\pi_1(G), \mathbf{Tw}^{\mathrm{reg}}(X)) &\xrightarrow{\sim} \mathrm{Hom}(\pi_1(G), \Gamma(X, \omega_X^{\mathrm{reg}}[1])) \\ &\xrightarrow{\sim} \mathrm{Hom}(\pi_1(G) \underset{\mathbb{Z}}{\otimes} k, \Gamma(X, \omega_X^{\mathrm{reg}}[1])) \xrightarrow{\sim} \mathbf{Ext}(\mathfrak{z}_G \otimes \mathcal{O}_X, \omega_X^{\mathrm{reg}}).\end{aligned}$$

Altogether, we have an isomorphism of k-linear groupoids:

$$\mathrm{Par}^\circ_G(\mathbf{Tw}^{\mathrm{reg}}) \xrightarrow{\sim} Q(\Lambda_T, k)^W \times \mathbf{Ext}(\mathfrak{z}_G \otimes \mathcal{O}_X, \omega_X^{\mathrm{reg}}).$$

Remark 3.6. Note that this space identifies with Par°_G for proper X.

3.2.8. The (conjectural-but-within-reach) parametrization theorem of factorization gadgets on Gr_G asserts that the following three vertical arrows are all equivalences:

$$\begin{array}{ccccc} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) & \longrightarrow & \mathbf{Tw}^{\mathrm{reg,fact}}(\mathrm{Gr}_G) & \longrightarrow & \mathbf{Ge}^{\mathrm{reg,fact}}(\mathrm{Gr}_G) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathrm{Par}^\circ(\mathbf{Pic}) & \longrightarrow & \mathrm{Par}^\circ(\mathbf{Tw}^{\mathrm{reg}}) & \longrightarrow & \begin{array}{c} Q(\Lambda_T, \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} k/\mathbb{Z} \\ \times \\ \mathrm{Hom}(\pi_1(G), \mathbf{Ge}^{\mathrm{reg}}(X)) \end{array} \end{array} \tag{3.6}$$

The fact that (3.2) is a fiber sequence for a curve X implies the same for the lower sequence in (3.6), whence also for the upper sequence.

Remark 3.7. One can view (3.6) as giving an intrinsic meaning to Par° when the curve X is *proper*. Namely, it classifies regular factorizable twistings on Gr_G. To remove the properness hypothesis, one may try to define a notion of *regularly factorizable* twistings which are only supposed to be regular "with respect to the factorization isomorphisms." We have not yet pursued this trend of thought.

Remark 3.8. The third isomorphism in (3.6) is a theorem of Ryan Reich [Re12]. An ongoing work of James Tao and the author tries to establish the first two isomorphisms.

4. The $\kappa \to \infty$ machine

4.1. What are we trying to do?

[8]Of course, the construction there gives a map $Q(\Lambda_T, k)^W \to \mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_G)$; to lift it to $\mathbf{Tw}^{\mathrm{reg,fact}}(\mathrm{Gr}_G)$, one needs to appeal to R. Reich's classification of regular gerbes [Re12].

4.1.1. We now describe a "machine" that takes as input a category $\mathcal{C}^{(\kappa,E)}$ for the quantum Langlands theory at parameter κ and produces its incarnation at $\kappa=\infty$.

In fact, the machine will do more—there is a "compactified" space of quantum parameters Par_G, and as soon as we know how to produce the category $\mathcal{C}^{(\kappa,E)}$ for an arbitrary (κ,E), we can view it as a sheaf of categories over Par_G whose fiber at a distinguished point $(\mathfrak{g}^\infty,0)\in\mathrm{Par}_G$ realizes its incarnation at $\kappa=\infty$.

The guideline of these constructions can be summarized in one line:

– *replace all* $\mathfrak{g}$ *by* $\mathfrak{g}^\kappa$.

We will explain what $\mathfrak{g}^\kappa$ means in §4.2.

4.1.2. Here are some examples of the degeneration behavior:

$\kappa<\infty$	$\kappa=\infty$	reference
$\widehat{\mathfrak{g}}^\kappa$-**Mod**	$\mathrm{QCoh}(\mathrm{Conn}(\overset{\circ}{D}_x))$	§4.2.5
$\mathrm{KL}^\kappa_{G,x}$	Rep_G	§4.3.4
$\mathcal{D}$-$\mathbf{Mod}^\kappa(\mathcal{L}_xG)$	$\mathrm{QCoh}(\mathcal{L}_xG\times\mathrm{Conn}(\overset{\circ}{D}_x))$	§4.4.1
$\mathcal{D}$-$\mathbf{Mod}^\kappa(\mathrm{Gr}_{G,x})$	$\mathrm{QCoh}(\mathrm{Gr}_{G,\nabla})$	§4.4.2
$\mathrm{Whit}_{G,x}$	$\mathrm{QCoh}(\mathrm{Op}^{\mathrm{unr}}_{G,x})$	§4.4.3 - §4.4.15
$\mathcal{D}$-$\mathbf{Mod}^\kappa(\mathrm{Gr}_{G,x})^{\mathcal{L}_xN}$	$\mathrm{QCoh}(\mathrm{LocSys}_G(D_x)\underset{\mathrm{LocSys}_G(\overset{\circ}{D}_x)}{\times}\mathrm{LocSys}_B(\overset{\circ}{D}_x))$	§4.4.16
$\mathcal{D}$-$\mathbf{Mod}^\kappa(\mathrm{Gr}_{G,x})^{\mathcal{L}_xN\cdot\mathcal{L}^+_xT}$	tl;dw	§4.4.16
$\mathcal{L}_xG$-$\mathbf{Mod}^\kappa$	$\mathrm{ShvCat}(\mathrm{LocSys}_G(\overset{\circ}{D}_x))$	§4.5 (sketch)
$\mathcal{D}$-$\mathbf{Mod}^\kappa(\mathrm{Bun}_G)$	$\mathrm{QCoh}(\mathrm{LocSys}_G)$	[Zh17, §6].

4.1.3. *Confession.* The current implementation of the machine has a drawback: we do not know how to *renormalize* in a systematic manner, i.e., we obtain categories such as $\mathrm{QCoh}(\mathrm{LocSys}_G)$ but not $\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}_G)$.

4.2. Compactifying $\overset{\circ}{\mathrm{Par}}_G$.

4.2.1. Consider the tautological symplectic form on $\mathfrak{g}\oplus\mathfrak{g}^*$, defined by the pairing:

$$\langle\xi\oplus\varphi,\xi'\oplus\varphi'\rangle:=\varphi(\xi')-\varphi'(\xi).$$

Let $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}\oplus\mathfrak{g}^*)$ denote the scheme parametrizing Lagrangian, G-invariant subspaces of $\mathfrak{g}\oplus\mathfrak{g}^*$. In other words, a k-point of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}\oplus\mathfrak{g}^*)$ is a G-invariant linear subspace $\mathfrak{g}^\kappa\subset\mathfrak{g}\oplus\mathfrak{g}^*$ such that $\varphi(\xi')=\varphi'(\xi)$ for every pair of elements $\xi\oplus\varphi,\xi'\oplus\varphi'\in\mathfrak{g}^\kappa$.

