INTRODUCTION TO QUANTUM LOCAL GEOMETRIC LANGLANDS.

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Contents

1. Introduction

Let me begin by saying a few (rough) words about classical local Langlands. In the classical setting, we consider irreducible, continuous $G(K)$-representations on vector spaces of characteristic 0. The classical local Langlands correspondence relates such representations to Galois representations $\mathrm{Gal}(K) \to G^\vee$.

For $G = \mathrm{GL}_n$, supercuspidal representations are indeed completely classified by irreducible Galois representations $\mathrm{Gal}(K) \to \mathrm{GL}_n$.

However, since representations naturally form a category, we might like a more categorical statement. A categorification of Galois representations is $\mathrm{QCoh}(\mathrm{Hom}(\mathrm{Gal}(K), G^\vee)/G^\vee)$. So one can ask if we have an equivalence of categories

$$G(K) - \mathrm{rep} \leftrightarrow \mathrm{QCoh}(\mathrm{Hom}(\mathrm{Gal}(K), G^\vee)/G^\vee.$$

Unfortunately there is no chance for this to be true, because the left side is too coarse. Work in progress of Genestier and V. Lafforgue, and Fargues and Scholze, studies an enrichment of the left hand side. The refinement of the left hand side is based on the idea that the category of the LHS is obtained as the "trace of Frobenius" of some 2-category. It is through the process of taking this trace that the coarseness appears.

We are going to *start* with the correct 2-categorical object. The basic object is "category equipped with an action of a group". We replace $G(K)$ by a "group ind-scheme", namely the *loop group* $\mathcal{L}G$ of G. Then (working over an algebraically closed ground field) we have $\mathcal{L}G(k) = G((t))$. Then the 2-category is "categories acted on by $\mathcal{L}G$". This notion will be reviewed more carefully later; for now we just want to give an overview.

2. Groups acting on categories

2.1. **Weak and strong actions.** For starters, let H be a (finite type) algebraic group. What do we mean by H acting on a category $\mathcal{C}$? The most naive notion would be that for all $h \in H$, you get a functor $h \colon C \to C$. But we want this in families since we are doing algebraic geometry. Then there is already a distinction between "weak" and "strong" actions.

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Notes by Tony Feng.

Example 2.1. If H acts on Y, then H acts on the category $\mathrm{QCoh}(Y)$. This is a "weak action".

Now we give an exampleof a "strong action".

Example 2.2. If H acts on Y, then H acts on the category $\mathrm{Dmod}(Y)$. This is a "strong action".

Example 2.3. The canonical action of H on $\mathfrak{h}$-mod is a strong action.

Example 2.4. If H acts on an associative algebra A, then H acts weakly on A-mod. To upgrade it to a strong action, you ask for the additional structure of a map $\mathfrak{h} \to A$ realizing the derivative of the H-action.

2.2. **Twisted actions.** We need the notion of a "twisted action". This begins with a central extension

$$0 \to k \to \widehat{\mathfrak{h}} \to \mathfrak{h} \to 0.$$

Moreover, you want an extension of the adjoint action of H on $\mathfrak{h}$ to an action on$\widehat{\mathfrak{h}}$. (Note that the action of $\mathfrak{h}$ automatically extends). We denote this extension by the symbol κ. There is a notion of "twisted strong action" of H on a category. Let me say a few words about what this means.

First we need to give a definition of a group action on a category. The category $\mathrm{QCoh}(H)$ has a monoidal structure. In fact it has two different monoidal structures. The action we consider now is "convolution", obtained by pushforward via

$$H \times H \to H.$$

There is a notion of an "action of a monoidal category".

Definition 2.5. A *weak action* of H on $\mathcal{C}$ is an action of $\mathrm{QCoh}(H)$ on $\mathcal{C}$.

Definition 2.6. A *strong action* of H on $\mathcal{C}$ is an action of $\mathrm{Dmod}(H)$ on $\mathcal{C}$.

It will be explained that κ leads to a notion of "twisted D-modules" $\mathrm{Dmod}_\kappa(H)$.

Definition 2.7. A *twisted strong action of H on $\mathcal{C}$* is an action of $\mathrm{Dmod}_\kappa(H)$ on $\mathcal{C}$ as a monoidal category.