Taking G-fixed points defines a morphism:

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}\oplus\mathfrak{g}^*)\to\mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z}\oplus\mathfrak{z}^*),\quad(\mathfrak{g}^\kappa)\rightsquigarrow(\mathfrak{g}^\kappa)^G.$$

The algebraic stack Par_G is defined as the space of pairs $(\mathfrak{g}^\kappa,E)$ where

– $\mathfrak{g}^\kappa\in\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}\oplus\mathfrak{g}^*)$, and
– E is an extension of $\mathcal{O}_X$-modules:

$$0\to\omega_X\to E\to(\mathfrak{g}^\kappa)^G\underset{k}{\otimes}\mathcal{O}_X\to0. \tag{4.1}$$

4.2.2. There is an immersion $\mathrm{Par}^{\circ}_G \to \mathrm{Par}_G$ sending (κ, E) to the pair where:

- $\mathfrak{g}^\kappa$ is the graph of the linear map $\mathfrak{g} \to \mathfrak{g}^*$ defined by κ;
- along the map $\mathrm{pr}_{\mathfrak{g}} : \mathfrak{g}^\kappa \xrightarrow{\sim} \mathfrak{g}$, we have an isomorphism $\mathrm{pr}_{\mathfrak{z}} : (\mathfrak{g}^\kappa)^G \to \mathfrak{z}$; thus E defines an extension as in (4.1).

The image of $\mathrm{Par}^{\circ}_G \to \mathrm{Par}_G$ is precisely the open substack of $(\mathfrak{g}^\kappa, E)$ where the projection $\mathrm{pr}_{\mathfrak{g}} : \mathfrak{g}^\kappa \to \mathfrak{g}$ is an isomorphism. To the contrary, we have points

$$(\mathfrak{g}^\infty, E) := (\mathfrak{g}^*, E) \in \mathrm{Par}_G$$

lying "at $\kappa = \infty$." Dennis likes to call these points "degenerate."

4.2.3. We note that $\mathfrak{g}^\kappa$ is itself a Lie algebra with bracket:

$$[\xi \oplus \varphi, \xi' \oplus \varphi'] := [\xi, \xi'] \oplus \mathrm{Coad}_\xi(\varphi').$$

Furthermore, it admits a G-action (inherited from $\mathfrak{g} \oplus \mathfrak{g}^*$.) There is a canonical symmetric bilinear form on $\mathfrak{g}^\kappa$ defined by:

$$(\xi \oplus \varphi, \xi' \oplus \varphi') := \varphi(\xi') = \varphi'(\xi).$$

4.2.4. All the constructions relevant for quantum geometric Langlands can (and should) be done for the parameter space Par_G rather than Par°_G. For instance, given $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}_G$, there is a central extension of Lie-$*$ algebras:

$$0 \to \omega_X \to \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}} \to (\mathfrak{g}^\kappa)_{\mathcal{D}} \to 0 \tag{4.2}$$

such that the $\mathrm{Jet}(G_X)$-action on $(\mathfrak{g}^\kappa)_{\mathcal{D}}$ extends to an action on $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}$. The construction of (4.2) is analogous to the one in §1.1 (and specializes to it when $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}^{\circ}_G$).

Applying the functor $\mathrm{H}^0_{\mathrm{dR}}(\overset{\circ}{D}_x, -)$ to (4.2), we obtain a central extension:

$$0 \to k\mathbf{1} \to \widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathfrak{g}^\kappa(\mathcal{K}_x) \to 0 \tag{4.3}$$

together with a splitting over $\mathfrak{g}^\kappa(\mathcal{O}_x)$ and an extension of the $\mathcal{L}_x G$-action on $\mathfrak{g}^\kappa(\mathcal{K}_x)$ to $\widehat{\mathfrak{g}}^{(\kappa,E)}$.

4.2.5. Specializing to the parameter $(\mathfrak{g}^\infty, 0)$, the extension (4.3) becomes an extension of *abelian* Lie algebras:

$$0 \to k\mathbf{1} \to \widehat{\mathfrak{g}}^{(\infty,0)} \to \mathfrak{g}^\infty(\mathcal{K}_x) \to 0$$

which is canonically split. The $\mathcal{L}_x G$-action on $\widehat{\mathfrak{g}}^{(\infty,0)}$ extends the co-adjoint action on $\mathfrak{g}^\infty(\mathcal{K}_x)$, and carries an element $\varphi \otimes f \in \mathfrak{g}^\infty(\mathcal{K}_x)$ to $\mathrm{Res}(\varphi(g^{-1}dg) \cdot f) \in k\mathbf{1}$.

Lemma 4.1. *There is an isomorphism of topological associative algebras acted on by G:*

$$\mathrm{U}(\widehat{\mathfrak{g}}^{(\infty,0)})/(1 - \mathbf{1}) \xrightarrow{\sim} \mathcal{O}_{\mathrm{Conn}(\overset{\circ}{D}_x)}$$

where $\mathrm{Conn}(\overset{\circ}{D}_x)$ is the ind-scheme of connections on the trivial G-torsor on $\overset{\circ}{D}_x$, equipped with the G-action by gauge transformations.

An immediate consequence of Lemma 4.1 is that the category of Kac-Moody modules $\widehat{\mathfrak{g}}^{(\kappa,E)}$-**Mod** degenerates to

$$\widehat{\mathfrak{g}}^{(\infty,0)}\text{-}\mathbf{Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Conn}(\overset{\circ}{D}_x)), \tag{4.4}$$

such that the $\mathcal{L}_x G$-action passes to gauge transformation.

4.3. Degeneration: $\mathrm{KL}_{G,x} \rightsquigarrow \mathrm{Rep}_G$.

4.3.1. Recall that the (unrenormalized) Kazhdan-Lusztig category at *non-degenerate* parameter $(\kappa, E) \in \mathrm{Par}^{\circ}_G$ is defined as the *strong* $\mathcal{L}^+_x G$-invariants of $\widehat{\mathfrak{g}}^{(\kappa,E)}$-**Mod**:

$$\mathrm{KL}^{(\kappa,E)}_{G,x} := \widehat{\mathfrak{g}}^{(\kappa,E)}\text{-}\mathbf{Mod}^{(\mathcal{L}^+_x G)_{\mathrm{dR}}}.$$

At a possibly degenerate level $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}_G$, we need to replace $(\mathcal{L}^+_x G)_{\mathrm{dR}}$ by the quotient:

$$(\mathcal{L}^+_x G)^\kappa := \mathcal{L}^+_x G / \exp(\mathfrak{g}^\kappa(\mathcal{O}_x)).$$

Since the extension (4.3) splits over $\mathfrak{g}^\kappa(\mathcal{O}_x)$, there is an action of $(\mathcal{L}^+_x G)^\kappa$ on the category $\widehat{\mathfrak{g}}^{(\kappa,E)}$-**Mod**, so we may set:

$$\mathrm{KL}^{(\kappa,E)}_{G,x} := \widehat{\mathfrak{g}}^{(\kappa,E)}\text{-}\mathbf{Mod}^{(\mathcal{L}^+_x G)^\kappa}. \tag{4.5}$$

4.3.2. *Digression: inert Lie algebroids.* To calculate $(\mathcal{L}^+_x G)^\kappa$-invariants at the fully degenerate parameter $(\mathfrak{g}^\infty, 0) \in \mathrm{Par}_G$, we need some additional tools. To a smooth scheme Y and a complex $\mathcal{F} \in \mathrm{QCoh}(Y)$, we may associate the abelian Lie algebroid $\mathcal{L}_\mathcal{F}$ with underlying (complex of) quasi-coherent sheaf $\mathcal{F}$. We call $\mathcal{L}_\mathcal{F}$ the *inert* Lie algebroid on $\mathcal{F}$.