The parameter κ is a "quantum parameter". Our whole perspective will fit into a quantum deformation. In the quantum formulation, Langlands duality becomes much more symmetric between G and $G^\vee$.

3. Actions of loop groups on categories

So far we have been discussing finite-dimensional algebraic groups. But actually we want to discuss the loop group, which is not such an object. Some technical issues need to be addressed, but they are not significant.

Let κ be an invariant symmetric bilinear form

$$\mathfrak{g} \otimes \mathfrak{g} \to k.$$

This gives rise to a central extension

$$0 \to k \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0$$

classified by the Lie bracket

$$[x_1 \otimes f_1, x_2 \otimes f_2] = ([x_1, x_2] \otimes f_1 f_2, \kappa(x_1, x_2) \cdot \mathrm{Res}(f_1 df_2)).$$

The totality of categories acted on by $\mathcal{L}G$ at level κ forms a $(\infty, 2)$-category $\mathcal{L}G - \mathrm{mod}_\kappa$. The word "quantum" in "quantum geometric Langlands" refers to a nondegeneracy condition on κ, which we will explain later.

We give some examples of objects in this category.

Example 3.1. The affine Grassmannian is $\mathrm{Gr}_G := \mathcal{L}G/\mathcal{L}^+G$. The datum of κ gives rise to a twisting on Gr_G. Then $\mathrm{Dmod}_\kappa(\mathrm{Gr}_G)$ is strongly acted on by $\mathcal{L}G$ at level κ.

The affine flag variety is $\mathrm{Fl}_G := \mathcal{L}G/I$, where I is the Iwahori subgroup. The datum of κ gives rise to a twisting on Gr_G. Then $\mathrm{Dmod}_\kappa(\mathrm{Gr}_G)$ is strongly acted on by $\mathcal{L}G$ at level κ.

In fact we can take $\mathcal{L}G/H$ for any subgroup H, and we will get an action of the loop group on its category of D-modules.

Example 3.2. We consider $\widehat{\mathfrak{g}}_\kappa - \mathrm{mod}$ (representations of $\mathfrak{g}_\kappa$ on which the center acts trivially).

Example 3.3. Here is an example which is important for the local-global interaction (but not for this workshop). Let X be a global curve. We consider $\mathrm{Bun}_G^{\mathrm{level}_x}$. Then $\mathrm{Dmod}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_x}) \in \mathcal{L}G - \mathrm{mod}_\kappa$.

4. Local quantum geometric Langlands

Note that there are two actions of $\mathcal{L}G$ on $\mathcal{L}G$, by multiplication on either side. We can view $\mathrm{Dmod}_\kappa(\mathcal{L}G)$ in $(\mathcal{L}G \times \mathcal{L}G) - \mathrm{mod}_{(\kappa,-\kappa)}$. We'll explain what we mean by $-\kappa$. There is actually a shift, as the "0" should be the "critical level" $\kappa_0 = -1/2\kappa_{\mathrm{killing}}$. With respect to this level the story is "untwisted" (you actually see the Galois group). Then $-\kappa$ means reflection with respect to κ_0.

From κ we get a bilinear form on the Cartan, by $\kappa|_\mathfrak{h} - \kappa_0|_\mathfrak{h}$. At some point we'll see the reason for this shift. Then the critical value gives the 0 form on $\mathfrak{h}$.

Definition 4.1. The situation is *quantum* when $\kappa|_\mathfrak{h} - \kappa_0|_\mathfrak{h}$ is nondegenerate.

With this definition in place, we can state what we mean by local geometric Langlands.

Definition 4.2. We say that κ is (*positive-rational*, *negative-rational*, or *irrational*) if for every simple factor of $\mathfrak{g}$, $\kappa = \kappa_{\mathrm{Killing}} \cdot c$ has $c + 1/2$ being a (positive-rational, negative-rational, or irrational) number.

The behavior of the Langlands correpondence is different depending on which of these three cases applies. When we write κ we are thinking positive-rational or irrational; for the negative rational case we write $-\kappa$.

We consider $\mathcal{L}G - \mathrm{mod}_\kappa$ and $\mathcal{L}G^\vee - \mathrm{mod}$. Then $\mathfrak{h}^\vee = \mathfrak{h}^*$. The invariant bilinear forms on $\mathfrak{g}$ are in bijection with W-invariant forms on $\mathfrak{h}$.