In particular, $\mathcal{L}_\mathcal{F}$-**Mod** is equivalent to quasi-coherent sheaves over $\mathbb{V}(\mathcal{F}) := \underline{\mathrm{Spec}}_Y(\mathrm{Sym}(\mathcal{F}))$, and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}_\mathcal{F}\text{-}\mathbf{Mod} & \longrightarrow & \mathrm{QCoh}(\mathbb{V}(\mathcal{F})) \\ \Big\downarrow{\scriptstyle \mathrm{oblv}} & & \Big\downarrow{\scriptstyle \pi_*} \\ \mathrm{QCoh}(Y) & \longrightarrow & \mathrm{QCoh}(Y) \end{array}$$

where $\pi : \mathbb{V}(\mathcal{F}) \to Y$ is the projection map.

4.3.3. We note that any Lie algebroid $\mathcal{L}$ determines a *formal moduli problem* $Y^\flat$ pointed by Y. The precise definition is unimportant[9], but we note:

- $Y^\flat$ is a prestack under Y such that the map $Y \to Y^\flat$ is an isomorphism on reduced part, and there is a well-behaved cotangent complex $\mathbb{T}_{Y/Y^\flat}$ that identifies with $\mathcal{L}$;
- The category $\mathrm{IndCoh}(Y^\flat)$ identifies with $\mathcal{L}$-**Mod**.

Let $\mathcal{F} \in \mathrm{QCoh}(Y)$, and $\mathcal{L}_\mathcal{F}$, $Y^\flat$ be the corresponding inert Lie algebroid and its formal moduli problem. Given a vector space $\mathfrak{k}$, the following data are equivalent:

$$\left\{ \begin{matrix} \text{maps } \eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{F} \\ \text{in } \mathrm{QCoh}(Y) \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \mathrm{B}\exp(\mathfrak{k})\text{-actions} \\ \text{on } Y^\flat \end{matrix} \right\}$$

where the formation of $\exp(\mathfrak{k})$ regards $\mathfrak{k}$ as an abelian Lie algebra. Furthermore, we have:

Lemma 4.2. *There is a canonical equivalence of DG categories:*

$$\mathrm{QCoh}(\mathbb{V}(\mathrm{Cofib}(\eta))) \xrightarrow{\sim} \mathrm{IndCoh}(Y^\flat)^{\mathrm{B}\exp(\mathfrak{k})}.$$

This gives us an easy way to calculate the $\mathrm{B}\exp(\mathfrak{k})$-invariants of

$$\mathrm{IndCoh}(Y^\flat) \xrightarrow{\sim} \mathcal{L}_\mathcal{F}\text{-}\mathbf{Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\mathcal{F})).$$

Remark 4.3. There is analogue of Lemma 4.2 in the twisted setting. Here we have a fiber sequence $\mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F}$ in $\mathrm{QCoh}(Y)$. This datum produces a $\widehat{\mathbb{G}}_m$-gerbe $\widehat{Y}^\flat$ over $Y^\flat$, together with a trivialization over Y. We have equivalence:

$$\mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1}) \xrightarrow{\sim} \mathrm{IndCoh}_{\widehat{Y}^\flat}(Y^\flat)$$

where $\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1}$ denotes the fiber of $\lambda : \mathbb{V}(\widehat{\mathcal{F}}) \to \mathbb{A}^1 \times Y$ (coming from $\mathcal{O}_Y \to \widehat{\mathcal{F}}$) at $\{1\} \times Y$.

[9] See [GR17] for the precise definitions, or [Zh17, §3,§4] for what we need here.

Again, a morphism $\widehat{\eta} : \mathfrak{k} \otimes \mathcal{O}_Y \to \widehat{\mathcal{F}}$ determines a $\mathrm{B}\exp(\mathfrak{k})$-action on both objects $\widehat{Y}^\flat \to Y^\flat$ compatibly, which further preserves Y. We have an equivalence of DG categories:

$$\mathrm{QCoh}(\mathbb{V}(\mathrm{Cofib}(\widehat{\eta}))_{\lambda=1}) \xrightarrow{\sim} \mathrm{IndCoh}_{\widehat{Y}^\flat}(Y^\flat)^{\mathrm{B}\exp(\mathfrak{k})}.$$

4.3.4. We now return to the Kazhdan-Lusztig category, and specialize (4.5) to the parameter $(\mathfrak{g}^\infty, 0)$. Note that we have an isomorphism:

$$(\mathcal{L}_x^+ G)^\infty \xrightarrow{\sim} \mathrm{B}\exp(\mathfrak{g}^*(\mathcal{O}_x)) \rtimes \mathcal{L}_x^+ G$$

where the semi-direct product is formed by the co-adjoint action. Using (4.4), we obtain:

$$\begin{aligned}\mathrm{KL}_{G,x}^{(\infty,0)} &\xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Conn}(\overset{\circ}{D}_x))^{\mathrm{B}\exp(\mathfrak{g}^*(\mathcal{O}_x))\rtimes\mathcal{L}_x^+G} \\ &\xrightarrow{\sim} (\mathrm{QCoh}(\mathrm{Conn}(\overset{\circ}{D}_x))^{\mathrm{B}\exp(\mathfrak{g}^*(\mathcal{O}_x))})^{\mathcal{L}_x^+G} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Conn}(D_x))^{\mathcal{L}_x^+G},\end{aligned}$$

where we used (the twisted version of) Lemma 4.2 for the last isomorphism. Now, note that $\mathrm{Conn}(D_x)/\mathcal{L}_x^+G$ identifies with pt/G. We find:

$$\mathrm{KL}_{G,x}^{(\infty,0)} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{pt}/G) \xrightarrow{\sim} \mathrm{Rep}_G.$$

4.4. Degeneration: $\mathrm{Whit}_G \rightsquigarrow \mathrm{QCoh}(\mathrm{Op}_G^{\mathrm{unr}})$.

4.4.1. We first study the degeneration behavior of $\mathcal{D}$-modules on $\mathcal{L}_xG$ and $\mathrm{Gr}_{G,x}$; this will essentially be performing the calculation of §4.3 "over $\mathcal{L}_xG$." Given a quantum parameter $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}_G$, we recall the central extension (4.3). It defines a *multiplicative* central extension of Lie algebroids on $\mathcal{L}_xG$:[10]

$$0 \to \mathcal{O}_{\mathcal{L}_xG}\mathbf{1} \to \widehat{\mathfrak{g}}^{(\kappa,E)}\widehat{\otimes}\mathcal{O}_{\mathcal{L}_xG} \to \mathcal{L}_x\mathfrak{g}^\kappa\widehat{\otimes}\mathcal{O}_{\mathcal{L}_xG} \to 0. \tag{4.6}$$

We write:

$$\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathcal{L}_xG) := \mathrm{U}(\widehat{\mathfrak{g}}^{(\kappa,E)}\widehat{\otimes}\mathcal{O}_{\mathcal{L}_xG})/(1-\mathbf{1})\text{-}\mathbf{Mod}.$$

Example 4.4. At the fully degenerate point $(\mathfrak{g}^\infty, 0) \in \mathrm{Par}_G$, the category $\mathcal{D}\text{-}\mathbf{Mod}^{(\infty,0)}(\mathcal{L}_xG)$ identifies with $\mathrm{QCoh}(\mathcal{L}_xG \times \mathrm{Conn}(\overset{\circ}{D}_x))$. Indeed, this follows immediately from Lemma 4.1.

4.4.2. We define:

$$\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x}) := \mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathcal{L}_xG)^{(\mathcal{L}_x^+G)^\kappa}.$$

Alternatively, we may consider the $(\mathcal{L}_x^+G)^\kappa$-quotient of the central extension (4.6), regarded as a central extension of Lie algebroids over $\mathrm{Gr}_{G,x}$, and $\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x})$ identifies with the category of modules over it.

Example 4.5. At the fully degenerate point $(\mathfrak{g}^\infty, 0) \in \mathrm{Par}_G$, we calculate using Example 4.4 and (the twisted version of) Lemma 4.2:

$$\begin{aligned}\mathcal{D}\text{-}\mathbf{Mod}^{(\infty,0)}(\mathrm{Gr}_{G,x}) &\xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_xG \times \mathrm{Conn}(\overset{\circ}{D}_x))^{\mathrm{B}\exp(\mathfrak{g}^*(\mathcal{O}_x))\rtimes\mathcal{L}_x^+G} \\ &\xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_xG \times \mathrm{Conn}(D_x))^{\mathcal{L}_x^+G} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}_{G,\nabla}),\end{aligned}$$

where $\mathrm{Gr}_{G,\nabla}$ classifies a G-bundle $\mathcal{P}_G$ on D_x, a trivialization thereof over $\overset{\circ}{D}_x$, and a connection ∇ on $\mathcal{P}_G$. The forgetful functor to $\mathrm{QCoh}(\mathrm{Gr}_G)$ identifies with the pushforward along $\mathrm{Gr}_{G,\nabla} \to \mathrm{Gr}_G$.

[10] We are careless about ∞-type issues, which makes us blind to the subtleties related to the Tate extension. However, we believe that a careful application of the ideas here can produce a fully accurate definition of the categories over Par_G.

4.4.3. *Twist by* $\overset{\circ}{\omega}{}_x^{1/2}$. In order to be completely canonical in defining the Whittaker category, we need to introduce a twist by the theta characteristic. From now on, we fix a square root of ω_x and call it $\omega_x^{1/2}$. We let ω_x^ρ denote the T-bundle induced from $\omega_x^{1/2}$ along $2\rho \in \Lambda_T$. As usual, its sections over the formal punctured disc will be denoted by $\overset{\circ}{\omega}{}_x^\rho$.

We let $\mathcal{L}_x N_\omega$ denote the group scheme over $\overset{\circ}{D}_x$ which classifies automorphisms of the induced B-bundle $(\overset{\circ}{\omega}{}_x^\rho)_B$, which preserve the further induced T-bundle $((\overset{\circ}{\omega}{}_x^\rho)_B)_T \xrightarrow{\sim} \overset{\circ}{\omega}{}_x^\rho$. Here are some variants of the geometric objects considered above:

- $\mathcal{L}_x G_\omega$ (respectively $\mathcal{L}_x^+ G_\omega$) denotes sections of $(\overset{\circ}{\omega}{}_x^\rho)_G$ (respectively $(\omega_x^\rho)_G$);
- $\mathrm{Gr}_{G,x,\omega}$ classifies a G-bundle over D_x, together with an isomorphism $\mathcal{P}_G\big|_{\overset{\circ}{D}_x} \xrightarrow{\sim} (\overset{\circ}{\omega}{}_x^\rho)_G$.

We can still realize $\mathrm{Gr}_{G,x,\omega}$ as the quotient $\mathcal{L}_x G_\omega / \mathcal{L}_x^+ G_\omega$. There is an analogue of the central extension (4.3), denoted by:

$$0 \to k\mathbf{1} \to \widehat{\mathfrak{g}}_\omega^{(\kappa,E)} \to \mathcal{L}_x \mathfrak{g}_\omega^\kappa \to 0.$$

It is formed by taking the $(\omega_X^\rho)_G$-twist of the Lie-$*$ algebra extension $\mathfrak{g}_{\mathcal{D}}^{(\kappa,E)}$ (see §1.1.4) and then taking de Rham cohomology over $\overset{\circ}{D}_x$.

In particular, $\mathcal{L}_x \mathfrak{g}_\omega^\kappa$ can be realized as sections of the twisted bundle $(\mathfrak{g}^\kappa)_{\overset{\circ}{\omega}{}_x^\rho}$, where we regard $\mathfrak{g}^\kappa$ as a T-representation.

Notation 4.6. Similar notations $\mathcal{L}_x(\cdot)_\omega$ and $\mathcal{L}_x^+(\cdot)_\omega$ will be applied to any T-representation. As a particular example, we have the twisted loop algebra $\mathcal{L}_x\mathfrak{g}_\omega$, which identifies with the Lie algebra of the group scheme $\mathcal{L}_x G_\omega$.

4.4.4. We have $\overset{\circ}{\omega}{}_x^{1/2}$-twisted analogues of the above categories:

- $\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathcal{L}_x G_\omega) := \mathrm{U}(\widehat{\mathfrak{g}}_\omega^{(\kappa,E)} \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G}) / (1 - \mathbf{1})\text{-}\mathbf{Mod}$;
- $\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega}) := \mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathcal{L}_x G_\omega)^{(\mathcal{L}_x^+ G_\omega)^\kappa}$.

The analogues of their degeneration behavior continue to hold. More precisely, we have:

$$\mathcal{D}\text{-}\mathbf{Mod}^{(\infty,0)}(\mathcal{L}_x G_\omega) \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(\overset{\circ}{D}_x)),$$

where $\mathrm{Conn}_\omega(\overset{\circ}{D}_x)$ denotes the space of connections on the G-bundle $(\overset{\circ}{\omega}{}_x^\rho)_G$. We use the notation $\mathrm{Conn}_\omega(D_x)$ in a similar way, and there holds:

$$\begin{aligned}\mathcal{D}\text{-}\mathbf{Mod}^{(\infty,0)}(\mathrm{Gr}_{G,x,\omega}) &\xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(D_x))^{\mathcal{L}_x^+ G_\omega} \\ &\xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}_{G,\nabla,\omega}),\end{aligned}$$

where $\mathrm{Gr}_{G,\nabla,\omega}$ classifies the data of $\mathrm{Gr}_{G,x,\omega}$ together with a connection on $\mathcal{P}_G$.