Conjecture 4.3 (Local Geometric Langlands)**.** *Assume that κ is positive rational or irrational. There is an equivalence*

$$\mathbb{L}_\kappa \colon \mathcal{L}G - \mathrm{mod}_\kappa \xrightarrow{\sim} \mathcal{L}G^\vee - \mathrm{mod}_{-\kappa^\vee}.$$

We will also formulate various expectations about this equivalence.

Example 4.4. Consider $\mathrm{Dmod}_\kappa(\mathrm{Gr}_G) \in \mathcal{L}G - \mathrm{mod}_\kappa$. What is its image under $\mathbb{L}_\kappa$? The expectation is that the answer is $\mathrm{Dmod}_{-\kappa}(\mathrm{Gr}_{G^\vee})$.

Example 4.5. Consider $\widehat{\mathfrak{g}}_\kappa - \mathrm{mod} \in \mathcal{L}G - \mathrm{mod}_\kappa$. The expectation is that

$$\mathbb{L}_\kappa(\widehat{\mathfrak{g}}_\kappa - \mathrm{mod}) = \mathrm{Whit}_{-\kappa^\vee}(\mathcal{L}G^\vee).$$

This is an instance of the philosophy: "Kac-Moody brane goes to Whittaker brane".

Example 4.6. Consider $\mathrm{Dmod}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_x}) \in \mathcal{L}G - \mathrm{mod}_\kappa$. The expectation is that

$$\mathbb{L}_\kappa(\mathrm{Dmod}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_x}) = \mathrm{Dmod}_{-\kappa^\vee}(\mathrm{Bun}_{G^\vee}^{\mathrm{level}_x}).$$

We can't even formulate these expectations as conjectures, without the functor $\mathbb{L}_\kappa$. We will try to formulate some more tangible conjectures, which are just about $(\infty, 1)$-categories (the equivalence in the main conjecture is about $(\infty, 2)$-categories).

Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{L}G - \mathrm{mod}_\kappa$. Suppose you have guessed $\mathcal{C}_1^\vee = \mathbb{L}_\kappa(\mathcal{C}_1)$ and $\mathcal{C}_2^\vee = \mathbb{L}_\kappa(\mathcal{C}_2)$. Consider $\mathrm{Funct}(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathrm{Funct}_{\mathcal{L}G^\vee}(\mathcal{C}_1^\vee, \mathcal{C}_2^\vee)$. These are only 1-categories, and they should be equivalent.

Remark 4.7. If κ is irrational, then we get a functor $\mathbb{L}_{-\kappa}$ in the opposite direction. Is it the inverse? No, as with the Fourier transform it is off by an involution - a Cartan involution.

Let $\mathcal{C} \in \mathcal{L}G - \mathrm{mod}_\kappa$ and $C^\vee = \mathbb{L}_\kappa(\mathcal{C}) \in \mathcal{L}G^\vee - \mathrm{mod}_{-\kappa^\vee}$. It is expected that

$$\mathrm{Whit}(\mathcal{C}) \cong \mathrm{KM}(\mathcal{C}^\vee) \qquad \text{and} \qquad \mathrm{KM}(\mathcal{C}) \cong \mathrm{Whit}(\mathcal{C}^\vee).$$

What does this mean? We define

$$\mathrm{Whit}(\mathcal{C}) := \mathcal{C}^{\mathcal{L}N, \chi}$$

(here $\mathcal{C}^{\mathcal{L}N,\chi}$ means invariants of $\mathcal{C}$ with respect to the action of $\mathcal{L}N, \chi$). We define

$$\mathrm{KM}(\mathcal{C}) := \mathrm{Funct}_{\mathcal{L}G}(\widehat{\mathfrak{g}} - \mathrm{mod}_\kappa, \mathcal{C})$$

or equivalently,

$$\mathrm{KM}(\mathcal{C}) := \mathcal{C}^{\mathcal{L}G, \mathrm{weak}}.$$

This already leads to a conjecture:

$$\mathrm{Whit}(\mathrm{Dmod}_\kappa(\mathrm{Gr}_G)) \cong \mathrm{KM}(\mathrm{Dmod}_{-\kappa^\vee}(\mathrm{Gr}_{\widehat{G}})).$$

This is known as the *fundamental local equivalence* (FLE), and will be the focus of the talks after the third day.