4.4.5. We now analyze the Whittaker/oper condition. Suppose $\mathfrak{g}^\kappa$ is a Lagrangian, G-invariant subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$. Associated to $\mathfrak{g}^\kappa$ is a subspace:

$$\mathfrak{n}^\kappa := \mathfrak{g}^\kappa \cap (\mathfrak{n} \oplus \mathfrak{b}^\perp) \hookrightarrow \mathfrak{n} \oplus \mathfrak{b}^\perp$$

where $\mathfrak{b}^\perp := (\mathfrak{g}/\mathfrak{b})^* \subset \mathfrak{g}^*$ consists of linear functionals vanishing on $\mathfrak{b}$. We write $\mathfrak{n}_{(1)}^\kappa$ for the subspace $[\mathfrak{n}, \mathfrak{n}^\kappa] \hookrightarrow \mathfrak{n}^\kappa$. Note that $\mathfrak{n}_{(1)}^\kappa$ is also the intersection of $\mathfrak{n}^\kappa$ with $\mathfrak{n}_{(1)} \oplus (\mathfrak{b}_{(-1)})^\perp$, where $\mathfrak{b}_{(-1)}$ is the sum of $\mathfrak{b}$ with the negative simple root spaces.

4.4.6. The weights of the $\mathfrak{t}$-action on $\mathfrak{n}^\kappa/\mathfrak{n}^\kappa_{(1)}$ identify with the simple roots $\{\check{\alpha}_i\}_{i\in\Delta}$.[11] Thus we may form the "canonical" character:

$$\chi : \mathcal{L}_x(\mathfrak{n}^\kappa/\mathfrak{n}^\kappa_{(1)})_\omega \xrightarrow{\sim} \bigoplus_{i\in\Delta} (\overset{\circ}{\omega}{}_x^{1/2})^{\langle 2\rho,\check{\alpha}_i\rangle} \xrightarrow{\sim} \bigoplus_{i\in\Delta} \overset{\circ}{\omega}_x \xrightarrow{\sum \mathrm{Res}} k. \tag{4.7}$$

where $\sum$ Res denotes the "sum of residue" map. The precomposition of (4.7) with the projection map $\mathcal{L}_x(\mathfrak{n}^\kappa)_\omega \to \mathcal{L}_x(\mathfrak{n}^\kappa/\mathfrak{n}^\kappa_{(1)})_\omega$ will again be denoted by χ (as no confusion should arise!)

Example 4.7. At the fully degenerate point $\mathfrak{g}^\infty$, we have:

$$\mathfrak{n}^\infty/\mathfrak{n}^\infty_{(1)} \xrightarrow{\sim} \mathfrak{b}^\perp/(\mathfrak{b}^\perp_{(-1)}) \xrightarrow{\sim} (\mathfrak{b}_{(-1)}/\mathfrak{b})^*$$

so χ defines an element in $\mathrm{Hom}_c(\mathcal{L}_x(\mathfrak{b}_{(-1)}/\mathfrak{b})^*_\omega, k)$ that we may call the "canonical" element.

4.4.7. Define a group prestack $(\mathcal{L}_x N_\omega)^\kappa$ by the quotient:

$$(\mathcal{L}_x N_\omega)^\kappa := \mathcal{L}_x N_\omega / \exp(\mathcal{L}_x \mathfrak{n}^\kappa_\omega),$$

where we use the tautological action of $\mathcal{L}_x N_\omega$ on $\mathcal{L}_x \mathfrak{n}^\kappa_\omega$.

Lemma 4.8. *Suppose H is a group prestack, and $\mathfrak{k}$ is a Lie algebra together with a morphism $\mathfrak{k} \to \mathfrak{h}$. Suppose the H-action on $\exp(\mathfrak{h})$ extends to $\exp(\mathfrak{k})$, so the quotient $H/\exp(\mathfrak{k})$ is again a group prestack. Then the following categories are equivalent:*

$$\left\{ \begin{matrix} H\text{-equivariant Lie} \\ \text{algebra character of } \mathfrak{k} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{multiplicative line bundle on} \\ H/\exp(\mathfrak{k}) \text{ with a trivialization over } H \end{matrix} \right\}. \tag{4.8}$$

Lemma 4.8 shows that the character χ (4.7) determines a multiplicative line bundle $(\mathcal{L}_x N_\omega)^\kappa$ together with a trivialization over $\mathcal{L}_x N_\omega$. Hence, if we have a map of prestacks $\mathcal{Y} \to \mathcal{Y}^\flat$ acted on compatibly by the group schemes $\mathcal{L}_x N_\omega \to (\mathcal{L}_x N_\omega)^\kappa$, we may form the category of $(\mathcal{L}_x N_\omega)^\kappa$-equivariant sheaves $\mathrm{IndCoh}(\mathcal{Y}^\flat)^{(\mathcal{L}_x N_\omega)^\kappa,\chi}$ against the character χ; it is equipped with a forgetful functor:

$$\mathrm{oblv} : \mathrm{IndCoh}(\mathcal{Y}^\flat)^{(\mathcal{L}_x N_\omega)^\kappa,\chi} \to \mathrm{IndCoh}(\mathcal{Y})^{\mathcal{L}_x N_\omega}.$$

Example 4.9. Suppose $\mathfrak{g}^\kappa$ is the graph of a bilinear form. Then we have an isomorphism $(\mathcal{L}_x N_\omega)^\kappa \xrightarrow{\sim} (\mathcal{L}_x N_\omega)_{\mathrm{dR}}$; thus the datum on the right is precisely a multiplicative local system on $\mathcal{L}_x N_\omega$ whose underlying line bundle is trivialized. The local system determined by (4.7) identifies with the pullback of exp under:

$$(\mathcal{L}_x N)_\omega \to (\mathcal{L}_x N)_\omega/[(\mathcal{L}_x N)_\omega, (\mathcal{L}_x N)_\omega] \xrightarrow{\sim} \bigoplus_{i\in\Delta} \overset{\circ}{\omega}_x \xrightarrow{\sum \mathrm{Res}} \mathbb{G}_a.$$

[12] Indeed, this follows from the fact that id : $\mathrm{Lie}(\mathbb{G}_a) \to k$ determines the exponential local system on $\mathbb{G}_a$, and the equivalence (4.8) is functorial.

[11] One may be tempted to fix "Chevalley generators" $\{e_i\}_{i\in\Delta}$ as a $\mathfrak{t}$-eigenbasis of $\mathfrak{n}^\kappa/\mathfrak{n}^\kappa_{(1)}$. However, this *cannot* be done compatibly over the entire space Par_G. For example, when $G = \mathrm{SL}_2$, such a choice amounts to a nonvanishing global section of $\mathcal{O}_{\mathbb{P}^1}(-1)$.

[12] The isomorphism in the middle is constructed as follows. Consider the exact sequence of B-representations (where $\mathfrak{n}_{\check{\alpha}_i}$ is the simple root space corresponding to $\check{\alpha}_i$, regarded as a *quotient* of $\mathfrak{n}/\mathfrak{n}^{(1)}$):

$$0 \to \mathfrak{n}_{\check{\alpha}_i} \to (k\mathbf{1} \oplus \mathfrak{n}_{\check{\alpha}_i}) \to k\mathbf{1} \to 0.$$

After we twist it by the B-bundle $(\overset{\circ}{\omega}{}_x^\rho)_B$, the first term becomes $\overset{\circ}{\omega}_x$ and the last term becomes $\mathcal{K}_x$. An element of $(\mathcal{L}_x N)_\omega$ thus determines a "shearing" map $\mathcal{K}_x \to \overset{\circ}{\omega}_x$, i.e., a section of $\overset{\circ}{\omega}_x$.

4.4.8. Let $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}_G$ be a quantum parameter. Recall that the category $\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega})$ from §4.4.4. It is equipped with a $(\mathcal{L}_x G_\omega)^\kappa$-action. We define

$$\mathrm{Whit}^{(\kappa,E)}_{G,x} := \mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega})^{(\mathcal{L}_x N_\omega)^\kappa,\chi}$$

i.e., the category of objects in $\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega})$ that are $(\mathcal{L}_x N_\omega)^\kappa$-equivariant against χ. From Example 4.9, we have:

Lemma 4.10. *Suppose $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}^\circ_G$. Then $\mathrm{Whit}^{(\kappa,E)}_{G,x}$ identifies with the usual Whittaker category $\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega})^{\mathcal{L}_x N_\omega,\chi}$.*

4.4.9. *Unramified opers.* We recall the definition of the placid ind-scheme $\mathrm{Op}^{\mathrm{unr}}_{G,x}$. It classifies triples $(\mathcal{P}_G, \nabla, \mathcal{P}_B, \alpha)$ where:

- $\mathcal{P}_G$ is a G-bundle over D_x, and ∇ is a connection on it;
- $\mathcal{P}_B$ is a reduction of $\mathcal{P}_G$ to B *over* $\overset{\circ}{D}_x$, and α is an isomorphism of its induced T-bundle $(\mathcal{P}_B)_T \xrightarrow{\sim} \overset{\circ}{\omega}{}^\rho_x$.

These data are suppose to satisfy the following *oper* condition. To state it, we note first that α gives rise to an isomorphism for each simple root $\check{\alpha}_i$:

$$\mathcal{P}_B^{\check{\alpha}_i} \xrightarrow{\sim} (\mathcal{P}_B)_T^{\check{\alpha}_i} \xrightarrow{\sim} \overset{\circ}{\omega}{}_x^{\langle \rho, \check{\alpha}_i\rangle} \xrightarrow{\sim} \overset{\circ}{\omega}_x. \tag{4.9}$$

On the other hand, we may consider the composition:

$$\mathcal{T}_{\overset{\circ}{D}_x} \xrightarrow{\nabla} \mathrm{At}(\mathcal{P}_G) \to \mathrm{At}(\mathcal{P}_G)/\,\mathrm{At}(\mathcal{P}_B) \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}. \tag{4.10}$$

We require that

- the image lands in $(\mathfrak{b}_{(-1)}/\mathfrak{b})_{\mathcal{P}_B}$, and
- the projection to each negative simple root space

$$\mathcal{T}_{\overset{\circ}{D}_x} \to (\mathfrak{b}_{-\check{\alpha}_i}/\mathfrak{b})_{\mathcal{P}_B} \xrightarrow{\sim} \mathcal{P}_B^{-\check{\alpha}_i} \tag{4.11}$$

  is the monoidal dual of (4.9).

Remark 4.11. If G is of adjoint type, then we may drop α from the definition, and simply require the maps (4.11) to be isomorphisms. Indeed, we may recover α as follows: the isomorphisms (4.11) tell us what $(\mathcal{P}_B)_T^{\check{\alpha}_i}$ is for each simple root, and the adjoint type hypothesis says that the simple roots span $\check{\Lambda}_T$.

4.4.10. We introduce a piece of (standard) notation. Given the data $(\mathcal{P}_G, \nabla, \mathcal{P}_B)$, we may form the composition (4.10). It is $\mathcal{K}_x$-linear, so may be regarded as an object $\nabla_{/\mathcal{P}_B}$ in any of the following vector spaces:

$$\nabla_{/\mathcal{P}_B} \in \mathrm{Hom}_{\mathcal{K}_x}(\mathcal{T}_{\overset{\circ}{D}_x}, (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}_x}((\mathfrak{b}^\perp)_{\mathcal{P}_B}, \overset{\circ}{\omega}_x) \xrightarrow{\sim} \mathrm{Hom}_c((\mathfrak{b}^\perp)_{\mathcal{P}_B}, k).$$

Given the additional datum α, the above requirements can be rephrased as:

- $\nabla_{/\mathcal{P}_B}$ belongs to the subspace $\mathrm{Hom}_c((\mathfrak{b}_{(-1)}/\mathfrak{b})^*_{\mathcal{P}_B}, k)$;
- since the B action on $(\mathfrak{b}_{(-1)}/\mathfrak{b})^*$ factors through T, we have

$$(\mathfrak{b}_{(-1)}/\mathfrak{b})^*_{\mathcal{P}_B} \xrightarrow{\sim} (\mathfrak{b}_{(-1)}/\mathfrak{b})^*_\omega$$

  so we require $\nabla_{/\mathcal{P}_B}$ to identify with the "canonical" element in $\mathrm{Hom}_c((\mathfrak{b}_{(-1)}/\mathfrak{b})^*_\omega, k)$ (see Example 4.7).

Remark 4.12. Of course, we can combine the two requirements into saying that $\nabla_{/\mathcal{P}_B}$ identifies with the "canonical" element in $\mathrm{Hom}_c((\mathfrak{b}_{(-1)}/\mathfrak{b})^*_\omega, k) \hookrightarrow \mathrm{Hom}_c((\mathfrak{g}/\mathfrak{b})^*_{\mathcal{P}_B}, k)$.

4.4.11. We can now state the degeneration result:

Lemma 4.13. *There is a canonical equivalence of DG categories:*

$$\mathrm{Whit}^{(\infty,0)}_{G,x} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{unr}}_{G,x}). \tag{4.12}$$

We first note from §4.4.4 the isomorphisms:

$$\mathcal{D}\textbf{-Mod}^{(\infty,0)}(\mathrm{Gr}_{G,x,\omega}) \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(D_x))^{\mathcal{L}^+_x G} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}_{G,\nabla,\omega})$$

so we tautologically have:

$$\mathrm{Whit}^{(\infty,0)}_{G,x} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}_{G,\nabla,\omega})^{\mathcal{L}_x N^\infty_\omega,\chi} \xrightarrow{\sim} (\mathrm{QCoh}(\mathrm{Gr}_{G,\nabla,\omega})^{\mathrm{B}\exp(\mathcal{L}_x \mathfrak{n}^\infty_\omega),\chi})^{\mathcal{L}_x N_\omega}.$$

4.4.12. We define the following auxiliary objects:

- let $\mathrm{Conn}^{\mathrm{Op}}_\omega(D_x)$ be the closed subscheme of $\mathrm{Conn}_\omega(D_x)$ consisting of connections ∇ on $(\omega^\rho_x)_G$ whose restriction to $\overset{\circ}{D}_x$ satisfies the oper condition.
- let $\mathrm{Gr}^{\mathrm{Op}}_{G,\nabla,\omega}$ be the closed subscheme of $\mathrm{Gr}_{G,\nabla,\omega}$, where the connection ∇ on $\mathcal{P}_G$ restricts to one on $\mathcal{P}_G\big|_{\overset{\circ}{D}_x} \xrightarrow{\sim} (\omega^\rho_x)_G$ that satisfies the oper condition (as above).

Clearly, we have a Cartesian square:

$$\begin{array}{ccc} \mathcal{L}_x G_\omega \times \mathrm{Conn}^{\mathrm{Op}}_\omega(D_x) & \hookrightarrow & \mathcal{L}_x G \times \mathrm{Conn}_\omega(D_x) \\ \downarrow{\scriptstyle \mathcal{L}^+_x G_\omega} & & \downarrow{\scriptstyle \mathcal{L}^+_x G_\omega} \\ \mathrm{Gr}^{\mathrm{Op}}_{G,\nabla,\omega} & \hookrightarrow & \mathrm{Gr}_{G,\nabla,\omega} \end{array}$$

where the vertical maps are $\mathcal{L}^+_x G_\omega$-torsors.

4.4.13. On the other hand, $\mathcal{L}_x N_\omega$ acts on $\mathrm{Gr}^{\mathrm{Op}}_{G,\nabla,\omega}$, and there is a canonical isomorphism:

$$\mathcal{L}_x N_\omega \backslash \mathrm{Gr}^{\mathrm{Op}}_{G,\nabla,\omega} \xrightarrow{\sim} \mathrm{Op}^{\mathrm{unr}}_{G,x}.$$

Thus we have reduced the statement of Lemma 4.13 to an $\mathcal{L}_x N_\omega$-equivariant equivalence:

$$\mathrm{QCoh}(\mathrm{Gr}_{G,\nabla,\omega})^{\mathrm{B}\exp(\mathcal{L}_x \mathfrak{n}^\infty_\omega),\chi} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}^{\mathrm{Op}}_{G,\nabla,\omega}). \tag{4.13}$$

The equivalence (4.13) will in turn follow from an $(\mathcal{L}_x N_\omega, \mathcal{L}^+_x G_\omega)$-bi-equivariant equivalence:

$$\mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(D_x))^{\mathrm{B}\exp(\mathfrak{n}^\infty_\omega),\chi} \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}^{\mathrm{Op}}_\omega(D_x)). \tag{4.14}$$

4.4.14. To prove (4.14), we note a generalization of Lemma 4.2. Let us be in the set-up of §4.3.3, together with the additional datum of a character (of abelian Lie algebras) $\chi : \mathfrak{k} \to k$. Note that the map $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{F}$ gives rise to a morphism:

$$\mathrm{char} : \mathbb{V}(\mathcal{F}) \to \mathfrak{k}^* \times Y \xrightarrow{\mathrm{pr}} \mathfrak{k}^*,$$

We let $\mathbb{V}(\mathcal{F})_{\mathrm{char}=\chi}$ denote its fiber at $\{\chi\}$.

Lemma 4.14. *There is a canonical equivalence of DG categories:*

$$\mathrm{IndCoh}(Y^\flat)^{\mathrm{B}\exp(\mathfrak{k}),\chi} \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\mathcal{F})_{\mathrm{char}=\chi})$$

Recall that $\mathrm{IndCoh}(Y^\flat) \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\mathcal{F}))$, so we have an easy way to calculate its $\mathrm{B}\exp(\mathfrak{k})$-invariants against a character.

Remark 4.15. Since $\mathbb{V}(\mathcal{F})_{\mathrm{char}=0}$ identifies with $\mathbb{V}(\mathrm{Cofib}(\eta))$, we recover Lemma 4.2 as the special case of taking $\chi = 0$.

Remark 4.16. Like Lemma 4.2, there is also a twisted version of Lemma 4.14 which asserts an equivalence of DG categories:

$$\mathrm{IndCoh}_{\widehat{Y}^\flat}(Y^\flat)^{\mathrm{B}\exp(\mathfrak{k}),\chi} \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1,\mathrm{char}=\chi})$$

where we recall $\mathrm{IndCoh}_{\widehat{Y}^\flat}(Y^\flat) \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1})$.

4.4.15. We now apply (the twisted version of) Lemma 4.14 to the following situation:

- Y is the loop group $\mathcal{L}_x G_\omega$;
- the central extension of inert Lie algebroids $\mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F}$ is given by:

$$\mathcal{O}_{\mathcal{L}_x G} \to \widehat{\mathfrak{g}}^\infty_\omega \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G_\omega} \to \mathfrak{g}^\infty_\omega \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G_\omega}$$

- $\mathfrak{k} = \mathfrak{n}^\infty_\omega \xrightarrow{\sim} \mathfrak{b}^\perp_\omega$;
- χ is the "canonical" element of $\mathrm{Hom}_c(\mathcal{L}_x(\mathfrak{b}_{(-1)}/\mathfrak{b})^*_\omega, k)$ (see Example 4.7), embedded in $\mathrm{Hom}_c(\mathcal{L}_x\mathfrak{b}^\perp_\omega, k)$.
- the $\mathrm{B}\exp(\mathfrak{k})$-action is supplied by the inclusion $\eta : \mathfrak{b}^\perp_\omega \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G} \to \widehat{\mathfrak{g}}^\infty_\omega \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G}$.

In particular, the morphism $\mathrm{char} : \mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1} \to \mathfrak{k}^*$ is given by:

$$\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(D_x) \to \mathrm{Hom}_c(\mathfrak{b}^\perp_\omega, k), \quad (g, \nabla) \rightsquigarrow \nabla_{/\mathcal{P}_B}.$$

Hence the object $\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1,\mathrm{char}=\chi}$ identifies with $\mathcal{L}_x G_\omega \times \mathrm{Conn}^{\mathrm{Op}}_\omega(D_x)$. The equivalence of Lemma 4.14 then gives produces (4.14). We omit checking that it is equivariant with respect to both $\mathcal{L}_x N_\omega$ and $\mathcal{L}^+_x G_\omega$-actions. □(Lemma 4.13)

4.4.16. *Variants.* As a variant of the Whittaker category construction, we may define the *principal series* category as $\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x})^{(\mathcal{L}_x N)^\kappa}$.[13] By a similar (but easier) calculation, we have:

$$\begin{aligned}\mathcal{D}\text{-}\mathbf{Mod}^{(\infty,0)}(\mathrm{Gr}_{G,x})^{(\mathcal{L}_x N)^\infty} &\xrightarrow{\sim} (\mathrm{QCoh}(\mathrm{Gr}_{G,\nabla})^{\mathrm{B}\exp(\mathcal{L}_x\mathfrak{b}^\perp)})^{\mathcal{L}_x N} \\ &\xrightarrow{\sim} \mathrm{QCoh}(\mathrm{LocSys}_G(D_x) \underset{\mathrm{LocSys}_G(\overset{\circ}{D}_x)}{\times} \mathrm{LocSys}_B(\overset{\circ}{D}_x)).\end{aligned}$$

A further variant defines the *semi-infinite* category $\mathcal{D}\text{-}\mathbf{Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x})^{(\mathcal{L}_x N)^\kappa(\mathcal{L}^+_x T)^\kappa}$ and we have:

$$\begin{aligned}&\mathcal{D}\text{-}\mathbf{Mod}^{(\infty,0)}(\mathrm{Gr}_{G,x})^{(\mathcal{L}_x N)^\infty(\mathcal{L}^+_x T)^\infty} \\ &\xrightarrow{\sim} \mathrm{QCoh}(\mathrm{LocSys}_G(D_x) \underset{\mathrm{LocSys}_G(\overset{\circ}{D}_x)}{\times} \mathrm{LocSys}_B(\overset{\circ}{D}_x) \underset{\mathrm{LocSys}_T(\overset{\circ}{D}_x)}{\times} \mathrm{LocSys}_T(D_x)).\end{aligned}$$

4.5. Degeneration: $G(\!(t)\!)$-Mod $\rightsquigarrow \mathrm{ShvCat}(\mathrm{LocSys}(\overset{\circ}{D}))$.

4.5.1. Given a prestack $\mathcal{Y}$ and a 3-gerbe $\mathcal{G}$ on $\mathcal{Y}$, i.e., a map $\mathcal{Y} \to \mathrm{B}^3\,\mathbb{G}_m$, we may form the "twisted" $(\infty,2)$-category $\mathrm{ShvCat}_{\mathcal{G}}(\mathcal{Y})$. Suppose that instead of $\mathbb{G}_m$, we have a 3-gerbe $\widehat{\mathcal{G}}$ for the group $\widehat{\mathbb{G}}_m$. We use the same notation $\mathrm{ShvCat}_{\widehat{\mathcal{G}}}(\mathcal{Y})$ for the sheaves of categories twisted by its induced 3-gerbe for $\mathbb{G}_m$.

[13] Note that we removed the $\overset{\circ}{\omega}{}^\rho_x$-twist as well as the character χ

4.5.2. Let $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}_G$ be a quantum parameter. Recall that we have an associated group prestack $(\mathcal{L}_x G)^\kappa$ and a multiplicative $\mathbb{G}_m$-gerbe $(\mathcal{L}_x G)^{(\kappa,E)}$ over it defined by the central extension $\widehat{\mathfrak{g}}^{(\kappa,E)}$ of $\mathcal{L}_x\mathfrak{g}^\kappa$. Delooping, we obtain a 3-gerbe $\mathrm{B}(\mathcal{L}_x G)^{(\kappa,E)}$ over $\mathrm{B}(\mathcal{L}_x G)^\kappa$. Write:

$$\mathcal{L}_x G\text{-}\mathbf{Mod}^{(\kappa,E)} := \mathrm{ShvCat}_{\mathrm{B}(\mathcal{L}_x G)^{(\kappa,E)}}(\mathrm{B}(\mathcal{L}_x G)^\kappa)$$

as an $(\infty, 2)$-category.

Example 4.17. When $\mathfrak{g}^\kappa$ is the graph of a bilinear form κ, there is an isomorphism $(\mathcal{L}_x G)^\kappa \xrightarrow{\sim} (\mathcal{L}_x G)_{\mathrm{dR}}$. Note that $\mathrm{ShvCat}(\mathrm{B}(\mathcal{L}_x G)_{\mathrm{dR}})$ identifies with the 2-category of categories with a strong $\mathcal{L}_x G$-action. A twisted version of this identification shows that our definition recovers the classical one at such levels.

4.5.3. We state the degeneration behavior of $\mathcal{L}_x G\text{-}\mathbf{Mod}^{(\kappa,E)}$:

Lemma 4.18. *There is a canonical equivalence of $(\infty, 2)$-categories:*

$$\mathcal{L}_x G\text{-}\mathbf{Mod}^{(\infty,0)} \xrightarrow{\sim} \mathrm{ShvCat}(\mathrm{LocSys}(\overset{\circ}{D}_x)).$$

Recall that $(\mathcal{L}_x G)^\infty \xrightarrow{\sim} \mathrm{B}\exp(\mathcal{L}_x\mathfrak{g}^*) \rtimes \mathcal{L}_x G$, and the $\widehat{\mathbb{G}}_m$-gerbe over it is given by $\mathrm{B}\exp(\mathcal{L}_x\widehat{\mathfrak{g}}^{(\kappa,E)}) \rtimes \mathcal{L}_x G$. Thus we may regard $\mathcal{L}_x G\text{-}\mathbf{Mod}^{(\infty,0)}$ as:

$$\mathcal{L}_x G\text{-}\mathbf{Mod}^{(\infty,0)} \xrightarrow{\sim} (\mathrm{ShvCat}_{\mathrm{B}^2\exp(\mathcal{L}_x\widehat{\mathfrak{g}}^{(\kappa,E)})}(\mathrm{B}^2\exp(\mathcal{L}_x\mathfrak{g}^*)))^{\mathcal{L}_x G}.$$

In other words, we reduce Lemma 4.18 to an $\mathcal{L}_x G$-equivariant equivalence:

$$\mathrm{ShvCat}_{\mathrm{B}^2\exp(\mathcal{L}_x\widehat{\mathfrak{g}}^{(\kappa,E)})}(\mathrm{B}^2\exp(\mathcal{L}_x\mathfrak{g}^*)) \xrightarrow{\sim} \mathrm{ShvCat}(\mathrm{Conn}(\overset{\circ}{D}_x)). \tag{4.15}$$

4.5.4. Suppose V is a finite dimensional vector space. Then we have a canonical equivalence:

$$\mathrm{ShvCat}(\mathrm{B}^2\exp(V)) \xrightarrow{\sim} \mathrm{ShvCat}(V^*). \tag{4.16}$$

Indeed, the left-hand-side identifies with categories together with a $\mathrm{B}\exp(V)$-action, i.e., an action of the monoidal category

$$\mathrm{QCoh}(\mathrm{B}\exp(V)) \xrightarrow{\sim} \mathbf{Rep}_V \xrightarrow{\sim} \mathrm{QCoh}(V^*),$$

which identifies with the right-hand-side. Since both sides of (4.16), regarded as functors $\mathbf{Vect} \to (\infty, 2)\text{-}\mathbf{Cat}$, commute with limits and filtered colimits, the same equivalence is valid for Tate vector spaces. Hence we obtain:

$$\mathrm{ShvCat}(\mathrm{B}^2\exp(\mathcal{L}_x\mathfrak{g}^*)) \xrightarrow{\sim} \mathrm{ShvCat}(\overset{\circ}{\omega}_x).$$

The equivalence (4.15) is a twisted version of this.

References

[BB93] Beilinson, Alexander, and Joseph Bernstein. "A proof of Jantzen conjectures." *Advances in Soviet mathematics* 16.Part 1 (1993): 1-50.

[BD04] Beilinson, Alexander, and Vladimir G. Drinfeld. *Chiral algebras.* Vol. 51. American Mathematical Soc., 2004.

[BD01] Brylinski, Jean-Luc, and Pierre Deligne. "Central extensions of reductive groups by K_2." *Publications mathématiques de l'IHÉS* 94.1 (2001): 5-85.

[GL16] Gaitsgory, D., and S. Lysenko. "Parameters and duality for the metaplectic geometric Langlands theory." *arXiv preprint* arXiv:1608.00284 (2016).

[GR17] Gaitsgory, Dennis, and Nick Rozenblyum. *A study in derived algebraic geometry.* American Mathematical Soc., 2017.

[Re12] Reich, Ryan. "Twisted geometric Satake equivalence via gerbes on the factorizable Grassmannian." *Representation Theory* 16.11 (2012): 345-449.

[Zh17] Zhao, Yifei. "Quantum parameters of the geometric Langlands theory." *arXiv preprint* arXiv:1708.05108 (2017).